SYMBOLIC ANALYSIS FOR SOME PLANAR PIECEWISE LINEAR MAPS

XIN-CHU FU AND PETER ASHWIN

Abstract. In this paper a class of linear maps on the 2-torus and some planar piecewise isometries are discussed. For these discontinuous maps, by introducing codings underlying the map operations, symbolic descriptions of the dynamics and admissibility conditions for itineraries are given, and explicit expressions in terms of the codings for periodic points are presented.

1. Introduction

In contrast to continuous dynamical systems, there is not yet a systematic theory and only limited effective methods available in the study of dynamical systems with discontinuities, but symbolic dynamics may be a useful tool in this study. For example, in [10] a symbolic dynamics approach for the study of fractal pattern in second-order non-linear digital filters is presented; in [25, 24, 17, 18] symbolic dynamics for some invertible 2-dimensional discontinuous hyperbolic maps is discussed; in [14] symbolic dynamics analysis for generalized piecewise isometries is given, and some interesting examples and open questions are presented; in [20] symbolic analysis for piecewise continuous and piecewise monotone transformations on the interval $I = [0, 1]$ is discussed, and by giving necessary and sufficient condition for an arbitrary symbol sequence to be an n-address, an algorithm to calculate the topological entropy is given. Most recently, in [2] some orbits of a class of non-ergodic piecewise affine maps of the 2-torus are described in terms of a symbol shift obtained from the "triadic odometer" substitution rule.

In this paper we show that symbolic dynamics method is helpful when considering a class of piecewise linear maps on the 2-torus, i.e., the maps $(x', y') = f(x, y)$ on $X = [0, 1)^2$ with the form below:

$$\left\{ \begin{array}{l} x' = ax + by \pmod{1} \\ y' = cx + dy \pmod{1} \end{array} \right. \tag{1}$$

where $a, b, c, d \in \mathbb{R}$, and this map can be thought of as a map $f = g \circ M$ where

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(we also write $M = (a, b; c, d)$ for convenience) and $g(x) = x - \lfloor x \rfloor$ is a map that takes modulo 1 in each component, and we assume the determinant $\det M \neq 0$. As pointed out by Adler in [1], the symbolic dynamics even for hyperbolic automorphisms of the 2-torus (i.e., all coefficients in (1) are integers, and the map

Date: February 8, 2008.
2000 *Mathematics Subject Classification.* 37D50.
Key words and phrases. Linear map on 2-torus, piecewise linear, piecewise isometry, discontinuous dynamical system, coding, symbolic dynamics.

is hyperbolic type) remains a fertile area for research, while there are some results in higher dimensions. In general there are still few general methods for the class of maps (1), the symbolic analysis in this paper represent some technical steps towards this. We focus on the symbolic dynamics for this class of maps, and we present some necessary and sufficient conditions for admissibility of sequences. We also give some results concerning the measure of the maximal invariant set of certain non-invertible maps of this type generalizing a result of [9].

In [5, 7] for planar piecewise isometries we have introduced symbolic codings underlying map operations. It has been shown that this kind of codings is helpful for revealing the dynamical properties of the maps. In this paper we use this idea to symbolically analyze the maps of the form (1).

In [9] it is shown that for the parabolic map of the form (1) (i.e., $\det M = 1$, and the trace $a + d = 2$) the 2-dimensional systems have 1-dimensional dynamical characteristics, that is, the system possesses invariant straight line segments along the eigen-direction. In [25] it is shown that the Tél map, a piecewise linear hyperbolic map defined on the whole plane, can be decomposed, along stable and unstable manifolds, into two coupled one-dimensional maps. Based on these ideas we perform a dimension reduction in certain cases of the general 2-torus maps (1).

In [25] for the Tél map, the orbit of a given point is encoded according to the sign of x in the n-th image and the sign of y in the m-th pre-image, and the symbolic sequences define an ordering of the contracting and expanding foliations of the phase plane. Then admissibility conditions for symbolic sequences can be given by ordering rules of the foliations. In this paper we generalize these ideas by introducing codings underlying the map operations, we give symbolic descriptions of the dynamics and admissibility conditions for itineraries, and present explicit expressions in terms of the codings for periodic points.

2. Partition and coding for linear maps on the 2-torus

The maps (1) can be classified into three types, depending on the eigenvalues $\lambda_{1,2}$ of M we refer to the map as type I ($\lambda_2 > \lambda_1$), type II ($\lambda_1 = \lambda_2$) or type III ($\lambda_1 = \overline{\lambda}_2 \neq \lambda_2$). When $\det M = 1$, i.e., the map is area-preserving, then we can refer to a type I, II or III map as being hyperbolic, parabolic or elliptic, respectively, as we discussed in [9].

We code the orbits of the torus map (1) by 2-d vector (s, t) based on the following partition of the phase space (See figure 1)

$$\begin{aligned} [0,1)^2 &= \bigcup_{s,t} P_{s,t},\ S \le s \le S',\ T \le t \le T', \\ P_{s,t} &= \{(x,y) \in [0,1)^2, s \le ax + by < s+1,\ t \le cx + dy < t+1\}, \end{aligned}$$

where $T = \min\{\lfloor ax + by \rfloor\},\ T' = \max\{\lfloor ax + by \rfloor\},\ S = \min\{\lfloor cx + dy \rfloor\},\ S' = \max\{\lfloor cx + dy \rfloor\}$.

This coding underlies the map operations, as the system (1) can be rewritten as

$$\left\{ \begin{array}{l} x' = ax + by - s \\ y' = cx + dy - t \end{array} \right. . \tag{2}$$

For all $(x, y) \in [0, 1)^2$ we can define a unique itinerary, this is a map $\iota : [0, 1)^2 \to \Sigma(N)$ with $\Sigma(N)$ the set of infinite words with N letters, i.e., the N pairs (s, t),

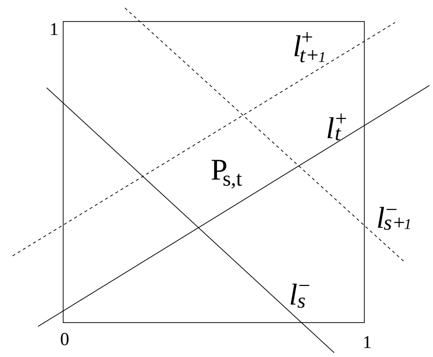

FIGURE 1. *The partition for the torus map (1), where l^-_n is the straight line $ax + by = n$ and l^+_m is the straight line $cx + dy = m$.*

where $N = (T' - T + 1)(S' - S + 1)$,

$$\iota(x, y) = \mathbf{r} = (r_0 r_1 \cdots), \text{ where } r_j = (s_j, t_j) \text{ if } f^j(x, y) \in P_{s_j,t_j}.$$

Recall that a symbolic sequence $\mathbf{r} \in \Sigma(N)$ is admissible if there exists a point $(x, y) \in [0, 1)^2$ such that $\iota(x, y) = \mathbf{r}$. Let $\Sigma_f \subseteq \Sigma(N)$ be the subset of all admissible sequences, then the following diagram commutes:

$$\begin{array}{ccc} [0,1)^2 & \xrightarrow{f} & [0,1)^2 \\ \iota \downarrow & & \downarrow \iota \\ \Sigma_f & \xrightarrow{\sigma} & \Sigma_f \end{array}$$

So characterizing the structure of Σ_f would be helpful to the understanding of the dynamics of f on $[0, 1)^2$.

