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Universality of Nash components.

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Abstract

We show that Nash equilibrium components are universal for the collection of connected polyhedral sets. More precisely for every polyhedral set we construct a so-called binary game—a common interest game whose common payoff to the players is at most equal to one—whose success set (the set of strategy profiles where the maximal payoff of one is indeed achieved) is homeomorphic to the given polyhedral set. Since compact semi-algebraic sets can be triangulated, a similar result follows for the collection of connected compact semi-algebraic sets.

We discuss implications of our results for the strategic stability of success sets, and apply the results to construct a Nash component with index k for any fixed integer k.

JEL Codes. C72, D44.
Keywords. Strategic form games, Nash equilibrium, Nash component, topology.

1 Introduction

In non-cooperative game theory the claim that Nash equilibrium components can have any conceivable shape they can reasonably be expected to have seems to have the status of a "folk conjecture". [1] Few researchers in the field doubt this claim, yet no proof seems to be available, and it is not entirely clear what is meant by "every conceivable shape". In this paper we provide a rigorous proof of a specific version of this "folk conjecture" on the topological structure of Nash equilibrium components.

Concretely, in this paper we establish four facts. Below we first briefly state each separate fact. Next, we explain the precise formulation and interpretation of each of these facts in more detail. In the remainder of the introduction we then discuss some of the implications of our results and briefly explain the organization of the paper.

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[‡]We would like to thank an anonymous editor for the very helpful and constructive comments on an earlier draft of this paper that helped us to improve the presentation of our results substantially.
[1]We thank the anonymous editor for coining this expression.

THE FACTS. The four facts we prove are

[1] The success set [2] of a binary game is a cubistic set.

[2] Conversely, for every cubistic set C in $\mathbb{R}^n$, there is an n-person binary game for which C equals its success set.

[3] The success set of a binary game is the union of Nash components of that binary game.

[4] Every simplicial set is homeomorphic to a cubistic set.

Explanation of [1]. A binary game is an n-person strategic form game with the following special features. First, each player has two pure strategies. Second, it is a common interest game, meaning that the payoffs to the players are all equal. Third, for every profile of pure strategies the common payoff to the players is either zero or one.

Since each player has two pure strategies, the strategy space of the mixed extension of a binary game can be identified with the standard hypercube in $\mathbb{R}^n$. The success set of a binary game is the collection of points in that hypercube that correspond to a strategy profile in which the (common) payoff to the players equals one. We argue that the success set C of a binary is in fact what we call a cubistic set, meaning that

[a] C can be written as the union of a number of faces of the hypercube, and

[b] a face F of the hypercube is a subset of C whenever each vertex of F is an element of C.

Explanation of [2]. Conversely, we show that for every cubistic set C in $\mathbb{R}^n$ there is an n-person binary game such that C is the success set of that particular binary game. In short, [1] and [2] together show that there is a one-to-one match between cubistic subsets of the hypercube in $\mathbb{R}^n$ and success sets of n-person binary games.

Explanation of [3]. Since the success set of a binary game is the collection of strategy profiles with (joint) payoff one—the maximal payoff possible—to the players, it is evident that the success set is a subset of the set of Nash equilibria of the game. We show the stronger statement that the success set of a binary game is the union of Nash components of that binary game. To be precise, given a binary game with success set C, there is an open set U containing C such that every Nash equilibrium in U is even an element of C.

[2] We also thank the anonymous editor for coining this expression.

Explanation of [4]. Finally we show that cubistic sets are what is often called "universal" in literature, meaning that for every simplicial set there is a cubistic set that is homeomorphic with the given simplicial set. In other words, the collection of cubistic sets encompasses—up to homeomorphisms—every conceivable shape a simplicial set might have. In again other words, every shape we can construct using simplicial sets, we can also achieve using cubistic sets.

The consequences. We discuss three direct consequences of our facts.

Our main contribution is the universality of Nash components. To be precise, our results imply that for every connected simplicial set there is a—binary—game whose success set is a Nash component that is homeomorphic to the given simplicial set. This main result is a direct consequence of facts [4], [2], and [3], respectively. Thus, a Nash equilibrium component has topologically no additional structure beyond what follows directly from the definition and the fundamental triangulation result of semi-algebraic sets by Llojasiewicz [9].

Secondly, we note that success sets are strict equilibrium sets in the sense of Balkenborg and Schlag [3]. Therefore we obtain from their Theorem 6 the result that every success set with non-zero Euler characteristic contains a stable set in the sense of Mertens [10].

Thirdly, we use our results to provide a simple alternative to the construction described in Govindan, von Schemde and von Stengel [7] of a Nash equilibrium component with index p for any given integer p.

Setup of the paper. Section 2 introduces some of the notation we need in this paper. In section 3 we prove facts [1] and [2]. In section 4 we prove fact [3]. In section 5 we prove fact [4]. In section 6 we use facts [2], [3], and [4] to show that Nash equilibrium components are universal. In sections 7 and 8 we discuss the implications of our results for strategic stability, and we provide an alternative for the construction of a Nash component of index p for any given integer p.

