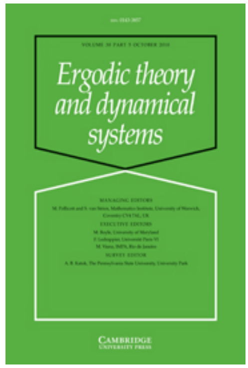

Cone exchange transformations and boundedness of orbits

PETER ASHWIN and AREK GOETZ

Ergodic Theory and Dynamical Systems / Volume 30 / Issue 05 / October 2010, pp 1311 - 1330
DOI: 10.1017/S0143385709000625, Published online: 07 September 2009

Ergod. Th. & Dynam. Sys. (2010), **30**, 1311–1330
doi:10.1017/S0143385709000625

Cone exchange transformations and boundedness of orbits

PETER ASHWIN† and AREK GOETZ‡

† *Mathematics Research Institute, University of Exeter, Exeter EX4 4QF, UK*
(e-mail: P.Ashwin@ex.ac.uk)
‡ *San Francisco State University, 1600 Holloway Avenue, San Francisco, CA 94132, USA*
(e-mail: goetz@sfsu.edu)

(*Received* 23 *September* 2008 *and accepted in revised form* 29 *May* 2009)

Abstract. We introduce a class of two-dimensional piecewise isometries on the plane that we refer to as cone exchange transformations (CETs). These are generalizations of interval exchange transformations (IETs) to 2D unbounded domains. We show for a typical CET that boundedness of orbits is determined by ergodic properties of an associated IET and a quantity we refer to as the 'flux at infinity'. In particular we show, under an assumption of unique ergodicity of the associated IET, that a positive flux at infinity implies unboundedness of almost all orbits outside some bounded region, while a negative flux at infinity implies boundedness of all orbits. We also discuss some examples of CETs for which the flux is zero and/or we do not have unique ergodicity of the associated IET; in these cases (which are of great interest from the point of view of applications such as dual billiards) it remains an outstanding problem to find computable necessary and sufficient conditions for boundedness of orbits.

1. *Introduction*

Several applications require one to investigate iterated maps of the plane, where the map is a piecewise isometry defined by rotations on each of a number of subsets of the plane. A topical application is that of dual billiards, where Schwartz [**15**] has recently resolved a long-standing conjecture by Neumann and Moser. In particular, Schwartz has constructed a dual billiard system (partition illustrated in Figure 1) for which he proves there is an unbounded orbit.

In another example, Ashwin and Goetz [**1**] show that for a different system (illustrated in Figure 2) all orbits are bounded, and they describe a partition into periodic and aperiodic orbits. The question of boundedness of orbits is nontrivial even for the simplest map in this class: a rotation on each of two half-planes. In this case Goetz and Quas [**10**] have shown that if such a map is invertible, then it has periodic points arbitrarily close to infinity.

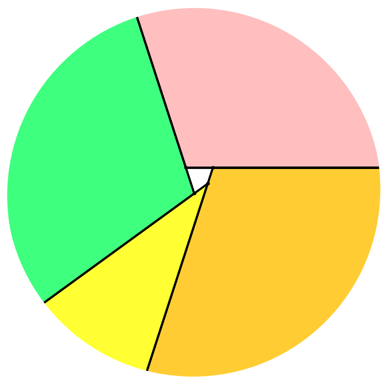

FIGURE 1. *Schwartz's dual billiard cones.* The Schwartz map T acts on the complement of the white quadrilateral that is split into four infinite cones. Each cone is rotated about its vertex by π. The quadrilateral has interior angles $4\pi/5$, $2\pi/5$, $2\pi/5$, $2\pi/5$. Schwartz proves the existence of unbounded orbits for T.

In general almost all orbits are recurrent, but boundedness of orbits even for this simple family of maps is yet to be proven or disproven. This paper aims to examine a class of cone exchange transformations (CETs) that gives a unified framework for discussing such questions and many more.

The structure of this article is as follows. In this section we define cone exchange transformations, a class of piecewise isometry maps that act on cones. The main results for this class are stated in §1.1. In §2 we parameterize CETs and discuss the properties of asymptotic radial and mass flux. We then state and prove a series of lemmas, all of which are building blocks for the proofs of Theorems 1 and 2. In §3, we discuss two examples in the literature for which our theorems hold. We also include details of some examples illustrating that the converse to the main theorem is particularly subtle; in §4 we discuss these cases. This includes discussion of necessary conditions for there to be unbounded oscillations or boundedness of orbits in a zero flux case. We finish with a discussion of possible extensions of the results in §5.

1.1. *Statement of main definitions and results.* Recall that an *interval exchange transformation* (IET) on a finite interval J is a piecewise linear invertible map $\theta \mapsto S(\theta) = \theta + \phi_k(\theta)$ such that ϕ_k is constant on a finite partition of J into intervals $\theta \in I_k$. Generic properties of IETs are relatively well understood [**4, 13, 17**]. We use some of these known results about IETs to understand boundedness of orbits of CETs, a particular case of a cone isometry transformation (CIT) that we now define.

Definition 1. (Cone isometry transformations) Let $X \subset \mathbb{C}$ be an unbounded polygonal region with a partition $\{P_j\}_{j\in N}$, a finite collection of mutually disjoint convex polygonal subsets of $\mathbb{C}$ indexed by N. Let $\{T_j\}_{j\in N}$ be a collection of isometries $z \mapsto u_j z + b_j$, where $b_j, u_j \in \mathbb{C}$, and $|u_j| = 1$. The collection of maps $T = \{T_j|_{P_j}\}_{j\in N}$ we call a *cone isometry transformation.*

The sets $\{P_j\}_{j\in N}$ are called the *domains* of T. The CIT acts on $X = \bigcup_{j\in N} P_j$ by

$$T(z) = T_j(z) \quad \text{if } z \in P_j.$$

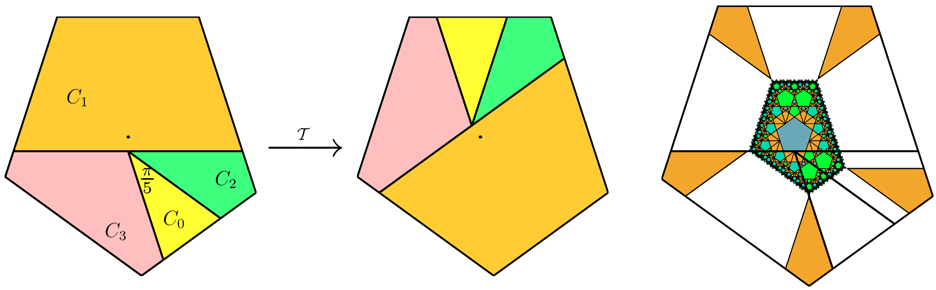

FIGURE 2. The cone exchange T from [**1**] restricted to a large invariant pentagon. The collection of periodic cells fills the smaller central pentagon (the right figure) and five trapezia, with the black lines indicating the boundaries between differently coded periodic cells. The set of all aperiodic cells has non-zero measure and comprises the white region. Outside of the periodic central pentagon, there is a foliation of one-dimensional invariant curves (not shown) that exhaust $\mathbb{C}$.

We define a *cone* (*strip*) to be the unbounded intersection of a finite number of open half-planes, such that the unbounded boundary lines are non-parallel (parallel). The unbounded pieces of the partition can include both cones and strips.

The action of T induces a map $\hat{T}$ on a subset of the circle (at ∞). We define the *projection* $\Pi(X)$ onto the circle by

$$\theta \in \Pi(X) \Leftrightarrow \{Re^{i\theta} \mid R > 0\} \cap X \text{ is unbounded.}$$

Consider a cone isometry $T : X \to X$ and define the *map induced from T by projection onto the circle at infinity* to be

$$\hat{T}(\theta) = \lim_{R\to\infty} \arg(T(Re^{i\theta})), \tag{1}$$

where $\hat{T}$ is defined on $\Pi(X) \subset S^1 = [0, 2\pi)$. Note that cones map to non-empty intervals, while strips map to points under Π. We show in Lemma 1 that this map is well defined apart from a finite number of angles $\Theta_{\text{dis}} \subset [0, 2\pi)$. In fact, it is well defined independent of choice of origin as well.

