The two-dimensional hydrogen atom revisited

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The bound-state energy eigenvalues for the two-dimensional Kepler problem are found to be degenerate. This ''accidental'' degeneracy is due to the existence of a two-dimensional analog of the quantum-mechanical Runge–Lenz vector. Reformulating the problem in momentum space leads to an integral form of the Schrödinger equation. This equation is solved by projecting the two-dimensional momentum space onto the surface of a three-dimensional sphere. The eigenfunctions are then expanded in terms of spherical harmonics, and this leads to an integral relation in terms of special functions which has not previously been tabulated. The dynamical symmetry of the problem is also considered, and it is shown that the two components of the Runge–Lenz vector in real space correspond to the generators of infinitesimal rotations about the respective coordinate axes in momentum space. © *2002 American Institute of Physics.* [DOI: 10.1063/1.1503868]

I. INTRODUCTION

A semiconductor quantum well under illumination is a quasi-two-dimensional system, in which photoexcited electrons and holes are essentially confined to a plane. The mutual Coulomb interaction leads to electron–hole bound states known as excitons, which are extremely important for the optical properties of the quantum well. The relative in-plane motion of the electron and hole can be described by a two-dimensional Schrödinger equation for a single particle with a reduced mass. This is a physical realization of the two-dimensional hydrogenic problem, which originated as a purely theoretical construction.[1] An important similarity with the three-dimensional hydrogen atom is the ''accidental'' degeneracy of the bound-state energy levels. This degeneracy is due to the existence of the quantum-mechanical Runge–Lenz vector, first introduced by Pauli[2] in three dimensions, and indicates the presence of a dynamical symmetry of the system.

The most important study relating to the hidden symmetry of the hydrogen atom was that by Fock in 1935.[3] He considered the Schrödinger equation in momentum space, which led to an integral equation. Considering negative-energy (bound-state) solutions, he projected the three-dimensional momentum space onto the surface of a four-dimensional hypersphere. After a suitable transformation of the wavefunction, the resulting integral equation was seen to be invariant under rotations in four-dimensional momentum space. Fock deduced that the dynamical symmetry of the hydrogen atom is described by the four-dimensional rotation group SO(4), which contains the geometrical symmetry SO(3) as a subgroup. He related this hidden symmetry to the observed degeneracy of the energy eigenvalues.

Shortly afterwards, Bargmann[4] made the connection between Pauli's quantum mechanical Runge–Lenz vector and Fock's discovery of invariance under rotations in four-dimensional momentum space. Fock's method was also extended by Alliluev[5] to the case of d dimensions ($d \geq 2$). A comprehensive review concerning the symmetry of the hydrogen atom was later given by Bander and Itzykson,[6,7] including a detailed group-theoretical treatment and extension to scattering states.

Improvements in semiconductor growth techniques over the subsequent decades, which enabled the manufacture of effectively two-dimensional structures, led to a resurgence of interest in

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the two-dimensional hydrogen atom. The Runge–Lenz vector for this case was defined for the first time,[8] and real-space solutions of the Schrödinger equation were applied to problems of atomic physics in two dimensions.[9]

Recent studies have focused on diverse aspects of the hydrogenic problem. The d-dimensional case has been reconsidered, leading to a generalized Runge–Lenz vector (see Ref. 10 and references therein). The algebraic basis of the dynamical symmetry has also been given a thorough mathematical treatment.[11,12]

In the present work we return to the two-dimensional problem, and use the method of Fock to obtain a new integral relation in terms of special functions. The dynamical symmetry of the system is also considered, and a new interpretation of the two-dimensional Runge–Lenz vector is presented.

II. PROBLEM FORMULATION

A. Preliminaries

The relative in-plane motion of an electron and hole, with effective masses m_e and m_h, respectively, may be treated as that of a single particle with reduced mass $\mu = m_e m_h/(m_e + m_h)$ and energy E, moving in a Coulomb potential $V(\rho)$. The wavefunction of the particle satisfies the stationary Schrödinger equation

$$\hat{H}\Psi(\boldsymbol{\rho}) = \left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) - \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + V(\rho) \right]\Psi(\boldsymbol{\rho}) = E\Psi(\boldsymbol{\rho}), \tag{1}$$

where (ρ,ϕ) are plane polar coordinates. Note that excitonic Rydberg units are used throughout this article, which leads to a potential of the form $V(\rho) = -2/\rho$.

