PACKINGS INDUCED BY PIECEWISE ISOMETRIES CANNOT CONTAIN THE ARBELOS

Marcello Trovati, Peter Ashwin and Nigel Byott

Mathematics Research Institute, School of Engineering, Computing and Mathematics
Harrison Building, University of Exeter, Exeter, EX4 4QF, UK

Abstract. Planar piecewise isometries with convex polygonal atoms that are piecewise irrational rotations can naturally generate a packing of phase space given by periodic cells that are discs. We show that such packings cannot contain certain subpackings of Apollonian packings, namely those belonging to a family of *Arbelos* subpackings. We do this by showing that the unit complex numbers giving the directions of tangency within such an isometric-generated packing lie in a finitely generated subgroup of the circle group, whereas this is not the case for the Arbelos subpackings. In the opposite direction, we show that, given an arbitrary disc packing of a polygonal region, there is a piecewise isometry whose regular cells approximate the given packing to any specified precision.

1. **Introduction.** Consider an orientation-preserving piecewise isometry T of a planar region M, namely a decomposition of M into the union of convex polygonal regions, or atoms, such that T is an orientation-preserving isometry on each atom. It has been recognised for some time that such piecewise isometries (PWIs) induce a full measure partition of the phase space into the union of the *regular set* of periodically coded points and its complement, the *exceptional set*, see for example [7]. Under generic assumptions on the piecewise isometry that we state in Section 2, the set of points with the same periodic coding is a disc and hence the regular set gives a *disc packing* of M that may or may not be dense in M. Previous work has suggested that the packing is typically very "loose", so that tangencies between the discs are rare: for example in [3] it is shown that almost all packings for a certain family of PWIs have no tangencies, and it is conjectured in [4] that the exceptional sets for the same family of maps have positive measure. Moreover, one can obtain a full measure disc packing which is tangent-free [3]. However, the possibility is not excluded that, for some parameterised families of PWIs, there may be dense set of parameters for which many tangencies occur.

This paper presents a further result in this direction. Apollonian packings are well known [1, 2, 6, 11] and are the most studied of dense disc packings. They are conjectured to be the "tightest possible" packings in that their complements are thought to have smallest possible Hausdorff dimension [10]. In [1] there are numerical estimations of the complement of Apollonian packings that indicate that they have dimension 1.30568, derived by examining scaling properties of a semigroup that generates the packing. Within an Apollonian packing, one may identify

sequences of discs which form a subpacking of a particularly simple kind, namely Arbelos packings, in which each disc is tangent to two fixed boundary discs. The main result of this article is that certain Arbelos packings (and hence the Apollonian packings which contain them) can never arise as subpackings of the disc packings associated to PWIs satisfying our assumptions.

We now describe the content of the paper in a little more detail. After recalling in Theorem 2.2 that periodically coded cells for PWIs are typically discs, we show in Theorem 2.3 that the unit vectors in the directions of the tangencies between discs lie in a finitely generated subgroup of the circle group S^1. By contrast we prove in Theorem 4.3 that the tangency directions of any of a family of Arbelos subpackings give rise to a subgroup of S^1 which is not finitely generated, and in Theorem 4.4 we conclude from this that such subpackings cannot be generated by a PWI. In the opposite direction, we show in Theorem 5.1 that, given any non-overlapping disc packing A of a polygonal region D, there is a PWI on D whose regular set is arbitrarily close to A.

We note that a stronger result than Theorem 4.4 has recently been given in [12], but for a more restrictive class of piecewise isometries than considered here: for such a PWI, any given disc in the associated disc packing can be tangent to at most finitely many other discs.

2. **Disc packings induced by PWIs.** Piecewise isometries arise in different contexts ranging from purely abstract geometrical problems to physical systems. In particular, when we consider 2-dimensional PWIs, these generate interesting geometrical configurations (see [7, 8, 9]). However there are several examples, such as bandpass Sigma-Delta modulators as depicted in Figure 1 (see [5]), that generate a disc packing which appears to be *tangent-free*, i.e. there are no tangencies between any two distinct discs.

In this Section, our aim is to investigate some properties of disc packings induced by PWIs, emphasising the bridge between dynamics on the one hand, and combinatorial properties on the other hand.

2.1. **Cells for PWIs.** A planar piecewise isometry is a map of the form

$$T : M \to M$$

where M is a closed bounded polygonal region within $\mathbb{C}$ (or $\mathbb{R}^2$) that has a finite partition into N convex polygons or *atoms*:

$$M = \bigcup_{k=1}^{N} \overline{M}_k$$

where each M_k is an open convex polygon such that $T|_{M_k} = T_k$ is an isometry. We write S^1 for the circle group in the complex plane $S^1 = \{z \in \mathbb{C} \ : \ |z| = 1\}$, and we only consider orientation-preserving isometries; these are necessarily of the form

$$T_k(z) = v_k z + w_k. \tag{1}$$

for $v_k \in S^1$ and $w_k \in \mathbb{C}$. We suppose that the boundaries of M_k are line segments of the form

$$V_{k,j}\mathbb{R} + W_{k,j} \tag{2}$$

with $V_{k,j} \in S^1$ and $W_{k,j} \in \mathbb{C}$, for $j = 1, \ldots, l(k)$, the positive integer $l(k)$ being the number of sides of the polygon M_k. We define $B_r(x) = \{y \in \mathbb{C} : |x - y| < r\}$.

Definition 2.1. We say that a set $X \subset M \subset \mathbb{C}$ is a *disc packing* of M if

$$X = \bigcup_{i \in I} B_{r_i}(x_i)$$

where $B_{r_i}(x_i)$ are disjoint (non-overlapping) open discs with radii $r_i > 0$ and centres $x_i \in \mathbb{C}$, indexed by a countable set I.

A disc packing X of M is said to be *dense* if $\overline{X} = \overline{M}$. The pairs $\{(x_i, r_i) : i \in I\}$ with $r_i > 0$ and $B_{r_i}(x_i) \subset M$ characterise the disc packing; if we want them to be non-overlapping we require that $|x_i - x_j| \geq r_i + r_j$ whenever $i \neq j$. There is a *tangency* between B_{r_i} and B_{r_j} if and only if $|x_i - x_j| = r_i + r_j$. Note that the Lebesgue measure of this packing is simply $\sum_i \pi r_i^2$.

