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Overconvergent Witt vectors

By *Christopher Davis* at Irvine, *Andreas Langer* at Exeter and *Thomas Zink* at Bielefeld

Abstract. Let A be a finitely generated algebra over a field K of characteristic $p > 0$. We introduce a subring $W^{\dagger}(A) \subset W(A)$, which we call the ring of overconvergent Witt vectors, and prove its basic properties. In a subsequent paper we use the results to define an overconvergent de Rham–Witt complex for smooth varieties over K whose hypercohomology is the rigid cohomology.

Introduction

The p-adic cohomology of algebraic varieties may be defined using rings of Witt vectors. Let X be a smooth quasi-projective scheme over a perfect field K of characteristic $p > 0$. The Witt vectors may be considered as a Zariski sheaf $W(\mathcal{O}_X)$. The de Rham–Witt complex of Deligne–Illusie is a complex of $W(\mathcal{O}_X)$-modules whose hypercohomology is the crystalline cohomology of X. In [2] we constructed a subcomplex of the de Rham–Witt complex whose hypercohomology is the rigid cohomology of X defined by Berthelot [1]. For this we put a growth condition on Witt vectors which is inspired by the work of Monsky and Washnitzer. Therefore we speak of overconvergent Witt vectors. In this paper we study the rings of overconvergent Witt vectors systematically and prove in particular all facts used in [2]. Similar growth conditions of Witt vectors were used by de Jong [5] in his work on homomorphisms of p-divisible groups and by Kedlaya [7] in his work on the Crew Conjecture. We describe the precise relation below.

Let A be a finitely generated algebra over a field K of characteristic p. Let $W(A)$ be the ring of Witt vectors with respect to p. We define a subring $W^{\dagger}(A) \subset W(A)$, which we call the ring of overconvergent Witt vectors. Let $A = K[T_1, \ldots, T_d]$ be the polynomial ring. We say that a Witt vector $(f_0, f_1, f_2, \ldots) \in W(A)$ is overconvergent if there is a real number $\varepsilon > 0$ and a real number C such that

$$m - \varepsilon p^{-m} \deg f_m \geqq C \quad \text{for all } m \geqq 0.$$

The overconvergent Witt vectors form a subring $W^{\dagger}(A) \subset W(A)$.

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There is a natural morphism from the ring of restricted power series

$$W(K)\{T_1, \dots, T_d\} \to W(A),$$

which maps T_i to its Teichmüller representative $[T_i]$.

The inverse image of $W^\dagger(A)$ is the set of those power series which converge in some neighborhood of the unit ball. This is the weak completion $A^\dagger$ of $W(K)[T_1, \dots, T_d]$ in the sense of Monsky and Washnitzer. We note that the bounded Witt vectors used by Lubkin [10] are different from the overconvergent Witt vectors.

If $A \to B$ is a surjection of finitely generated K-algebras, we obtain by definition a surjection of the rings of overconvergent Witt vectors

$$W^\dagger(A) \to W^\dagger(B).$$

A much deeper fact that we use in [2] concerns subalgebras. Let $A \subset B$ be two smooth K-algebras. Then

$$W^\dagger(A) = W(A) \cap W^\dagger(B)$$

(see Proposition 2.16). In particular, this implies that the functor $A \mapsto W^\dagger(A)$ is a sheaf for the smooth topology.

Further, we show (see Corollary 2.46): Let A be a finitely generated algebra over K. Let $B = A[T]/\big(f(T)\big)$ be a finite étale A-algebra, where $f(T) \in A[T]$ is a monic polynomial of degree n such that $f'(T)$ is a unit in B. We denote by t the residue class of T in B. Then $W^\dagger(B)$ is finite and étale over $W^\dagger(A)$, and the elements $1, [t], [t]^2, \dots, [t]^{n-1}$ form a basis of the $W^\dagger(A)$-module $W^\dagger(B)$.

Finally, we prove that $W^\dagger(A) \to A$ satisfies Hensel's lemma (see Proposition 2.30) if A is a finitely generated algebra over a perfect field K. The essential fact is that $W^\dagger(A)$ is a weakly complete algebra over $W(K)$ in the sense of Monsky–Washnitzer.

Now we decribe the relation with the work of de Jong and Kedlaya, who studied the slope filtration of isocrystals over overconvergent Witt vectors of perfect fields with a valuation. Let K be a finite extension of $\mathbb{Q}_p$. Let $O = O_K$ be the ring of integers, $\pi \in O$ be a prime element and $q = p^e$ be the number of elements in $O/(\pi)$.

Let R be an O-algebra. Using the Witt polynomials

$$\mathbf{w}_n = X_0^{q^n} + \pi X_1^{q^{n-1}} + \cdots + \pi^{n-1} X_{n-1}^q + \pi^n X_n,$$

one [3] defines the ring of ramified Witt vectors $W_O(R)$.

Assume that $\pi R = 0$ and we are given a valuation $v : R \to \mathbb{R} \cup \{\infty\}$. A Witt vector $(x_0, x_1, x_2, \dots) \in W_O(R)$ is called overconvergent if there are real numbers $\epsilon > 0$ and C such that

$$i + \epsilon v(x_i) \geqq C \quad \text{for } i \in \mathbb{Z}_{\geqq 0}.$$

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The overconvergent Witt vectors form a ring $W_O^{\dagger}(R)$. If R is a perfect field L of characteristic p, this ring is denoted by $W_{\mathrm{con}}(L,O)$ in Kedlaya [7]. If $K=\mathbb{Q}_p$ and R is a polynomial algebra over a field with its degree valuation, we obtain the ring $W^{\dagger}(R)$ described above. Since our aim is to lay the foundations for the overconvergent de Rham–Witt complex, we consider only the case $K=\mathbb{Q}_p$. But all our results hold for ramified Witt vectors as well.

1. Pseudovaluations

We set $\overline{\mathbb{R}}=\mathbb{R}\cup\{\infty\}\cup\{-\infty\}$ with its natural order.

Definition 1.1. Let A be an abelian group. An order function is a function

$$v : A \to \overline{\mathbb{R}}$$

such that $v(0)=\infty$ and such that for arbitrary $a,b\in A$:

$$v(a\pm b)\geqq \min\{v(a),v(b)\}.$$

An order function is the same thing as a descending filtration of A by subgroups F^rA indexed by $r\in\overline{\mathbb{R}}$, with the property $\bigcap\limits_{s<r} F^sA=F^rA$. The relation of both notions is $F^r(A)=\{a\in A\,|\,v(a)\geqq r\}$.

In particular, the above inequality is an equality if $v(a)\neq v(b)$. Moreover, we have $v(a)=v(-a)$.

Let $\phi : A\to B$ be a surjective homomorphism of abelian groups. Then we define the quotient $\bar{v} : B\to\overline{\mathbb{R}}$ by

$$\bar{v}(b)=\sup\{v(a)\,|\,a\in A,\ \phi(a)=b\}. \tag{1.2}$$

This is again an order function.

We define an order function v^n on the direct sum A^n as follows:

$$v^n((a_1,\dots,a_n))=\min_i\{v(a_i)\}. \tag{1.3}$$

Definition 1.4. Let A be a ring with 1. A pseudovaluation v on A is an order function on the additive group

$$v : A\to\overline{\mathbb{R}}$$

such that the following properties hold:

(1) $v(1)=0$,

(2) $v(ab)\geqq v(a)+v(b)$ if $v(a)\neq-\infty$ and $v(b)\neq-\infty$.

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We call v proper if it does not take the value $-\infty$. We call v negative if v is proper and $v(a) \leqq 0$ for all $a \in A$, $a \neq 0$. If v is proper and (2) is an equality, v is called a valuation. On each ring A we have the trivial valuation $v(a) = 0$ for $a \neq 0$.

Let $\phi : A \to B$ be a surjective ring homomorphism. Let v be a pseudovaluation on A. Let $\bar{v}$ the induced order function on B. If $\bar{v}(1) \neq \infty$, then $\bar{v}$ is a pseudovaluation. In particular, this is the case if v is negative.

Example 1.5. Let R be a ring with a negative pseudovaluation μ. Consider the polynomial ring $A = R[T_1, \ldots, T_m]$. Let $d_1 > 0, \ldots, d_m > 0$ be real numbers. Then we define a valuation on A as follows: For a polynomial

$$f = \sum_k c_k T_1^{k_1} \cdot \ldots \cdot T_m^{k_m}$$

we set

$$v(f) = \inf\{\mu(c_k) - k_1 d_1 - \cdots - k_m d_m\}. \tag{1.6}$$

This is a valuation if μ is a valuation. We often consider the case where R is an integral domain and μ is the trivial valuation. If moreover $d_i = 1$, we call v the standard degree valuation.

We are interested in pseudovaluations up to equivalence:

Definition 1.7. Let $v_1, v_2 : A \to \mathbb{R} \cup \{\infty\}$ be two functions such that $v_i \neq 0$ for all $a \in A$. We say that they are linearly equivalent if there are real numbers $c_1 > 0$, $c_2 > 0$, $d_1 \geqq 0$, $d_2 \geqq 0$ such that for all $a \in A$:

$$v_1(a) \geqq c_2 v_2(a) - d_2, \quad v_2(a) \geqq c_1 v_1(a) - d_1.$$

In Example 1.5, (1.6), we obtain for different choices of the numbers d_i linearly equivalent negative pseudovaluations. The equivalence class of v does not change if we replace μ by an equivalent negative pseudovaluation.

Let v_1 and v_2 be two negative pseudovaluations on A. If $A \to B$ is a surjective ring homomorphism, then the quotients $\bar{v}_1$ and $\overline{v_2}$ are again linearly equivalent.

Proposition 1.8. *Let μ be a negative pseudovaluation on a ring R. We consider a surjective ring homomorphism $\phi : R[T_1, \ldots, T_m] \to R[S_1, \ldots, S_n]$. Let v_T be a pseudovaluation on $R[T_1, \ldots, T_m]$ and let v_S be a pseudovaluation on $R[S_1, \ldots, S_n]$ as defined by* (1.6). *Then the quotient of v_T with respect to ϕ is a pseudovaluation which is linearly equivalent to the valuation v_S.*

We omit the straightforward proof, which is essentially contained in [13]. A proof in a more general context is given in [2].

Definition 1.9. Let μ be a negative pseudovaluation on a ring R. Let B be a finitely generated R-algebra. Choose an arbitrary surjection $R[T_1, \ldots, T_m] \to B$ and an arbitrary

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degree valuation v on $R[T_1, \ldots, T_m]$. Then the quotient $\bar{v}$ on B is up to linear equivalence independent of these choices. We call any negative pseudovaluation in this equivalence class admissible.