Related literature. Bubelis [4] shows that, given any real algebraic number a, there exists a 3-person game with rational data in which a is the payoff to at least one player in the unique Nash equilibrium point of that game. He also presents a method which reduces an arbitrary n-person game to a 3-person game. Finally, a completely mixed game, where the equilibrium set is a manifold of dimension one, is constructed.

Datta [5] shows that every real algebraic variety is isomorphic to the set of totally mixed Nash equilibria of some three-person game, and also to the set of totally mixed Nash equilibria of an n-person game in which each player has two pure strategies. It follows that every compact differentiable manifold can be written as the set of totally mixed Nash equilibria of some game. Moreover, there exist isolated Nash equilibria of arbitrary topological degree.

McKelvey and McLennan [12] provide a tight upper bound on the maximal number of regular totally mixed Nash equilibria for generic finite games in strategic form.

2 Preliminaries

A finite normal form game consists of a finite set of players $N = \{1, \dots, n\}$, and for each player $i \in N$ a finite pure strategy set S_i and a payoff function $u_i \colon S \to \mathbb{R}$ on the set $S := \prod_{i \in N} S_i$ of pure strategy profiles. A mixed strategy σ_i of player i is a vector $(\sigma_i(s_i))_{s_i \in S_i}$ that assigns a probability $\sigma_i(s_i) \geq 0$ to each pure strategy $s_i \in S_i$. The support of a mixed strategy σ_i is the set of all pure strategies s_i with $\sigma_i(s_i) > 0$. The multilinear extension of the payoff function u_i of player i is the function that assigns to each strategy profile $\sigma = (\sigma_i)_{i \in N}$ the expected payoff

$$u_i(\sigma) = \sum_{s \in S} \prod_{j \in N} \sigma_j(s_j) u_i(s).$$

By $u_i(\sigma \mid s_i)$ we denote the payoff to player i when player i plays pure strategy $s_i \in S_i$ while his opponents adhere to the mixed strategy profile σ. A strategy profile $\sigma \in \Sigma$ is a Nash equilibrium when $u_i(\sigma) \geq u_i(\sigma \mid s_i)$ holds for every player i and every pure strategy s_i of player i.

BINARY GAMES. In this paper we deal with binary games only. A binary game is a finite normal form game $(N, (u_i)_{i \in N})$ such that

[1] $S_i = \{A, B\}$ for every player $i \in N$,

[2] there is a common payoff function u such that $u_1 = \cdots = u_n = u$, and

[3] $u(s) \in \{0, 1\}$ for all strategy profiles $s = (s_1, \dots, s_n)$.

To simplify notation for binary games we write σ_i for the probability that player i plays pure strategy A. So, the probability that player i plays pure strategy B equals $1 - \sigma_i$. We write $\varepsilon_i = 1 - \sigma_i$. We also write $\sigma = (\sigma_i)_{i \in N}$ for a strategy profile, and we denote the space of strategy profiles by Σ. So, geometrically speaking Σ is a hypercube of dimension n.

cubistic sets. We also need the following terminology. For a set $T \subset N$, define the characteristic vector $e_T \in \mathbb{R}^N$ by

$$e_{Ti} = \begin{cases} 1 & \text{if } i \in T \\ 0 & \text{otherwise.} \end{cases}$$

The (standard) hypercube Σ in $\mathbb{R}^N$ is the convex hull of all characteristic vectors in $\mathbb{R}^N$. A subset $F \subset \Sigma$ of the hypercube is a face of Σ if there are sets Z and P in N such that

$$F = \{\sigma \in \Sigma \mid \sigma_i = 0 \text{ for all } i \in Z \text{ and } \sigma_j = 1 \text{ for all } j \in P\}.$$

The characteristic vectors that are elements of a face F are called the vertices of F. It is straightforward to check that a face F is the convex hull of its vertices.

A subset C of Σ is called a *cubistic set* if

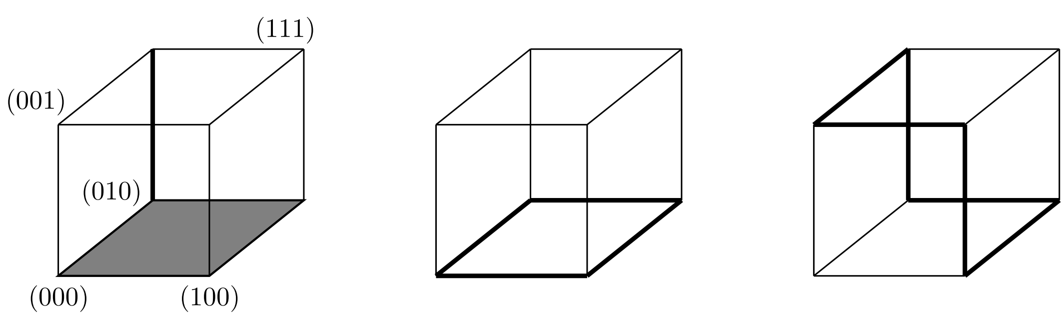
[1] C is the union of faces of Σ

[2] if for a face F of Σ all vertices of F are elements of C, then $F \subseteq C$. [3]

