Applications of transfer operator methods to the dynamics of low-dimensional piecewise smooth maps

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I certify that all material in this thesis which is not my own work has been identified and that no material has previously been submitted and approved for the award of a degree by this or any other University.

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Abstract

This thesis primarily concentrates on stochastic and spectral properties of the transfer operator generated by piecewise expanding maps (PWEs) & piecewise isometries (PWIs). We also consider the applications of the transfer operator in thermodynamic formalism. The original motivation stems from studies of one-dimensional PWEs. In particular, any one dimensional mixing PWE admits a unique absolutely continuous invariant probability measure (ACIP) and this ACIP has a bounded variation density. The methodology used to prove the existence of this ACIP is based on a so-called functional analytic approach and a key step in this approach is to show that the corresponding transfer operator has a spectral gap. Moreover, when a PWE has Markov property this ACIP can also be viewed as a Gibbs measure in thermodynamic formalism.

In this thesis, we extend the studies on one-dimensional PWEs in several aspects. First, we use the functional analytic approach to study piecewise area preserving maps (PAPs) in particular to search for the ACIPs with multidimensional bounded variation densities. We also explore the relationship between the uniqueness of ACIPs with bounded variation densities and topological transitivity/ minimality for PWIs.

Second, we consider the mixing and corresponding mixing rate properties of a collection of piecewise linear Markov maps generated by composing $x \mapsto mx \mod 1$ with permutations in S_N. We show that typical permutations preserve the mixing property under the composition. Moreover, by applying the Fredholm determinant approach, we calculate the mixing rate via spectral gaps and obtain the max/min spectral gaps when m, N are fixed. The spectral gaps can be made arbitrarily small when the permutations are fully refined.

Finally, we consider the computations of fractal dimensions for generalized Moran constructions, where different iteration function systems are applied on different levels. By using the techniques in thermodynamic formalism, we approximate the fractal dimensions via the zeros of the Bowen's equation on the pressure functions truncated at each level.

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Nomenclature

a.e. Almost everywhere

ACIP Absolutely continuous invariant probability measure

BV Space of functions with bounded variation

IET Interval exchange transformation

PAP Piecewise area preserving map

PWE Piecewise expanding map

PWI Piecewise isometry

$\mathcal{M}(X)$ Set of Borel probability measures on a compact set X

$\mathcal{M}_I(T)$ Set of probability invariant measures with respect to a transformation T

$\mathcal{M}_{IA}(T)$ Set of ACIPs with respect to a transformation T

$\mathcal{M}_{IB}(T)$ Set of ACIPs having bounded variantion densities with respect to a transformation T

$\mathcal{M}_{IC}(T)$ Set of ACIPs having a.e. continuous densities with respect to a transformation T

GMSC Generalized Moran structure conditions

GMS Generalized Moran set

UE/LE Upper/lower estimation

Chapter 1

Introduction

1.1 Background on piecewise smooth systems

Dynamical systems theory has proved to be a powerful tool with which to analyze and understand the behaviours of a diverse range of systems. For a system that is defined by a smooth function of its arguments, there has been a well developed qualitative approach in understanding the behaviours of many important physical phenomena such as fluid flows, elastic deformation and nonlinear optical systems. However, there are many systems in nature for which the functional dependence on the arguments are not smooth or even discontinuous. Examples include electrical circuits with switches, mechanical devices where components impact with each other and so on. These systems are event driven in the sense that smoothness or continuity is lost at instantaneous events (for example, upon application of a switch), and can be characterized by piecewise smooth functions. They have fascinating dynamics and a rich underlying mathematical structure, but are not easily described in terms of the modern qualitative theory of dynamical systems.

In the work presented, we will primarily provide a detailed study on two classes of piecewise smooth maps with chaotic behaviour. They are *piecewise expanding maps* (PWEs) which are locally and uniformly expanding at any directions and *piecewise isometries* (PWIs) where Euclidean distance is locally preserved. When acting on a compact interval I, a map f is piecewise expanding if its derivative is uniformly away from ± 1, i.e., there exists a κ such that for any $x \in I$, $|f'(x)| > \kappa > 1$; a map f is piecewise isometry if $|f'(x)| = 1$ for any $x \in I$. Both PWEs and PWIs have been shown individually to exhibit beautiful dynamical properties [14, 43]. Moreover, there are also increasing evidences showing that a combination of PWEs and PWIs further creates many interesting systems with non-trivial dynamical behaviours, for instance intermittency maps [12, 55, 57, 101], or strange non-chaotic attractors [51, 98].

Generally, PWEs and PWIs share a common feature in terms of chaos, i.e., initial-condition sensitivity. We exemplify this by two text-book popular examples:

$$g(x) = 2x \mod 1 \text{ (doubling map) and } h(x) = x + \frac{\sqrt{2}}{2} \mod 1 \text{ (irrational rotation).} \tag{1.1}$$

Basically, a universal moral is that typical orbits under g (or h) tend to wander around the phase space in an unpredictable manner. Hence both systems are termed "chaotic" as it is numerically impossible to forecast the long-term evolutions of typical trajectories near an initial condition.

This initial-condition sensitivity indicates that these piecewise smooth dynamics are hard to understand in a deterministic manner. Instead, an ergodic analysis on an individual invariant measure or on the set of all invariant measures has shown to be a more reasonable and fruitful approach and this analysis will be our focal point throughout this thesis.

The structure of invariant measures reveals the profound and intrinsic distinctions on the principles of chaos between PWEs and PWIs. Let us first recall some basic and necessary terminologies from [121]. For a measure-preserving dynamical system $(X, \mathfrak{B}, \mu, T)$, it is said to be *strongly mixing* if for any nonempty sets $A, B \in \mathfrak{B}$, one has

$$\lim_{n \to +\infty} |\mu(A \cap T^{-n}B) - \mu(A)\mu(B)| = 0; \tag{1.2}$$

and it is said to be *weakly mixing* if for any nonempty sets $A, B \in \mathfrak{B}$, one has

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\mu(A \cap T^{-n}B) - \mu(A)\mu(B)| = 0. \tag{1.3}$$

Strongly mixing implies weakly mixing and means that successive pre-images of B eventually spread uniformly over A. Similarly, there is also a topological version of mixing without involving a measure μ. A topological dynamical system $(X, \mathfrak{B}, T)$ is said to be *topologically mixing*, if for any nonempty open subsets $U, V \in \mathfrak{B}$, there exists an $N \geq 0$ such that for all $n > N$

$$T^n(U) \cap V \neq \emptyset. \tag{1.4}$$

Additionally, a topological dynamical system $(X, \mathfrak{B}, T)$ is said to be *topologically transitive*, if for any pair of nonempty open sets $U, V \in \mathfrak{B}$, there exists an $n \geq 0$ such that

$$T^n(U) \cap V \neq \emptyset. \tag{1.5}$$

It is straightforward to see that topologically mixing implies topological transitivity.

Meanwhile, a map T is said to be *uniquely ergodic* if the set of all invariant probability measures $\mathcal{M}_I(T)$ comprises a singleton. Unique ergodicity is usually related to *minimality*, i.e., any point $x \in X$ is dense in X. Minimality also implies topological transitivity. However, it seems hard to illustrate the relationship between topologically mixing and minimality.

For the doubling map g and irrational rotation h in (1.1), the normalized Lebesgue measure m is one of their common invariant probability measures. By elementary arguments in [121], g is strongly mixing but not uniquely ergodic; in contrast, h is uniquely ergodic but not strongly mixing. Simultaneously, from topological view, both g and h are topologically transitive, but g is topologically mixing while in contrast h is minimal.

These differences can be extended for general PWEs and PWIs in some sense. In particular, for one dimensional PWE f acting on an interval I, Viana has shown that if the map f is topologically mixing, then f admits a unique absolutely continuous invariant measure (ACIP) μ which is strongly mixing and fully supported [118]. But as a contrast, for *interval exchange transformations* (IETs), i.e., a class of one dimensional PWIs, Katok asserts that they are never strongly mixing [67]. Recently, Avila & Forni [4] further prove that every transitive interval exchange transformation which is not a rotation, is weakly mixing. Regarding unique ergodicity, it is clearly that PWEs are never uniquely ergodic. However, for IETs, Masur [86] and Veech [117] independently prove that almost all IETs are uniquely ergodic; simultaneously, Keane & Rauzy [71] reveal that unique ergodicity holds for a Baire residual subset of the space of IETs. To the best of my knowledge, it is still unclear whether there are examples with coexistence of unique ergodicity (with the ergodic measure non-atomic) and strongly mixing in the piecewise smooth interval (surjective) systems. Mixing properties and unique ergodicity in higher dimensional PWEs and PWIs have also made great progress. For multi-dimensional PWEs, they still admit a strongly mixing ACIP under slightly more technical assumptions besides being topologically mixing, see [78, 110, 115] for details. Unique ergodicity of rectangle exchange transformations and cone exchange transformations are also discussed in the works [2, 52]. However, it is still unclear whether unique ergodicity is a generic property (in either topological or measurable sense) for general PWIs, the obstacle lies in the facts that unlike IETs, general PWIs may have infinite number of ergodic invariant measures concentrated on the invariant set of periodic domains [48].

Measure properties of PWEs and PWIs are coherently related with their topological properties. Take the doubling map g and the irrational rotation h in (1.1) as examples again. The doubling map g has periodic orbits of any period as well as points having dense orbits, which leads to the diversity of $\mathcal{M}_I(g)$. For the irrational map h, unique ergodicity and minimality are equivalent. This equivalence is less clear and may fail in general IETs. For example, minimal but non-uniquely ergodic IETs are constructed [26]. To fully understand the structure of $\mathcal{M}_I$ for these non-uniquely ergodic but minimal examples, it is natural to explore equivalent conditions to minimality in IETs in terms of certain appropriate subsets in $\mathcal{M}_I$. These will be further discussed in Chapter 2.

As a derivation of the ergodic analysis on invariant measures, the *Kolmogorov-Sinai entropy* h_μ of the pair (T, μ) is a nonnegative number (see [121] for the definition) which measures in some sense the "complexity" or "information production" of T, viewed through the invariant measure μ. The notion of *topological entropy* h_{top} of T is a nonnegative number measuring the topological-dynamical complexity of T. In particular, when $T : X \circlearrowleft$ is a continuous map on a compact manifold, the topological entropy h_{top} have the Bowen's definition using (n, ϵ)-separated subsets. By using this definition, the (un-weighted) *variational principle* (see [121, Section8.2]) holds:

$$h_{\text{top}} = \sup\{h_\mu | \ \mu \in \mathcal{M}_I(T)\}. \tag{1.6}$$

The quantifications on entropies are also useful in distinguishing their differences in the sense of chaos between PWIs and PWEs. A word of caution though, piecewise version of Bowen's definition on topological entropy as well as its equivalences with variational principle require non-trivial adaption while Kolmogorov-Sinai entropy is still adequate for piecewise smooth systems. It has been shown that PWIs have zero topological entropy [20], while PWEs admit a unique ACIP of maximal positive Kolmogorov-Sinai entropy [21, 22, 118]. These differences on entropy values intuitively show that the chaos from PWIs exhibit "weak" initial-condition sensitivity and "slowly" separating nearby starting orbits; this phenomenon is referred to as *pseudo-chaos* [81]. However, the dynamics of pseudo-chaotic systems are still far from trivial and predictable [45].

Iterations of piecewise smooth maps can generate strange attractors (or repellers) which usually possess fractal structures. The ergodic analysis on the probability measures of these attractors are helpful in estimating the *Hausdorff dimension* as well as other fractal dimensions (e.g., upper box dimension and packing dimension; see Section 1.3.4 for definitions). The methodology of the estimation introduces "weights" in the idea of entropies and variational principle in equation (1.6). More precisely, one can define the notion of *topological pressure* as the weighted extensions of topological entropy, and the notion of *Gibbs measure or equilibrium states* (see Section 1.3.4 for definitions) as the weighted extensions the maximum entropy measure. In particular, for a conformal expanding f with a corresponding repeller Λ, *Bowen's formalism* states that: $\dim_H \Lambda$ is the smallest value of t satisfying the pressure $P(-t \log f') = 0$, and there exists a Gibbs measure as a unique equilibrium state [95]. Moreover, the study of (existence, uniqueness, etc of) Gibbs measures and their relationships with equilibrium states in general dynamical systems contribute the subject of *thermodynamical formalism*. We discuss thermodynamic formalism in Section 1.3.4 and also refer to books [106, 99, 35] for a more detailed overview.

In summary, PWEs and PWIs show distinct dynamics, and extensive studies have been made individually. For PWEs, the dynamics are usually regarded as an essentially random sequence with strongly mixing ACIPs being the corresponding probabilities, and focus on determining their stochastic properties [118]. In particular, such dynamics usually exhibit "orbit instability" and "structural stability" [68]. Conversely, unique ergodicity in transitive components for PWIs is an important paradigm for elliptic dynamics which have the "orbit stability" and "structural instability" [68]. Nevertheless, functional analysis approaches turn out to be particularly useful for studying both classes of systems. Indeed, many operators (e.g., renormalization operator [28], Veech operator [119], transfer operator [100, 118] and etc) are induced, and determine the original dynamical behaviours of the generating systems.

1.2 Overview

This thesis will particularly concentrate on stochastic and spectral properties of the transfer (or Ruelle) operator generated by different piecewise smooth maps.

In Chapter 2, we investigate the properties of absolutely continuous invariant probability measures (ACIPs), especially those measures with bounded variation densities, for piecewise area preserving maps (PAPs) on $\mathbb{R}^d$, by employing the corresponding transfer operator. The class of PAPs unifies PWIs and piecewise hyperbolic maps where Lebesgue measure is locally preserved. We explore the relationship between topological transitivity and uniqueness of ACIPs, and then give an approach to construct invariant measures with bounded variation densities for PWIs. Our results 'partially' answer one of the fundamental questions posed by Goetz - to determine all invariant non-atomic probability Borel measures in piecewise rotations. When restricting PAPs to interval exchange transformations (IETs), our results imply that for non-uniquely ergodic IETs with two or more ACIPs, these ACIPs have very irregular densities, i.e., they have unbounded variation. This is a joint work with Congping Lin, published in Ergodic Theory and Dynamical Systems [123].

In Chapter 3, we consider the effect on the mixing properties of a piecewise smooth interval map f when its domain is divided into N equal subintervals and f is composed with a permutation of these. The case of the stretch-and-fold map $f(x) = mx \bmod 1$ for integers $m \geq 2$ is examined in detail. We give a combinatorial description of those permutations σ for which $\sigma \circ f$ is still (topologically) mixing, and show that the proportion of such permutations tends to 1 as $N \to \infty$. We then investigate the mixing rate of $\sigma \circ f$ (as measured by the modulus of the second largest eigenvalue of the transfer operator). In contrast to the situation for continuous time diffusive systems, we show that composition with a permutation cannot improve the mixing rate of f, but typically makes it worse. Under some mild assumptions on m and N, we obtain a precise value for the worst mixing rate as σ ranges through all permutations; this can be made arbitrarily close to 1 as $N \to \infty$ (with m fixed). We illustrate the geometric distribution of the second largest eigenvalues in the complex plane for small m and N, and propose a conjecture concerning their location in general. Finally, we give examples of other interval maps f for which composition with permutations produces different behaviour other than that obtained from the stretch-and-fold map. This is joint work with Nigel Byott and Mark Holland, which is accepted in Discrete and Continuous Dynamical Systems-series A [23].

In Chapter 4, we consider the computation of Hausdorff, upper box and packing dimensions for general Moran set constructions, where the contraction transformations applied at each step may be different. We allow the transformations to be nonlinear and contraction rates to have an infimum of zero. Furthermore, the basic sets of the construction can have a complicated topology. We use the thermodynamic formalism approach to show the existence of a non-atomic measure that is supported on the resulting fractal limit set, and using this measure we compute the corresponding estimations of the dimensions. In addition, we also consider dimension estimations for stochastic Moran set constructions, where chaotic dynamical systems are used to generate the contraction ratios at each step.

The dependencies between the chapters are illustrated in Figure 1.1.

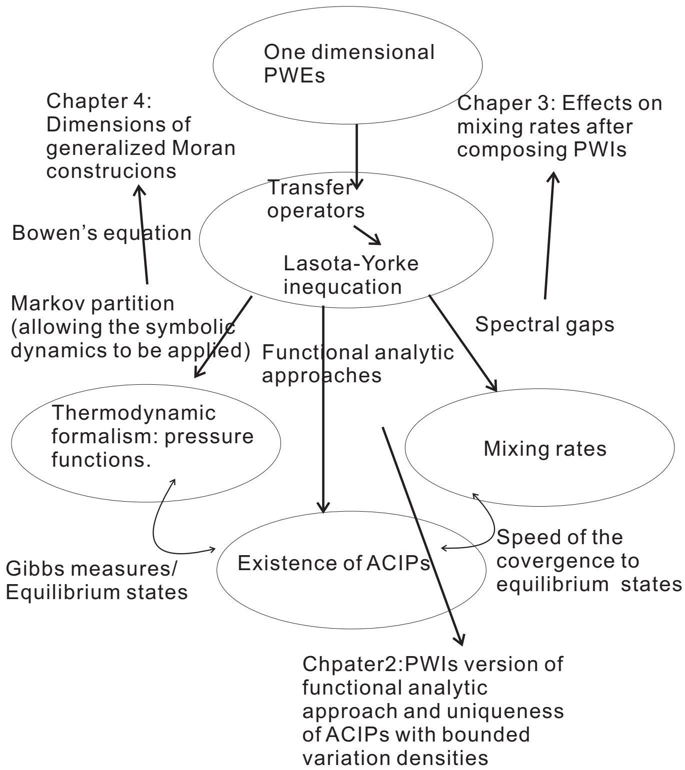

Figure 1.1: A leitfaden on the dependencies between the chapters.

1.3 Preliminaries

1.3.1 Basic measure theory

Measure theory will play an essential role throughout this thesis. We denote by m the normalized Lebesgue measure on a compact manifold $(X, \mathfrak{B})$. If the measure is not specified then Lebesgue measure is meant. We also denote by $\mathcal{M}(X)$ the set of all Borel probability measures with support on X. It is known from [121, Theorem6.1] and [61, Theorem 5.3] that if $\mu \in \mathcal{M}(X)$ then μ is a *Radon measure* [87], i.e.,

1. $\mu(K) < \infty$ for any compact subset $K \subset X$;
2. $\mu(V) = \sup\{\mu(K) : \ K \subset V \text{ is compact}\}$ for any open set $V \subset X$;
3. $\mu(E) = \inf\{\mu(V) : \ E \subset V, \ V \text{ is open}\}$ for any $E \subset X$.

For a measure $\mu \in \mathcal{M}(X)$, a statement is said to hold for μ almost every (a.e.) if a set A which consists of all values where the statement fails, has a zero measure, i.e., $\mu(A) = 0$. Given two measures $\mu, \nu \in \mathcal{M}(X)$, the measure μ is said to be *absolutely continuous* with respect to ν (denoted by $\mu \ll \nu$) if and only if for any Borel set $A \subset X$ where $\nu(A) = 0$, there is $\mu(A) = 0$. The measure μ is said to be *singular* with respect to ν (denoted by $\mu \perp \nu$) if and only if there exists a Borel set A, such that $\nu(A) = 0$ and $\mu(A) = 1$.

Lemma 1.1 (Radon-Nikodym)**.** *[107] If $\mu, \nu \in \mathcal{M}(X)$ with $\mu \ll \nu$ then there exists a unique measurable map $\varphi : X \to \mathbb{R}^+$ (up to ν-a.e. equivalence) such that*

$$\mu(A) = \int_A \varphi d\nu, \quad A \in \mathfrak{B}. \tag{1.7}$$

The map φ is called as a Radon Nikodym derivative (density) and is commonly written as $d\mu/d\nu \in L^1(\nu)$.

The *image* of a measure μ under a measurable map T is defined by:

$$T_\sharp \mu(A) = \mu(T^{-1}(A)), \quad A \in \mathfrak{B}. \tag{1.8}$$

It is straightforward to see $\mu \in \mathcal{M}_I(T)$ if and only if $T_\sharp \mu = \mu$. Moreover, if the map $T : (X, \mu) \circlearrowleft$ is also *non-singular* (with respect to μ), then $T_\sharp \mu \ll \mu$. By non-singularity, we mean that f is measurable and if $\mu(A) = 0$ then $\mu(T^{-1}(A)) = \mu(T(A)) = 0$.

Next we consider the convergence of measures in $\mathcal{M}(X)$. Let $\mu, \mu_1, \mu_2, \cdots \in \mathcal{M}(X)$, we say that the sequence $\{\mu_i\}$ *weakly converges* to μ, i.e.,

$$\mu_i \rightharpoonup \mu,$$

if for all $\phi \in C^0(X)$, i.e., the space of real-valued continuous functions on X.

$$\int \phi d\mu_i \to \int \phi d\mu.$$

This weak convergence can be linked with density functions via:

Lemma 1.2. *[87] Suppose that $\nu, \mu_1, \mu_2, \cdots \in \mathcal{M}(X)$ and $\mu_i \ll \nu$ with densities $\varphi_i := \frac{d\mu_i}{d\nu}$ for each i. If there exists a map φ such that $||\varphi_n - \varphi||_1 \to 0$ as $n \to \infty$, then there exists a measure μ with $d\mu = \varphi d\nu$ such that $\mu_n \rightharpoonup \mu$.*

It is well known that every sequence of measures in $\mathcal{M}(X)$ has a weakly convergent subsequence [87]. Moreover, if we define a *weak* topology* on $\mathcal{M}(X)$ as the smallest topology making each of the maps $\mu \mapsto \int_X \phi d\mu$ $(\forall \phi \in C^0(X))$ continuous, then the following holds.

Lemma 1.3 (Compactness)**.** *[121] $\mathcal{M}(X)$ is compact under the weak* topology.*

By the *Riesz representation theorem* [107], one can embed $M(X)$ into the dual space of $C^0(X)$, i.e., the set of all continuous complex-valued linear functionals. More precisely, there exists an isomorphism $\alpha : \mu \mapsto \int (\cdot) d\mu$ between $M(X)$ and

$$\{v \in C^0(X)^* \mid v(1) = 1, v(\varphi) \geq 0 \text{ if } \varphi \geq 0\}.$$

For a PWE or a PWI T, it is straightforward to see that the set of invariant probability measures $\mathcal{M}_I(T)$ is a convex and compact subset of $\mathcal{M}(X)$. However, characterizing the elements of $\mathcal{M}_I(T)$ is far from trivial in general. In particular, it is feasible to consider some proper subsets of $\mathcal{M}_I(T)$, such as a subset consisting of absolutely continuous invariant probability measures (ACIPs). The ACIPs are more feasible to be approximated by computers (see [14, Chapter 5] for the explanations) and one way of investigating these measures is via transfer operators which is reviewed in the following.

1.3.2 Transfer operator

Let $(X, \mathfrak{B}, m)$ be a probability space where m is the normalized Lebesgue measure and let $T : X \to X$ be a non-singular map. The *transfer operator* $\mathcal{L}_T : L^1(m) \to L^1(m)$ associated with f is defined up to $m - a.e.$ equivalence as follows [14]:

$$\int_A \mathcal{L}_T \varphi dm = \int_{T^{-1}(A)} \varphi dm, \ \ \varphi \in L^1(m), \ \ A \in \mathfrak{B}. \tag{1.9}$$

An explicit and equivalent expression is to view $\mathcal{L}_T$ as a positive weighted composition operator as:

$$\mathcal{L}_T \varphi(x) := \sum_{y \in T^{-1}(x)} \frac{\varphi(y)}{|\det DT(y)|}. \tag{1.10}$$

with the weight map $y \mapsto \frac{1}{|\det DT(y)|}$. This transfer operator is a special case of *Ruelle operator* where the weight map is any measurable map $g : X \to \mathbb{R}$ such that for each $x \in X$ the sum $\sum_{y \in f^{-1}(x)} g(y)$ is convergent. Note for further use the following expression for the nth iterations, $(n \geq 1)$ of the transfer operator $\mathcal{L}_f$:

$$\mathcal{L}_T^n \varphi(x) := \sum_{y \in T^{-n}(x)} \frac{\varphi(y)}{|\det D^n T(y)|}. \tag{1.11}$$

For the n-th iterations of the Ruelle operator, one can replace the weight map $y \mapsto \frac{1}{|\det D^n T(y)|}$ by the notation $g^{(n)}(y) := \prod_{k=0}^{n-1} g(T^k(y))$.

Moreover, the transfer operator also possesses the following dual property [118]:

$$\int (\mathcal{L}_T \varphi) \psi dm = \int \varphi \cdot (\psi \circ T) dm, \;\; \varphi \in L^p(m), \psi \in L^q(m) \text{ with } 1/p + 1/q = 1. \tag{1.12}$$

This dual property associates fixed points of $\mathcal{L}_T$ with ACIPs via the following lemma.

Lemma 1.4 (Invariant density). *[14] For a non-singular map T and $\varphi \in L^1$ with $||\varphi||_1 = 1$, $\mathcal{L}_T \varphi = \varphi$ if and only if $d\mu = \varphi dm$ is an invariant measure under T.*

Spectral properties of the transfer operators play a central role in the sequel. It is necessary to recall a few definitions and facts on the basic spectral theory of a bounded linear operator as a parenthesis before we state a case study in the next section. We also refer to [66] for a more detailed overview.

Suppose that $(\mathcal{B}, ||\cdot||)$ is a Banach space, and the transfer operator $\mathcal{L}_T : \mathcal{B} \mapsto \mathcal{B}$ is a *bounded linear operator* (i.e., there exists a constant C such that $||\mathcal{L}_T \varphi|| \leq C||\varphi||$ for all $\varphi \in \mathcal{B}$, and the smallest such constant is the *operator norm* of $\mathcal{L}_T$). The *resolvent set* of $\mathcal{L}_T$ is the set of complex numbers z which make $\mathcal{L}_T - zId : \mathcal{B} \mapsto \mathcal{B}$ an invertible operator with a bounded inverse $(\mathcal{L}_T - zId)^{-1} : \mathcal{B} \mapsto \mathcal{B}$. The *Spectrum* $\operatorname{Spec} \mathcal{L}_T$ is the complement of resolvent set, and the *spectral radius* $r(\mathcal{L}_T)$ is

$$r(\mathcal{L}_T) := \sup\{|z| \mid z \in \operatorname{Spec} \mathcal{L}_T\}. \tag{1.13}$$

In particular, we recall the *spectral radius formula* acting on a Banach space $(\mathcal{B}, ||\cdot||)$:

$$r(\mathcal{L}_T) = \lim_{n \to \infty} (||\mathcal{L}_T^n||)^{1/n}.$$

An element $z \in \operatorname{Spec} \mathcal{L}_T$ is an eigenvalue of $\mathcal{L}_T$. The *geometric multiplicity* of an eigenvalue z is the dimension $1 \leq m_1(z) \leq \infty$ of the eigenspace $\{v \in \mathcal{B} | (\mathcal{L}_T - z)v = 0\}$. The *algebraic multiplicity* of z is the dimension $m_2 \leq \infty$ of the generalized eigenspace $\{v \in \mathcal{B} | \; \exists \, m_0 \geq 1, \; (\mathcal{L}_T - z)^{m_0} v = 0\}$. Clearly, $m_2 \geq m_1$. We then define the *essential spectral radius* $r_{ess}(\mathcal{L}_T|_{\mathcal{B}})$:

$$\begin{aligned} r_{ess}(\mathcal{L}_T|_{\mathcal{B}}) := & \quad \inf\{r \geq 0 : \lambda \in \operatorname{Spec}(\mathcal{L}_T|_{\mathcal{B}}), \, |\lambda| > r \\ & \Rightarrow \lambda \text{ is an isolated eigenvalue of finite multiplicity}\}. \end{aligned} \tag{1.14}$$

It is clear that $r_{ess}(\mathcal{L}_T) \leq r(\mathcal{L}_T)$. Moreover, let $\mathcal{B}^*$ be the dual space of the Banach space $(\mathcal{B}, ||\cdot||)$ and set the *dual operator* $\mathcal{L}^* : \mathcal{B}^* \to \mathcal{B}^*$ via:

$$(\mathcal{L}^*\mu)\varphi := \mu(\mathcal{L}_T\varphi), \quad \forall\mu \in \mathcal{B}^* \text{ and } \forall\varphi \in \mathcal{B}. \tag{1.15}$$

Then $\mathcal{L}_T$ and $\mathcal{L}^*$ share the same spectrum, essential spectral radius and isolated eigenvalues.

The easiest case to analysis the spectrum is of course when $r_{ess}(\mathcal{L}_T) < r(\mathcal{L}_T)$, and this indeed happens when T is an one-dimensional PWE.

1.3.3 Case study: one-dimensional PWEs

In this section, we state the spectral analysis of one-dimensional PWEs, which is well understood in the space of maps with bounded variation.

Definition 1.1 (One-dimensional PWEs)**.** *[74] Let $I = [0, 1)$, we say $f : I \circlearrowleft$ is a piecewise expanding map(PWE) if:*

- *there are $0 = \zeta_1 < \zeta_2 < \cdots < \zeta_{N-1} = 1$ which define subintervals $I_i = [\zeta_{i-1}, \zeta_i)$ such that each $f_i := f|_{I_i} \in C^2$ and monotone, and*
- *$|f'| \geq 2$ and f' has a bounded variation.*

An important technical remark is that: due to the discontinuities of the map f or its derivative, the transfer operator $\mathcal{L}_f$ (defined in Section 1.3.2) does not preserve the space of (Hölder) continuous functions. Instead, $\mathcal{L}_f$ preserves the space of maps of *bounded variation* [6, 118] which is defined below. Given a function $\varphi : I \to \mathbb{R}$, we define the total variation of φ as

$$\text{var}(\varphi) = \sup\{\sum_{k=1}^{n} |\varphi(x_k) - \varphi(x_{k-1})| : 0 \leq x_0 \leq \ldots \leq x_n = 1\}, \tag{1.16}$$

where the sup is taken over all partitions of I. We say that φ has bounded variation (i.e., $\varphi \in \text{BV}$) if $\text{var}(\varphi) < \infty$. Since $\text{var}(\cdot)$ is a semi-norm, we equip the norm $\|\cdot\|_{\text{BV}}$ by

$$||\varphi||_{\text{BV}} := ||\varphi||_1 + \text{var}(\varphi). \tag{1.17}$$

A bounded variation space which is endowed with this norm is a Banach space.

The Banach space $(\text{BV}, ||\cdot||_{BV})$ is compactly embedded into the L^1 space. That is, each sequence $\{\varphi_n\}$ of L^1 functions with uniformly bounded BV$-$norm has a subsequence which converges (in L^1-norm) to an element in BV. This is known as *Helly's selection theorem* [10].

Additionally, a key inequality for $\mathcal{L}_f$ is as follows.

Lemma 1.5 (Lasota-Yorke inequality)**.** *[14] Suppose $f : I \circlearrowleft$ is a PWE, then there exist constants $C_0 > 0$ and $0 < \theta < 1$, such that for any $\varphi \in L^1$, and any $n \geq 0$*

$$||\mathcal{L}_f^n \varphi||_{BV} \leq \theta^n ||\varphi||_{BV} + C_0 |\varphi|_1. \tag{1.18}$$

Combining the compact embedding property of BV into L^1 and the Lasota-Yorke inequality gives the following result.

Proposition 1.1 (Existence of invariant ACIPs)**.** *[14] Suppose $f : I \circlearrowleft$ is a PWE, then f has only finitely many ergodic ACIPs. Moreover, all the densities of these ACIPs are functions of bounded variation and strictly positive. In particular, if f is also topologically mixing then f admits a unique ACIP, which is also strongly mixing.*

The main idea in proving Theorem 1.1 involves the Birkoff average of a sequence induced by the transfer operator $\mathcal{L}_f$, which is motivated by the Kryloff-Bogolioubоff Theorem for continuous maps [77]. A brief sketch of this idea is stated as follows. Let

$$\varphi_n := \frac{1}{n} \sum_{k=0}^{n-1} \mathcal{L}_f^k \mathbf{1}. \tag{1.19}$$

Then the Lasota-Yorke inequality (1.18) implies $\sup_n ||\varphi_n||_{BV} < C_1 < \infty$ (i.e., the sequence $\{\varphi_n\}$ is uniformly bounded in BV-norm). Hence, by the compactly embedding property of BV into L^1 (or Helly's Theorem), φ_n converges to a map $\varphi \in$ BV in L^1-norm. By elementary arguments of the transfer operator in Section 1.3.2, this map φ is a probability density and invariant under $\mathcal{L}_f$. Moreover, it is worth noticing that due to $\varphi \in$ BV, the support set of measure μ does not appear to have an irregular geometric structure. This fact together with topologically mixing property eventually implies the strongly mixing property of μ, which completes the proof. A word of caution, though the density always has bounded variation, it is still hard to derive a universal explicit formula in general. Alternative approaches have focused on approximating the density, see works [41, 63].

The above protocol on searching ACIPs with regular densities can be regarded as a classical example of a so-called functional analytic approach [77]. Indeed, this approach has numerous subsequent generalizations, which have been used to search ACIPs for more general frameworks, e.g., coupled map lattice, hyperbolic maps and geodesic flow on hyperbolic plane [77].

We adapt the above functional analytic approach for multidimensional PWIs in Chapter 2. This class of maps has completely no hyperbolicity, which leads to the lack of a Lasota-Yorke inequity. In fact, the corresponding transfer operator is a unitary operator. Therefore, non-trivial adaption is required to search an appropriate subspace of bounded variation in order to obtain the L^1-norm convergence. Moreover, we show that some invariant densities for PWIs may mutually intermingle, which means that the support sets are highly pathological.

Combining the Lasota-Yorke inequality and compactly embedding property further allows the applications of the Ionescu-Tulcea/Marinescu Theorem [59] to obtain the following result:

Theorem 1.1 (Quasi-compactness). *[74] If $f : I \circlearrowleft$ is a PWE and topologically mixing, then there exists* $0 < \tau := \max\{|\lambda|, 1 \neq \lambda \in \mathrm{Spec}(\mathcal{L}_f|_{BV})\} < 1$ *such that:*

$$\exp\left\{-\liminf_{k\to\infty} \mathrm{essinf}_{x\in[0,1]} \frac{1}{k}\log|(f')^k(x)|\right\} = r_{ess}(\mathcal{L}_f|_{BV}),$$

and

$$r_{ess}(\mathcal{L}_f|_{BV}) \leq \tau < r(\mathcal{L}_f|_{BV}) = 1. \tag{1.20}$$

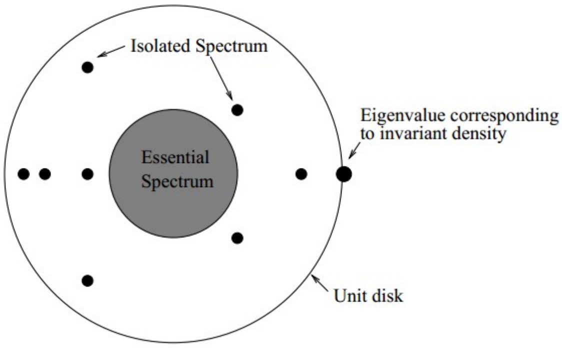

Figure 1.2: Schematic representation of the spectrum (as a subset in complex plane) for the transfer operator $\mathcal{L}_f|_{BV}$. Image is taken from [40].

Inequality (1.20) is equivalent to say $\mathcal{L}_f|_{BV}$ is *quasi-compact*. This name stems from the following fact. Recall that a linear bounded operator $\mathcal{K}$ is *compact* if the closure of the image of the unit ball is compact. A well known result states that the essential spectral radius of a compact operator is always zero, and its spectrum consists of isolated eigenvalues of finite multiplicity and accumulates at zero [6]. Figure 1.2 intuitively illustrates that there is a similar scenario in Theorem 1.1: except for the spectrum inside the closed disc of radius r_{ess}, the operator behaves just like a compact operator as its spectrum consists of isolated eigenvalues of finite multiplicity. One can thus write a spectral decomposition for $\mathcal{L}_f|_{BV}$. Therefore, the operator $\mathcal{L}_f|_{BV}$ is called quasi-compact.

Inequality (1.20) also implies that $\mathcal{L}_f|_{BV}$ has a spectral gap. In this sense, Theorem 1.1 for $\mathcal{L}_f|_{BV}$ is also commonly regarded as the infinite-dimensional extension of the *Perron-Frobenius theorem* for non-negative matrices [6].