The eigenfunctions of Eq. (1) are derived in Appendix A. It is well known that the bound-state energy levels are of the form[1]

$$E = -\frac{1}{(n+\frac{1}{2})^2}, \quad n = 0,1,2,..., \tag{2}$$

where n is the principal quantum number. Notably, Eq. (2) does not contain explicitly the azimuthal quantum number m, which enters the radial equation [see Appendix A, Eq. (A4)]. Each energy level is $(2n+1)$-fold degenerate, the so-called accidental degeneracy.

It is convenient to introduce a vector operator corresponding to the z-projection of the angular momentum, $\hat{\mathbf{L}}_z = \mathbf{e}_z \hat{L}_z$, where $\mathbf{e}_z$ is a unit vector normal to the plane of motion of the electron and hole. We now introduce the two-dimensional analog of the quantum-mechanical Runge–Lenz vector as the dimensionless operator

$$\hat{\mathbf{A}} = (\hat{\mathbf{q}} \times \hat{\mathbf{L}}_z - \hat{\mathbf{L}}_z \times \hat{\mathbf{q}}) - \frac{2}{\rho}\boldsymbol{\rho}, \tag{3}$$

where $\hat{\mathbf{q}} = -i\nabla$ is the momentum operator. Note that $\hat{\mathbf{A}}$ lies in the plane and has Cartesian components $\hat{A}_x$ and $\hat{A}_y$.

$\hat{L}_z$, $\hat{A}_x$, and $\hat{A}_y$ represent conserved quantities and therefore commute with the Hamiltonian:

$$[\hat{H},\hat{L}_z] = [\hat{H},\hat{A}_x] = [\hat{H},\hat{A}_y] = 0. \tag{4}$$

They also satisfy the following commutation relations:

$$[\hat{L}_z,\hat{A}_x] = i\hat{A}_y, \tag{5}$$

$$[\hat{L}_z,\hat{A}_y] = -i\hat{A}_x, \tag{6}$$

$$[\hat{A}_x,\hat{A}_y]=-4i\hat{L}_z\hat{H}. \tag{7}$$

B. Derivation of energy eigenvalues from $\hat{A}$

The existence of the noncommuting operators $\hat{A}_x$ and $\hat{A}_y$, representing conserved physical quantities, implies that the Runge–Lenz vector is related to the accidental degeneracy of the energy levels in two dimensions.[13] We now present a simple interpretation of the hidden symmetry underlying this degeneracy.

For eigenfunctions of the Hamiltonian we can replace $\hat{H}$ by the energy E, and defining

$$\hat{\mathbf{A}}'=\frac{\hat{\mathbf{A}}}{2\sqrt{-E}}, \tag{8}$$

we obtain the new commutation relations:

$$[\hat{L}_z,\hat{A}'_x]=i\hat{A}'_y, \tag{9}$$

$$[\hat{L}_z,\hat{A}'_y]=-i\hat{A}'_x, \tag{10}$$

$$[\hat{A}'_x,\hat{A}'_y]=i\hat{L}_z. \tag{11}$$

If we now construct a three-dimensional vector operator

$$\hat{\mathbf{J}}=\hat{\mathbf{A}}'+\hat{\mathbf{L}}_z, \tag{12}$$

then the components of $\hat{\mathbf{J}}$ satisfy the commutation rules of ordinary angular momentum:

$$[\hat{J}_j,\hat{J}_k]=i\epsilon_{jkl}\hat{J}_l, \tag{13}$$

where ϵ_{jkl} is the Levi-Civita symbol.

Noting that $\hat{\mathbf{A}}'\cdot\hat{\mathbf{L}}_z=\hat{\mathbf{L}}_z\cdot\hat{\mathbf{A}}'=0$, we have

$$\hat{\mathbf{J}}^2=(\hat{\mathbf{A}}'+\hat{\mathbf{L}}_z)^2=\hat{\mathbf{A}}'^2+\hat{\mathbf{L}}_z^2, \tag{14}$$

where the operator $\hat{\mathbf{J}}^2$ has eigenvalues $j(j+1)$ and commutes with the Hamiltonian.