examples. We give three examples in $\mathbb{R}^3$. Consider the sets

$$\begin{aligned}
C_1 &= \mathrm{ch}\{(0,0,0),(1,0,0),(0,1,0),(1,1,0)\} \bigcup \mathrm{ch}\{(0,1,0),(0,1,1)\} \\
C_2 &= \mathrm{ch}\{(0,0,0),(1,0,0)\} \bigcup \mathrm{ch}\{(1,0,0),(1,1,0)\} \bigcup \\
&\quad\ \mathrm{ch}\{(1,1,0),(0,1,0)\} \bigcup \mathrm{ch}\{(0,1,0),(0,0,0)\} \\
C_3 &= \mathrm{ch}\{(1,0,0),(1,1,0)\} \bigcup \mathrm{ch}\{(1,1,0),(0,1,0)\} \bigcup \mathrm{ch}\{(0,1,0),(0,1,1)\} \bigcup \\
&\quad\ \mathrm{ch}\{(0,1,1),(0,0,1)\} \bigcup \mathrm{ch}\{(0,0,1),(1,0,1)\} \bigcup \mathrm{ch}\{(1,0,1),(1,0,0)\}
\end{aligned}$$

depicted below as subsets of the unit cube in $\mathbb{R}^3$.



The set C_1 *The set C_2* *The set C_3*

[3] This is equivalent to *n-convexity* of C. C is n-convex if for all points $(\rho_i, \sigma_{-i}) \in C$, $(\tau_i, \sigma_{-i}) \in C$ and all scalars $0 \leq \alpha \leq 1$ the convex combination $(1-\alpha)(\rho_i, \sigma_{-i}) + \alpha(\tau_i, \sigma_{-i})$ is in C. The notion of n-convexity extends the notion of biconvexity in Aumann and Hart [1].

The set C_1 is cubistic. The set C_2 is not cubistic, since the vertices $(000), (100), (010)$, and (110) are elements of C_2, while the convex hull of these four vertices is not a subset of C_2. The set C_3 is again cubistic.

EULER CHARACTERISTIC. A collection $\mathcal{D}$ of polytopes in $\mathbb{R}^n$ is called a polyhedral complex when

[a] for every polytope $P \in \mathcal{D}$ and every face F of P, F is an element of $\mathcal{D}$, and

[b] for every $P, Q \in \mathcal{D}$, $P \cap Q$ is a face of both P and Q.

The set $D = \bigcup_{P \in \mathcal{D}} P$ is the carrier of the complex $\mathcal{D}$. A set D in $\mathbb{R}^n$ that is the carrier of a polyhedral complex is called a polyhedral set.

Let $\mathcal{D}$ be a polyhedral complex. For $k \in \mathbb{N}$, let $\mathcal{D}_k$ denote the collection of polytopes $P \in \mathcal{D}$ of dimension k. Since $\mathcal{D}$ is a polyhedral complex in $\mathbb{R}^n$, it is clear that $\mathcal{D}_k = \phi$ for $k > n$. The Euler characteristic $\chi(\mathcal{D})$ of the polyhedral complex is defined by

$$\chi(\mathcal{D}) = \sum_{k=0}^{n} (-1)^k \cdot |\mathcal{D}_k|,$$

where $|\mathcal{D}_k|$ denotes the number of elements of $\mathcal{D}_k$. It can be shown that the Euler characteristic is a topological invariant. [4] In particular, for any two polyhedral complexes $\mathcal{D}$ and $\mathcal{E}$ whose carriers are homeomorphic, we have $\chi(\mathcal{D}) = \chi(\mathcal{E})$. This further implies that for any polyhedral set D, and any two complexes $\mathcal{D}$ and $\mathcal{E}$ whose carrier equals D, we have $\chi(\mathcal{D}) = \chi(\mathcal{E})$. Hence, the Euler characteristic depends only on the carrier, and the Euler characteristic $\chi(D)$ of the polyhedral set D is a well-defined integer number.

3 Binary games and success sets

In this section we prove the first two facts mentioned in the introduction. We show that the success set of a binary game is a cubistic set, and conversely, that for every cubistic set C there is a binary game whose success set equals C.

Consider a binary game (N, u), where u is the payoff function common to all players. As said before, Σ is the set of strategy profiles $\sigma = (\sigma_i)_{i \in N}$, where $0 \leq \sigma_i \leq 1$ for all $i \in N$. The *success set* of the binary game (N, u) is the subset C of Σ defined by

$$C = \{\sigma \in \Sigma \mid u(\sigma) = 1\}.$$

[4] It is a known fact that the Euler characteristic is a homotopy invariant, a stronger notion than topological invariance.