3. Type I and type II linear torus maps

We discuss in this section the type I and II cases of (1), namely, the cases where the eigenvalues are real.

Suppose the n-th iteration (x_n, y_n) of the map with coding (s_n, t_n) is located on the straight line l_n:

$$y = \frac{1}{\tau}x + \gamma_n,$$

where $n \geq 0$, and $(\tau, 1)$ is an eigenvector for $M = (a, b; c, d)$. Then we have:

Proposition 1. *A type I or type II map (1) sends points on line l_n to points on line l_{n+1}, where $\gamma_{n+1} = \frac{1}{\tau}s_n - t_n + \frac{d\tau - b}{\tau}\gamma_n$, $n \geq 0$. In this sense, the 2-dimensional system (1) possesses 1-dimensional dynamics.*

Proof. We have

$$x_{n+1} = ax_n + by_n - s_n \tag{3}$$

$$y_{n+1} = cx_n + dy_n - t_n \tag{4}$$

$$y_n = \frac{1}{\tau}x_n + \gamma_n \tag{5}$$

$$y_{n+1} = \frac{1}{\tau}x_{n+1} + \gamma_{n+1} \tag{6}$$

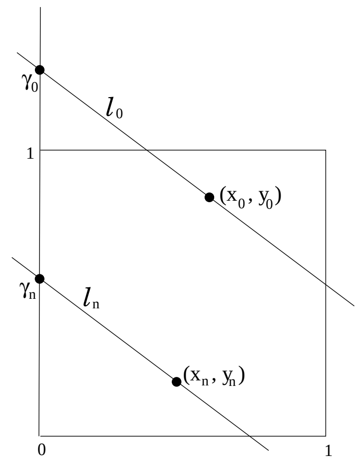

FIGURE 2. *The n-th iteration (x_n, y_n) with coding (s_n, t_n) is located on the straight line $l_n:\ y = \frac{1}{\tau}x + \gamma_n$.*

Substituting (3) and (4) into (6), we have:

$$cx_n + dy_n - t_n = \frac{1}{\tau}(ax_n + by_n - s_n) + \tau_{n+1}, \tag{7}$$

and substituting (5) into (7), we get:

$$\gamma_{n+1} = \frac{1}{\tau}s_n - t_n + \frac{d\tau - b}{\tau}\gamma_n,\ \ n \geq 0 \tag{8}$$

That is, a type I or type II map (1) sends points on line l_n to points on line l_{n+1}, and line l_{n+1} can be determined through (8) by l_n and coding (s_n, t_n). If we identify all points on line l_i to be a single point, $i \geq 0$, then there is a factor system that is 1-dimensional. □

Rewrite (8) as

$$\gamma_n = -\frac{1}{\tau}\frac{\tau}{d\tau - b}s_n + \frac{\tau}{d\tau - b}t_n + \frac{\tau}{d\tau - b}\gamma_{n+1},$$

so we get the relation among γ_j and the codings (s_j, t_j) for $0 \leq n \leq j \leq n + J + 1$:

$$\gamma_n = -\frac{1}{\tau}\sum_{j=0}^{J}\left(\frac{\tau}{d\tau - b}\right)^{j+1} s_{n+j} + \sum_{j=0}^{J}\left(\frac{\tau}{d\tau - b}\right)^{j+1} t_{n+j} + \left(\frac{\tau}{d\tau - b}\right)^{J+1} \gamma_{n+J+1} \tag{9}$$

Under the condition that $|\frac{\tau}{d\tau - b}| < 1$, the last term in (9) vanishes as J goes to infinity, and we have:

$$\gamma_n = -\frac{1}{\tau}\sum_{j\geq 0}\left(\frac{\tau}{d\tau - b}\right)^{j+1} s_{n+j} + \sum_{j\geq 0}\left(\frac{\tau}{d\tau - b}\right)^{j+1} t_{n+j} \tag{10}$$

Whenever $\mathbf{r}$ is a periodic sequence, that is,

$$\mathbf{r} = (r_0 r_1 \cdots) = P(r_0 r_1 \cdots r_{n-1}),$$

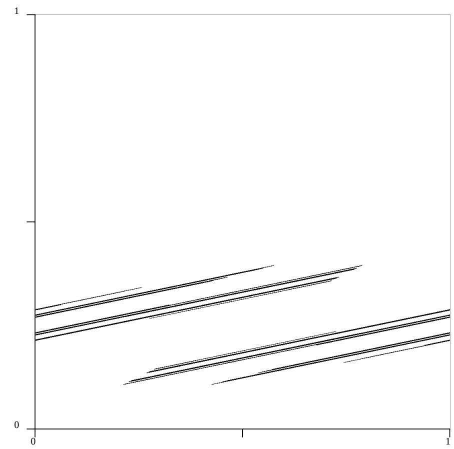

FIGURE 3. *Numerical result for hyperbolic system with* $a = \frac{3}{2}, b = 1, c = \frac{1}{4}, d = \frac{1}{2}$. *For big* n, γ_n *takes a small subset of its possible values, the whole phase space is shrunk into a Cantor set consisting of uncountable parallel straight line segments.*

where P denotes the periodic concatenation of its argument, and if this is admissible then there will be some points (x, y) with $\iota(x, y) = \mathbf{r}$ and $\{\gamma_j,\ j \geq 0\}$ calculated by (10) is also a periodic sequence, and

$$(\gamma_0\gamma_1 \cdots \gamma_j \cdots) = P(\gamma_0\gamma_1 \cdots \gamma_{n-1}).$$

Therefore, if the itinerary of (x_0, y_0) is periodic, its orbit $\{(x_j, y_j),\ j \geq 0\}$ will jump periodically among n straight line segments $l_j:\ y = \frac{1}{\tau}x + \gamma_j,\ 0 \leq j \leq n-1$ with the same slope. This is similar to quasi-periodic motion in elliptic systems in which quasi-periodic points jump periodically among finite number of circles with the same radius. For system (1) in type I and type II cases we call an orbit starting from (x_0, y_0) periodically coded if $\iota(x_0, y_0)$ is a periodic sequence.