We will make a technical assumption that the CIT is *regular*, i.e. we assume that all directions of strip boundaries are mapped under T to directions of one or other of the bordering cone boundaries†. We concentrate from here on a special class of CITs such that $\hat{T}$ is invertible and $X = \mathbb{C}$, although the definitions can clearly be adapted to more general convex unbounded $X \subset \mathbb{C}$.

Definition 2. (Cone exchange transformation) We say that a regular CIT $T : \mathbb{C} \to \mathbb{C}$ is a *cone exchange transformation* if $\hat{T}$ is invertible.

While in this paper will not use it, it is worth noting that the family of CETs is closed on taking the first return map to an infinite cone.

An important quantity for CETs is the *average radial flux at infinity* (we sometimes just refer to this as the *flux*) defined by

$$\Phi_r^\infty = \lim_{R\to\infty} \frac{1}{2\pi} \int_0^{2\pi} [|T(Re^{i\theta})| - R]\, d\theta. \tag{2}$$

† Regularity implies that for typical T, far enough from the origin all points enter a cone after a single iterate and then have angles that are approximately determined by iterates of the interval exchange $\hat{T}$.

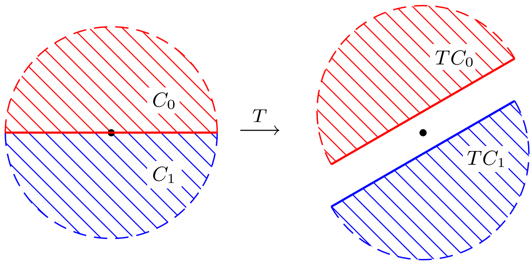

FIGURE 3. *A two half-plane map* $T : \mathbb{C} \to \mathbb{C}$. In this illustration, the upper and lower half-planes C_0 and C_1 are separated by distinct translation vectors and then they are rotated by θ. The mass flux $\Phi_m(R) > 0$ and the radial flux is also $\Phi_r^\infty > 0$. It is shown in [**5**] that this system is globally repelling.

The asymptotic dynamics of the CET is determined by the following theorem. For clarity we state the main results; proofs, examples and further discussion are given in later sections.

THEOREM 1. (Flux determines asymptotic behaviour) *Suppose that* $T : \mathbb{C} \to \mathbb{C}$ *is a CET and* $\hat{T}$ *is uniquely ergodic.*

(a) *If* $\Phi_r^\infty > 0$ *then there is an* $\rho > 0$ *such that:*

  (a1) *for all* $|z| > \rho$ *we have*

$$\lim_{k\to\infty} |T^k(z)| = \infty;$$

  (a2) *the ergodic averages of* $\arg(T^k(z))$ *are determined by the induced transformation at infinity* $\hat{T}$*; namely, for any integrable observable* $F : S^1 \to \mathbb{R}$ *and all* $|z| > \rho$*, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} F(\arg(T^k(z))) = \frac{1}{2\pi} \int_0^{2\pi} F(\theta)\, d\theta.$$

(b) *If* $\Phi_r^\infty < 0$ *then there is a* $\rho > 0$ *such that the system has a global attractor; namely, for all* $z \in \mathbb{C}$,

$$\limsup_{k\to\infty} |T^k(z)| < \rho.$$

Theorem 1 shows that, subject to an assumption on $\hat{T}$, $\Phi_r^\infty > 0$ implies unboundedness of all forward trajectories that start sufficiently far from the origin, while $\Phi_r^\infty < 0$ implies boundedness of forward trajectories. The proof of Theorem 1 is based on some detailed estimates and the uniform ergodicity theorem; in essence it generalizes the idea of the proof in [**5**] for the specific map shown in Figure 3. As we show in §3, our results can be used to recover several results from the literature. By using generic unique ergodicity for the induced map $\hat{T}$ we obtain the following result, precise details and proof of which are also deferred to §2.

THEOREM 2. *There is a natural parameterization of the set of CETs on* $\mathbb{C}$ *with* $n \geq 2$ *cones, such that* $\hat{T}$ *is uniquely ergodic and* $\Phi_r^\infty \neq 0$ *on a set of parameters with full Lebesgue measure. Hence there is a full Lebesgue measure set of parameters such that precisely one of (a) or (b) from Theorem 1 holds.*

However, there are many CETs that do not satisfy either hypothesis (a) or (b) of Theorem 1 and many of these are important in applications such as dual billiards. These include cases of CETs that are non-uniquely ergodic and/or have zero flux, $\Phi_r^\infty = 0$; see §4.

2. *Properties of cone exchange transformations*

We start with the observation that for any CET, at least one of the domains of X must be unbounded and convex, and hence it must be a cone or a strip outside a bounded region.

Assume the unbounded regions are comprised of $n > 0$ cones and $m \geq 0$ strips. We write the set of unbounded domains P_k where $k \in \{1, \ldots, n\}$ are the cones and P_{n+k} for $k \in \{1, \ldots, m\}$ the strips. The unbounded boundaries of P_k are subsets of $(iw_k + R)e^{i\alpha_k}$ and $(iv_k + R)e^{i\beta_k}$ for $R > 0$, where $w_k, v_k, \alpha_k, \beta_k$ are constants. For the cones P_k we can assume without loss of generality that $\alpha_k < \beta_k = \alpha_{k+1} \leq \alpha_k + \pi$, while for the strips we have $\alpha_k = \beta_k$. We write

$$\Theta_{\mathrm{dis}} = \{\alpha_k : 1 \leq k \leq m\}$$

to denote the discontinuity set of $\hat{T}$; this is clearly of zero measure. The CET on each P_k can be written

$$T_k(z) = e^{i\phi_k} z + u_k e^{i\tau_k} \quad \text{for } z \in P_k \tag{3}$$

for some $\phi_k \in [0, 2\pi)$ and $t_k = u_k e^{i\tau_k} \in \mathbb{C}$.

We think of the far-field behaviour of any CET as being determined by the data

$$\{(\alpha_k, \beta_k, \tau_k, \phi_k, w_k, v_k, u_k)\}_{k=1}^n \in ([0, 2\pi)^4 \times \mathbb{R}^3)^n \tag{4}$$

subject to the constraint that $\hat{T}(\theta) = \theta + \phi_k$ for $\theta \in [\alpha_k, \beta_k)$ is invertible and the P_k for $k = 1, \ldots, m+n$ form a partition of X outside a bounded region. One can measure how far this is from a simple exchange on angles by defining

$$U_{\max} = \max\{|u_k|, |v_k|, |w_k| : k = 1, \ldots, m+n\}$$

and noting that $U_{\max} = 0$ implies that $T(Re^{i\theta}) = Re^{i\hat{T}(\theta)}$, i.e. the CET is simply the IET applied to the angle. We show in the next lemma that $\hat{T}$ can be written using these coordinates.

LEMMA 1. *For any CET $T : \mathbb{C} \to \mathbb{C}$ the map $\hat{T}(\theta)$ (1) is defined on $S^1 = [0, 2\pi)$ for $\theta \notin \Theta_{\mathrm{dis}}$. The map is equal to the IET defined for $\theta \in \Pi(P_k)$ by*

$$\hat{T}(\theta) = \theta + \phi_k$$

whenever $\theta \in (\alpha_k, \beta_k)$.

Proof. Fix any $\theta \notin \Theta_{\mathrm{dis}}$; then $\{Re^{i\theta} | R > 0\} \cap P_k$ is unbounded for some k corresponding to $\theta \in (\alpha_k, \beta_k)$, and for this k we have

$$\hat{T}(\theta) = \lim_{R \to \infty} \arg(e^{i\phi_k}(w + Re^{i\theta}) + t_k) = \theta + \phi_k.$$

The map $\hat{T}$ is an interval exchange transformation if T is almost everywhere invertible (although the converse is not necessarily true). □

Any planar isometry will locally preserve 2D Lebesgue measure, and if T is injective then ℓ is invariant on the range of T: $\ell(T^{-1}(A)) = \ell(A)$ for all measurable $A \subset T(X)$. However, if X has infinite Lebesgue measure, $T : X \to \mathbb{C}$ can be injective but not surjective†.