We say $\mathbf{s} = s_0 s_1 \cdots s_k \cdots$ with $s_k \in \{1, \cdots, N\}$ is an *itinerary* (which may be a finite or infinite word) if there is an $x \in M$ such that $T^n(x) \in M_{s_n}$ for all n. For an itinerary $\mathbf{s} = s_0 \cdots s_n$ of length n (where we allow $n = \infty$), we define the n-cell $C(\mathbf{s})$ by

$$x \in C(\mathbf{s}) \text{ if and only if } T^j(x) \in M_{s_j} \text{ for all } 0 \leq j \leq n.$$

For each n there is clearly a full measure partition of M into n-cells; the excluded points are those that land on the boundary of one of the atoms after at most n iterates (after any number of iterates if $n = \infty$); the latter has zero measure, although its closure may have positive measure [3]. We define an *infinite coding* as $\mathbf{s} = \{s_i\}_{i \in \mathbb{Z}^+}$, and we say that $\mathbf{s}$ is *periodic* with period n if it can be written

$$\mathbf{s} = \overline{(s_0 s_1 \cdots s_{n-1})} = s_0 \cdots s_{n-1} s_0 \cdots s_{n-1} \cdots.$$

We now give a slightly strengthened version of results of Goetz [7, Prop 2] and Ashwin & Fu [3, Props 1 and 2].

Theorem 2.2. *Let $C(\mathbf{s})$ be a cell of a piecewise isometry T, corresponding to a finite or infinite itinerary $\mathbf{s}$. Then $C(\mathbf{s})$ is a convex region, and is open if $\mathbf{s}$ is finite.*

If $\mathbf{s}$ is a periodic (infinite) itinerary such that $C(\mathbf{s})$ is non-empty, then $C(\mathbf{s})$ has non-empty interior. Moreover, if the rotation factors $v_1, \ldots, v_N$ of T in (1) satisfy the condition

$$v_1^{n_1} v_2^{n_2} \cdots v_N^{n_N} = 1 \text{ with integers } n_k \geq 0 \Rightarrow n_k = 0 \text{ for all } k, \qquad (3)$$

then the periodic ∞-cell $C(\mathbf{s})$ is a closed disc of strictly positive radius, possibly with countably many boundary points excluded.

Proof. First consider a finite word $\mathbf{s} = s_0 \ldots s_n$. For $0 \leq k \leq n$ set $N_k = T_{s_0}^{-1} \circ \cdots \circ T_{s_{k-1}}^{-1}(M_{s_k})$. Then each N_k is a convex open set, being an isometric image of M_{s_k}. We will show by induction on n that

$$C(\mathbf{s}) = \bigcap_{k=0}^{n} N_k,$$

so that in particular, $C(\mathbf{s})$ is a convex open set. The induction starts with $C(s_0) = M_{s_0} = N_0$. Now assume the result for $C(s_0 \ldots s_{n-1})$. The restriction of T^{n-1} to this $(n-1)$-cell is $T_{s_{n-2}} \circ \ldots \circ T_{s_0}$ and its image lies in $M_{s_{n-1}}$. Thus, for $x \in C(s_0 \ldots s_{n-1})$, we have

$$x \in C(s_0 \ldots s_n) \Leftrightarrow T_{s_{n-1}}(T^{n-1}(x)) \in M_{s_n} \Leftrightarrow x \in N_n.$$

We therefore have

$$C(s_0 \ldots s_n) = C(s_0 \ldots s_{n-1}) \cap N_n = \left(\bigcap_{k=0}^{n-1} N_k \right) \cap N_n,$$

completing the induction. For an infinite word $\mathbf{s} = s_0 s_1 \ldots$, we then have

$$C(\mathbf{s}) = \bigcap_{n=0}^{\infty} C(s_0 \ldots s_n)$$

so that $C(\mathbf{s})$ is convex in this case also.

Next consider a non-empty ∞-cell $C(\mathbf{s})$ corresponding to the periodic coding $\mathbf{s} = \overline{(s_0 s_1 \cdots s_{n-1})}$. We have $T^n(C(\mathbf{s})) = C(\mathbf{s})$, since $\mathbf{s}$ has period n, so the map $\tilde{T} = T^n|_{C(\mathbf{s})}$ is an isometry from the convex set $C(\mathbf{s})$ to itself. Therefore $C(\mathbf{s})$ contains a fixed point of $\tilde{T}$; denote this by p. For $x \in C(\mathbf{s})$, we have

$$\tilde{T}(x) = v_{s_{n-1}}(v_{s_{n-2}}(\cdots(v_{s_0} x + w_{s_0})\cdots) + w_{s_{n-2}}) + w_{s_{n-1}}) = \tilde{v}\,x + \tilde{w}$$

where $\tilde{v} = v_{s_{n-1}} v_{s_{n-2}} \cdots v_{s_0} = e^{i\theta}$, say, and $\tilde{w}$ can be similarly computed. As $\tilde{T}(p) = p$, it follows that

$$\tilde{T}(x) = e^{i\theta}(x - p) + p \text{ for } x \in C(\mathbf{s}).$$

Moreover, the restriction of T^n to the n-cell $C(s_0 s_1 \ldots s_{n-1} s_0) \supset C(\mathbf{s})$ is given by the same formula. As this n-cell is open, $B_\epsilon(p) \subset C(s_0 s_1 \ldots s_{n-1} s_0)$ for some $\epsilon > 0$. Thus $T^n(B_\epsilon(p)) = B_\epsilon(p)$, so that $B_\epsilon(p) \subseteq C(\mathbf{s})$. This shows that $C(\mathbf{s})$ has non-empty interior.

For the remainder of the proof, we assume that T satisfies the hypothesis (3). Then in particular, $\tilde{v}^n \neq 1$ for all $n \geq 1$, so that $\theta \notin \pi\mathbb{Q}$. We claim that if $y \in C(\mathbf{s})$ and $|x - p| < |y - p|$ then $x \in C(\mathbf{s})$, showing that the interior of $C(\mathbf{s})$ is a disc with centre p. Let $r = |y - p|$. Then, by the irrationality of θ/π, we know that $\{\tilde{T}^n(y) : n \in \mathbb{Z}^+\}$ is dense in $A_r = \{|z - p| = r\}$. We assume without loss of generality that $p = 0$ and $x \in \mathbb{R}^+$. Let $r_1 = r$, $r_2 = ir$ and $r_3 = -ir$. Then x is in the interior of the triangle $\triangle(r_1, r_2, r_3)$, and the vertices of this triangle lie in A_r. Given $\epsilon > 0$, there exist n_1, n_2, n_3 such that

$$|\tilde{T}^{n_k}(y) - r_k| < \epsilon \quad \text{for } k = 1, 2, 3.$$

Choosing ϵ small enough, we can ensure that x lies in $\triangle(\tilde{T}^{n_1}(y), \tilde{T}^{n_2}(y), \tilde{T}^{n_3}(y))$. Since the vertices of this triangle are in the convex set $C(\mathbf{s})$, it follows that $x \in C(\mathbf{s})$, as claimed. Finally, note that points on the boundary of $C(\mathbf{s})$ that never hit a point of tangency between $C(\mathbf{s})$ and the boundary of any atom M_k will be included in $C(\mathbf{s})$; the countable set that does hit a point of tangency may be excluded from $C(\mathbf{s})$. $\qquad\square$

Remark 1. The hypothesis (3) is satisfied with $v_k = \exp(i\theta_k)$ for a full measure set of rotations $(\theta_1, \ldots, \theta_N)$ with respect to the Lebesgue measure on $[0, 2\pi)^N$. In particular, if the θ_k/π are all irrational and independent over the rationals, then (3) holds.