We now make use of a special expression relating $\hat{\mathbf{A}}^2$ and $\hat{\mathbf{L}}_z^2$, the derivation of which is given in Appendix B:

$$\hat{\mathbf{A}}^2=\hat{H}(4\hat{\mathbf{L}}_z^2+1)+4. \tag{15}$$

Substituting in Eq. (14) and again replacing $\hat{H}$ with E, we obtain

$$\hat{\mathbf{J}}^2=-\frac{1}{4E}[E(4\hat{\mathbf{L}}_z^2+1)+4]+\hat{\mathbf{L}}_z^2. \tag{16}$$

Because $[\hat{H},\hat{\mathbf{J}}^2]=0$, an eigenfunction of the Hamiltonian will also be an eigenfunction of $\hat{\mathbf{J}}^2$. Operating with both sides of Eq. (16) on an eigenfunction of the Hamiltonian, we obtain for the eigenvalues of $\hat{\mathbf{J}}^2$:

$$j(j+1)=-\left(\frac{1}{4}+\frac{1}{E}\right). \tag{17}$$

Rearranging, and identifying j with the principal quantum number n, we obtain the correct expression for the energy eigenvalues:

$$E = -\frac{1}{(n+\frac{1}{2})^2}, \quad n=0,1,2,\dots. \tag{18}$$

Note that the z-component of $\hat{\mathbf{J}}$ is simply $\hat{L}_z$. Recalling that the eigenvalues of $\hat{L}_z$ are denoted by m, there are $(2j+1)$ values of m for a given j. However, as $j=n$, we see that there are $(2n+1)$ values of m for a given energy, which corresponds to the observed $(2n+1)$-fold degeneracy.

III. FOCK'S METHOD IN TWO DIMENSIONS

A. Stereographic projection

The method of Fock,[3] in which a three-dimensional momentum space is projected onto the surface of a four-dimensional hypersphere, may be applied to our two-dimensional problem. We begin by defining a pair of two-dimensional Fourier transforms between real space and momentum space:

$$\Phi(\mathbf{q}) = \int \Psi(\boldsymbol{\rho}) e^{i\mathbf{q}\cdot\boldsymbol{\rho}} \, d\boldsymbol{\rho}, \tag{19}$$

$$\Psi(\boldsymbol{\rho}) = \frac{1}{(2\pi)^2} \int \Phi(\mathbf{q}) e^{-i\mathbf{q}\cdot\boldsymbol{\rho}} \, d\mathbf{q}. \tag{20}$$

We shall restrict our interest to bound states, and hence the energy $E = -q_0^2$ will be negative.
Substitution of Eq. (20) in Eq. (1) yields the following integral equation for $\Phi(\mathbf{q})$:

$$(q^2+q_0^2)\Phi(\mathbf{q}) = \frac{1}{\pi} \int \frac{\Phi(\mathbf{q}') \, d\mathbf{q}'}{|\mathbf{q}-\mathbf{q}'|}. \tag{21}$$

The two-dimensional momentum space is now projected onto the surface of a three-dimensional unit sphere centered at the origin, and so it is natural to scale the in-plane momentum by q_0. Each point on a unit sphere is completely defined by two polar angles, θ and ϕ, and the Cartesian coordinates of a point on the unit sphere are given by

$$u_x = \sin\theta\cos\phi = \frac{2q_0 q_x}{q^2+q_0^2}, \tag{22}$$

$$u_y = \sin\theta\sin\phi = \frac{2q_0 q_y}{q^2+q_0^2}, \tag{23}$$

$$u_z = \cos\theta = \frac{q^2-q_0^2}{q^2+q_0^2}. \tag{24}$$

An element of surface area on the unit sphere is given by

$$d\Omega = \sin\theta \, d\theta \, d\phi = \left(\frac{2q_0}{q^2+q_0^2}\right)^2 d\mathbf{q}, \tag{25}$$

and the distance between two points transforms as

$$|\mathbf{u}-\mathbf{u}'| = \frac{2q_0}{(q^2+q_0^2)^{1/2}(q'^2+q_0^2)^{1/2}}|\mathbf{q}-\mathbf{q}'|. \tag{26}$$

If the wavefunction on the sphere is expressed as

$$\chi(\mathbf{u}) = \frac{1}{\sqrt{q_0}} \left(\frac{q^2 + q_0^2}{2q_0} \right)^{3/2} \Phi(\mathbf{q}), \tag{27}$$

then Eq. (21) reduces to the simple form

$$\chi(\mathbf{u}) = \frac{1}{2\pi q_0} \int \frac{\chi(\mathbf{u}') \, d\Omega'}{|\mathbf{u} - \mathbf{u}'|}. \tag{28}$$