As we will compare the action of $T(z)$ to that of $\hat{T}(\arg(z))$ we define the set of 'good angles' on the circle of radius R by

$$\Theta_{\text{good}}(R) = \{\theta : \text{ there is a } k \text{ so that } Re^{i\theta} \in P_k \text{ and } \theta \in [\alpha_k, \beta_k)\}$$

and its complement $\Theta_{\text{bad}}(R)$. Similarly define X_{good} by

$$X_{\text{good}} = \bigcup_k (P_k \cap \Pi^{-1}((\alpha_k, \beta_k))) = \bigcup_{R>0} Re^{i\Theta_{\text{good}}(R)}.$$

We note the following lemma.

LEMMA 2. *For all $R' > R$, the set of bad angles $\Theta_{\text{bad}}(R')$ is contained within a δ-neighbourhood of Θ_{dis} where*

$$\delta < \frac{\pi U_{\max}}{R}.$$

Proof. Note that the bad angles consist of θ such that $Re^{i\theta}$ lies within the strips

$$\{(is + r)e^{i\alpha_k} : r > 0, |s| < |w_k|\} \cup \{(is + r)e^{i\beta_k} : r > 0, |s| < |v_k|\}.$$

Hence, as $|w_k|, |v_k| \le U_{\max}$, by elementary trigonometry the set of bad angles on a circle of radius R is at most a $(\pi U_{\max}/R)$-neighbourhood of Θ_{dis}. □

The next lemma relates the dynamics of T to that of $\hat{T}$. Suppose that $T : \mathbb{C} \to \mathbb{C}$ is a CET. Let $F_T(\theta)$ be the piecewise continuous function defined by

$$F_T(\theta) = u_k \cos(\tau_k - \theta - \phi_k) \tag{5}$$

for $\theta \in [\alpha_k, \beta_k)$.

LEMMA 3. *There is an R_0 such that if $R > R_0$ then one of the following holds:*

(a) *if $\theta \in \Theta_{\text{good}}(R)$ then*

$$||T(Re^{i\theta})| - R - F_T(\theta)| < \left(\frac{\pi^2}{8} + \frac{\pi}{2}\right)\frac{U_{\max}^2}{R} \tag{6}$$

and

$$|\arg(T(Re^{i\theta})) - \hat{T}(\theta)| < \frac{\pi U_{\max}}{2R}; \tag{7}$$

(b) *if $\theta \in \Theta_{\text{bad}}(R)$, then*

$$||T(Re^{i\theta})| - R - F_T(\theta)| < 2U_{\max}. \tag{8}$$

† Contrast this with the case where X has finite Lebesgue measure where, up to a set of zero measure, T is injective if and only if it is surjective.

Proof. First assume that $\theta \in \Theta_{\mathrm{good}}(R)$ and we pick k such that $Re^{i\theta} \in P_k$. It follows that

$$\hat{T}(\theta) = \theta + \phi_k \tag{9}$$

and

$$Se^{i\theta} = T(Re^{i\theta}) = e^{i\phi_k} Re^{i\theta} + u_k e^{i\tau_k}.$$

Multiplying the above equation by $e^{-i\psi}$ and taking real and imaginary parts, we have

$$S = R\cos(\theta + \phi_k - \psi) + u_k \cos(\tau_k - \psi) \tag{10}$$

$$0 = R\sin(\theta + \phi_k - \psi) + u_k \sin(\tau_k - \psi). \tag{11}$$

From (11) we have

$$\sin(\theta + \phi_k - \psi) = -\frac{u_k}{R}\sin(\tau_k - \psi).$$

Fixing an $R_0 > U_{\max}$, then for any $R > R_0$, by the intermediate value theorem there is a solution ψ to this equation with

$$|\theta + \phi_k - \psi| < \pi/2 \tag{12}$$

such that

$$\frac{2}{\pi}|\theta + \phi_k - \psi| < \sin|\theta + \phi_k - \psi| < \frac{U_{\max}}{R} \tag{13}$$

meaning that $|\theta + \phi_k - \psi| < \pi U_{\max}/(2R)$. Hence, using (9), we have (7). From (10) we have

$$S - R - u_k\cos(\tau_k - \psi) = R\cos(\theta + \phi_k - \psi) - R$$

so that

$$|S - R - u_k\cos(\tau_k - \psi)| = R|1 - \cos(\theta + \phi_k - \psi)| = R(1 - \cos(\theta + \phi_k - \psi))$$

where the last equality follows from (12). From $1 - \cos x \le x^2/2$ we have

$$|S - R - u_k\cos(\tau_k - \psi)| \le R\frac{|\theta + \phi_k - \psi|^2}{2}$$

so, using (7), we have

$$|S - R - u_k\cos(\tau_k - \psi)| < R\frac{\pi^2 U_{\max}^2}{2(2R^2)} = \frac{\pi^2 U_{\max}^2}{8R}. \tag{14}$$

From the mean value theorem and (13) we have

$$|u_k\cos(\tau_k - \psi) - u_k\cos(\tau_k - \theta - \phi_k)| < U_{\max}|\theta + \phi_k - \psi| < \frac{\pi U_{\max}^2}{2R}. \tag{15}$$

Applying the triangle inequality to (14), (15) we get

$$|S - R - u_k\cos(\tau_k - \theta - \phi_k)| < \frac{\pi^2 U_{\max}^2}{8R} + \frac{\pi U_{\max}^2}{2R}$$

which can be expressed as (6).

In the case of $\theta \in \Theta_{\mathrm{bad}}(R)$ one cannot make a uniformly better estimate than

$$|\arg(T(Re^{i\theta})) - \hat{T}(\theta)| \le \pi$$

(by suitable choice of the argument). From this and (10) one obtains (8). □

2.1. *Flux for cone exchanges.* Suppose that $T:\mathbb{C}\to\mathbb{C}$ is a CET. Define the *asymptotic radial flux* of T to be

$$\Phi_r^\infty = \lim_{R\to\infty} \Phi_r(R) \tag{16}$$

where the *radial flux* is given by

$$\Phi_r(R) = \frac{1}{2\pi}\int_0^{2\pi} [|T(Re^{i\theta})| - R]\, d\theta.$$

One can verify that the limit in (16) exists and Φ_r^∞ is finite for any cone exchange; it is the average change in radius on a circle of radius R.

Using the estimate (6) and the definition of F_T (5), we have

$$\begin{aligned}\Phi_r(R) &= \frac{1}{2\pi}\sum_{k=1}^{n}\int_{\theta=\alpha_k}^{\beta_k} u_k \cos(\tau_k - \theta - \phi_k)\, d\theta + E_1 + E_2\\ &= \frac{1}{2\pi}\int_{\theta=0}^{2\pi} F_T(\theta)\, d\theta + E_1 + E_2\end{aligned}$$

where the error term E_1 is from integrating the error from (6) and E_2 is the error from those angles that lie in $\Theta_{\mathrm{bad}}(R)$. One can verify that both of these can be estimated by C/R uniformly in z, where C is a constant; hence

$$\Phi_r^\infty = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} F_T(\theta)\, d\theta. \tag{17}$$

Let $B_R(x)$ denote the closed ball of radius R centred on x, and let $B_R = B_R(0)$. Suppose that $T:\mathbb{C}\to\mathbb{C}$ is a CET and define the *mass flux* of T to be the total net loss of area by B_R under iteration by T; i.e. let

$$\Phi_m(R) = \mu(T^{-1}(B_R)\cap B_R^c) - \mu(T^{-1}(B_R^c)\cap B_R), \tag{18}$$

where μ denotes Lebesgue measure restricted to X. We can relate the mass flux and the radial flux as follows.