Transfer operators coincide with non-negative matrices in the class of piecewise linear Markov maps acting on locally constant-weighted maps. Here, the *Markovian property* means that if each branch f_i is a homeomorphism from a subinterval I_i onto some connected union of other subintervals I_j. The Markovian property directly induces a transition matrix $A_f := (a_{ij})_{1\leq i,j\leq n}$ as

$$a_{ij} = \begin{cases} 1, & \text{if } I_j \subset f(I_i); \\ 0, & \text{otherwise.} \end{cases}$$

Based on this transition matrix, the coincidence is interpreted via the following proposition:

Proposition 1.2 (Matrix representation of the transfer operator). *[14] Let $f : I \circlearrowleft$ be a piecewise linear Markov map on the finite partition $\mathcal{P} = \{I_i\}_{i=1}^n$, then there exists an $n \times n$ matrix $M^{(f)}$ associated with f, such that for every piecewise (with respect to the same partition $\mathcal{P}$) constant function ψ, one has $\mathcal{L}_f\psi = (M^{(f)})^T \cdot \pi^{(f)}$, where $\pi^{(f)} = (\pi_1, \cdots, \pi_n)^T$ is the column representative vector. The matrix $M^{(f)}$ is of form $M^{(f)} = (m_{ij})_{1\leq i,j\leq n}$, where*

$$m_{ij} = \frac{a_{ij}}{|f'|} = \frac{m(I_i \cap f^{-1}(I_j))}{m(I_i)}, \quad 1 \leq i, j \leq n. \tag{1.21}$$

Using this matrix representation, one can transform the verification of properties of the transfer operator to that of the matrix $M^{(f)}$ for piecewise linear Markov maps. In many situations, such a transformation reduces the infinite dimensional phase space into finite, and is useful to simplify many discussions of the ergodic properties on the invariant measures. We state the following as an example.

For a PWE f, if f is topologically mixing then the quasi-compactness of $\mathcal{L}_f$ is inherently related with the *speed of convergence to the equilibrium* in BV norm [72]. Suppose φ is the corresponding invariant probability density and τ is the second largest eigenvalue of $\mathcal{L}_f|_{BV}$. On one hand, for any $\epsilon > 0$, the spectral decomposition of $\mathcal{L}_f|_{\mathrm{BV}}$ implies that there exists a constant $C_{BV} > 0$ such that for all $n \geq 0$ and $\phi \in \mathrm{BV}$ with $\|\phi\|_1 = 1$,

$$||\mathcal{L}_f^n(\phi) - \varphi||_{\mathrm{BV}} \leq C_{BV} \cdot (\tau + \epsilon)^n ||\phi||_{\mathrm{BV}}. \tag{1.22}$$

On the other hand, for any $\epsilon > 0$, there exists $\phi \in \mathrm{BV}$ with $\|\phi\|_1 = 1$ such that for any sufficiently large n and for some $C_\phi > 0$,

$$\|\mathcal{L}_f^n(\phi) - \varphi\|_{BV} \geq C_\phi(\tau - \epsilon)^n. \tag{1.23}$$

Therefore, τ determines the optimal speed of convergence to the equilibrium. A word of caution, the exact value of τ is far from trivial to estimate in general. However, by transforming this problem into the analysis of the matrix $M^{(f)}$ again, the estimation on τ in piecewise linear Markov maps becomes comparably feasible. Indeed, the Fredholm matrix/determinant approach is the main tool for piecewise linear Markov maps, see [89, 90, 91].

Consider a piecewise linear Markov map $f : I \to I$ with a finite partition $\mathcal{P} = \{I_i\}_{i=1}^q$ and a representative transition matrix B. Here B is a $q \times q$ matrix with $B_{ij} = 1$ if $I_j \subset f(I_i)$, and $B_{ij} = 0$ if $f(I_i) \cap I_j = \emptyset$. We assume that f is differentiable on the interior of each element of $\mathcal{P}$. If $\mathcal{L}_f$ is the Perron-Frobenius operator, and $J \subset I$, we consider the power series defined on $\mathbb{C} \times D$ with $D \subset \mathbb{C}$:

$$s^J(z,x) := \sum_{n=0}^{\infty} z^n \mathcal{L}_f^n(\mathcal{X}_J)(x) = \mathcal{X}_J(x) + \sum_{n=1}^{\infty} z^n \mathcal{L}_f^n(\mathcal{X}_J)(x), \tag{1.24}$$

where $\mathcal{X}_J(x)$ is the indicator function of J. When $J = I_i \in \mathcal{P}$, we write $s^J(z,x)$ as $s^{(i)}(z,x)$. We let $\underline{s}(z,x)$ be the vector $(s^{(i)}(z,x))_{i=1}^q$, and similarly $\underline{\mathcal{X}}(x) = (\mathcal{X}_{(i)}(x))_{i=1}^q$. For a Markov system we have the following result.

Proposition 1.3 (Fredholm matrix)**.** *For a piecewise linear Markov map $f : I \to I$ with a finite partition $\mathcal{P} = \{I_i\}_{i=1}^q$, there exists a $q \times q$ matrix $\Phi(z)$ and such that*

$$\underline{s}(z,x) = (I - \Phi(z))^{-1} \underline{\mathcal{X}}(x). \tag{1.25}$$

The matrix $\Phi(z)$ in Proposition 1.3 is called a *Fredholm matrix*.

Proof. We only consider the Markov case where the slope is constant on each I_i (but not constant globally). Our proof is a slight adaption of the calculations in [89, 90]. In particular we obtain an explicit form of $\Phi(z)$. First of all, by the definition of $\mathcal{L}_f$ we have

$$s^J(z,x) = \mathcal{X}_J(x) + \sum_{n=1}^{\infty} z^n \sum_{f^n(y)=x} \frac{\mathcal{X}_J(y)}{|(f^n)'(y)|}.$$

If $J = I_i \in \mathcal{P}$, the following hold:

$$\begin{aligned} s^{(i)}(z,x) &= \mathcal{X}_{(i)}(x) + z \sum_{n=1}^{\infty} z^{n-1} \sum_{f^{n-1}(f(y))=x} \frac{\mathcal{X}_{(i)}(y)}{|(f^{n-1})'(f(y)) f'(y)|} \\ &= \mathcal{X}_{(i)}(x) + z \sum_{n=1}^{\infty} \frac{z^{n-1}}{|(f' \mid I_i)|} \sum_{f^{n-1}(\tilde{y})=x} \sum_{\substack{i,j \\ B_{ij}=1}} \frac{\mathcal{X}_{(j)}(\tilde{y})}{|(f^{n-1})'(\tilde{y})|} \\ &= \mathcal{X}_{(i)}(x) + \left(\frac{z}{|(f' \mid I_i)|} \sum_j B_{ij} \right) s^{(j)}(z,x). \end{aligned}$$

Hence we obtain a $q \times q$ matrix $\Phi(z)$ with $\Phi(z)_{ij} = \{z/|(f' \mid I_i)|\} B_{ij}$ and

$$\underline{s}(z,x) = (I - \Phi(z))^{-1} \underline{\mathcal{X}}(x). \tag{1.26}$$

This completes the proof. □

Given the Fredholm matrix $\Phi(z)$, we define the *Fredholm determinant* to be the quantity $D(z) := \det(I - \Phi(z))$. For piecewise-linear expanding (Markov) systems, the Fredholm matrix and Fredholm determinant have the following properties (see [89, 90]) which are useful in the sequel:

1. The number of ergodic components of f is equal to the dimension of the eigenspace of $I - \Phi(1)$ associated to the eigenvalue of value zero. Moreover, the number of ergodic components is also equal to the order of the zero at $z = 1$ in the equation $D(z) = 0$.

2. If zero is a simple eigenvalue of $I - \Phi(1)$ then the system is ergodic. Moreover if $\{|z| = 1\} \cap \mathrm{Spec}(\mathcal{L}_f|_{\mathrm{BV}}) = \{1\}$ then the system is mixing

3. If $\lambda \in \mathbb{C}$ and $|\lambda| > r_{ess}$, then $\lambda \in \mathrm{Spec}(\mathcal{L}_f|_{\mathrm{BV}})$ if and only if $z = \lambda^{-1}$ is a zero root of $D(z)$, i.e., $D(1/\lambda) = 0$.

4. If $D(1/\lambda) = 0$ then λ is an eigenvalue of $\mathcal{L}_f|_{\mathrm{BV}}$.

In addition, the (ergodic) invariant density $\rho(x)$ can also be computed, and is given by the following formula:

$$\rho(x) = \sum_i \frac{v_i}{\sum_j |I_j| v_j} \mathcal{X}_{(i)}(x), \tag{1.27}$$

where $\underline{v} := (v_i)_{i=1}^q$ is a left-eigenvector of $\Phi(1)$ associated to eigenvalue 1. i.e., $\sum_i v_i \Phi(1)_{i,j} = v_j$. The proof of equation (1.27) is quite straightforward, i.e., by checking $(\mathcal{L}_f \rho)(x) = \rho(x)$. Thus, the Fredholm matrix $\Phi(z)$ at $z = 1$ can be viewed as the dual operator to $\mathcal{L}_f$ (relative to the vector space generated by the indicator functions on the Markov partition $\mathcal{P}$).

It can be further shown that for a piecewise linear (Markov) expanding system, the Fredholm determinant $D(z)$ is also related to the *dynamical zeta function* $\zeta(z)$ via: [6, 89]

$$\zeta(z) = \frac{1}{D(z)} \text{ where, } \zeta(z) = \exp\left\{ \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{p: f^n(p) = p} \frac{1}{(f^n)'(p)} \right\}, \tag{1.28}$$

Thus, within the region $|z| > r_{ess}$, $\zeta(z)$ is meromorphic with singularities (poles) at the zeros of $D(z)$. In particular $\zeta(z)$ is analytic in the region $|z| < r_{ess}$. To see the relationship between $\zeta(z)$ and $D(z)$, the Fredholm determinant can be calculated as:

$$\det\{(I - \Phi(z))^{-1}\} = \exp\{-\mathrm{tr}\,(\log(I - \Phi(z)))\} = \exp\left\{ \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr}(\Phi(z)^n) \right\},$$

where tr is the trace operation. If, for example $\Phi(z) = \frac{1}{m} B$, where $f' \mid I_i = m$ for all $I \in \mathcal{P}$, then

$$\det(I - \Phi(z))^{-1} = \exp\left\{\sum_{n=1}^{\infty} \frac{z^n}{n} \cdot \frac{\mathrm{tr}(B^n)}{m^n}\right\},$$

and $\mathrm{tr}(B^n)$ is precisely the number of fixed points of f^n.

In Chapter 3, we use this Fredholm matrix/determinant approach to study dynamics generated by the family of maps $x \mapsto mx \bmod 1$, $(2 \leq m \in \mathbb{N})$ with composition of a permutation $\sigma \in S_N$, and show how the mixing rate $\tau(\sigma)$ varies with the permutation σ.

Instead of considering the speed of convergence to equilibrium, a related quantity is that of *decay of correlations* [6, 89, 118]. We say that a mixing measure-preserving system (f, M, μ) has decay of correlations for some real-valued test functions $\phi, \psi \in L^2(\mu)$ with rate function $r(n)$ for positive n if there exists some constant $C_{\phi,\psi}$ such that

$$\mathcal{C}_n(\phi, \psi, \mu) := \left| \int \phi(\psi \circ f^n)\, d\mu - \int \phi\, d\mu \int \psi\, d\mu \right| \leq C_{\phi,\psi} r(n). \tag{1.29}$$

Notice that due to $f-$invariance of μ, we can rewrite:

$$\mathcal{C}_n(\phi, \psi, \mu) = \int (\phi - E\phi)((\psi - E\psi) \circ f^n) d\mu,$$

where $E\phi = \int \phi d\mu$ and $E\psi = \int \psi d\mu$. Therefore we can view $\mathcal{C}_n(\phi, \psi, \mu)$ as $n-$th correlation in the language of probability theory.

The *Schwartz inequality* and the L^2 assumption on the observations ϕ, ψ imply that $\phi(\psi \circ f^n)$ is integrable. It is also worth mentioning that if $\phi = \chi_A$ and $\psi = \chi_B$ are characteristic functions of two Borel sets, then the correlation function $\mathcal{C}_n(\phi, \psi, \mu)$ reduces to the left hand side of Equation (1.2) in the definition of strongly mixing. Thus, by Lebesgue dominated convergence theorem [107], it follows that $\lim_{n\to\infty} |\mathcal{C}_n(\phi, \psi, \mu)| = 0$ for all real-valued L^2 test functions ϕ, ψ. Generally, people are interested in the speed at which the correlation function of suitable mixing dynamical systems decays to zero. Usually, it is necessary to restrict to a subset of functions in $L^2(\mu)$ satisfying some smoothness property, so that one can show the relationship between the correlation function and the speed of convergence to equilibrium via applying the results of the corresponding transfer operators having a spectral gap. In particular, if f is a one-dimensional PWE, $\psi \in L^\infty$ and $\phi \in \mathrm{BV}$ with $\int \phi d\mu = 1$, then

$$\mathcal{C}_n(\phi, \psi, \mu) = \left| \int \psi\{\mathcal{L}_f^n(\phi\varphi) - \varphi\} dm \right| \leq \|\psi\|_\infty \|\mathcal{L}_f^n(\phi\varphi) - \varphi\|_1$$

$$\leq \|\psi\|_\infty \|\mathcal{L}_f^n(\phi\varphi) - \varphi\|_{\mathrm{BV}},$$

where φ is the invariant probability density of an ACIP μ and the last step is from

$\|\cdot\|_1 \leq \|\cdot\|_{\mathrm{BV}}$. Hence the rate of decay of correlations depends on the eigenvalues of $\mathcal{L}_f|_{\mathrm{BV}}$. More precisely, for any $\epsilon > 0$:

$$\lim_{n\to\infty} (|\lambda| + \epsilon)^{-n} \mathcal{C}(\phi, \psi, \mu) = 0, \quad \lambda^{-1} = \inf\{z : z^{-1} \in \mathrm{Spec}(\mathcal{L}_f|_{\mathrm{BV}}) \setminus \{1\}\}. \tag{1.30}$$

Due to the Markov property again, a piecewise Markov linear map satisfying the topologically mixing condition can be conjugated with a sub-shift of finite type. Therefore, the unique ACIP of the map in Theorem 1.1 can also be viewed as a *Markov measure* (see [6] for the definition), which is a special *Gibbs measure* with a local constant weight potential [6]. This relationship shows the link between the spectral properties of transfer operators with the thermodynamic formalism approach which is stated in next section.

1.3.4 The thermodynamic formalism approach

The thermodynamic formalism approach can be described as a set of ideas and techniques brought from statistical mechanics to dynamical systems in the pioneering work of Sinai, Bowen and Ruelle in the early seventies and eighties, see e.g., [13, 112, 113, 108]. This approach has been shown as a very powerful tool for a wide range of problems. We give a short review here of its applications, particularly in dimension theory (through Bowen's formalism see e.g., [13, 95, 99]).

Fractal dimensions: We first recall some quantities of fractal dimensions (see e.g., [34, 87] for a more general discussion).

Among the fractal dimensions, the notion of Hausdorff dimension is probably the oldest and most important. Suppose F is a non-empty subset in $\mathbb{R}^d$. For any nonnegative number s and $\epsilon > 0$, let

$$\mathcal{H}^s_\epsilon(F) := \inf \left(\sum_i (\mathrm{diam}(U_i))^s \right), \tag{1.31}$$

where the infimum is taken over all covers $\{U_i\}$ with $\mathrm{diam}(U_i) < \epsilon$. As ϵ decreases, the class of permissible covers of F in (1.31) is reduced. Therefore, the infimum $\mathcal{H}^s_\epsilon$ increases. This fact implies that the quantity $\mathcal{H}^s(F) := \lim_{\epsilon\to 0} \mathcal{H}^s_\epsilon(F)$ exists which is called as *Hausdorff measure*. The *Hausdorff dimension* of F is defined by

$$\dim_H(F) := \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(F) = \infty\}. \tag{1.32}$$

and is interpreted as a unique critical value of s at which the Hausdorff measure of F "jumps" from $+\infty$ to 0. The fact that Hausdorff dimension is based on Hausdorff measure makes this dimension mathematically convenient to be manipulated. However, a major disadvantage of Hausdorff dimension lies in the calculation difficulties in many situations. Therefore, to understand the geometrical properties of fractal sets, other dimensions are also essential.

(Upper) box dimension is another important fractal dimension which is relatively easier to estimate than Hausdorff dimension. Given a non-empty set $F \subset \mathbb{R}^d$, and $\epsilon > 0$, let $N(\epsilon)$ denote the smallest number of ϵ-balls needed to cover F. The *(upper) box dimension* of F is defined by:

$$\overline{\dim}_B(F) := \limsup_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)}. \tag{1.33}$$

Analogous to Hausdorff dimension, there is an alternative description of upper box dimension [95]: for any nonnegative number s, let

$$W^s(F) := \lim_{\epsilon \to 0} \sup\{\textstyle\sum_i \operatorname{diam}(B_i)^s : \begin{array}{l} B_i \text{ is a ball with } \operatorname{diam}(B_i) \le \epsilon, \\ B_i^\circ \cap B_j^\circ = \emptyset (i \neq j), B_i \cap F \neq \emptyset\}, \end{array} \tag{1.34}$$

then

$$\overline{\dim}_B(F) := \sup\{s : W^s(F) = \infty\} = \inf\{s : W^s(F) = 0\}. \tag{1.35}$$

It is worth mentioning that $W^s(\cdot)$ in equation (1.34) usually does not define a measure (due to lack of subadditivity). Hence, upper box dimension is not based on a measure.

Comparing equation (1.31) with equation (1.34), we have

$$\dim_H(F) \le \overline{\dim}_B(F). \tag{1.36}$$

In order to have a better understanding on the gap between Hausdorff dimension and upper box dimension in (1.36), we introduce packing dimension and packing measure. Let

$$\mathcal{P}^s_\epsilon(F) := \sup\{\sum_i \operatorname{diam}(B_i)^s\}$$

where the supremum is taken over a collection of disjoint balls $\{B_i\}$ of radius at most ϵ and with centers in F. Similarly, by the monotonicity of $\mathcal{P}^s_\epsilon$ on ϵ, the limit $\mathcal{P}^s_0(F) = \lim_{\epsilon \to 0} \mathcal{P}^s_\epsilon(F)$ exists. However, by considering countable dense sets, it is easy to see that $\mathcal{P}^s_0$ is not a measure (due to lack of subadditivity again). Hence, we modify $\mathcal{P}^s_0$ to

$$\mathcal{P}^s(F) := \inf\left\{\sum_i \mathcal{P}^s_0(F_i) : F \subset \bigcup_{i=1}^\infty F_i\right\}, \tag{1.37}$$

which is a measure, and is called an s-dimensional *packing measure*. The *packing dimension* is naturally defined as

$$\dim_P(F) := \sup\{s : \mathcal{P}^s(F) = \infty\} = \inf\{s : \mathcal{P}^s(F) = 0\}. \tag{1.38}$$

For a general set $F \subset \mathbb{R}^d$, the following relations hold:

$$\dim_H(F) \leq \dim_P(F) \leq \overline{\dim}_B(F), \quad \text{and} \quad \mathcal{H}^s(F) \leq \mathcal{P}^s(F). \tag{1.39}$$

Suitable examples show that none of inequalities in (1.39) can be replaced by equalities [35].

The following lemma is useful for studying packing and box dimension, particularly for fractal sets with some kinds of self similarity.

Lemma 1.6. *[34, Corollary 3.9] Let $F \subset \mathbb{R}^n$ be compact and such that*

$$\overline{\dim}_B(F \cap V) = \overline{\dim}_B(F)$$

for all open sets V that intersect with F. Then $\dim_P(F) = \overline{\dim}_B(F)$.

It is important to note that most of Hausdorff dimension calculations involve an upper and a lower estimate, which are hopefully equal. For the upper estimate, the inequalities (1.39) provide a useful approach while the lower estimate is usually more difficult to deal with. As far as I am concerned, Frostman Lemma along with the notion of local dimension probably is one of the most important approaches and has been widely studied. We briefly review this approach as follows.

Local dimension is a local study on a Borel probability measure μ. More precisely, suppose a Borel probability measure μ supports on a fractal set F, then for any point $x \in F$, the local dimension is defined as

$$\dim_{loc}(\mu, x) := \lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}, \tag{1.40}$$

which can be viewed as a function of the point x. As the limit in the definition (1.40) does not always exist, one can always speak of $\overline{\dim}_{loc}$ or $\underline{\dim}_{loc}$ via replacing lim into $\overline{\lim}$ or $\underline{\lim}$ respectively. The local dimension is strongly related to the Hausdorff and packing dimension of a Borel probability measure μ supported on the compact set F (i.e., $\mu \in \mathcal{M}(F)$) via following lemma.

Lemma 1.7. *[87] Let*

$$\dim_H(\mu) := \inf\{\dim_H(E), E \subseteq F, \ \textit{with } \mu(E) = 1\}, \tag{1.41}$$

$$\dim_P(\mu) := \inf\{\dim_P(E), E \subseteq F \ \textit{with } \mu(E) = 1\}; \tag{1.42}$$

then

$$\dim_H(F) \geq \dim_H(\mu) = \operatorname{esssup}_x \underline{dim}_{loc}(\mu, x),$$

$$\dim_P(F) \geq \dim_P(\mu) = \operatorname{esssup}_x \overline{dim}_{loc}(\mu, x),$$

where the essential supremum is taken with respect to the measure μ.

In particular, if $\mu \in \mathcal{M}(F)$ is Ahlfors-regular [87, Theorem5.7], i.e., there exist some constants $C \geq 1$ and $s \geq 0$ such that for all points $x \in F$,

$$C^{-1}r^s \leq \mu(B(x,r)) \leq Cr^s,$$

then $s = \dim_H(\mu) = \dim_H(F) = \dim_P(F) = \overline{\dim}_B(F)$. This is directly from Lemma 2.1, and is sometimes called as *uniform mass distribution principle* [34].

The Frostman lemma below shows an inverse relationship between a Borel probability measure supported on F and the Hausdorff dimension of F.

Lemma 1.8 (Frostman Lemma). *[87] Let F be a Borel set in $\mathbb{R}^d$ and $s \geq 0$, then the following are equivalent:*

1. $\mathcal{H}^s(F) > 0$.

2. *There is a Borel probability measure supported in F with $\mu(F) = 1$ such that $\mu(B(x,r)) \leq r^s$ holds for all $x \in F$ and $r > 0$.*

After the preparations in fractal dimensions, we now review the thermodynamic formalism approach and begin with the concept of pressure function.

Pressure function: There are various equivalent definitions of pressure function, among which, we particularly consider the following two definitions. Suppose T is a continuous map from a compact set X to itself, and $\mathcal{M}_I(T)$ is the set of all invariant Borel probability measures on X. We endow $\mathcal{M}_I(T)$ with the weak* topology, and for each $\mu \in \mathcal{M}_I(T)$ we denote by h_μ the measure theoretic entropy of μ. Given an $\alpha-$Hölder continuous potential function $\phi : X \to \mathbb{R}^+$ and a real number t, the corresponding *free energy* $\mathcal{F}_t(\mu)$ of $\mu \in \mathcal{M}_I$ is defined as

$$\mathcal{F}_t(\mu) := h_\mu - t\int \phi d\mu, \tag{1.43}$$

and the *(geometrical) pressure function* $P(t)$ is defined as:

$$P(t) := \sup\{\mathcal{F}_t(\mu) | \mu \in \mathcal{M}_I\}. \tag{1.44}$$

For each $t \in \mathbb{R}$, we have $P(t) < \infty$ and the function $t \to P(t)$ is convex, non-increasing and $\alpha-$Hölder continuous . Therefore, there is a unique solution t_0 where $P(t_0) = 0$. A measure $\mu \in \mathcal{M}_I$ is called an *equilibrium state*, if the supremum (1.44) is attained for this measure. The central problems in the thermodynamic formalism approach are the studies of the (non-)existence of equilibrium states, and of the real analytic dependence of $P(t)$ on the parameter t.

In particular, if the potential function $\phi = \log|T'|$ (if T is also differentiable), then we say $\chi_\mu = \int \phi d\mu = \int \log|T'|$ is the *Lyapunov exponent* of μ, and such particular choice of ϕ leads to a close connection between the corresponding pressure function and the dimension spectrum of pointwise dimensions of the *Gibbs measure*. To state this connection, we particular consider the finite symbolic dynamical systems $(\sum_p^+, \sigma)$, where $\sum_p^+ = \{0, \cdots, p-1\}^{\mathbb{N}}$ and $\sigma : \sum_p^+ \to \sum_p^+$ as the left-shift map. We remark that all the results of the thermodynamic formalism approach on finite symbolic dynamical systems in the Introduction can be extended into a *finite type sub-shift systems* [6, 93], but for simplification, we only concentrate on the full-shift case.

Consider another version of the pressure function. Given an $\alpha-$Hölder continuous function $\phi : \Sigma_p \to \mathbb{R}^+$ and $t \in \mathbb{R}$, let $S_k(t\phi) := \sum_{i=0}^{k-1}(-t\phi) \circ \sigma^i$, then the *(topological) pressure* $\overline{P}(t)$ is defined by

$$\overline{P}(t) := \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\omega^{(n)}} \inf_{\xi \in \omega^{(n)}} \exp(S_n(-t\phi)(\xi)) \right), \tag{1.45}$$

where the sum is taken over all words $\omega^{(n)}$ of length n, and within the sum the infimum is taken over all elements ξ belonging to $\omega^{(n)}$.

Weighted Variational Principle: For any $\alpha-$Hölder continuous function $\phi : \sum_p^+ \to \mathbb{R}^+$ and $t \in \mathbb{R}$, $P(t) = \overline{P}(t)$. This is referred to as a *weighted variational principle* (in comparison to the un-weighted version, recall equation (1.6)) [121]. Under this assertion, the two definitions of pressure function coincide. Moreover, one can say that it is at the heart of the thermodynamic formalism approach. The standpoint is that (un-)weighted variational principle is important to select "interesting" invariant probability measures in a dynamical system. In particular, Gibbs measures are "selected" in this criterion for a shift map of a symbolic space.

Gibbs measure: For any $\alpha-$Hölder continuous map $\phi : \sum_p^+ \to \mathbb{R}^+$, a measure $\mu \in \mathcal{M}_I$ is called a *Gibbs measure* for the potential ϕ if there exists a constant $D > 1$ such that

$$D^{-1} \leq \frac{\mu\{y : y_i = x_i, i = 0, \cdots, n-1\}}{\exp(-nP(\phi) + \sum_{k=0}^{n-1} \phi(\sigma^k(x)))} < D \tag{1.46}$$

for all $x = (x_1, x_2, \cdots) \in \sum_p^+$ and $n \geq 0$.

In fact, for the shift map σ on a finite symbolic space, the hypothesis of the $\alpha-$Hölder continuity of the potential ϕ ensures the existence and uniqueness of the Gibbs measure and its coincidence with the equilibrium state for ϕ. Indeed, we can motivate the reasons of Gibbs measures (equation (1.46)) maximizing the pressure (equation (1.44)) with an exquisitely simple fact [6]: Suppose $n \geq 1$ is a fixed integer and consider n fixed real numbers $a_1, \cdots, a_n$. Then the maximum over all probability vectors $(p_1, \cdots, p_n)$ of the expression

$$-\sum_{i=1}^{n} p_i \log p_i + \sum_{i=1}^{n} p_i a_i$$

is equal to $\sum_{i=1}^{n} e^{a_i}$ and is attained if and only if

$$p_i = \frac{e^{a_i}}{\sum_{j=1}^{n} e^{a_j}}, \quad i = 1, \cdots, n.$$

It is worth remarking that the existence of Gibbs measures and their coincidence with equilibrium states of the shift maps σ can be extended to differentiable dynamical systems acting on a compact set with uniform hyperbolicity conditions in higher dimensions; c.f., the discussions on *Sinai-Ruelle-Bowen (SRB) measures* for diffeomorphisms [6, 13, 109]. However, lack of uniform hyperbolicity (sometimes, called as non-uniform hyperbolicity) or compactness would make extensions of the thermodynamic formalism approach much more sophisticated. Indeed, there are examples where the pressure functions $t \to P(t)$ are not necessarily analytic at every real value. The value t_c at which the pressure function is not real analytic is called *phase transition* [80, 92]. The verification on the existence or the uniqueness of equilibrium state is far to be trivial. To the best of my knowledge, there are no general theorems describing the thermodynamic formalism approach for arbitrary non-uniformly hyperbolic systems even in one dimension. Alternatively, researchers turn to extended the properties on the thermodynamic formalism approach case by case such as complex rational maps [84, 85, 105, 102, 103, 104] and real unimodal/multimodal maps [15, 17, 16, 96].

Transfer operator (Ruelle operator) in Section 1.3.2 is the main tool in constructing and studying Gibbs measures on a finite symbolic space. Let $C^{\alpha}(\sum_p^+)$ be a space of all $\alpha-$Hölder continuous functions on $\sum_p^+$ and define the Ruelle operator $\mathcal{L}_g : C^{\alpha}(\sum_p^+) \to C^{\alpha}(\sum_p^+)$ as:

$$\mathcal{L}_g \phi(x) := \sum_{y \in \sigma^{-1}(x)} \exp(g(y))\phi(y) = \sum_k \exp(g(kx))\phi(kx),$$

along with the dual operator $\mathcal{L}_g^* : \mathcal{M}(\sum_p^+) \to \mathcal{M}(\sum_p^+)$. Recall that $\mathcal{M}(\sum_p^+)$ denotes the space of all Borel probability measures on $\sum_p^+$. The following proposition states the link between the Ruelle operator and the corresponding Gibbs measures.

Proposition 1.4. *[93] Let $(\sum_p^+, \sigma)$ be a mixing shift on a finite symbolic space, then there exists a number $\lambda := \exp(P(\phi))$, a continuous function $h : \sum_p^+ \to \mathbb{R}^+$ and a measure $v \in \mathcal{M}(\sum_p^+)$ for which $\mathcal{L}_g h = \lambda h$, $\mathcal{L}_g^* v = \lambda v$, and $v(h) = 1$. Furthermore, $\mu := hv \in \mathcal{M}_I(\sigma)$ and μ is a Gibbs measure for the potential ϕ.*

A special class of potential ϕ that only depends on the first coordinate (i.e., $\phi(x) = \phi(x_1)$) plays an important role in the dimension estimation.

Proposition 1.5. *[93] Given numbers $0 < \lambda_i < 1, i = 1, \cdots, p$, define the function $\phi : \sum_p^+ \to \sum_p^+$ by $\phi(x) = \phi(x_1, x_2, \cdots) = \log \lambda_{x_1}^{-1}$, then ϕ is Hölder continuous.*

Furthermore, there exists a unique $s \geq 0$ *such that* $P(-s\phi) = 0$.

From *Bowen's formalism* [95], this unique solution s will become the Hausdorff, packing and box dimension of the repeller generated by the corresponding iterated function system, when all the λ_i satisfy some geometrical restrictions. In Chapter 4, we consider the fractal dimensions for generalized Moran constructions. These constructions are varied in the iterated function systems at each level. The intuitive idea is to approximate the fractal dimensions via a sequence of $\{s_k\}$, where each s_k is the unique solution of the pressure function for the symbolic space truncated at k-th level.

Chapter 2

Invariant measures with bounded variation densities

Conservative systems are often used as models of the physical world, where conservative is usually understood as energy preserving (i.e., where energy is invariant under the time evolution). In this chapter we consider conservative systems that are governed by discrete time dynamical systems. In particular, we focus on multidimensional *piecewise area preserving maps* (PAPs), which is a general extension of interval exchange transformations (IETs) into $\mathbb{R}^d$. Regarding IETs, Keane conjectured that minimality implies unique ergodicity in [69] and this conjecture holds for IETs with two or three intervals. However, counterexamples have been constructed; see [70, 75]. Thereafter, Masur [86] and Veech [117] have independently demonstrated that almost every topologically transitive IET (with respect to Lebesgue measure) is uniquely ergodic; simultaneously, Keane & Rauzy [71] revealed that unique ergodicity holds for a Baire residual subset of the space of IETs. To fully understand the densities of absolutely continuous invariant measures (ACIPs) for these non-uniquely ergodic counterexamples, it is natural to explore equivalent conditions to the topological transitivity in IETs in terms of ACIPs.

When extending to the multidimensional PAPs, we are facing at least two technical obstacles: complicated topology in high dimensions and non-local preservation of distance. For the class of PAPs which preserve distance locally, a special case of interest is the class of piecewise isometries (PWIs). Establishing the properties of their ACIPs will partially contribute to answering a fundamental question posed in [48], i.e., to determine all invariant non-atomic probability Borel measures for piecewise rotations. This question is still open so far.

For the class of PAPs that do not preserve distance locally, a particular case is piecewise hyperbolic maps. For these maps, properties that have been studied include transitivity and possession of a unique physical measure (e.g., see works of Boyarsky & Góra [14] and Viana [118]). These studies use a functional analytic approach by choosing a “reasonable” function space and applying a transfer operator on this space. They study statistical properties of the system by looking at the operator fixed point and determining if there is a spectral gap. In one-dimensional piecewise expanding maps, the space of bounded variation functions has been demonstrated to be such a "reasonable" space [14, 118]. In higher dimensions, the space of multidimensional bounded variation functions can still be chosen under certain assumptions [19, 73, 116] and contains a classical anisotropic Sobolev space of Triebel-Lizorkin type [7].

In this chapter, our interest is to explore the structure of ACIPs and the relationship between the uniqueness of such measures and topological properties, e.g., the existence of dense orbits, topological transitivity and minimality for multidimensional PAPs (particularly for PWIs) by applying the functional analytic approach. Definitions of PAPs and PWIs are given below.

Let X be a compact subset of $\mathbb{R}^d$ and $(X, \mathfrak{B}, m)$ be a probability space. For convenience, m always denotes $d-$dimensional normalized Lebesgue measure on X, and $\mathfrak{B}$ is the Borel σ-field. We say $\mathcal{P} = \{\omega_i\}_{i=0}^{r-1}$ is a *topological partition* of X if: (**i**) $\omega_i \cap \omega_j = \emptyset$, for $i \neq j$; (**ii**) $\bigcup_{i=0}^{r-1} \omega_i = X$; and (**iii**) for each ω_i, $\mathrm{int}(\omega_i) \neq \emptyset$ and $m(\partial\omega_i) = 0$. Here each ω_i is called an atom; $\mathrm{int}\, A$ and ∂A are the interior and boundary of A respectively.

Definition 2.1. *A nonsingular map $f : (X, \mathfrak{B}, m) \to (X, \mathfrak{B}, m)$ with a topological partition $\mathcal{P} = \{\omega_i\}_{i=0}^{r-1}$ is called a* piecewise area preserving map *(PAP) if $f|_{\mathrm{int}(\omega_i)} \in C^1$ for each ω_i and $|\det Df(x)| \equiv 1$ for $x \in \bigcup_{i=0}^{r-1} \mathrm{int}(\omega_i)$. Here non-singularity means that f is measurable (with respect to $\mathfrak{B}$) and $m(A) = 0$ implies $m(f^{-1}(A)) = m(f(A)) = 0$ for any $A \in \mathfrak{B}$; and Df refers to the Jacobian matrix. We say a PAP f is piecewise-invertible-area-preserving if $f|_{\omega_i}$ is invertible for each ω_i, and say f is an invertible PAP if f is globally invertible. In particular, if each $f|_{\mathrm{int}(\omega_i)}$ is isometry (i.e., preserving Euclidean distance) then we say f is a* piecewise isometry *(PWI).*

Our definition of PAPs include piecewise hyperbolic maps with determinant ± 1, e.g., baker's map, Arnold's cat map, area preserving Hénon map and standard map [83]. However, we will mainly concentrate on methods working on non-hyperbolic maps such as PWIs.

For a PAP $f : X \to X$, the ACIPs are classified based on the density properties as [1]

$$\begin{aligned}
\mathcal{M}_{IA}(f) : &= \{\mu \text{ is an ACIP with respect to } f\},\\
\mathcal{M}_{IB}(f) : &= \{\mu \in \mathcal{M}_{IA}(f) : \frac{d\mu}{dm} = \eta|_X \text{ for some } \eta \in BV(\Omega), \text{where } \Omega \supset X \text{ is an open ball}\},\\
\mathcal{M}_{IC}(f) : &= \{\mu \in \mathcal{M}_{IA}(f) : \frac{d\mu}{dm} \text{ is } m - a.e. \text{ continuous}\},
\end{aligned}$$

[1] $\frac{d\mu}{dm} \in L^1(m)$ is $m - a.e.$ continuous means that its equivalence class contains a representative which is an $m - a.e.$ continuous map.

where $BV(\Omega)$ is the space of bounded variation functions (see Definition 2.2). We choose to work with $\mathcal{M}_{IB}$ and $\mathcal{M}_{IC}$ for the following reasons.

- The function spaces mentioned in $\mathcal{M}_{IB}$ and $\mathcal{M}_{IC}$ are "large enough" Banach subspaces of $L^1(m)$, i.e., they contain discontinuous functions [110].

- Functions in these spaces have "good" geometric properties, e.g., $\chi_E \in BV(\Omega)$ implies that the measurable subset $E \subset \Omega$ has finite perimeter [33].

- These function spaces coincide with those chosen in piecewise hyperbolic maps in [7, 19, 116].

- These function spaces are invariant under the *transfer operator* (recall Section 1.3.2 for the definition) for PWIs.

It is clear that $\mathcal{M}_{IB} \subset \mathcal{M}_{IC} \subset \mathcal{M}_{IA}$ for one dimensional invertible PAPs [120], while in higher dimensions, $\mathcal{M}_{IB} \cup \mathcal{M}_{IC} \subset \mathcal{M}_{IA}$. Additionally, for non-invertible PAPs, the set $\mathcal{M}_{IA}$ is possibly empty and conditions for which $\mathcal{M}_{IA} \neq \emptyset$ are discussed in Section 2.2.2.