As a consequence of Theorem 2.2, under the hypothesis (3) there is a full measure partition of M i.e.

$$M = \mathcal{E} \cup \mathcal{R} \cup \mathcal{Z}$$

where $\mathcal{R}$ is the *regular* set of periodically coded points, $\mathcal{E}$ is the *exceptional* set $\mathcal{E}$ consisting of those ∞-cells with aperiodic itineraries, and $\mathcal{Z}$ is a set of Lebesgue measure $\ell(\mathcal{Z}) = 0$ consisting of the points whose orbits hit the boundary of one of the atoms. The discs comprising $\mathcal{R}$ may or may not give a dense packing of M.

In Figure 1 we show an example of part of a numerically computed disc packing induced by the periodically coded cells for a piecewise isometry arising in signal
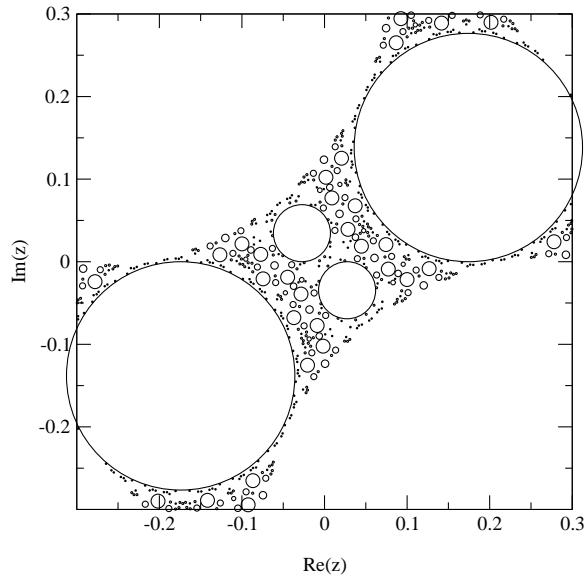
FIGURE 1. Example of part of the disc packing for a map (describing the behaviour of a bandpass Sigma-Delta modulator) that can be seen as a piecewise rotation by an irrational angle (see [5] for details).

processing: a bandpass Sigma-Delta modulator (see [5] for details). Note that the packing in the figure does not appear to have any obvious tangencies between any of the larger discs; at smaller scales there is also apparently an absence of tangencies.

2.2. **Tangencies of periodic cells.**

Theorem 2.3. *The set of tangency lines between any two discs in $\mathcal{R}$ for an orientation-preserving piecewise isometry have directions given by elements of the circle group S^1 of the form*

$$\pm v_{i_1}^{-1} \, v_{i_2}^{-1} \, \cdots v_{i_k}^{-1} \, (V_{p,q})$$

for all choices of the indices. In particular, the elements of S^1 corresponding to the directions of such tangencies all lie in the finitely generated subgroup $\langle \pm v_k^{-1}, V_{k,j} : 1 \le k \le n, 1 \le j \le l(k) \rangle$ of S^1.

Proof. We first note that the element of S^1 corresponding to a given tangency line is ambiguous up to sign: namely $\pm z \in S^1$ both determine the same tangency direction.

Suppose we have two discs C_1 and C_2 within $\mathcal{R}$ that have itineraries $\mathbf{s^1}$ and $\mathbf{s^2}$ with periods n_1 and n_2. Suppose that they are tangent at some point $p = \overline{C_1} \cap \overline{C_2}$; then it follows that for $n = \mathrm{lcm}(n_1, n_2)$ the discs must lie within different n-cells. (If we consider the partition into n-cells for this n then C_1 and C_2 must lie within different n-cells, otherwise $s_k^1 = s_k^2$ for all $0 \le k \le n$).

Hence if L_p is a line segment that forms a mutual tangent between any two discs in $\mathcal{R}$, L_p must be a preimage of one of the lines in (2) under an iterate of T. If we write
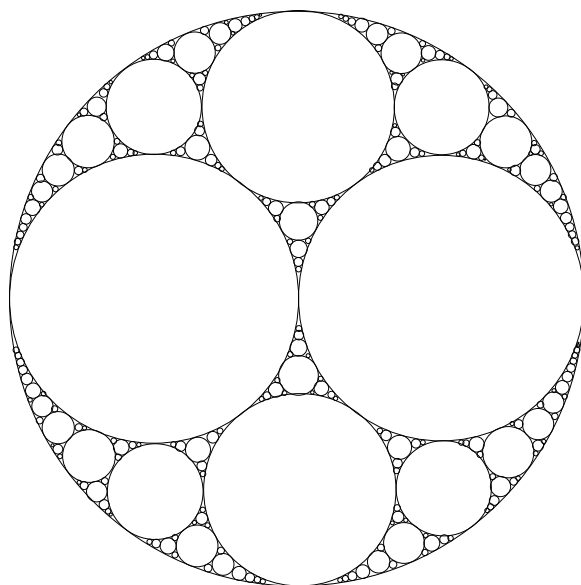
$$L_p = \chi \mathbb{R} + p$$

FIGURE 2. An Apollonian packing.

for direction $\chi \in S^1$, χ must be within $K_n\{V_{k,j}\}$ where K_n is the set of finite words in $\langle v_1^{-1}, \cdots, v_N^{-1} \rangle$ of length n. $\qquad \square$

3. **An excluded Arbelos packing.** One might expect that in a dense disc packing of some region M of the complex plane, each disc would necessarily be tangent to other discs in the packing. This turns out to be far from the case. Indeed, a dense disc packing of an equilateral triangle is exhibited in [3], based on the construction of the Sierpinski gasket, in which no tangencies occur between the discs.