LEMMA 4. *Suppose that $T:\mathbb{C}\to\mathbb{C}$ is a CET, then the asymptotic radial flux and the mass flux are related by*

$$\Phi_r^\infty = \lim_{R\to\infty}\frac{\Phi_m(R)}{2\pi R}.$$

Proof. We wish to compute

$$\Phi_m(R) = \mu(T^{-1}(B_R)\cap B_R^c) - \mu(T^{-1}(B_R^c)\cap B_R),$$

the net mass flux away from 0; note that

$$\Phi_m(R) = \sum_k \Phi_{m,k}(R)$$

where

$$\Phi_{m,k} = \mu(T^{-1}(B_R)\cap B_R^c\cap P_k) - \mu(T^{-1}(B_R^c)\cap B_R\cap P_k),$$

hence

$$\Phi_{m,k} = \int_{\theta=\alpha_k}^{\beta_k}\int_{r=R}^{|T_k(Re^{i\theta})|} r\, dr\, d\theta + E_k(R)$$

where $E_k(R)$ is an error term caused by terms in $\Theta_{\mathrm{bad}}(R)$. We can bound this by a square whose side is the largest translation; namely, $|E_k(R)| < U_{\max}^2$ uniformly in R. Hence

$$\Phi_{m,k} = \int_{\theta=\alpha}^{\beta} \frac{1}{2}[|T(Re^{i\theta})|^2 - R^2]\, d\theta + E$$

with $E < nU_{\max}^2$, meaning that

$$\Phi_m = \sum_k \int_{\theta=\alpha_k}^{\beta_k} \frac{1}{2}[|Re^{i\theta+i\phi_k} + u_k e^{i\tau_k}|^2 - R^2]\, d\theta + O(1)$$

and so, by applying (6),

$$\begin{aligned} &= \sum_k \int_{\theta=\alpha_k}^{\beta_k} R\, u_k \cos(\tau_k - \theta - \phi_k)\, d\theta + O(1) \\ &= R \int_{\theta=0}^{2\pi} F_T(\theta)\, d\theta + O(1) = 2\pi R\, \Phi_r(R) + O(1), \end{aligned}$$

and hence $\Phi_r(R) = \Phi_m(R)/(2\pi R) + O(1/R)$. In particular

$$\Phi_r^\infty = \lim_{R\to\infty} \frac{\Phi_m(R)}{2\pi R}. \qquad \square$$

LEMMA 5. *Suppose that $T : \mathbb{C} \to \mathbb{C}$ is an invertible cone exchange and $\Phi_m(R) = 0$ outside some finite R; then the asymptotic radial flux is also zero. Moreover, suppose that there are radii $0 < R_1 < R_0$ such that:*

(a) *T is bijective from the preimage of the annulus $A = B_{R_0} \cap B_{R_1}^c$ onto itself;*

(b) *there are no points that skip over the annulus A, i.e.*

$$T(B_{R_1}) \cap B_{R_0}^c = T(B_{R_0}^c) \cap B_{R_1} = \emptyset. \tag{19}$$

Then it follows that

$$\Phi_m(R_0) = \Phi_m(R_1).$$

Proof. Let $A = B_{R_1} \cap B_{R_0}^c$ be the annulus and suppose that T restricted to $T^{-1}(A)$ is both injective and surjective onto A. Since T is piecewise isometric and invertible on A, it is measure preserving in the sense that for any measurable subset $X \subset A$ we have $\mu(T^{-1}(X)) = \mu(X)$. Note that

$$\mu(T^{-1}(A)) = \mu(T^{-1}(A) \cap (A \,\dot\cup\, A^c)) = \mu(T^{-1}(A) \cap A) + \mu(T^{-1}(A) \cap A^c)$$

while also

$$\begin{aligned} \mu(A) &= \mu(A \cap (T^{-1}(A) \,\dot\cup\, T^{-1}(A^c) \,\dot\cup\, Z)) \\ &= \mu(A \cap T^{-1}(A)) + \mu(T^{-1}(A) \cap A^c) + \mu(A \cap Z) \end{aligned}$$

where Z is the set of points that are not a preimage of T and $\dot\cup$ indicates disjoint union. Hence

$$\mu(T^{-1}(A) \cap A^c) = \mu(T^{-1}(A^c) \cap A).$$

If we write (using assumptions (19))

$$T^{-1}(A) \cap A^c = (T^{-1}(B_{R_0}) \cap B_{R_0}^c) \,\dot\cup\, (T^{-1}(B_{R_1}^c) \cup B_{R_1})$$

and

$$T^{-1}(A^c) \cap A = (T^{-1}(B_{R_0}^c) \cap B_{R_0}) \dot{\cup} (T^{-1}(B_{R_1}) \cup B_{R_1}^c),$$

then it follows that

$$\mu(T^{-1}(B_{R_0}) \cap B_{R_0}^c) + \mu(T^{-1}(B_{R_1}^c) \cap B_{R_1})$$
$$= \mu(T^{-1}(B_{R_0}^c) \cap B_{R_0}) + \mu(T^{-1}(B_{R_1}) \cap B_{R_1}^c)$$

which can be expressed as

$$\Phi_m(R_0) = \Phi_m(R_1).$$ □

2.2. *Boundedness and radial flux.* In this section we give a proof of Theorem 1. We start with a lemma that gives a bound on minimum return times to the set of bad angles.

LEMMA 6. *Suppose that $T : \mathbb{C} \to \mathbb{C}$ is a CET and $\hat{T}$ is minimal. Given any $N > 0$, there is a $\rho > 0$ such that for any $|z| > \rho$ and $z \in X_{\text{bad}}$, $T^k(z) \notin X_{\text{bad}}$ for all except at most n values of $k \in \{0, 1, \ldots, N\}$.*

Proof. Fix on some $N > 0$ we can think of as arbitrarily large. As there are no periodic orbits for $\hat{T}$, all $\theta \in \Theta_{\text{dis}}$ must leave Θ_{dis} after at most n iterates and then never return. To simplify the exposition we assume that they do leave immediately and never return. Consider the closest approach to Θ_{dis} made by any starting point in Θ_{dis} after N iterates; i.e. for fixed N we define

$$\epsilon(N) = \min\{|\hat{T}^k(\theta) - \hat{\theta}| : k = 1, \ldots, N, \ (\theta, \hat{\theta}) \in \Theta_{\text{dis}}^2\}$$

and note that $\epsilon > 0$. Lemma 2 means we can pick a $\rho_1 > 0$ such that for all $R > \rho_1$, $\Theta_{\text{bad}}(R)$ is within an $\epsilon/4$-neighbourhood of Θ_{dis}. Lemma 3 means we can find a $\rho_2 > \rho_1$ such that if $R > \rho = \rho_2 + NU_{\max}$ and $\theta \in \Theta_{\text{bad}}(\rho_1)$ (writing $T^k(Re^{i\theta}) = R_k e^{i\theta_k}$) then

$$R_{k+1} > \rho_2 \quad \text{and} \quad |\theta_{k+1} - \hat{T}(\theta_k)| < \frac{\epsilon}{4N}$$

for $k = 1, \ldots, N-1$, so

$$|\theta_{k+1} - \hat{T}^k(\theta_1)| < \frac{\epsilon}{4}.$$

Hence for any $R > \rho$ and $\theta \in \Theta_{\text{bad}}(R)$, $T^k(Re^{i\theta})$ cannot return to within $\epsilon/2$ of X_{bad} during its first N iterates. For the more general case, we need to exclude the first n iterates. □

We will need a uniform ergodic theorem **[14]** stated here (without proof) for convenience.

THEOREM 3. (Uniform ergodic theorem) *Pick any integrable $F : S^1 \to \mathbb{R}$ and suppose $\hat{T} : S^1 \to S^1$ is uniformly ergodic for Lebesgue measure. Then for any $\epsilon > 0$ there is an M such that*

$$\left| \frac{1}{N} \sum_{k=0}^{N-1} F(\hat{T}^k(\theta_0)) - \frac{1}{2\pi} \int_0^{2\pi} F(\theta)\, d\theta \right| < \epsilon$$

for all $\theta_0 \in S^1$ and all $N > M$.