Numerical results show that for some parameters, asymptotically, γ_n takes values within a quite narrow range, that means under the dynamics the whole phase space can be shrunk into a very small subset. (See Figure 3).

In some cases, γ_n may only take a finite number of values for each orbit. For example, we have

Proposition 2. *When* $M = (1 + A, -A; A, 1 - A)$, A *is any real number, then* γ_n *takes at most two values for each orbit.*

Proof. This can be shown as follows. Now f is an area preserving parabolic map with eigenvalue $\lambda = 1$. So $\tau = 1$ and therefore $\frac{d\tau - b}{\tau} = 1$. According to (8),

$$\gamma_{n+1} = s_n - t_n + \gamma_n,\ n \geq 0,$$

so $\gamma_{n+1} - \gamma_n,\ n \geq 0$ are all integers. But for $\tau = 1,\ -1 < \gamma_n < 1$, therefore γ_n takes at most two values:

$$\gamma_n = \begin{cases} \gamma_0 \\ \gamma_0 - \operatorname{sgn}\gamma_0 \end{cases}.$$

When $\gamma_0 = 0$, then $\gamma_n = 0$ for all n. In fact, all points on the line $y = x$ are fixed points of f. For $x_0 \neq y_0$, the motion of the orbit from the initial point (x_0, y_0) will just restrict on two lines $l: \ y = x + \gamma_0$ and $l': \ y = x + \gamma_0 - \operatorname{sgn}\gamma_0$, where $\gamma_0 = y_0 - x_0$. ☐

For a general area preserving torus map f discussed in [9] , we have $M = M_{A,\alpha} = (1 + A, \alpha^{-1}A; -\alpha A, 1 - A)$. Then $\tau = -\alpha^{-1}$ as the eigenvalue $\lambda = 1$, and therefore

$$\frac{d\tau - b}{\tau} = \frac{-(1-A)\alpha^{-1} - \alpha^{-1}A}{-\alpha^{-1}} = 1.$$

According to (8),

$$\gamma_{n+1} = -\alpha s_n - t_n + \gamma_n, \ n \geq 0.$$

so we have

$$\gamma_{n+1} - \gamma_n = -\alpha s_n - t_n, \ n \geq 0. \tag{11}$$

Proposition 3. *When $M = M_{A,\alpha}$, and α is rational, then γ_n takes a finite number of values for each orbit.*

Proof. Now f is an area-preserving semi-rational parabolic map. Since α is rational, from (11) there is a integer N such that $N(\gamma_{n+1} - \gamma_n), n \geq 0$ are all integers. Note that $\gamma_n, n \geq 0$ are bounded, so γ_n can only take a finite number of values for each orbit. ☐

In [9] it is shown that for a rational parabolic torus map (A, α are both rational) the maximal invariant set X^+ has positive Lebesgue measure. Here we further have

Proposition 4. *For a semi-rational parabolic torus map (α is rational) the maximal invariant set X^+ has positive Lebesgue measure.*

Proof. From Proposition 3 an orbit of f is contained in finite number of straight lines $l_n : y = -\alpha x + \gamma_n$. Note that f is piecewise continuous and preserves Lebesgue measure locally. So the intersection of the maximal invariant set X^+ with these lines contains at least one line segment L with positive length (in fact, the intersection usually contains countable line segments possessing positive length). By similar argument used in this paper, the location of L can be determined *linearly* by the codings of the orbit, γ_n and α. Let γ_0 vary continuously in a certain small range such that the codings keep unchanged and (11) keep valid, then the location of L and its length changes continuously. In this way we get a small 2-dimensional piece with positive Lebesgue measure. Because this piece is contained in X^+, X^+ therefore has positive Lebesgue measure. ☐

The above proposition partially solved an open problem in [9]. It still remains unknown whether X^+ has positive measure for irrational parabolic torus maps.

3.1. **The cases for $c = 1, d = 0$.** We can check that we can take $\tau = (l - d)/c$ where l is an eigenvalue of M. When $c = 1, d = 0$, then $\tau = l$, and the system (1) becomes

$$\begin{cases} x' = ax + by \pmod{1} \\ y' = x \end{cases} \tag{12}$$

We suppose $a, b > 0$. When $a + b \leq 1$, the map (12) is continuous, so we further suppose $a + b > 1$. Therefore system (12) is type I with two eigenvalues $l_{1,2}$, and $-b < l_1 < 0, \ l_2 > 1$.

We can code the orbits of the torus map (12) by just a number s, rather than by a 2-d vector (s,t), based on the following partition of the phase space

$$\begin{aligned}[0,1)^2 &= \bigcup_s P_s, \ s \in \{0,1,\cdots,[a+b]\} \\ P_s &= \{(x,y) \in [0,1)^2, s \le ax+by < s+1\},\end{aligned}$$

Suppose an orbit from $(x_0,y_0) \in [0,1)^2$ with the symbolic sequence $(s_0 s_1 \cdots s_j \cdots)$ as its itinerary, and suppose the n-th iteration (x_n,y_n) is on the straight line l_n

$$y = \frac{1}{l_1}x + \alpha_n, \ n \ge 0,$$

which relates to the stable manifold direction $(\lambda_1, 1)$, then from (10) we have

$$\alpha_n = \frac{1}{b}\sum_{j\ge 0}\frac{s_{n+j}}{l_2^j}. \tag{13}$$

Here the infinite series does converge as $l_2 > 1$.