At the other extreme, there are disc packings in which every disc is tangent to infinitely many others. This occurs in the best-known family of dense disc packings, namely the Apollonian packings. These packings were introduced by Apollonius of Perga and known in classical Greek times because of their intrinsic beauty and elegance. They have been extensively studied ever since, and several famous mathematicians, from Descartes to Coxeter, have discovered interesting properties (see [11] for details). An example of an Apollonian packing of a circle is depicted in Figure 2. We may regard this as a packing of the Riemann sphere, with the region exterior to the bounding circle viewed as a disc containing the point at infinity [1].

We define an Apollonian packing of a disc D to be an infinite union of closed discs in D such that

- their interiors are disjoint;
- each disc is in the space formed by, and is tangent to, either three other discs or two other discs and the boundary of D;
- given any three mutually tangent discs (or two discs tangent with each other and the bounding circle) there is another disc of the packing, tangent to all three of the given discs (or to both given discs and the bounding circle).
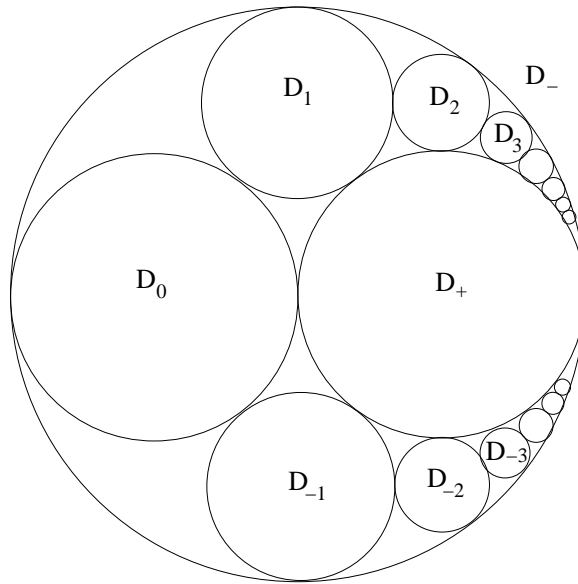
FIGURE 3.   The Arbelos packing of D_- defined in Section 3 with $z_0 = 0$. We show in Theorem 3.2 that the angles of tangency of the packing to the disc D_+ do not lie in a finitely generated subgroup of S^1.

Any such packing is necessarily dense. An Apollonian packing of D is uniquely determined by any two mutually tangent discs of the packing which are both internally tangent to the boundary of D.

Any Möbius transformation of the Riemann sphere takes an Apollonian packing to another Apollonian packing. If a point on the boundary of one of the discs is mapped to the point at infinity, the disc is mapped to a half-plane.

In [1] different kinds of Apollonian subpackings are studied, depending on the configuration they produce. Three basic configurations are the Spiral, the Ring, and the Knife or *Arbelos*; these are sequences of discs generated by iterating a certain transformation.

Our goal now is to deduce from Theorem 2.3 that one particular Arbelos packing $\mathcal{A}$ of the unit disc $D_- := \{z : |z| \leq 1\}$ can never arise from a piecewise isometry satisfying the condition (3). This is the content of Corollary 1 below. Later on, in Theorem 4.4, we will generalise this result to cover an infinite family of Arbelos packings of D_-.

The Arbelos packing we consider is adapted from [1], and is illustrated in Figure 3. We can generate it using the Möbius transformation

$$f(z) = \zeta \frac{z - Z}{1 - z\overline{Z}} \tag{4}$$

where

$$\zeta = \frac{3 - 4i}{5}, \ Z = \frac{1 - 2i}{5}.$$

For any $|\zeta| = 1$ and $Z \in D_-$, $f(z)$ maps the unit circle to itself, and we have chosen Z and ζ in such a way that $f(1) = 1$ and $f(0) = (1 + 2i)/5$. Note that other values of Z and ζ will define other Arbelos packings of D_-.

This f fixes the discs D_- and $D_+ := \{z : |z - \frac{1}{2}| < \frac{1}{2}\}$, and maps the disc $D_0 := \{z : |z + \frac{1}{2}| < \frac{1}{2}\}$ onto an infinite family of discs

$$D_k := f^{k-1}(D_0)$$

for $k \geq 1$. Note that each disc D_k (for $k > 1$) is tangent to the disc D_- (internally) and the discs D_+, D_{k-1} and D_{k+1} (externally). However we will see that the configuration generated by f on $\mathbb{C}$ cannot be produced by any piecewise isometry.

We write this Arbelos packing

$$\mathcal{A} := \{D_k \ : \ k \in \mathbb{Z}\}$$

and note that any Apollonian packing of the unit disc will contain a packing of this form up to a Möbius transformation (see [1] for details). We focus on the sequence of tangencies of D_k with D_+ and aim to show that this cannot appear within the regular set for any piecewise isometry. Writing the tangency points

$$z_k = \overline{D}_k \cap \overline{D}_+$$

we obtain an explicit formula for z_k (note that f maps a neighbourhood of z_k onto a neighbourhood of z_{k+1}).

Lemma 3.1. *Let z_k be defined as above. Then*

$$z_k = \frac{k}{k - 2i} = \frac{k^2 + 2ki}{k^2 + 4}.$$

Moreover the slope V_k of the common tangent of D_k and D_+ is given by

$$V_k = -\frac{k^2 - 4}{4k}$$

and the unit complex number v_k giving the direction of tangency at z_k, is given by

$$v_k = \frac{i(k + 2i)}{k - 2i} = \frac{(k^2 - 4)i - 4k}{k^2 + 4}.$$

Proof. We first prove the formula for z_k by induction. Note that the formula stated gives $z_0 = 0$, which is the tangent point between D_0 and D_+. Assuming the formula for z_k, we have

$$\begin{aligned}
f(z_k) &= \frac{3 - 4i}{5} \times \frac{5k - (k - 2i)(1 - 2i)}{5(k - 2i) - k(1 + 2i)} \\
&= \frac{(2 - i)^2}{(2 - i)(2 + i)} \times \frac{(2 + i)(2k + 2)}{(2 - i)(2k + 2 - 4i)} \\
&= \frac{(k + 1)}{k + 1 - 2i}.
\end{aligned}$$

We easily verify that this point lies on the boundary of D_+, so it is indeed the tangent point between D_{k+1} and D_+, as asserted.