The novelty of this chapter is that we introduce multidimensional bounded variation functions to analyze ACIPs for PAPs, especially for PWIs. In Theorem 2.1, we explore the relationship between the set of nomadic points and the sets $\mathcal{M}_{IC}$, $\mathcal{M}_{IB}$ for invertible multidimensional PAPs. In particular, we demonstrate that when the set of nomadic points has a positive Lebesgue measure, both $\mathcal{M}_{IB}$ and $\mathcal{M}_{IC}$ are singletons. This can be applied to non-uniquely ergodic IETs constructed in [70, 75] to show the irregularity of densities of their ACIPs. For invertible PWIs, in Theorem 2.2 we give an approach to construct invariant measures with bounded variation densities. These results partially answer one of Goetz's questions in [48].

The chapter is organized in the following way. Preliminaries and the main results are stated in Section 2.1, then applications along with discussions are in Section 2.2 and finally proofs are given in Section 3.3.

2.1 Preliminaries and Main results

In this section, we give the formal definitions of *multidimensional bounded variation*, and then state the main results which are connected to one of the open questions in [48].

2.1.1 Multidimensional bounded variation

There are various definitions of multidimensional bounded variation functions, e.g., see Appendix A.1 and [33, 120]. These definitions can be reduced to the usual notation of bounded variation in one dimension; see Appendix A.1. We state one of these as follows.

Definition 2.2. *[33] Let Ω be an open set of $\mathbb{R}^d$. A function $\eta \in L^1(\Omega)$ is a* bounded variation *function ($\eta \in BV(\Omega)$) if*

$$\mathrm{var}(\eta) := \sup \left\{ \int_\Omega \eta \cdot \mathrm{div} \overrightarrow{\phi}\, dm : \overrightarrow{\phi} \in C_c^1(\Omega, \mathbb{R}^d), ||\overrightarrow{\phi}||_\infty \leq 1 \right\} < \infty. \tag{2.1}$$

Here $\overrightarrow{\phi} = (\phi_i)_{i=1}^d$, $\mathrm{div}\, \overrightarrow{\phi} = \sum_{i=1}^d \frac{\partial \phi_i}{\partial x_i}$, $||\overrightarrow{\phi}||_\infty := \sup_x |\overrightarrow{\phi}(x)|$, and $C_c^1(\Omega, \mathbb{R}^d)$ is the set of $\overrightarrow{\phi} \in C^1(\Omega, \mathbb{R}^d)$ with compact support. We define a norm on $BV(\Omega)$ by $||\eta||_{BV} := ||\eta||_1 + \mathrm{var}(\eta)$.

For functions of bounded variation, we state the corresponding Helly's Theorem [33] below.

Helly's Theorem [33] *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded domain with Lipschitz boundary. Assume that $\{\eta_n\}_{n=1}^\infty$ is a sequence in $BV(\Omega)$ satisfying $\sup_n ||\eta_n||_{BV} < \infty$, then there exists a subsequence $\{\eta_{n_k}\}_{k=1}^\infty$ and a function $\eta \in BV(\Omega)$ such that $\eta_{n_k} \to \eta$ in $L^1(\Omega)$ as $k \to \infty$.*

2.1.2 Main results

Let X be a compact subset of $\mathbb{R}^d$ and $f : X \to X$ be an invertible map. We say $x \in X$ is a *nomadic point* of f if $O_f(x) := \{f^i(x) | i \in \mathbb{Z}\}$ is dense in X. We denote $\mathrm{nom}(f)$ to be the set of all nomadic points of the map f. If $\mathrm{nom}(f) = X$ then f is called *minimal*.

Theorem 2.1. *Let $(X, \mathfrak{B}, m)$ be a probability space where m is the normalized Lebesgue measure and $f : X \to X$ be an invertible PAP with a topological partition $\mathcal{P} = \{\omega_i\}_{i=0}^{r-1}$. Then the following hold:*

(i) if $m(\mathrm{nom}(f)) > 0$ then $\mathcal{M}_{IB}(f) \cup \mathcal{M}_{IC}(f) = \{m\}$;

(ii) if $f|_{\omega_i}$ is a homeomorphism for each ω_i, then $\mathcal{M}_{IC}(f) = \{m\}$ implies $\mathrm{nom}(f) \neq \emptyset$.

We remark here that even for area preserving diffeomorphisms, $nom(f) \neq \emptyset$ does not necessarily imply $m(nom(f)) > 0$, see e.g. Fayad & Katok [36].

Corollary 2.1. *Suppose f is an invertible PAP with $m(\mathrm{nom}(f)) > 0$ and there exists a measure $m \neq \mu \in \mathcal{M}_{IA}(f)$, then $\varphi := \frac{d\mu}{dm} \notin BV(\Omega)$ and the set of discontinuities of φ has positive Lebesgue measure.*

The following Theorem 2.2 aims to construct ACIPs with bounded variation densities for invertible PWIs, say $f : X \to X$ with a topological partition $\mathcal{P} = \{\omega_0, \cdots, \omega_{r-1}\}$. As a bounded variation function is defined on an open set, we choose an open ball $\Omega \supset X$ and extend f to $\overline{f} : \Omega \to \Omega$ by

$$\overline{f}(x) = \begin{cases} f(x), & x \in X \\ x, & x \in \Omega \backslash X. \end{cases} \tag{2.2}$$

Given any $\eta \in BV(\Omega)$, the sequence of variations $\{\text{var}(\mathcal{L}_f^n \eta)\}_{n=0}^{\infty}$ are not necessarily uniformly bounded [73]. Therefore, we alternatively work with functions η which lie in a plausible proper subset $BV^*(\Omega)$ (see below). This subset is associated with a Sobolev space.

Let $\omega_r := \Omega \backslash X$ and without ambiguity we still write $\mathcal{P} = \{\omega_0, \cdots, \omega_r\}$ as a topological partition of Ω. Moreover, we denote

$$\partial \mathcal{P}^{\infty} := \{x \in \Omega : \overline{f}^n(x) \in \partial \mathcal{P} \text{ for some } n \geq 0\},$$

where $\partial \mathcal{P} := \cup_{i=0}^{r} \partial \omega_i$, and define a $\delta-$neighborhood of $\partial \mathcal{P}^{\infty}$ by

$$N_\delta := \{x \in \Omega, \text{dist}(x, \partial \mathcal{P}^{\infty}) < \delta\}. \tag{2.3}$$

For a given invertible PWI f, the function subspace $BV^*(\Omega)$ that we consider is defined as

$$BV^*(\Omega) := \{\eta \in BV(\Omega) : \eta|_{N_\delta} \in W^{1,2} \text{ for some } \delta > 0\}, \tag{2.4}$$

where $W^{1,2}$ is a Sobolev space (see Appendix A.2). We remark that different invertible PWI f determines different $BV^*(\Omega)$ individually, but in all cases $W^{1,2}(\Omega) \subseteq BV^*(\Omega) \subseteq BV(\Omega)$.

Theorem 2.2. *Suppose $f : X \to X$ is an invertible PWI and $\Omega \supset X$ an open ball. Then given any $\eta \in BV^*(\Omega)$ with $\eta|_X \geq 0$ and $||\eta|_X||_1 > 0$, there exists a subsequence of the Birkhoff averages of the transfer operator $\mathcal{L}_{\overline{f}}$ that converges to a function $\overline{\eta} \in BV(\Omega)$ in $L^1(m)$, i.e.,*

$$\frac{1}{n_k} \sum_{i=0}^{n_k-1} \mathcal{L}_{\overline{f}}^i \eta \to \overline{\eta} \in BV(\Omega), \text{ as } k \to \infty,$$

and by normalization $d\mu := \overline{\eta}|_X dm \in \mathcal{M}_{IB}(f)$.

Concerning the open question [48] in piecewise rotations (defined in Appendix A.3), the following corollaries give a universal approach to partially determine the ACIPs.

Corollary 2.2. *Suppose $f : X \to X$ is an invertible piecewise rotation, then*

(*i*) $\mathcal{M}_{IA}(f) = \{\varphi dm : \varphi = \mathbb{E}(\varphi|\mathcal{I}), \varphi \in L^1(m)\}$, *where* $\mathcal{I} = \{B \in \mathcal{B} : f^{-1}(B) = B \bmod m\}$;

(*ii*) *given any $\eta \in BV^*(\Omega)$ satisfying $\eta|_X \geq 0$ and $||\eta|_X||_1 > 0$, any accumulation point of $\{\frac{1}{n}\sum_{i=0}^{n-1} \mathcal{L}_{\overline{f}}^i \eta\}_{n=1}^{\infty}$ is an invariant density in $\mathcal{M}_{IB}(f)$. Furthermore, if $m(\text{nom}(f)) > 0$ then $\mathcal{M}_{IB}(f) = \{m\}$.*

Corollary 2.3. *Suppose $f : X \to X$ is a non-invertible piecewise rotation. Let $X^+ := \bigcap_{i=0}^{\infty} f^i(X)$ and define $f^+ : \overline{X^+} \to \overline{X^+}$ as in equation (2.5) in Section 2.2.2. Then f^+ is $m-a.e.$ invertible. Furthermore,*

(i) *if $m(\overline{X^+}) > 0$, then the statements in Corollary 2.2 hold for f^+ and*

$$\mathcal{M}_{IA}(f) = \{\mu(\cdot) := \nu(\cdot \cap \overline{X^+}), \forall \nu \in \mathcal{M}_{IA}(f^+)\};$$

(ii) *if $m(\overline{X^+}) = 0$, then $\mathcal{M}_{IA}(f) = \mathcal{M}_{IB}(f) = \mathcal{M}_{IC}(f) = \emptyset$.*

Proof of Corollary 2.2 is based on Theorems 2.1, 2.2 and Lemma 2.8 while proof of Corollary 2.3 is based on Lemma 2.1 and Proposition 2.2. We remark that ACIPs only give a subset of non-atomic probability Borel measures. Therefore, to fully answer the question in piecewise rotations [48], we have to explore singular non-atomic probability invariant measures. For instance, when $m(\overline{X^+}) = 0$, it is natural to consider Hausdorff measure. This is discussed at the end of Section 2.2.2.

2.2 Applications and Discussions

In this section, we consider two main applications. We first consider IETs (see Appendix A.4 for the definition) which are non-uniquely ergodic. We apply Theorem 2.1 to show that the densities of their ACIPs can be irregular. We then consider multidimensional piecewise invertible area preserving maps and apply Theorem 2.1 and Theorem 2.2 to study their invariant densities. At the end of this section, we give a short discussion on the open question posed in [48] for piecewise rotations.

2.2.1 Interval exchange transformations

For an IET f, the set $\mathcal{M}_{IC}(f)$ can be refined to

$$\mathcal{M}'_{IC}(f) := \{\mu \in \mathcal{M}_{IA}(f) : \frac{d\mu}{dm} := \varphi \text{ has at most countably many discontinuity points}\},$$

where m is the normalized Lebesgue measure. Observe that $\mathcal{M}_{IB}(f) \subset \mathcal{M}'_{IC}(f) \subset \mathcal{M}_{IC}(f)$. Moreover, it is known that topological transitivity[1] implies minimality for IETs (see e.g., Corollary 14.5.11 in [68]). Hence by applying Theorem 2.1, we have the following corollary which characterizes the minimality properties of IETs.

Corollary 2.4. *For any IET $f : [0, 1) \to [0, 1)$,*

$$f \textit{ is minimal} \Leftrightarrow \mathcal{M}_{IC}(f) = \{m\} \Leftrightarrow \mathcal{M}'_{IC}(f) = \{m\}.$$

[1]Topological transitivity means that for any open sets U and V, there exists $n \in \mathbb{Z}$ such that $f^n(U) \cap V \neq \emptyset$.

This corollary can be used to investigate Keane's conjecture, namely that minimality implies unique ergodicity for IETs [69]. This conjecture was shown to be false and we review two well-known counterexamples here.

Keynes and Newton [75] considered the following map. Let $\widehat{T}_{\gamma\beta} : [0, 1+\beta) \to [0, 1+\beta)$ be

$$\widehat{T}_{\gamma\beta}(x) = \begin{cases} x+1, & \text{if } 0 \le x < \beta \\ x+\gamma \pmod 1, & \text{if } \beta \le x < 1+\beta. \end{cases}$$

By choosing appropriate β and γ (see [26, 75]), the map $T_{\gamma\beta}(x) := \frac{1}{1+\beta}\widehat{T}_{\gamma\beta}(x(1+\beta))$ is minimal and has an eigenvalue -1. This implies that $T^2_{\gamma\beta}$ is not uniquely ergodic and its ergodic measures belong to $\mathcal{M}_{IA}$ (see [75] for details).

Keane [70] also constructed an IET with four intervals satisfying a strong irrationality condition that implies minimality. Under certain conditions, there exist two different ergodic measures μ_1 and μ_2. Moreover, such ergodic measures are either both in $\mathcal{M}_{IA}$ or one is Lebesgue measure and the other is singular. For the measure that is singular, the Hausdorff dimension has been recently estimated [24]. Together with our results, we could obtain a better understanding of ergodic measures for non-uniquely ergodic IETs.

For the examples above, there are no explicit formulae for their densities (even if these densities belong to $\mathcal{M}_{IA}$). One of the difficulties in constructing counterexamples can be seen from the fact that these ergodic measures are in $\mathcal{M}_I(f)\backslash\mathcal{M}_{IC}(f)$ from Corollary 2.4. In the following proposition, we provide a more explicit description of the invariant densities for non-uniquely ergodic IETs.

Proposition 2.1. *Let f be any topologically transitive IET on $[0,1)$. Suppose $m \neq \mu \in \mathcal{M}_{IA}(f)$, then the following hold:*

(i) the density of μ is a simple function (i.e., a linear combination of finitely many characteristic functions);

(ii) for any representative from the equivalence class $\varphi := \frac{d\mu}{dm}$, φ is discontinuous everywhere and $\operatorname{supp}\mu = [0,1)$ *(recall that if $x \in \operatorname{supp}\mu$, then for any open ball B_x containing x, $\mu(B_x) > 0$ holds [68, page 141]).*

Remark 2.1. *For the two ergodic measures $\mu_1, \mu_2 \in \mathcal{M}_{IA}$ (i.e., $\mu_1, \mu_2 \ll m$) in the examples of Keynes & Newton [75] and Keane [70], it can be derived that $\mu_1 \perp \mu_2$ (i.e., there exists a subset $E \subset [0,1)$ such that $\mu_1(E) = 1$ but $\mu_2(E) = 0$) [121] and moreover,* $\operatorname{supp}\mu_1 = \operatorname{supp}\mu_2 = [0,1)$ *from Proposition 2.1. Hence, the measures μ_1 and μ_2 intermingle with each other in some sense.*

2.2.2 Piecewise invertible area preserving maps

In this subsection, we aim to understand the structure of ACIPs for piecewise invertible area preserving maps $f : X \to X$. For such a map f, the set $X^+ := \bigcap_{i=0}^{\infty} f^i(X)$

is invariant under the map, i.e., $f(X^+) = X^+$ [44]. In particular for PWIs, X^+ is almost closed, i.e., $m(X^+) = m(\overline{X^+})$ [1]. Here we show that such almost closedness of X^+ is valid for a broad range of piecewise invertible area preserving maps.

Lemma 2.1. *Let $f : X \to X$ be a piecewise invertible area preserving map with a topological partition $\mathcal{P} = \{\omega_0, \cdots, \omega_{r-1}\}$. Suppose $f_i := f|_{\mathrm{int}\,\omega_i}$ is Lipschitz for each ω_i, then $m(X^+) = m(\overline{X^+})$ and $f|_{X^+}$ is $m-a.e.$ invertible.*

Under the conditions of Lemma 2.1, it is not necessary to have the property $f(\overline{X^+}) \subseteq \overline{X^+}$, but we can define a map f^+ that is $m-a.e.$ equal to f and for which $f^+(\overline{X^+}) \subset \overline{X^+}$ (see below). Since each f_i is Lipschitz, then there exists a continuous extension $\widehat{f_i} : \overline{\mathrm{int}\,\omega_i} \to \overline{f_i(\mathrm{int}\,\omega_i)}$. For any $x \in (\bigcup_{i=0}^{r-1} \partial\omega_i) \cap \overline{X^+}$, let $g(x) := \widehat{f_{i^*}}(x)$ where $i^* := \min\{i : x \in \partial\omega_i\}$. Then we can define $f^+ : \overline{X^+} \to \overline{X^+}$ to be

$$f^+(x) = \begin{cases} f(x), & x \in \mathrm{int}(\omega_i) \cap X^+ \\ g(x), & \text{otherwise.} \end{cases} \tag{2.5}$$

Moreover, if f_i is bi-Lipschitz, the map f^+ can be shown to be non-singular and to be $m-a.e.$ invertible. The non-singularity and $m-a.e.$ invertibility of f^+ allow to obtain Theorem 2.1 and Theorem 2.2 for $f^+ : \overline{X^+} \to \overline{X^+}$. The ACIPs of f^+ can further be used to determine the ACIPs of f as follows.

Proposition 2.2. *Let $f : X \to X$ be a piecewise invertible area preserving map with a topological partition $\mathcal{P} := \{\omega_0, \cdots, \omega_{r-1}\}$ and $f^+ : \overline{X^+} \to \overline{X^+}$ be defined as in equation (2.5). Suppose that $f_i := f|_{\mathrm{int}\,\omega_i}$ is bi-Lipschitz continuous. Then f^+ is non-singular and $m-a.e.$ invertible. Moreover, the following hold:*

(i) if $m(\overline{X^+}) > 0$, then $\mathcal{M}_{IA}(f) = \{\mu(\cdot) := \nu(\cdot \cap \overline{X^+}), \forall \nu \in \mathcal{M}_{IA}(f^+)\}$;

(ii) if $m(\overline{X^+}) = 0$, then $\mathcal{M}_{IA}(f) = \emptyset$.

When $m(\overline{X^+}) = 0$, it is natural to consider invariant measures that are absolutely continuous with respect to Hausdorff measure. If we let $s = \dim_H \overline{X^+}$ and furthermore, if f^+ satisfies the following conditions:

(1) $0 < \mathcal{H}^s(X^+) = \mathcal{H}^s(\overline{X^+}) < \infty$;

(2) $\mathcal{H}^s$ is an invariant measure for f^+;

(3) f^+ is non-singular with respect to $\mathcal{H}^s$, i.e., $\mathcal{H}^s((f^+)^{-1}(A)) = \mathcal{H}^s(f^+(A)) = 0$ whenever $\mathcal{H}^s(A) = 0$;

then by the arguments analogous to those used in Proposition 2.2, we can show that f^+ is $\mathcal{H}^s - a.e.$ invertible.

The above three conditions can be achieved for some piecewise invertible area preserving maps. We take interval translation maps (see Appendix A.4) as examples. Condition (2) is demonstrated in [18] while condition (3) can be inferred by combining condition (2) and the definition of Hausdorff measure. The $\mathcal{H}^s - a.e.$ closedness of X^+ can be shown by adapting the proof of Lemma 2.1. Moreover, by [? , Theorem 9.3], condition (1) hold for particular interval translation maps where $\overline{X^+}$ are self similar sets satisfying an open set condition and positive Hausdorff dimension [18].

Concerning the open question in [48], for non-invertible piecewise rotations in the case of $m(\overline{X^+}) = 0$, we consider absolutely continuous (with respect to $\mathcal{H}^s$) invariant probability measures. We denote

$$\mathcal{H}^s_I(f) := \{\nu \text{ probability invariant measure of f } : \nu \ll \mathcal{H}^s\}.$$

Proposition 2.3. *Suppose $f : X \to X$ is a two-dimensional piecewise rotation with $m(\overline{X^+}) = 0$ and $s := \dim_H \overline{X^+} > 1$, then X^+ is $\mathcal{H}^s - a.e.$ closed and f^+ is non-singular with respect to $\mathcal{H}^s$. Moreover, the following hold:*

(i) if $0 < \mathcal{H}^s(\overline{X^+}) < \infty$, then $\mathcal{H}^s$ is an invariant measure of f^+ and f^+ is $\mathcal{H}^s - a.e.$ invertible; furthermore, $\mathcal{H}^s_I(f) = \left\{\mu(\cdot) := \nu(\cdot \cap \overline{X^+}), \forall \nu \in \mathcal{H}^s_I(f^+)\right\}$;

(ii) if $\mathcal{H}^s(\overline{X^+}) = 0$ then $\mathcal{H}^s_I(f) = \emptyset$.

Under the condition $s := \dim_H \overline{X^+} > 1$, the proof of Proposition 2.3 is analogous to the proofs of Proposition 2.2 and Lemma 2.1. However, this condition does not always hold. For instance, the Cartesian product of interval translation maps (as defined in [18]) with themselves provides some examples of piecewise rotations with Hausdorff dimension ranging $0 \leq s \leq 1$. It might be interesting to explore conditions for $s > 1$.

Regarding the determination of all the invariant non-atomic probability measures for piecewise rotations, it might be necessary to consider the structure of $\mathcal{H}^s_I(f^+)$ in the case of $\mathcal{H}^s(\overline{X^+}) = \infty$. In this case, f^+ is not necessarily $\mathcal{H}^s - a.e.$ invertible, however, we note that by [35, Theorem 6.2], there exists a compact subset $E \subset \overline{X^+}$ with $0 < \mathcal{H}^s(E) < \infty$. We suggest to establish a non-atomic probability invariant measure of f^+ induced by E as a reference measure and leave this for further studies.

2.3 Proofs

We first state a basic lemma regarding $\mathcal{M}_{IA}(f)$, i.e., the set of all ACIPs with respect to a map f, with a standard proof.

Lemma 2.2. *Let $f : X \to X$ be a nonsingular map and $\varphi \in L^1(m)$, then $\mathcal{L}_f\varphi = \varphi$ if and only if $d\mu := \varphi dm \in \mathcal{M}_{IA}(f)$.*

Proof: Suppose $\mathcal{L}_f\varphi = \varphi$. Let $d\mu := \varphi dm$, then for each $A \subset X$,

$$\mu(A) = \int_A \varphi dm = \int_A \mathcal{L}_f\varphi dm = \int_{f^{-1}(A)} \varphi dm = \mu(f^{-1}(A)),$$

which implies $\mu \in \mathcal{M}_{IA}(f)$.

On the other hand, if $d\mu := \varphi dm$ is an invariant measure of f, i.e., $\mu(f^{-1}(A)) = \mu(A)$ for any Borel set $A \subset X$, then $\int_A \mathcal{L}_f \varphi dm = \int_{f^{-1}(A)} \varphi dm = \mu(f^{-1}(A)) = \mu(A) = \int_A \varphi dm$, which implies $\mathcal{L}_f \varphi = \varphi$. □

Proof of Theorem 2.1

To prove statement (i), we first show that $\mathcal{M}_{IC} = \{m\}$ followed by a proof of $\mathcal{M}_{IB} = \{m\}$. For statement (ii), we start with a lemma showing the equivalence between topological transitivity and the existence of a nomadic point.

Proof of the statement (i) in Theorem 2.1: Consider any $\mu \in \mathcal{M}_{IC}$. Take a representative $\varphi = \frac{d\mu}{dm}$ from the equivalence class such that φ is $m - a.e.$ continuous. Furthermore, take any point $x' \in \bigcup_{i=0}^{r-1} \operatorname{int} \omega_i$ where φ is continuous. Since $m(nom(f)) > 0$, we can choose a nomadic point x^* such that for any $n \in \mathbb{Z}$, the equality $\varphi \circ f^{-n}(x^*) = \varphi(x^*)$ holds. Then there exists a subsequence $\{f^{k_t}(x^*)\}$ such that as $|k_t| \to \infty$, $f^{k_t}(x^*) \to x'$ because x^* is a nomadic point. By the continuity of φ at x', we have

$$\varphi(x') = \varphi\left(\lim_{|k_t|\to\infty} f^{k_t}(x^*)\right) = \varphi(x^*).$$

This implies $\varphi \equiv 1$, i.e., $\mathcal{M}_{IC} = \{m\}$.

Consider the case $\mu \in \mathcal{M}_{IB}(f)$, i.e., $\varphi = \frac{d\mu}{dm}$ with $\varphi = \eta|_X$ for some $\eta \in BV(\Omega)$. Hence, by [120, page 178], $m - a.e.$ $x \in \Omega$ are *regular points* of η. By a regular point x_0, we mean there exists a unit vector $a \in \mathbb{R}^d$ such that the limits

$$\lim_{x\to x_0, \langle x-x_0, a\rangle>0} \eta(x) \quad \text{and} \quad \lim_{x\to x_0, \langle x-x_0, a\rangle<0} \eta(x)$$

exist, where $\langle \cdot, \cdot \rangle$ is the inner product. Therefore, by analogous arguments to that used in $\mathcal{M}_{IC}(f) = \{m\}$, it follows that for any regular point $x_0 \in \bigcup_{i=0}^{r-1} \operatorname{int}(\omega_i)$,

$$\lim_{x\to x_0, \langle x-x_0, a\rangle>0} \varphi(x) = \lim_{x\to x_0, \langle x-x_0, a\rangle<0} \varphi(x).$$

Hence, by [120, page 168], we have $\lim_{x\to x_0} \varphi(x) = \varphi(x^*)$ where x^* is a nomadic point. In addition, $\varphi \in L^1(m)$ implies that $m - a.e.$ $x \in X$ is a Lebesgue point of φ, i.e.,

$$\lim_{r\to 0^+} \frac{1}{m(B(x,r))} \int_{B(x,r)} |\varphi(y) - \varphi(x)| dm(y) = 0.$$

Therefore, if a regular point x_0 is also a Lebesgue point, then

$$\begin{aligned}0 \le |\varphi(x_0) - \varphi(x^*)| &= \lim_{r\to 0^+} \frac{1}{m(B(x_0,r))}\int_{B(x_0,r)} |\varphi(x_0) - \varphi(x^*)| dm(y) \\ &\le \lim_{r\to 0^+} \frac{1}{m(B(x_0,r))}\int_{B(x_0,r)} |\varphi(y) - \varphi(x_0)| dm(y) \\ &+ \lim_{r\to 0^+} \frac{1}{m(B(x_0,r))}\int_{B(x_0,r)} |\varphi(y) - \varphi(x^*)| dm(y) = 0.\end{aligned}$$

This implies that $\varphi(x_0) = \varphi(x^*)$, meaning that $\varphi \equiv 1$. □

Before proving statement (ii), we first formulate an equivalent condition of topological transitivity.

Lemma 2.3. *Let $f : X \to X$ be an invertible PAP with a topological partition $\mathcal{P} = \{\omega_i\}_{i=0}^{r-1}$. Suppose $f|_{\omega_i}$ is a homeomorphism for each ω_i, then the following are equivalent.*

(i) f has a nomadic point;

(ii) f is strongly topologically transitive, i.e., for any open sets U, V, there exists $n \in \mathbb{Z}$ such that int$(f^n(U)) \cap V \ne \emptyset$;

(iii) f is topologically transitive.

Since PAPs are not necessarily continuous at every point, the proof of Lemma 2.3 will not be standard (see [121] for the continuous version). Therefore, we provide the details of the proof here.

Proof of Lemma 2.3: "(ii) implies (iii)" is direct and we only need to prove (i) implies (ii) and (iii) implies (i). For convenience, we denote $\mathcal{P}^{(n)} := \bigvee_{i=0}^{n} f^{-i}(\mathcal{P})$ for $n \ge 0$ and $\mathcal{P}^{(n)} := \bigvee_{i=-1}^{n} f^{-i}(\mathcal{P})$ for $n < 0$, and denote $\omega^{(n)}$ one of the atoms in the topological partition $\mathcal{P}^{(n)}$ for any $n \in \mathbb{Z}$.

"$(i) \Rightarrow (ii)$." We prove by contradiction. Suppose there exist open sets $U, V \ne \emptyset$ such that for any $n \in \mathbb{Z}$, $\operatorname{int}(f^n(U)) \cap V = \emptyset$. Given a nomadic point x^* of f, there exist $n_1, n_2 \in \mathbb{Z}$ such that $f^{n_1}(x^*) \in U$ and $f^{n_2}(x^*) \in V$. Let $t := n_2 - n_1$, then there exists an atom $\omega^{(t)} \in \mathcal{P}^{(t)}$ such that $f^{n_1}(x^*) \in \omega^{(t)}$.

Suppose $f^{n_1}(x^*) \in \operatorname{int}\omega^{(t)}$. Since $f^t|_{\operatorname{int}\omega^{(t)}}$ is homeomorphic, we have $f^t(U \cap \operatorname{int}\omega^{(t)}) = \operatorname{int} f^t(U \cap \operatorname{int}\omega^{(t)})$. Therefore,

$$f^{n_2}(x^*) = f^t(f^{n_1})(x^*) \in f^t(U \cap \operatorname{int}\omega^{(t)}) \subset \operatorname{int} f^t(U),$$

which implies $f^{n_2}(x^*) \in \operatorname{int}(f^t(U)) \cap V$. This is a contradiction.

Suppose $f^{n_1}(x^*) \in \partial\omega^{(t)} \cap \omega^{(t)}$. Let $l := n_1 + n_2$, then there exists an atom $\omega^{(l)}$ such that $x^* \in \omega^{(l)}$. Since $f^{n_1}|_{\omega^{(l)}}, f^{n_2}|_{\omega^{(l)}}$ are continuous, there exists $x' \in \omega^{(l)}$ sufficiently close to x^* such that $f^{n_1}(x') \in \operatorname{int}\omega^{(t)} \cap U$ and $f^{n_2}(x') \in V$. Repeat the same process as the above case, using x' in place of x^* and this completes the proof.

"$(iii) \Rightarrow (i)$." Suppose $\{U_i\}_{i=1}^{\infty}$ is a countable base for X. By (iii), there exists an $n_1 \in \mathbb{Z}$ such that $f^{n_1}(U_1) \cap U_2 \neq \emptyset$. Let $y := f^{n_1}(x) \in f^{n_1}(U_1) \cap U_2$ where $x \in \omega^{(n_1)} \in \mathcal{P}^{(n_1)}$.

If $x \in \operatorname{int} \omega^{(n_1)}$, then there exists an open ball $x \in B_x \subset \omega^{(n_1)}$ such that $f^{n_1}(B_x) \subset \operatorname{int} f^{n_1}(U_1)$. Therefore, $y \in \operatorname{int}(f^{n_1}(U_1)) \cap U_2$.

Otherwise, if $x \in \partial\omega^{(n_1)} \cap \omega^{(n_1)}$, by using the approach analogous to that used in "$(i) \Rightarrow (ii)$", then there exists $x' \in \operatorname{int} \omega^{(n_1)} \cap U_1$ and $y' := f^{n_1}(x') \in \operatorname{int}(f^{n_1}(U_1)) \cap U_2$. Hence

$$\operatorname{int}(f^{n_1}(U_1)) \cap U_2 \neq \emptyset.$$

Therefore, there exists a closed ball B_2 such that $B_2 \subset \operatorname{int}(f^{n_1}(U_1)) \cap U_2 \cap \operatorname{int} \omega^{(n_1)}$. Moreover, since $f^{n_1}|_{B_2}$ is a homeomorphism, then $V_1 := f^{-n_1}(B_2)$ is closed. Analogously, for open sets $\operatorname{int} B_2$ and U_3 there exist $n_2 \in \mathbb{Z}$ and a closed ball $B_3 \subset \operatorname{int}(f^{n_2}(B_2)) \cap U_3$, with $f^{n_2}|_{B_3}$ being a homeomorphism. Let $V_2 := f^{-n_2}(B_3) \subset B_2$, then V_2 is closed and $f^{-n_1}(V_2) \subset V_1$.

If one continues this process, there will exist $\{n_i\}_{i=1}^{\infty}$ and a sequence of nonempty closed sets $\{V_i\}_{i=1}^{\infty}$ such that $f^{-n_i}(V_{i+1}) \subset V_i$ for each i. Therefore, $\bigcap_{i=1}^{\infty} f^N(V_{i+1}) \neq \emptyset$, where $N = -\sum_{j=1}^{i} n_j$. We note that $V_i \subset U_i$ for each i. Fix any $\bar{x} \in \bigcap_{i=1}^{\infty} f^N(V_{i+1})$, then $\bar{x}$ is nomadic. This completes the proof. □

Remark 2.2. *There does exist a map f that is topologically transitive but has no nomadic points [94]. However, Lemma 2.3 does not apply to this case as f does not extend continuously to the boundary from its interior.*

Proof of statement (ii) in Theorem 2.1: We argue by contradiction. Suppose f has no nomadic points, so by Lemma 2.3 there will exist two open sets $U, V \subset X$ such that $\operatorname{int}(f^n(U)) \cap V = \emptyset$ for all $n \in \mathbb{Z}$. Therefore, $\operatorname{int}(f^{i+n}(U)) \cap \operatorname{int} f^i(V) = \emptyset \mod m$, for any $n, i \in \mathbb{Z}$. Let

$$U^* := \bigcup_{i=-\infty}^{\infty} \operatorname{int} f^i(U) \text{ and } V^* := \bigcup_{i=-\infty}^{\infty} \operatorname{int} f^i(V).$$

Then $U^* \cap V^* = \emptyset \mod m$ and both U^* and V^* are invariant under f up to $m-a.e.$. Hence both $m|_{U^*}$ and $m|_{V^*}$ are invariant measures and $m|_{U^*} \neq m|_{V^*}$. Since $\partial U^* \subset \bigcup_{i=-\infty}^{\infty} \partial f^i(U)$ and $\partial V^* \subset \bigcup_{i=-\infty}^{\infty} \partial f^i(V)$, then this implies $m(\partial U^*) = m(\partial V^*) = 0$. I.e., the discontinuity points of χ_{U^*} and χ_{V^*} lie in a set of zero Lebesgue measure. Therefore, $m|_{U^*}, m|_{V^*} \in \mathcal{M}_{IC}$, which contradicts the uniqueness of measures in $\mathcal{M}_{IC}$. □

Remark 2.3. *When restricting PAPs to IETs, each open set of X is a union of countably many open intervals, therefore, χ_{U^*} and χ_{V^*} have at most countably many discontinuities. This fact is used to prove Corollary 2.4.*

Proof of Theorem 2.2

We prove Theorem 2.2 by the following lemmas. Recall that given an invertible PWI $f : X \to X$ and an open ball $\Omega \supset X$, we can extend f into an $\overline{f} : \Omega \to \Omega$ as in (2.2) and define $BV^*(\Omega) \subset BV(\Omega)$ as in (2.4).