Proposition 5. *The admissibility condition for symbolic sequences* $(s_0 s_1 \cdots s_j \cdots)$ *is*

$$\sum_{j\ge 0}\frac{s_j}{l_2^j} < b - \frac{b}{l_1} = b + l_2;$$

and the condition for a symbolic sequence $(s_0 s_1 \cdots s_j \cdots)$ *to be the itinerary of a given point* (x_0,y_0) *is*

$$\sum_{j\ge 0}\frac{s_j}{l_2^j} = by_0 + l_2 x_0.$$

Proof. For an admissible symbolic sequence $(s_0 s_1 \cdots s_j \cdots)$, a point (x_n,y_n) with coding s_n on the straight line segment

$$y = \frac{1}{l_1}x + \frac{1}{b}\sum_{j\ge 0}\frac{s_{n+j}}{l_2^j}$$

is mapped to a point (x_{n+1},y_{n+1}) with coding s_{n+1} on the straight line segment

$$y = \frac{1}{l_1}x + \frac{1}{b}\sum_{j\ge 0}\frac{s_{n+j+1}}{l_2^j}.$$

So whenever an initial point (x_0,y_0) is on the straight line segment l_0, then (x_n,y_n) would be on the straight line segment l_n. This process can continue if l_0 intersects with $[0,1)^2$, so we have one of the admissibility conditions for symbolic sequences as follows

$$\sum_{j\ge 0}\frac{s_j}{l_2^j} < b - \frac{b}{l_1} = b + l_2. \tag{14}$$

For a given point $(x_0,y_0) \in [0,1)^2$, let its itinerary be

$$\iota(x_0,y_0) = (s_0 s_1 \cdots s_j \cdots).$$

Suppose $y_0 = \frac{1}{l_1}x_0 + K$, then $K = y_0 - \frac{1}{l_1}x_0$, so we should have $\alpha_0 = K$, so we have the conditions for a symbolic sequence to be the itinerary of the given point (x_0,y_0) as follows:

$$\sum_{j\ge 0}\frac{s_j}{l_2^j} = by_0 + l_2 x_0. \tag{15}$$

□

The set of points $C(\mathbf{s}) = \iota^{-1}(\mathbf{s})$ with the same itinerary $\mathbf{s}$ we refer to as a *cell.*

For an admissible symbolic sequence $\mathbf{s} = (s_0 s_1 \cdots s_j \cdots)$, the cell $C(\mathbf{s})$ is the union of straight line segments.

For a fixed point of f defined by (12), its itinerary is also a fixed point for the shift map. We can list all admissible symbolic sequences for fixed points as

$$\mathbf{s_j} = (jj \cdots j \cdots), \quad 0 \le j \le [a+b] - 2,$$

then

$$\alpha_n(\mathbf{s_j}) = \frac{1}{b} \sum_{i \ge 0} \frac{j}{l_2^i} = \frac{j}{b} \frac{\lambda_2}{\lambda_2 - 1} = \alpha(j),$$

therefore a fixed point (x_j, y_j) should satisfy

$$\begin{cases} y_j = \frac{1}{l_1} x_j + \alpha(j) \\ y_j = x_j \end{cases},$$

solve this we get

$$x_j = y_j = \frac{j}{b} \frac{\lambda_1 \lambda_2}{\lambda_1 \lambda_2 - (\lambda_1 + \lambda_2) + 1} = \frac{j}{a+b-1}.$$

So all fixed points of f are

$$F_j = (\frac{j}{a+b-1}, \frac{j}{a+b-1}), \quad j = 0, 1, \cdots, [a+b] - 2.$$

The above discussion show that there is a one-to-one correspondence between admissible fixed sequences and fixed points of f. Fixed points are period-1 points. For general periodic points, we have

Proposition 6. *There is a one to one correspondence between admissible periodic symbolic sequences and periodic points of f defined by (12).*

Proof. To see this, suppose $\mathbf{s}$ is an admissible n-periodic sequence,

$$\mathbf{s} = (s_0 s_1 \cdots s_j \cdots) = P(s_0 s_1 \cdots s_{n-1}),$$

and there exists a point $(x_0, y_0) \in \iota^{-1}(\mathbf{s})$ that is a n-periodic point of f. From $s_{j+n} = s_j$ we have

$$\alpha_i = \frac{1}{b} \frac{\lambda_2^n}{\lambda_2^n - 1} \sum_{j=0}^{n-1} \frac{s_{j+i}}{\lambda_2^j}, \; i \ge 0, \tag{16}$$

therefore $(\alpha_0 \alpha_1 \cdots \alpha_i \cdots) = P(\alpha_0 \alpha_1 \cdots \alpha_{n-1})$. Use the facts that $y_i = x_{i-1}, \; i \ge 1, \; y_0 = y_n = x_{n-1}$, we have

$$\begin{cases} x_i = \frac{1}{l_1} x_{i+1} + \alpha_{i+1} \\ x_{i+n} = x_i \end{cases}, \; i \ge 0,$$

solve this we obtain

$$\begin{cases} x_i = \frac{\lambda_1^n}{\lambda_1^n - 1} \sum_{k=0}^{n-1} \frac{\alpha_{i+1+k}}{\lambda_1^k} \\ y_i = \frac{\lambda_1^n}{\lambda_1^n - 1} \sum_{k=0}^{n-1} \frac{\alpha_{i+k}}{\lambda_1^k} \end{cases}, \; i \ge 0 \tag{17}$$

We can check that $\{(x_i, y_i), i \ge 0\}$ determined by (17) is a n-periodic orbit. And from (16) and (17), the periodic point (x_0, y_0) in $\iota^{-1}(\mathbf{s})$ is uniquely determined by the codings $s_0, s_1, \cdots, s_{n-1}$. □

3.2. **The cases for $a = b = 1$.** Below we discuss a more specific case, $a = b = 1$. Now the map f is invertible, and is continuous on the 2-torus topology. In this case Adler's method in [1] can be applied to discuss its symbolic dynamics.

The method in [1] is for hyperbolic linear torus automorphism, that is, the map is hyperbolic and the entries of the matrix M are integers, and so the map is invertible and continuous. We can also apply our method described in section 3 to give a symbolic description for the dynamics of the torus automorphism. Moreover, the method discussed below can be also applicable to all invertible hyperbolic linear torus maps of the form (1) which are not necessarily continuous.

For $(x, y) \in [0, 1)^2$, there is a unique bi-infinite symbolic sequence

$$\mathbf{s} = (\cdots s_{-j} \cdots s_{-1} \cdot s_0 s_1 \cdots s_j \cdots)$$

such that

$$\mathbf{s} = \iota(x, y) \ \text{ if } f^j \in P_{s_j}, \ j \in \mathbb{Z}.$$

The two eigenvalues of M are $\lambda_{1,2} = \frac{1}{2}(1 \pm \sqrt{5})$, and $\mathbf{v}_{1,2} = (\lambda_{1,2}, 1)$ are two corresponding eigenvectors.