We prove the following Theorem which characterizes cubistic sets as maximizers of common payoff functions. Fix the set of players N. So, also the hypercube Σ of profiles of mixed strategies is fixed.

Theorem 3.1 *For a set $C \subset \Sigma$ the following two statements are equivalent.*

[1] *C is a cubistic set*

[2] *there is a common payoff function u such that the set C is the success set of the binary game (N, u).*

Proof. We show both directions separately.

A. Suppose that C is cubistic. Define $T \subseteq \{0,1\}^n$ by

$$T = \{s \in \{0,1\}^n \mid s \in C\}.$$

Let $u \colon \Sigma \to \mathbb{R}$ be the multilinear extension of the function $u \colon \{0,1\}^n \to \mathbb{R}$ defined by

$$u(s) = \begin{cases} 1 & \text{for} \quad s \in T \\ 0 & \text{for} \quad s \notin T \end{cases}$$

We show that C is the success set of (N, u). Take $\sigma \in C$. Since C is cubistic, there is a face F of Σ with $\sigma \in F \subseteq C$. Since for every vertex $s \in F$ we have $u(s) = 1$, it is straightforward to check that also $u(\sigma) = 1$.

Conversely, take $\sigma \notin C$. Let F be the smallest face of Σ that contains σ. Since $\sigma \notin C$, we know that there is at least one vertex e^T in F with $e^T \notin C$. Then, $u(e^T) = 0$, so that by multilinearity $u(\sigma) < 1$. Hence, C is the success set of (N, u).

B. Suppose that C is the success set of the binary game (N, u). Take $\sigma \in C$. So, $u(\sigma) = 1$. Let F be the minimal face of Σ that contains σ. Then, since $u(s) \in \{0, 1\}$ for every vertex of Σ, necessarily $u(s) = 1$ for every vertex s of F by the multilinearity of u. Hence, C is the union of faces of Σ.

Further, if $u(s) = 1$ for all vertices of a face F of C, then also $u(\sigma) = 1$ for all $\sigma \in F$ by the multilinearity of u. ■

4 Binary games and Nash equilibria

In this section we prove Fact 3 stated in the introduction. We show that the success set of a binary game is the union of Nash components of that game. More precisely, there exists an open neighborhood of the success set such that every Nash equilibrium in the open neighborhood is an element of the success set.

Our proof relies on a rather lengthy argument using higher-order polynomial expansions (claim 2 in the proof). One of the authors (Balkenborg [2]) already proved a slightly more general statement, which is published as Proposition 4 in Balkenborg and Schlag [3].

Let (N, u) be a binary game, and let C be its success set. Since in any strategy profile $\sigma \in C$ every player receives the maximal payoff 1, it is clear that every $\sigma \in C$ is a Nash equilibrium.

Theorem 4.1 *There exists an open subset $V \supseteq C$ of Σ such that for every strategy profile $\sigma \in V$ it holds that σ is a Nash equilibrium of (N, u) precisely when $\sigma \in C$.*

Proof. We assume $C \neq \Sigma$. After a few preparatory observations in part A, we construct the set V in part B, and argue that it is open. We then show in part C that a strategy profile $\sigma \in V$ that is not in C is not a Nash equilibrium.

A. We first need to make a few observations. Since C is the success set of the binary game (N, u), we know by Theorem 3.1 that C is cubistic. In particular, when $\mathcal{F}$ is the collection of faces F of Σ with $F \subseteq C$, then $C = \bigcup_{F \in \mathcal{F}} F$. Take $F \in \mathcal{F}$. Write

$$K = \{i \in N \mid \tau_i \in \{0, 1\} \text{ for all } \tau \in F\}.$$

Assume wlog that $\tau_i = 1$ for all $i \in K$. For $\eta \in (0, \frac{1}{2})$, define

$$F(\eta) = \{\tau \in F \mid \eta \leq \tau_i \leq 1 - \eta \text{ for all } i \notin K\}.$$

For a set $A \subseteq K$ and $\tau \in F(\eta)$ we define strategy profile τ_A by

$$\tau_{Ai} = \begin{cases} 0 & \text{if } i \in A \\ \tau_i & \text{if } i \notin A. \end{cases}$$

Let D denote the collection of sets $A \subseteq K$ with $u(\tau_A) = 1$. Let E denote the collection of sets $B \subseteq K$ with $u(\tau_B) < 1$. Note that, due to the fact that C is cubistic, the set D, and hence also E, does not depend on the specific element $\tau \in F(\eta)$, but only on F. Write

$$m = \frac{1}{2} \cdot \min\{1 - u(\tau_B) \mid \tau \in F(\eta), B \in E\}.$$

We first argue that m is well-defined. Since C is cubistic and $C \neq \Sigma$, there is $B \subseteq K$ with $u(\tau_B) < 1$. Then $B \in E$. So, m is properly defined. Since

[1] for each $\tau \in F(\eta)$ and $B \in E$ we have $1 - u(\tau_B) > 0$,

[2] for each $B \in E$ the function $\tau \mapsto 1 - u(\tau_B)$ is continuous on $F(\eta)$, and

[3] $F(\eta)$ is compact,

we know that $m > 0$.