Lemma 2.4. *Let Ω be an open subset of $\mathbb{R}^d$. Then $\eta \in BV^*(\Omega)$ if and only if $\eta = \eta^{(1)} + \eta^{(2)}$ where $\eta^{(1)} \in BV(\Omega)$ with $\eta^{(1)}|_{N_{\delta'}} = 0$ for some $\delta' > 0$ (recall that $N_{\delta'}$ is defined in equation (2.3)), and $\eta^{(2)} \in W^{1,2}(\Omega)$.*

Proof: We only prove the sufficiency. Suppose $\eta \in BV^*(\Omega)$, then there exists a $\delta > 0$ such that $\eta|_{N_\delta} \in W^{1,2}$. Let $\delta' = \delta/2$, then $\overline{N_{\delta'}} \subset N_\delta$. Hence there exists a bump function $B(x) \in C_c^\infty(\Omega)$ such that $B|_{\overline{N_{\delta'}}} \equiv 1$ and $B|_{N_\delta^c} \equiv 0$ (here N_δ^c denotes the complement of N_δ). We define

$$\eta^{(2)} := \eta \cdot B, \quad \eta^{(1)} := \eta - \eta^{(2)}.$$

It is clear the $\eta^{(2)} \in W^{1,2}(\Omega)$ and $\eta^{(1)} \in BV(\Omega)$ with $\eta^{(1)}|_{N_{\delta'}} = 0$. □

For convenience, we denote the sequence of the Birkhoff average of $\mathcal{L}_{\overline{f}}$ on a L^1 function η by $\{\eta_n\}_{n=1}^\infty$, i.e.,

$$\eta_n := \frac{1}{n} \sum_{i=0}^{n-1} \mathcal{L}_{\overline{f}}^i \eta.$$

Lemma 2.5. *Suppose $f : X \to X$ is an invertible PWI and $\Omega \supset X$ an open ball. Then for any $\eta \in W^{1,2}(\Omega)$, there exists a subsequence $\{n_k\}_{k=1}^\infty$ such that $\eta_{n_k} \to \overline{\eta} \in W^{1,2}(\Omega)$ in L^1 as $k \to \infty$.*

Proof: Since $\overline{f} : \Omega \to \Omega$ is a PWI, it follows that $\overline{f}_i := \overline{f}|_{\operatorname{int}\omega_i} = A_i x + c_i$, where A_i is an orthogonal matrix and c_i is a translation vector. Hence, by using the orthogonality and the definition of $||\cdot||_{W^{1,2}}$, we have:

$$||\eta_2 \circ \overline{f}^{-i}||_{W^{1,2}} = ||\eta_2||_{W^{1,2}}.$$

Moreover, since $W^{1,2}(\Omega)$ is a Hilbert space, the lemma can be shown directly by using the Banach-Saks Theorem [64] (see Appendix A.2). □

Lemma 2.6 (Change of Coordinates). *[32, Page 535] Let $\psi : W \to \Omega$ be a C^2- diffeomorphism where W and Ω are open subsets of $\mathbb{R}^d$. Given $\overrightarrow{\phi} \in C^1(\Omega, \mathbb{R}^d)$, let*

$$\overrightarrow{\phi}^\psi(y) := D\psi^{-1}(y)\overrightarrow{\phi}(\psi(y))$$

then

$$\operatorname{div}(|\det D\psi|\overrightarrow{\phi}^\psi) = (\operatorname{div}\overrightarrow{\phi}) \circ \psi \cdot |\det D\psi|. \tag{2.6}$$

Lemma 2.7. *Let $f : X \to X$ be an invertible piecewise isometry with a topological partition $\mathcal{P} = \{\omega_0, \omega_1, \cdots, \omega_{r-1}\}$. Suppose that $\eta \in BV(\Omega)$ with $\eta|_{N_\delta} = 0$ for some $\delta > 0$, then $\operatorname{var}(\mathcal{L}_{\overline{f}}\eta) \le \operatorname{var}(\eta) < \infty$.*

Proof: Let $\omega_r := \Omega\backslash X$ and $\overline{f}_i := \overline{f}|_{\omega_i}$ for each $\omega_i, i = 0, \cdots, r$. For any $\eta \in BV(\Omega)$, we have $\mathcal{L}_{\overline{f}}\eta \in BV(\Omega)$. Recall that

$$\mathrm{var}(\mathcal{L}_{\overline{f}}\eta) = \sup\left\{\int_\Omega (\eta \circ \overline{f}^{-1} \cdot \mathrm{div}\,\overrightarrow{\phi})dm \;:\; \overrightarrow{\phi} \in C_c^1(\Omega, \mathbb{R}^d), ||\overrightarrow{\phi}||_\infty \leq 1\right\}.$$

Hence, given any $\epsilon > 0$, there exists $\overrightarrow{\phi} \in C_c^1(\Omega, \mathbb{R}^d)$ with $||\overrightarrow{\phi}||_\infty \leq 1$ such that

$$\mathrm{var}(\mathcal{L}_{\overline{f}}\eta) - \epsilon \leq \int_\Omega (\eta \circ \overline{f}^{-1} \cdot \mathrm{div}\,\overrightarrow{\phi})dm = \sum_{i=0}^{r} \int_{\overline{f}_i(\mathrm{int}\,\omega_i)} (\eta \circ \overline{f}_i^{-1} \cdot \mathrm{div}\,\overrightarrow{\phi})dm.$$

Let $x = \overline{f}_i^{-1}(y)$ on each ω_i, we have

$$\int_{\overline{f}_i(\mathrm{int}\,\omega_i)} \eta(\overline{f}_i^{-1}(y)) \cdot \mathrm{div}\,\overrightarrow{\phi}(y)dm(y) = \int_{\mathrm{int}\,\omega_i} \eta(x) \cdot \mathrm{div}(\overrightarrow{\phi})(\overline{f}_i(x))dm(\overline{f}_i(x))$$

$$= \int_{\mathrm{int}\,\omega_i} \eta \cdot (\mathrm{div}\,\overrightarrow{\phi}) \circ \overline{f}_i dm = \int_{\mathrm{int}\,\omega_i} \eta \cdot \mathrm{div}\,\overrightarrow{\phi}^{\overline{f}_i} dm,$$

where $\overrightarrow{\phi}^{\overline{f}_i} := (D\overline{f}_i)^{-1} \cdot \overrightarrow{\phi} \circ \overline{f}_i$. The second equality is due to $|\det D\overline{f}_i(x)| \equiv 1$ while the third is due to equation (2.6) in Lemma 2.6. Hence,

$$\mathrm{var}(\mathcal{L}_{\overline{f}}\eta) - \epsilon \leq \sum_{i=1}^{r} \int_{\mathrm{int}\,\omega_i} \eta(x) \cdot \mathrm{div}\,\overrightarrow{\phi}^{\overline{f}_i} dm.$$

Moreover, each $\overline{f}_i$ can be written in the form of $\overline{f}_i(x) = A_i \cdot x + c_i$ where the orthogonal matrix A_i preserves the Euclidean metric. Then

$$\sup_{x \in \mathrm{int}\,\omega_i} |\overrightarrow{\phi}^{\overline{f}_i}(x)| = \sup_{x \in \mathrm{int}\,\omega_i} |A_i^{-1}\overrightarrow{\phi}(\overline{f}_i(x))| = \sup_{x \in \mathrm{int}\,\omega_i} |\overrightarrow{\phi}(\overline{f}_i(x))| = \sup_{y \in \overline{f}_i(\mathrm{int}\,\omega_i)} |\overrightarrow{\phi}(y)| \leq 1.$$

Since $\eta|_{N_\delta} = 0$ for some $\delta > 0$, for each ω_i, we let $\omega_i^{(\delta)} := N_\delta^c \cap \omega_i$ which is a compact subset of ω_i. Moreover, $\omega_i^{(\delta)} \subset \omega_i^{(\delta/2)} \subset \mathrm{int}\,\omega_i$. Therefore, there exists a bump function $B_i(x) \in C_c^\infty(\Omega)$ such that $0 \leq B_i(x) \leq 1$, $B_i(x) = 1$ on $\omega_i^{(\delta)}$ and $B_i(x) = 0$ on $(\mathrm{int}\,\omega^{(\delta/2)})^c$. We extend the function $\overrightarrow{\phi}^{\overline{f}_i}$ to be zero outside ω_i and define

$$\overrightarrow{\psi}(x) := \sum_{i=0}^{r} B_i(x) \cdot \overrightarrow{\phi}^{\overline{f}_i}(x), \; x \in \Omega.$$

It is apparent to see that $\overrightarrow{\psi} \in C_c^1(\Omega, \mathbb{R}^d)$ and $\sup_{x\in\Omega} |\overrightarrow{\psi}| \leq \sup_{x\in\Omega} |\overrightarrow{\phi}^{\overline{f}_i}(x)| \leq 1$ for each i. Moreover, since $\overrightarrow{\psi}|_{\omega_\delta} = \overrightarrow{\phi}^{\overline{f}_i}|_{\omega_\delta}$ and $\eta|_{N_\delta} = 0$, we have

$$\mathrm{var}(\mathcal{L}_{\overline{f}}\eta) - \epsilon \leq \sum_{i=1}^{r} \int_{\mathrm{int}\,\omega_i} \eta(x) \cdot \mathrm{div}\,\overrightarrow{\phi}^{\overline{f}_i} dm = \int_\Omega \eta \cdot \mathrm{div}\,\overrightarrow{\psi}\,dm \leq \mathrm{var}(\eta).$$

Since ϵ is arbitrary, it follows that $\mathrm{var}(\mathcal{L}_{\overline{f}}\eta) \leq \mathrm{var}(\eta) < \infty$. $\square$

Proof of Theorem 2.2: Given any $\eta \in BV^*(\Omega)$, from Lemma 2.4, we can write $\eta = \eta^{(1)} + \eta^{(2)}$ where $\eta^{(1)}|_{N_\delta} = 0$ for some $\delta > 0$ and $\eta^{(2)} \in W^{1,2}$. Moreover, by Lemma 2.5, there exists a subsequence $\{\eta^{(2)}_{n_k}\}$ of $\{\eta^{(2)}_n\}$ which converges to a function $\overline{\eta}^{(2)} \in W^{1,2}$ in $L^1(m)$.

Consider the function $\eta^{(1)}$. Given any $n \geq 1$, by analogous arguments used in Lemma 2.7, we have $\text{var}(\mathcal{L}^n_{\overline{f}}\eta^{(1)}) \leq \text{var}(\eta^{(1)})$. It follows that $||\mathcal{L}^n_{\overline{f}}\eta^{(1)}||_{BV} \leq ||\eta^{(1)}||_{BV}$. Hence, $||\eta^{(1)}_n||_{BV} \leq ||\eta^{(1)}||_{BV} < \infty$. By Helly's Theorem [33], there exists a subsequence, for convenience say, $\{\eta^{(1)}_{n_k}\}$ converging to a function $\overline{\eta}^{(1)} \in BV$ in $L^1(m)$.

By the triangle inequality, for each $\overline{\eta}^{(i)}$, $i = 1, 2$,

$$||\mathcal{L}_{\overline{f}}\overline{\eta}^{(i)} - \overline{\eta}^{(i)}||_1 \leq ||\mathcal{L}_{\overline{f}}\overline{\eta}^{(i)} - \mathcal{L}_{\overline{f}}\eta^{(i)}_{n_k}||_1 + ||\mathcal{L}_{\overline{f}}\eta^{(i)}_{n_k} - \eta^{(i)}_{n_k}||_1 + ||\eta^{(i)}_{n_k} - \overline{\eta}^{(i)}||_1. \tag{2.7}$$

It is clear that the first and third term tend to 0 as $k \to \infty$. Moreover, we note that

$$||\mathcal{L}_{\overline{f}}\eta^{(i)}_{n_k} - \eta^{(i)}_{n_k}||_1 = \left|\left| \frac{1}{n_k}\sum_{j=1}^{n_k} \eta^{(i)} \circ \overline{f}^{-j} - \frac{1}{n_k}\sum_{j=0}^{n_k-1} \eta^{(i)} \circ \overline{f}^{-j} \right|\right|_1 = \frac{1}{n_k}||\eta^{(i)} \circ \overline{f}^{-n_k} - \eta^{(i)}||_1 \leq \frac{2}{n_k}||\eta^{(i)}||_1,$$

hence the second term tends to 0 as $k \to \infty$. Therefore, $\mathcal{L}_{\overline{f}}\overline{\eta}^{(i)} = \overline{\eta}^{(i)}$, implying that $\overline{\eta} := \overline{\eta}^{(1)} + \overline{\eta}^{(2)} \in BV(\Omega)$ in an invariant density of $\overline{f}$ from Lemma 2.2. Moreover, since $\eta|_X \geq 0$ and $||\overline{\eta}|_X||_1 = ||\eta|_X||_1 > 0$, by normalization $d\mu = \overline{\eta}|_X dm \in \mathcal{M}_{IB}(f)$.
□

Remark 2.4. *For a given invertible PWI, if the set N_δ satisfies $f^{-1}(N_\delta) \subseteq N_\delta$, then the above accumulation point $\overline{\eta}$ can show to be in $BV^*(\Omega)$ for the following reason. Since the sequence $\{\eta^{(1)}_{n_k}\}$ converges to $\overline{\eta}^{(1)}$ in L^1, there will exist a subsequence which pointwise converges to $\overline{\eta}^{(1)}$. Moreover, since $\eta^{(1)}|_{N_\delta} = 0$ and $\overline{f}^{-1}(N_\delta) \subseteq N_\delta$, it follows that $\eta^{(1)}_n|_{N_\delta} = 0$ for any $n \geq 0$. Therefore, $\overline{\eta}^{(1)}|_{N_\delta} = 0$. By Lemma 2.4, this implies $\overline{\eta} = \overline{\eta}^{(1)} + \overline{\eta}^{(2)} \in BV^*(\Omega)$.*

Proof of Proposition 2.1

Lemma 2.8. *Let $(X, \mathfrak{B}, m)$ be a probability space and $f : X \to X$ be an invertible PAP, then $d\mu := \varphi dm \in \mathcal{M}_{IA}(f)$ if and only if $\varphi = \mathbb{E}(\varphi|\mathfrak{I})$, where $\mathbb{E}(\varphi|\mathfrak{I})$ is the conditional expectation on the $\sigma-$field $\mathfrak{I} := \{B \in \mathfrak{B} | f^{-1}(B) = B \mod m\}$.*

Proof: Suppose $d\mu = \varphi dm \in \mathcal{M}_{IA}(f)$, then by Lemma 2.2 we have $\mathcal{L}_f\varphi = \varphi$, which implies $\varphi \circ f = \varphi$. Combining with the Birkhoff Ergodic Theorem, we obtain

$$\varphi = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \varphi \circ f^i = \mathbb{E}(\varphi|\mathfrak{I}).$$

For the converse, given any fixed $L \in \mathbb{N}$, we define $\varphi_L(x) := \min\{\varphi(x), L\}$. By the Birkhoff Ergodic Theorem, for $m-a.e.\ x\in X$,

$$\mathcal{L}_f(\mathbb{E}(\varphi_L|\mathcal{I}))(x)=\mathbb{E}(\varphi_L|\mathcal{I})\circ f^{-1}(x)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=-1}^{n-2}\varphi_L\circ f^i(x)$$
$$=\lim_{n\to\infty}\left[\frac{1}{n}(\sum_{i=0}^{n-1}\varphi_L\circ f^i)(x)+\frac{1}{n}\left(\varphi_L\circ f^{-1}(x)-\varphi_L\circ f^{n-1}(x)\right)\right].$$

Since

$$\lim_{n\to\infty}\frac{1}{n}\left(\varphi_L\circ f^{-1}(x)-\varphi_L\circ f^{n-1}(x)\right)\leq\lim_{n\to\infty}\frac{2L}{n}=0,$$

we have $\mathcal{L}_f\mathbb{E}(\varphi_L|\mathcal{I})=\mathbb{E}(\varphi_L|\mathcal{I})$. Then by the Monotone convergence theorem,

$$\mathcal{L}_f\mathbb{E}(\varphi|\mathcal{I})=\mathbb{E}(\varphi|\mathcal{I}).$$

Therefore, $\varphi=\mathbb{E}(\varphi|\mathcal{I})$ implies $d\mu:=\varphi dm\in\mathcal{M}_I(f)$ from Lemma 2.2. □

We say a $\sigma-$field $\mathcal{I}$ is *finitely generated*, if there exists a partition $\mathcal{A}:=\{A_i\}_{i=0}^{r-1}\subseteq\mathcal{I}$, and for each $B\in\mathcal{I}$, there exist finitely many $A_{i_1},\cdots,A_{i_l}\in\mathcal{A}$ such that $B=\bigcup_{k=1}^{l}A_{i_k}\bmod m$.

Proof of Proposition 2.1: We first show statement (i). For a topologically transitive IET f, since there are only finitely many ergodic measures, say $\{\nu_i\}_{i=0}^{r-1}\in\mathcal{M}_{IA}(f)$ [69], the $\sigma-$field $\mathcal{I}$ is finitely generated by a partition, say $\mathcal{A}:=\{A_0,\cdots,A_{r-1}\}$ with $0<m(A_i)<1$ and $\nu_i=m|_{A_i}\in\mathcal{M}_I(f)$. Therefore, for any $d\mu=\varphi dm\in\mathcal{M}_{IA}$, by Lemma 2.8 and [14],

$$\varphi=\mathbb{E}(\varphi|\mathcal{I})=\sum_{i=0}^{r-1}\frac{1}{m(A_i)}\int_{A_i}\varphi dm\cdot\chi_{A_i},\quad\forall\ \varphi\in L^1(m).\tag{2.8}$$

This implies that φ is a simple function.

Now we show statement (ii). Since each $m|_{A_i}\in\mathcal{M}_{IA}(f)$ and $m(A_i)>0$, it is clear that for $m-a.e.\ x\in A_i$, the orbit $O_f(x)\subset A_i$. Moreover, since f is minimal, each A_i is dense in X. Hence $\operatorname{int}A_i=\emptyset$. Based on (2.8), if $\mu:=\varphi dm\in\mathcal{M}_{IA}(f)$ and $\mu\neq m$ then there exists $i\neq j$ such that $\int_{A_i}\varphi dm/m(A_i)\neq\int_{A_j}\varphi dm/m(A_j)$. Therefore, φ is discontinuous everywhere. Moreover, for any open set $U\subset[0,1)$, we have $m(A_i\cap U)>0$. Hence $\mu(U)>0$. It follows that $\operatorname{supp}(\mu)=[0,1)$. □

Proofs of Lemma 2.1 and Proposition 2.2

Proof of Lemma 2.1: We first show almost closedness of X^+. Since f_i is Lipschitz continuous, it can be continuously extended from $\operatorname{int}\omega_i$ onto $\overline{\operatorname{int}\omega_i}$. If we denote its continuous extension by $\widehat{f_i}:\overline{\operatorname{int}\omega_i}\to\widehat{f_i}(\overline{\operatorname{int}\omega_i})$, then each $\widehat{f_i}$ is also Lipschitz continuous. Together with the non-singularity of f, we know [1]

$$\begin{aligned}\overline{X^+} := \text{closure}\left(\bigcap_{j=0}^{\infty} f^j(\bigcup_{i=0}^{r-1}\omega_i)\right) &\subseteq \bigcap_{j=0}^{\infty}\text{closure}\left(\bigcup_{i=0}^{r-1}\widehat{f}_i^j(\text{int}\,\omega_i)\right) \text{ mod } m\\ &\subseteq \bigcap_{j=0}^{\infty}\bigcup_{i=0}^{r-1}\widehat{f}_i^j(\overline{\text{int}\,\omega_i}) \text{ mod } m\\ &= \bigcap_{j=0}^{\infty}\bigcup_{i=0}^{r-1} f_i(\omega_i) \text{ mod } m\\ &= X^+ \text{ mod } m.\end{aligned}$$

This implies $m(X^+) = m(\overline{X^+})$.

Denote $\omega_i^+ := \omega_i \cap X^+$, then $X^+ = \cup_{i=0}^{r-1}\omega_i^+$. Consequently,

$$\sum_{i=0}^{r-1} m(f(\omega_i^+)) = \sum_{i=0}^{r-1} m(\omega_i^+) = m(\bigcup_{i=0}^{r-1}\omega_i^+) = m(X^+) = m(f(X^+)) = m(\cup_{i=0}^{r-1} f(\omega_i^+)).$$

This follows that $m(f(\omega_i^+) \cap f(\omega_j^+)) = 0$, implying $f|_{X^+}$ is $m-a.e.$ invertible. □

Proof of Proposition 2.2: We note that f^+ is piecewise Lipschitz continuous, then for any Borel subset $A \subset \overline{X^+}$ with $m(A) = 0$, we have $m(f^+(A)) = 0$. Moreover, since $f|_{\omega_i}$ is bi-Lipschitz, it follows $m((f^+)^{-1}(A)) = 0$. Hence f^+ is non-singular. The $m-a.e.$ invertibility of f^+ is directly from Lemma 2.1.

When $m(\overline{X^+}) > 0$, $f|_{\overline{X^+}}$ can be viewed as a first return map of f on $\overline{X^+}$. Therefore,

$$\left\{\mu(\cdot) := \nu(\cdot \cap \overline{X^+}), \forall \nu \in \mathcal{M}_{IA}(f^+)\right\} \subseteq \mathcal{M}_{IA}(f).$$

Moreover, for any $\mu \in \mathcal{M}_{IA}(f)$, since $\mu(X) = 1$ and $f^{-1} \circ f(X) = X$, $\mu(f(X)) = 1$. This implies that $\mu(X^+) = 1$ and completes the proof of statement (i).

For statement (ii), we argue by contradiction. Suppose that there exists $\mu \in \mathcal{M}_{IA}(f)$, since $m(\overline{X^+}) = 0$, it follows that $\mu(\overline{X^+}) = 0$. This is a contradiction with $\mu(X^+) = 1$. □

[1] By $A \subseteq B$ mod m we mean that for $m-a.e.$ $x \in A$, we have $x \in B$.

Chapter 3

On mixing and mixing rate properties for piecewise expanding maps composing with permutations

Mixing processes of various kinds occur throughout nature and are vital in many technological applications, such as quantum chaos and statistical mechanics [109]. It is therefore an important and interesting problem to understand the properties of these processes from a mathematical perspective. In the context of discrete time dynamical systems, transfer operator methods provide a method of investigating such questions, and this approach has been developed in a variety of settings [8, 6, 9, 14, 50, 54, 72, 118]. The transfer operator $\mathcal{L}$ acts on a suitable Banach space of real-valued functions (or distributions), and its spectrum provides a powerful tool for analyzing many mixing properties of the system, e.g. whether or not the system is indeed mixing [14], the mixing rate of the system [6, 72, 118], and the existence of almost-invariant sets [30, 42].

We restrict our attention to a piecewise smooth map f on a compact interval I. We briefly recall some facts about this situation; for more details see, for instance, [29]. We consider the transfer operator $\mathcal{L}_f$ of f restricted to the Banach space BV of functions of bounded variation. The spectrum of $\mathcal{L}_f|_{BV}$ is contained in the unit disk in the complex plane. If f is piecewise expanding, with the expansion factor uniformly bounded away from 1, then the essential spectral radius r_{ess} can be interpreted as the slowest *local* mixing rate of the system. The spectrum certainly contains the eigenvalue 1, corresponding to the equilibrium state of the system, but may also contain further isolated points of modulus greater than r_{ess}. These isolated eigenvalues come from resonances in the system, and the corresponding eigenfunctions will converge to equilibrium at a slower rate than would be predicted from r_{ess}. Thus the *global* mixing behaviour of the system is determined by the quantity $\sup\{|\lambda| : \lambda \in \mathrm{Spec}(\mathcal{L}_f|_{\mathrm{BV}})\backslash\{1\}\}$. For brevity, we will refer to this quantity as the *mixing rate* of the system. Thus, good mixing is achieved when the mixing rate is small, while a mixing rate close to 1 indicates that there are eigenfunctions for which convergence to equilibrium is very slow.

Even in the case of one-dimensional PWEs, there seems to be no general technique known for calculating the isolated eigenvalues and hence finding the mixing rate. A number of examples have nevertheless been investigated in detail. Baladi [5] constructed an expanding Markov map of constant slope for which the transfer operator has a complex-conjugate pair of isolated eigenvalues, and this was used by Collet and Eckmann [27] to construct a two dimensional piecewise hyperbolic function which is the (skew-)product of piecewise expanding interval map with similar behaviour to Baladi's map. Dellnitz et. al. [29] described a parameterized family of expanding interval maps for which the location of a nontrivial real, positive isolated eigenvalue may be controlled.

In this chapter we study the effect on mixing of dividing I into N equal subintervals and composing f with a permutation of these. As far as we are aware, this is the first attempt to investigate the effect of permutations on mixing for discrete dynamical systems. In the continuous setting, Ashwin et al. [3] considered a 1-dimensional diffusion process, and showed that the mixing rate is typically improved if subintervals of its domain are permuted at regular timesteps. They considered permutations of various simple kinds (such as simple rotation, secondary rotation and interleaving), and investigated numerically the effect of certain permutations for small N. As well as treating the discrete setting, the novelty of our approach is that we use combinatorial and group-theoretic arguments to treat all permutations systematically for arbitrarily large N.

We now describe more precisely the situation we investigate and the main results we obtain. Let f again be a piecewise smooth map on the compact interval I, let I be divided into N subintervals of equal length, and let $\sigma : I \to I$ be a piecewise smooth map which simply permutes these intervals. (Thus we may identify σ with an element of the symmetric group S_N, which consists of the $N!$ permutations of N objects.) Then the composite function $\sigma \circ f$ is again a piecewise smooth map on I, and we wish to compare the global mixing behaviour of $\sigma \circ f$ and f. The main focus of our study will be the stretch-and-fold map $f(x) = mx \bmod 1$ on the interval $I = [0, 1]$, where $m \geq 2$ is an integer. This map is a standard (very simple) example of a piecewise expanding interval map, and is itself often taken as the canonical mixing protocol for polymers and pastes [76]. It can also be regarded as the prototype for the much-studied family of maps $x \mapsto \beta x + \alpha \bmod 1$ (see for example [38, 46, 53]). Our functions $\sigma \circ f$ provide a generalisation of the basic map f in a different direction.

For each choice of the two integer parameters m, N, we are interested in the mixing behaviour of the collection of maps $\sigma \circ f$ as σ ranges through S_N. The mixing behaviour of f itself is easy to describe. The essential spectral radius of $\mathcal{L}_f$ is $1/m$, and there are no isolated eigenvalues λ with $|\lambda| > 1/m$ apart from the simple eigenvalue 1. Thus the mixing rate for f is $1/m$. To investigate the mixing behaviour of $\sigma \circ f$, we must first address the issue of whether $\sigma \circ f$ is indeed mixing at all. This is essentially a combinatorial question, depending on m and N as well as on the particular permutation σ. We will show in Theorem 3.1 that, if m is fixed, then for many values of N, the function $\sigma \circ f$ is mixing for *all* permutations $\sigma \in S_N$; for the remaining values of N, the map $\sigma \circ f$ will fail to be mixing for some permutations σ, but the proportion of such permutations tends to 0 as $N \to \infty$.

We will see that the essential spectral radius of $\mathcal{L}_{\sigma\circ f}$ is again $1/m$, so that, when $\sigma \circ f$ is mixing, its mixing rate can be no better than that of f. For simplicity, we assume that $N > m$ and $\gcd(m, N) = 1$. In particular, this guarantees that $\sigma \circ f$ is mixing for all $\sigma \in S_N$. Its mixing rate is

$$\tau_\sigma := \sup\{|\lambda| : \lambda \in \mathrm{Spec}(\mathcal{L}_{\sigma\circ f}|_{\mathrm{BV}})\backslash\{1\}\} \geq 1/m,$$

and composition with σ results in a *worse* mixing rate than for f alone unless we have equality. We determine in Theorem 3.2 how bad the mixing rate can become: the maximal value of τ_σ as σ ranges through S_N is $\sin(m\pi/N)/m\sin(\pi/N)$, which can be made arbitrarily close to 1 by taking N sufficiently large. The function $\sigma \circ f$ is a Markov map, and an argument using Fredholm determinants [89, 90] shows that τ_σ is the modulus of the second largest eigenvalue of the probability transition matrix for $\sigma \circ f$, which is a doubly stochastic matrix. It is this which enables us to prove Theorem 3.2, the maximal value of τ_σ being obtained when the matrix is conjugate to a circulant matrix. Eigenvalues of various classes of stochastic matrices have been discussed by many authors (see for instance [11, 31, 60, 65, 124]). We give a result (Lemma 3.2) on the effect of permuting the columns of such a matrix, which seems to be new and may be of independent interest. A natural question is how, as σ varies, the second largest isolated eigenvalues of $\mathcal{L}_{\sigma\circ f}$ (and not just their moduli) are distributed in the complex plane. We propose a conjecture on their distribution, on the basis of some numerical investigations.

The results just described relate to the particular maps $f(x) = mx \bmod 1$, which are amenable to detailed combinatorial analysis. We also briefly discuss two further cases, which exhibit different types of behaviour. Firstly, we give an example to show that, for a non-uniformly expanding map f with intermittent behaviour, the composition with permutations may speed up the mixing rate. Secondly, we exhibit a Markov map f which is mixing, but where the proportion of permutations $\sigma \in S_N$ with $\sigma \circ f$ mixing does *not* tend to 1 as $N \to \infty$. These two examples indicate that one cannot expect general results along the lines of our Theorems 3.1 and 3.2 to hold for arbitrary interval maps f. Nevertheless, the results we have obtained suggest that the effect of composition with permutations is fundamentally different for discrete and continuous dynamical systems: it typically results in improved mixing in the continuous case, but frequently leads to worse mixing in the discrete case.

The organisation of this chapter is as follows. In §3.1, we give the necessary background and then state our main results. We also briefly discuss the location of the isolated eigenvalues in the complex plane. Since the proof of Theorem 3.1 is essentially combinatorial in character, it is given in Appendix B, along with some explicit formulae for the proportion of non-mixing permutations in special cases. Theorem 3.2 is proved in §3.3. Finally, the two additional examples mentioned above are presented in §3.2.

3.1 Background and statement of results

3.1.1 Mixing versus non-mixing

In this section we state our main result in relation to the question of mixing versus non-mixing of $\sigma \circ f$. Given a measure preserving system (f, M, μ), we recall that the system is (strongly) mixing if

$$|\mu(f^{-n}A \cap B) - \mu(A)\mu(B)| \to 0 \text{ as } n \to \infty, \tag{3.1}$$

where A, B are μ-measurable sets. We also recall (f, M) is topologically mixing if for all open $U, V \subset M$, there exists a constant $n_0 = n_0(U, V)$ such that $\forall\ n \geq n_0$, $f^n(U) \cap V \neq \emptyset$. To show that f is *not* mixing, it is usually easier to show that f is not topologically mixing.

For the examples that we consider, it will be also true that topologically mixing implies strongly mixing, see [118].

We will consider maps on the unit interval I, dividing I into N equal subintervals. To avoid the problem of functions being undefined, or multiply defined, at endpoints of these subintervals, we work with (non-compact) intervals which are closed on the left and open on the right. Thus we consider piecewise continuous maps $f\colon [0, 1) \longrightarrow [0, 1)$. We can of course regard f as a map on the compact interval $[0, 1]$ (by stipulating $f(1) = f(0)$) or on the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$.

We divide the unit interval as follows. Fix $N \geq 2$, and let $I_j = [j/N, (j+1)/N)$, $0 \leq j < N$. For any permutation σ of $\{0, 1, \ldots, N-1\}$ we write σ also for the corresponding interval exchange map:

$$\sigma(x) = x + (\sigma(j) - j)/N \text{ mod } 1 \text{ for } x \in I_j.$$

We write S_N for the group of all permutations of $\mathbb{Z}/N\mathbb{Z}$.

The specific map $f\colon [0, 1) \longrightarrow [0, 1)$ we consider is $f(x) = mx$ mod 1 for a fixed integer $m \geq 2$. Our first result shows that the composite $\sigma \circ f$ is mixing for *almost all* permutations σ when N is large enough.

Theorem 3.1. *Let f be as above. Then*

(i) if N is not a multiple of m then $\sigma \circ f$ is mixing for all $\sigma \in S_N$;

(ii) if $N > m$ and N is a multiple of m, say $N = m\ell$, then there will be some $\sigma \in S_N$ for which $\sigma \circ f$ is not mixing. As $\ell \to \infty$ (with m fixed), however, the proportion of permutations σ with $\sigma \circ f$ mixing tends to 1.

3.1.2 The main result on mixing rates

In our setting we consider specifically the map $f(x) = mx \mod 1$. When f is composed with a permutation $\sigma \in S_N$, we have seen in Theorem 3.1 that the resulting piecewise linear transformation $\sigma \circ f$ is usually (but not always) mixing. When $\sigma \circ f$ is mixing, we consider its mixing rate

$$\tau_\sigma := \sup \{|\lambda| : \lambda \in \operatorname{Spec}(\mathcal{L}_{\sigma\circ f}|_{\mathrm{BV}}) \backslash \{1\}\}\,. \tag{3.2}$$

We state the following result.

Theorem 3.2. *Fix m, $N \geq 2$ and consider the transformations $\sigma \circ f$ where $f(x) = mx \mod 1$ and $\sigma \in S_N$. Then the following hold.*

(i) For all $\sigma \in S_N$, the essential spectral radius is given by $r_{ess}(\mathcal{L}_{\sigma\circ f}|_{\mathrm{BV}}) = 1/m$.

(ii) If $N > m$ and $\gcd(m, N) = 1$, then, for each $\sigma \in S_N$, we have

$$\tau_\sigma \leq \tau_{\max} := \frac{\sin(m\pi/N)}{m\sin(\pi/N)}.$$

Moreover, each of the values $(-1)^{m-1}e^{2\pi ij/N}\tau_{\max}$ for $0 \leq j < N$ and $(-1)^m\tau_{\max}$ occurs as an isolated eigenvalue of $\mathcal{L}_{\sigma\circ f}$ for an appropriate choice of σ. Thus $\tau_\sigma = \tau_{\max}$ for these σ.

The proof of Theorem 3.2 is given in §3.3.

3.2 Further examples

We give two examples to demonstrate that the conclusions of Theorems 3.1 and 3.2 do not necessarily hold if we replace our standard map $f(x) = mx \mod 1$ by other interval maps. In §3.2, we give an example of a Markov map f where the proportion of permutations $\sigma \in S_N$ with $\sigma \circ f$ non-mixing is bounded away from 0 as $N \to \infty$. Thus the conclusion of Theorem 3.1 does not hold. In §3.2, we give a family of interval maps f for which composition with permutations typically improves the mixing rate, in contrast to Theorem 3.2(ii).

An example with many non-mixing permutations

Consider the piecewise continuous function $f : [0,1) \longrightarrow [0,1)$ given by

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x < \frac{1}{2}, \\ x - \frac{1}{2} & \text{if } \frac{1}{2} \leq x < 1. \end{cases} \tag{3.3}$$

Fix $\ell \geq 1$ and divide $[0,1)$ into $N = 2\ell$ equal subintervals

$$I_j = \left[\frac{j}{2\ell}, \frac{j+1}{2\ell}\right), \quad 0 \leq j \leq 2\ell - 1.$$

For a permutation $\sigma \in S_{2\ell}$ of these subintervals, let $g = \sigma \circ f$. We have the following result.

Proposition 3.1. *The proportion of permutations σ for which g is non-mixing is bounded away from 0 as $\ell \to \infty$.*

Proof. For any subset $A \subseteq \{0, \ldots, 2\ell - 1\}$, define $\tilde{g}(A) \subseteq \{0, \ldots, 2\ell - 1\}$ by

$$\tilde{g}(A) = \{\sigma(2j), \sigma(2j+1) \ : \ j \in A, \ j < \ell\} \cup \{\sigma(j - \ell) \ : j \in A, j \geq \ell\}.$$

Then, analogously to Proposition B.1, we have

$$g\left(\bigcup_{a \in A} I_a\right) = \bigcup_{b \in \tilde{g}(A)} I_b.$$

Note however that Proposition B.2 no longer holds: for example, if $A = \{0, \ell, \ell+1\}$ (with $\ell \geq 2$) then $\tilde{g}(A) = \{0, 1\}$ has fewer elements than A.

Now if there is some non-empty subset A such that $\tilde{g}^r(A) \neq \{0, \ldots, 2\ell - 1\}$ for all $r \geq 0$ then g is non-mixing. But if σ has the property that $\sigma(j - \ell) = j$ for some $j \geq \ell$ then, taking $A = \{j\}$, we have $\tilde{g}^r(A) = A$ for all r. Thus g is non-mixing. We therefore need to investigate the proportion of permutations with the above property.

Let $1 \leq m \leq \ell$ and let S be a subset of $\{\ell, \ldots, 2\ell - 1\}$ of size m. There are $(2\ell - m)!$ permutations $\sigma \in S_{2\ell}$ such that $\sigma(j - \ell) = j$ for all $j \in S$. Moreover, the number of such sets S of size m is $\binom{\ell}{m}$. Thus, by the Inclusion-Exclusion Principle, the proportion of permutations $\sigma \in S_{2\ell}$ with $\sigma(j - \ell) = j$ for at least one $j \geq \ell$ is

$$\sum_{m=1}^{\ell} (-1)^{m-1} a_m$$

where

$$a_m = \binom{\ell}{m} \frac{(2\ell - m)!}{(2\ell)!}.$$

Now the terms in the alternating series are decreasing: for $1 \leq m < \ell$ we have

$$\frac{a_{m+1}}{a_m} = \frac{m!(\ell-m)!(2\ell-m-1)!}{(m+1)!(\ell-m-1)!(2\ell-m)!} = \frac{\ell-m}{(m+1)(2\ell-m)} < \frac{1}{2(m+1)}.$$

Hence the required proportion is bounded below by

$$a_1 - a_2 = \frac{1}{2} - \frac{\ell(\ell-1)}{2(2\ell)(2\ell-1)} > \frac{3}{8}.$$

So we have proved that, for each $\ell \geq 1$, the function $\sigma \circ f$ is non-mixing for more than $3/8$ of the permutations $\sigma \in S_{2\ell}$. □

We remark that, although the map f in (3.3) is not expanding throughout its domain, its second iterate f^2 is piecewise expanding with expansion factor at least 2 everywhere, so our general discussion of mixing rates can still be applied.

An example where permutations speed up mixing

Consider the following family of *intermittency maps* $f_\alpha : [0,1] \to [0,1]$, $\alpha \in (0,1)$ given by

$$f_\alpha(x) = \begin{cases} x(1+2^\alpha x^\alpha) & \text{if } x \in [0,1/2], \\ 2x-1 & \text{if } x \in (1/2,1]. \end{cases} \tag{3.4}$$

This family has been widely studied [57, 79, 122] and optimal decay of correlations/speed of convergence to equilibrium has been established in [49]. In particular, it is shown that there exists a Banach space $\mathcal{B}$, (e.g., the space of Lipchitz continuous functions) such that

$$||\mathcal{L}_f^n(\phi) - \rho||_{\mathcal{B}} \leq C n^{-(1-\frac{1}{\alpha})} ||\varphi||_{\mathcal{B}}, \tag{3.5}$$

for all $\phi \in \mathcal{B}$ with $\|\phi\|_1 = 1$. Moreover this asymptotic in n is optimal within $\mathcal{B}$. The sub-exponential mixing rate arises since each f_α admits a neutral fixed point at $x = 0$, namely $f'(0) = 1$. Thus f is expanding but not uniformly expanding, and the existence of the neutral fixed point inhibits the mixing. In particular, the functional analytic methods discussed in introduction do not apply since $\lambda = 1$ is no longer an isolated eigenvalue of $\mathcal{L}_f$. i.e. there is no spectral gap.

We now consider f_α composed with a permutation $\sigma \in S_N$ such that $\sigma \circ f_\alpha$ is topologically mixing. Most choices of σ will not fix the interval $[0, 1/N]$, so that $\sigma \circ f_\alpha$ no longer has a neutral fixed point and is uniformly expanding on $[0,1]$. Thus $\sigma \circ f_\alpha$ has bounded variation on $[0,1]$, and it follows from [72, 118] that $\sigma \circ f$ has absolutely continuous invariant measure, with density in BV. Since the system is uniformly expanding, the operator $\mathcal{L}_{\sigma \circ f}$ now has a spectral gap, so that rate of convergence to equilibrium is exponentially fast.