Proposition 7. *The admissibility conditions for a symbolic sequence* $(\cdots s_{-1} \cdot s_0 s_1 \cdots)$ *are*

$$\begin{cases} 0 \leq \displaystyle\sum_{j \leq -1} (-1)^j \lambda_2^j s_j + \sum_{j \geq 0} \lambda_2^{-j} s_j < \lambda_2 - \lambda_1 \\ 0 \leq \displaystyle\sum_{j \geq 0} \lambda_2^{-j} s_j - \lambda_2^2 \sum_{j \leq -1} (-1)^j \lambda_2^j s_j < 1 + \lambda_2^2 \end{cases};$$

and the conditions for a symbolic sequence $(\cdots s_{-1} \cdot s_0 s_1 \cdots)$ *to be the itinerary of a given point* (x_0, y_0) *are*

$$\begin{cases} \displaystyle\sum_{j \geq 0} \lambda_2^{-j} s_j = y_0 - \frac{1}{l_1} x_0 \\ \displaystyle\sum_{j \leq -1} \lambda_2^j s_j = \frac{1}{l_2} x_0 - y_0 \end{cases}.$$

And there is a one-to-one correspondence between all points in the phase space (and the orbits from them) and all admissible symbolic sequences (and their shifts).

Proof. Suppose the n-th iteration (x_n, y_n) is the intersection of two straight lines $l_n^- : \ y = \frac{1}{\lambda_1} x + \alpha_n$ and $l_n^+ : \ y = \frac{1}{\lambda_2} x + \beta_n$ (See figure 4). Then from (13) we have

$$\alpha_n = \sum_{j \geq 0} \frac{s_{n+j}}{l_2^j}. \tag{18}$$

Similar to (3)-(6) we have

$$x_n = x_{n-1} + y_{n-1} - s_{n-1} \tag{19}$$

$$y_n = x_{n-1} \tag{20}$$

$$y_{n-1} = \frac{1}{\lambda_2} x_{n-1} + \beta_{n-1} \tag{21}$$

$$y_n = \frac{1}{\lambda_2} x_n + \lambda_n \tag{22}$$

Substitute (19) and (20) into (22), we have:

$$x_{n-1} = \frac{1}{\lambda_2}(x_{n-1} + y_{n-1} - s_{n-1}) + \beta_n, \tag{23}$$

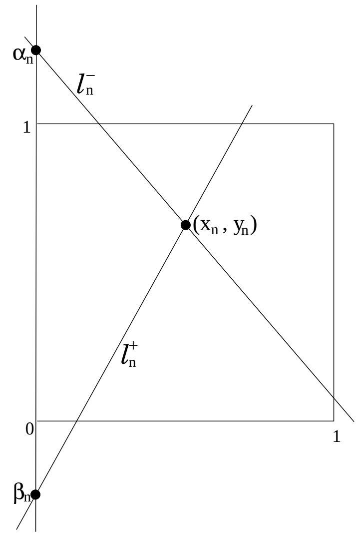

FIGURE 4. *The stable and unstable manifolds* $l_n^- :\ y = \frac{1}{\lambda_1}x + \alpha_n$ *and* $l_n^+ :\ y = \frac{1}{\lambda_2}x + \beta_n$ *intersect at* $(x_n, y_n) = f^n(x_0, y_0)$.

and substitute (21) into (23), we get:

$$\beta_n = \frac{1}{\lambda_2}s_{n-1} - \frac{1}{\lambda_2}\beta_{n-1}$$

so we have:

$$\beta_n = -\sum_{j\geq 1}(-1)^j \frac{s_{n-j}}{\lambda_2^j}. \tag{24}$$

Whenever an initial point (x_0, y_0) is on the intersection of two straight lines l_0^- and l_0^+, then (x_n, y_n) would be on the intersection of straight lines $l_n^\pm$. This process can continue if the intersection is within $[0,1)^2$, so we have two of the admissibility conditions for symbolic sequences as follows

$$\begin{cases} 0 \leq \quad \alpha_0 - \beta_0 = \sum\limits_{j\leq -1}(-1)^j\lambda_2^j s_j + \sum\limits_{j\geq 0}\lambda_2^{-j}s_j < \lambda_2 - \lambda_1 \\ 0 \leq \quad \alpha_0 + \lambda_2^2\beta_0 = \sum\limits_{j\geq 0}\lambda_2^{-j}s_j - \lambda_2^2\sum\limits_{j\leq -1}(-1)^j\lambda_2^j s_j < 1 + \lambda_2^2 \end{cases}, \tag{25}$$

For $(x_0, y_0) \in [0,1)^2$, let its itinerary be

$$\iota(x_0, y_0) = (\cdots s_{-j}\cdots s_{-1}\cdot s_0 s_1\cdots s_j\cdots).$$

Suppose

$$\begin{cases} y_0 = \frac{1}{l_1}x_0 + K_1 \\ y_0 = \frac{1}{l_2}x_0 + K_2 \end{cases},$$

then $K_1 = y_0 - \frac{1}{l_1}x_0$ and $K_2 = y_0 - \frac{1}{l_2}x_0$, so we should have $\alpha_0 = K_1$ and $\beta_0 = K_2$, so we have the conditions for a symbolic sequence to be the itinerary of the given point (x_0, y_0) as follows

$$\left\{\begin{array}{l} \sum\limits_{j\geq 0} \lambda_2^{-j} s_j = y_0 - \frac{1}{l_1} x_0 \\ \sum\limits_{j\leq -1} \lambda_2^{j} s_j = \frac{1}{l_2} x_0 - y_0 \end{array}\right. \tag{26}$$

From the discussion above, the n-th iteration (x_n, y_n) is the intersection of two straight lines l_n^- and l_n^+, then

$$\left\{\begin{array}{ll} x_n = & \frac{\alpha_n - \beta_n}{\lambda_2 - \lambda_1} \\ y_n = & \frac{\lambda_2 \beta_n - \lambda_1 \alpha_n}{\lambda_2 - \lambda_1} \end{array}\right. , \quad n \geq 0 \tag{27}$$

By substituting (18) and (24) into (27), we know that there is a one-to-one correspondence between all points in the phase space (and the orbits from them) and all admissible symbolic sequences (and their shifts):

$$(x_n, y_n) \longleftrightarrow (\cdots s_{n-1} \cdot s_n s_{n+1} \cdots), \; n \geq 0.$$

□

Note that in the case for $a = b = 1$, the partition only contains two elements $P_0 = \{(x, y) \in [0, 1)^2, x + y < 1\}$ and $P_1 = \{(x, y) \in [0, 1)^2, x + y \geq 1\}$. So coding $s_n = 0$ or 1.