Let μ be an admissible pseudovaluation on a finitely generated R-algebra B. Let v be the pseudovaluation on a polynomial algebra $B[T_1, \ldots, T_m]$ given by Example 1.5. Then v is admissible. This is easily seen if we write B as a quotient of a polynomial algebra.

Lemma 1.10. *Let (R, μ) be a ring with a negative pseudovaluation. Let A be an R-algebra which is finite and free as an R-module. Let τ be an admissible pseudovaluation on A.*

Choose an R-module isomorphism $R^n \cong A$. With respect to this isomorphism τ is linearly equivalent to the order function μ^n given by (1.3).

Proof. Let $e_1, \ldots, e_n$ be a basis of A as an R-module. Consider the natural surjection

$$\alpha : R[T_1, \ldots, T_n] \to A$$

such that $\alpha(T_i) = e_i$. We have the equations

$$e_i e_j = \sum_{l=1}^{n} c_{ij}^{(l)} e_l, \quad c_{ij}^{(l)} \in R.$$

We choose a number d such that for all coefficients $c_{ij}^{(l)}$:

$$\mu(c_{ij}^{(l)}) + d \geqq 0.$$

Let $\tilde{\tau}$ be the pseudovaluation (1.6) on $R[T_1, \ldots, T_n]$ such that $\tilde{\tau}(X_i) = -d$. We can take for τ the quotient of $\tilde{\tau}$.

Consider an element $a \in A$. We choose a representative of a:

$$f = \sum_k r_k T_1^{k_1} \cdot \ldots \cdot T_n^{k_n}.$$

We claim that there is a linear polynomial f_1 which maps to a such that

$$\tilde{\tau}(f_1) \geqq \tilde{\tau}(f).$$

Indeed, assume that some of the monomials $r_k T_1^{k_1} \cdot \ldots \cdot T_n^{k_n}$ are divisible by $T_i T_j$. We will pretend in our notation that $i \neq j$, but the other case is the same. We find an equation

$$r_k T^k = \sum_l r_k c_{ij}^{(l)} T^{k(l)},$$

where $|k(l)| = |k| - 1$. For any fixed l we have

$$\tilde{\tau}(r_k c_{ij}^{(l)} T^{k(l)}) = \mu(r_k c_{ij}^{(l)}) - d(|k| - 1) \geqq \mu(a_k) + \mu(c_{ij}^{(l)}) + d - d|k| \geqq \tilde{\tau}(f).$$

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We conclude that

$$(1.11) \qquad \tau(a) = \sup\{\tilde{\tau}(f) \mid f = r_0 + r_1 T_1 + \cdots + r_n T_n,\ \alpha(f) = b\}.$$

By construction, a has a unique representative

$$g = \sum_{i=1}^{n} s_i T_i.$$

Clearly, $\tilde{\tau}$ restricted to linear forms as above is linearly equivalent to the order function μ^n defined by (1.3). We need to compare $\tilde{\tau}(g)$ and $\tau(a)$.

We have the following relation in A:

$$1 = \sum_{i=1}^{n} c_i e_i, \quad c_i \in R.$$

Given a representative f as in (1.11), we find

$$g = \sum_{i=1}^{n} (r_i + c_i r_0) T_i.$$

Then

$$\begin{aligned} \tilde{\tau}(g) = \min\{\mu(r_i + c_i r_0) - d\} &\geqq \min_i\{\min\{\mu(r_i) - d, \mu(c_i) - d + \mu(r_0)\}\} \\ &\geqq \tilde{\tau}(f) - d', \end{aligned}$$

where d' is chosen such that $-d' < \mu(c_i) - d$. Since this is true for arbitrary f, we find $\tilde{\tau}(g) \geqq \tau(b) - d'$. Since τ is the quotient norm, we have the obvious inequality

$$\tau(b) \geqq \tilde{\tau}(g).$$

This completes the proof. $\square$

Example 1.12. Let v be a negative pseudovaluation on A. Let $d > 0$ be a real number. Then we have defined a pseudovaluation on the polynomial algebra $A[X]$:

$$(1.13) \qquad \mu\left(\sum a_i X^i\right) = \min\{v(a_i) - id\}.$$

Let $f \in A$ be such that f is not nilpotent. Then we define a pseudovaluation v' on the localization A_f by taking the quotient under the map

$$A[X] \to A_f$$

which sends X to f^{-1}. As we remarked above, v' depends only on the linear equivalence class of v on A.

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Let $z \in A_f$. Consider all possible representations of z in the form

$$z = \sum_l a_l/f^l. \tag{1.14}$$

Then v' is the supremum over all these representations of the following numbers:

$$\min_l \{v(a_l) - ld\}. \tag{1.15}$$

If the supremum is assumed, we call the representation optimal.

Lemma 1.16. *Let (A, v) be a ring with a negative pseudovaluation. Let $f \in A$ be a non-zero divisor. Let v' be the induced pseudovaluation on A_f which is associated to a fixed number $d > 0$.*

We are going to define a function $\tau : A_f \to \mathbb{R} \cup \{\infty\}$. For $z \in A_f$ we consider the set of all possible representations

$$z = a/(f^m). \tag{1.17}$$

We define $\tau(z)$ to be the maximum of the numbers $v(a) - md$ for all possible representations (1.17).

Then there is a real constant $Q > 0$ such that for all $z \in A_f$

$$v'(z) \geqq \tau(z) \geqq Qv'(z).$$

Proof. The first of the asserted inequalities is trivial. Consider any representation

$$z = \sum_{l=0}^{m} u_l/(f^l) \quad \text{such that } u_m \neq 0.$$

We set

$$-C = \min_l \{v(u_l) - ld\}.$$

Note that this implies that $-C \leqq -md$. We find a representation of the form (1.17):

$$z = \left(\sum_{l=0}^{m} u_l f^{m-l}\right)/(f^m) = a/f^m.$$

Then

$$\begin{aligned} v(a) - md = v\left(\sum_{l=0}^{m} u_l f^{m-l}\right) - md &\geqq \min_l \{v(u_l) + (m-l)v(f) - md\} \\ &\geqq \min_l \{v(u_l) - ld - md + mv(f) + l(d - v(f))\} \\ &\geqq \min_l \{-C - C + mv(f)\}. \end{aligned}$$

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We further have

$$mv(f) = (-dm)\frac{v(f)}{-d} \geqq (-C)\frac{v(f)}{-d}.$$

Together we obtain

$$v(a) - md \geqq -C\left(2 + \frac{v(f)}{-d}\right).$$

This implies

$$\tau(a/f^m) \geqq \left(2 + \frac{v(f)}{-d}\right)v'(z). \quad \square$$

The motivation for the following definition is Lemma 2.14 below.

Definition 1.18. Let (A, v) be a ring with a negative pseudovaluation. We say that a non-zero divisor $f \in A$ is localizing with respect to v if there are real numbers $C > 0$ and $D \geqq 0$ such that for all natural numbers n:

$$v(f^n x) \leqq Cv(x) + nD \quad \text{for all } x \in A. \tag{1.19}$$

If μ is a negative pseudovaluation on A which is linearly equivalent to v, then f is localizing with respect to v if and only if it is localizing with respect to μ. Indeed, any function ρ which is linearly equivalent satisfies an inequality (1.19).

It is helpful to remark that making C smaller we may always arrange that D is smaller than any given positive number. It is also easy to see that a unit of the ring A is always localizing.

Let $A = R[T_1, \dots, T_e]$ be a polynomial ring over an integral domain R with a degree valuation v. Then we have the equation

$$v(f^n x) = v(x) + nv(f), \quad x \in A.$$

Therefore (1.19) holds with $C = 1$ and $D = 0$.

More generally, let (B, μ) be a ring with a negative pseudovaluation. We endow $A = B[T]$ with the natural extension v of μ such that $v(T) = -d$. Assume that $f = T^m + a_{m-1}T^{m-1} + \dots + a_0$ is a monic polynomial with $a_i \in B$.

Definition 1.20. We say that f is regular with respect to T if

$$\min_{0 \leqq i < m} \{\mu(a_i) - id\} > -md. \tag{1.21}$$

For a regular polynomial we have $v(f) = -md$. We remark that each monic polynomial f becomes regular for a suitable choice of d.

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Proposition 1.22. *Let $f(T) \in B[T]$ be a regular polynomial* (1.20). *Then we have for an arbitrary polynomial $g(T) \in B[T]$ that*

$$\nu\big(f(T)g(T)\big) = \nu\big(f(T)\big) + \nu\big(g(T)\big).$$

In particular, any monic polynomial in $B[T]$ is localizing.

Proof. We write

$$g = \sum_{k=0}^{n} b_k T^k.$$

Let k_0 be the largest index such that $\nu(g) = \nu(b_{k_0}) - k_0 d$. Note that fg contains the monomial

$$(b_{k_0} + b_{k_0+1}a_{m-1} + \cdots)T^{m+k_0}.$$

We find by (1.21) that

$$\mu(b_{k_0+i}a_{m-i}) \geqq \mu(b_{k_0}) + \mu(a_{m-i}) \geqq \mu(b_{k_0}) - id.$$

On the other hand, we have by the choice of k_0 that

$$\mu(b_{k_0}) - k_0 d < \mu(b_{k_0+i}) - (i + k_0)d.$$

This proves that $\mu(b_{k_0+i}a_{m-i}) > \mu(b_{k_0})$. Therefore we obtain

$$\mu(b_{k_0} + b_{k_0+1}a_{m-1} + \cdots) = \mu(b_{k_0}).$$

This shows the inequality

$$\nu(fg) \leqq \nu(b_{k_0}) - d(m + k_0) = \nu(g) - md = \nu(g) + \nu(f).$$

The opposite inequality is obvious. The last assertion follows because any monic polynomial is regular for a suitable chosen d. $\square$

Proposition 1.23. *Assume that $f = T^m + a_{m-1}T^{m-1} + \cdots + a_0 \in B[T]$ is a polynomial which is regular with respect to T. Each $z \in A_f$ has a unique representation*:

$$z = \sum_l u_l/f^l, \quad u_l \in B[T], \tag{1.24}$$

where u_l is for $l > 0$ a polynomial of degree strictly less than $m = \deg f$. Then the representation (1.24) *is optimal* (*compare* (1.15)).

Proof. The first assertion follows from the Euclidean division. Consider any other representation

$$z = \sum_i v_i/f^i, \quad v_i \in B[T].$$

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Assume that $n = \deg v_i \geqq m$ for some $i > 0$. Let $c \in B$ be the highest coefficient of the polynomial v_i and set $t = n - m$. Then we conclude

$$v(cT^t f) \geqq \mu(c) + v(T^t) + v(f) = \mu(c) + v(T^t) + v(T^m) = v(cT^n) \geqq v(v_i).$$

We write

$$v_i/f^i = \big((v_i - cT^t f)/f^i\big) - (cT^t/f^{i-1}).$$

If we insert this into the representation (1.14), the number (1.15) becomes bigger because

$$v(v_i - cT^t f) \geqq v(v_i), \quad v(cT^t) \geqq v(cT^n) \geqq v(v_i).$$

Continuing this process proves the lemma. □

The statement of the last proposition applies in particular to a polynomial ring over a field $A = K[T_1, \dots, T_e]$ with the standard degree valuation. By Noether normalization, any polynomial becomes regular with respect to some variable after a coordinate change.