3.3 Proof of the Theorem 3.2

Since the proof of Theorem 3.1 is purely combinatorics and not quite related with the spectral property of the transfer operator, we move these proofs into appendix.

In this section, we prove Theorem 3.2. The computation of the essential spectral radius r_{ess} for $\sigma \circ f$ is straightforward since $\sigma \circ f$ is piecewise linear with constant slope $1/m$. Hence Theorem 3.2(i) is a consequence of [72].

We now turn to Theorem 3.2(ii). This requires a detailed study of the eigenvalues of the Fredholm matrices $\Phi(z)$ associated to $\sigma \circ f$. We first give the required background on Fredhom matrices, see [89, 90].

Computation of Fredholm matrix eigenvalues

We now consider Fredholm matrices for our maps $\sigma \circ f$ with $f(x) = mx \bmod 1$ and $\sigma \in S_N$, where we assume that $N > m$ and $\gcd(m, N) = 1$. Recall that the definition and basic properties of the Fredholm matrices are stated in Proposition 1.3. These matrices are attached to a partition of $[0,1]$ on which $\sigma \circ f$ is Markov, so we first need to determine such a partition. For $k \geq 1$, consider the partition

$$\mathcal{P}_k := \{[(j/k, (j+1)/k) \ : \ 0 \leq j \leq k-1\}$$

of $[0,1)$ into k equal subintervals. Then the map f is Markov w.r.t. $\mathcal{P}_m$, while the map σ is Markov w.r.t. $\mathcal{P}_N$. The map $\sigma \circ f$, however, is in general not Markov w.r.t. either of these partitions. For example consider $m = 2, N = 3$. Clearly any $\sigma \in S_3$ is Markov on the partition

$$\mathcal{P}_3 = \{[0, 1/3), [1/3, 2/3), [2/3, 1)\}.$$

However, if we take the permutation σ interchanging the last two subintervals, then we have

$$\sigma \circ f(x) = \begin{cases} 2x & \text{if } 0 \leq x < 1/6, \\ 2x + 1/3 & \text{if } 1/6 \leq x < 1/3, \end{cases}$$

so that $\sigma \circ f$ is not continuous on $[0, 1/3]$ and hence not Markov on $\mathcal{P}$. In general, to ensure that $\sigma \circ f$ is Markov for all $\sigma \in S_N$, we must work with the partition $\mathcal{P}_{Nm}$.

Due to our specific choice $f(x) = mx \bmod 1$, the $Nm \times Nm$ Fredholm matrix $\Phi(1)$ is precisely the probability-transition matrix between the Markov states, and has all its entries in $\{0, 1/m\}$. (Recall that the definition and properties of the Fredholm matrix were stated in Section 1.3.3). If $\lambda \in \operatorname{Spec}(\mathcal{L}_f|_{\mathrm{BV}})$ then we know that $z = 1/\lambda$ is a solution to $D(z) = \det(I - \Phi(z)) = 0$. It is therefore an equivalent problem to consider the corresponding equation (in λ) to $\det(B - \lambda I)$=0, where B is the state transition matrix (with entries in $\{0, 1\}$). Hence if $\tilde{\lambda}$ is an eigenvalue of

B, then $\lambda = \tilde{\lambda}/m \in \mathrm{Spec}(\mathcal{L}_f|_{\mathrm{BV}})$.

Note that in our case, $m\Phi(1)$ is precisely the state transition matrix B. We will show that the eigenvalues of $\Phi(1)$ can in fact be determined from the $N \times N$ transition matrix associated with the partition $\mathcal{P}_N$. We remark that the image under $\sigma \circ f$ of a subinterval in $\mathcal{P}_N$ may have more than one connected component, so $\sigma \circ f$ is not necessarily Markov with respect to $\mathcal{P}_N$.

We must first define some notation. Following the conventions of Section 3.1.1, we index the subintervals in $\mathcal{P}_k$ by $\{0, 1, \ldots, k-1\}$. We therefore begin the numbering of the rows and columns in the associated matrices from 0. We define $A(m, N)$ and $B(m, N)$ to be the state transition matrices for f w.r.t. $\mathcal{P}_N$ and $\mathcal{P}_{Nm}$ respectively. Thus for $0 \le i, j \le N-1$ we have

$$A(m,N)_{ij} = \begin{cases} 1 & \text{if } j \equiv mi + d \bmod N \text{ with } 0 \le d \le m-1, \\ 0 & \text{otherwise,} \end{cases}$$

and for $0 \le i, j \le Nm - 1$ we have

$$B(m,N)_{ij} = \begin{cases} 1 & \text{if } j \equiv mi + d \bmod Nm \text{ with } 0 \le d \le m-1, \\ 0 & \text{otherwise.} \end{cases}$$

For example, when $m = 2$ and $N = 3$, we have

$$A(2,3) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad B(2,3) = \left(\begin{array}{cc|cc|cc} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right).$$

The eigenvalues for $A(2,3)$ are $\{\pm 1, 2\}$, while those for $B(2,3)$ are $\{\pm 1, 2, 0\}$, where the eigenspace for the eigenvalue 0 has dimension 3. In the case $m = 3, N = 5$ we have:

$$A(3,5) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix},$$

and the eigenvalues for $A(3,5)$ are $\{3, \pm i, \pm 1\}$. Note that all row sums and column sums in both $A(m, N)$ and $B(m, N)$ are m. Each row in either matrix consists of m consecutive occurrences of 1 (where, in the case of $A(m, N)$ these may "wrap around" from the last column to the first). The rows of $B(m, N)$ naturally fall into

m identical blocks each consisting of N rows, and the columns into N blocks each consisting of m identical columns, as indicated for $B(2,3)$ above.

The corresponding state transition matrices for $\sigma \circ f$ are obtained by permuting the columns of $A(m,N)$ and $B(m,N)$. More precisely, given a permutation σ of $\{0,\dots,N-1\}$, let $P(\sigma)$ be the $N \times N$ permutation matrix given by

$$P(\sigma)_{ij} = \begin{cases} 1 & \text{if } j = \sigma(i), \\ 0 & \text{otherwise,} \end{cases}$$

and let $Q(\sigma)$ be the $Nm \times Nm$ matrix obtained by replacing each entry 1 (respectively, 0) in $P(\sigma)$ by an $m \times m$ identity matrix (respectively, zero matrix). Then the state transition matrices for $\sigma \circ f$ w.r.t. the partitions $\mathcal{P}_N$ and $\mathcal{P}_{Nm}$ are $A(m,N)P(\sigma)$ and $B(m,N)Q(\sigma)$ respectively. For example, if $m = 2$, $N = 3$ and σ is the 3-cycle $(0,1,2)$ then

$$P(\sigma) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad Q(\sigma) = \left(\begin{array}{cc|cc|cc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right),$$

so that

$$A(2,3)P(\sigma) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad B(2,3)Q(\sigma) = \left(\begin{array}{cc|cc|cc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right).$$

Note also that $P(\sigma)A(m,N)$ is the matrix obtained by applying the *inverse* permutation σ^{-1} to the *rows* of $A(m,N)$.

To determine the mixing rate of $\sigma \circ f$, we need to investigate the eigenvalues of the Fredholm matrix $\Phi(1) = m^{-1}B(m,N)Q(\sigma)$ corresponding to the partition $\mathcal{P}_{Nm}$ on which $\sigma \circ f$ is Markov. Clearly λ is an eigenvalue of $\Phi(1)$ if and only if $m\lambda$ is an eigenvalue of $B(m,N)Q(\sigma)$, so it suffices to find the eigenvalues of the latter $Nm \times Nm$ matrix. In fact we only need consider $N \times N$ matrices.

Lemma 3.1. *For all m, N and all $\sigma \in S_N$, the nonzero eigenvalues of $B(m,N)Q(\sigma)$ are the same as those of $A(m,N)P(\sigma)$.*

Proof. For brevity, we write $A = A(m,N)$, $B = B(m,N)$, $P = P(\sigma)$ and

$Q = Q(\sigma)$.

We view BQ as determining a linear endomorphism θ on the space $V = \mathbb{C}^{Nm}$ of column vectors. Clearly BQ has rank N, since the first N rows are linearly independent and the remaining rows merely repeat these. The kernel W of θ therefore has dimension $N(m-1)$, and θ induces an endomorphism $\overline{\theta}$ on the quotient space V/W of dimension N. The eigenvalues of θ (that is, of BQ) are therefore the eigenvalues of $\overline{\theta}$, together with the eigenvalue 0 of multiplicity $N(m-1)$ coming from W. The result will therefore follow if we show that the matrix AP represents $\overline{\theta}$.

We define vectors $\mathbf{v}^{r,s}$ for $0 \le r \le N-1$, $0 \le s \le m-1$ (independent of σ) as follows. For $s = 0$, set

$$\mathbf{v}_i^{r,0} = \begin{cases} 1 & \text{if } i = mr, \\ 0 & \text{otherwise,} \end{cases}$$

and for $s > 0$,

$$\mathbf{v}_i^{r,s} = \begin{cases} -1 & \text{if } i = mr, \\ 1 & \text{if } i = mr + s, \\ 0 & \text{otherwise.} \end{cases}$$

For example, if $m = 2$ and $N = 3$ we have

$$\mathbf{v}^{0,0} = \begin{pmatrix} 1 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}^{0,1} = \begin{pmatrix} -1 \\ 1 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}^{1,0} = \begin{pmatrix} 0 \\ 0 \\ \hline 1 \\ 0 \\ \hline 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}^{1,1} = \begin{pmatrix} 0 \\ 0 \\ \hline -1 \\ 1 \\ \hline 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}^{2,0} = \begin{pmatrix} 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}^{2,1} = \begin{pmatrix} 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 1 \\ -1 \end{pmatrix},$$

where the horizontal lines correspond to the division of the columns of $B(2,3)Q(\sigma)$ into blocks.

It is clear that the $\mathbf{v}^{r,s}$ form a basis for V, and that if $s \ne 0$ then $BQ\mathbf{v}^{r,s} = 0$. Hence the $N(m-1)$ vectors $\mathbf{v}^{r,s}$ for $s \ne 0$ form a basis for W. Thus the N cosets $\mathbf{v}^{r,0} + W$ form a basis for V/W. If we partition BQ into $m \times m$ blocks (as in the above example), the matrix of $\overline{\theta}$ with respect to this basis is then obtained by replacing each block with the sum of one of its (identical) columns. This gives precisely the matrix AP. □

We next consider a matrix related to $A(m, N)$ but with eigenvalues that are easy to determine. By permuting the *rows* of $A(m, N)$, we can obtain a symmetric circulant matrix $C(m, N)$. Its explicit description depends on the parity of m. Let

$$\delta = \begin{cases} (1-m)/2 & \text{if } m \text{ is odd;} \\ (1-m+N)/2 & \text{if } m \text{ is even.} \end{cases}$$

Then $\delta \in \mathbb{Z}$ in both cases since $\gcd(m, N) = 1$, and $C(m, N)$ has entries

$$C(m,N)_{ij} = \begin{cases} 1 & \text{if } j \equiv i + \delta + r \bmod N \text{ with } 0 \le r \le m-1, \\ 0 & \text{otherwise} \end{cases} \tag{3.6}$$

for $0 \le i, j \le N - 1$. Observe that $C(m, N)$ is indeed symmetric since

$$j \equiv i + \delta + r \bmod N \Leftrightarrow i \equiv j + \delta + (m - 1 - r) \bmod N.$$

For example,

$$C(2,5) = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}; \qquad C(3,5) = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Since $C(m, N)$ is a real symmetric matrix, its eigenvalues are real. Since $C(m, N)$ is a circulant matrix, we can write these eigenvalues down explicitly. Let $\omega_j = e^{2\pi i j/N}$ for $0 \le j < N$, and let

$$\mathbf{v}_j = \left(1, \omega_j, \omega_j^2, \ldots, \omega_j^{N-1}\right)^T. \tag{3.7}$$

Then $\mathbf{v}_j$ is an eigenvector for $C(m, N)$ with eigenvalue

$$\lambda_j = \sum_{r=0}^{m-1} \omega_j^{\delta + r}.$$

Although the λ_j are not necessarily distinct, the N eigenvectors $\mathbf{v}_j$ are linearly independent since

$$\det(\omega_j^k)_{0 \le j,k < N} = \prod_{j<k} (\omega_k - \omega_j) \neq 0,$$

so there are no further eigenvalues. Trivially $\lambda_0 = m$. For $j \neq 0$, we have

$$\lambda_j = \frac{\omega_j^\delta(\omega_j^m - 1)}{\omega_j - 1}.$$

Writing $\zeta_j = e^{\pi i j/N}$, so that $\omega_j = \zeta_j^2$, we then have

$$\lambda_j = \zeta_j^{2\delta + m - 1} \left(\frac{\zeta_j^m - \zeta_j^{-m}}{\zeta_j - \zeta_j^{-1}} \right) = (-1)^{(m-1)j} \frac{\sin(mj\pi/N)}{\sin(j\pi/N)}, \tag{3.8}$$

since $2\delta + m - 1 = 0$ (resp. N) if m is odd (resp. even). In particular,

$$\det(C) = \prod_{j=0}^{N-1} \lambda_j = \pm m \prod_{j=1}^{N-1} \frac{\sin(mj\pi/N)}{\sin(j\pi/N)} = \pm m, \tag{3.9}$$

using the fact that the residues mj mod N are just the residues j mod N in some order because $\gcd(m, N) = 1$. It follows easily from (3.8) that, for $j \neq 0$, we have

$$\lambda_{N-j} = (-1)^{(N-1)(m-1)} \lambda_j = \lambda_j,$$

where the second equality holds since N and m cannot both be even.

We also mention some variants of $C(m, N)$. Firstly, cyclically permuting the rows of $C(m, N)$ gives circulant matrices $C^{(h)}(m, N)$ for which the $\mathbf{v}_j$ are eigenvectors with eigenvalues $\omega_j^h \lambda_j$. Secondly, we may permute the rows of $C(m, N)$ to obtain the anticirculant matrix $C'(m, N)$ with the same first row as $C(m, N)$; for example

$$C'(2,5) = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}; \qquad C'(3,5) = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Explicitly, the entries of $C'(m, N)$ are

$$C'(m, N)_{ij} = \begin{cases} 1 & \text{if } j \equiv -i + \delta + r \bmod N \text{ with } 0 \leq r \leq m-1, \\ 0 & \text{otherwise} \end{cases}$$

for $0 \leq i, j \leq N - 1$. The eigenvalues of $C'(m, N)$ are real since any anticirculant matrix is symmetric. Using [25, Theorem 2], we can write down these eigenvalues explicitly: they are m, $\lambda_{N/2}$ (for N even), and both values of $\pm\sqrt{\lambda_j \lambda_{N-j}} = \pm\lambda_j$ for $j \neq 0$, $N/2$.

The maximum value of $|\lambda_j|$ for $1 \leq j \leq N-1$ is attained at $j = 1$. Although this is essentially elementary, it is trickier to verify than it might appear, so we include a proof.

Proposition 3.2.

$$\max_{1 \leq j \leq N-1} \left| \frac{\sin(mj\pi/N)}{\sin(j\pi/N)} \right| = \frac{\sin(m\pi/N)}{\sin(\pi/N)}.$$

Proof. Since $|\sin(\pi k \pm x)| = |\sin x|$ for all $k \in \mathbb{Z}$, we may assume that $1 \leq m \leq N/2$, and moreover it suffices to take $1 \leq j \leq N/2$. We consider the two functions $u(x) = \sin mx / \sin x$ and $v(x) = 1/\sin x$ on the interval $(0, \pi)$. Now $u(x)$ has precisely $m-1$ zeros on this interval, at $x = h\pi/m$ for $1 \leq h \leq m-1$. Since $u(x)$

may be written as a polynomial of degree $m-1$ in $\cos x$, and $\cos x$ is monotonically decreasing on this interval, it follows that $u(x)$ has precisely $m-2$ stationary points, one in each of the intervals $(h\pi/m, (h+1)\pi/m)$ for $1 \le h \le m-2$. In particular, as $\lim_{x\to 0} u(x) = m$, it follows that $u(x)$ is positive and decreasing on $(0, \pi/m)$, so that $u(\pi/N) > u(j\pi/N) \ge 0$ if $2 \le j \le N/m$. On the other hand, as $v(x)$ is positive and decreasing throughout $(0, \pi/2)$, we have for $N/m \le j \le N/2$ that $|u(j\pi/N)| \le v(j\pi/N) \le v(\pi/m) < v(\pi/2m)$. But $v(\pi/2m) = u(\pi/2m) \le u(\pi/N)$ as $m \le N/2$. Hence $|u(j\pi/N)| < u(\pi/N)$ for $2 \le j \le N/2$, as required. □

We now seek to relate the eigenvalues of the matrices $A(m,N)P(\sigma)$ to those of $C(m,N)$. After scaling by $1/m$, these matrices become doubly stochastic. Our next result gives some information on the behaviour of the eigenvalues of a column stochastic matrix under permutation of its columns (or, more generally, under right multiplication by an orthogonal, column stochastic matrix).

Recall that an $N \times N$ matrix is *row* (respectively, *column*) *stochastic* if its entries are non-negative real numbers and the sum of each row (respectively, column) is 1. It is *doubly stochastic* if it is both row and column stochastic. The product of two row (respectively, column, doubly) stochastic matrices is again row (respectively, column, doubly) stochastic. For $A(m,N)$ as above, the probability transition matrices $m^{-1}A(m,N)$ are doubly stochastic. Any permutation matrix $P(\sigma)$ is doubly stochastic and orthogonal.

We view our matrices as linear maps on the space $\mathbb{C}^N$ of column vectors, endowed with the usual complex inner product $(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^{N} x_j \overline{y}_j$ for $\mathbf{x} = (x_1, \ldots, x_N)^T$, $\mathbf{y} = (y_1, \ldots, y_N)^T$, and we write $||\mathbf{x}|| = \sqrt{(\mathbf{x}, \mathbf{x})}$ for $\mathbf{x} \in \mathbb{C}^N$. Any row stochastic matrix has the obvious eigenvector $\mathbf{e} = (1, \ldots, 1)^T$ with eigenvalue 1. It is well-known that any eigenvalue λ satisfies $|\lambda| \le 1$. If B is a column stochastic matrix then $\mathbf{e}$ is not necessarily an eigenvector for B, but if $(\mathbf{x}, \mathbf{e}) = 0$ then $(B\mathbf{x}, \mathbf{e}) = 0$, so that B preserves the subspace V_0 of vectors in $\mathbb{C}^N$ perpendicular to $\mathbf{e}$.

Lemma 3.2. *Let B be an $N \times N$ column stochastic matrix. Then the eigenvalues of B^TB on V_0 are real and non-negative. Let η be the largest of these, and let P be an $N \times N$ orthogonal, column stochastic matrix (e.g. a permutation matrix). Then every eigenvalue λ of BP on V_0 satisfies*

$$|\lambda| \le \sqrt{\eta}.$$

Moreover, if B is a circulant matrix then

$$\sqrt{\eta} = \max\{\ |\lambda|\ :\ \lambda \textit{ is an eigenvalue of } B \textit{ on } V_0\}.$$

Proof. Since B^TB is a real symmetric matrix, its eigenvalues are real. Moreover, for any $\mathbf{x} \in \mathbb{C}^N$, we have $(B^TB\mathbf{x}, \mathbf{x}) = (B\mathbf{x}, B\mathbf{x}) \ge 0$, so these eigenvalues are nonnegative. We have

$$\eta = \max\{\ (B^T B\mathbf{x}, \mathbf{x})\ :\ \mathbf{x} \in V_0,\ ||\mathbf{x}|| = 1\} = \max\{\ (B\mathbf{x}, B\mathbf{x})\ :\ \mathbf{x} \in V_0,\ ||\mathbf{x}|| = 1\}. \tag{3.10}$$

Now let $\mathbf{y} \in V_0$ be an eigenvector of BP, corresponding to the eigenvalue λ, and normalised so that $||\mathbf{y}|| = 1$. Then

$$|\lambda|^2 = (\lambda\mathbf{y}, \lambda\mathbf{y}) = (BP\mathbf{y}, BP\mathbf{y}) = (B\mathbf{z}, B\mathbf{z}),$$

where $\mathbf{z} = P\mathbf{y}$. But $\mathbf{z} \in V_0$ since P is column stochastic, and $||\mathbf{z}|| = 1$ since P is orthogonal, so that $|\lambda|^2 \leq \eta$ as claimed.

Now suppose that B is also a circulant matrix. Let $\mathbf{y}_j = N^{-1/2}\mathbf{v}_j$, where the $\mathbf{v}_j$ are defined in (3.7). Then the $\mathbf{y}_j$ for $1 \leq j \leq N-1$ form an orthonormal basis of eigenvectors for B on V_0. Let λ_j be the eigenvalue for $\mathbf{y}_j$, and let k be an index such that $|\lambda_k| = \max_{1\leq j\leq N-1} |\lambda_j|$. For any $\mathbf{x} \in V_0$ with $||\mathbf{x}|| = 1$, we may write $\mathbf{x} = \sum_{j=1}^{N-1} c_j\mathbf{y}_j$ with $\sum_{j=1}^{N-1} |c_j|^2 = 1$. Then

$$(B\mathbf{x}, B\mathbf{x}) = \sum_{j=1}^{N-1} |c_j|^2|\lambda_j|^2 \leq |\lambda_k|^2 = (B\mathbf{y}_k, B\mathbf{y}_k),$$

so the maximum in (3.10) is attained at $\mathbf{x} = \mathbf{y}_k$, giving $\eta = |\lambda_k|^2$. □

Proof of Theorem 3.2(ii). For a given $\sigma \in S_N$, we are interested in the eigenvalues of the matrix $\Phi(1) = m^{-1}B(m,N)Q(\sigma)$, since these are the eigenvalues of $\Phi(1)$ (where $\Phi(z)$ is the Fredholm matrix of $\sigma \circ f$) and therefore the isolated eigenvalues in $\mathrm{Spec}(\mathcal{L}_f|_{\mathrm{BV}})$. By Lemma 3.1, it suffices to consider the eigenvalues of $m^{-1}A(m,N)P(\sigma)$.

The matrix $C(m,N)$ was obtained from $A(m,N)$ by applying some permutation ρ to the *rows*. Thus $P(\rho^{-1})A(m,N) = C(m,N)$. For any $\sigma \in S_N$, the matrix $m^{-1}A(m,N)P(\sigma) = m^{-1}P(\rho)C(m,N)P(\sigma)$ is conjugate to $m^{-1}C(m,N)P(\sigma\rho^{-1})$, therefore, it suffices to consider the eigenvalues of the doubly stochastic matrices $m^{-1}C(m,N)P(\sigma)$ for all $\sigma \in S_N$. We exclude the eigenvalue 1 associated to the trivial eigenvector $\mathbf{e}$, so consider only the eigenvalues on its orthogonal complement V_0.

We apply Lemma 3.2 to the doubly stochastic circulant matrix $B = m^{-1}C(m,N)$, so that $\eta = m^{-1}|\lambda_1|$ by Proposition 3.2. This shows that, for any $\sigma \in S_N$, each eigenvalue λ of $m^{-1}C(m,N)P(\sigma)$ satisfies $|\lambda| \leq m^{-1}\lambda_1$. Thus, in the notation of Theorem 3.2, we have shown that $\tau_\sigma \leq \tau_{\max}$. Moreover, $\tau_{\max} = m^{-1}|\lambda_1| = (-1)^{m-1}m^{-1}\lambda_1$.

Finally, we must show that each of the values $(-1)^{m-1}e^{2\pi ij/N}\tau_{\max}$ and $(-1)^m\tau_{\max}$ occurs as an eigenvalue of $m^{-1}A(m,N)P(\sigma)$ for some σ. But each of $m^{-1}C^{(j)}(m,N)$ (for $0 \leq j \leq N-1$) and $m^{-1}C'(m,N)$ is conjugate to one of these matrices, since

$C^{(j)}(m,N)$ and $C'(m,N)$ can be obtained by permuting the rows of $A(m,N)$. In particular, we have matrices whose eigenvalues include $m^{-1}\omega_1^j\lambda_1$ and $\pm m^{-1}\lambda_1$, as claimed. □

We finish this section by noting a further consequence of our discussion of circulant matrices.

Proposition 3.3. *For any $\sigma \in S_N$, the matrix $A(m,N)P(\sigma)$ has eigenvalue m with (algebraic) multiplicity 1. All its other eigenvalues are algebraic integers of norm ± 1. Composition with σ preserves the mixing rate of f (that is, $\tau_\sigma = 1/m$ in the notation of Section 3.1.2) if and only if these algebraic integers are roots of unity.*

Proof. Clearly the characteristic polynomial of $A(m,N)P(\sigma)$ has integer coefficients and has leading coefficient 1, i.e. its roots are algebraic integers. If λ is any one of these eigenvalues, then its conjugates are also eigenvalues, and its norm (i.e. the product of its conjugates) must be a rational integer. Now the product of the eigenvalues is $\pm\det(A(m,N)P(\sigma)) = \pm\det(C(m,N))\det(P(\rho\sigma)) = \pm m$ since any permutation matrix has determinant ± 1. We have the obvious eigenvalue m (with eigenvector $\mathbf{e}$), so m has multiplicity 1 as a root of the characteristic polynomial, and all the other roots must have norm ± 1.

Now if all the eigenvalues $\lambda \neq m$ of $A(m,N)P(\sigma)$ are roots of unity, we have $|\lambda| = 1$. Thus no element of $\mathrm{Spec}(\mathfrak{L}_f|_{\mathrm{BV}})$ has modulus between m^{-1} and 1, and $\sigma \circ f$ has the same mixing rate as f. Conversely, suppose that $\sigma \circ f$ and f have the same mixing rate. Then we must have $|\lambda| \leq 1$ for all eigenvalues $\lambda \neq m$ of $A(m,N)P(\sigma)$. But then all the conjugates λ' of λ are again eigenvalues, and hence satisfy $|\lambda'| \leq 1$. In fact each $|\lambda'| = 1$, since the product of the λ' is ± 1. Now any algebraic integer all of whose conjugates have modulus 1 must be a root of unity (see e.g. [39, IV, (4,5a)]). Hence all the eigenvalues $\lambda \neq m$ of $A(m,N)P(\sigma)$ are roots of unity. □

3.4 Discussion: mixing-rate distributions

Let us make some further observations, many of which are based on numerical computations, and the rigorous proofs are still open. For each $\sigma \in S_N$, we define two maps $U : S_N \to \mathbb{R}$ and $V : S_N \to \mathbb{C}$ as follows:

$$U : \sigma \in S_N \mapsto \tau_\sigma := \max\{\ |\lambda|\ :\ \lambda \in \mathrm{Spec}(\mathfrak{L}_{\sigma\circ f}|_{\mathrm{BV}}),\ |\lambda| \neq 1\}, \qquad (3.11)$$

and

$$V : \sigma \in S_N \mapsto \Lambda_\sigma := \{\lambda \in \mathrm{Spec}(\mathfrak{L}_{\sigma\circ f}|_{\mathrm{BV}}) \text{ and } |\lambda| = \tau_\sigma\}. \qquad (3.12)$$

The dynamics of U and V determine the distribution of the mixing rates for $\{\sigma \circ f\}_{\sigma\in S_N}$. From the proof of Theorem 3.2, we have τ_σ equals to modulus of $\lambda_2(\sigma)$

the second largest eigenvalue of $M(\sigma) := m^{-1}A(m,N)P(\sigma)$ (Here, "second largest" is in terms of the modulus, the largest eigenvalue always being 1, and a given matrix may have more than one second largest eigenvalue since they may be distinct eigenvalues with the same modulus.) In order to know more about the properties on the dynamics of U and V beyond Theorem 3.2, we pose the following conjectures from several different perspectives.

Convex hull problem on the geometric location

We would like to understand the geometric properties of the (finite) set $\bigcup_{\sigma\in S_N}\Lambda_\sigma$ in the complex plane. The elements of this set are the isolated eigenvalues of $\mathcal{L}_{\sigma\circ f}$ which determine the mixing rates of the maps $\sigma\circ f$ for all permutations $\sigma\in S_N$. For small values of m and N with $\gcd(m,N)=1$, these sets are shown in Figure 3.1. In particular, these eigenvalues are located between the inner circle with radius $1/m$ and the outer circle with radius $\tau_{\max}$. This is in agreement with Theorem 3.2. As a special case, when $m = N-1 = 4$ which follows $\sin(m\pi/N) = \sin(\pi/N)$, the two circles coincide.

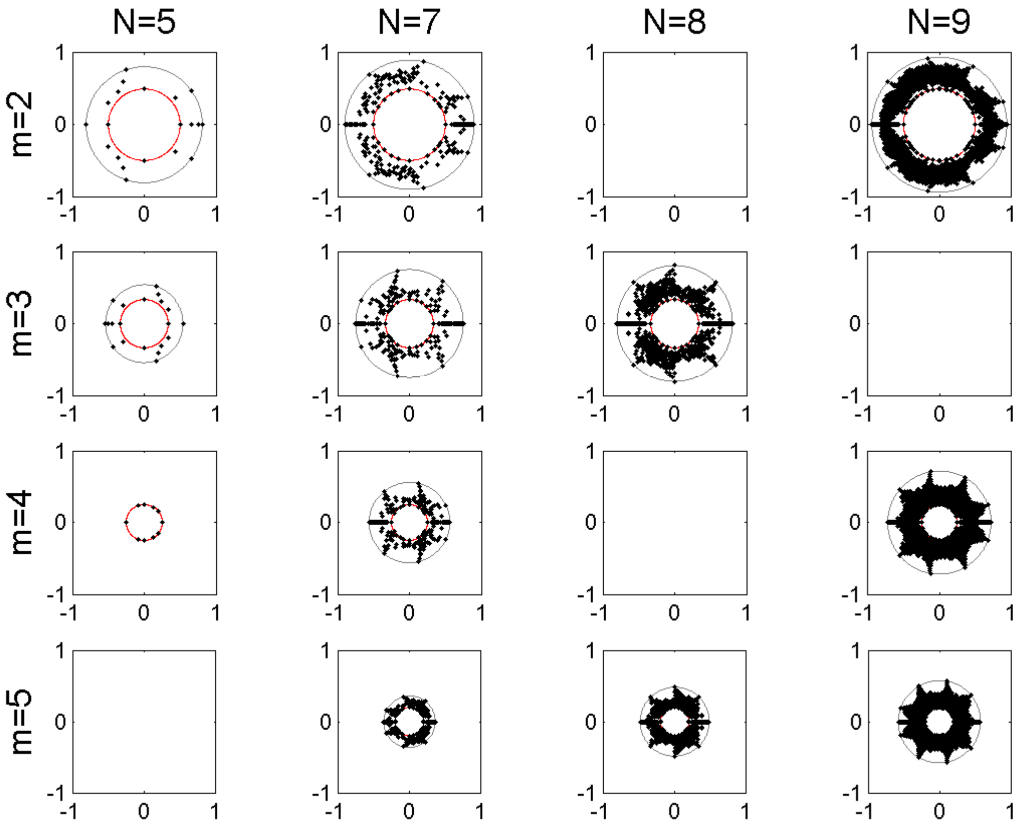

Figure 3.1: Distribution of mixing rates for the composition $\sigma\circ f$ where $f(x) := mx$ mod 1 and $\sigma\in S_N$ with $\gcd(N,m)=1$. These eigenvalues are located between the inner circle with radius $1/m$ and the outer circle with radius $\tau_{\max}$. We leave the graphs empty when $\gcd(m,N)>1$.

The set $\mathcal{D}_N$ of doubly stochastic matrices of order N has good convexity properties [111, Chapter I, §5]. By a well-known result of Birkhoff, $\mathcal{D}_N$ is precisely the convex hull of the permutation matrices of order N. Moreover, the eigenvalues of matrices in $\mathcal{D}_N$ lie in the convex hull of all roots of unity of order at most N.

Together with Figure 3.1, this suggests the following conjecture:

Conjecture 3.1. *Suppose that* $\gcd(m, N) = 1$. *Then* $\bigcup_{\sigma \in S_N} \Lambda_\sigma$ *is contained in the convex hull of the points* $(-1)^{m-1} e^{2\pi i j/N} \tau_{\max}$ *for* $0 \le j < N$ *and* $(-1)^m \tau_{\max}$. *In particular, these are the only points* $\lambda \in \bigcup_{\sigma \in S_N} \Lambda_\sigma$ *with* $|\lambda| = \tau_{\max}$.

We note that the convex hull in Conjecture 3.1 is a regular N-gon if N is even, and an irregular $(N+1)$-gon (obtained by adding one extra vertex to a regular N-gon) if N is odd; c.f. Figure 3.1.

It might be worth mentioning that the case "$m = 2$" seems to be relatively easy to begin with. Our heuristic reason is that there always exists a decomposition $M(\sigma) = \frac{1}{2}P_1 + \frac{1}{2}P_2$ where P_1, P_2 are permutation matrices, and particularly this decomposition is uniquely determined when $m = 2$. Moreover, whenever P_1 and P_2 commute, such $M(\sigma)$ possesses a second largest eigenvalue λ_2 with the maximum modulus $\cos(\pi/N)$. Hence, we suspect that Conjecture 3.1 in the case $m = 2$ might be related to the problem on controlling the second largest eigenvalue of the sum (or convex combination) of two non-commutative permutation matrices. In addition, we remark that the assumption on "gcd" is critical in Conjecture 3.1. Indeed, Conjecture 3.1 usually fails when $gcd(m, N) > 1$; see Figure 3.2 as a counterexample.

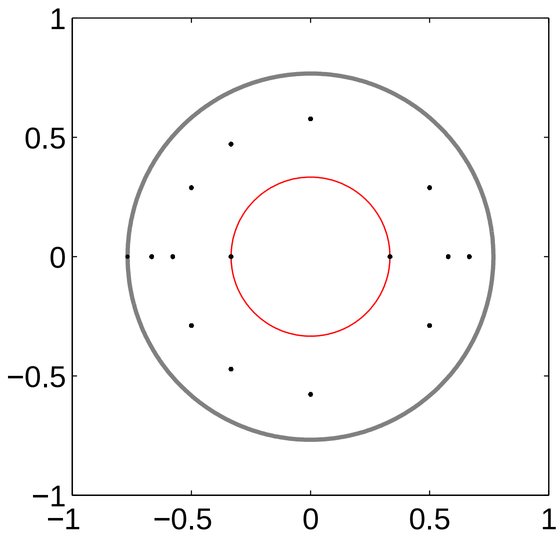

Figure 3.2: Distribution of mixing rates for the composition $\sigma \circ f$ where $f(x) := 3x \mod 1$ and $\sigma \in S_9$. The red inner circle is of radius $1/3$ and the out gray circle is of radius $\frac{\sin \pi/3}{3 \sin \pi/9}$.

Distribution on τ_σ

Apart from the discussions on Λ_σ in the complex plane, the analysis of the mixing rate τ_σ in the real line also appears interesting. For fixed sufficiently large N, we treat the map $V : \sigma \longrightarrow \tau_\sigma$ as a random variable (with respect to the *Haar measure* on the permutation group S_N with an equal probability $1/N!$ for each permutation) and consider its probability distribution.

The cardinality of S_N increases exponentially as N increases, therefore it would be very difficult to numerically enumerate all possible permutations for large N. Alternatively, we randomly (with equal probability) choose a proportion of permutations σ from the group S_N and investigate the distribution of the corresponding mixing rates. For example, Figure 3.3 shows the cumulative and density functions for fixed parameters $m = 3, N = 50$ from a sample size 10^6. Note that the value

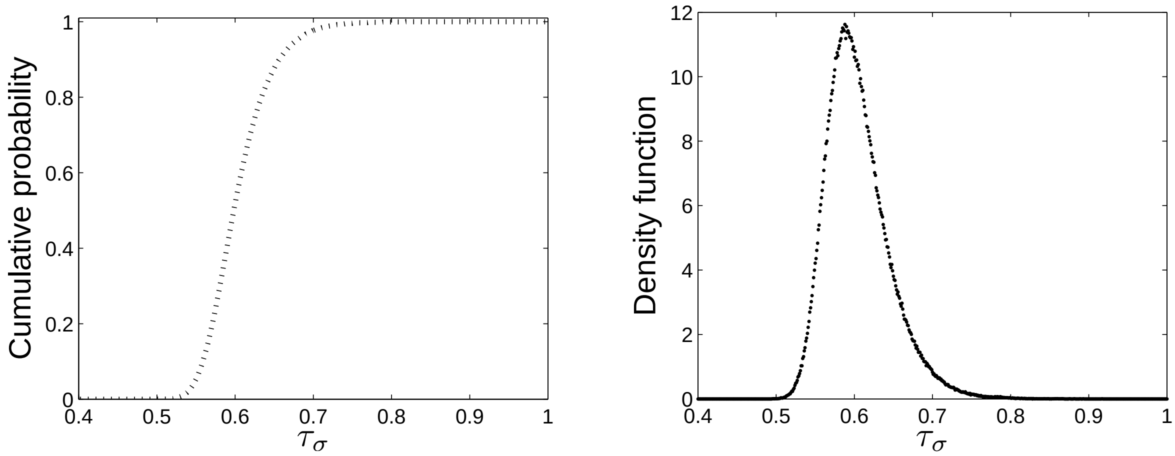

Figure 3.3: Left: Numerical accumulation function for fixed parameters $N = 50, m = 3$ from a sample size 10^6. Right: Density function for the corresponding cumulative function in the left panel with a bin size $h = 0.001$.

for the density function at each bin (with bin size $h = 0.001$) is multiplied by $1/h$ such that the curve is independent of bin size. In particularly, both cumulative and density functions are seemingly regular with high smoothness.