Proof of Theorem 1. We commence with a proof of (a1). Pick any $\epsilon > 0$; by Theorem 3 there is an $M(\epsilon)$ such that

$$\left| \frac{1}{N} \sum_{k=0}^{N-1} F_T(\hat{T}^k(\theta)) - \Phi_r^\infty \right| < \epsilon \tag{20}$$

for all $N > M$ and all θ. We will pick an $N > M$ such that in addition

$$\frac{N\Phi_r^\infty}{3} > 2\epsilon + 2U_{\max}. \tag{21}$$

By Lemma 6 we can find a $\rho_1 > 0$ such that $|z| > \rho_1$ and $z \in X_{\text{bad}}$ implies that $T^k(z) \notin X_{\text{bad}}$ for $k = 1, \ldots, N-1$.

Let us write $T^k(z) = R_k e^{i\theta_k}$. By Lemma 3 there is a $\rho_2 > \rho_1$ such that for all $|z| > \rho_2$ and $k = 1, \ldots, N-1$ we have

$$|\theta_k - \hat{T}^{k-1}(\theta_1)| < \frac{\epsilon}{NU_{\max}}. \tag{22}$$

Defining $d_k = R_{k+1} - R_k$ and assuming that $R_k > \rho_2$ for $k = 1, \ldots, N-1$, we have

$$|d_k - F_T(\theta_k)| < \frac{\epsilon}{N} \tag{23}$$

for all $R_0 > \rho_2$ and $k = 1, \ldots, N-1$, while for the 'bad angle' θ_0 we have

$$|d_0 - F_T(\theta_0)| < 2U_{\max}. \tag{24}$$

From (22) we have

$$|F_T(\theta_k) - F_T(\hat{T}^{k-1}(\theta_1))| < \frac{\epsilon}{N}. \tag{25}$$

We now proceed by estimating the change in R over an orbit segment of length N that remains outside a disk of radius ρ_2. From (23) we have, defining $D_N = R_N - R_0$, that

$$\begin{aligned} D_N &= d_{N-1} + d_{N-2} + \cdots + d_1 + d_0 \\ &= \sum_{k=0}^{N-1} (d_k - F_T(\theta_k)) + \sum_{k=0}^{N-1} F_T(\theta_k). \end{aligned}$$

Choosing $\hat{\theta}$ such that $\hat{T}(\hat{\theta}) = \theta_1$, we write

$$D_N = \sum_{k=0}^{N-1} (d_k - F_T(\theta_k)) + \sum_{k=0}^{N-1} (F_T(\theta_k) - F_T(\hat{T}^k(\hat{\theta}))) + \sum_{k=0}^{N-1} F_T(\hat{T}^k(\hat{\theta})). \tag{26}$$

Note that from (25) we have

$$\left| \sum_{k=0}^{N-1} (F_T(\theta_k) - F_T(\hat{T}^k(\hat{\theta}))) \right| < \epsilon,$$

while from (23), (24) we have

$$\left| \sum_{k=0}^{N-1} (d_k - F_T(\theta_k)) \right| < 2U_{\max} + \epsilon.$$

(Note that the $2U_{\max}$ term is absent if θ_k remains in Θ_{good} for all k.) Finally, from (20) we have

$$\sum_{k=0}^{N-1} F_T(\hat{T}^k(\hat{\theta})) > N(\Phi_r^\infty - \epsilon).$$

Putting this together into (26), we have

$$D_N > N(\Phi_r^\infty - \epsilon) - 2\epsilon - 2U_{\max}. \tag{27}$$

Hence, taking $\epsilon = \Phi_r^\infty/3 > 0$ and using assumption (21), there is an $\epsilon > 0$ such that $D_N > N\Phi_r^\infty/3$ and so

$$R_N > R_0 + N\Phi_r^\infty/3 \tag{28}$$

implying that

$$R_{kN} > R_0 + kN\Phi_r^\infty/3. \tag{29}$$

Hence $|T^{Nk}(z)|$ must tend to infinity as $k \to \infty$.

For the proof of (a2), first pick any $F : S^1 \to \mathbb{R}$ that is integrable. For any $M > 0$, there is an $R > \rho > 0$ such that (i) if $|T^k(z)| > R$ for all k then the orbit of z visits the bad set at most once every M iterates and (ii) we have uniform convergence of Φ to its ergodic mean. Hence for any integrable F we have

$$\left| \lim_{M\to\infty} \frac{1}{M} \sum_{k=0}^{M-1} F(\arg(T^k(z))) - \frac{1}{2\pi} \int_0^{2\pi} F(\theta)\, d\theta \right| < \frac{C}{M}.$$

By using (a1) we have that the proportion of visits to Θ_{bad} goes to zero as $k \to \infty$; hence one can choose M arbitrarily large and obtain the result.

The proof of (b) is obtained by noting that in a similar way to the proof of (a1) we have

$$D_N < N(\Phi_r^\infty + \epsilon) + 2\epsilon + U_{\max}, \tag{30}$$

so if $\Phi_r^\infty < 0$ one can find a $\rho > 0$ such that all orbits starting in $|z| > \rho$ have

$$|T^N(z)| < |z| - \frac{\Phi_r^\infty}{3}.$$

Hence any initial condition starting outside a ball of radius ρ must enter it in finite time and thereafter remain there. A consequence of this is that there is a bounded set that contains the ω-limits for almost initial conditions. □

We now prove the second main result, Theorem 2; we first state and prove a lemma inspired by Boshernitzan (personal communication). Recall (e.g. [**17**]) that any $(n+1)$-interval exchange J on $[\beta_1, \beta_1 + 2\pi)$ with partition

$$\beta_1 < \cdots < \beta_{n+1} < \beta_{n+2} := \beta_1 + 2\pi \tag{31}$$

that permutes the order of the $(n+1)$ intervals according to the permutation $\Pi \in S(\{1, \ldots, n+1\})$ can be written explicitly as

$$J(\theta) = \theta + \phi_k \quad \text{for } \theta \in [\beta_k, \beta_{k+1}),$$

where $\phi_k = \beta'_k - \beta_k$ for $\beta'_1 = \beta_{\Pi^{-1}(1)}$ and

$$\beta'_k = \beta_1 + \sum_{j=1}^{\Pi^{-1}(k)-1} (\beta_{\Pi^{-1}(j)+1} - \beta_{\Pi^{-1}(j)})$$

for $k > 1$†.

LEMMA 7. *Suppose that $I : S^1 \to S^1$ is an n-interval exchange on a circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ with partition*

$$\alpha_1 < \cdots < \alpha_n < \alpha_{n+1} := \alpha_1 + 2\pi. \tag{32}$$

Then I can be written as an n-interval exchange on $[\alpha_1, \alpha_1 + 2\pi)$ with partition $\{\alpha_k\}_{k=1}^n$ that fixes the interval $[\alpha_1, \alpha_2)$, composed with a rotation by $\xi \in S^1$. For Lebesgue almost all $(\alpha_1, \ldots, \alpha_n, \xi)$ the map I is uniquely ergodic.

Proof. Consider the preimage α' such that $I_+(\alpha') = \alpha_1$ and set $\alpha' + \xi = \alpha_1$. Note that I can be uniquely written as the composition of an IET on $[\alpha_1, \alpha_1 + 2\pi)$ with partition (32) and permutation $\tilde{\Pi}$ that fixes $\tilde{\Pi}(1) = 1$, composed with a rotation by $\xi \in S^1$. Hence I is parameterized uniquely by $\tilde{\Pi}$, $\xi \in S^1$ and $(\alpha_1, \ldots, \alpha_n)$ satisfying (32). Boshernitzan (personal communication) has a proof based on [**13, 17**] that an interval exchange on S^1 parameterized in this way will be uniquely ergodic for Lebesgue almost all ξ, independent of choice of α_k in (32). We prove here a weaker result than this.