B. Expansion in spherical harmonics

Any function on a sphere can be expressed in terms of spherical harmonics, so for $\chi(\mathbf{u})$ we have

$$\chi(\mathbf{u}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} Y_l^m(\theta, \phi), \tag{29}$$

where $Y_l^m(\theta, \phi)$ are basically defined as in Ref. 14:

$$Y_l^m(\theta, \phi) = c_{lm} \sqrt{\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}} P_l^{|m|}(\cos\theta) e^{im\phi}, \tag{30}$$

where $P_n^{|m|}(\cos\theta)$ is an associated Legendre function as defined in Ref. 15. The constant c_{lm} is an arbitrary "phase factor." As long as $|c_{lm}|^2 = 1$ we are free to choose c_{lm}, and for reasons which will become clear we set

$$c_{lm} = (-i)^{|m|}. \tag{31}$$

The kernel of the integral in Eq. (28) can also be expanded in this basis as[16]

$$\frac{1}{|\mathbf{u} - \mathbf{u}'|} = \sum_{\lambda=0}^{\infty} \sum_{\mu=-\lambda}^{\lambda} \frac{4\pi}{2\lambda+1} Y_\lambda^\mu(\theta, \phi) Y_\lambda^{\mu*}(\theta', \phi'). \tag{32}$$

Substituting Eqs. (29) and (32) into Eq. (28) we have

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} Y_l^m(\theta, \phi) = \frac{2}{q_0} \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\infty} \sum_{m_1=-l_1}^{l_1} \sum_{m_2=-l_2}^{l_2} \int \frac{1}{2l_2+1} A_{l_1 m_1} Y_{l_1}^{m_1}(\theta', \phi')$$
$$\times Y_{l_2}^{m_2}(\theta, \phi) Y_{l_2}^{m_2*}(\theta', \phi') \, d\Omega'. \tag{33}$$

We now make use of the orthogonality property of spherical harmonics to reduce Eq. (33) to the following:

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} Y_l^m(\theta, \phi) = \frac{2}{q_0} \sum_{l_1=0}^{\infty} \sum_{m_1=-l_1}^{l_1} \frac{1}{2l_1+1} A_{l_1 m_1} Y_{l_1}^{m_1}(\theta, \phi). \tag{34}$$

Multiplying both sides of Eq. (34) by $Y_n^{m'*}(\theta, \phi)$ and integrating over $d\Omega$ gives

$$A_{nm'} = \frac{2}{q_0(2n+1)} A_{nm'}, \tag{35}$$

where we have again used the orthogonality relation for spherical harmonics. The final step is to rearrange for q_0 and identify the index n with the principal quantum number. This enables us to find an expression for the energy in excitonic Rydbergs:

$$E = -q_0^2 = -\frac{1}{(n+\frac{1}{2})^2}, \quad n=0,1,2,\dots . \tag{36}$$

This is seen to be identical to Eq. (2).

For a particular value of n, the general solution of Eq. (28) can be expressed as

$$\chi_n(\mathbf{u}) = \sum_{m=-n}^{n} A_{nm} Y_n^m(\theta,\phi). \tag{37}$$

Each of the functions entering the sum in Eq. (37) satisfies Eq. (28) separately. So, for each value of n we have $(2n+1)$ linearly independent solutions, and this explains the observed $(2n+1)$-fold degeneracy.

We are free to choose any linear combination of spherical harmonics for our eigenfunctions, but for convenience we simply choose

$$\chi_{nm}(\mathbf{u}) = A_{nm} Y_n^m(\theta,\phi). \tag{38}$$

If we also require our eigenfunctions to be normalized as follows:

$$\frac{1}{(2\pi)^2}\int |\chi(\mathbf{u})|^2\, d\Omega = \frac{1}{(2\pi)^2}\int \frac{q^2+q_0^2}{2q_0^2}|\Phi(\mathbf{q})|^2\, d\mathbf{q} = \int |\Psi(\boldsymbol{\rho})|^2\, d\boldsymbol{\rho} = 1, \tag{39}$$

then Eq. (38) reduces to

$$\chi_{nm}(\mathbf{u}) = 2\pi Y_n^m(\theta,\phi). \tag{40}$$

Applying the transformation in Eq. (27), we can obtain an explicit expression for the orthonormal eigenfunctions of Eq. (21):

$$\Phi_{nm}(\mathbf{q}) = c_{nm}\sqrt{2\pi\frac{(n-|m|)!}{(n+|m|)!}}\left(\frac{2q_0}{q^2+q_0^2}\right)^{3/2} P_n^{|m|}(\cos\theta)e^{im\phi}, \tag{41}$$

where we have used the fact that $q_0=(n+\frac{1}{2})^{-1}$, and θ and ϕ are defined by Eqs. (22)–(24).