Based on this statistical analysis in Figure 3.3, it is natural to investigate the asymptotic behaviours on the average of mixing rate

$$\lim_{N\to\infty} \frac{1}{N!} \sum_{\sigma\in S_N} \tau_\sigma \tag{3.13}$$

which has a clear lower bound $1/m$ and upper bound 1.

From computational results for fixed m with randomly chosen σ, when increasing N, the average mixing rates seem to converge (i.e., satisfy the law of large numbers); see Figure 3.4 as an example. Thus, we ask:

Question 3.1. *1. Does the limit (3.13) exist? If not, are there any non-trivial*

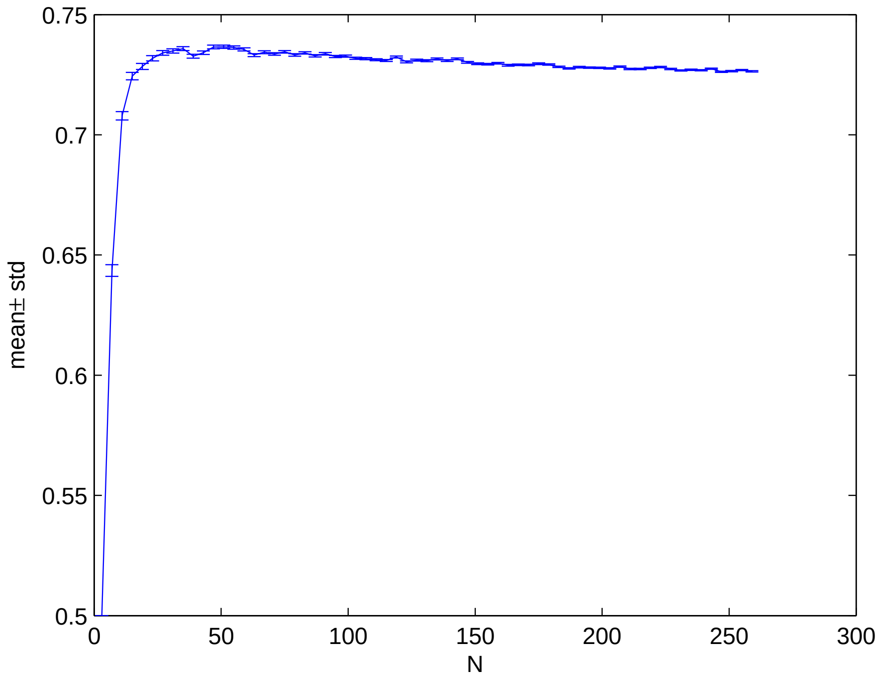

Figure 3.4: The average mixing rate changes with parameter N for fixed $m = 2$ (sample size of 1000 is used here in randomly chosen σ).

upper or lower bounds for $\limsup$ *or* $\liminf$*?*

2. *Is*

$$\lim_{N\to\infty} \frac{\sharp\left\{\sigma \in S_N : 1/m < \tau_\sigma < \frac{\sin(m\pi/N)}{m\sin(\pi/N)}\right\}}{N!} = 1?$$

3. *Is*

$$\text{closure}\left(\bigcup_{N\in\mathbb{N},\gcd(m,N)=1}\ \bigcup_{\sigma\in S_N} \tau_\sigma\right) = [1/m, 1]?$$

Best/worst-mixing permutations

Theorem 3.2 estimates the best/worse mixing rates, but which permutations possess the best/worst are still unclear. As an example, Figure 3.5 and 3.6 enumerate all the permutations for which the best/worst mixing rates are attained for the case of "$m = 2, N = 5$" respectively. We notice that any $\sigma \circ f$ has same topological entropy $\log m$. Therefore, the mixing rate seems an index which is a refinement of entropy. Hence, we are asking:

Question 3.2. 1. *What is the algebraic structure of the set of the best/worst-mixing permutations?*

2. *Does the fact that $\sigma_1 \circ f$ and $\sigma_2 \circ f$ have the same mixing rate imply the existence of a conjugacy of high smoothness?*

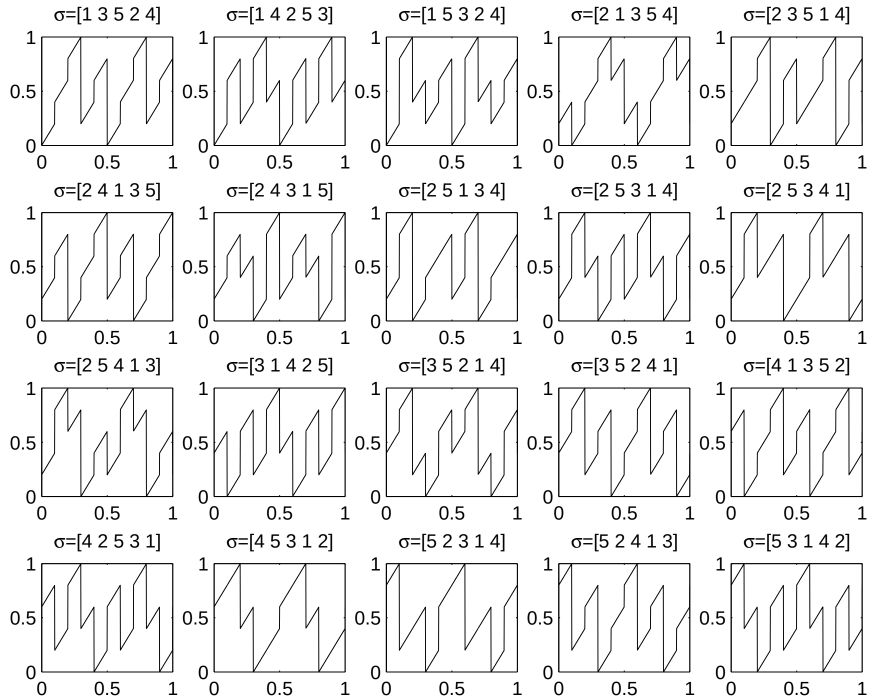

Figure 3.5: Graphs of $\sigma \circ f$ for which permutations have worst mixing rate are shown in each plot. The parameters $N = 5, m = 2$.

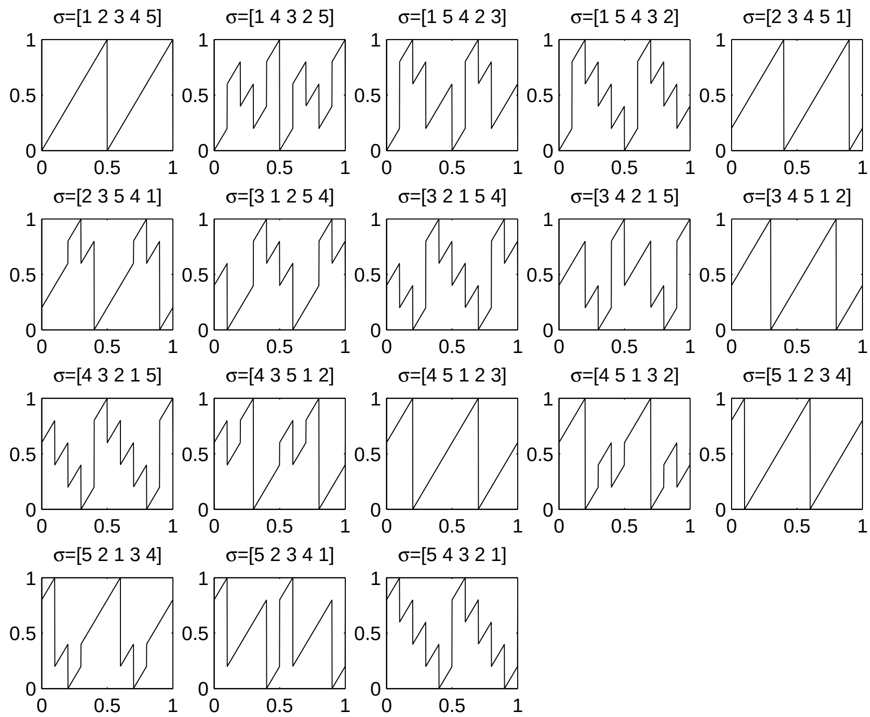

Figure 3.6: Graphs of $\sigma \circ f$ for which permutations have best mixing rate are shown in each plot. The parameters $N = 5, m = 2$.

Chapter 4

Dimension results of generalized Moran sets

This chapter considers geometrical constructions of fractals where different contractions are applied at each level of the construction. We allow the basic sets of the construction to have wild topological properties (such as fractal boundaries), and we permit arbitrary placement of the basic sets within the construction. Following [58], the limiting sets that arise from these geometrical constructions will be called *Moran sets* [88]. We define precisely how these sets are constructed in Section 4.1, and call these constructions *general Moran set constructions*. Dimension results for Moran set constructions are mainly considered in [58], especially for constructions generated by similarities. Certain one-dimensional, non-linear (cookie-cutter) Moran set constructions are considered in [82]. In the latter case, dimension estimates are computed via proving the existence of a *Gibbs-like* measure supported on the Moran set.

A comprehensive paper by [97] produces techniques for computing fractal dimensions for quite general self-similar, non-linear constructions where the basic sets have wild topologies. They show that if the geometric construction was bounded in a suitable way then dimension estimates could be obtained. These bounds involved upper estimates on the contraction rates of the construction, and bounds on the geometry of the *Moran coverings*; see Section 4.1. In particular, they use the thermodynamic formalism approach via solving the Bowen's equation (as stated in Section 1.3.4) to compute bounds on the fractal dimension.

The aim of this chapter is to provide a unified approach to study the dimension properties of general Moran sets arising out of non-linear constructions, thus extending the approach in [97] to such constructions. We do this by taking successive approximations of self-similar constructions which limit upon the general (non-self-similar) Moran set construction. This allows us to obtain dimension estimates for sets that are more general than those considered in [58, 82], for example in the former where the constructions are linear, and in the latter where the basic sets have controlled topology (e.g., the basic sets are intervals). In particular, the novelty of our approach is that we weaken the classical conformal condition into the lower and upper estimating vector conditions; see Section 4.1. For illustration, we show how to recover the same estimates in these references when using the thermodynamic formalism approach. We also consider situations where the non-linear contraction rates have an infimum of zero. This is considered in [58] for similarities and we show how the methods can be adapted to the nonlinear setting.

As a particular investigation in this chapter, we also consider Moran set constructions where the contraction rates at each step are generated by chaotic maps. More precisely, we consider the contraction rates at each step of the construction to be *stochastic*, where the stochasticity is generated by utilizing the ergodic properties of these maps. We consider various scenarios of applying stochasticity, namely constructions that are either *homogeneous* or *non-homogeneous*. For such constructions we consider conditions that allow the fractal dimensions to be computed, and consider questions concerning *typical* fractal dimensions. We see that the recurrence properties of the chaotic maps play a role in determining the typical values of the dimensions. To our knowledge this investigation is new and brings forward ergodic theory techniques to understand dimension results for stochastic Moran sets.

This chapter is organized as follows. In Section 4.1, we explain how to encode the general Moran sets using symbolic constructions. To calculate explicit bounds on the fractal dimensions, we give the main geometric assumptions on the basic sets. In Section 4.2, the main theorems are presented, while in Section 4.3 the main applications are presented, and these include dimension results for generalized iterated function systems, and dimension results for stochastically generated Moran sets. The formal proofs are presented in Section 4.4. The results presented in this chapter come from my collaboration with Holland [56].

4.1 Geometric and symbolic constructions

We define the following symbolic space. For a sequence of positive integers $\{n_k\}_{k\geq 1}$ and any $k \in \mathbb{N}$, let

$$D_k = \{(i_1, i_2, \cdots, i_k); \quad 1 \leq i_j \leq n_j, 1 \leq j \leq k\} \quad \text{with} \quad D_0 = \emptyset, \tag{4.1}$$

and define

$$D = \bigcup_{k=0}^{\infty} D_k. \tag{4.2}$$

The level set D_k contains all words of length k. The collection D is a countable collection of level sets. The set

$$D^* = \{(i_1, i_2, \cdots, i_k, \cdots) : 1 \leq i_j \leq n_j, j \geq 1\}$$

is the (uncountable) set of infinite strings.

Given words $\omega, \omega' \in D$ we define $\omega * \omega'$ as the concatenation of the two words. We remark that such concatenation is not always well defined. Additionally, for a infinite string $\underline{\omega} \in D^*$, we denote $\underline{\omega}|_k := \omega^{(k)} = \omega \in D_k$ as a truncated word of length k.

Let us begin with several definitions as a preparation.

Definition 4.1. *Given a map $f : \mathbb{R}^d \to \mathbb{R}^d$, we define the class $\Im$ such that if $f \in \Im$ then*

(H1): There exists a compact forward invariant set A, such that $f(A) \subset A$;

(H2): For any compact set $B \subset A$, $\operatorname{diam}(f^n(B)) \to 0$ as $n \to \infty$.

We remark that for any $f \in \Im$, then $\bigcap_{n=1}^{\infty} f^n(A)$ is a singleton. If $f_1, f_2 \in \Im$ share the same forward invariant set A, then both $f_2 \circ f_1$ and $f_1 \circ f_2 \in \Im$. We say that a map f is *contracting* if there exists a $0 < c < 1$ such that for all $x, y \in \mathbb{R}^d$, $d(f(x), f(y)) \le c \cdot d(x, y)$. If f is contracting then $f \in \Im$, but the converse need not be true.

Definition 4.2. *A family of compact sets are called* basic sets $\Omega = \{\triangle_\omega \subset \mathbb{R}^d,\ \omega \in D\}$*, if this family of sets satisfies:* $\lim_{k\to\infty} \max_{\omega \in D_k} \operatorname{diam}(\triangle_\omega) = 0$.

Based on D and the class $\Im$ of maps, we now consider the following *generalized Moran structure conditions* (GMSC) for the basic sets $\Omega = \{\triangle_\omega,\ \omega \in D\}$.

Definition 4.3. *Given basic sets $\Omega = \{\triangle_\omega,\ \omega \in D\}$ and a family of maps $\{f_{j,i} \in \Im : i \le n_j, j \ge 1\}$, they satisfy GMSC (with respect to D) if the following hold.*

(A1) Suppose $k \ge 1$, $\omega \in D_{k-1}$ and $\omega * j \in D_k$ (for $1 \le j \le n_k$). Then elements of $\triangle_{\omega * j}$ are completely determined by elements of $\triangle_\omega$ and the vector of maps $\Xi_k = (f_{k,1}, f_{k,2}, \dots, f_{k,n_k}) \in \Im$, i.e., $\triangle_{\omega*j} \subseteq \triangle_\omega$ and $\triangle_{\omega*j} = f_{k,j}(\triangle_\omega)$;

(A2) The strong separation condition holds: given any k and $\omega, \omega' \in D_k$ with $\omega \ne \omega'$ then

$$\triangle_\omega \cap \triangle_{\omega'} = \emptyset.$$

We denote $f_\omega = f_{1,i_1} \circ \cdots \circ f_{k,i_k}$ for a word $\omega = (i_1, \cdots, i_k) \in D_k$. By taking the composition of maps in condition ($A1$), it follows that $f_\omega(\triangle) = \triangle_\omega$.

For a given Ω, we define

$$F = \bigcap_{k\in\mathbb{N}} \bigcup_{\omega \in D_k} \triangle_\omega. \tag{4.3}$$

The set F is a compact set, and by the strong separation condition (A2) is totally disconnected. So far we have made no assumptions on the topology of the basic sets $\{\triangle_\omega, \omega \in D\}$, other than these sets being compact. In particular they need not to be connected, and their boundaries could be fractal. It is sufficient for our purposes to work with a weaker version of (A2), and we say that the *weak separation condition holds* if

(A2') For any $\omega, \omega' \in D$ with $\omega \neq \omega'$:

$$\{\triangle_\omega \cap \triangle_{\omega'}\} \cap F = \emptyset.$$

Definition 4.4. *Given F as in equation (4.3), we call F a generalized Moran set (GMS) if F satisfies* (A1), (A2').

See Fig 4.1 for the geometrical interpretation.

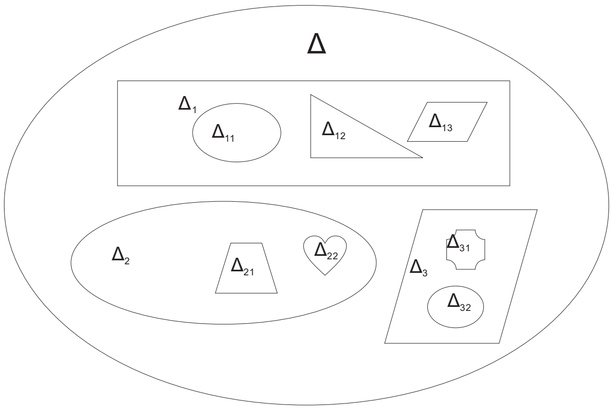

Figure 4.1: Schematic representation of the geometric construction of a general Moran set F.

Given a GMS $F \subset \mathbb{R}^d$, there is a canonical projection map $\mathcal{X} : D^* \to F$ which assigns to each $\underline{\omega} = (i_n)_{n=1}^{\infty}$ the point $x \in F$ given by $\bigcap_k \triangle_{\omega^{(k)}}$, where $\omega^{(k)} = \underline{\omega}|_k = (i_1, \cdots, i_k) \in D_k$ is an infinite string $\underline{\omega}$ truncated to the first k places. Due to the weak separation condition $(A2')$, this canonical projection map χ is a bijection except for a countable set.

We can turn D^* into a metric space by assigning the distance function $d(\underline{\omega}, \underline{\omega'})$ to points $\underline{\omega}$, $\underline{\omega'} \in D^*$ as follows:

$$n(\underline{\omega}, \underline{\omega'}) := \min\{i | \omega_j = \omega'_j \text{ for } 0 < j < i \text{ but } \omega_i \neq \omega'_i\}, \text{ if } \omega \neq \omega',$$

and $n(\underline{\omega}, \underline{\omega}) := \infty$. For a given $0 < p_i < n_i^{-1}$ we set $d(\underline{\omega}, \underline{\omega'}) = \prod_{i=1}^{n(\underline{\omega}, \underline{\omega'})} p_i$. Then (D^*, d) is a compact metric space. It is easy to see that D^* is a generalization of traditional symbolic space, since if $n_k = p$ is a constant, then $D^* = \Sigma_p^+$, where $\Sigma_p^+ = \{1, \ldots, p\}^{\mathbb{N}}$. However, if n_k is not a constant, then the shift map σ on D^* does not necessarily preserve D^*. Alternatively, we consider a sequence of symbolic spaces that can be thought of as approximations to D^*. These symbol spaces are generated from the sets D_k.

Definition 4.5. *Given* $D = \bigcup_k D_k$*, the symbol space* $[D_k]$ *is defined as the set of infinite strings, with indices corresponding to elements of* D_k*. That is*

$$[D_k] = \{\underline{\mathbf{w}} = (\omega_i)_{i=1}^{\infty} = (\omega_1, \omega_2 \ \ldots,), \, \omega_i \in D_k\}.$$

The associated shift map $\sigma_k : [D_k] \to [D_k]$ *is defined by*

$$\sigma_k(\omega_1, \omega_2 \ \ldots,) = (\omega_2 \, \omega_3, \ldots,), \quad \text{with } (\omega_i)_{i=1}^{\infty} \in [D_k].$$

In addition, we denote $[D_k]_n := \{\underline{\mathbf{w}}|_n : \underline{\mathbf{w}} \in [D_k]\}$ as the level set of length n in $[D_k]$. Notice that $[D_k]$ is isomorphic to $\Sigma_{p_k}^+$ with $p_k = \text{card}(D_k)$.

Conformal properties and upper/lower estimating vectors

To obtain explicit estimates on the Hausdorff dimension of F, some restrictions on the basic sets $\{\triangle_\omega\}$ are required. In particular we require control on the diameter of $\triangle_\omega$ with respect to the level set D_k that ω belongs to. In particular we require that their diameters shrink exponentially fast with k. We also require control of the geometry of $\triangle_\omega$ via a technical condition restricting the number of $\triangle_\omega$ (of a certain size-scale) that can intersect with a given ball $B(x, r) \in \mathbb{R}^d$ where $x \in F$. For self similar constructions, control on the geometry is specified in [97] by use of lower, and upper estimating vectors. We adapt these methods for the non-self similar constructions. Let $\overline{\Psi} = \{\overline{\Psi}^{(k)}\}$ denote a countable collection of vectors $\overline{\Psi}^{(k)}$ with

$$\overline{\Psi}^{(k)} = (\Psi^{(k)}(\omega))_{\omega \in D_k}.$$

Here $\omega = (i_1, \ldots, i_k) \in D_k$. Given $\omega \in D_k$, we assume that there is a sequence of constants $c_{i_1}^{(1)}, \ldots c_{i_k}^{(k)}$ such that

$$\Psi^{(k)}(\omega) = c_{i_1}^{(1)} c_{i_2}^{(2)} \ldots c_{i_k}^{(k)} = \prod_{j=1}^{k} c_{i_j}^{(j)}.$$

For notational simplicity we sometimes write $\Psi_\omega^{(k)} := \Psi^{(k)}(\omega)$.

Definition 4.6 (Basic vectors)**.** *The collection of vectors* $\overline{\Psi} = \{\overline{\Psi}^{(k)}\}_{k=1}^{\infty}$ *is called a* basic collection *of vectors if for all* $k \geq 1$ *and all* $\omega = (i_1, \ldots, i_k) \in D_k$*, the sequence* $\Psi^{(k)}(\omega)$ *satisfies*

$$\sup_{k \in \mathbb{N}, 1 \leq j \leq n_k} c_j^{(k)} < 1. \tag{4.4}$$

Definition 4.7 (UE vectors)**.** *A basic collection of vectors* $\overline{\Psi} = \{\overline{\Psi}^{(k)}\}_{k=1}^{\infty}$ *is called an* upper estimating *(UE) vector if for any* k *and* $\omega \in D_k$*:*

$$\text{diam}(\triangle_\omega) \leq C \Psi_\omega^{(k)},$$

and the constant $C > 0$ is independent of ω and k.

To get bounds on the Hausdorff dimension we require further control of the geometry of each Δ_ω. We introduce two definitions: the first is that of conformality, while the second introduces the notion of lower-estimating vectors for a geometric construction.

Definition 4.8 (Conformal vectors)**.** *Given a basic collection of vectors $\overline{\Psi} = \{\overline{\Psi}^{(k)}\}_{k=1}^{\infty}$, we say that a symbolic construction $\{\triangle_\omega\}$ is* conformal *(w.r.t. $\overline{\Psi}$) if $\exists C > 0$ such that for each $k \geq 1$, $\omega \in D_k$, $\exists x \in \Delta_\omega$:*

$$B\left(x, \frac{1}{C}\Psi_\omega^{(k)}\right) \subset \triangle_\omega \subset B\left(x, C\Psi_\omega^{(k)}\right). \tag{4.5}$$

An alternative geometric constraint is considered in [97] and we formulate this condition for the GMSC as follows. Given $0 < r < 1$ fixed, let $\overline{\Psi}$ be a basic collection of vectors. For any $\underline{\omega} \in D^*$, let $n := n(\underline{\omega})$ be the unique positive integer such that $\Psi_\omega^n > r$ and $\Psi_\omega^{n+1} < r$, where $\omega = \underline{\omega}|_n$. Fix this $\underline{\omega}$, and consider the *cylinder set* $C_{\underline{\omega}|_n} := \{\underline{\omega'} \in D^*, \underline{\omega'}|_n = \underline{\omega}|_n\} \in D^*$. We have the fact that if $\underline{\omega'} \in C_\omega|_n$ and $n(\underline{\omega'}) \geq n(\underline{\omega})$ then $C_{\underline{\omega'}|_{n(\underline{\omega'})}} \subseteq C_{\underline{\omega}|_{n(\underline{\omega})}}$.

Let $C(\underline{\omega})$ be the largest cylinder set containing $\underline{\omega}$ which satisfies $C(\underline{\omega}) := C_{\underline{\omega''}|_n}$ for some $\underline{\omega''} \in C(\underline{\omega})$ and $C_{\underline{\omega'}|_n} \subseteq C(\underline{\omega})$ for any $\underline{\omega'} \in C(\underline{\omega})$. Therefore, the sets $C(\underline{\omega})$ according to different infinite strings $\underline{\omega} \in D^*$ either coincide or are totally disjoint. We collect these sets as $C_r^j, j = 1, \cdots, N_r$ and there sets form a cover of D^*. Moreover, there always exist $\underline{\omega}^{(j)}$ such that $C_r^{(j)} = C_{\underline{\omega}^{(j)}|_{n(\underline{\omega}^{(j)})}}$. By projecting $\{\triangle_r^{(j)}\} := \{\chi(C_r^{(j)})\}$, it also forms a cover of the GMS F. We call $\{\triangle_r^{(j)}\}$ the *Moran cover*.

Consider the open ball $B(x, r)$ of the radius r centered at the point $x \in F$, and let $N(x, r)$ denote the number of the Moran cover $\{\triangle_r^{(j)}\}$ that have non-empty intersection with $B(x, r)$. We define the lower estimating vector below:

Definition 4.9 (LE vectors)**.** *If there exists a constant M such that the above $N(x, r) < M$ for all $x \in F$, then we say $\overline{\Psi}$ is a* lower estimating *(LE) vector.*

Thermodynamic formalism

Consider a sequence of pressure functions $\{P_k\}_k$ and a sequence of potentials $\{\Phi_{k,s}\}_k$ defined as follows. Recall that $\overline{\Psi} = \{\overline{\Psi}^{(k)}\}_k$ with $\overline{\Psi}^{(k)} = \{\Psi_\omega^{(k)}\}$, $\omega \in D_k$. Consider the symbolic space $[D_k]$ together with the shift map $\sigma_k : [D_k] \to [D_k]$. For any $\Phi \in \text{Lip}([D_k])$ we can study the associated pressure function $P_k(\Phi)$ as stated in equation (1.45) of Chapter 1. For $\underline{\mathbf{w}} \in [D_k]$ we consider the specific function $\Phi_{k,s}(\underline{\mathbf{w}}) := s \log \Psi^{(k)}(\mathbf{w}^{(1)})$, where $\mathbf{w}^{(1)}$ is the first symbol of $\underline{\mathbf{w}}$ in $[D_k]$. This function can be extended to a function on F_k via $\Phi_{k,s}(x) = s \log \Psi^{(k)}(\mathbf{w}^{(1)})$, where $\mathcal{X}(\underline{\mathbf{w}}) = x$. Thus

$P_k : \text{Lip}(F_k) \to \mathbb{R}$ is given by

$$P_k(\Phi_{k,s}) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\mathbf{w}^{(n)} \in [D_k]_n} \inf_{x \in \Delta_{\mathbf{w}^{(n)}}} \exp\left\{ S_n(\Phi_{k,s}(\mathbf{w}^{(n)})) \right\} \right), \tag{4.6}$$

where the sum is taken over all n-th words $\mathbf{w}^{(n)}$ in $[D_k]_n$. (Recall $[D_k]_n$ is the level set of length k in $[D_k]$).

To study the fractal dimension of F, we consider the sequence s_k, where s_k is the value of s which solves $P_k(\Phi_{k,s}) = 0$. In particular we consider the (lim)-inf and (lim)-sup of this sequence. We define:

$$s^* := \limsup s_k, \quad \text{and} \quad s_* := \liminf s_k. \tag{4.7}$$

The main focus of this paper is to consider when s^* is the upper-box dimension of F, and when s_* is the Hausdorff dimension of F. We need a further hypothesis on the existence of a *Gibbs-like* measure on F.

(A3) Given $\beta > 0$, there exists a measure m_Ψ supported on F, and $L > 0$ such that for all $k \geq 1, \omega \in D_k$,

$$\frac{L^{-1}}{\sum_{\omega' \in D_k} (\Psi^{(k)}(\omega'))^\beta} \leq \frac{m_\Psi(\triangle_\omega)}{(\Psi^{(k)}(\omega))^\beta} \leq \frac{L}{\sum_{\omega' \in D_k} (\Psi^{(k)}(\omega'))^\beta} \tag{4.8}$$

For the range of applications that we consider, we verify directly the hypothesis (A3). For the similarity transformations considered in [58], m_Ψ is obtained by taking a weak limit of a sequence measures $\{m_k\}$. Each m_k is defined on each $\omega \in D_\ell, \ell \leq k$ by

$$m_k(\triangle_\omega) = \sum_{i_{\ell+1},\ldots,i_k} \frac{(c_{1,i_1} c_{2,i_2} \ldots c_{k,i_k})^\beta}{\prod_{j=1}^k \sum_{i=1}^{n_k} c_{j,i}^\beta}, \quad \omega = (i_1, \ldots i_\ell).$$

By direct calculation this reduces to $m_\ell(\omega)$. For iterated function systems defined by expanding maps, we can show (A3) holds by using bounded distortion estimates, see Section 4.3.1.

4.2 Statement of main results

The main theorem is the following:

Theorem 4.1. *Consider a GMSC admitting a GMS F. Suppose $\overline{\Psi} = \{\overline{\Psi}^{(k)}\}$ is a (UE) and (LE) vector, and suppose that there exists a Gibbs-like measure m_Ψ satisfying (A3). Assume further that*

$$d := \inf_{k \in \mathbb{N}, 1 \leq j \leq n_{k+1}} c_j^{(k+1)} > 0. \tag{4.9}$$

Then

1. $\dim_H F = \dim_H m_\Psi = s_*$.
2. $\dim_P F, \overline{\dim}_B F \leq s^*$.

Furthermore if F also satisfies the conformality condition, as in Definition 4.8 then

$$\dim_P F = \overline{\dim}_B F = s^*.$$

In general if

$$\inf_{k\in\mathbb{N},\omega\in D_k} c_j^{(k+1)} = 0, \tag{4.10}$$

then the conclusions of Theorem 4.1 will fail; see [37, 58] for the examples regarding the failure of Theorem 4.1 under the condition 4.9. To consider more general situations where (4.10) holds we have to impose conditions on how fast the $\Psi_\omega^{(k)}$ decay. For fixed k, we denote

$$M_k := \max_{\omega\in D_k} \Psi_\omega^{(k)},\ d_k := \min_{1\leq j\leq n_{k+1}} c_j^{(k+1)}. \tag{4.11}$$

Theorem 4.2. *Consider a GMSC admitting a GMS F. Suppose $\overline{\Psi} = \{\overline{\Psi}^{(k)}\}$ is a (UE) and (LE) vector, and suppose that there exists a Gibbs-like measure m_Ψ satisfying (A3). Assume further that*

$$\lim_{k\to\infty} \frac{\log d_k}{\log M_k} = 0, \tag{4.12}$$

then $\dim_H F = \dim_H(m_\Psi) = s_*$ and $\dim_P F, \overline{\dim}_B F \leq s^*$, where d_k, M_k are defined in (4.11). Furthermore if F also satisfies the conformality condition, then

$$\dim_P F = \overline{\dim}_B F = s^*.$$

Remark 4.1. *Under the assumption (4.9), the Conformality vector implies the (LE) property; see the proof of Lemma 4.7. However, such a relationship might fail in general.*

4.3 Applications

We consider applications of the theorems to a range of examples. We first consider iterated function systems, and then explore stochastic Moran set constructions.

4.3.1 Iterated function systems

Connection with the bounded distortion

In this section we consider iterated function systems defined by a collection of expanding maps.

Suppose that we are given basic sets $\Omega = \{\triangle_w \in \mathbb{R}^d : \omega \in D\}$ satisfying GMSC with respect to D as stated in Definition 4.3. Moreover, we further suppose that $T_{j,i_j} : \triangle_{\omega^{(j+1)}} \to \triangle_{\omega^{(j)}}, \forall i_j = 1, \cdots, n_k$ satisfies the following three assumptions:

$C^{1+\alpha}$ ***diffeomorphism:*** The derivative DT_{j,i_j} from the tangent bundle is an $\alpha-$Hölder continuous function, i.e., there exists a constant $C := C_{j,i_j}$ such that $||DT_{j,i_j}(x) - DT_{j,i_j}(y)|| \leq C||x-y||^\alpha$;

(distance) expanding: there exists a $\beta := \beta_{j,i_j} > 1$ such that $||T_{j,i_j}(x) - T_{j,i_j}(y)|| \geq \beta||x-y||, \ \forall x, y \in \triangle_{\omega^{(j+1)}}$;

full branch: $T_{j,i_j}(\triangle_{\omega^{(j+1)}}) = \triangle_{\omega^{(j)}}$.

We can therefore take $\Xi_k = (f_{k,1}, \ldots, f_{k,n_k})$ to be the vector of contractions associated to the inverse branches of $(T_{k,1}, \ldots, T_{k,n_k})$ at $k-$th level. For $\omega = (i_1, \ldots, i_k) \in D_k$ we have $\triangle_{\omega^{(k)}} = f_{\omega^{(k)}}(\triangle)$ where $f_{\omega^{(k)}} = f_{1,i_1} \circ \cdots, \circ f_{k,i_k}$. Using the theory developed in section 4.1, the following corollary holds:

Corollary 4.1. *Consider a GMSC with $\{\Xi_k\}_{k=1}^\infty$ a collection of vectors associated to the inverse branches, and admitting a GMS F. We further assume that the $C^{1+\alpha}$ (distance)-expansivity of $\{\Xi_k\}_{k=1}^\infty$ is* uniformly bounded, *(i.e., the sequence $\{\beta_{j,i_j}\}$ is uniformly bounded away from 1, the sequence $\{\det(Df_{j,i_j})\}$ is uniformly bounded away from zero, and the sequence of Hölder constants $\{C_{j,i_j}\}$ is uniformly bounded).*

Then there exists a conformal vector $\overline{\Psi}$ satisfying the (A3) condition. Moreover,

$$\dim_H F = s_*, \quad \dim_P F = \overline{\dim}_B F = s^*,$$

where s_* and s^* are defined in equation (4.7).

Proof. We first verify the *bounded distortion result*: there exists $D > 0$, independent of k such that for all $x, y \in \triangle$:

$$\frac{1}{D} \leq \frac{|\det(Df_{\omega^{(k)}}(x))|}{|\det(Df_{\omega^{(k)}}(y))|} \leq D. \tag{4.13}$$

The proof of the distortion result is based on the chain rule, for the same iterated function system at each level; see [35, 62, 106]. More precisely, we have:

$$\begin{aligned}
& \left|\log|\det Df_{\omega^{(k)}}(x)| - \log|\det Df_{\omega^{(k)}}(y)|\right| \\
= & \sum_{j=1}^{k} \left|\log|\det Df_{j,i_j}(f_{\omega^{(j)}}(x))| - \log|\det Df_{j,i_j}(f_{\omega^{(j)}}(y))|\right| \\
\leq & \sum_{j=1}^{k} C_1 \left|\det D_{j,i_j}(f_{\omega^{(j)}}(x)) - \det D_{j,i_j}(f_{\omega^{(j)}}(y))\right| \\
\leq & \sum_{j=1}^{k} C_2 ||Df_{j,i_j}(f_{\omega^{(j)}})(x) - Df_{j,i_j}(f_{\omega^{(j)}})(y)|| \\
\leq & \sum_{j=1}^{k} C_3 ||f_{\omega^{(j)}}(x) - f_{\omega^{(j)}}(y)||^{\alpha} \\
\leq & C_3 \sum_{j=1}^{k} \beta^{-j\alpha} ||x-y||^{\alpha} \leq \frac{C_3 \beta^{-\alpha}}{1-\beta^{-\alpha}} ||x-y||^{\alpha}.
\end{aligned}$$

Due to the uniform boundness assumption, these constants $C_i, i = 1, 2, 3$ and β are independent of the choice of k, which implies (4.13).