We next prove the formula for the unit complex numbers v_k giving the direction of tangency at the point z_k. Noting that $v_0 = i$, we see that the formula stated is correct for $k = 0$. Now

$$v_{k+1} = \frac{(\frac{df}{dz}(z_k))}{|\frac{df}{dz}(z_k)|} v_k$$

where

$$\frac{df}{dz} = \frac{4(3 - 4i)}{(z + 2iz - 5)^2}.$$

Assuming the stated expressions for z_k and v_k, we calculate

$$v_{k+1} = \frac{4(3-4i)(k-2i)^2}{(2i-4)^2(k+1-2i)^2} \times \frac{20|k+1-2i|^2}{20|k-2i|^2} \times \frac{i(k+2i)}{k-2i}.$$

Simplifying the above expression gives the value stated for v_{k+1}, and the result follows by induction.

Finally, we obtain the formula for the slope V_k of the tangent by observing that if $v_k = x + iy$ then $V_k = y/x$. □

Theorem 3.2. *For the Arbelos packing $\mathcal{A}$, the sequence v_n does not lie within any finitely generated subgroup of the group S^1.*

Proof. It suffices to show that the subgroup $H = \langle v_1, v_2, \ldots \rangle$ of S^1 is not finitely generated. We have $v_n = (p_n + q_n i)/r_n$ where $p_n = -4n$, $q_n = n^2 - 4$ and $r_n = n^2 + 4$. Any odd prime factor of p_n must divide n, and therefore cannot divide r_n. Thus p_n and r_n have no odd prime factors in common.

If H were finitely generated, we could choose N large enough that $v_1, \ldots, v_N$ generate H. Then, for each m, every odd prime factor of r_m must occur in one of $r_1, \ldots, r_N$. Take m to be the product of all odd prime factors of $r_1, \ldots, r_N$. Then $r_m = m^2 + 4$ is odd, and has at least one odd prime factor which does not occur amongst the prime factors of $r_1, \ldots, r_N$. This contradicts the choice of N, showing that the group H cannot be finitely generated. □

Corollary 1. *The Arbelos packing $\mathcal{A}$ cannot arise as a subpacking of the regular set $\mathcal{R}$ of cells of any orientation-preserving piecewise isometry satisfying the condition (3).*

Proof. This follows on noting from Theorem 3.2 that the subgroup of S^1 generated by the tangency directions of $\mathcal{A}$ is not finitely generated, whereas for any piecewise isometry with polygonal atoms, Theorem 2.3 states that the allowable tangencies between periodic cells must be contained in a finitely generated subgroup of S^1. □

4. **An infinite family of excluded Arbelos packings.** In §3 above, we considered one special Arbelos packing $\mathcal{A}$. We now work more generally, and consider a one-parameter family of Arbelos packings. As before, these will be packings of the region between the circles $D_- = \{z \in \mathbb{C} : |z| = 1\}$ and $D_+ = \{z \in \mathbb{C} : |z - \frac{1}{2}| = \frac{1}{2}\}$, but we will now start with an arbitrary circle D_0, internally tangent to D_- and externally tangent to D_+. The choice of D_0 then uniquely determines the packing: the discs of the packing are $f^k(D_0)$ for $k \in \mathbb{Z}$, with f as in (4). An example of such a packing is shown in Figure 4.

To find the tangency points $z_k = \overline{D_+} \cap \overline{D_k}$ in such a packing, we transform our original Arbelos packing $\mathcal{A}$ into a packing of an infinite vertical strip, as shown in Figure 5.

Lemma 4.1. *Let $I : \mathbb{C} \to \mathbb{C}$ be the Möbius transformation given by*

$$I(z) = \frac{1}{z} + 1. \tag{5}$$

Then $I^{-1}(D_+)$ and $I^{-1}(D_-)$ are the straight lines $x = -1$ and $x = -\frac{1}{2}$ respectively. In particular, I^{-1} takes the tangency point z_k of $\mathcal{A}$ to

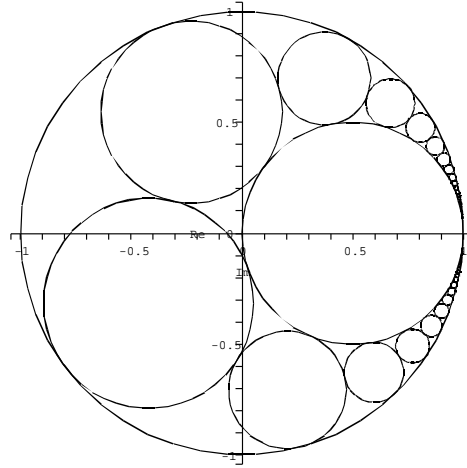$$\tilde{I}_k := I^{-1}(z_k) = -1 - \frac{ik}{2} \tag{6}$$

FIGURE 4.   The modified Arbelos packing, defined by taking $z_0 = \frac{1+3i}{10}$ as a point of tangency to the disc centred at $z = \frac{1}{2}$.

for $k = 0, 1, 2, \ldots$.

Proof. Clearly I is an analytic function that preserves orientation and tangencies, and takes the common point $z = 1$ of D_- and D_+ to ∞. We can write points on D_- or D_+ as $z = e^{i\theta}$ or $z = \frac{1}{2}(e^{i\theta} + 1)$ respectively, for $\theta \in [0, 2\pi)$. A straightforward calculation yields that $I^{-1}(w) = \frac{1}{w-1}$. Then by direct computation we obtain

$$I^{-1}(e^{i\theta}) = -\frac{1}{2} - i\frac{\sin\theta}{2(1 - \cos\theta)} \tag{7}$$

and

$$I^{-1}\left(\frac{1}{2}(e^{i\theta} + 1)\right) = -1 - i\frac{\sin\theta}{1 - \cos\theta} \tag{8}$$

Therefore, I^{-1} maps D_- onto the line (7) and D_+ onto the line (8), as shown in Figure 5.

Using the expression for z_k in Lemma 3.1, we then have

$$\tilde{I}_k = \frac{1}{z_k - 1} = \frac{k - 2i}{k - (k - 2i)},$$

which simplifies to (6).   □

An arbitrary Arbelos packing of the region between D_- and D_+ corresponds under I^{-1} to a packing of the strip $\{-1 \leq x \leq -\frac{1}{2}\}$ in which successive discs are mutually tangent. Any such packing must be obtained from the packing in Figure 5 by a vertical shift. To obtain the preimages under I of the tangency points z_k in the new Arbelos packing, we therefore simply replace (6) by

$$\tilde{I}_k = -1 - \left(\frac{k + \beta}{2}\right)i, \tag{9}$$

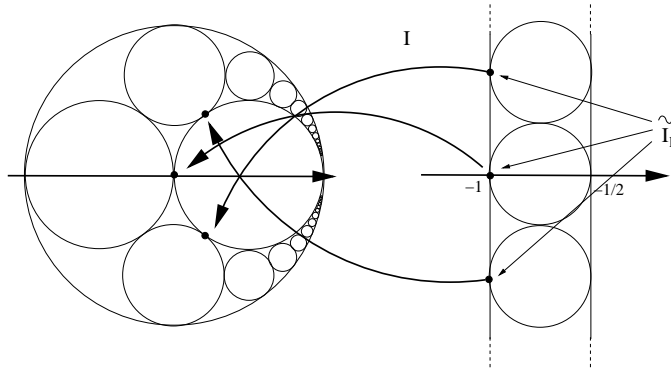where the parameter β is an arbitrary real number depending on the choice of D_0.