As applications of Proposition 7, in the following we look at fixed points and 2-period points.

For fixed points, all their codings are 0 or 1. But $(\cdots 11{\cdot}11 \cdots)$ is not admissible; for $(\cdots 00 \cdot 00 \cdots)$, $\alpha_n = \beta_n = 0$, $-\infty < n < +\infty$, from (27) we know for all iterations $x_n = y_n = 0$, so the only fixed point is $(0, 0)$.

For a 2-period point $(x^{(2)}, y^{(2)})$, the possible itineraries are:

$$\iota(x^{(2)}, y^{(2)}) = \left\{\begin{array}{l} \mathbf{s_0} = (\cdots 0101 \cdot 0101 \cdots) \\ \mathbf{s_1} = (\cdots 1010 \cdot 1010 \cdots) \end{array}\right.$$

therefore

$$\left\{\begin{array}{l} \alpha_0(\mathbf{s_0}) = \sum\limits_{j\geq 0} \frac{1}{\lambda_2^{2j+1}} = \frac{\lambda_2}{\lambda_2^2 - 1} \\ \alpha_0(\mathbf{s_1}) = \sum\limits_{j\geq 0} \frac{1}{\lambda_2^{2j}} = \frac{\lambda_2^2}{\lambda_2^2 - 1} \\ \beta_0(\mathbf{s_0}) = -\sum\limits_{j\geq 0} (-1)^{2j+1} \frac{1}{\lambda_2^{2j+1}} = \frac{\lambda_2}{\lambda_2^2 - 1} \\ \beta_0(\mathbf{s_1}) = -\sum\limits_{j\geq 1} \frac{1}{\lambda_2^{2j}} = \frac{1}{1 - \lambda_2^2} \end{array}\right. .$$

Check these with the admissibility conditions (25), we find $\alpha_0(\mathbf{s_1}) - \beta_0(\mathbf{s_1}) = \lambda_2 - \lambda_1$, so $\mathbf{s_1}$, and therefore $\mathbf{s_0}$ as well, are not admissible. This means that there is no 2-period point for the system.

This procedure can be applied to the analysis of periodic points of higher periods.

4. Type III linear torus maps

The partition and coding introduced in Section 2 can be applied to all the three types of linear torus maps, however, from the above discussions, the partition and coding are more helpful to type I and II maps, as these linear systems on the 2-torus can be viewed as 1-dimensional systems in the sense that they keep eigen-directions invariant. In Section 3 by using this property we have given symbolic descriptions about their dynamics. In this section we will introduce another coding which is more helpful to describe type III maps.

For the area-preserving cases $\det M = 1$, as introduced in [4] and [5], the Sigma-Delta modulator map, which is a piecewise linear elliptic system on the plane, and the overflow oscillation map, which is an elliptic linear system on the 2-torus, are equivalent to piecewise rotations, which can be viewed as a complex system with one variable, by appropriate transformations of the linearized parts into Jordan normal form.

In general, a type III linear system on the 2-torus

$$\left\{ \begin{array}{l} x' = ax + by \pmod{1} \\ y' = cx + dy \pmod{1} \end{array} \right.$$

can be viewed as a 1-dimensional complex system in another sense, i.e., the system can be transformed to complex system with one variable by linear shear. That is, let N be the matrix such that

$$N^{-1}(a, b; c, d)N = \sqrt{\Delta}(\cos\theta, -\sin\theta; \sin\theta, \cos\theta),$$

where $\Delta = \det M$ (> 0) is the determinant, and $\sqrt{\Delta}\mathrm{e}^{\pm i\theta}$ are two eigenvalues, then let

$$\left(\begin{array}{c} x \\ y \end{array} \right) = N \left(\begin{array}{c} u \\ v \end{array} \right), \quad \text{and } z = u + iv,$$

the linear system on the 2-torus becomes the complex system:

$$z' = \sqrt{\Delta}\mathrm{e}^{i\theta} z + W_j, \tag{28}$$

where $z \in P'_j \subset X$, $X = \bigcup P'_j$, P'_j are parallelogram partition elements obtained by shearing the original partition elements $P_{s,t}$ (See figure 1), and $W_j \in \mathbb{C}$ are translation terms induced by the transformation.

For the type of system (28), we can introduce a coding underlying map operations, which generalizes the one introduced for the overflow oscillation system in [5], and give a symbolic description about the periodic points. We will present some details in the next section for planar piecewise isometries.

4.1. **Planar piecewise isometries.** From (28) a type III torus map with unit determinant ($\det M = 1$) can be represented in Jordan normal form as a anti-clockwise rotation by θ followed by a translation, that is, the map is equivalent to a planar piecewise isometry after a linear shearing.

In general, we consider planar piecewise isometries $f : X \to X$ defined on a partition $\{P_i\}_{i=1}^n$ into (possibly unbounded) open, convex polygonal atoms, such that

$$X = \cup_i \overline{P}_i \text{ with } P_i \cap P_j = \emptyset \text{ for } i \neq j,$$

and f restricted to each P_i is an isometry. We suppose here that f is a oriented piecewise isometry, and has a common rotation angle θ, and the family of PWIs can be parameterized by the θ. So f can be written as

$$f(z) = \omega z + W_j, \quad \text{if } z \in P_j,\ j = 0, 1, \cdots, N-1. \tag{29}$$

Where $\omega = \mathrm{e}^{i\theta}$. For rational θ/π, usually f either perform a periodic permutation or polygon packing where all points in the polygons are periodic. We will be interested in the cases where θ/π is irrational. i.e., f is an irrational rearrangement.

By using the coding introduced in Appendix B of [11], for a periodic point $z_{\mathbf{k}}$ of period n with itinerary $\iota(z_{\mathbf{k}}) = P(k_0 k_1 \cdots k_{n-1})$, where

$$k_j = (k_j^{(0)}, k_j^{(1)}, \cdots, k_j^{(N-1)}) \in \{e_0, e_1, \cdots, e_{N-1}\},\ e_j = (\delta_{0j}, \delta_{1j}, \cdots, \delta_{(N-1)j}),$$

we have

$$z_{\mathbf{k}} = \frac{1}{1-\omega^n} \sum_{j=0}^{n-1} \omega^j \sum_{s=0}^{N-1} W_s k_{n-1-j}^{(s)}. \tag{30}$$

So we have the following result (there is a version of the result below in [16], here we give a new proof by using (30)):

Proposition 8. *There is a one-to-one correspondence between admissible periodic symbolic sequences and periodic points of f as long as $\frac{\theta}{\pi} \notin \mathbb{Q}$.*

Proof. We can use (30) to prove this. Let

$$\iota(z_{\mathbf{k}}) = \iota(z'_{\mathbf{k}}) = P(k_0 k_1 \cdots k_{n-1}),$$

then from (30) we have

$$(1-\omega^n)(z_{\mathbf{k}} - z'_{\mathbf{k}}) = 0,$$

but $1 - \omega^n \neq 0$ since $\theta/\pi \notin \mathbb{Q}$, so it must have $z_{\mathbf{k}} = z'_{\mathbf{k}}$. □

In [5], [7] and Appendix B of [11] the explicit expression (30) is used to prove that tangencies between discs in the invariant disc packings for the overflow oscillation, Sigma-Delta modulator, and some general planar piecewise isometries are very rare. We expect this can be applied to other problems for PWIs.