From this bounded distortion property (4.13), we can directly construct a collection of vectors $\overline{\Psi}$ and verify the conformality and (A3). More precisely, for any fixed $x \in \triangle$, let $\Psi_{\omega}^{(k)} = |\det Df_{\omega^{(k)}}(x)|, \forall \omega = \omega^{(k)} \in D_k$. Then, for all $y \neq x \in \triangle$, we have

$$\frac{1}{D} \leq \frac{|\det Df_{\omega^{(k)}}(y)|}{\Psi_{\omega}^{(k)}} \leq D.$$

Therefore, the pressure function (4.6) is equal at two different initial points $x \neq y$, which implies that $\Psi_{\omega^{(k)}}$ is well defined. In addition, due to the uniform boundedness assumption again, the conformality and equation (4.9) are easy to be verified, while the verification of $(A3)$ is similar to [82, Prop 2.7]. Therefore, the results are directly followed by Theorem 4.1. □

Corollary 4.1 extends the results of [82] to higher dimensions as well as to complex conformal holomorphic expanding case in the Riemann sphere $\overline{\mathbb{C}}$ (by letting $\Psi_{\omega}^{(k)} := |\operatorname{Arg}((f_{\omega})')|$), and to scenarios where the basic sets have complicated topology (e.g., the basic sets themselves are fractal sets). A special case where Corollary 4.1 applies is the sequence of affine contractions on compact intervals: $\Xi_k = (c_{k,1}, \ldots, c_{k,n_k})$, where $c_{k,j}$ is the derivative of the contraction $f_{k,j}$ along the j-th branch. We show directly that this result is consistent with the linear theory considered in [58]. In this case, we have the conformal vector $\overline{\Psi} = \{\overline{\Psi}^{(k)}\}$, where $\overline{\Psi}^{(k)} = (\Psi_{\omega}^{(k)})_{\omega \in D_k}$. When $\omega = (i_1, \ldots, i_k) \in D_k$ then $\Psi_{\omega}^{(k)} = \prod_{j=1}^{k} c_{j,n_j}$. From equation (4.6) we have

$$P_k(\Phi_{k,s}) = \lim_{n\to\infty} \frac{1}{n} \log\left(\sum_{\mathbf{w}^{(n)} \in [D_k]_n} \inf_{x \in \Delta_{\mathbf{w}^{(n)}}} \exp\left\{S_n(\Phi_{k,s}(\mathbf{w}^{(n)}))\right\} \right), \tag{4.14}$$

where the sum is taken over all n-th words $\mathbf{w}^{(n)}$. We take piecewise constant potential $\Phi_{k,s}(x) = s_k \log \Psi^{(k)}(\mathbf{w}^{(1)}), \forall x \in \triangle_{\mathbf{w}^{(n)}}$. This reduces to

$$P_k(\Phi_{k,s}(x)) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\mathbf{w}^{(n)} \in [D_k]_n} \inf_{x \in \triangle_{\mathbf{w}^{(n)}}} \exp\left(\sum_{j=0}^{n-1} s_k \log([\sigma_k^j \underline{\mathbf{w}}(x)]^{(1)}) \right), \tag{4.15}$$

and hence we obtain

$$\begin{aligned}
P_k(\Phi_{k,s}(x)) &= \lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{\mathbf{w}^{(n)}\in[D_k]_n}\exp\left(\sum_{j=0}^{n-1}\log\Psi_j^{s_k}\right)\right)\\
&= \lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{\mathbf{w}^{(n)}\in[D_k]_n}\prod_{j=0}^{n-1}\Psi_j^{s_k}\right)\\
&= \log\left(\sum_{\omega\in D_k}(\Psi_\omega^{(k)})^{s_k}\right),
\end{aligned}$$

where $\mathbf{w}^{(n)} = (\omega_1,\ldots,\omega_n)$ and the last step is from the binomial theorem. Thus $P(\Phi_{k,s}(x)) = 0$ is equivalent to $\sum_{\omega\in D_k}(\Psi_\omega^{(k)})^{s_k} = 1$, that is $\sum_{i=1}^k(\Pi_{j=1}^{n_k}c_{i,j})^{s_k} = 1$ as in [58]. Moreover $\liminf_{k\to\infty} s_k = s_* = \dim_H(F)$ and $\limsup_{k\to\infty} s_k = s^* = \dim_P(F) = \overline{\dim}_B(F)$.

□

4.3.2 The homogeneous and stochastic Moran set constructions

We begin with a definition.

Definition 4.10 (Homogeneous GMS)**.** *Consider a GMSC admitting a GMS F. Suppose $\overline{\Psi}$ is a (UE),(LE) vector and satisfy the* (A3) *condition. We further assume that the (UE) and (LE) vector $\overline{\Psi}$ has the property that $\Psi_\omega^k = \Psi_{\omega'}^{(k)}$ for all $\omega,\omega'\in D_k$, then we call F a* homogeneous GMS.

The homogeneous construction is perhaps the simplest example of a GMSC. A natural exploration is to consider ways of generating the GMS F via stochastic sequences of contractions. For example, we consider the vector $\overline{\Psi}$ generated stochastically via chaotic maps in the following sense: Let (T, M, μ) be a measure preserving system, where $T : M \to M$ is a map preserving an ergodic measure μ. Given a test function (observable) $\phi : M \to [0, 1]$ and initial condition $x \in M$, we let $\Psi_\omega^{(k)} = \prod_{j=1}^k \phi(T^j(x))$ for any $\omega \in D_k$. We assume that $n_k = q$ is fixed, and the conditions of Definition 4.3 apply. In this case the vector Ξ_k consists of q components each with value $\phi(T^k(x))$. Thus the GMS F (and hence its dimension) depends on the initial value $x \in M$. In this section we primarily investigate the Hausdorff dimension of F, and it's dependency on x. The results are obtained by using methods in ergodic theory.

Theorem 4.3. *Suppose that (T, M, μ) is an ergodic system, and suppose that $\phi : M \to [0, 1/q]$ is such that $\log\phi \in L^1(\mu)$ with $\int \log\phi\, d\mu > 0$. Suppose further that F is the homogeneous GMS arising from a GMSC with a (UE) and (LE) vector*

$\Psi_\omega^{(k)} = \prod_{j=1}^k \phi(T^j(x))$. *Then for* μ*-a.e.* $x \in M$

$$\dim_H(F) = \dim_P(F) = \overline{\dim_B}(F) = \frac{-\log q}{\int \log \phi d\mu}.$$

Proof. We consider the first case where $\inf_{x\in M} \phi(x) > 0$. From the Main Theorem 4.1, we have that

$$\dim_H(F) = s_*,$$

where

$$s_* = \liminf_{k\to\infty} \left(\frac{\log\left(\prod_{j=1}^k \phi \circ T^j(x)\right)}{-k \log q} \right)^{-1}.$$

By the Birkhoff Ergodic Theorem we have for μ-a.e. $x \in M$:

$$\lim_{k\to\infty} \frac{1}{k} \log\left(\prod_{j=1}^k \phi \circ T^j(x)\right) = \lim_{k\to\infty} \frac{1}{k} \sum_{j=1}^k \log \phi \circ T^j(x) = \int \log \phi d\mu.$$

Now consider the case where $\inf_{x\in M} \phi(x) = 0$, which implies that $\inf_{k\in\mathbb{N}, 1\le j\le n_{k+1}} c_j^{(k+1)} = 0$. Therefore, we need to show that equation (4.12) applies for μ-typical orbits. If so, then Theorem 4.2 will establish the corresponding result.

Proceeding, and using the notation of equation (4.12) we have that $d_k = \phi(T^k(x))$ and $M_k = \prod_{i=0}^k \phi(T^i(x))$. Hence,

$$\frac{\log d_k}{\log M_k} = \frac{\log \phi(T^k(x))}{\sum_{i=0}^k \log \phi(T^i(x))} = \frac{k^{-1} \log \phi(T^k(x))}{k^{-1} \sum_{i=0}^k \log \phi(T^i(x))}. \tag{4.16}$$

Again, by Birkhoff Ergodic Theorem, $k^{-1} \sum_{i=0}^k \log \phi(T^i(x)) \to \int \log \phi \, d\mu \neq 0$. We show $k^{-1} \log \phi(T^k(x)) \to 0$, (again for μ-a.e. $x \in M$) by the following lemma:

Lemma 4.1. *Suppose* $a_k \to \infty$ *is monotone with the property that* $\lim_{k\to\infty} a_k/a_{k+1} = 1$. *Suppose* b_k *is a series such that* $\frac{1}{a_k} \sum_{j=1}^k b_j$ *converges to* $L \neq 0$.

Then $b_k / \sum_{j=1}^k b_k \to 0$.

Before proving Lemma 4.1 we show how the Theorem follows. In the notation of equation (4.16), let $a_n = n$, and $b_n = \log(\phi(T^n(x))$. The ergodic theorem implies that $\frac{1}{a_k} \sum_{j=1}^k b_j$ converges for μ-a.e. $x \in M$, and moreover $\lim_{k\to\infty} a_k/a_{k+1} = 1$. Hence the conditions of Lemma 4.1 are satisfied. It follows that $\log d_k / \log M_k \to 0$ for μ-a.e. $x \in M$, and thus the conditions of Theorem 4.2 are satisfied.

Now to prove the lemma, let $S_k := \frac{1}{a_k} \sum_{j=1}^k b_k$. We have the following relation:

$$S_{k+1} - \frac{a_k}{a_{k+1}} S_k = \frac{b_{k+1}}{a_{k+1}}.$$

If we take limits of both sides then we obtain $\lim_{k\to\infty} b_k/a_k = 0$ as required. □

4.3.3 Stochastic Moran constructions using multiple generating maps

Consider a family of maps $\{(T_i, M, \mu_i)\}_{i=1}^t$ with $T_i : M \to M$ (M compact), and each T_i preserves an ergodic measure $\mu_i \in L^p$ for some $p > 1$ (i.e., $\mu \ll m$ and the density is L^p integrable). Given $\mathbf{x} \in M^t$, we can generate a GMS F via a GMSC in the following way. Take Hölder continuous functions $\phi_i : M \to [0, 1]$, and suppose that the basic vector $\Psi_\omega^{(k)}$ has the form:

$$\Psi_\omega^{(k)} = \prod_{j=1}^{k} \phi_{i_j}(T_{i_j}^j(x_{i_j})) : \ \omega = (i_1, \ldots, i_k), \ \mathbf{x} = (x_1, \ldots, x_t). \tag{4.17}$$

The following integral I_s and in particular how it varies with the parameter s are also of interest to us:

$$I_s := \int_{M^t} \log \left\{ \sum_{i=1}^{t} \phi_i(x_i)^s \right\} d\mu, \text{ where } \mu = \otimes_{i=1}^t \mu_i, \tag{4.18}$$

We assume the following condition on the zeros of I_s.

(A4) The integral I_s has a unique zero root.

Based on these conditions, we have the following result.

Theorem 4.4. *Suppose that $\{T_i, M, \mu_i\}_{i=1}^t$ form an ergodic system (i.e., each measure μ_i is ergodic w.r.t. T_i) and each $\phi_i : M \to [0, 1]$ is positive Hölder continuous with $\int -\log \phi_i \, d\mu_i < \infty$. Suppose that the GMS F results from a GMSC with a (UE) and (LE) vector $\overline{\Psi}$ generated via the vectors $\Xi_k = (\phi_1(T_1^k(x_1)), \ldots, \phi_t(T_t^k(x_t)))$ satisfying condition (A3). In addition, suppose further that condition (A4) holds.*

Then for μ-a.e. $\mathbf{x} \in M^t$,

$$\dim_H(F) = \dim_P(F) = \overline{\dim}_B(F) = s_*,$$

where s_* is the solution of the functional equation:

$$\int_{M^t} \log \left\{ \sum_{i=1}^{t} \phi_i(x_i)^{s_*} \right\} d\mu = 0, \ \textit{where } \mu = \otimes_i^t \mu_i \tag{4.19}$$

Proof. The proof is analogous to that of Theorem 4.3. We first consider case where $\inf_i \inf_{x_i} \phi_i(x_i) > 0$. Since the set F results from a GMSC, it is implicit that the ϕ_i are contractions. In particular the contractions satisfy:

$$\sum_{i=1}^{t} \phi_i(T_i^j(x_i)) < 1, \ (\forall\, j), \quad \text{and} \quad \sup_i \sup_{x_i} \phi_i(x_i) < 1/t. \tag{4.20}$$

Moreover, ergodicity of the product system implies that equation (4.20) holds for μ-a.e $\mathbf{x} \in M^t$. The vector of maps is given by $\Xi_k = (\phi_1(T_1^k(x_1)), \ldots, \phi_t(T_t^k(x_t))$. It suffices to compute the dimensions s_k and calculate the limit $\liminf s_k$. From the notation of section 4.1, we have:

$$\sum_{\omega \in D_k} \left(\prod_{j=1}^{k} \phi_{i_j}(T_{i_j}^j(x_{i_j})) \right)^{s_k} = 1. \tag{4.21}$$

A simple application of the binomial theorem implies that this expression is equivalent to:

$$\prod_{j=1}^{k} \left(\sum_{i=1}^{t} \{\phi_i(T_i^j(x_i)\}^{s_k} \right) = 1, \tag{4.22}$$

and so

$$\sum_{j=1}^{k} \log \left(\sum_{i=1}^{t} \{\phi_i(T_i^j(x_i)\}^{s_k} \right) = 0. \tag{4.23}$$

Now for *fixed* s, from the ergodic theorem, we have

$$\lim_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} \log \left(\sum_{i=1}^{t} \{\phi_i(T_i^j(x_i)\}^{s} \right) = \int_{M^t} \log \left\{ \sum_{i=1}^{t} \phi_i(x_i)^s \right\} d\mu. \tag{4.24}$$

Clearly, the value s_* which is the solution of (4.19) gives the right hand side of (4.24) as zero. We now justify that $s_* = \liminf s_k$ by showing that for large k, $s_k = s_* + o(1)$. For finite (but large) k, we have

$$\sum_{j=1}^{k} \log \left(\sum_{i=1}^{t} \{\phi_i(T_i^j(x_i)\}^{s} \right) = k \left(\int_{M^t} \log \left\{ \sum_{i=1}^{t} \phi_i(x_i)^s \right\} d\mu + o(1) \right). \tag{4.25}$$

Based on the condition $(A4)$ together with the continuity of each ϕ_i, it follows that: $\forall\, \epsilon > 0$, there exists a K such that $\forall k \geq K$, we can choose s_k with $|s_* - s_k| < \delta$ and s_k satisfying (4.23). Hence $s_* = \liminf s_k$.

Suppose now there exists an i such that $\inf_{x_i} \phi_i(x_i) = 0$, but $\int \phi_i \, d\mu_i \neq 0$. In this case, it implies that $\inf_{k\in\mathbb{N}, 1\leq j\leq n_{k+1}} c_j^{(k+1)} = \inf_{k\in\mathbb{N}, 1\leq j\leq n_{k+1}} \phi_i \circ T_i^k(x_i) = 0$. Therefore, we need to show that equation (4.12) applies for μ-typical orbits. If so, then Theorem 4.2 will establish the corresponding result. Proceeding, and using the notation of equation (4.12) we have that

$$\begin{aligned} d_k &= \min\nolimits_{1\leq i\leq t}\{\phi_i(T_i^k(x_i))\}, \\ M_k &= \max\nolimits_{\omega} \left\{ \prod_{j=1}^{k} \phi_{i_j}(T_{i_j}^j(x_{i_j})) \right\}, \quad \omega = (i_1, \ldots i_k). \end{aligned} \tag{4.26}$$

To show that equation (4.12) applies for μ-a.e. $\mathbf{x} \in M^t$, if letting $\alpha := \sup_i \sup_{x_i} \phi_i(x_i)$, we first notice that

$$M_k \leq \alpha^k \leq (1/t)^k,$$

and so it directly implies that

$$\frac{\log d_k}{\log M_k} \leq \frac{\max_{1 \leq i \leq t}\{-\log \phi_i(T_i^k(x_i))\}}{-k \log \alpha}. \tag{4.27}$$

We have to show that for μ-a.e. $\mathbf{x} \in M^t$, the right hand term of equation (4.27) goes to zero. We use a Borel-Cantelli argument as follows. For any fixed i, let

$$A_k^{(i)} := \{x_i \in M : \phi_i(T_i^k(x_i)) \leq \alpha^{\sqrt{k}}\}.$$

Due to the invariance of μ_i and the conditions on $\mu_i, \phi_i \in L^p$, we have:

$$\begin{aligned}\mu_i(A_k^{(i)}) &= \mu_i\{x_i \in M : \phi_i(x_i) \leq \alpha^{\sqrt{k}}\} \\ &\leq \text{Leb}\{x_i \in M : \phi_i(x_i) \leq \alpha^{\sqrt{k}}\}^{\frac{1-p}{p}} \leq \alpha^{\gamma\sqrt{k}},\end{aligned}$$

where γ is a constant independent of the choice of i. Therefore, $\sum_{k=1}^{\infty} \mu_k(A_k^{(i)}) < \infty$, by Borel-Cantelli arguments, $\mu_i(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k^{(i)}) = 0$ for any i. Hence $\mu - a.e.\mathbf{x} \in M^t$, we have $\max_{1 \leq i \leq t}\{-\log \phi_i(T_i^k(x_i)) \leq -\sqrt{k} \log \alpha$. So

$$(4.27) \leq \frac{1}{\sqrt{k}} \to 0, \quad \text{as } k \ \to 0,$$

which implies the corresponding theorem.

□

4.4 Proofs

Proof of Theorem 4.1

The proof of Theorem 4.1 is given in several steps. We first obtain an estimate on how the measure m_Ψ scales on balls of radius r, as $r \to 0$. A second step is to show $\dim_H(F) = s_*$ and the arguments used here require the existence of a (LE) and (UE) vector.

Lemma 4.2. *Suppose $\overline{\Psi} = \{\overline{\Psi}^{(k)}\}$ is a (LE) vector and satisfies (A3) condition. Then for any $x \in F$, and any open ball $B(x,r)$, $(0 < r < 1)$ and any $\epsilon > 0$, there exists a Gibbs measure m_Ψ such that*

$$m_\Psi(B(x,r)) \leq C r^{s_*+\epsilon}. \tag{4.28}$$

The constant $C > 0$ is independent of r.

Proof. For $\omega \in D_k$ and for any $\beta < s_* := \liminf s_k$, by property (A3), we have

$$m_\Psi(\triangle_\omega) \leq L_1(\Psi_\omega^{(k)})^\beta. \tag{4.29}$$

Consider $x \in F$ and the ball $B(x,r)$ with $r \in (0,1)$. Since $\overline{\Psi}$ is a (LE) vector, there exists an $M > 0$ such that the number $N(x,r)$ of $\triangle^{(j)}$ (in the Moran cover of F) with $\triangle^{(j)} \cap B(x,r) \neq \emptyset$ is bounded by M. Hence

$$m_\Psi(B(x,r)) \leq \sum_{j=1}^{N(x,r)} m_\Psi(\triangle^{(j)}) \leq \sum_{j=1}^{N(x,r)} L_1(\Psi_\omega^{(k)})^\beta \tag{4.30}$$

where ω corresponds to those for which $\triangle_\omega = \triangle^{(j)}$, and $\triangle^{(j)} \cap B(x,r) \neq \emptyset$. Since $d \leq c_j^{(k+1)}(\omega)$ by (4.9) and $N(x,r) \leq M$, therefore

$$m_\Psi(B(x,r)) \leq L_1 M(\Psi_\omega^{(k)})^\beta \leq \frac{L_1 M}{d^\beta}(\Psi_\omega^{(k+1)})^\beta \leq \frac{L_1 M}{d^\beta} r^\beta. \tag{4.31}$$

This proves equation (4.28).

□

Lemma 4.3. *Consider a GMSC admitting a GMS F. Suppose that condition (A3) holds, $\overline{\Psi} = \{\overline{\Psi}^{(k)}\}$ is a (UE) and (LE) vector and*

$$d = \inf_{k \in \mathbb{N}, 1 \leq j \leq n_{k+1}} c_j^{(k+1)} > 0. \tag{4.32}$$

Then

$$\dim_H F = \dim_H(m_\Psi) = s_*.$$

Proof. We first show that $\dim_H F \geq s_*$. Recall

$$\dim_H(m_\Psi) = \inf\{\dim_H(E) : \text{ with } m_\Psi(E) = 1\}.$$

Moreover, from the proof of Lemma 4.2, we have $m_\Psi(B(x,r)) \leq \frac{L_1 M}{d^\beta} r^\beta$, where $\beta < s_*$ is arbitrary. By the uniform mass distribution (namely Lemma 2.1 in Introduction), it follows that $s_* \leq \dim_H(m_\Psi) \leq \dim_H F$, since we can choose β arbitrarily close to s_*.

We now show that $\dim_H(F) \leq s_*$. Choose any $\beta > s_*$ then for Hausdorff measure $\mathcal{H}^\beta$ we have

$$\begin{aligned} \mathcal{H}^\beta(F) &\leq \liminf_{k\to\infty} \sum_{\omega \in D_k} \operatorname{diam}(\triangle_\omega)^\beta \leq \liminf_{k\to\infty} \sum_{\omega \in D_k} C(\Psi_\omega^{(k)})^\beta \\ &\leq \liminf_{k\to\infty} \sum_{\omega \in D_k} C(\Psi_\omega^{(k)})^{s_k} \leq CL_1 \left(\liminf_{k\to\infty} \sum_{\omega \in D_k} m_\Psi(\Delta_\omega) \right) < \infty, \end{aligned} \tag{4.33}$$

where in the second line we take the infimum along the subsequence s_k such that $s_k < \beta$ (which holds infinitely often). Thus $\mathcal{H}^\beta(F) \le C$ and so $\dim_H F \le \beta$. Since $\beta > s_*$ is arbitrary, it follows that $\dim_H(F) \le s_*$. This completes the proof.

□

Lemma 4.4. *Consider a GMSC admitting a GMS F. Suppose that $\overline{\Psi} = \{\overline{\Psi}^{(k)}\}$ is a (UE), (LE) vector and*

$$d = \inf_{k\in\mathbb{N}, 1\le j\le n_{k+1}} c_j^{(k+1)} > 0. \tag{4.34}$$

Then

$$\dim_P F \le \overline{\dim_B} F \le s^*.$$

Proof. We prove by contradiction. Suppose that $\overline{\dim_B}(F) > s^*$. Then by the definition $s^* = \limsup_{k\to\infty} s_k$, there exists a constant $\delta > 0$ sufficiently small, such that $\overline{\dim_B}(F) - 3\delta > s_k$ (eventually in k). By the definition of s_k and the pressure $P_k : t \mapsto P_k(t\log\phi^{(k)})$ is a decreasing analytic function for any $k > 0$, we have

$$P_k((\overline{\dim_B}(F) - 3\delta)\log\Phi^{(k)}) < 0, (\text{eventually in } k). \tag{4.35}$$

Denote $\overline{\dim_B} F = \beta$, then by the definition of upper box dimension we have

$$\limsup_{\epsilon\to 0} \frac{\log N_\epsilon(F)}{-\log\epsilon} = \beta.$$

Therefore, we have the following scaling rules for estimating the number of balls:

$$\forall\, \delta_0 > 0,\ \exists\ \epsilon > 0:\ N_\epsilon(F) \ge \epsilon^{\delta_0-\beta}, \tag{4.36}$$

Particularly choose $\delta_0 = \delta$ and take the corresponding ϵ-scale ($\epsilon = \epsilon(\delta)$) Moran covering $\{\Delta_\omega^{(j)}\}, j = 1, \cdots, N^\epsilon(F)$ of F. Note that $N^\epsilon(F) \ge N_\epsilon(F)$. Since $0 < d \le \sup_{k\in\mathbb{N}, 1\le j\le n_{k+1}} c_j^{k+1} < 1$, there exists $A > 0$ such that for $j = 1, \cdots, N^{\epsilon_0}(F)$:

$$\frac{\epsilon}{A} \le \Psi_\omega^{(n(\omega))} \le \epsilon. \tag{4.37}$$

Hence there exist uniform constants C_1 and C_2 such that

$$C_1 \log(\frac{1}{\epsilon}) \le n(x_j) \le C_2 \log(\frac{A}{\epsilon}).$$

In the Moran covering the $n(\omega)$ can take on at most $C_3 := C_2\log(\frac{A}{\epsilon}) - C_1\log(\frac{1}{\epsilon}) > 0$ possible values.

By having $N^\epsilon(F)$ balls and C_3 baskets, for $N^\epsilon(F) \ge C_3$ there exists a basket containing at least $\frac{N^\epsilon}{C_3}$ balls. This implies that there exists a positive integer $\alpha :=$

$\alpha(\delta)$ with $C_1\log(\frac{1}{\epsilon})\le\alpha\le C_2\log(\frac{A}{\epsilon})$ such that for a sufficient small ϵ,

$$\sharp\{\omega \text{ such that } n(\omega)=\alpha\}\ge\frac{N^{\epsilon}(F)}{C_3}\ge\frac{N_{\epsilon}(F)}{C_3}\ge\frac{\epsilon^{\delta-\beta}}{C_3}\ge\epsilon^{2\delta-\beta}.$$

Recall that for any fixed number s, the pressure function $\phi(x):=s\log\Psi_{\omega_1}(\underline{\mathbf{w}})$, (where $\underline{\mathbf{w}}:=(\omega_1,\omega_2,\dots)\in[D_k]$ for any $k>K$) is only dependent on the first coordinate ω_1. Therefore,

$$(S_n\Phi)(x)=\sum_{j=1}^{n}\Phi(\sigma_k^j x)=s\log\prod_{j=1}^{n}\Psi_{\omega_j}^{(k)},$$

and hence $\exp(S_n\Phi)(x)=(\prod_{j=1}^{n}\Psi_{\omega_j}^{(k)})^s$. We now compute:

$$P_k(\Phi(x))=\lim_{n\to\infty}\frac{1}{n}\log\sum_{\mathbf{w}^{(n)}\in[D_k]_n}\inf_{x\in\Delta_{\mathbf{w}^{(n)}}}\exp\left(\sum_{j=0}^{n-1}s\log([\sigma_k^j\underline{\mathbf{w}}(x)]^{(1)})\right)\tag{4.38}$$

and

$$\begin{aligned}P_k(\Phi(x))&=\lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{\mathbf{w}^{(n)}\in[D_k]_n}\exp\left(\sum_{j=0}^{n-1}\log((\Psi_{\omega_j}^{(k)})^s)\right)\right)\\&=\lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{\mathbf{w}^{(n)}\in[D_k]_n}\prod_{j=0}^{n-1}(\Psi_{\omega_j}^{k})^s\right)\\&=\log\left(\sum_{\omega\in D_k}(\Psi_{\omega}^{(k)})^s\right),\end{aligned}$$

where $\mathbf{w}^{(n)}=(\omega_1,\dots,\omega_n)$ and $\omega_i\in D_k$.

Particular choosing $k=\alpha$ and $s=\overline{\dim}_B(F)-3\delta=\beta-3\delta$, we can then estimate the right hand side from below using the Moran cover. This gives:

$$\begin{aligned}P_{\alpha}((\beta-3\delta)\Psi_{\omega_1}^{(\alpha)})&\ge\log\left(\sum_{\omega\in\{\Delta^{(j)}\}}(\Psi_{\omega}^{(\alpha)})^{\beta-3\delta}\right)\\&\ge\log\left(\left(\frac{\epsilon}{A}\right)^{\beta-3\delta}\epsilon^{2\delta-\beta}\right)\\&=\log(\epsilon^{-\delta}A^{3\delta-\beta})\ge 0.\end{aligned}\tag{4.39}$$

The last positive inequality is based on the facts that the constant A is independent of δ and $\epsilon^{-\delta}\ge\log(1/\epsilon)$.

Repeating the above algorithm on a sequence $\{\delta(k)\}$, we can then obtain countably many equations as (4.39). This contradicts with (4.35), and implies the results. □

Computation of lower bounds for box/packing dimensions

We now provide a lower bound for the box/packing dimensions when the construction is conformal.

Lemma 4.5. *Suppose a GMS F satisfies the conformality condition (say the vector $\overline{\Psi}$ is the corresponding* conformal vector*, as in Definition 4.8), then*

$$\overline{\dim}_B(F) = s^*.$$

Proof. Based on Lemma 4.4, it suffices to prove the lower bound. For each $\beta < s^*$, there exists a subsequence $\{s_k\}$ such that for each k, $\beta < s_k$. Moreover, the conformal condition implies that $B(x, C^{-1}\Psi_\omega^{(k)}) \subseteq \triangle_\omega$, for each $\omega \in D_k$. We recall the equivalent definition of upper boxing dimension as (1.35) in Chapter 1. If let

$$W^s(F) := \lim_{r\to 0}\sup\{\sum_i \mathrm{diam}(B_i)^s : \mathrm{diam}(B_i) \le r, B_i^\circ \cap B_j^\circ = \emptyset (i \neq j), B_i \cap F \neq \emptyset\},$$

then

$$\overline{\dim}_B(F) := \sup\{s : W^s(F) = \infty\} = \inf\{s : W^s(F) = 0\}.$$

Hence, we have:

$$\begin{aligned} W^\beta(F) &\ge \limsup_{k\to\infty} \sum_{\omega\in D_k} \mathrm{diam}(B(x))^\beta \\ &\ge \limsup_{k\to\infty} \sum_{\omega\in D_k} C^{-1}(\Psi_\omega^{(k)})^{s_k} \\ &\ge \limsup_{k\to\infty} \sum_{\omega\in D_k} C^{-1}L_1^{-1}m_\Psi(\triangle_\omega) > 0, \end{aligned}$$

which implies that $\overline{\dim}_B(F) > \beta$. Since β is arbitrarily close to s^*, it follows that $\overline{\dim}_B(F) = s^*$.

□

Lemma 4.6. *Suppose a GMS F satisfies the conformality condition, then* $\dim_P(F) = \overline{dim}_B(F)$.

Proof. By Lemma 1.6, it suffices to show that for any open set V, $\overline{\dim}_B(F \cap V) \le \overline{\dim}_B(F) = s^*$, provided $F \cap V \neq \emptyset$. We do this as follows. Clearly $\overline{\dim}_B(F \cap V) \le \overline{\dim}_B(F)$. Moreover, for any open set V with $F \cap V \neq \emptyset$, there exists a $\tilde{\omega} \in D_N$ such that $\triangle_{\tilde{\omega}} \subset V$. Analogous to the proof in Lemma 4.5, take any $\beta < s^*$ then there exists a subsequence s_k with $\beta < s_k$, such that

$$\begin{aligned} W^{\beta}(F\cap V) &\geq \limsup_{k\to\infty} \sum_{\omega\in D_k, \triangle_\omega\subseteq\triangle_{\tilde{\omega}}} \operatorname{diam}(B(x, \frac{1}{C}\Psi^{(k)}))^{\beta} \\ &\geq \limsup_{k\to\infty} \sum_{\omega\in D_k, \triangle_\omega\subseteq\triangle_{\tilde{\omega}}} C^{-1}(\Psi^{(k)})^{s_k} \\ &\geq \limsup_{k\to\infty} \sum_{\omega\in D_k, \triangle_\omega\subseteq\triangle_{\tilde{\omega}}} C^{-1}L_1^{-1}m_{\Psi}(\triangle_\omega) \\ &= C^{-1}L_1^{-1}m_{\Psi}(\triangle_{\tilde{\omega}}) > 0. \end{aligned}$$

Hence we obtain $s^* = \overline{\dim_B}(F\cap V) = \overline{\dim_B}(F)$, which completes the proof. □

Conformality implies (LE) property

The following lemma gives the relationship between conformality and (LE) property.

Lemma 4.7. *If a vector* $\overline{\Psi}$ *is conformal, and*

$$\inf_{k\in\mathbb{N}, 1\leq j\leq n_{k+1}} c_j^{(k+1)}(\omega) > 0,$$

then the vector $\overline{\Psi}$ *satisfies (LE) property.*

Proof. For any fixed $0 < r < 1$, and any $x \in F$, consider the open ball $B(x,r)$ centered in x with radius of r, and let $N(x,r)$ be the number of Moran covering $\{\triangle^{(j)}\}$ that have nonempty intersection with $B(x,r)$. Hence

$$B(x,r)\bigcup\left(\cup_{j=1}^{N(x,r)}\triangle^{(j)}\right)\subseteq B(x,R),$$

where

$$R = 2r + \sup_j \operatorname{diam}(\triangle^{(j)}).$$

Using conformality condition and recalling from the definition of $\triangle^{(j)}$, we can choose elements $\triangle_{\omega^{(n)}}$ such that $\Psi_\omega^{(n)} \geq r$ and $\Psi_\omega^{(n+1)} \leq r$. Therefore,

$$R \leq 2r + \sup_j C\Psi_{\omega_j}^{(k)} \leq 2r + d^{-1}r,$$

and

$$\operatorname{diam}(\triangle^{(j)}) \geq C^{-1}\Psi_{\omega_j}^{(k)} \geq C^{-1}r.$$

Hence it follows that for each $x \in F$ and $0 < r < 1$

$$N(x,r) \leq \frac{2r + d^{-1}r}{C^{-1}r} = \frac{2 + d^{-1}}{C^{-1}} < \infty.$$

Therefore the vector $\overline{\Psi}$ satisfies (LE) property.

□

Proof of Theorem 4.2

The proof of Theorem 4.2 is as follows. We claim first of all that $\dim_H F \leq s_*$. The proof of this claim follows step by step the proof of Lemma 4.3 via equation (4.33). Hence it suffices to show only that $\dim_H(F) \geq s_*$.

Suppose $\beta < s_*$, then there exists a $K \in \mathbb{N}$ such that for all $k \geq K$, $s_k > \beta$. Moreover for any $\omega \in D$, $(\Psi_\omega^{(k)})^{s_k} < (\Psi_\omega^{(k)})^\beta$.

Since $\log d_k / \log M_k \to 0$, there exists an $\epsilon > 0$ such that for all $k \geq K$, $M_k^{\epsilon/2} < d_k^{\beta+\epsilon}$ and hence that $M_k^{\epsilon/2}/d_k^{\beta+\epsilon} < 1$.

Now take the Moran cover $\triangle^{(j)}$ such that $m_k(\triangle^{(j)}) \leq L_1(\Psi_\omega^{(k)})^{s_k} < (\Psi_\omega^{(k)})^\beta$. Given $r > 0$, the Moran cover $\triangle_\omega^{(j)}$ has the property that $(\Psi_\omega^{(n(\omega))}) \geq r$ and $(\Psi_\omega^{(n(\omega)+1)}) < r$. Since $s_k > \beta$ we choose ϵ sufficiently small so that $s_k > \beta + \epsilon$. Therefore, we obtain the following series of estimates:

$$\begin{aligned} m_\Psi(B(x,r)) &\leq \sum_{j=1}^{N(x,r)} m_\Psi(\triangle^{(j)}) \leq \sum_{j=1}^{N(x,r)} L_1(\Psi_\omega^{(k)})^{\beta+\epsilon} \\ &\leq \sum_{j=1}^{N(x,r)} \frac{L_1}{d_{k(\omega)}^{\beta+\epsilon}}(\Psi_\omega^{(k+1)})^{\beta+\epsilon} \leq \sum_{j=1}^{N(x,r)} \frac{L_1 M^{\epsilon/2}}{d_{k(\omega)}^{\beta+\epsilon}}(\Psi_\omega^{(k+1)})^{\beta+\frac{\epsilon}{2}} \\ &\leq L_1 M r^{\beta+\frac{\epsilon}{2}}. \end{aligned} \tag{4.40}$$

By the uniform mass distribution it follows that $\dim_H(m_\Psi) \geq s_*$, since we can choose β arbitrarily close to s_* and ϵ arbitrarily close to 0. It follows that $\dim_H(F) \geq \dim_H(m_\Psi) \geq s_*$ and hence we obtain $\dim_H(F) = s_*$

We now turn to the box dimension. First consider the case where the vector $\overline{\Psi}$ is upper-estimating, but the construction is not conformal. We can repeat the proof of Lemma 4.4, but we note that the constant A appearing in equation (4.37) is now dependent on ϵ. Taking again the Moran covering $\{\Delta_\omega^j\}, j = 1, \cdots, N^\epsilon(F)$ of F at scale ϵ, we have $\Psi_\omega^{(n(\omega)+1)} < r \leq \Psi_\omega^{(n(\omega))}$. Recalling that $d_k = \min_{1 \leq j \leq n_{k+1}} c_j^{(k+1)}$, we obtain

$$\epsilon > \Psi_\omega^{(n(\omega)+1)} = \Psi_\omega^{(n(\omega))} \left(\frac{\Psi_\omega^{(n(\omega)+1)}}{\Psi_\omega^{(n(\omega))}} \right) \geq d_{n(\omega)+1} \Psi_\omega^{(n(\omega))},$$

and hence $\Psi_\omega^{(n(\omega))} \leq \epsilon d_{n(\omega)+1}^{-1}$. Since $\lim_{k\to\infty} \frac{\log d_k}{\log M_k} = 0$ it follows that for all $\eta > 0$, there exists a K, such that $\forall\, k \geq K$, $1 > d_k > M_k^\eta > 0$, and therefore

$$\Psi_\omega^{(n(\omega))} \leq \epsilon M_k^{-\eta}.$$

Hence by definition of M_k we obtain (for arbitrary $\eta > 0$): $\epsilon < \Psi_\omega^{(n(\omega))} \leq \epsilon^{\frac{1}{1+\eta}}$. Following step by step the proof of Lemma 4.4, we obtain $\overline{\dim}_B(F) \leq s^*$.