B. Construct the set $V \supseteq C$ as follows. Fix $F \in \mathcal{F}$ and $\eta > 0$. Given $F(\eta)$, define m as above. Define $V(F, \eta)$ as the set of strategy profiles σ with

(1) $\sigma_i > \frac{2^n}{2^n + m \cdot n}$ for all $i \in K$
(2) $\eta < \sigma_i < 1 - \eta$ for all $i \notin K$.

Clearly $V(F, \eta)$ is open. Moreover, $F(2 \cdot \eta) \subseteq V(F, \eta)$. Therefore

$$V = \bigcup_{F,\eta} V(F, \eta)$$

is an open set that contains C.

C. Take $\sigma \in V$ with $u(\sigma) < 1$. We show that σ is not a Nash equilibrium. Since $\sigma \in V$, we can take $F \in \mathcal{F}$ and $\eta > 0$ with $\sigma \in V(F, \eta)$. Define $\kappa \in \Sigma$ by

$$\kappa_i = \begin{cases} 1 & \text{if } i \in K \\ \sigma_i & \text{if } i \notin K. \end{cases}$$

Then $\kappa \in F(\eta)$. We show that

$$\sum_{j \in K} [u(\sigma|\kappa_j) - u(\sigma)] > 0.$$

Then $u(\sigma|\kappa_j) - u(\sigma) > 0$ for at least one player $j \in K$, so that σ is not a Nash equilibrium. We prove the above inequality in 3 separate claims.

Let $\varepsilon_i = 1 - \sigma_i$ be the weight that player i puts on pure strategy B in the strategy profile σ under consideration. For $A \subseteq K$ and $j \in A$ define

$$\Delta(A, j) = \left[\prod_{i \in A} \varepsilon_i \prod_{i \in K \setminus A} \sigma_i \right] \cdot \left(u(\kappa_{A \setminus j}) - u(\kappa_A) \right).$$

Claim 1. It holds that

$$\sum_{j \in K} [u(\sigma|\kappa_j) - u(\sigma)] = \sum_{j \in K} \sum_{\substack{A \subseteq K \\ j \in A}} \Delta(A, j).$$

Proof of claim 1. For every $j \in K$,

$$\begin{aligned} u(\sigma) &= \sum_{A\subseteq K} \left[\prod_{i\in A} \varepsilon_i \prod_{i\in K\setminus A} \sigma_i \right] \cdot u(\kappa_A) \\ &= \varepsilon_j \cdot \sum_{\substack{A\subseteq K \\ j\in A}} \left[\prod_{\substack{i\in A \\ i\neq j}} \varepsilon_i \prod_{i\in K\setminus A} \sigma_i \right] \cdot u(\kappa_A) + \sigma_j \cdot \sum_{\substack{A\subseteq K \\ j\notin A}} \left[\prod_{i\in A} \varepsilon_i \prod_{\substack{i\in K\setminus A \\ i\neq j}} \sigma_i \right] \cdot u(\kappa_A). \end{aligned}$$

Note that the only difference between σ and $(\sigma|\kappa_j)$ is that $\kappa_j = 1$. So, using the above expression and the fact that $1 - \sigma_j = \varepsilon_j$, we get that

$$\begin{aligned} u(\sigma|\kappa_j) - u(\sigma) &= \varepsilon_j \cdot \sum_{\substack{A\subseteq K \\ j\notin A}} \left[\prod_{i\in A} \varepsilon_i \prod_{\substack{i\in K\setminus A \\ i\neq j}} \sigma_i \right] \cdot u(\kappa_A) - \varepsilon_j \cdot \sum_{\substack{A\subseteq K \\ j\in A}} \left[\prod_{\substack{i\in A \\ i\neq j}} \varepsilon_i \prod_{i\in K\setminus A} \sigma_i \right] \cdot u(\kappa_A) \\ &= \sum_{\substack{A\subseteq K \\ j\in A}} \left[\prod_{i\in A} \varepsilon_i \prod_{i\in K\setminus A} \sigma_i \right] \cdot u(\kappa_{A\setminus j}) - \sum_{\substack{A\subseteq K \\ j\in A}} \left[\prod_{i\in A} \varepsilon_i \prod_{i\in K\setminus A} \sigma_i \right] \cdot u(\kappa_A) \\ &= \sum_{\substack{A\subseteq K \\ j\in A}} \left[\prod_{i\in A} \varepsilon_i \prod_{i\in K\setminus A} \sigma_i \right] \cdot \big(u(\kappa_{A\setminus j}) - u(\kappa_A) \big) \\ &= \sum_{\substack{A\subseteq K \\ j\in A}} \Delta(A, j) \end{aligned}$$

Hence,

$$\sum_{j\in K} [u(\sigma|\kappa_j) - u(\sigma)] = \sum_{j\in K} \sum_{\substack{A\subseteq K \\ j\in A}} \Delta(A, j).$$

This ends the proof of claim 1.