FIGURE 5.   The transformation described in Section 4. In fact
each tangency point z_k in the Arbelos between D_0 and all the
other circles $D_1, \ldots$ is associated to a tangency point between a
family of circles of unit radius and a vertical line $x = -1$.

Theorem 4.2. *Consider any Arbelos packing $\{D_k : k \in \mathbb{Z}\}$ of the region between*
D_- and D_+, and let $\beta \in \mathbb{R}$ be the corresponding parameter as above. Then the
tangency points $z_k = \overline{D}_+ \cap \overline{D}_k$ of the packing are given by

$$z_k = \frac{k + \beta}{k + \beta - 2i} = \frac{(k+\beta)^2 + 2i(k+\beta)}{4 + (k+\beta)^2},$$

and the unit complex numbers in the directions of tangency are

$$v_k = \frac{-4(k+\beta)}{(k+\beta)^2 + 4} + i\frac{(k+\beta)^2 - 4}{(k+\beta)^2 + 4}.$$

In particular, both z_k and v_k have the form

$$\frac{p(k)}{r(k)} + i\frac{q(k)}{r(k)}$$

where $p(k)$, $q(k)$, $r(k)$, are real polynomials (depending on β) of degree at most 2.

Proof. The formula for z_k is immediate from (9); we have

$$z_k = I(\tilde{I}_k) = \frac{1}{\tilde{I}_k} + 1 = \frac{k + \beta}{k + \beta - 2i}.$$

The radius of D_+ passing through z_k is represented by the complex number

$$z_k - \frac{1}{2} = \frac{k + \beta + 2i}{2(k + \beta - 2i)} = \frac{(k+\beta)^2 - 4}{4((k+\beta)^2 + 4)} + i\frac{4(k+\beta)^2}{4((k+\beta)^2 + 4)},$$

with modulus $\frac{1}{2}$. To obtain a unit complex number perpendicular to this, we simply
multiply by $2i$, which gives the stated expression for v_k.   □

Taking $\beta = 0$ in Theorem 4.2, we obtain the formulae of Lemma 3.1. The example
shown in Figure 4 corresponds to $\beta = 2/3$.

We now generalise Theorem 3.2 to the family of Arbelos packings in Theorem
4.2.

Theorem 4.3 (Non-Finite Generation of Subgroups of Circle Groups)**.** *Let*

$$v_k = \frac{p(k)}{r(k)} + i\frac{q(k)}{r(k)}, \qquad k = 0, 1, 2, \ldots$$

be the sequence of unit complex numbers giving the tangency directions for any Arbelos packing as in Theorem 4.2. Then the subgroup

$$H = \langle v_0, v_1, \ldots, \rangle$$

of the circle group $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is not finitely generated.

Proof. We adapt the proof of Theorem 3.2, using the explicit formula for v_k (in terms of the parameter β) given in Theorem 4.2. Recall that to prove Theorem 3.2, we found a large prime factor occurring in the sequence of denominators of the v_k, and therefore implicitly used the uniqueness of prime factorisations in $\mathbb{Z}$. This time, however, we do not in general have $\beta \in \mathbb{Z}$, and we will need to use an appropriate result on the uniqueness of factorisations in a less elementary setting.

From Theorem 4.2 we have $p(k) = -4(k + \beta)$, $q(k) = (k + \beta)^2 - 4$ and $r(k) = 4 + (k + \beta)^2$.

Suppose for a contradiction that H is finitely generated, so for some $N \in \mathbb{N}$ we have

$$H = \langle v_0, v_1, v_2, \ldots, v_N \rangle.$$

Then for each k, the real and imaginary parts $p(k)/r(k)$ and $q(k)/r(k)$ of v_k lie in the ring

$$R = \mathbb{Z}\left[\beta, \frac{1}{4 + \beta^2}, \frac{1}{4 + (1 + \beta)^2}, \ldots, \frac{1}{4 + (N + \beta)^2}\right],$$

obtained by first adjoining β to $\mathbb{Z}$ and then inverting the denominators $r(1), \ldots r(N)$. In particular, writing $p(k)$ as an element of R and "clearing denominators", we have the equation in $\mathbb{Z}[\beta]$:

$$4(k + \beta) \prod_{j=0}^{N} \left(4 + (j + \beta)^2\right)^{e_j} = \left(4 + (k + \beta)^2\right) f \tag{10}$$

for some integers $e_0, \ldots, e_N \geq 0$ and some $f \in \mathbb{Z}[\beta]$.

We will show that k can be chosen so that (10) gives rise to a contradiction. To do so, we must distinguish the cases when β is transcendental or algebraic over $\mathbb{Q}$.

Case 1: β is transcendental.
Then $\mathbb{Q}[\beta] \cong \mathbb{Q}[X]$ is a Unique Factorisation Domain: up to multiplication by an element of $\mathbb{Q} \setminus \{0\}$, any non-zero polynomial can be written as product of monic irreducible polynomials, and this factorisation is unique up to the order of the irreducible factors. Now if $k > N$ then $k + \beta$, $4 + (j + \beta)^2$ for $0 \leq j \leq N$, and $4 + (k + \beta)^2$ are distinct monic irreducible polynomials. So $4 + (k + \beta)^2$ is an irreducible factor on the right in (10) which does not occur on the left. This contradicts the unique factorisation property, showing that H cannot be finitely generated.

Case 2: β is algebraic over $\mathbb{Q}$.
We work in the real algebraic number field $F = \mathbb{Q}(\beta)$, and use some standard facts from algebraic number theory. Let O_F denote the ring of algebraic integers in F. In O_F we do not necessarily have unique factorisation of elements into prime elements, but we do have unique factorisation of ideals into prime ideals. In particular, for $\alpha \in O_F \setminus \{0\}$, we may define $v_{\mathfrak{p}}(\alpha) \geq 0$ for each non-zero prime ideal $\mathfrak{p}$ of O_F as the exponents in the factorisation of the principal ideal generated by α:

$$\alpha O_F = \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(\alpha)},$$

where the product is over all non-zero prime ideals $\mathfrak{p}$. Then $\alpha \in \mathfrak{p}$ if and only if $v_\mathfrak{p}(\alpha) > 0$.