Calculation of $\overline{\dim}_B(F)$ in the conformal case

We now prove that $\dim_P F = \overline{\dim}_B F = s^*$ when the construction is conformal. First of all note that for any $\eta > s^*$, there exists a $K > 0$ such that $\forall k > K, \quad s_k < \eta$. Since $\lim_{k\to\infty} \frac{\log d_k}{\log M_k} = 0$ it follows that for all $\epsilon > 0$, there exists a K such that $\forall\, k \geq K$, $1 > d_k > M_k^{\epsilon} > 0$. In particular, for any given η and ϵ, we can choose K such that $d_k^{\eta} > M_k^{\epsilon}$. At a scale $r > 0$, we take the Moran cover by using sets $\triangle_{\omega}^{(j)}$ with the property $\Psi_{\omega}^{(n(\omega))} > r$ and $\Psi_{\omega}^{n(\omega)+1} \leq r$. Since the construction is conformal there exists a $C > 0$ such that

$$\triangle_{\omega}^{(j)} \subseteq B(x, Cr),$$

where $B(x,r)$ is the open ball of radius r with the center x. Hence we have the following series of estimates for k sufficiently large:

$$\begin{aligned} m_{\Psi}(B(x,Cr)) &\geq m_k(\triangle^{(j)}) \geq L_1^{-1}(\Psi_{\omega}^{(k)})^{s^*+\frac{\epsilon}{2}} \\ &\geq L_1^{-1}(\Psi_{\omega}^{(k)})^{s^*+\frac{\epsilon}{2}} \left(\frac{M_k^{\epsilon/2}}{d_k^{s^*+\epsilon}}\right) \\ &\geq L_1^{-1}\frac{(\Psi_{\omega}^{(k)})^{s^*+\epsilon}}{d_k^{s^*+\epsilon}} \\ &\geq L_1^{-1}(\Psi_{\omega}^{(k-1)})^{s^*+\epsilon/2} \geq L_5 r^{s^*+\epsilon/2}. \end{aligned}$$

By the uniform mass distribution principle, it follows that $\dim_P(F) < s^* + \epsilon/2$. Since ϵ is arbitrary, combining with the fact that $\dim_P F = \overline{\dim}_B(F)$, it follows that $\overline{\dim}_B(F) < s^*$. The lower bound for the upper-box dimension follows from Lemma 4.5.

□

Appendix A

Multidimensional bounded variation functions

This appendix introduces the notions of multidimensional bounded variation, Sobolev space, piecewise rotations and interval translation maps, which are particularly used in Chapter 2.

A.1 Bounded variation

We introduce the usual notion of bounded variation in one dimension followed by definitions of multidimensional bounded variation.

Let $\eta \in L^1(\mathbb{R})$ and $[a, b]$ be an interval outside of which $\eta(x) = 0$. The *total variation* of η is defined to be

$$V(\eta) := \sup \sum_{i=0}^{r-1} |\eta(x_{i+1}) - \eta(x_i)|,$$

where "sup" is taken over all possible finite partitions of the interval $[a, b]$ by points $x_0 = a < x_1 < \cdots < x_r = b$. The *essential total variation* of η is defined as $\bar{V}(\eta) := \inf_u V(\eta + u)$, where the "inf" is taken over all functions u that equal zero almost everywhere on $[a, b]$ [120].

Next, we proceed to describe a definition of multidimensional bounded variation. Let Ω be an open subset of $\mathbb{R}^d$ and $\eta \in L^1(\Omega)$ be a function with compact support. We regard $\eta(x_i)$ as a function of the variable x_i for the other variables fixed and denote by $\bar{V}_i(x_i')$ the essential total variation of the function η with respect to x_i for a fixed point $x_i' = \{x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_d\}$.

Definition A.1. *[120] Suppose Ω is an open subset of $\mathbb{R}^d$. A function $\eta \in L^1(\Omega)$ with compact support is said to be of* bounded variation *if the integrals*

$$\int \bar{V}_i(x_i') dx_i' < \infty, \qquad (i = 1, 2, \cdots, n).$$

We give a further definition of bounded variation below.

Definition A.2. *[120, p158] Suppose Ω is an open subset of $\mathbb{R}^d$ and a function $\eta \in L^1(\Omega)$, then η is said to be of* bounded variation *if there exists a constant K such that*

$$\left|\int_\Omega \frac{\partial \phi}{\partial x_i}\eta dx\right| \leq K \sup_{x\in\Omega}|\phi(x)|, \quad (i = 1, 2, \cdots, d), \tag{A.1}$$

for all $\phi \in C_c^1(\Omega, \mathbb{R})$.

Definition A.1 and Definition A.2 coincide for functions $\eta \in L^1(\Omega)$ with compact support [120]. Moreover, we show the equivalence between Definition A.2 and Definition 2.2 in Section 2.1.1 via the following proposition.

Proposition A.1. *Suppose Ω is an open subset of $\mathbb{R}^n$ and $\eta \in L^1(\Omega)$. Then $\mathrm{var}(\eta) < \infty$ (recall that $\mathrm{var}(\cdot)$ is defined in Definition 2.2) if and only if the inequality (A.1) holds for all $\phi \in C_c^1(\Omega, \mathbb{R})$.*

Proof: "$\Rightarrow$". Suppose that $\mathrm{var}(\eta) < \infty$, then by Definition 2.2,

$$\left|\int_\Omega \eta(x) \operatorname{div} \vec{\phi}(x)dx\right| \leq \mathrm{var}(\eta)||\vec{\phi}||_\infty, \quad \forall \vec{\phi} \in C_c^1(\Omega, \mathbb{R}^d), \tag{A.2}$$

For any $\phi \in C_c^1(\Omega, \mathbb{R})$, let $\vec{\phi_i} = (\overbrace{0, \cdots}^{i-1}, \phi, \cdots, 0)$, then the inequality (A.2) holds for $\vec{\phi}_i$. This implies that inequality (A.1) holds for all $C_c^1(\Omega, \mathbb{R}^d)$.

"$\Leftarrow$". By the inequality (A.1) and Definition 2.2, it is clear that $\mathrm{var}(\eta) \leq d \cdot K < \infty$ where d is the dimension constant. □

A.2 Sobolev space $W^{1,2}$

Definition A.3. *[64] Suppose Ω is an open set in $\mathbb{R}^d$. The Sobolev space $W^{1,2}(\Omega)$ is defined to be the set of all functions $u \in L^2(\Omega)$ such that for every multi-index $\alpha = (\alpha_1, \cdots, \alpha_d)$ with $\alpha_i \geq 0$ and $|\alpha| := \sum_{j=1}^d \alpha_j \leq 1$, the weak partial derivative $D_\alpha u$ belongs to $L^2(\Omega)$, i.e.,*

$$W^{1,2}(\Omega) = \{u \in L^2(\Omega) : D_\alpha u \in L^2(\Omega), \forall\ |\alpha| \leq 1\}.$$

We define a norm on $W^{1,2}(\Omega)$ by

$$||u||_{W^{1,2}} := (\sum_{|\alpha|\leq 1} \int_\Omega |D_\alpha u|^2 dm)^{1/2}.$$

The Sobolev space $W^{1,2}(\Omega)$ with norm $||\cdot||_{W^{1,2}}$ is a Hilbert subspace of $BV(\Omega)$. We state the following Banach-Saks theorem which is applied to the Sobolev space $W^{1,2}$ in Lemma 2.5.

Banach-Saks Theorem [64] Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence from a Hilbert space H with $||x_n||_H \le K$ (independent of n), then there exists a subsequence $\{x_{n_j}\}_{j\in\mathbb{N}}$ and an $x \in H$ such that

$$\frac{1}{k}\sum_{j=1}^{k} x_{n_j} \to x, \quad \text{as } k \to \infty$$

in $||\cdot||_H$.

A.3 Piecewise rotations

Definition A.4. *[47] Let X be a compact subset of $\mathbb{C}$. A map $T : X \to X$ is called a* piecewise rotation *with a partition $\mathcal{P} := \{\omega_0, \cdots, \omega_{r-1}\}$ if*

$$T|_{\omega_j} x = \rho_j x + z_j, \quad x \in \omega_j$$

for some complex numbers z_j and ρ_j such that $|\rho_j| = 1$ for all $j = 0, 1, \cdots, r-1$. The atoms are assumed to be mutually disjoint convex polygons.

It is clear that piecewise rotations are PWIs in $\mathbb{R}^2$ with a topological partition $\mathcal{P}$ and are homeomorphisms when restricting on each atom.

A.4 Interval translation maps and interval exchange transformations

Definition A.5. *Let $I = [0, 1)$ be an interval and $0 = \beta_0 < \beta_1 < \cdots < \beta_r = 1$ be a finite partition of I. An interval map $T : I \to I$ is said to be an* interval translation map *[18] if*

$$T(x) = x + \gamma_i, \quad \beta_{i-1} \le x < \beta_i,$$

where each γ_i is a fixed real number. Particularly, if T maps I onto itself then T is called an interval exchange transformation.

Appendix B

Permutations preserving mixing for mx mod 1

Our main goal in this appendix is the proof of Theorem 3.1. In §B.1, we prove statement (i), and, in the setting of statement (ii), give a group theoretic interpretation of those σ for which $\sigma \circ f$ is not mixing. The asymptotic analysis needed to complete the proof of Theorem 3.1 is given in §B.2. In §B.3 we give explicit formulae for the proportion of non-mixing permutations for small values of ℓ.

The calculations are mainly combinatorial. It is to acknowledge that Nigel Byott contributes to these results.

B.1 When is $\sigma \circ f$ non-mixing?

Recall that $f(x) = mx \bmod 1$ and $\sigma \in S_N$, and that we partition the unit interval into subintervals $I_a = [a/N, (a+1)/N)$ for $a \in \{0, 1, \ldots, N-1\}$. We identify the indexing set $\{0, \ldots, N-1\}$ with the the ring $\mathbb{Z}/N\mathbb{Z}$ of integers modulo N, so that arithmetic in this indexing set is to be interpreted as arithmetic modulo N.

To begin with, we allow arbitrary m, $N \geq 2$. We set $g = \sigma \circ f$.

Definition B.1. *For any subset $A \subseteq \mathbb{Z}/N\mathbb{Z}$, we define*

$$\tilde{f}(A) = \bigcup_{d=0}^{m-1} (mA + d) \subseteq \mathbb{Z}/N\mathbb{Z}, \qquad \tilde{g}(A) = \sigma(\tilde{f}(A)).$$

Proposition B.1. *For each $A \subseteq \mathbb{Z}/N\mathbb{Z}$, we have*

$$f\left(\bigcup_{a \in A} I_a\right) = \bigcup_{b \in \tilde{f}(A)} I_b, \qquad g\left(\bigcup_{a \in A} I_a\right) = \bigcup_{b \in \tilde{g}(A)} I_b.$$

Proof. This is immediate since for each $j \in \mathbb{Z}/N\mathbb{Z}$ we have

$$f(I_j) = \bigcup_{d=0}^{m-1} I_{mj+d}, \qquad \sigma(I_j) = I_{\sigma(j)}.$$

□

Proposition B.2. *For each $A \subseteq \mathbb{Z}/N\mathbb{Z}$, we have*

$$\sharp A \leq \sharp\tilde{f}(A) \leq m\sharp A.$$

Moreover, suppose that $0 < \sharp A < N$. Then we have $\sharp\tilde{f}(A) = \sharp A$ if and only if the following two conditions hold:

(i) $N = m\ell$ for some integer ℓ;

(ii) A is a union of cosets of $\ell\mathbb{Z}/N\mathbb{Z}$ (that is, $j \in A \Rightarrow j+\ell \in A$ for all $j \in \mathbb{Z}/N\mathbb{Z}$).

Proof. If $b \in \tilde{f}(A)$ then $b \equiv ma + d \pmod{N}$ for at least one of the $m\sharp A$ pairs (a, d) with $a \in A$ and $0 \leq d < m$. Hence $\sharp\tilde{f}(A) \leq m\sharp A$. Now fix one pair (a_0, d_0). If another pair (a, d) gives the same element b then

$$ma + d \equiv ma_0 + d_0 \pmod{N}. \tag{B.1}$$

Thus $d \equiv d_0 \pmod{s}$, where $s = \gcd(m, N)$. This gives m/s possibilities for d. For each of these, (B.1) has s solutions a in $\mathbb{Z}/N\mathbb{Z}$, all congruent mod N/s (but in general not all in A). So each b arises from at most m of the pairs (a, d), giving $\sharp\tilde{f}(A) \geq \sharp A$. This proves the first assertion.

If $\sharp\tilde{f}(A) = \sharp A$, then each b must arise from exactly m pairs (a, d). Thus given $a_0 \in A$, we may take $d = d_0 = 0$, and the s solutions a to (B.1) must all lie in A. This shows that $a_0 + N/s \in A$, so that A is stable under addition of N/s.

First suppose (i) holds. Then $N/s = \ell$, so that if $\sharp\tilde{f}(A) = \sharp A$ then (ii) holds. Conversely, if (i) and (ii) hold, then each $b \in \tilde{f}(A)$ arises from m pairs $(a + j\ell, d)$ with $0 \leq j < m$, so that $\sharp\tilde{f}(A) = \sharp A$.

It remains to show that if (i) does not hold and $\sharp\tilde{f}(A) = \sharp A > 0$ then $A = \mathbb{Z}/N\mathbb{Z}$. So let $m = es$ with $e > 1$, and let $a_0 \in A$. Since $s < m$, we may take $d_0 = s$ in (B.1). But (B.1) must have s solutions for each of the possible values $d \equiv d_0 \pmod{s}$ with $0 \leq d < m$, so we can find $a_1 \in A$ with $ma_1 \equiv ma_0 + s \pmod{N}$. Then $ea_1 \equiv ea_0 + 1 \pmod{N/s}$. Iterating, we can find $a_j \in A$ with $ea_j \equiv ea_{j-1} + 1 \equiv ea_0 + j \pmod{N/s}$ for $j \geq 1$. As $\gcd(e, N/s) = 1$, we have $a_e \equiv a_0 + 1 \pmod{N/s}$. Since we already know that A is stable under addition of N/s, it follows that A is stable under addition of 1, so that $A = \mathbb{Z}/N\mathbb{Z}$. □

Corollary B.1. *For any $A \subseteq \mathbb{Z}/N\mathbb{Z}$, we have*

$$\sharp\tilde{g}(A) \geq \sharp A.$$

Moreover, if A is a proper subset of $\mathbb{Z}/N\mathbb{Z}$ then equality can only occur if $N = \ell m$ for some integer ℓ.

Proof. This is clear since $\sharp\tilde{g}(A) = \sharp\tilde{f}(A)$. □

Lemma B.1. *g fails to be (topologically) mixing if and only if there is some proper subset A of $\mathbb{Z}/N\mathbb{Z}$ such that $\sharp\tilde{g}^r(A) = \sharp A$ for all $r \geq 0$.*

Proof. Let A be a subset with $0 < \sharp A < N$ and $\sharp\tilde{g}^r(A) = \sharp A$ for all r. As there are only finitely many subsets of $\mathbb{Z}/N\mathbb{Z}$, we may choose $s \geq 0$ and $t \geq 1$ with $\tilde{g}^{s+t}(A) = \tilde{g}^s(A)$. Set $B = \tilde{g}^s(A)$ and take non-empty open sets $U \subset I_j$ and $V \subset I_k$ where $j \in B$ and $k \notin B$. Then for all $n \geq 0$ we have $g^{nt}(U) \subseteq \bigcup_{b\in B} I_b$ so that $g^{nt}(U) \cap V = \emptyset$. Hence g is not mixing.

Conversely, suppose there is no proper subset A with $\sharp\tilde{g}^r(A) = \sharp A$ for all r. To see that g is mixing, we show that for any non-empty open subset U of $[0, 1)$ we have $g^n(U) = [0, 1)$ for large enough n. Without loss of generality, U is an interval of length $\delta > 0$. Since $m > 1$, we can choose h large enough that $g^h(U)$ contains the initial point j/N of some interval I_j. Then for some $\epsilon > 0$, we have $[j/N, j/N + \epsilon) \subseteq g^h(U) \cap I_j$. Choose k so that $m^k\epsilon > 1/N$ and let $g^k(j/N) = j'/N$. Then $I_{j'} \subset g^k(I_j) \subset g^{h+k}(U)$. Now let $B = \{j'\}$ and take $s \geq 0$, $t \geq 1$ with $\tilde{g}^{s+t}(B) = \tilde{g}^s(B)$. The non-empty set $A = \tilde{g}^s(B)$ then satisfies the condition $\tilde{g}^{nt}(A) = A$ for all $t \geq 0$. Hence, by Corollary B.1, we have $\sharp\tilde{g}^r(A) = \sharp A$ for all $r \geq 0$. Thus our hypothesis forces $A = \mathbb{Z}/N\mathbb{Z}$, so that $g^q(I_{j'}) = [0, 1)$ for all $q \geq s$. It follows that $g^n(U) = [0, 1)$ for all $n \geq h + k + s$, as required. □

Proof of Theorem 3.1(i). Suppose that N is not a multiple of m, and let $g = \sigma \circ f$ with $\sigma \in S_N$. By Corollary B.1 there is no proper subset A with $\sharp\tilde{g}(A) = \sharp A$. Hence by Lemma B.1, g is mixing. □

We now suppose that $N = m\ell$ for some integer $\ell \geq 1$.

Proposition B.3. *There exists a permutation $\delta \in S_N$ such that*

$$f(I_j) \supseteq I_{\delta(j)} \text{ for all } j \in \mathbb{Z}/N\mathbb{Z}. \tag{B.2}$$

For any such δ, and any $A \subseteq \mathbb{Z}/N\mathbb{Z}$, the following are equivalent:

(i) $\sharp\tilde{g}(A) = \sharp A$;

(ii) *A is a union of cosets of the subgroup $\ell\mathbb{Z}/N\mathbb{Z}$ of $\mathbb{Z}/N\mathbb{Z}$;*

(iii) $\sigma\delta(A) = \tilde{g}(A)$.

Proof. To prove the first assertion, we exhibit a permutation δ with the required property. For $0 \leq i < N$, write $i = j + c\ell$ with $0 \leq c < m$ and $0 \leq j < \ell$, and set $\delta(i) = mj + c$. It is routine to verify that $\delta \in S_N$, and, as

$$f(I_j) = \bigcup_{d=0}^{m-1} I_{mj+d},$$

the condition (B.2) holds.

Now fix a choice of $\delta \in S_N$ satisfying (B.2). Since $\sharp\tilde{g}(A) = \sharp\tilde{f}(A)$, the equivalence of (i) and (ii) follows from Proposition B.2. Since $\sigma\delta \in S_N$, it is immediate that (iii)⇒(i). It remains to show that (ii)⇒(iii).

Since $f(I_j) = f(I_{j+\ell})$ for each j, it follows from (B.2) that δ takes the m elements $j + c\ell$, $0 \leq c < m$ to the m elements $mj + d$, $0 \leq d < m$ in some order. Thus, if (ii) holds, δ takes each coset $a + \ell\mathbb{Z}/N\mathbb{Z}$ contained in A to $\tilde{f}(\{a\})$. Thus $\delta(A) = \tilde{f}(A)$, and applying σ gives (iii). □

We consider partitions $\mathbb{Z}/N\mathbb{Z}$ into disjoint non-empty sets: $\mathbb{Z}/N\mathbb{Z} = A_1 \cup \ldots \cup A_t$. We call the set $\mathbb{B} = \{A_1, \ldots, A_t\}$ of subsets of $\mathbb{Z}/N\mathbb{Z}$ a *block decomposition* of $\mathbb{Z}/N\mathbb{Z}$, and refer to the A_i as *blocks*. We say that $\mathbb{B}$ is *trivial* if $t = 1$, and that $\mathbb{B}$ is ℓ-stable if, for any $j \in \mathbb{Z}/N\mathbb{Z}$ and $1 \leq r \leq t$, we have $j \in A_r \Rightarrow j + \ell \in A_r$. Thus $\mathbb{B}$ is ℓ-stable if and only if each A_r is a union of cosets of the subgroup $\ell\mathbb{Z}/N\mathbb{Z}$ of $\mathbb{Z}/N\mathbb{Z}$. If $\mathbb{B} = \{A_1, \ldots, A_t\}$ is a block decomposition and $\sigma \in S_N$, then $\sigma\mathbb{B} = \{\sigma(A_1), \ldots, \sigma(A_t)\}$ is also a block decomposition, and we define the stabiliser $G_{\mathbb{B}}$ of $\mathbb{B}$ as

$$G_{\mathbb{B}} = \{\sigma \in S_N \ : \ \sigma(\mathbb{B}) = \mathbb{B}\}.$$

Then $G_{\mathbb{B}}$ is a subgroup of S_N.

Lemma B.2. *Let $f(x) = mx \bmod 1$ and let $N = m\ell$. Let δ be as in Proposition B.3. Then, for any $\sigma \in S_N$, the composite $g = \sigma \circ f$ fails to be mixing if and only if there is some non-trivial ℓ-stable block decomposition $\mathbb{B}$ of $\mathbb{Z}/N\mathbb{Z}$ such that $\sigma\delta \in G_{\mathbb{B}}$.*

Proof. Let $\sigma\delta \in G_{\mathbb{B}}$ for some non-trivial, ℓ-stable block decomposition $\mathbb{B}$, and let A be a block of $\mathbb{B}$. Then A is a proper subset of $\mathbb{Z}/N\mathbb{Z}$ which is a union of cosets of $\ell\mathbb{Z}/N\mathbb{Z}$. Thus $\tilde{g}(A) = \sigma\delta(A)$ by Proposition B.3, and this set is also a block of $\mathbb{B}$. Inductively, we then have $\tilde{g}^r(A) = (\sigma\delta)^r(A)$, and hence $\sharp\tilde{g}^r(A) = \sharp(\sigma\delta)^r(A) = \sharp A$, for all $r \geq 0$. It then follows from Lemma B.1 that g is non-mixing.

Conversely, suppose that g is non-mixing. By Lemma B.1, there is a proper subset A of $\mathbb{Z}/N\mathbb{Z}$ such that $\sharp\tilde{g}^r(A) = \sharp A$ for all $r \geq 0$. By Proposition B.3 and induction, $\tilde{g}^r(A) = (\sigma\delta)^r(A)$ for all $r \geq 0$. Moreover, each $(\sigma\delta)^r(A)$ is a union of cosets of $\ell\mathbb{Z}/N\mathbb{Z}$. Since $\sigma\delta$ is a permutation, it follows that $(\sigma\delta)^s(A^c)$ is also a union of cosets for each $s \geq 0$, where A^c is the complement of A. Let $\widetilde{\mathbb{B}}$ be set of all intersections of the sets $(\sigma\delta)^r(A)$, $(\sigma\delta)^s(A^c)$ for r, $s \geq 0$. Thus $\widetilde{\mathbb{B}}$ is a collection of subsets of $\mathbb{Z}/N\mathbb{Z}$, each of which is a union of cosets of $\ell\mathbb{Z}/N\mathbb{Z}$. Let $\mathbb{B}$ be the collection of minimal non-empty sets in $\widetilde{\mathbb{B}}$. Then $\mathbb{B}$ is an ℓ-stable block decomposition and $\sigma\delta \in G_{\mathbb{B}}$. Moreover, $\mathbb{B}$ is non-trivial since A is a union of blocks of $\mathbb{B}$. □

Remark B.1. *A similar argument shows that $f \circ \sigma$ is non-mixing if and only if $\delta\sigma \in G_{\mathbb{B}}$ for some non-trivial ℓ-stable block decomposition.*

B.2 Asymptotic behaviour as $\ell \to \infty$

We continue to assume $N = m\ell$. We shall investigate the proportion of permutations which do not preserve mixing:

$$p(\ell, m) = \frac{\sharp\{\sigma \in S_{m\ell} \ : \ \sigma \circ f \text{ is not mixing } \}}{(m\ell)!}. \tag{B.3}$$

By Lemma B.2, this is the proportion of permutations such that $\sigma\delta$ is in the stabiliser of at least one non-trivial ℓ-stable block decomposition.

The following Lemma will complete the proof of Theorem 3.1.

Lemma B.3. *When $N = m\ell$ with $\ell \geq 6$, we have*

$$p(\ell, m) < 11 \left(\frac{2e}{\ell}\right)^{m-1}.$$

In particular, for each fixed $m \geq 2$ we have $p(\ell, m) \to 0$ as $\ell \to \infty$.

From Lemma B.2 we have

$$p(\ell, m) \leq \frac{1}{(m\ell)!} \sum_{\mathbb{B}} \sharp G_{\mathbb{B}}, \tag{B.4}$$

where the sum is over all non-trivial ℓ-stable block decompositions $\mathbb{B}$. (This is not an equality since the $G_{\mathbb{B}}$ are not disjoint.) Given integers $1 \leq r_1 \leq \ldots \leq r_j$ with $r_1 + \cdots + r_j = \ell$, we consider the contribution to (B.4) from all block decompositions $\mathbb{B}$ with block sizes $mr_1, \ldots, mr_j$. The number of such block decompositions can be found as follows. Let us set

$$n_i(r_1, \ldots, r_j) = \sharp\{h \ : r_h = i\}$$

and

$$d(r_1, \ldots, r_j) = \prod_{i=1}^{\ell} n_i(r_1, \ldots, r_j)!.$$

Then the number of ℓ-stable block decompositions $\mathbb{B}$ of $\{1,\dots,m\ell\}$ with block sizes $mr_1,\dots,mr_j$ is

$$\frac{1}{d(r_1,\dots,r_j)}\binom{\ell}{r_1,\dots,r_j},$$

where

$$\binom{\ell}{r_1,\dots,r_j}=\frac{\ell!}{r_1!\dots r_j!}$$

is the multinomial coefficient. Moreover, any such $\mathbb{B}$ is preserved by a group of permutations $S_{mr_1}\times\cdots\times S_{mr_j}$ permuting the elements within each block, but we can also permute the blocks of any given size amongst themselves. Thus we have

$$\sharp G_{\mathbb{B}}=d(r_1,\dots,r_j)\left(\prod_{h=1}^{j}(mr_h)!\right).$$

The contribution to (B.4) from block decompositions with block sizes $mr_1,\dots,mr_j$ is therefore

$$\frac{1}{(m\ell)!}\left[d(r_1,\dots,r_j)\left(\prod_{h=1}^{j}(mr_h)!\right)\right]\left[\frac{1}{d(r_1,\dots,r_j)}\binom{\ell}{r_1,\dots,r_j}\right]$$

which simplifies to

$$\binom{\ell}{r_1,\dots,r_j}\binom{m\ell}{mr_1,\dots,mr_j}^{-1}.$$

Thus we may rewrite (B.4) as

$$p(\ell,m)\le\sum_{j=2}^{\ell}b_j(\ell),\tag{B.5}$$

where

$$b_j(\ell)=\sum_{\substack{1\le r_1\le\ldots\le r_j\\ r_1+\ldots+r_j=\ell}}\binom{\ell}{r_1,\dots,r_j}\binom{m\ell}{mr_1,\dots,mr_j}^{-1}.$$

The definition of $b_j(\ell)$ makes sense for $j=1$, giving $b_1(\ell)=1$.

Proposition B.4. *For $2\le j\le\ell$, we have*

$$b_j(\ell)\le\sum_{r=1}^{\lfloor\ell/j\rfloor}\binom{\ell}{r}\binom{m\ell}{mr}^{-1}b_{j-1}(\ell-r).$$

Proof. Separating out r_1 in the definition of $b_j(\ell)$, we may write

$$b_j(\ell)\le\sum_{r_1=1}^{\lfloor\ell/j\rfloor}\sum_{\substack{1\le r_2\le\ldots\le r_j\\ r_2+\cdots+r_j=\ell-r_1}}\binom{\ell}{r_1,\dots,r_j}\binom{m\ell}{mr_1,\dots,mr_j}^{-1}.$$

(Note that we have "$\le$" rather than "$=$" since the condition $r_2 \ge r_1$ has been weakened to $r_2 \ge 1$.) The result then follows on using the (easily verified) identity

$$\binom{\ell}{r_1,\ldots,r_j} = \binom{\ell}{r_1}\binom{\ell - r_1}{r_2,\ldots,r_j},$$

together with the corresponding identity where all the arguments are multiplied by m. ☐

Proposition B.5. *Suppose that $m \ge 2$ and $\ell \ge 3$. Then, for $1 \le j \le \ell$, we have*

$$b_j \le \left(\frac{2e}{\ell}\right)^{(m-1)(j-1)}. \tag{B.6}$$

Proof. We argue by induction on j. The result holds for $j = 1$ since $b_1(\ell) = 1$. Suppose that $2 \le j \le \ell$ and that the result holds for $j - 1$. From Proposition B.4, we have

$$b_j(\ell) \le \binom{\ell}{1}\binom{m\ell}{m}^{-1} b_{j-1}(\ell - 1) + \sum_{r=2}^{\lfloor \ell/j \rfloor} \binom{\ell}{r}\binom{m\ell}{mr}^{-1} b_{j-1}(\ell - r). \tag{B.7}$$

For the first term, we have the estimate

$$\begin{aligned}
\binom{\ell}{1}\binom{m\ell}{m}^{-1} b_{j-1}(\ell-1) &\le \frac{\ell\,(m!)}{(m\ell)(m\ell-1)\ldots(m\ell-\ell+1)}\left(\frac{2e}{\ell-1}\right)^{(m-1)(j-2)} \\
&\le \frac{(m-1)!}{m^{m-1}(\ell-1)^{m-1}}\left(\frac{2e}{\ell}\cdot\frac{\ell}{\ell-1}\right)^{(m-1)(j-2)} \\
&= \frac{1}{2}\left(\frac{1}{2}\cdot\frac{\ell}{\ell-1}\cdot\frac{2}{\ell}\right)^{m-1}\left(\frac{2e}{\ell}\cdot\frac{\ell}{\ell-1}\right)^{(m-1)(j-2)} \\
&\le \frac{1}{2^m e^{m-1}}\left(\frac{2e}{\ell}\right)^{(m-1)(j-2)}\left(\frac{\ell}{\ell-1}\right)^{(m-1)(j-1)}.
\end{aligned}$$

But

$$\left(\frac{\ell}{\ell-1}\right)^{(m-1)(j-1)} \le \left(\frac{\ell}{\ell-1}\right)^{(m-1)(\ell-1)} < e^{m-1}$$

since $(1+\frac{1}{n})^n$ is an increasing function of n and $(1+\frac{1}{n})^n \to e$ as $n \to \infty$. As $m \ge 2$, it follows that

$$\binom{\ell}{1}\binom{m\ell}{m}^{-1} b_{j-1}(\ell-1) \le \frac{1}{4}\left(\frac{2e}{\ell}\right)^{(m-1)(j-1)}. \tag{B.8}$$

We now consider each term in the sum in (B.7). For $2 \le r \le \lfloor \ell/j \rfloor$, we have

$$\binom{\ell}{r}^m \le \binom{m\ell}{mr}.$$

(This is obvious combinatorially: some of the ways of choosing mr objects from $m\ell$

are given by choosing r objects from the first ℓ, then another r from the second ℓ, and so on.) Also, since $2 \leq r \leq \ell/2$, we have

$$\binom{\ell}{2} \leq \binom{\ell}{r}.$$

Thus

$$\binom{\ell}{r}\binom{m\ell}{mr}^{-1} \leq \binom{\ell}{r}^{1-m} \leq \binom{\ell}{2}^{1-m} = \frac{2^{m-1}}{\ell^{m-1}(\ell-1)^{m-1}}.$$

From the induction hypothesis, we have

$$b_{j-1}(\ell-r) \leq \left(\frac{2e}{\ell-r}\right)^{(m-1)(j-2)} \leq \left(\frac{2e}{\ell}\right)^{(m-1)(j-2)} \left(\frac{j}{j-1}\right)^{(m-1)(j-2)},$$

and $\big(j/(j-1)\big)^{j-2} < \big(j/(j-1)\big)^{j-1} < e$. Thus

$$\begin{aligned} \sum_{r=2}^{\lfloor \ell/j \rfloor} \binom{\ell}{r}\binom{m\ell}{mr}^{-1} b_{j-1}(\ell-r) &< \frac{\ell}{j}\frac{2^{m-1}}{\ell^{m-1}(\ell-1)^{m-1}}\left(\frac{2e}{\ell}\right)^{(m-1)(j-2)} e^{m-1} \\ &= \frac{\ell}{j(\ell-1)^{m-1}}\left(\frac{2e}{\ell}\right)^{(m-1)(j-1)}. \end{aligned}$$

But as $j \geq 2$, $m \geq 2$ and $\ell \geq 3$, we have

$$\frac{\ell}{j(\ell-1)^{m-1}} \leq \frac{\ell}{2(\ell-1)} \leq \frac{3}{4}.$$

Substituting the last estimate and (B.8) into (B.7), we therefore obtain

$$b_j \leq \left(\frac{2e}{\ell}\right)^{(m-1)(j-1)},$$

which completes the induction. □

Proof of Lemma B.3. Since we are assuming $\ell \geq 6$, we have $2e/\ell < 1$. It then follows from (B.5) and Proposition B.5 that

$$\begin{aligned} p(\ell,m) &< \sum_{j=2}^{\infty}\left(\frac{2e}{\ell}\right)^{(m-1)(j-1)} \\ &= \left(\frac{2e}{\ell}\right)^{m-1}\left[1-\left(\frac{2e}{\ell}\right)^{m-1}\right]^{-1}. \end{aligned}$$

As $m \geq 2$ and $2e/\ell < \frac{10}{11}$, this gives

$$p(\ell,m) < \left[1-\left(\frac{2e}{\ell}\right)\right]^{-1}\left(\frac{2e}{\ell}\right)^{m-1} < 11\left(\frac{2e}{\ell}\right)^{(m-1)},$$

as required. □

B.3 The proportion of non-mixing permutations

In this section, we will use the Inclusion-Exclusion Principle (see e.g. [114, p. 21]) to give explicit formulae for the proportion $p(\ell, m)$ of non-mixing permutations when $N = m\ell$ with ℓ small.

The stabiliser of any non-trivial ℓ-stable block decomposition contains the subgroup $H \cong S_m \times \ldots \times S_m$ of order $(m!)^\ell$ which permutes the m elements of each coset amongst themselves. In order to refer to specific block decompositions, we let $C_1, \ldots, C_\ell$ denote the cosets of $\ell\mathbb{Z}/N\mathbb{Z}$ in $\mathbb{Z}/N\mathbb{Z}$ (in some order). Giving an ℓ-stable block decomposition amounts to giving a partition of $\{C_1, \ldots, C_\ell\}$, and we denote the block decomposition by the corresponding partition of the set of indices $\{1, \ldots, \ell\}$. Thus $\{1, \ldots, \ell-1\}, \{\ell\}$ represents the ℓ-stable block decomposition consisting of the two blocks $C_1 \cup \ldots \cup C_{\ell-1}$ of size $(\ell-1)m$ and C_ℓ of size m.

$\ell = 1$

The only ℓ-stable block decomposition is the trivial one, so $g = \sigma \circ f$ is mixing for all σ, and $p(1, m) = 1$.

$\ell = 2$

There is only one non-trivial ℓ-stable block decomposition. This has two blocks, each of size m. Its stabiliser contains H and also contains elements swapping the two blocks, so has order $2\sharp H$. Thus

$$p(2, m) = \frac{2\sharp H}{(2m)!} = \binom{2m-1}{m}^{-1}.$$

In particular, taking $m = 2$, we get $p(2, 2) = 1/3$. Thus, when the doubling map $f(x) = 2x \bmod 1$ is composed with permutations σ of the 4 equal subintervals of $[0, 1)$, those $\sigma \in S_4$ for which $f \circ \sigma$ is not mixing form a single coset of a subgroup of index 3 in S_4. (Any such subgroup is dihedral of order 8.)

$\ell = 3$

There are 4 non-trivial ℓ-stable block decompositions:

(i) $\{1, 2\}, \{3\}$; (ii) $\{1, 3\}, \{2\}$; (iii) $\{2, 3\}, \{1\}$; (iv) $\{1\}, \{2\}, \{3\}$.

The stabiliser of any one of the block decompositions (i), (ii), (iii) has order $(2m)!m! = \binom{2m}{m}\sharp H$ since it contains any permutation of the $2m$ elements in the block consisting of 2 cosets. The stabiliser of the block decomposition (iv) has order $6\sharp H$ since we may permute the 3 blocks amongst themselves in $3! = 6$ ways.

We now consider the stabilisers of any of the $\binom{4}{2} = 6$ pairs of the block decompositions. First consider the 3 pairs consisting of any two of (i), (ii) or (iii). Any permutation fixing such a pair must fix each coset, so the stabiliser of any of these 3 pairs is just H. A permutation stabilising (say) (i) and (iv) could also swap the cosets C_1 and C_2, so the stabilisers of the other 3 pairs have orders $2\sharp H$. The stabiliser of any 3 (or all 4) block decompositions is again just H. Thus the precise number of permutations in S_{3m} fixing at least one of the block decompositions is

$$\left(3\binom{2m}{m} + 6 - 3 - 3\times 2 + \binom{4}{3} - 1\right)\sharp H = 3\binom{2m}{m}\sharp H = 3(2m)!m!.$$

Hence

$$p(3,m) = \frac{1}{(3m)!}\times 3(2m)!m! = \binom{3m-1}{2m}^{-1}.$$

In particular, $p(3,2) = 1/5$.

$\ell = 4$

There are 14 non-trivial ℓ-stable block decompositions, but we only need to consider the 4 block decompositions with block sizes 3, 1 and the 3 block decompositions with block sizes 2, 2, since any permutation stabilising a non-trivial block decomposition must stabilise one of these. We can then apply to the Inclusion-Exclusion Principle to the stabilisers of these 7 block decompositions, by considering all possible pairs, and, for each pair, considering any ways of extending the pair to a larger subset of the blocks with stabiliser larger than H. After some simplification, we obtain the formula

$$p(4,m) = \left[4\binom{3m}{m,m,m} + 6\binom{2m}{m}^2 - 12\binom{2m}{m}\right]\frac{(m!)^4}{(4m)!}.$$

In particular, we find

$$p(4,2) = \frac{1}{5}, \qquad p(4,3) = \frac{37}{1540}.$$

Note that, in contrast to the cases $\ell = 2$ and $\ell = 3$, $p(4,m)$ is not in general the reciprocal of an integer.

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