C. New integral relations

To obtain the real-space eigenfunctions $\Psi(\boldsymbol{\rho})$ we make an inverse Fourier transform:

$$\Psi(\boldsymbol{\rho}) = \frac{1}{(2\pi)^2}\int \Phi(\mathbf{q})e^{-i\mathbf{q}\cdot\boldsymbol{\rho}}\, d\mathbf{q} = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^\infty \Phi(\mathbf{q})e^{-iq\rho\cos\phi'}q\, dq\, d\phi', \tag{42}$$

where ϕ' is the azimuthal angle between the vectors $\boldsymbol{\rho}$ and $\mathbf{q}$. However, if we now substitute Eq. (41) into this expression, we have to be careful with our notation. The angle labeled ϕ in Eq. (41) is actually related to ϕ' via

$$\phi = \phi' + \phi_\rho, \tag{43}$$

where ϕ_ρ is the azimuthal angle of the vector $\boldsymbol{\rho}$, which can be treated as constant for the purposes of our integration.

Taking this into account, the substitution of Eq. (41) into Eq. (42) yields

$$\Psi(\boldsymbol{\rho}) = \frac{c_{nm}}{(2\pi)^{3/2}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}\, e^{im\phi_\rho}\int_0^{2\pi}\int_0^\infty \left(\frac{2q_0}{q^2+q_0^2}\right)^{3/2} P_n^{|m|}(\cos\theta)e^{i(m\phi'-q\rho\cos\phi')}q\, dq\, d\phi'. \tag{44}$$

From Eq. (24) we obtain

$$P_n^{|m|}(\cos\theta) = P_n^{|m|}\left(\frac{q^2 - q_0^2}{q^2 + q_0^2}\right), \tag{45}$$

and we use the following form of Bessel's integral:[16]

$$\int_0^{2\pi} e^{i(m\phi' - q\rho\cos\phi')} \, d\phi' = 2\pi(-i)^m J_m(q\rho), \tag{46}$$

where $J_m(q\rho)$ is a Bessel function of the first kind of order m. Substituting Eqs. (45) and (46) into Eq. (44) leads to

$$\Psi(\boldsymbol{\rho}) = \frac{c_{nm}(-i)^m}{\sqrt{2\pi}} \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} \, e^{im\phi_\rho} \int_0^\infty \left(\frac{2q_0}{q^2 + q_0^2}\right)^{3/2} P_n^{|m|}\left(\frac{q^2 - q_0^2}{q^2 + q_0^2}\right) J_m(q\rho) q \, dq. \tag{47}$$

We now make a change of variables, $x = q_0\rho$ and $y = q^2/q_0^2$, so that Eq. (47) becomes

$$\Psi(\boldsymbol{\rho}) = c_{nm}(-1)^{n+m}(-i)^m \sqrt{\frac{q_0(n-|m|)!}{\pi(n+|m|)!}} \, e^{im\phi_\rho} \int_0^\infty P_n^{|m|}\left(\frac{1-y}{1+y}\right) \frac{J_m(x\sqrt{y})}{(1+y)^{3/2}} \, dy, \tag{48}$$

where we have used the fact that[16]

$$P_n^{|m|}\left(\frac{y-1}{y+1}\right) = (-1)^{n+m} P_n^{|m|}\left(\frac{1-y}{1+y}\right). \tag{49}$$

If we now equate the expression for $\Psi(\boldsymbol{\rho})$ in Eq. (48) with that derived in Appendix A, we obtain the following:

$$c_{nm}(-1)^{n+m}(-i)^m \int_0^\infty P_n^{|m|}\left(\frac{1-y}{1+y}\right) \frac{J_m(x\sqrt{y})}{(1+y)^{3/2}} \, dy = \frac{(2x)^{|m|}e^{-x}}{n+1/2} L_{n-|m|}^{2|m|}(2x). \tag{50}$$

The value of c_{nm} chosen earlier in Eq. (31) ensures that both sides of Eq. (50) are numerically equal. If we restrict our interest to $m \geq 0$, then the relation simplifies to

$$\int_0^\infty P_n^m\left(\frac{1-y}{1+y}\right) \frac{J_m(x\sqrt{y})}{(1+y)^{3/2}} \, dy = \frac{(-1)^n(2x)^m e^{-x}}{n+1/2} L_{n-m}^{2m}(2x), \quad n,m = 0,1,2,\ldots; \quad m \leq n. \tag{51}$$