On identifying S^1 with the interval $[\alpha_1, \alpha_1 + 2\pi)$, I can also be viewed as an interval exchange J on at most $n + 1$ intervals, introducing an additional discontinuity at α' if $\alpha' \notin \{\alpha_1, \ldots, \alpha_n\}$. The trajectories of I and J are dynamically equivalent, as are their respective ergodic invariant measures. More precisely, for any partition $(\alpha_1, \ldots, \alpha_n)$ there is a full measure set of ξ for which the preimage α' of α_1 under I has $\alpha' \in (\alpha_k, \alpha_{k+1})$ for some $1 \le k \le n$. Hence, for a full measure set of ξ, I can be written as an $(n + 1)$-interval exchange J on $[\alpha_1, \alpha_1 + 2\pi)$, with a partition defined by

$$\beta_j = \begin{cases} \alpha_j & \text{for } j < k+1, \\ \alpha' = \alpha_1 - \xi & \text{for } j = k+1, \\ \alpha_{j-1} & \text{for } j > k+1, \end{cases} \tag{33}$$

for $j = 1, \ldots, n + 1$, which clearly satisfies (31).

Consider the permutation Π of the $n + 1$ intervals by J (note that this is not $\tilde{\Pi}$ unless $\xi = 0$). If Π is reducible in the sense of [**17**] then there is an invariant subinterval $[\beta_1, \beta_k)$ for some $1 < k < n + 1$. This means there is an invariant subinterval $[\alpha_1, \alpha_{k'})$ for some $1 < k' < n$ (only the α_k are points of discontinuity for the map on S^1) which means in particular that $\alpha' = \alpha_\ell$ for some $1 \le \ell < k'$. However, this can only happen for a finite set of ξ; hence for almost all $(\alpha_1, \ldots, \alpha_n, \xi)$, the permutation Π for J is irreducible.

Given irreducibility of Π, [**17**] implies that for almost all choices of $(\beta_1, \ldots, \beta_{n+1})$, J (and hence I) is uniquely ergodic. However, a set has zero measure for $(\beta_1, \ldots, \beta_{n+1})$ if and only if it has zero measure for $(\alpha_1, \ldots, \alpha_n, \xi)$, as there is an invertible linear map (33) that relates the two $(n + 1)$-dimensional real spaces. Hence there is a full measure set of $(\alpha_1, \ldots, \alpha_n, \xi)$ for which I is uniquely ergodic. □

† Note that β'_k is such that $J_+(\beta_k) := \lim_{x \searrow \beta_k} J(x) = \beta'_k$.

Proof of Theorem 2. Take the parameterization of the far-field behaviour given by (4) subject to the constraint that the induced map $\hat{T}$ at infinity is an IET, which means that we can choose α_k as in the statement of Lemma 7, $\beta_k = \alpha_{k+1}$ and $\phi_k = \alpha'_k - \alpha_k + \xi$ for some $\xi \in [0, 2\pi)$ and permutation Π. By Lemma 7, for any $\tilde{\Pi}$ and almost all choices of α_k and ξ, the IET $\hat{T}$ is uniquely ergodic.

Now the flux of any CET that projects to $\hat{T}$ can be written as

$$\begin{aligned}\Phi_r^\infty &= \frac{1}{2\pi}\int_0^{2\pi} F_T(\theta)\, d\theta \\ &= \frac{1}{2\pi}\sum_k \int_{\alpha_k}^{\alpha_{k+1}} u_k \cos(\tau_k - \theta - \phi_k)\, d\theta \\ &= \frac{1}{2\pi}\sum_k u_k V_k\end{aligned}$$

where $V_k = \sin(\tau_k - \alpha_k - \phi_k) - \sin(\tau_k - \alpha_{k+1} - \phi_k)$. We fix an arbitrary k and claim that there is a full Lebesgue measure set of α_k, ξ, τ_k such that V_k is non-zero. To see this, observe that for a full measure of α_k there will be a countable set of $\tau_k - \phi_k$ and hence of ξ that gives $V_k = 0$. Hence $\Phi_r^\infty = 0$ on a non-degenerate linear subspace of $\{u_k\}$ and so the set of parameters of the CET for which $\Phi_r^\infty \neq 0$ has full measure. Hence for a full Lebesgue measure set of these parameters precisely one of (a) or (b) of Theorem 1 holds. □

3. *Examples*

We summarize some examples of planar CETs and some fundamental questions related to boundedness of orbits under iteration of the map. We also show how Theorem 1 covers results previously obtained in [**5, 7**].

Two half-plane map. Fix $\theta \in [0, 2\pi)$, $a, b \in \mathbb{C}$ and define $T : \mathbb{C} \to \mathbb{C}$ by

$$T(z) = \begin{cases} e^{i\theta}(z + a) & \text{if } z \in C_0 = \{z \mid \mathrm{Im}(z) \geq 0\}, \\ e^{i\theta}(z + b) & \text{if } z \in C_1 = \{z \mid \mathrm{Im}(z) < 0\}. \end{cases} \tag{34}$$

The general case of this map was studied in [**5**] where it was found that the map T is surjective (the rotated half-planes overlap) if and only if $\mathrm{Im}(a - b) \leq 0$ and it is injective (the rotated half-planes get separated) if and only if $\mathrm{Im}(a - b) \geq 0$, hence bijective if and only if $a - b \in \mathbb{R}$. Boshernitzan and Goetz found that in the non-injective case ($\mathrm{Im}(a - b) < 0$), the map is *globally attracting*, i.e. there exists $M > 0$ such that for all $z \in \mathbb{C}$, $|T^n z| \leq M$ for all sufficiently large n. In the non-surjective case ($\mathrm{Im}(a - b) > 0$), the map is *globally repelling*, i.e. there exists $M > 0$ such that $\lim_{n\to\infty} |T^n z| = \infty$ for all z satisfying $|z| \geq M$.

For the map (34) the flux can be computed to be simply

$$\Phi_r^\infty = 2(|a| + |b|) \cdot \mathrm{Im}(a - b).$$

The sign of the flux is the sign of $\mathrm{Im}(a - b)$. Thus the map acts by separating the upper and lower half-planes ($\mathrm{Im}(a - b) > 0$) if $\Phi_r^\infty > 0$. If the map acts by forcing the half-planes to overlap ($\mathrm{Im}(a - b) > 0$), then $\Phi_r^\infty < 0$. This recovers the main result from [**5**] in the case where θ is an irrational multiple of π.

The bijective case was more recently addressed in [**10**]. In that case, there exist periodic points in every neighbourhood of infinity. Moreover, almost all points are recurrent (i.e. for almost all Lebesgue $z \in \mathbb{C}$, an arbitrary neighbourhood of z contains an iterate of z). Whereas for θs that are rational multiples of π, all orbits are bounded, if $\theta/\pi \notin \mathbb{Q}$, then it remains an open question to exhibit an orbit escaping to infinity, or to show that all or almost all orbits remain bounded.

One can verify for the bijective case that the radial flux is zero and so Theorem 1 cannot be applied. By [**10**] infinity is neither a forward nor backward attractor; in every neighbourhood of it there are periodic points.

Bandpass sigma–delta map. Consider the map illustrated in Figure 4 defined on the plane $z \in \mathbb{C}$, written as

$$z' = e^{-i\phi} z + w(z) e^{-i\phi} \tag{35}$$

where ϕ is a parameter and w is a piecewise constant real-valued function

$$w(z) = \frac{2\cos\phi \,\operatorname{sgn}(\operatorname{Im}(z)) - \operatorname{sgn}(\operatorname{Im}(z e^{i\phi}))}{\sin\phi}$$

taking the values $(\pm 2\cos\phi \pm 1)/\sin\phi$. This map arises in a model of a bandpass sigma–delta modulator [**3, 8**] and was shown in [**7**] to possess a non-trivial global attractor. Note that this map is a rotation everywhere except on the lines r and $re^{-i\phi}$ for $r \in \mathbb{R}$. This splits the plane into four cones and the map becomes a cone exchange on these four cones. Assuming that $0 < \phi < \pi$, we compute the radial flux for this map. We note that