Write

$$Y = \{(A, j)|A \subseteq K, j \in A, \Delta(A, j) < 0\} \quad \text{and} \quad Z = \{(B, k)|B \subseteq K, k \in B, \Delta(B, k) > 0\}.$$

Claim 2. For every $(A, j) \in Y$ there is $(B, k) \in Z$ with

$$-\Delta(A, j) < n \cdot 2^{-n} \cdot \Delta(B, k).$$

Proof of claim 2. Take $(A, j) \in Y$. Then in particular

$$u(\kappa_{A\setminus j}) - u(\kappa_A) < 0.$$

So, $u(\kappa_{A\setminus j}) < u(\kappa_A) \leq 1$, which implies that $A\setminus j \notin D$. Since $\phi \in D$, there must be a $B \subseteq A\setminus j$ and a $k \in B$ with $B \notin D$ and $B \setminus k \in D$. We first show that

$$-\Delta(A, j) \leq n \cdot 2^{-(n+1)} \cdot \Delta(B, k).$$

Since $0 < u(\kappa_A) - u(\kappa_{A\setminus j}) \leq 1$, we have

$$\begin{aligned} -\Delta(A,j) &= \left[\prod_{i\in A}\varepsilon_i \prod_{i\in K\setminus A}\sigma_i\right]\cdot\big(u(\kappa_A)-u(\kappa_{A\setminus j})\big) \leq \prod_{i\in A}\varepsilon_i \cdot \prod_{i\in K\setminus A}\sigma_i \\ &\leq \frac{\varepsilon_j}{\sigma_j}\cdot\prod_{i\in B}\varepsilon_i\cdot\prod_{i\in K\setminus B}\sigma_i. \end{aligned}$$

Since $\sigma \in V(F,\eta)$, we know that $\sigma_j > \frac{2^n}{2^n+m\cdot n}$. So, $\frac{\varepsilon_j}{\sigma_j} < m\cdot n\cdot 2^{-n}$. Also, since $\kappa\in F(\eta)$ and $B\in E$, we know that $m \leq \frac{1}{2}\cdot(1-u(\kappa_B))$. So, using the above inequality,

$$\begin{aligned} -\Delta(A,j) &\leq \frac{\varepsilon_j}{\sigma_j}\cdot\prod_{i\in B}\varepsilon_i\cdot\prod_{i\in K\setminus B}\sigma_i \\ &\leq m\cdot n\cdot 2^{-n}\cdot\prod_{i\in B}\varepsilon_i\cdot\prod_{i\in K\setminus B}\sigma_i \\ &\leq n\cdot 2^{-(n+1)}\cdot\left[\prod_{i\in B}\varepsilon_i\cdot\prod_{i\in K\setminus B}\sigma_i\right](1-u(\kappa_B)) \\ &= n\cdot 2^{-(n+1)}\cdot\left[\prod_{i\in B}\varepsilon_i\cdot\prod_{i\in K\setminus B}\sigma_i\right](u(\kappa_{B\setminus k})-u(\kappa_B)) \\ &= n\cdot 2^{-(n+1)}\cdot\Delta(B,k). \end{aligned}$$

Then, since $\Delta(A,j) < 0$, automatically $\Delta(B,k) > 0$. So, $(B,k)\in Z$ and also

$$-\Delta(A,j) \leq n\cdot 2^{-(n+1)}\cdot\Delta(B,k) < n\cdot 2^{-n}\cdot\Delta(B,k).$$

End proof of claim 2.

Claim 3. The expression

$$\sum_{j\in K}\sum_{\substack{A\subseteq K\\ j\in A}}\Delta(A,j)$$

is strictly positive.

Proof of claim 3. Note that

$$\sum_{j\in K}\sum_{\substack{A\subseteq K\\ j\in A}}\Delta(A,j) = \sum_{(A,j)\in Y}\Delta(A,j) + \sum_{(B,k)\in Z}\Delta(B,k).$$

We consider two cases.

3a. If $Y = \phi$. We show that Z is not empty. Define $L = \{i\in N | \sigma_i \neq \kappa_i\}$. Recall that $u(\sigma) < 1$. Further, $\kappa\in F$, so that $u(\kappa) = 1$. So, $\sigma\neq\kappa$, which implies that $L\neq\phi$. Since $u(\sigma) < 1$, there must be a $B\subseteq L$ and a $k\in B$ such that $u(\kappa_B) < 1$ and $u(\kappa_{B\setminus k}) = 1$. We show that $(B,k)\in Z$.