As far as we can ascertain, this integral relation between special functions has not previously been tabulated. For $n,m = 0$ we recover the known integral relation[15]

$$\int_0^\infty \frac{J_0(x\sqrt{y})}{(1+y)^{3/2}} \, dy = 2e^{-x}. \tag{52}$$

IV. DYNAMICAL SYMMETRY

A. Infinitesimal generators

Consider now a vector $\mathbf{u}$ from the origin to a point on the three-dimensional unit sphere defined in Sec. III A. If this vector is rotated through an infinitesimal angle α in the $(u_x u_z)$ plane, we have a new vector

$$\mathbf{u}' = \mathbf{u} + \delta\mathbf{u}, \tag{53}$$

where the components of **u** are given in Eqs. (22)–(24), and

$$\delta\mathbf{u} = \alpha\mathbf{e}_y \times \mathbf{u}. \tag{54}$$

This rotation on the sphere corresponds to a change in the two-dimensional momentum from **q** to **q′**. The Cartesian components of Eq. (53) are then found to be

$$u_x' = \frac{2q_0 q_x'}{q'^2 + q_0^2} = \frac{2q_0 q_x}{q^2 + q_0^2} + \alpha\frac{q^2 - q_0^2}{q^2 + q_0^2}, \tag{55}$$

$$u_y' = \frac{2q_0 q_y'}{q'^2 + q_0^2} = \frac{2q_0 q_y}{q^2 + q_0^2}, \tag{56}$$

$$u_z' = \frac{q'^2 - q_0^2}{q'^2 + q_0^2} = \frac{q^2 - q_0^2}{q^2 + q_0^2} - \alpha\frac{2q_0 q_x}{q^2 + q_0^2}, \tag{57}$$

where $q^2 = q_x^2 + q_y^2$.

After some manipulation we can also find the components of $\delta\mathbf{q} = \mathbf{q}' - \mathbf{q}$:

$$\delta q_x = \alpha\frac{q^2 - q_0^2 - 2q_x^2}{2q_0}, \tag{58}$$

$$\delta q_y = -\alpha\frac{q_x q_y}{q_0}. \tag{59}$$

The corresponding change in $\Phi(\mathbf{q})$ is given by

$$\delta\Phi(\mathbf{q}) = \frac{\alpha}{(q^2 + q_0^2)^{3/2}}\left(\frac{q^2 - q_0^2 - 2q_x^2}{2q_0}\frac{\partial}{\partial q_x} - \frac{q_x q_y}{q_0}\frac{\partial}{\partial q_y}\right)[(q^2 + q_0^2)^{3/2}\Phi(\mathbf{q})]. \tag{60}$$

We can write this as

$$\delta\Phi(\mathbf{q}) = -\frac{i}{2q_0}\alpha\hat{\mathcal{A}}_x\Phi(\mathbf{q}), \tag{61}$$

where the infinitesimal generator is given by

$$\hat{\mathcal{A}}_x = \frac{i}{(q^2 + q_0^2)^{3/2}}\left[(q^2 - q_0^2 - 2q_x^2)\frac{\partial}{\partial q_x} - 2q_x q_y\frac{\partial}{\partial q_y}\right](q^2 + q_0^2)^{3/2}. \tag{62}$$

We now make use of the following operator expression in the momentum representation:

$$\hat{\boldsymbol{\rho}} = \mathbf{e}_x\hat{x} + \mathbf{e}_y\hat{y} = i\nabla_q, \tag{63}$$

and the commutation relation

$$[\hat{\boldsymbol{\rho}}, f(\mathbf{q})] = i\nabla_q f, \tag{64}$$

to derive a more compact expression for $\hat{\mathcal{A}}_x$:

$$\hat{\mathcal{A}}_x = (q^2 - q_0^2)\hat{x} - 2q_x(\mathbf{q}\cdot\hat{\boldsymbol{\rho}}) - 3iq_x. \tag{65}$$

By considering an infinitesimal rotation in the $(u_y u_z)$ plane we can obtain a similar expression for $\hat{\mathcal{A}}_y$:

$$\hat{\mathcal{A}}_y = (q^2 - q_0^2)\hat{y} - 2q_y(\mathbf{q} \cdot \hat{\boldsymbol{\rho}}) - 3iq_y. \tag{66}$$

These expressions operate on a particular energy eigenfunction with eigenvalue $-q_0^2$. If we move the constant $-q_0^2$ to the right and replace it with the Hamiltonian in momentum space, $\hat{\mathcal{H}}$,

$$\hat{\mathcal{A}}_x = q^2\hat{x} + \hat{x}\hat{\mathcal{H}} - 2q_x(\mathbf{q} \cdot \hat{\boldsymbol{\rho}}) - 3iq_x, \tag{67}$$

$$\hat{\mathcal{A}}_y = q^2\hat{y} + \hat{y}\hat{\mathcal{H}} - 2q_y(\mathbf{q} \cdot \hat{\boldsymbol{\rho}}) - 3iq_y, \tag{68}$$

then $\hat{\mathcal{A}}_x$ and $\hat{\mathcal{A}}_y$ can operate on any linear combination of eigenfunctions.