$$F_T(\theta) = \begin{cases} (2\cos\phi - 1)\cos(\theta)/\sin(\phi), & 0 < \theta < \pi - \phi, \\ (2\cos\phi + 1)\cos(\theta)/\sin(\phi), & \pi - \phi < \theta < \pi, \\ (-2\cos\phi + 1)\cos(\theta)/\sin(\phi), & \pi < \theta < 2\pi - \phi, \\ (-2\cos\phi - 1)\cos(\theta)/\sin(\phi), & 2\pi - \phi < \theta < 2\pi, \end{cases} \tag{36}$$

and integrating this gives

$$\Phi_r^\infty = \frac{1}{2\pi}\int_0^{2\pi} F_T(\theta)\, d\theta = -\frac{2}{\pi} \tag{37}$$

independent of ϕ. The induced map at infinity for this map is simply the rotation

$$\hat{T}(\theta) = \theta - \phi \pmod{2\pi}$$

and so for all irrational values of θ/π, $\hat{T}$ is uniquely ergodic. Hence we can apply Theorem 1 to prove that the system has a bounded attractor for all irrational ϕ/π with $0 < \phi < \pi$. It was shown in [**7**] that the geometric structure of the attractor is as shown in Figure 4 for any $\phi \in (2\pi/3, 4\pi/3)$; this result cannot be obtained from our Theorem 1. However, one should be able to weaken the statement of Theorem 1 to show that there is a bounded attractor in cases where ϕ/π is not irrational by showing that the average of F_T along any orbit of $\hat{T}$ is negative.

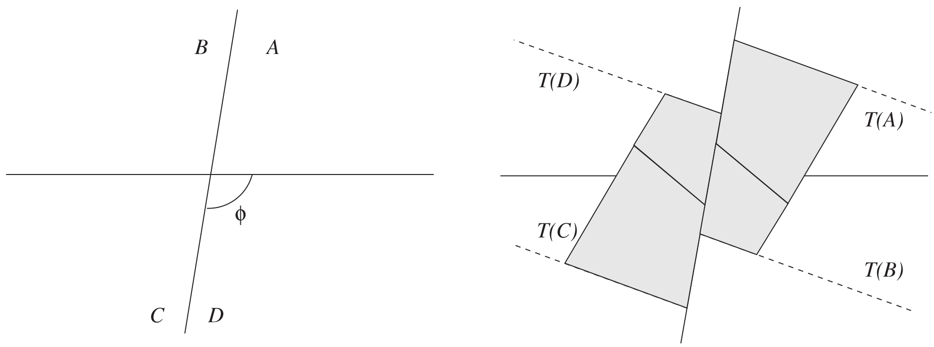

FIGURE 4. Illustration of (left) the partition and (right) the global attractor for the bandpass sigma–delta map (35) with $\phi = 1.8$. The map T is a rotation on each of the four cones A, B, C, D shown, and for this parameter value [7] shows that the global attractor is a union of two shaded parallelograms.

4. *Partial converse results and examples*

4.1. *An example with zero flux and unbounded orbits.*

The negation of Theorem 1 would mean that a necessary condition for an invertible map to have all orbits bounded is that $\Phi_r^\infty = 0$ and/or $\hat{T}$ is not uniquely ergodic.

In the next example we show the conditions for Theorem 1(a) are sufficient but not necessary for unboundedness of orbits.

The check-board quadrant exchange. This is an example of a CET that is injective, but not surjective on a bounded set. We partition the plane into four Cartesian quadrants $C_0, \ldots, C_3$ and define the map T to be rotation by $\pi/2$ followed by translations as follows:

$$T(z) = \begin{cases} iz - 1 - i & \text{if } z \in C_0 = \{z \mid \mathrm{Re}(z) \geq 0 \text{ and } \mathrm{Im}(z) \geq 0\}, \\ iz + 1 - i & \text{if } z \in C_1 = \{z \mid \mathrm{Re}(z) < 0 \text{ and } \mathrm{Im}(z) \geq 0\}, \\ iz + 1 + i & \text{if } z \in C_2 = \{z \mid \mathrm{Re}(z) < 0 \text{ and } \mathrm{Im}(z) < 0\}, \\ iz + i - 1 & \text{if } z \in C_3 = \{z \mid \mathrm{Re}(z) > 0 \text{ and } \mathrm{Im}(z) < 0\}. \end{cases} \tag{38}$$

The images of $C_0, \ldots, C_3$ are illustrated in Figure 5.

PROPOSITION 1. *The asymptotic radial flux of T defined by (38) is zero. However, all orbits of T diverge to infinity.*

Proof. One can verify the zero flux by direct calculation. Note that T acts symmetrically on each of the four quadrants. By identifying all points that can be reached by rotations of $\pi/2$ about the origin (i.e. identifying all quadrants with C_0), we obtain the factor map H on C_0 with a quadrant and a half-strip as atoms of H:

$$H : C_0 \to C_0, \quad z \mapsto \begin{cases} -1 + i + z & \text{if } \mathrm{Re}(z) \in [1, \infty), \\ 1 + i - iz & \text{if } \mathrm{Re}(z) \in [0, 1). \end{cases}$$

Pick any $z \in \mathbb{C}$ and let $l(z) = |\mathrm{Re}(z)| + |\mathrm{Im}(z)|$. Note that $l(H(z)) = l(z)$ if $z \in C_0 + 1$ and $l(H(z)) = l(z) + 2$ otherwise (where $C_0 + 1 = \{z + 1 \mid z \in C_0\}$). Since on the cone $C_0 + 1$, H acts as translation by $(-1 + i)$, every orbits spends only a finite time in $C_0 + 1$

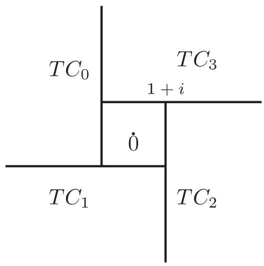

FIGURE 5. The images of the quadrants under the check-board quadrant exchange CET T. The square has vertices $\pm 1 \pm i$.

before leaving it, as the translation vector is perpendicular to the symmetry axis of the cone $C_0 + 1$. It follows that $l(H^n(z)) \to \infty$ as $n \to \infty$. Thus the H-orbit of z and hence the T-orbit of z diverges to infinity. □

We remark that one can modify $T(z)$ to obtain an injective but non-surjective CET for which some but not all orbits diverge to infinity. Let $V(z) = T(z) + 1 + i$. It follows from Proposition 2 (below) that almost all points in

$$G = \mathbb{C} - V(\mathbb{C}) = \{z \mid \text{Re}(z), \text{Im}(z) \in [0, 2]\}$$

have unbounded orbits. Since G consists of four lattice squares and V maps lattice squares onto lattice squares, it follows that none of the points in G can accumulate except at infinity. Hence, the orbit of G diverges to infinity. On the other hand, the reader may check that the lattice squares whose vertices are $3 + 4k + i$ ($k = 0, 1, \ldots$) are all periodic under $V(z)$.

While for V we were able to guarantee that orbits accumulating at infinity must actually diverge to infinity, in a more general setting we have a weaker proposition. In the case of injective cone exchanges with bounded gaps we can conclude that almost all points in G are unbounded.