Since by definition $\kappa_i = \sigma_i$ for all $i \notin K$, we know that $L \subseteq K$. So, also $B \subseteq K$. We show that $\Delta(B, k) > 0$. Since $B \subseteq K$, $\kappa_i = 1$ for all $i \in B$. So, since $\sigma_i \neq \kappa_i$ for all $i \in B$, $\varepsilon_i = 1 - \sigma_i > 0$ for all $i \in B$. Also, since $\sigma \in V(F, \eta)$, $\sigma_i > \frac{2^n}{2^n + m \cdot n} > 0$ for all $i \in K \setminus B$. Hence,

$$\Delta(B, k) = \left[\prod_{i \in B} \varepsilon_i \prod_{i \in K \setminus B} \sigma_i \right] \cdot \big(u(\kappa_{B \setminus k}) - u(\kappa_B) \big) > 0.$$

3b. If $Y \neq \phi$. Then by claim 2 there is a map $F \colon Y \to Z$ with

$$-\Delta(A, j) < n \cdot 2^{-n} \cdot \Delta(F(A, j))$$

for all $(A, j) \in Y$. From this it easily follows that

$$\sum_{(A,j) \in Y} -\Delta(A, j) < \sum_{(B,k) \in Z} \Delta(B, k).$$

Hence

$$\sum_{j \in K} \sum_{\substack{A \subseteq K \\ j \in A}} \Delta(A, j) = \sum_{(A,j) \in Y} \Delta(A, j) + \sum_{(B,k) \in Z} \Delta(B, k) > 0.$$

This concludes the proof. ∎

5 Cubistic and simplicial sets

In this section we prove Fact 4 of the introduction. We show that for every simplicial set there exists a cubistic set that is homeomorphic to the given simplicial set.

A collection $\{v_1, \dots, v_k\}$ of points in $\mathbb{R}^n$ is called independent if for all $a_1, \dots, a_k$ in $\mathbb{R}$

$$a_1 \cdot v_1 + \cdots + a_k \cdot v_k = 0$$

implies $a_1 = \cdots = a_k = 0$. A collection $\{v_0, \dots, v_k\}$ of points in $\mathbb{R}^n$ is geometrically (or affine) independent if the collection $\{v_1 - v_0, \dots, v_k - v_0\}$ is independent in $\mathbb{R}^n$. Note that a subset of a geometrically independent collection is also geometrically independent. The convex hull

$$\{t_0 \cdot v_0 + \cdots + t_k \cdot v_k \mid t_i \leq 0 \text{ for all } i \text{ and } \sum_i t_i = 1\}$$

of a geometrically independent set $\{v_0, \dots, v_k\}$ is called a simplex. We also say that $\{v_0, \dots, v_k\}$ spans the simplex. A subset F of a simplex S is a face of S if F is spanned by a subset of the (unique) geometrically independent set $\{v_0, \dots, v_k\}$ that spans S.

A polyhedral complex $\mathcal{S}$ in $\mathbb{R}^n$ whose elements are all simplices is called a simplicial complex. The set $S = \bigcup_{D \in \mathcal{S}} D$ is called the carrier of the simplicial complex $\mathcal{S}$. A set $S \subset \mathbb{R}^n$ is called a simplicial set if it is the carrier of a simplicial complex. Let e_i denote the i^{th} unit vector in $\mathbb{R}^N$. The standard simplex Δ in $\mathbb{R}^N$ is the simplex spanned by the collection $\{e_i \mid i \in N\}$ of unit vectors in $\mathbb{R}^N$. A simplicial set in $\mathbb{R}^N$ is called standard if it is a union of faces of the standard simplex Δ. We first need the following observation.

Proposition 5.1 *Any simplicial set is homeomorphic to a standard simplicial set.*

Proof. Let S be a simplicial set, and let $\mathcal{S}$ be any simplicial complex whose carrier is S. Let V be the collection of vertices (simplices of dimension zero) in $\mathcal{S}$. For $v \in V$, let $e(v) \in \mathbb{R}^V$ denote the unit vector defined by, for every $w \in V$,

$$e(v)_w = \begin{cases} 1 & \text{if } w = v \\ 0 & \text{if } w \neq v. \end{cases}$$

For every $D \in \mathcal{S}$, let $F(D)$ denote the convex hull of the set

$$\{e(v)|v \text{ is a vertex of } D\}.$$

It is straightforward to check that $\mathcal{T} = \{F(D)|D \in \mathcal{S}\}$ is a simplicial complex. So, $T = \bigcup_{E \in \mathcal{T}} E$ is a simplicial set. Further, since each $F(D)$ is a face of the standard simplex in $\mathbb{R}^V$, T is also a union of faces of that standard simplex. Hence, T is a standard simplicial set. Further, the piecewise linear extension of the map $v \mapsto e(v)$ is a homeomorphism from S to T. ∎

We show that for any standard simplicial set there is a cubistic set that is homeomorphic to it. For $x = (x_i)_{i \in N} \in \mathbb{R}^N$, define

$$\|x\|_1 = \sum_{i \in N} |x_i| \quad \text{and} \quad \|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

the Manhattan norm—also called ℓ_1 norm—and the supremum norm respectively. For $\kappa \in \Delta$, consider the ray

$$L_\kappa = \{\lambda \cdot \kappa \mid \lambda \geq 0\}$$

it generates. This ray intersects the standard simplex Δ in κ itself, and the hypercube Σ in the point $\frac{\kappa}{\|\kappa\|_\infty}$. This relationship defines a homeomorphism Φ between Δ and $\Phi(\Delta) \subseteq \Sigma$ by $\Phi(\kappa) := \frac{\kappa}{\|\kappa\|_\infty}$. Using the homeomorphism Φ we obtain the following result.