B. Relation to Runge–Lenz vector

Recall the definition of the two-dimensional Runge–Lenz vector in real space:

$$\hat{\mathbf{A}} = (\hat{\mathbf{q}} \times \hat{\mathbf{L}}_z - \hat{\mathbf{L}}_z \times \hat{\mathbf{q}}) - \frac{2}{\rho}\boldsymbol{\rho}. \tag{69}$$

Using $\hat{\mathbf{L}}_z = \boldsymbol{\rho} \times \hat{\mathbf{q}}$, and the following identity for the triple product of three vectors:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}, \tag{70}$$

we can apply the commutation relation $[\boldsymbol{\rho}, \hat{\mathbf{q}}] = i$ to rewrite Eq. (69) in the form:

$$\hat{\mathbf{A}} = \hat{\mathbf{q}}^2\boldsymbol{\rho} + \boldsymbol{\rho}\left(\hat{\mathbf{q}}^2 - \frac{2}{\rho}\right) - 2\hat{\mathbf{q}}(\hat{\mathbf{q}} \cdot \boldsymbol{\rho}) - 3i\hat{\mathbf{q}}. \tag{71}$$

If we now return to the expression for the real-space Hamiltonian in Eq. (1), it is apparent that we may substitute

$$\hat{\mathbf{q}}^2 - \frac{2}{\rho} = \hat{H} \tag{72}$$

in Eq. (71) to yield

$$\hat{\mathbf{A}} = \hat{\mathbf{q}}^2\boldsymbol{\rho} + \boldsymbol{\rho}\hat{H} - 2\hat{\mathbf{q}}(\hat{\mathbf{q}} \cdot \boldsymbol{\rho}) - 3i\hat{\mathbf{q}}. \tag{73}$$

Comparing this with Eqs. (67) and (68), it is evident that the two components of the Runge–Lenz vector in real space correspond to the generators of infinitesimal rotations in the $(u_x u_z)$ and $(u_y u_z)$ planes.

V. CONCLUSION

We have shown that the accidental degeneracy in the energy eigenvalues of the two-dimensional Kepler problem may be explained by the existence of a planar analog of the familiar three-dimensional Runge–Lenz vector. By moving into momentum space and making a stereographic projection onto a three-dimensional sphere, a new integral relation in terms of special functions has been obtained, which to our knowledge has not previously been tabulated. We have also demonstrated explicitly that the components of the two-dimensional Runge–Lenz vector in real space are intimately related to infinitesimal rotations in three-dimensional momentum space.

APPENDIX A: SOLUTION OF REAL-SPACE SCHRÖDINGER EQUATION

We apply the method of separation of variables to Eq. (1), making the substitution

$$\Psi(\boldsymbol{\rho}) = R(\rho)\Phi(\phi). \tag{A1}$$

Introducing a separation constant m^2, we can obtain the angular equation

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0, \tag{A2}$$

with the solution

$$\Phi(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}. \tag{A3}$$

The corresponding radial equation (with $E = -q_0^2$) is

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(\frac{2}{\rho} - q_0^2 - \frac{m^2}{\rho^2}\right)R = 0. \tag{A4}$$

We make the substitution

$$R(\rho) = C\rho^{|m|} e^{-q_0\rho} w(\rho), \tag{A5}$$

where C is a normalization constant. This leads to the equation

$$\rho\frac{d^2w}{d\rho^2} + (2|m| + 1 - 2q_0\rho)\frac{dw}{d\rho} + (2 - 2|m|q_0 - q_0)w = 0. \tag{A6}$$

Making a final change of variables $\beta = 2q_0\rho$, we obtain

$$\beta\frac{d^2w}{d\beta^2} + (2|m| + 1 - \beta)\frac{dw}{d\beta} + \left(\frac{1}{q_0} - |m| - \frac{1}{2}\right)w = 0. \tag{A7}$$

This is the confluent hypergeometric equation,[15] which has two linearly independent solutions. If we choose the solution which is regular at the origin, then this becomes a polynomial of finite degree if $q_0 = (n + \frac{1}{2})^{-1}$ with $n = 0,1,2,\ldots$. Equation (A7) then becomes the associated Laguerre equation,[16] the solutions of which are the associated Laguerre polynomials:

$$w = L_{n-|m|}^{2|m|}(\beta) = L_{n-|m|}^{2|m|}(2q_0\rho). \tag{A8}$$

We can now write the real-space wavefunction in the form

$$\Psi_{nm}(\boldsymbol{\rho}) = \frac{C}{2\pi}\rho^{|m|} e^{-q_0\rho} L_{n-|m|}^{2|m|}(2q_0\rho) e^{im\phi_\rho}, \tag{A9}$$

where the reason for the subscript on ϕ is explained in Sec. III C.