Proposition 1.25. *Let (A, v) be a ring with a pseudovaluation. Let $f, g \in A$. Then fg is localizing if and only if f and g are localizing.*

Proof. Assume fg is localizing. Then we find the inequality

$$v(f^n g^n x) \leqq Cv(x) + nD.$$

On the other hand, we have the inequality

$$nv(f) + v(g^n x) \leqq v(f^n g^n x).$$

This shows that

$$v(g^n x) \leqq Cv(x) + n\big(D - v(f)\big).$$

We leave the opposite implication to the reader. □

Proposition 1.26. *Let (A, v) be a ring with a pseudovaluation. Assume that A is an integral domain such that each non-zero element of A is localizing. Let $A \to B$ be a finite ring homomorphism such that B is a free A-module. Let μ be an admissible pseudovaluation on B. Then any non-zero divisor in B is localizing with respect to μ.*

Proof. We choose an isomorphism of A-modules: $A^r \cong B$. By Lemma 1.10, the order function v^r on A^r is linearly equivalent to an admissible pseudovaluation on B. Let $f \in A$, $f \neq 0$. Then inequality (1.19) holds. It follows that for each $z \in A^r$:

$$v^r(f^n z) \leqq Cv^r(z) + nD.$$

This shows that f is localizing in B. More generally, consider a non-zero divisor $b \in B$, and an equation of minimal degree:

$$b^t + a_{t-1}b^{t-1} + \cdots + a_1 b + a_0 = 0, \quad a_i \in A.$$

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Then $a_0 \neq 0$ and therefore a_0 is localizing. But a_0 is a multiple of b in the ring B. Therefore b is localizing in B by Proposition 1.25. $\square$

Corollary 1.27. *Let $X \to \operatorname{Spec} K$ be a smooth scheme over a field K of characteristic p. Then any point of X has an affine neighborhood* $\operatorname{Spec} A$ *such that any non-zero element in A is localizing.*

Proof. This is immediate from a result of [6] which says that each point admits a neighborhood which is finite and étale over an affine space $\mathbb{A}^n_K$. $\square$

Let us assume that $f \in A$ is localizing with constants C, D given by (1.19). Then we will assume that the constant d used in the definition of v' on A_f is bigger than D. This can be done without loss of generality because the equivalence class of v' does not depend on d.

Proposition 1.28. *Let (A, v) be a ring with a negative pseudovaluation. Let $f \in A$ be a localizing element. Each $z \in A_f$ has a unique representation*

$$z = a/f^m, \quad \text{where } a \in A,\ f \nmid a.$$

We define a real valued function σ on A_f by

$$\sigma(z) = v(a) - md.$$

Then there exists a real constant $E > 0$ such that

$$v'(z) \geqq \sigma(z) \geqq Ev'(z).$$

In particular, the restriction of v' to A is linearly equivalent to v.

Proof. By Lemma 1.16, it suffices to show the last inequality with v' replaced by τ. All representations (1.17) of z are of the form

$$af^r/f^{m+r}.$$

Since f is localizing, there are real numbers $1 > C > 0$ and $D \geqq 0$ such that

$$\begin{aligned} v(af^r) - (m+r)d &\leqq Cv(a) + rD - md - rd \\ &\leqq C\big(v(a) - md\big) + (D-d)r \leqq C\sigma(z) + (D-d)r. \end{aligned}$$

We may assume that $d \geqq D$. Then the inequality above implies

$$\tau(z) \leqq C\sigma(z). \quad \square$$

Corollary 1.29. *Let (B, μ) be an integral domain with a pseudovaluation μ. Assume that each non-zero element is localizing. We endow $B[T]$ with a pseudovaluation of Example* 1.5.

Then each non-zero element in $B[T]$ is localizing.

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Proof. Clearly, each $b \in B$, $b \neq 0$, is localizing in $B[T]$. By the proposition, it suffices to find for a given $f \in B[T]$ an element $b \in B$ such that f is localizing in $B_b[T]$. By the remark preceding Proposition 1.22, we may assume that f is a regular polynomial. Then we can apply this proposition. □

The following corollary would allow us to prove Corollary 1.27 more generally by considering standard étale neighborhoods instead of Kedlaya's result.

Corollary 1.30. *Let (A, v) be a noetherian ring with a negative pseudovaluation. Let $a, f \in A$ be two localizing elements. Then a is localizing in A_f.*

Proof. By the Lemma of Artin–Rees, there is a natural number r such that for $m \geqq r$

$$ax \in f^m A \quad \text{implies} \quad x \in f^{m-r} A.$$

Assume that $x \in A$, but $x \notin fA$. Then we conclude that for each $n \in \mathbb{N}$

$$a^n x \in f^m A \quad \text{implies} \quad m \leqq nr.$$

Consider a reduced fraction $x/f^m \in A_f$. To show that a is localizing, it suffices to find an estimation for

$$\sigma\big(a^n(x/f^m)\big),$$

where σ is the function of Proposition 1.28:

$$\sigma(x/f^m) = v(x) - md.$$

By the remarks above, we may write with $y \notin fA$:

$$\frac{a^n x}{f^m} = \frac{y f^s}{f^m}, \quad s \leqq nr.$$

Using this equation, we obtain

$$\begin{aligned} v(y) &\leqq v(yf^s) - v(f^s) \leqq v(a^n x) - sv(f) \\ &\leqq Cv(x) + nD - nrv(f). \end{aligned}$$

Here $C \leqq 1$, D are positive real constants, which exist because a is localizing in A.

Now it is easy to give an estimation for

$$\sigma\big(a^n(x/f^m)\big) = \sigma(yf^s/f^m).$$

We omit the details. □

We reformulate Proposition 1.28 in the case where $A = R[T_1, \dots, T_e]$ is a polynomial algebra over an integral domain R with the standard degree valuation v. It extends to a val-

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uation on the quotient field of A, which we denote by v too. Let $f \in A$ be a non-zero element. We define v' on A_f associated to $d > 0$ as before (cf. (1.15)).

We define a second pseudovaluation μ on the ring A_f as follows. Let $\vartheta(z)$ be the smallest integer $n \geqq 0$ such that $f^n z \in A$. We set

$$\mu(z) = \min\{v(z), -d\vartheta(z)\}. \tag{1.31}$$

Proposition 1.32. *Let A be a polynomial ring with the standard degree valuation v. Let $f \in A$ be a non-constant polynomial. Let us define pseudovaluations v' resp. μ on A_f by the formulas* (1.15) *resp.* (1.31)*. Then there are constants Q_1 and Q_2 such that*

$$Q_1\mu \geqq v' \geqq Q_2\mu.$$

Proof. We write an element $z \in A_f$ as a reduced fraction

$$z = (a/f^m)$$

such that $m = \vartheta(z)$. By Proposition 1.28, it is enough to compare μ with the function σ. The inequality $\sigma(z) \leqq \mu(z)$ is obvious. We show that for a sufficiently big number $C > 1$:

$$C\mu(z) \leqq v(a) - md.$$

This is obvious if $-Cmd \leqq -md + v(a)$. Therefore we can make the assumption

$$-(C-1)md \geqq v(a).$$

We have to find C such that the following inequality is satisfied:

$$C\bigl(v(a) - mv(f)\bigr) \leqq v(a) - md.$$

By assumption, we have

$$(C-1)v(a) \leqq -(C-1)^2 md.$$

Therefore it suffices to show that for big C:

$$-(C-1)^2 md \leqq m\bigl(Cv(f) - d\bigr).$$

But this is obvious. □

2. Overconvergent Witt vectors

Let us fix a prime number p. We are going to introduce the ring of overconvergent Witt vectors. Let A be a ring with a proper pseudovaluation v. We assume that $pA = 0$.

Let $W(A)$ be the ring of Witt vectors. For any Witt vector

$$\alpha = (a_0, a_1, a_2, \ldots) \in W(A)$$

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we consider the following set $\mathcal{T}(\alpha)$ in the x-y-plane:

$$\left(p^{-i}v(a_i), i\right), \quad v(a_i) \neq \infty.$$

For $\varepsilon, c \in \mathbb{R}$, $\varepsilon > 0$, we define the half plane

$$H_{\varepsilon,c} = \{(x, y) \in \mathbb{R}^2 \mid y \geqq -\varepsilon x + c\}.$$

Moreover, we consider for all $c \in \mathbb{R}$ the half plane

$$H_c = \{(x, y) \in \mathbb{R}^2 \mid x \geqq c\}.$$

Let $\mathcal{H}$ the set of all half planes of the two different types above. We define the Newton polygon $\mathrm{NP}(\alpha)$ as follows:

$$\mathrm{NP}(\alpha) = \bigcap_{H \in \mathcal{H}, \mathcal{T}(\alpha) \in H} H.$$

Definition 2.1. We say that a Witt vector α has radius of convergence $\varepsilon > 0$ if there is a constant $c \in \mathbb{R}$ such that

$$i \geqq -\varepsilon p^{-i} v(a_i) + c.$$

We denote the set of these Witt vectors by $W^{\varepsilon}(A)$.

Equivalently, one may say that the Newton polygon $\mathrm{NP}(\alpha)$ lies above a line of slope $-\varepsilon$.

We define the Gauss norm $\gamma_\varepsilon : W(A) \to \mathbb{R}$ by

$$\gamma_\varepsilon(\alpha) = \inf\{i + \varepsilon p^{-i} v(a_i)\}. \tag{2.2}$$

Convergence of radius $\varepsilon > 0$ means that $\gamma_\varepsilon(\alpha) \neq -\infty$. We will denote the set of Witt vectors of radius of convergence ε by $W^{\varepsilon}(A)$.