By the way, a piecewise isometric system can have attractors although its all Lyapunov exponents are zero ([6, 21]).

5. Some remarks

5.1. **Linear torus maps in higher dimensions.** The techniques and ideas used in this paper may be generalized to higher dimensional piecewise linear maps, e.g., the similar maps on the m-torus, while the classification may be more complicated than the 2-torus case, as we have to deal with matrices with mixed real and complex eigenvalues, leading to some kind of mixed types.

For the following class of maps $({x'}^{(1)}, \cdots, {x'}^{(m)}) = f(x^{(1)}, \cdots, x^{(m)})$ of the m-torus $X = [0,1]^m$ of the form

$$\left\{ \begin{array}{rcll} {x'}^{(1)} & = & a_{11}x^{(1)} + \cdots + a_{1m}x^{(m)} & \pmod 1 \\ & \vdots & & \\ {x'}^{(m)} & = & a_{m1}x^{(1)} + \cdots + a_{mm}x^{(m)} & \pmod 1 \end{array} \right. \tag{31}$$

at first we suppose $M = (a_{11}, \cdots, a_{1m}; \cdots; a_{m1}, \cdots, a_{mm})$ has a real eigenvalue λ, and the n-th iteration $(x_n^{(1)}, \cdots, x_n^{(m)})$ of the map with coding $(s_n^{(1)}, \cdots, s_n^{(m)})$ is located on the hyperplane l_n:

$$x^{(m)} = \alpha_1 x^{(1)} + \cdots + \alpha_{m-1} x^{(m-1)} + \delta_n,$$

where $s_n^{(j)} = \lfloor a_{j1}x^{(1)} + \cdots + a_{jm}x^{(m)} \rfloor, n \geq 0$, and $(\alpha_1, \cdots, \alpha_{m-1}, -1)$ is an eigenvector for the transposed matrix M^T with respect to the eigenvalue λ. Then similar to the discussion in Section 3 we have:

$$\delta_{n+1} = \sum_{i=1}^{m} \alpha_i s_n^{(i)} + \lambda \delta_n,\ n \geq 0 \tag{32}$$

where $\alpha_m = -1$.

That is, the map (31) sends points on hyperplane l_n to points on hyperplane l_{n+1}, and hyperplane l_{n+1} can be determined through (32) by l_n and coding $(s_n^{(1)}, \cdots, s_n^{(m)})$. In this sense, the m-dimensional system (31) possesses $(m-1)$-dimensional dynamics.

(32) can be rewritten as

$$\delta_n = -\frac{1}{\lambda} \sum_{i=1}^{m} \alpha_i s_n^{(i)} + \frac{1}{\lambda} \delta_{n+1},$$

so we have the relation among δ_j and the codings $(s_j^{(1)}, \cdots, s_j^{(m)})$ for $0 \leq n \leq j \leq n+J+1$:

$$\delta_n = -\sum_{j=0}^{J} \frac{1}{\lambda^{j+1}} \sum_{i=1}^{m} \alpha_i s_{n+j}^{(i)} + \frac{1}{\lambda^{J+1}} \delta_{n+J+1} \tag{33}$$

Under some conditions, say $|\lambda| > 1$, we have:

$$\delta_n = -\sum_{j=0}^{+\infty} \frac{1}{\lambda^{j+1}} \sum_{i=1}^{m} \alpha_i s_{n+j}^{(i)} \tag{34}$$

Now we suppose $m = 2m'$ be even and all eigenvalues are complex, and suppose that there exists a matrix N such that

$$N^{-1} M N = \Delta^{\frac{1}{m}} \mathrm{diag}(M_1, \cdots, M_{m'}),$$

where $M_k = (\cos\theta_k, -\sin\theta_k; \sin\theta_k, \cos\theta_k)$. Let

$$\begin{pmatrix} x^{(1)} \\ \vdots \\ x^{(m)} \end{pmatrix} = N \begin{pmatrix} u^{(1)} \\ v^{(1)} \\ \vdots \\ u^{(m')} \\ v^{(m')} \end{pmatrix}, \quad \text{and } z^{(k)} = u^{(k)} + i v^{(k)},\ k = 1, \cdots, m',$$

then similar to the 2-dimensional case, the linear system (31) on the m-torus can be transformed to a complex system with $\frac{m}{2}$ variables, which is a rotation on each component.

5.2. **Admissibility conditions for planar PWIs.** For the elliptic cases, admissibility conditions for periodic itineraries are crucial in estimating the measure of the exceptional sets. To see what are the situations we may face, here as an example we remark briefly on the piecewise isometry of the torus obtained from the overflow oscillation problem (see [6] for details).

The piecewise isometry $f : M_\theta \to M_\theta$ is defined by

$$f(z) = \omega z + W k(z),$$

where $\theta \in (0, \pi/2)$, M_θ is the rhombus unit cell for a torus:

$$M_\theta = \{z \in \mathbf{C}\ :\ |\mathrm{Re}(z)| \leq 1 \ \text{ and } \ |\mathrm{Re}(z e^{i\theta})| \leq 1\}$$

and $\omega = \mathrm{e}^{i\theta}, W = -\frac{2i}{\sin\theta}$ is a constant, and

$$k(z) = \begin{cases} -1 & \text{if } z \in M_{-1} \\ +1 & \text{if } z \in M_{+1} \\ 0 & \text{if } z \in M_0 \end{cases}$$

is the coding, where

$$\begin{aligned} M_{-1} &= \{z \in M_\theta, \mathrm{Re}(ze^{2i\theta}) > 1\}, \\ M_{+1} &= \{z \in M_\theta, \mathrm{Re}(ze^{2i\theta}) \le -1\}, \\ M_0 &= M_\theta \setminus (M_{-1} \bigcup M_{+1}). \end{aligned}$$

