Hat problem on a graph

Submitted by

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I certify that all material in this thesis which is not my own work has been identified and that no material has previously been submitted and approved for the award of a degree by this or any other University.

I dedicate to my Mom

whose son is to be a doctor, but not the kind who cures people.*

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*The idea taken from Professor Randy Pausch**: "After I got my PhD, my mother took great relish in introducing me by saying: «This is my son. He's a doctor, but not the kind who helps people»", R. Pausch, *Last Lecture*, p. 24, Hyperion 2008.

**Randolph Frederick Pausch (October 23, 1960 – July 25, 2008) was an American professor of computer science, human-computer interaction and design at Carnegie Mellon University in Pittsburgh, Pennsylvania, and a best-selling author who achieved worldwide fame for his "The Last Lecture" speech on September 18, 2007 at Carnegie Mellon. The lecture was conceived after Pausch learned, in summer 2007, that his previously known pancreatic cancer was terminal. In May 2008, Pausch was listed by *Time* as one of the World's Top-100 Most Influential People.

ABSTRACT

The topic of this thesis is the hat problem. In this problem, a team of n players enters a room, and a blue or red hat is randomly placed on the head of each player. Every player can see the hats of all of the other players but not his own. Then each player must simultaneously guess the color of his own hat or pass. The team wins if at least one player guesses his hat color correctly and no one guesses his hat color wrong, otherwise the team loses. The aim is to maximize the probability of winning.

This thesis is based on publications, which form the second chapter. In the first chapter we give an overview of the published results.

In Section 1.1 we introduce to the hat problem and the hat problem on a graph, where vertices correspond to players, and a player can see the adjacent players.

To the hat problem on a graph we devote the next few sections. First, we give some fundamental theorems about the problem. Then we solve the hat problem on trees, cycles, and unicyclic graphs. Next we consider the hat problem on graphs with a universal vertex. We also investigate the problem on graphs with a neighborhood-dominated vertex. In addition, we consider the hat problem on disconnected graphs. Next we investigate the problem on graphs such that the only known information are degrees of vertices. We also present Nordhaus-Gaddum type inequalities for the hat problem on a graph.

In Section 1.6 we investigate the hat problem on directed graphs.

The topic of Section 1.7 is the generalized hat problem with $q \geq 2$ colors.

A modified hat problem is considered in Section 1.8. In this problem there are $n \geq 3$ players and two colors. The players do not have to guess their hat colors simultaneously and we modify the way of making a guess. We give an optimal strategy for this problem which guarantees the win.

Applications of the hat problem and its connections to different areas of science are presented in Section 1.9. We also give there a comprehensive list of variations of the hat problem considered in the literature.

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