Proposition 2.3. *Let (A, v) be a ring with a proper pseudovaluation such that $pA = 0$. Then for any $\varepsilon > 0$ the Gauss norm γ_ε is a pseudovaluation on $W(A)$. In particular, $W^{\varepsilon}(A)$ is a ring.*

If we moreover assume that v is a valuation, we have the following equality for arbitrary $\xi, \eta \in W^{\varepsilon}(A)$:

$$\gamma_\varepsilon(\xi\eta) = \gamma_\varepsilon(\xi) + \gamma_\varepsilon(\eta). \tag{2.4}$$

Proof. Clearly, we may assume $\varepsilon = 1$. We set $\gamma = \gamma_1$. The first two requirements of Definition 1.4 are clear. Consider two Witt vectors

$$\xi = (a_0, a_1, \ldots) \in W(A), \quad \eta = (b_0, b_1, \ldots) \in W(A).$$

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We begin by showing the inequality

$$\gamma(\xi+\eta) \geqq \min\{\gamma(\xi), \gamma(\eta)\}.$$

To this end, we may assume that there is $g \in \mathbb{R}$ such that

$$i + p^{-i}v(a_i) \geqq g, \quad i + p^{-i}v(b_i) \geqq g.$$

We write

$$\xi + \eta = (s_0, s_1, \ldots).$$

Let S_m be the polynomials which define the addition of the Witt vectors:

$$s_m = S_m(a_0, \ldots, a_m, b_0, \ldots, b_m).$$

We know that S_m is a sum of monomials

$$M = \pm a_0^{e_0} \cdot \ldots \cdot a_m^{e_m} b_0^{f_0} \cdot \ldots \cdot b_m^{f_m}$$

such that

$$\sum_{i=0}^{m} p^i e_i + \sum_{i=0}^{m} p^i f_i = p^m.$$

We have to show that $p^{-m}v(m) + m \geqq g$. For this, we compute

$$\begin{aligned}
p^{-m}v(M) + m &\geqq p^{-m}\left(\sum_{i=0}^{m} e_i v(a_i) + \sum_{i=0}^{m} f_i v(b_i) + \sum_{i=0}^{m} p^i e_i m + \sum_{i=0}^{m} p^i f_i m\right) \\
&\geqq p^{-m}\left(\sum_{i=0}^{m} p^i e_i\big(p^{-i}v(a_i) + i\big) + \sum_{i=0}^{m} p^i f_i\big(p^{-i}v(b_i) + i\big)\right) \\
&\geqq p^{-m}\left(\sum_{i=0}^{m} p^i e_i g + \sum_{i=0}^{m} p^i f_i g\right) \geqq g.
\end{aligned}$$

This proves the fourth requirement of Definition 1.4.

Next we prove the inequality

$$\gamma(\xi\eta) \geqq \gamma(\xi) + \gamma(\eta). \tag{2.5}$$

By the inequality already shown, we are reduced to the case

$$\xi = {}^{V^i}[a], \quad \text{and} \quad \eta = {}^{V^j}[b].$$

Since, by assumption, F and V commute on $W(A)$, we find

$$\xi\eta = {}^{V^{i+j}}[a^{p^j} b^{p^i}].$$

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We obtain

$$(2.6)\qquad \gamma(\xi\eta)=\frac{v(a^{p^j}b^{p^i})}{p^{i+j}}+i+j\geqq\frac{p^jv(a)+p^iv(b)}{p^{i+j}}+i+j=\gamma(\xi)+\gamma(\eta).$$

This proves that γ is a pseudovaluation.

Finally, we prove equality (2.4) if v is a valuation. We remark that (2.6) is an equality in this case. From this we obtain (2.4) in the case where

$$\xi={}^{V^i}[a]+\xi_1,\quad \eta={}^{V^j}[b]+\eta_2,$$

where $\xi_1\in V^{i+1}W(A)$, $\eta\in V^{j+1}W(A)$ and

$$\gamma(\xi_1)\geqq\gamma({}^{V^i}[a]),\quad \gamma(\eta_2)\geqq\gamma({}^{V^j}[b]).$$

Next we consider the case that there are i and j such that

$$p^{-i}v(a_i)+i=\gamma(\xi),\quad p^{-j}v(b_j)+j=\gamma(\eta).$$

We assume that i and j are minimal with this property. Then we write

$$\begin{aligned}\xi&=(a_0,\dots,a_{i-1},0,\dots)+\xi_1=\xi'+\xi_1,\\ \eta&=(b_0,\dots,b_{j-1},0,\dots)+\eta_1=\eta'+\eta_1.\end{aligned}$$

We have by our choice

$$\gamma(\xi')>\gamma(\xi)=\gamma(\xi_1),\quad \gamma(\eta')>\gamma(\eta)=\gamma(\eta_1).$$

By the case already treated, we have $\gamma(\xi_1\eta_1)=\gamma(\xi_1)+\gamma(\eta_1)$. Then we obtain

$$(2.7)\qquad \begin{aligned}\gamma(\xi\eta)&=\gamma(\xi_1\eta_1+\xi_1\eta'+\xi'\eta_1+\xi'\eta')\\ &\geqq\min\{\gamma(\xi_1\eta_1)+\gamma(\xi_1\eta')+\gamma(\xi'\eta_1)+\gamma(\xi'\eta')\}.\end{aligned}$$

But by inequality (2.5), this minimum is assumed only for $\gamma(\xi_1\eta_1)$ and therefore (2.7) is an equality.

Finally, if i and j as above do not exist, this becomes true if we replace ε by any δ which is a little smaller. If δ approaches ε, we obtain the desired result. $\square$

We have the formulas

$$(2.8)\qquad \begin{aligned}\gamma_\varepsilon({}^V\alpha)&=1+\gamma_{\varepsilon/p}(\alpha),\\ \gamma_\varepsilon({}^F\alpha)&\geqq\gamma_{p\varepsilon}(\alpha),\\ \gamma_\varepsilon(p)&=1.\end{aligned}$$

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Definition 2.9. The union of the rings $W^{\varepsilon}(A)$ for $\varepsilon > 0$ is called the ring of overconvergent Witt vectors $W^{\dagger}(A)$.

Corollary 2.10. *Let $\alpha \in W^{\dagger}(A)$ and let $\delta > 0$ be a real number. Then there is an $\varepsilon > 0$ such that $\gamma_{\varepsilon}(\alpha) > -\delta$.*

Proof. Take some negative line of slope $-\tau$ below the Newton polygon of α. If this line does not meet the negative x-axis, we conclude that $\gamma_{\tau}(\alpha) \geqq 0$. In the other case, we rotate the line around the intersection point to obtain the desired slope $-\varepsilon$. $\square$

We will from now on assume that the pseudovaluation v on A is negative. Then we have

$$W^{\delta}(A) \subset W^{\varepsilon}(A) \quad \text{if } \delta > \varepsilon.$$

By Proposition 2.3, this is a subring.

The ring $W^{\dagger}(A)$ does not change if we replace v by a linearly equivalent pseudovaluation. More generally, let $f : A \to \mathbb{R} \cup \{\infty\}$ be any function which is linearly equivalent to v. Then a Witt vector $(x_0, x_1, \ldots) \in W(A)$ is overconvergent with respect to the v if and only if there is an $\varepsilon > 0$ and a constant $C \in \mathbb{R}$ such that for all $i \geqq 0$:

$$i + p^{i} f(x_i)\varepsilon \geqq -C.$$

With the notation of Definition 1.9 let A be a finitely generated algebra over (R, μ). Any admissible pseudovaluation on A leads us to the same ring $W^{\dagger}(A)$. Let $\alpha : A \to B$ be a homomorphism of finitely generated R-algebras. Then the induced homomorphism on the rings of Witt vectors respects overconvergent Witt vectors:

$$W(\alpha) : W^{\dagger}(A) \to W^{\dagger}(B). \tag{2.11}$$

This is seen by choosing a diagram

$$\begin{array}{ccc} R[T_1, \ldots, T_n] & \longrightarrow & R[T_1, \ldots, T_n, S_1, \ldots, S_m] \\ \downarrow & & \downarrow \\ A & \longrightarrow & B. \end{array}$$

On the truncated Witt vectors we consider the functions $\gamma_{\varepsilon}[n]$ given by

$$\gamma_{\varepsilon}[n] : W_{n+1}(A) \to \mathbb{R} \cup \{\infty\},$$
$$\gamma_{\varepsilon}[n](\alpha) = \min\{i + \varepsilon p^{-i} v(a_i) \mid i \leqq n\}.$$

This is the quotient of γ_{ε} under the natural map $W(A) \to W_{n+1}(A)$ in the sense of (1.2). We conclude that $\gamma_{\varepsilon}[n]$ is a proper pseudovaluation.

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The following is obvious: Let $\sum_{m=0}^{\infty} \alpha_m$ be an infinite sum of Witt vectors $\alpha_m \in W(A)$ which converges in the V-adic topology to $\sigma \in A$. Let $\varepsilon > 0$ and $C \in \mathbb{R}$ be such that

$$\gamma_\varepsilon(\alpha_m) \geqq C.$$

Then σ is overconvergent, and we have $\gamma_\varepsilon(\sigma) \geqq C$.

More generally, we can consider families of pseudovaluations $\delta_\varepsilon[n]$ of $W(A)$ which are indexed by real numbers $\varepsilon > 0$ and $n \in \mathbb{N} \cup \{\infty\}$. Write $\delta_\varepsilon = \delta_\varepsilon[\infty]$. We require that

$$\delta_{\varepsilon_1}[n] \geqq \delta_{\varepsilon_2}[n], \quad \varepsilon_1 \leqq \varepsilon_2,$$
$$\delta_\varepsilon[n] \geqq \delta_{\varepsilon_2}[m], \quad n \leqq m.$$

Definition 2.12. Two families $\delta_\varepsilon[n]$ and $\delta'_\varepsilon[n]$ given as above are called equivalent if there are constants $c_1, c_2, d_1, d_2 \in \mathbb{R}$, where $c_1 > 0$, $c_2 > 0$, such that for sufficiently small ε the following inequalities hold:

$$\delta_{c_1\varepsilon}[n] \geqq \delta'_\varepsilon[n] - d_1,$$
$$\delta'_{c_2\varepsilon}[n] \geqq \delta_\varepsilon[n] - d_2.$$

Let v and v' be negative pseudovaluations on A which are linearly equivalent. Then the families γ_ε and γ'_ε of Gauss norms defined by (2.2) are equivalent.

We obtain from Lemma 1.10 the following:

Proposition 2.13. *Let (R, μ) be a ring with a negative pseudovaluation. Let A be an R-algebra which is free as an R-module. Let τ be an admissible pseudovaluation on A given by Proposition* 1.8.

We transport μ^n to A by an isomorphism $R^n \cong A$. Then a Witt vector $(a_0, a_1, \ldots) \in W(A)$ is overconvergent with respect to τ if and only if there is an $\varepsilon > 0$ and a constant $C \in \mathbb{R}$ such that

$$i + p^{-i}\mu^n(a_i) \geqq -C.$$

In particular, a Witt vector $\underline{r} = (r_0, r_1, \ldots) \in W(R)$ is overconvergent if and only if its image in $W(A)$ is overconvergent.