For a point z_0 with itinerary $\mathbf{k} = (k_0 k_1 \cdots k_j \cdots)$, we have

$$z_{n+1} = \omega z_n + W k_n, \ n \ge 0.$$

Therefore we have

$$\begin{aligned} z_n &= -W \frac{k_n}{\omega} + \frac{z_{n+1}}{\omega} \\ &= -W \sum_{j=0}^{J} \frac{k_{n+j}}{\omega^{j+1}} + \frac{z_{n+J+1}}{\omega^{J+1}}. \end{aligned}$$

In form we can write

$$\begin{aligned} z_0 &= -W \sum_{j=0}^{+\infty} \frac{k_j}{\omega^{j+1}} \\ &= -W \left(\sum_{j=1}^{+\infty} k_{j-1} \cos j\theta - i \sum_{j=1}^{+\infty} k_{j-1} \sin j\theta \right). \end{aligned}$$

It is noted that the convergence of the above trigonometric series is indefinite. The above analysis is applicable to all planar piecewise isometries. So we can say that this is the elementary but the main difficulty involved in finding admissibility conditions for itineraries in the symbolic dynamics analysis for planar piecewise isometries. Recently Yu and Galias studied similar problem and obtained some numerical results on admissible periodic symbolic sequences ([23], see also [12, 22] for earlier works).

Acknowledgement. The authors would like to thank the UK Engineering and Physical Science Research Council for support via grant GR/M36335. XF also thanks School of Mathematical Sciences of Exeter University for generous support when the earlier version of this paper was written.sd

References

[1] R. L. Adler, Symbolic Dynamics and Markov Partitions. *MSRI* Publications, **53**, 1996. Also in *Bull. Amer. Math. Soc. (New Series)*, **35** (1998) 1-56.

[2] R.L. Adler, B. Kitchens and C. Tresser, Dynamics of non-ergodic piecewise affine maps of the torus. *Ergod. Th. & Dynam. Sys.* **21** (2001) 959-999.

[3] P. Ashwin, Elliptic behaviour in the sawtooth standard map, *Phys. Lett.* **A 232**, 409 (1997).

[4] P. Ashwin, J.H.B. Deane and X.-C. Fu, Dynamics of a bandpass sigma-delta modulator as a piecewise isometry, *Proceedings, ISCAS 2001*, (Proceedings of the IEEE International Symposium on Circuits and Systems, 2001), Sydney, Australia, 2001, pp.III-811–III-814.

[5] P. Ashwin and X.-C. Fu, Tangencies in invariant disk packings for certain planar piecewise isometries are rare. *Dynamical Systems*, **16** (2001) 333-345.

[6] P. Ashwin and X.-C. Fu, On the dynamics of some nonhyperbolic area-preserving piecewise linear maps. in *Mathematics in Signal Processing V*, Oxford Univ Press IMA Conference Series, 2002.

[7] P. Ashwin, X.-C. Fu and J.H.B. Deane, Properties of the invariant disk packing in a model bandpass sigma-delta modulator. To appear in *Int. J. of Bif. and Chaos*, Vol.**13** (2003).

[8] P. Ashwin, X.-C. Fu and J.R. Terry, Riddling of invariant sets for some discontinuous maps preserving Lebesgue measure. *Nonlinearity*, **15** (2002) 633-645.

[9] P. Ashwin, X.-C. Fu, T. Nishskawa and K. Zyczkowski, Invariant sets for discontinuous parabolic maps of the torus. *Nonlinearity*, **13** (2000) 818-835.

[10] L. Chua and T. Lin, Fractal pattern of second-order non-linear digital filters: a new symbolic analysis. *Int. J. of Circ. Th. and Appl.*, **18** (1990) 541-550.

[11] Xin-Chu Fu, Dynamical Behaviour of a Class of Discontinuous Maps and Related Topics. *PhD Thesis*, Exeter University, October 2001.

[12] Z. Galias and M. Ogorzalek, On symbolic dynamics of a chaotic second order digital filter, *Intern. J. Circ. Th. Appl.*, **20** (1992) 401-409.

[13] A. Goetz, Dynamics of a piecewise rotation. *Continuous and Discrete Dynamical Systems* **4**:593–608 (1998).

[14] A. Goetz, Sofic subshifts and piecewise isometric systems. *Ergodic Theory and Dyn. Sys.*, **19** (1999) 1485-1501.

[15] A. Goetz, Stability of piecewise rotations and affine maps. *Nonlinearity* **14** (2001) 205–219.

[16] A. Goetz, Dynamics of a piecewise isometries. *Illinois J. Math.* **44**:465–478 (2000).

[17] B.-L. Hao, *Elementary Symbolic Dynamics and Chaos in Dissipative Systems*, World Scientific, 1989.

[18] B.-L. Hao, and W.-M. Zheng, *Applied Symbolic Dynamics and Chaos*, World Scientific, 1998.

[19] L. Kocarev, C.W. Wu and L.O. Chua, Complex behaviour in Digital filters with overflow nonlinearity: analytical results. *IEEE Trans CAS-II* **43** (1996) 234–246.

[20] C. Kopf, Symbol sequences and entropy for piecewise monotone transformations with discontinuities. *Discrete and Continuous Dynamical Systems*, **6** (2000) 299–304.

[21] M. Mendes, Quasi-invariant attractors of piecewise isometric systems. *Discrete and Continuous Dynamical Systems*, **9** (2003) 323-338.

[22] C.J. Vowden, *PhD thesis*, Department of Engineering, Univ. of Warwick, 1999.

[23] X. Yu and Z. Galias, Periodic behaviour in a digital filter with two's complement arithmetic, *IEEE Trans. Circuits Systems I Fund. Theory Appl.*, **48** (2001) 1177–1190.

[24] W.-M. Zheng and B.-L. Hao, *Applied symbolic dynamics.* Advanced series in Nonlinear Science, Shanghai Scientific and Technological Education Publishing House, Shanghai, China, 1994.

[25] W.-M. Zheng, Symbolic dynamics for the Tél map. *Comm. Theor. Phys.*, **17** (1992) 167–174.

Xin-Chu Fu, Department of Mathematics, Shanghai University, Shanghai 200436, P. R. CHINA

E-mail address: x.c.fu@163.com

Peter Ashwin, School of Mathematical Sciences, University of Exeter, Exeter EX4 4QE, U. K.

E-mail address: P.Ashwin@ex.ac.uk