There is some integer $m \neq 0$ so that $m\beta \in O_F$. Let $\gamma = m\beta$. Multiplying (10) by a large enough power of m, and taking the ideal generated by each side, we obtain the following equality between ideals of O_F:

$$4m^t(mk + \gamma) \prod_{j=0}^{N} \left(4 + (mj + \gamma)^2\right)^{e_j} O_F = \left(4m^2 + (mk + \gamma)^2\right) \tilde{f} O_F \qquad (11)$$

for some $\tilde{f} \in O_F$ and some $t \geq 0$. Let

$$\theta = 2m \prod_{j=0}^{N} \left((mj + \gamma)^2 + 4m^2\right) \in O_F.$$

Then θ is independent of k, and any prime ideal $\mathfrak{p}$ dividing the left-hand side of (11) must divide either θO_F or $(mk + \gamma)O_F$.

Now let $k \in \mathbb{Z}^+$ lie in the ideal $\theta^2 O_F$ of O_F: we may for instance take k to be any positive multiple of $N_{F/\mathbb{Q}}(\theta^2)$, where $N_{F/\mathbb{Q}}$ denotes the norm map from F to $\mathbb{Q}$. Let $\mathfrak{p}$ be a prime ideal dividing $\left((mk + \gamma)^2 + 4m^2\right) O_F$. Then $\mathfrak{p}$ divides the right-hand side of (11), and hence divides either θO_F or $(mk + \gamma)O_F$. But if $\mathfrak{p}$ contains both $(mk + \gamma)^2 + 4m^2$ and $mk + \gamma$ then it also contains contains $4m^2$, and therefore (being prime) contains $2m$, which divides θ. It follows that *every* prime ideal $\mathfrak{p}$ dividing $\left((mk + \gamma)^2 + 4m^2\right) O_F$ in fact divides θO_F. But

$$(mk + \gamma)^2 + 4m^2 \equiv 4m^2 + \gamma^2 \pmod{\theta^2 O_F}$$

by the choice of k, and

$$v_\mathfrak{p}(4m^2 + \gamma^2) \leq v_\mathfrak{p}(\theta) < v_\mathfrak{p}(\theta^2)$$

because $4m^2 + \gamma^2$ divides θ. It follows that

$$v_\mathfrak{p}\left((mk + \gamma)^2 + 4m^2\right) = v_\mathfrak{p}(4m^2 + \gamma^2)$$

for all prime factors $\mathfrak{p}$ of $\left((mk + \gamma)^2 + 4m^2\right) O_F$. Hence this ideal divides the ideal $(\gamma^2 + 4m^2)O_F$. Taking norms, it follows that $N_{F/\mathbb{Q}}\left((mk + \gamma)^2 + 4m^2\right)$ divides $N_{F/\mathbb{Q}}(\gamma^2 + 4m^2)$ in $\mathbb{Z}$.

Since $N_{F/\mathbb{Q}}\left((mk + \gamma)^2 + 4m^2\right)$ is a non-constant polynomial in k (it is the product of the distinct conjugates of the algebraic number $(mk + \gamma)^2 + 4m^2$), we may choose $k \in \theta^2 O_F \cap \mathbb{Z}^+$ large enough to ensure that

$$|N_{F/\mathbb{Q}}\left((mk + \gamma)^2 + 4m^2\right)| > |N_{F/\mathbb{Q}}(\gamma^2 + 4m^2)|.$$

This contradicts that fact that $N_{F/\mathbb{Q}}(\gamma^2 + 4m^2)$ divides $|N_{F/\mathbb{Q}}\left((mk + \gamma)^2 + 4m^2\right)|$, showing that H cannot be finitely generated. $\qquad \square$

Hence we arrive at the following result, of which Corollary 1 is a special case:

Theorem 4.4. *No Arbelos packing of the region between $D_- = \{z \in \mathbb{C} : |z| < 1\}$ and $D_+ = \{z \in \mathbb{C} : |z - \frac{1}{2}| < \frac{1}{2}\}$ can appear as a subpacking of the regular set $\mathcal{R}$ of any orientation-preserving piecewise isometry.*

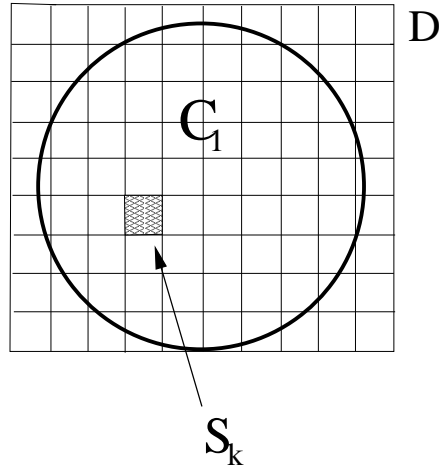*Proof.* This is immediate from Theorems 4.3 and 2.3. $\qquad \square$

FIGURE 6.   Example of Theorem 5.1: we can approximate the circle C_1 by defining a suitable region D.

As pointed out to us by one of the referees, the Arbelos packings considered in Theorem 4.4 are not entirely arbitrary: to obtain a general Arbelos packing (up to affine change of coordinates) we would need to allow arbitrary values not only for the shift parameter β but also for the radius R of D_+, here fixed at $1/2$. The expressions for v_k in Theorem 4.2 would then involve the two arbitrary parameters β and R. Instead of the two cases in the proof of Theorem 4.3, there would be a number of possibilities to consider, including the case where β and R are both transcendental over $\mathbb{Q}$ but satisfy an algebraic relation amongst themselves. We believe that our techniques can be extended to handle this more general situation, but, due to the additional technical difficulties involved, we have not been able to treat arbitrary Arbelos packings in the present paper.