Proof. Only the last sentence needs a justification. Assume $\underline{r}$ is overconvergent in A. By the first part of the proposition, this means the following:

Let e_i be a basis of the R-module A. We write

$$1 = \sum_m c_m e_m, \quad c_m \in R.$$

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Then overconvergence means that there are constants $\varepsilon > 0$ and $C \in \mathbb{R}$ such that for $1 \leqq m \leqq n$ and $i \geqq 0$:

$$i + p^{-i}\mu(c_m r_i)\varepsilon \geqq C.$$

By Cohen–Seidenberg, it is clear that c_m generate the unit ideal in R:

$$1 = \sum_m c_m u_m.$$

This gives

$$\mu(r_i) \geqq \min\{\mu(c_m r_i) + \mu(u_i)\} \geqq \min\{\mu(c_m r_i)\} - C'$$

for some constant C' which depends only on the elements u_m. Therefore we see that $\underline{r} \in W^\dagger(R)$. We leave the proof of inclusion $W^\dagger(R) \subset W^\dagger(A)$ to the reader. $\square$

Lemma 2.14. *Assume that $f \in A$ is localizing. Let $\underline{c} \in W(A)$ be a Witt vector such that $\underline{c} \in W^\dagger(A_f)$. Then $\underline{c} \in W^\dagger(A)$.*

Proof. We write $\underline{c} = (c_0, c_1, c_2, \ldots)$, where $c_i \in A$. By Lemma 1.16, we find representations $c_i = a_i/f^{m_i}$ and real numbers $\varepsilon > 0$ and U such that

$$i + p^{-i}\varepsilon\big(v(a_i) - m_i d\big) \geqq -U.$$

Since f is localizing, we obtain

$$v(a_i) = v(f^{m_i} c_i) \leqq Cv(c_i) + m_i D,$$

and therefore

$$-U \leqq i + p^{-i}\varepsilon\big(Cv(c_i) + m_i(D - d)\big) = i + p^{-i}\varepsilon Cv(c_i) + p^{-i}\varepsilon m_i(D - d).$$

By our choice $D < d$, the last summand is not positive. This shows that $\underline{c} \in W^\dagger(A)$. $\square$

Proposition 2.15. *Let (A, v) be an integral domain with a negative pseudovaluation such that any non-zero element is localizing. Let $\alpha : A \to B$ be an injective ring homomorphism of finite type which is generically finite. Then we have*

$$W(A) \cap W^\dagger(B) = W^\dagger(A).$$

Proof. Indeed, we find an element $c \in A$, $c \neq 0$, such that $A_c \to B_c$ is finite, and B_c is a free A_c-module. Clearly, it suffices to show the proposition if we replace B by B_c. We consider the maps $A \to A_c \to B_c$ and apply the last lemma and Proposition 2.13. $\square$

Proposition 2.16. *Let $A \to B$ be a smooth morphism of finitely generated algebras over a field K of characteristic p. We endow them with admissible pseudovaluations. Then we have*

$$W(A) \cap W^\dagger(B) = W^\dagger(A).$$

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Proof. By Proposition 3.2, $W^{\dagger}$ is a sheaf in the Zariski topology. Therefore the question is local on $\operatorname{Spec} A$. We therefore may assume by Corollary 1.29 that any non-zero element of A is localizing. Obviously, the question is local on $\operatorname{Spec} B$. By the definition of smoothness, we may therefore assume that the morphism factors

$$A \to A[T_1, \dots, T_d] \to B,$$

where the last arrow is étale and in particular generically finite. We show the proposition for both arrows separately.

We know by the remark after Definition 1.9 that there is an admissible pseudovaluation on $A[T_1, \dots, T_d]$ whose restriction to A is an admissible pseudovaluation. This shows the assertion for the first arrow.

For the second arrow we use Proposition 2.15. It is enough to show that any element in $C = A[T_1, \dots, T_d]$ is localizing. But this is Corollary 1.29. $\square$

Let R be an integral domain and endow it with the trivial valuation. Consider on the polynomial ring $A = R[T_1, \dots, T_d]$ a degree valuation v such that $v(T_i) = -\delta_i < 0$. Let γ_ε be the associated Gauss norms on $W(A)$. In the following we need the dependence on δ. Therefore we set

$$\gamma^{(\delta)} = \gamma_1, \quad \text{and then } \gamma^{(\varepsilon\delta)} = \gamma_\varepsilon.$$

Let us denote by $[1, d]$ the set of natural numbers between 1 and d. A weight k is a function $k : [1, d] \to \mathbb{Z}_{\geqq 0}[1/p]$. Its values are denoted by k_i. The denominator of k is the smallest number u such that $p^u k$ takes values in $\mathbb{Z}$. Set $\delta(k) = k_1\delta_1 + \cdots + k_d\delta_d$. We write $X_i = T_i$ for the Teichmüller representative and we set $X^k = X_1^{k_1} \cdot \ldots \cdot X_d^{k_d}$.

By [8], any element $\alpha \in W(A)$ has a unique expansion

$$\alpha = \sum_k \xi_k X^k, \quad \xi_k \in V^u W(R). \tag{2.17}$$

Here u denotes the denominator of k. This series is convergent in the V-adic topology, i.e. for a given $m \in \mathbb{N}$ we have $\xi_k \in V^m W(R)$ for almost all k.

For $\xi \in W(R)$ we define

$$\operatorname{ord}_V \xi = \min\{m \mid \xi \in V^m W(R)\}.$$

Proposition 2.18. *The Gauss norm of $\gamma^{(\delta)}$ is given by the formula*

$$\gamma^{(\delta)}(\alpha) = \inf\{\operatorname{ord}_V \xi_k - \delta(k)\}, \tag{2.19}$$

and the truncated Gauss norm is given by

$$\begin{aligned} \gamma^{(\delta)}[n](\alpha) &= \min\{\infty, \operatorname{ord}_V \xi_k - \delta(k) \mid \xi_k \notin V^{n+1} W(R)\} \\ &= \min_k\{\gamma^{(\delta)}[n](\xi_k X^k)\}. \end{aligned} \tag{2.20}$$

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Proof. It is enough to show equation (2.20). The formula is obvious if $\alpha = \xi_k X^k$ for a particular k. This implies (2.20) if the minimum is attained exactly once on the right-hand side.

Let $\delta^{(l)} \in \mathbb{R}^d_{>0}$, $l \in \mathbb{N}$, be a sequence which converges to the given δ. We denote by $\gamma^{(l)}[n]$ the truncated Gauss norm on $W_{n+1}(A)$ associated to numbers $\delta^{(l)}$. We easily see that

$$\lim_{l \to \infty} \gamma^{(l)}[n](\alpha) = \gamma^{(\delta)}[n](\alpha).$$

Clearly, the right-hand side of (2.20) is also continuous with respect to δ. Therefore it suffices for the proof to construct a sequence $\delta^{(l)}$ such that for each l the minimum

$$\min\{\gamma^{(l)}[n](\xi_k X^k)\}$$

is assumed exactly once. This is the case for $\alpha \neq 0$. Indeed, on the right-hand side of (2.20) all but finitely many $\gamma_\varepsilon[n](\xi_k X^k)$ are equal to ∞. We denote by g the smallest of these values and by g_1 the next greater value, which may be ∞. Let T be the set of weights where the value g is assumed.

The set of linear functions $\eta : \mathbb{R}^d \to \mathbb{R}$ such that

$$\operatorname{ord}_V \xi_k + \eta(k) \neq \operatorname{ord}_V \xi_{k'} + \eta(k')$$

for two different weights involved of T is dense. We find an η in this set whose matrix has positive entries. Moreover, we may assume that $\eta(k) < (q_1 - q)/2$ if $\gamma^{(\delta)}[n](\xi_k X^k) \neq \infty$. Then $\delta(l) = \delta + l^{-1}\eta$ meets our requirements. $\square$

Remark. In the case of a polynomial algebra A, it is useful to consider a stronger version of overconvergence, which makes only sense for rings of Witt vectors. With the above notation we define

$$\breve{\gamma}_\varepsilon(\alpha) = \inf\{\operatorname{ord}_V \xi_k - \varepsilon|k| - u\}. \tag{2.21}$$

This is clearly a pseudovaluation for each ε. If this inf is not $-\infty$, we call α overconvergent with respect to $\breve{\gamma}_\varepsilon$. One easily verifies

$$\begin{aligned} \breve{\gamma}_\varepsilon(\alpha) &\leqq \gamma_\varepsilon(\alpha), \\ \breve{\gamma}_\varepsilon(\alpha^r) &\geqq (r-1)\gamma_\varepsilon(\alpha) + \breve{\gamma}_\varepsilon(\alpha). \end{aligned} \tag{2.22}$$

It is important to note that the Teichmüller representative $[f]$ of an element $f \in A$ is $\breve{\gamma}_\varepsilon$-overconvergent. This is an immediate consequence of the following

Lemma 2.23. *Let R be a $\mathbb{Z}_p$-algebra. Let A be an R-algebra. Let $x_1, \dots, x_n \in R$ and $t_1, \dots, t_d \in A$ be elements. We denote by $k = (k_1, \dots, k_d) \in \mathbb{Z}_{\geqq 0}[1/p]$ a weight. Then we have in $W(A)$ the following relation*:

$$[x_1 t_1 + \cdots + x_d t_d] = \sum_{k, |k|=1} \alpha_k [t_1]^{k_1} \cdot \ldots \cdot [t_d]^{k_d}, \tag{2.24}$$

where $\alpha_k \in V^u W(R)$ and p^u is the denominator of k.

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Proof. Clearly, it is enough to show this lemma in the case where $x_1 = 1, \dots, x_d = 1$. Moreover, we may restrict ourselves to the case where $R = \mathbb{Z}_p$ and A is the polynomial algebra over $\mathbb{Z}_p$ in the variables $t_1, \dots, t_d$. Then $W(A)$ is a $\mathbb{Z}_{\geqq 0}[1/p]$-graded such that the monomial $[t_1]^{k_1} \cdot \ldots \cdot [t_d]^{k_d}$ has degree $|k|$ (note that this monomial is in general not in $W(A)$). More precisely, a Witt vector of polynomials $(p_0, p_1, p_2, \dots) \neq 0$ is homogeneous of degree $m \in \mathbb{Z}_{\geqq 0}[1/p]$ if each polynomial p_i has degree $p^i m$, if $p_i \neq 0$. In the case where $p^i m$ is not an integer, the condition says that $p_i = 0$.