To normalize this wavefunction we need to make use of the integral[16]

$$\int_0^\infty e^{-2q_0\rho}(2q_0\rho)^{2|m|+1} L_{n-|m|}^{2|m|}(2q_0\rho) L_{n-|m|}^{2|m|}(2q_0\rho)\, d(2q_0\rho) = \frac{(n+|m|)!}{(n-|m|)!}(2n+1). \tag{A10}$$

The normalized wavefunctions are therefore

$$\Psi_{nm}(\boldsymbol{\rho}) = \sqrt{\frac{q_0^3(n-|m|)!}{\pi(n+|m|)!}}(2q_0\rho)^{|m|} e^{-q_0\rho} L_{n-|m|}^{2|m|}(2q_0\rho) e^{im\phi_\rho}, \tag{A11}$$

satisfying the following orthogonality condition:

$$\int \Psi^*_{n_1 m_1}(\boldsymbol{\rho}) \Psi_{n_2 m_2}(\boldsymbol{\rho}) \, d\boldsymbol{\rho} = \delta_{n_1 n_2} \delta_{m_1 m_2}. \tag{A12}$$

APPENDIX B: DERIVATION OF EQ. (15)

From Eq. (3) we have

$$\hat{\mathbf{A}}^2 = \left[(\hat{\mathbf{q}} \times \hat{\mathbf{L}}_z - \hat{\mathbf{L}}_z \times \hat{\mathbf{q}}) - \frac{2}{\rho} \boldsymbol{\rho} \right]^2 = [2(\hat{\mathbf{q}} \times \hat{\mathbf{L}}_z) - i\hat{\mathbf{q}}]^2 - \frac{2}{\rho} \boldsymbol{\rho} \cdot [2(\hat{\mathbf{q}} \times \hat{\mathbf{L}}_z) - i\hat{\mathbf{q}}]$$

$$- \frac{2}{\rho}[2(\hat{\mathbf{q}} \times \hat{\mathbf{L}}_z) - i\hat{\mathbf{q}}] \cdot \boldsymbol{\rho} + 4. \tag{B1}$$

We further expand as follows:

$$[2(\hat{\mathbf{q}} \times \hat{\mathbf{L}}_z) - i\hat{\mathbf{q}}]^2 = 4(\hat{\mathbf{q}} \times \hat{\mathbf{L}}_z)^2 - 2i\hat{\mathbf{q}} \cdot (\hat{\mathbf{q}} \times \hat{\mathbf{L}}_z) - 2i(\hat{\mathbf{q}} \times \hat{\mathbf{L}}_z) \cdot \hat{\mathbf{q}} - \hat{\mathbf{q}}^2$$

$$= 4\hat{\mathbf{q}}^2 \hat{\mathbf{L}}_z^2 + 2\hat{\mathbf{q}}^2 - \hat{\mathbf{q}}^2 = \hat{\mathbf{q}}^2 (4\hat{\mathbf{L}}_z^2 + 1), \tag{B2}$$

and

$$- \frac{2}{\rho} \boldsymbol{\rho} \cdot [2(\hat{\mathbf{q}} \times \hat{\mathbf{L}}_z) - i\hat{\mathbf{q}}] - \frac{2}{\rho}[2(\hat{\mathbf{q}} \times \hat{\mathbf{L}}_z) - i\hat{\mathbf{q}}] \cdot \boldsymbol{\rho} = - \frac{2}{\rho}(4\hat{\mathbf{L}}_z^2 + 1). \tag{B3}$$

Substituting Eqs. (B2) and (B3) into Eq. (B1) gives

$$\hat{\mathbf{A}}^2 = \hat{\mathbf{q}}^2(4\hat{\mathbf{L}}_z^2 + 1) - \frac{2}{\rho}(4\hat{\mathbf{L}}_z^2 + 1) + 4, \tag{B4}$$

which, from Eq. (72), is just

$$\hat{\mathbf{A}}^2 = \hat{H}(4\hat{\mathbf{L}}_z^2 + 1) + 4. \tag{B5}$$

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