5. **Approximating a packing by a piecewise isometry.** Although Theorem 4.4 shows that a given Arbelos packing may not arise as the regular set of a piecewise isometry, we can define a piecewise isometry whose regular set "approximates" the Arbelos as closely as we wish. In fact we have the following:

Theorem 5.1. *Let A be a non-overlapping disc packing of a polygonal region D. Given $\epsilon > 0$ there exists an orientation-preserving piecewise isometry (T, D) on D with regular set B such that $\ell(A \triangle B) < \epsilon$, where $A \triangle B$ denotes the symmetric difference of the sets A and B.*

Proof. Cover D with non-overlapping open squares $S_1, \ldots S_N$ of rational side length δ, so $D \subseteq \cup_{j=1}^N \overline{S}_k$ (see Figure 6). We assume, without loss of generality, that $S_j \cap D \neq \emptyset$ for each j, and that the squares are numbered so that $S_j \subseteq A$ for $j = 1, \ldots p$; $S_j \subseteq D$ but $S_j \not\subseteq A$ for $j = p+1, \ldots q$; and $S_j \not\subseteq D$ for $j = q+1, \ldots, N$. We set

$$B = B^{(1)} \cup B^{(2)}$$

where

$$B^{(1)} = \bigcup_{j=1}^p S_j, \qquad B^{(2)} = \left(\bigcup_{j=q+1}^N S_j \right) \cap D = \bigcup_{j=q+1}^N \left(S_j \cap D \right).$$

We now construct a piecewise isometry $T : D \to D$ with partition

$$S_1, \ldots S_q, S_{q+1} \cap D, \ldots S_N \cap D$$

and regular set B.

We define T to be the identity on $S_1, \ldots, S_p$ and also on $S_{q+1} \cap D, \ldots, S_N \cap D$. Then B is contained in the regular set of T. Consider now the squares $S_{p+1}, \ldots, S_q$. We define T on each of these squares as an irrational interval exchange in one coordinate corresponding to some chosen irrational number λ: without loss of generality, we may assume the square in question is $[0, \delta]^2$, and we then define T on this square by $T(x, y) = T((x + \lambda) \bmod \delta, y)$. This is a piecewise isometry on the square, with two rectangular atoms, and (for example, on setting $\lambda = \delta/\sqrt{2}$) examination of the first coordinate shows that the orbit of any point has an aperiodic itinerary. The squares $S_{p+1}, \ldots, S_q$ therefore belong to the exceptional set of T.

It remains to show that δ can be chosen small enough to ensure that $\ell(A \triangle B) < \epsilon$. Since $B = B^{(1)} \cup B^{(2)}$ and $B^{(1)} \subseteq A$, we have

$$A \triangle B = (A \setminus B) \cup (B \setminus A) \subseteq (A \setminus B^{(1)}) \cup B^{(2)},$$

so that

$$\ell(A \triangle B) \leq \ell(A \backslash B^{(1)}) + \ell(B^{(2)}).$$

Take $\tilde{A} \subset A$, where $\tilde{A} = \cup_{k=1}^M D_k$ consists of a finite number of discs of A and is chosen so that $\ell(A \backslash \tilde{A}) < \epsilon/2$. For each disc D_k in $\tilde{A}$, let $B_k = D_k \cap B$. Thus B_k is the union of those squares S_j in the grid for which $S_j \subseteq D_k \subset A$, so that $B_k \subseteq B^{(1)}$. The diameter of each square S_j is $\sqrt{2}\delta$. So if the radius r_k of D_k satisfies $r_k > \sqrt{2}\delta$ then B_k contains a disc of radius $r_k - \sqrt{2}\delta$. Thus

$$\ell(D_k \setminus B_k) \leq \pi r_k^2 - \pi(r_k - \sqrt{2}\delta)^2 = 2\pi r_k \sqrt{2}\delta - 2\pi \delta^2 < 2\pi r_k \sqrt{2}\delta.$$

On the other hand, if $r_k \leq \sqrt{2}\delta$ then

$$\ell(D_k \setminus B_k) \leq \ell(D_k) = \pi r_k^2 < 2\pi r_k^2 \leq 2\pi r_k \sqrt{2}\delta.$$

Hence

$$\ell(\tilde{A} \setminus B^{(1)}) = \sum_{k=1}^M \ell(D_k \setminus B_k) < \sum_{k=1}^M 2\pi r_k \delta \sqrt{2} = C\sqrt{2}\delta$$

where $C = \sum_{k=1}^M 2\pi r_k$ is the total circumference of the discs in $\tilde{A}$.

Since $B^{(2)}$ is contained in the union of those squares overlapping the boundary of D, and each square has diameter $\sqrt{2}\delta$, we have

$$\ell(B^{(2)}) \leq 2\sqrt{\delta}R$$

where R is the length of the boundary of D.

Putting all this together, we have

$$\begin{aligned}
\ell(A \triangle B) &\leq \ell(A \setminus \tilde{A}) + \ell(\tilde{A} \setminus B^{(1)}) + \ell(B^{(2)}) \\
&< \epsilon/2 + \sqrt{2}\delta C + \sqrt{2}\delta R.
\end{aligned}$$

Choosing (rational) $\delta < \epsilon/2\sqrt{2}(C + R)$, we have $\ell(A \triangle B) < \epsilon$ as required. $\square$

REFERENCES

[1] D. Aharonov and K. Stephenson, *Geometric sequences of discs in the Apollonian packing,* Algebra i Analiz, **9** (1997), 1004–140.

[2] P. Agarwal and J. Pach, "Combinatorial Geometry," New York: Wiley-Interscience, 1995.

[3] P. Ashwin and X.-C. Fu, *Tangencies in invariant disc packings for certain planar piecewise isometries are rare,* Dynamical Systems, **16** (2001), 333–345

[4] P. Ashwin and X.-C. Fu, *On the geometry of orientation-preserving planar piecewise isometries,* J. Nonlinear Sci., **12** (2002), 207–240.

[5] P. Ashwin and X.-C. Fu and J. Deane, *Properties of the invariant disk packing in a model bandpass Sigma-Delta modulator,* Internat. J. Bifur. Chaos Appl. Sci. Engrg., (2003), 631–641.

[6] C. Collins and K. Stephenson, *A circle packing algorithm,* Comput. Geom., **25** (2003), 233–256.

[7] A. Goetz, *Stability of piecewise rotations and affine maps,* Nonlinearity, **14** (2001), 205–219.

[8] A. Goetz, *Dynamics of piecewise isometries,* Illinois J. Math., **44** (2000), 465–478.

[9] A. Goetz, *A self-similar example of a piecewise isometric attractor,* Dynamical Systems (Luminy-Marseille), (1998), 248–258.

[10] Z. A. Melzak, *Infinite Packing of Disks,* Canad. J. Math., **18** (1966), 838–852.

[11] K. Stephenson, *Circle packing: a mathematical tale,* Notices of the AMS, **50** (2003), 1376–1388.

[12] R.-Z. Yu, X.-F. Fu and S.-L. Shui, *Density of invariant disk packings for planar piecewise isometries,* Dynamical Systems, **22** (2007), 65–72.

E-mail address: `M.Trovati@ex.ac.uk`

E-mail address: `P.Ashwin@ex.ac.uk`

E-mail address: `N.P.Byott@ex.ac.uk`