Since $[t_1 + \cdots + t_d]$ is homogeneous of degree 1, the lemma follows from [8], Proposition 2.3. □

Lemma 2.25. *Let (A, v) be a ring with a proper pseudovaluation. Let $\alpha \in VW(A)$. Assume that $\gamma_\varepsilon(\alpha) \geqq 0$. Then the element $1 - \alpha$ is a unit in $W(A)$ and we have*

$$\gamma_\varepsilon(1 - \alpha)^{-1} \geqq 0. \tag{2.26}$$

Assume moreover that $A = R[T_1, \dots, T_d]$ is a polynomial ring with a degree valuation. Then

$$\breve{\gamma}_\varepsilon(1 - \alpha)^{-1} \geqq \min\{0, \breve{\gamma}_\varepsilon(\alpha)\}.$$

In particular, $(1 - \alpha)^{-1}$ is $\breve{\gamma}_\varepsilon$-overconvergent if so is α.

Proof. Write $\alpha = {}^V\eta$. We find $\gamma_{\varepsilon/p}(\eta) > -1$. In $W(A)$ we have the identity

$$(1 - {}^V\eta)^{-1} = 1 + \sum_{i>0} p^{i-1\,V}(\eta^i) = \sum_{i \geqq 0} \alpha^i.$$

The middle term shows that the series converges V-adically, and the last sum proves inequality (2.26). The last assertion is obvious from (2.22). □

Proposition 2.27. *Let (A, v) be a ring with a proper pseudovaluation. Furthermore, let $\mathbf{w}_n : W(A) \to A$ denote the Witt polynomials. An element $\alpha \in W^\dagger(A)$ is a unit if and only if $\mathbf{w}_0(\alpha)$ is a unit in A.*

Assume moreover that $A = R[T_1, \dots, T_d]$ is a polynomial ring with a degree valuation. If α is $\breve{\gamma}_\varepsilon$-overconvergent, then α^{-1} is $\breve{\gamma}_\delta$-overconvergent for some $\delta > 0$.

Proof. We write $\alpha = [a] + {}^V\eta$, with $a \in A$ and $\eta \in W(A)$. To prove the first assertion, we may assume that $a = 1$. Applying Corollary 2.10, we assume that $\gamma_\varepsilon(V\eta) > 0$. Then the assertion follows from Lemma 2.25.

Now we prove the second assertion: Since every Teichmüller representative is $\breve{\gamma}_\varepsilon$-overconvergent, it suffices to show that the inverse of $1 + [a^{-1}]^V\eta = 1 + {}^V([a^{-p}]\eta)$ is $\breve{\gamma}_\varepsilon$-overconvergent. Since $\breve{\gamma}_\varepsilon$ is a pseudovaluation, we see that ${}^V([a^{-p}]\eta)$ is $\breve{\gamma}_\varepsilon$-overconvergent too. By Corollary 2.10, we find ε/p such that

$$\gamma_{\varepsilon/p}([a^{-p}]\eta) > -1.$$

Therefore we may apply Lemma 2.25. □

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Proposition 2.28. *Let A be an algebra over a perfect field K. Let v be an admissible pseudovaluation on A. Then $W^{\dagger}(A)$ is an algebra over the complete local ring $W(K)$.*

The $W(K)$-algebra $W^{\dagger}(A)$ is weakly complete in the sense of [13].

Proof. Let $z_1, \dots, z_r \in W^{\dagger}(A)$. Consider an infinite series

$$\sum a_k z^k, \quad a_k \in W(K),\ z^k = z_1^{k_1} \cdot \dots \cdot z_r^{k_r}. \tag{2.29}$$

We assume that there are real numbers $\delta > 0$ and c such that

$$\operatorname{ord}_p a_k \geqq \delta|k| + c.$$

This implies that the series (2.29) converges in $W(A)$. We have to show that the series converges to an element $W^{\dagger}(A)$. We choose a common radius ε of convergence for $z_1, \dots, z_r$. Making ε smaller, we may assume that

$$\gamma_\varepsilon(z_i) \geqq -\delta.$$

Then we find

$$\gamma_\varepsilon(a_k a^k) \geqq \operatorname{ord}_p a_k - \delta|k| \geqq c.$$

Therefore (2.29) converges to an element of $W^{\dagger}(A)$. $\square$

We will point out that by Monsky and Washnitzer the last proposition implies Hensel's lemma for the overconvergent Witt vectors:

Proposition 2.30. *Let A be an algebra over a perfect field K. Let v be an admissible pseudovaluation on A. Let $f(T) \in W^{\dagger}(A)[T]$ be a polynomial. We consider the homomorphism $\mathbf{w}_0 : W^{\dagger}(A) \to A$.*

Let $a \in A$ be an element such that

$$f(a) = 0 \quad \textit{and} \quad f'(a) \textit{ is a unit in } A.$$

Then there is a unique $\alpha \in W^{\dagger}(A)$ such that $f(\alpha) = 0$ and such that $a \equiv \alpha \bmod VW^{\dagger}(A)$.

Proof. The kernel of the natural morphism $W^{\dagger}(A)/pW^{\dagger}(A) \to A$ is an ideal whose square is zero. Therefore there is an $\bar{\alpha} \in W^{\dagger}(A)/pW^{\dagger}(A)$ which reduces to a and such that $f(\bar{\alpha}) = 0$. The rest of the proof is a general fact about weakly complete algebras explained below. $\square$

For the explanation we follow the notation of [13]: Let (R, I) be a complete noetherian ring. Let A be a weakly complete finitely generated (w.c.f.g.) algebra over (R, I). We write $\bar{A} = A/IA$. Let $A \to B$ be a morphism of w.c.f.g. algebras such that

$$\bar{B} = \bar{A}[X_1, \dots, X_n]/(\bar{F}^{(1)} \cdots \bar{F}^{(s)}), \quad s \leqq n,$$

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and the $s \times s$ subdeterminants of $\left(\frac{\partial F^{(i)}}{\partial X_j}\right)$ generate the unit ideal in $\bar{B}$. Then by [13], p. 195, the morphism $A \to B$ is very smooth. As an example we may take for B the weak completion of

$$A[X,T]/\big(f(X), 1 - f'(X)T\big),$$

where $f(X) \in A[X]$ is a polynomial.

Proposition 2.31. *Let C be a weakly complete (not necessarily finitely generated but p-adically separated) algebra over (R, I). Let $f(X) \in C[X]$ be a polynomial and let $\bar{\gamma} \in \bar{C}$ be an element such that $f(\bar{\gamma}) = 0$ and $f'(\bar{\gamma})$ is a unit in $\bar{C}$. Then there is a unique element $\gamma \in C$ such that $f(\gamma) = 0$ and $\gamma \equiv \bar{\gamma} \bmod IC$.*

Proof. By Hensel's lemma applied to the completion of C, the uniqueness of the solution is clear.

For the existence we write $f(X) = s_d X^d + s_{d-1} X^{d-1} + \cdots + s_1 X + s_0$, where $s_i \in C$. Let $A = R[S_d, \ldots, S_0]^\dagger$ be the weak completion of the polynomial algebra. We set

$$F(X) = S_d X^d + \cdots + S_1 X + S_0 \in A[X]$$

and we let B be the weak completion of

$$A[X,T]/\big(F(X), 1 - TF'(X)\big).$$

Let $A \to C$ be the homomorphism defined by $S_i \mapsto s_i$. The solution $\bar{\gamma}$ defines a homomorphism

$$R/I[S_d, \ldots, S_0, X, T]/\big(\bar{F}(X), 1 - T\bar{F}'(X)\big) \to \bar{C},$$

where $S_i \mapsto s_i \bmod IC$ and $X \mapsto \bar{\gamma}$, $T \mapsto f'(\bar{\gamma})^{-1}$. Hence we obtain a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & \bar{C}. \end{array} \tag{2.32}$$

Since $A \to B$ is very smooth by the example above, we find a morphism $B' \to C$ making (2.32) commutative. The image of X is the desired solution $\gamma \in C$. □

We will now study the behavior of overconvergent Witt vectors in finite étale extensions. Let A be a finitely generated K-algebra. Let B be a finite étale A-algebra which is free as an A-module. Let e_i, $1 \leqq i \leqq r$, be a basis of the A-module B. Then the natural map

$$W(A)^r \to W(B), \tag{2.33}$$

which maps the standard basis of the free module $W(A)^r$ to the Teichmüller representatives $[e_i]$, is an isomorphism. Moreover, $W(B)$ is an étale algebra over $W(A)$.

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Indeed, from [8], A8, it follows that the $W_n(A)$-algebra $W_n(B)$ is étale for each n. We set $I_n = VW_{n-1}(A) \subset W_n(A)$. Then by loc. cit. we have $I_n W_n(B) \subset VW_n(B)$. From this we conclude by the Lemma of Nakayama that

$$W_n(A)^r \to W_n(B)$$

is an isomorphism. Taking the projective limit, we obtain (2.33). If we tensor (2.33) with $A \otimes_{\mathbf{w}_0}$, we obtain that $A \otimes_{\mathbf{w}_0} W(B) = B$.

We will now assume that B is monic,

$$B = A[T]/f(T)A[T],$$

where

$$f(T) = T^m - c_{m-1}T^{m-1} - \cdots - c_1 T - c_0. \tag{2.34}$$

Let v be a negative pseudovaluation on A. We endow B with the equivalence class of admissible pseudovaluations defined by Proposition 1.8.

Lemma 2.35. *Let $d \in \mathbb{R}$ be such that $d > v(c_i)$ for $i = 1, \dots, m$. An element $b \in B$ has a unique representation*

$$b = \sum_{i=0}^{m-1} a_i T^i.$$

We set

$$\tilde{v}(b) = \min_{i=1,\dots,m-1} \{v(a_i) - id\}. \tag{2.36}$$

Then $\tilde{v}$ is an admissible pseudovaluation on B.

Proof. We consider on $A[T]$ the pseudovaluation μ (cf. (1.13)). We will show that with d given as above $\tilde{v}$ is the quotient of μ.

Let

$$\tilde{b} = \sum_{j=0}^{s} u_j T^j$$

be an arbitrary representative of b. We need to show that $\mu(\tilde{b})$ is smaller than the right-hand side of (2.36). We prove this by induction on s. For $s < m$ there is nothing to show. For $s \geqq m$ we obtain another representative of b:

$$\tilde{b}' = \sum_{j=0}^{m-1} u_j T^j + \sum_{k \geqq m} u_k \left(\sum_{l=0}^{m-1} c_l T^l \right) T^{k-m}. \tag{2.37}$$

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On the right-hand side, there is a polynomial of degree at most $s-1$. Therefore it suffices, by induction, to show that

$$\mu(\tilde{b}') \geqq \mu(\tilde{b}).$$

The last inequality is a consequence of the following:

$$\begin{aligned} &\mu(u_j T^j) \geqq \mu(\tilde{b}) \quad \text{for } j = 0, \dots, m-1, \\ &\mu(u_k c_l T^{k-m+l}) \geqq \mu(\tilde{b}) \quad \text{for } k \geqq m,\ 0 \leqq l \leqq m-1. \end{aligned} \tag{2.38}$$

The first set of these inequalities is trivial. For the second set we compute

$$\begin{aligned} \mu(u_k c_l T^{k-m+l}) &\geqq v(u_k) + v(c_l) - kd + (m-l)d \\ &\geqq \mu(u_k) - kd \geqq \mu(\tilde{b}). \end{aligned}$$

The last equation holds because by the choice of d we have

$$v(c_l) + (m-l)d \geqq 0.$$

This shows the second set of inequalities. □

Because $\tilde{v}$ restricted to A coincides with v, we simplify the notation by setting $\tilde{v} = v$. The Gauss norms (2.2) induced by the pseudovaluation v on $W(B)$ and $W(A)$ will be also denoted by the same symbols γ_ε.