PROPOSITION 2. *Let $T : \mathbb{C} \to \mathbb{C}$ be an injective and forward Lebesgue measure preserving map. Then the orbits of almost all points in $D = \mathbb{C} - T(\mathbb{C})$ are unbounded.*

Proof. Suppose that $G_R \subset D$ is the subset of all z such that $|T^n z| < R$ for all $n > 0$. We claim that

$$T^i(G_R) \cap T^j(G_R) = \emptyset \quad \text{for all } i, j \geq 0, i \neq j. \tag{39}$$

Since G_R does not have preimages, the equation holds for $i = 0$. In general, equation (39) holds by induction. Therefore the sets $\{G_R, T(G_R), T^2(G_R), \ldots\}$ are mutually disjoint. These sets are contained in a ball of radius R and they all have the same measure, meaning that $\mu(G_R) = 0$. □

4.2. *A case where $\hat{T}$ is not uniquely ergodic.* We consider a family of CETs with three parameters $(\alpha, \beta, w) \in S^1 \times S^1 \times \mathbb{C}$ such that $0 < \beta < \alpha < \pi$. This example is built around an IET $\hat{T}$ that is minimal but non-uniquely ergodic for Lebesgue measure, for certain parameter values [12]. We define four cones

$$\begin{aligned} P_0 &= C_{[0,\pi+\beta-\alpha)}, & P_1 &= C_{[\pi+\beta-\alpha,\pi)}, \\ P_2 &= C_{[-\pi,\beta-\alpha)}, & P_3 &= C_{[\beta-\alpha+2\pi)}, \end{aligned} \tag{40}$$

where $C_{[a,b)} = \{z \mid \arg(z) \in [a, b)\}$ and define $T : \mathbb{C} \to \mathbb{C}$ by

$$T(z) = \begin{cases} e^{i\alpha} z + w & \text{if } z \in P_0 \cup P_2, \\ -e^{i\alpha} z + w & \text{if } z \in P_1 \cup P_3. \end{cases} \tag{41}$$

We note that for $w = 0$ we have $T(Re^{i\theta}) = Re^{i\hat{T}(\theta)}$. Although we do not have a detailed proof, we suggest that there exist parameters α, β and w such that orbits of (41) have unbounded oscillation, as defined below. This is because $\hat{T}(\theta)$ is the family of maps with a subset I of parameters $(\alpha, \beta) \in I$ such that the map is non-uniquely ergodic and minimal [12]. The radial flux for this T can be seen to be 0 by explicit calculation. For a non-Lebesgue invariant measure μ' for $\hat{T}$ one can verify that the radial flux is typically non-zero relative to this μ' even if it is zero for Lebesgue measure. Now define the *asymptotic radial oscillation* of the orbit of z to be

$$\operatorname{osc}(z) = \limsup_{n\to\infty} |T^n z| - \liminf_{n\to\infty} |T^n z|.$$

We believe, but have not yet been able to prove, that this map has orbits with unbounded radial oscillation, i.e. one can choose z such that $\operatorname{osc}(z)$ is arbitrarily large.

5. *Discussion and open questions*

The CETs introduced here give a convenient framework in which the ergodic properties of IETs interplay with the two-dimensional behaviour of orbits of CET. The CETs offer a rich set of examples for which one can investigate many fundamental questions about the dynamics of piecewise isometries on unbounded domains.

One can generalize IET dynamics to two dimensions by taking direct or semidirect products of IETs, and this may give rise to unexpected phenomena such as unbounded return times to atoms, illustrated for example in [9]. Nonetheless, this will produce examples that do not behave like typical two-dimensional piecewise isometries. For example, a typical direct product of two IETs will be minimal while typical piecewise isometries are known to have many 'periodic islands'. The construction of CETs allows one to derive examples of PWIs from IETs with known properties, while retaining much of the typical behaviour of a two-dimensional piecewise isometry.

There are many open questions concerning the dynamics and asymptotic behaviour of orbits for CETs; the following lists some of these.

- *Can one extend Theorem 1 to cases where $\hat{T}$ is not uniquely ergodic?* For example, one should be able to define a flux for each invariant measure μ of $\hat{T}$ by

$$\Phi_r^\infty(\mu) = \frac{1}{2\pi} \int_0^{2\pi} F_T(\theta)\, d\mu(\theta)$$

  and then obtain similar results to (a) under the assumption that $\Phi_r^\infty(\mu) > 0$ for all invariant measures μ for $\hat{T}$ (one can clearly refine this to just ergodic invariant measures).

- *In the case where $\Phi_r^\infty = 0$, can one relate higher-order terms in the asymptotic expansion of $\Phi_m(R)$ to dynamical boundedness of orbits of T?* We note from Lemma 4 that the asymptotic expansion of $\Phi_m(R)$ has $2\pi R\, \Phi_r^\infty$ as the leading term; can one obtain information from the next terms in this expansion?
- *Can one characterize the structure of aperiodic and periodic orbits for CETs?* For example, can one show that typical zero flux CETs have at least one periodic orbit? The dynamics of piecewise isometries in two-dimensional bounded domains suggests that this may be the case; see for example [**1, 9, 16, 18**], where questions focus on the geometric structure of the partition into aperiodic and periodic orbits.
- *Can one generalize the results to unbounded cone isometries with overlap between cones?* In such cases, the map on the circle at infinity that is not an IET but rather an interval translation map (see for example [**6, 11**] and references therein). We expect that results such as Theorem 1 can be fairly easily generalized. Such maps arise in a model defined on four quadrants [**2**].
- *Can one understand local properties of return maps using CETs?* Note that the class of CETs is closed under taking the first return to map to any cone; the first return can have only a finite number of unbounded atoms, though it may have an infinite number of bounded atoms. Moreover, if $T \in \mathcal{P}$ and T_Δ denotes the return map of T, then $\hat{T}_\Delta$ is the interval exchange transformation that is the first return of the interval exchange map $\hat{T}_\Delta$; in this sense the class of maps we consider here remain a sensible choice to discuss renormalization of cone exchange transformations and should allow a good understanding of the dynamics near infinity.

Acknowledgements. We thank the Royal Society for the support of this research via a visitor grant that enabled AG to visit Exeter during 2007, and BIRS (Canada) for providing an opportunity to continue working on this in 2008. The authors benefited from numerous conversations with Pascal Hubert and Anthony Quas, and from some insightful comments and criticisms from a referee. We thank M. Boshernitzan for permission to mention an unpublished result related to Lemma 7.

REFERENCES

[1] P. Ashwin and A. Goetz. Polygonal invariant curves for a planar piecewise isometry. *Trans. Amer. Math. Soc.* **348** (2006), 373–390.

[2] P. Ashwin and A. Goetz. Invariant curves and explosion of periodic islands for piecewise isometries. *SIAM J. Appl. Dyn. Syst.* **4** (2005), 437–458.

[3] P. Ashwin, X.-C. Fu and J. Deane. Properties of the invariant disk packing in a model bandpass sigma–delta modulator. *Internat. J. Bifur. Chaos* **13** (2003), 631–641.

[4] A. Avila and G. Forni. Weak mixing for interval exchange transformations and translation flows. *Ann. of Math.* (2) **165** (2007), 637–664.

[5] M. Boshernitzan and A. Goetz. A dichotomy for a two parameter piecewise rotation. *Ergod. Th. & Dynam. Sys.* **23** (2003), 759–770.

[6] M. Boshernitzan and I. Kornfeld. Interval translation mappings. *Ergod. Th. & Dynam. Sys.* **15** (1995), 821–831.

[7] J. H. B. Deane. Global attraction in the sigma–delta piecewise isometry. *Dyn. Syst.* **17** (2002), 377–388.

[8] O. Feely. Nonlinear dynamics of bandpass sigma–delta modulation. *Proceedings of NDES (Dublin, 1995)*, pp. 33–36.

[9] A. Goetz and G. Poggiaspalla. Rotations by $\pi/7$. *Nonlinearity* **17** (2004), 1787–1802.

[10] A. Goetz and A. Quas. Global properties of a family of piecewise isometries. *Ergod. Th. & Dynam. Sys.* **29** (2009), 545–568.

[11] H. Bruin. Renormalization in a class of interval translation maps of d branches. *Dyn. Syst.* **22** (2007), 11–24.

[12] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, Cambridge, 2004.

[13] H. Masur. Interval exchange transformations and measured foliations. *Ann. of Math.* (2) **115** (1982), 169–200.

[14] J. C. Oxtoby. Ergodic sets. *Bull. Amer. Math. Soc.* **58** (1952), 116–136.

[15] R. E. Schwartz. Unbounded orbits for outer billiards. *J. Modern Dynam.* **3** (2007), 371–424.

[16] M. Trovati and P. Ashwin. Tangency properties of a pentagonal tiling generated by a piecewise isometry. *Chaos* **17** (2007), 043129.

[17] M. Veech. Gauss measures for transformations on the space of interval exchange maps. *Ann. of Math.* (2) **115** (1982), 201–242.

[18] F. Vivaldi. Aperiodic orbits of piecewise rational rotations of convex polygons with recursive tiling. *Dyn. Syst.* **22** (2007), 25–63.