Lemma 2.39. *With the notation of Lemma* 2.35 *we assume that B is ètale over A. We will denote the residue class of T in B by t.*

Then there is a constant $G \in \mathbb{R}$ with the following property: Each $b \in B$ has for each integer $n \geqq 0$ a unique representation

$$b = \sum_{i=0}^{m-1} a_{ni} t^{ip^n}.$$

Then we have the following estimates for the pseudovaluations of a_{ni}:

$$v(a_{ni}) \geqq v(b) - p^n G. \tag{2.40}$$

Proof. Since B is étale over A, the elements

$$1, t^{p^n}, t^{2p^n}, \dots, t^{(m-1)p^n}$$

are for each n a basis of the A-module B. We write

$$t^i = \sum_{j=0}^{m-1} u_{ji} t^{jp}. \tag{2.41}$$

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We introduce the matrix $U = (u_{ji})$ and we set for matrices

$$v(U) = \min_{i,j}\{v(u_{ji})\}.$$

We deduce the relation

$$a_{1j} = \sum_i u_{ji} a_{0i}$$

and we will write the last equality in matrix notation:

$$a(1) = Ua(0).$$

Let $U^{(p^n)}$ be the matrix obtained from U by raising all entries of U to the p^n-th power. Then we obtain with the obvious notation:

$$a(n+1) = U^{(p^n)} a(n).$$

It is obvious that for two matrices U_1, U_2 with entries in B

$$v(U_1 U_2) \geqq v(U_1) + v(U_2).$$

We choose a constant C such that

$$v(U) \geqq -C.$$

Therefore we obtain:

$$\begin{aligned} v\big(a(n)\big) &= v\big(U^{(p^{n-1})} \cdot \ldots \cdot Ua(0)\big) \\ &\geqq -(p^{n-1}C + \cdots + pC + C) + v\big(a(0)\big). \end{aligned}$$

By Lemma 2.33, we have

$$v(b) = \min\{v(a_{0i}) - id\} \leqq v\big(a(0)\big).$$

Therefore we obtain

$$v\big(a(n)\big) \geqq -p^n \frac{C}{p-1} + v(b).$$

We thus found the desired constant. □

Proposition 2.42. *Let $B = A[t]$ be a finite étale A-algebra as in Lemma 2.39. Let $G > 0$ be the constant of this lemma. Let $x = [t] \in W(B)$ be the Teichmüller representative. By (2.33), $1, x, \ldots, x^{m-1}$ is a basis of the $W(A)$-module $W(B)$. We write an element $\eta \in W(B)$ as follows:*

$$\eta = \sum_{i=0}^{m-1} \xi_i x^i, \quad \xi_i \in W(A).$$

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There is a real number $\delta > 0$ such that for $\varepsilon \leqq \delta$

$$\gamma_\varepsilon(\eta) \geqq -C \quad \textit{implies} \quad \gamma_\varepsilon(\xi_i) \geqq -C - \varepsilon G.$$

Proof. We choose a constant $G' > 0$ such that

$$v(t^i) \geqq -G', \quad \text{for } i = 0, \dots, m-1.$$

Let δ be such that $\delta(G + G') \leqq 1$. Write

$$\xi_i = \sum_{s \geqq 0} {}^{V^s}[a_{s,i}] \quad \text{with } a_{s,i} \in A$$

and define

$$\zeta_i(n) = \sum_{s \geqq n} {}^{V^{s-n}}[a_{s,i}].$$

We will show by induction on n the following two assertions:

$$\gamma_\varepsilon\left(\sum_{i=0}^{m-1} {}^{V^n}\zeta_i(n) x^i\right) \geqq -C, \tag{2.43}$$

$$\gamma_\varepsilon({}^{V^n}[a_{n,i}]) \geqq -C - \varepsilon G. \tag{2.44}$$

We begin by showing that the first inequality for a given n implies the second. To this end, we set

$$\theta(n) = \sum_{i=0}^{m-1} {}^{V^n}\zeta_i(n) x^i.$$

The first non-zero component of this Witt vector is

$$y_n = \sum_{i=0}^{m-1} a_{n,i} t^{ip^n}$$

in place $n+1$. We conclude that

$$n + \varepsilon p^{-n} v(y_n) \geqq \gamma_\epsilon\big(\theta(n)\big) \geqq -C,$$

where the last inequality is (2.43). This shows that

$$v(y) \geqq -\varepsilon p^n (C + n).$$

We conclude by Lemma 2.39 that

$$v(a_{n,i}) \geqq -(p^n/\varepsilon)(C + n) - p^n G, \tag{2.45}$$

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and therefore

$$\gamma_\varepsilon({}^{V^n}[a_{n,i}]) = n + \varepsilon p^{-n} v(a_{n,i}) \geqq -C - \varepsilon G.$$

Therefore the proposition follows if we show the assertion (2.43) by induction. The assertion is trivial for $n = 0$ and we assume it for n. With the above notation we write

$$\begin{aligned}\theta(n+1) &= \theta(n) - \sum_{i=0}^{m-1} {}^{V^n}[a_{n,i}]x^i \\ &= \bigl(\theta(n) - {}^{V^n}[y_n]\bigr) - \left(\sum_{i=0}^{m-1} {}^{V^n}[a_{n,i}]x^i - {}^{V^n}[y_n]\right).\end{aligned}$$

The Witt vector in the first brackets has only entries which also appear in $\theta(n)$ and therefore has Gauss norm $\gamma_\varepsilon \geqq -C$. The assertion follows if we show the same inequality for the Witt vector in the second brackets:

$${}^{V^{n+1}}\tau = \left(\sum_{i=0}^{m-1} {}^{V^n}[a_{n,i}]x^i - {}^{V^n}[y_n]\right).$$

We set

$$[y_n] = \sum_{i=0}^{m-1} [a_{n,i} t^{ip^n}] = (s_0, s_1, s_2, \ldots).$$

Then we find

$${}^{V}\tau = (0, s_1, s_2, \ldots).$$

We know that s_l is a homogeneous polynomial of degree p^l in the variables $a_{n,i}t^{ip^n}$ for $i = 1, \ldots, m-1$. By the choice of G', we find $v(t^{ip^n}) \geqq p^n v(t^i) \geqq -p^n G'$. Using (2.45) we find

$$v(s_l) \geqq -p^l\bigl((p^n/\varepsilon)(C+n) + p^n G\bigr) - p^l p^n G' = -p^{n+l}\bigl((1/\varepsilon)(C+n) + G + G'\bigr).$$

We have

$${}^{V^{n+1}}\tau = \sum_{l \geqq 1} {}^{V^{n+l}}[s_l].$$

For the Gauss norms of the entries of this vector we find for $l \geqq 1$:

$$\begin{aligned}\gamma_\varepsilon({}^{V^{n+l}}[s_l]) &= n + l + \varepsilon p^{-n-l} v(s_l) \geqq n + l - \varepsilon\bigl((1/\varepsilon)(C+n) + G + G'\bigr) \\ &= l - C - \varepsilon(G + G') \geqq -C.\end{aligned}$$

The last inequality follows since $l \geqq 1$ by the choice of δ. We conclude that

$$\gamma_\varepsilon({}^{V^{n+1}}\tau) \geqq -C. \quad \square$$

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Corollary 2.46. *Let A be a finitely generated algebra over K. Let $B = A[T]/(f(T))$ be a finite étale A-algebra, where $f(T) \in A[T]$ is a monic polynomial of degree n. We denote by t the residue class of T in B, and set $x = [t] \in W^{\dagger}(B)$.*

Then $W^{\dagger}(B)$ is finite and étale over $W^{\dagger}(A)$ with basis $1, x, \ldots, x^{n-1}$.

3. The sheaf property

We will prove that the overconvergent Witt vectors are a sheaf for the Zariski topology. This is done for overconvergent Witt differentials over a perfect field in [2]. For Witt vectors the argument given here is more elementary and works over an integral domain R. The basic idea due to Meredith is the same as for Witt differentials.

Let R be an integral domain of characteristic p. Consider the ring

$$L = R[T_1, \ldots, T_g, S_1^{\pm 1}, \ldots, S_m^{\pm 1}].$$

We define a pseudovalution v on L. Write $a \in L$ as a Laurent polynomial

$$a = \sum_{k,l} \alpha_{kl} T^k S^l, \quad k \in \mathbb{Z}_{\geqq 0}^g,\ l \in \mathbb{Z}^m.$$

We set

$$v(a) = -\max\{|k| + |l| \mid \alpha_{kl} \neq 0\}$$

and $v(0) = \infty$. By Proposition 1.32, this is an admissible pseudovaluation (Definition 1.9). The degree of $a \in L$ is by definition $\deg a = -v(a)$. We should note that in general we have only $\deg(ab) \leqq \deg a + \deg b$, but not equality.

Consider a Witt vector

$$\sigma = (s_0, s_1, s_2, \ldots) \in W(L). \tag{3.1}$$

The normalized degree of σ is defined by

$$\operatorname{Ndeg} \sigma = \sup\{\deg s_i / p^i \mid i \in \mathbb{Z}_{\geqq 0}\}.$$

If $\operatorname{Ndeg} \sigma < \infty$, we call σ bounded. The bounded Witt vectors are a subring of the overconvergent Witt vectors which are defined by the Gauss norms

$$\gamma_\delta(\sigma) = \inf_i \{i - \delta \deg s_i / p^i\}.$$

We will say that a Witt vector (3.1) is concentrated in an interval $[c, d] \subset \mathbb{Z}_{\geqq 0}$ if $s_i = 0$ for $i \notin [c, d]$.

Let A be a finitely generated algebra over R. We consider $X = \operatorname{Spec} A$ as a Grothendieck topology with objects being the open set $D(f)$, with $f \in A$, and the usual coverings. We denote by $W^{\dagger}\mathcal{O}$ the presheaf $D(f) \mapsto W^{\dagger}(A_f)$.

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Proposition 3.2. *Let $f_1, \dots, f_m \in A$ be elements which generate the ideal A. Let $\mathcal{U}$ be the covering of* Spec A *by the open sets $U_i = D(f_i)$. Then we have for the Cech cohomology*

$$H^i(\mathcal{U}, W^\dagger\mathcal{O}) = \begin{cases} W^\dagger(A) & \text{if } i = 0, \\ 0 & \text{if } i \geqq 1. \end{cases} \tag{3.3}$$

In particular, the presheaf $W^\dagger\mathcal{O}$ extends uniquely to a sheaf on the topological space Spec A.

Proof. We augment the Cech complex by $W^\dagger(A)$ and show that the cohomology of the augmented complex is 0. Let us first assume that f_i is not a zero-divisor for $i = 1, \dots, m$.

We represent A as a quotient

$$R[T_1, \dots, T_g, S_1, \dots, S_m] \to A$$

such that the elements S_i are mapped to f_i. If we speak of the degree $\deg a$ of an element $a \in A$, we mean the degree of a given representative $\tilde{a} \in L$ which will be clear from the context, e.g. for f_i we take always the representative S_i and write $\deg f_i = 1$. In the same sense we will speak of the normalized degree of an element in $W(A)$.

For a localization $A_{f_{i_0} \cdot \dots \cdot f_{i_r}}$ we consider the representation

$$R[T_1, \dots, T_g, S_1, \dots, S_m, S_{i_0}^{-1}, \dots, S_{i_r}^{-1}] \to A_{f_{i_0} \cdot \dots \cdot f_{i_r}}.$$

Degrees of elements in $A_{f_{i_0} \cdot \dots \cdot f_{i_r}}$ are meant with respect to this representation.

Then the Teichmüller representatives $[f_1], \dots, [f_m]$ generate the unit ideal in $W^\dagger(A)$. We find the relation

$$\sum_{i=1}^{m} r_i[f_i] = 1, \quad r_i \in W^\dagger(A).$$

If we raise this equation to the $m\ell$-th power, we find the relation

$$Q_{\ell,i}(r_1, \dots, r_m, [f_1], \dots, [f_m])[f_i]^\ell = 1.$$

Here $Q_{\ell,i}$ are polynomials which are homogeneous of degree $m\ell$ in $r_1, \dots, r_m$ and homogeneous of degree $(m-1)\ell$ in $[f_1], \dots, [f_m]$.

Let $\sigma = (\sigma_{i_0,\dots,i_r})$ be an alternating r-cocycle with values in $\sigma_{i_0,\dots,i_r} \in W^\dagger(A_{f_{i_0} \cdot \dots \cdot f_{i_r}})$. We will show that σ is a coboundary.

We write $\sigma = \big(s(0), s(1), s(2), \dots\big)$. Since this Witt vector is overconvergent, we find $C \in \mathbb{N}$ such that

$$\ell - \frac{1}{C} \frac{\deg s(\ell)_{i_0,\dots,i_r}}{p^\ell} > -1.$$

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Here we have chosen representatives of $s(\ell)_{i_0,\ldots,i_r}$ which are fixed throughout our discussion. We rewrite the last inequality in the form

$$\deg s(\ell) < C(\ell+1)p^\ell. \tag{3.4}$$

Define the truncated cochain

$$\sigma^{<u} = \bigl(s(0), s(1), \ldots, s(2^u-1), 0, 0, \ldots\bigr).$$

We deduce from (3.4) that

$$\mathrm{Ndeg}\,\sigma^{<u} < C2^u. \tag{3.5}$$

In particular, this implies that

$$[f_\alpha]^{C2^u}\sigma^{<u}_{\alpha,i_1,\ldots,i_r} \in W(A_{f_{i_1}\cdot\ldots\cdot f_{i_r}}).$$

This Witt vector is concentrated in $[0, 2^u)$.

We will consider "generalized" cochains γ that have values in the polynomial rings $W^\dagger(A_{f_{i_0}\cdot\ldots\cdot f_{i_r}})[r_1,\ldots,r_m]$. If we speak of the normalized degree of such a cochain, we mean the maximum of the normalized degrees of the coefficients of these polynomials in $r_1,\ldots,r_n$, while degree means degree of the polynomials. This will be denoted by Pdeg.

Now we construct inductively "generalized" cochains

$$\tau^{<u}_{i_1,\ldots,i_r} \in W(A_{f_{i_1}\cdot\ldots\cdot f_{i_r}})[r_1,\ldots,r_m]$$

which are concentrated in $[0, 2^u)$ and which have the following properties:

(1) $\mathrm{Pdeg}\,\tau^{<u}_{i_1,\ldots,i_r} \leqq 2Cm2^u$.

(2) $\mathrm{Ndeg}\,\tau^{<u}_{i_1,\ldots,i_r} < 2Cm2^u$.

(3) $[f_{i_k}]^{C2^u}\tau^{<u}_{i_1,\ldots,i_k,\ldots,i_r} \in W(A_{f_{i_1}\cdot\ldots,\widehat{f_{i_k}},\ldots\cdot f_{i_r}})[r_1,\ldots,r_m]$.

(4) $\sigma^{<u} - \partial\tau^{<u} = 0$ modulo $V^{2^u}W^\dagger(A_{f_{i_0}\cdot\ldots\cdot f_{i_r}})$.

(5) $\tau^{<u+1} = \tau^{<u} + \tau[u+1]$, where $\tau[u+1]$ is concentrated in the interval $[2^u, 2^{u+1})$.

We note that in (4) we have evaluated the polynomials in r_i.

Assume that $\tau^{<u}$ is constructed. Then we consider the r-cochain

$$\gamma = (\sigma^{<u+1} - \partial\tau^{<u})^{<u+1}.$$

Here we truncate the difference in the place 2^{u+1}. Then γ is concentrated in $[2^u, 2^{u+1})$ and has normalized degree $\mathrm{Ndeg}\,\gamma < 2Cm2^u$. By property (1), we have $\mathrm{Pdeg}\,\gamma \leqq 2Cm2^u$. Then

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we define the "generalized" cochain

$$\tau[u+1]_{i_1,\ldots,i_r} = \left(\sum Q_{C2^{u+1},\alpha}[f_\alpha]^{C2^{u+1}}\gamma_{\alpha,i_1,\ldots,i_r}\right)^{<u+1},$$

where we truncate again the sum at the place 2^{u+1}. Then $\tau[u+1]$ has normalized degree

$$\mathrm{Ndeg}\,\tau[u+1] < mC2^{u+1} + 2Cm2^u = 2Cm2^{u+1}.$$

The degree of $\tau[u+1]$ as a polynomial in r_i is $\leqq mC2^{u+1} + mC2^{u+1} = 2Cm2^{u+1}$.

If we restrict our cochains to $W_{2^{u+1}}(A_{f_{i_1}\cdot\ldots\cdot f_{i_r}})$, then γ becomes a cocycle and a standard formal computation yields

$$\gamma - \partial\tau[u+1] = 0 \text{ modulo } V^{2^{u+1}}.$$

This implies

$$\sigma^{<u+1} - (\partial\tau^{<u} + \partial\tau[u+1]) = 0 \text{ modulo } V^{2^{u+1}}.$$

By property (5), this shows (4). The properties (1) and (2) follow from the corresponding properties for $\tau[u+1]$ proven above. Finally, we obtain

$$[f_{i_k}]^{C2^{u+1}}\tau[u+1]_{i_1,\ldots,i_r} \in W(A_{f_{i_1}\cdot\ldots,\widehat{f_{i_k}},\ldots\cdot f_{i_r}})[r_1,\ldots,r_m]$$

since this holds for $\sigma^{<u+1}$ and for $\partial\tau^{<u}$.

Therefore we have constructed data with the properties (1)–(5) as required.

We consider the polynomial

$$\tau[u]_{i_1,\ldots,i_r} = \sum_{|I|\leqq Cm2^{u+1}} \tau[u]_{i_1,\ldots,i_r}(I)r^I \in W^\dagger(A_{f_{i_1}\cdot\ldots\cdot f_{i_r}}),$$

i.e. we evaluate the polynomials. Then σ is the boundary of $\sum\limits_u \tau[u]$. We have to show that the last element converges to an element in $W^\dagger(A_{f_{i_1}\cdot\ldots\cdot f_{i_r}})$.

For this we compute the Gauss norms. Since $\tau[u](I)$ is concentrated in $[2^{u-1}, 2^u)$, we find by property (2)

$$\gamma_\delta\big(\tau[u](I)\big) \geqq 2^{u-1} - \delta 2Cm2^u \geqq 2^{u-2}$$

if $\delta < 1/8mC$.

On the other hand, we find for arbitrarily small $e > 0$ a $\delta > 0$ such that $\gamma_\delta(r_i) > -e$. Then we have

$$\gamma_\delta\big(\tau[u]_{i_1,\ldots,i_r}(I)r^I\big) > 2^{u-2} - |I|e \geqq 2^{u-2} - mC2^{u+1}e \geqq 0$$

if $e < 1/8mC$. Therefore the terms of the sum $\sum\limits_u \tau[u]_{i_1,\ldots,i_r}$ are uniformly bounded in the Gauss norm γ_δ and therefore this sum converges to an element of $\tau_{i_1,\ldots,i_r} \in W^\dagger(A_{f_{i_1}\cdot\ldots\cdot f_{i_r}})$, such that $\partial\tau = \sigma$.

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Finally, we treat the case where zero-divisors among the f_i are allowed. Then we find a surjective algebra homomorphism $B \to A$ such that B is a finitely generated algebra over R and such that there are preimages $g_i \in B$ of f_i which are not zero-divisors in B and generate the unit ideal: $(g_1, \ldots, g_m) = B$.

Let $\mathfrak{a}$ be the kernel of $B \to A$. Then we obtain an exact sequence of presheaves on $\operatorname{Spec} B$:

$$0 \to W^{\dagger}(\mathfrak{a}_g) \to W^{\dagger}(B_g) \to W^{\dagger}(A_g) \to 0. \tag{3.6}$$

Let $\tilde{\mathscr{U}}$ be the covering of $\operatorname{Spec} B$ given by the $D(g_i)$ for $i = 1, \ldots, m$. We have shown that the Cech cohomology of the presheaf $W^{\dagger}(B_g)$ is trivial with respect to $\tilde{\mathscr{U}}$. But the same argument shows that the Cech cohomology of the presheaf $W^{\dagger}(\mathfrak{a}_g)$ is trivial as well. Since $A_{g_i} = A_{f_i}$ we obtain (3.3) from the cohomology sequence. $\square$

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University of California, Department of Mathematics, Irvine, CA 92697, USA
e-mail: davis@math.uci.edu

University of Exeter, Mathematics, Exeter EX4 4QF, Devon, UK
e-mail: a.langer@exeter.ac.uk

Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany
e-mail: zink@math.uni-bielefeld.de

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