Theorem 5.2 *Every standard simplicial set is homeomorphic to a cubistic set.*

Proof. Let S be a standard simplicial set in Δ. Define $C = \Phi(S)$. We show that C is a cubistic set. Clearly, Φ maps each face $F \subseteq S$ of the standard simplex onto a union of faces of the standard cube, so that C is a union of faces of Σ. In order to show that C is cubistic, take a face G in Σ whose vertices are all in C. We show that G is in C. Since G is a face of Σ there are sets P and Z in N with

$$G = \{\sigma \in \Sigma \mid \sigma_i = 0 \text{ for all } i \in Z \text{ and } \sigma_j = 1 \text{ for all } j \in P\}.$$

Take $\sigma \in G$. We show that $\sigma \in C$. Define $\tau \in \Sigma$ by

$$\tau_i = \begin{cases} 1 & \text{if } \sigma_i > 0 \\ 0 & \text{if } \sigma_i = 0. \end{cases}$$

Then clearly τ is a vertex of G. So, $\tau \in C = \Phi(S)$. So, we can take $\kappa \in S$ with $\Phi(\kappa) = \tau$. Let F be the smallest face of Δ in S that still contains κ. Then, since by the definition of Φ we have that $\kappa_i > 0$ precisely when $\tau_i = 1$ precisely when $\sigma_i > 0$ precisely when $\frac{\sigma}{\|\sigma\|_1} > 0$, we can conclude that $\mu = \frac{\sigma}{\|\sigma\|_1} \in F$. So, $\mu \in F \subseteq S$. Hence, since $\|\sigma\|_\infty = \|\tau\|_\infty = 1$, we obtain $\sigma = \Phi(\mu) \in C$. ■

6 Universality

The first, and main, consequence of our facts is the universality of Nash components. To be precise, a collection $\mathcal{C}$ of sets is universal for a collection $\mathcal{D}$ of sets if

[a] $\mathcal{C}$ is a subset of $\mathcal{D}$, and more importantly

[b] for every set $D \in \mathcal{D}$ there is a set $C \in \mathcal{C}$ that is homeomorphic to D.

Our results imply the following Theorem.

Theorem 6.1 *The collection of success sets is universal for the collection of simplicial sets. Consequently, the collection of Nash components of binary games is universal for the collection of connected compact semi-algebraic sets.*

Proof. Since a success set is cubistic, it is polyhedral. Therefore, by Lemma 4 of Vermeulen and Jansen [13], any success set is also simplicial.

Take a polyhedral set D. As we just noted, D is also simplicial. So, by Theorems 5.1 and 5.2, the set D is homemorphic to a cubistic set. Then, by Theorem 3.1, the set D is homeomorphic to the success set of a binary game. This concludes the proof of the first statement.

In order to show the second claim, first note that a Nash component is a connected compact semi-algebraic set. Take a connected, compact semi-algebraic set K. According to Llojasiewicz

[9], there exists a simplicial set C that is homeomorphic to K. Then, according to the claim we just showed, there is a success set D of a binary game that is homeomorphic to C, and hence also to K. According to Theorem 4.1 there exists an open set V that contains D such that D equals the set of Nash equilibria in V. Hence, since D—being homeomorphic to K—is connected, it is a Nash component. ■

The celebrated result in geometry by Llojasiewicz [9] on analytic sets shows that the homeomorphisms involved in the above proof of universality are piecewise analytic. An alternative proof by Hironaka in Algebraic Geometry [8], specifically designed for semi-algebraic sets, shows that the homeomorphisms involved can even be taken to be semi-algebraic.

an example. We construct a binary game with a Nash component homeomorphic to the connected and compact semi-algebraic set

$$S_1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}.$$

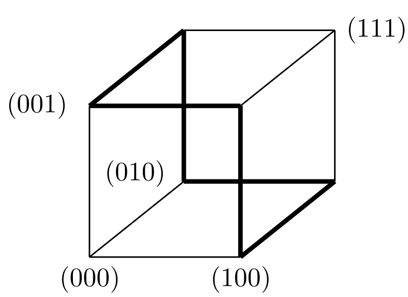
We first construct a standard simplicial set that is homeomorphic to S_1. Consider the standard simplex

$$\Delta = \{(\sigma_1, \sigma_2, \sigma_3) | \sigma_1 + \sigma_2 + \sigma_3 = 1, \sigma_1 \geq 0, \sigma_2 \geq 0, \sigma_3 \geq 0\}.$$

Its boundary

$$B = \{(\sigma_1, \sigma_2, \sigma_3) \in \Delta | \sigma_1 = 0 \vee \sigma_2 = 0 \vee \sigma_3 = 0\}$$

is a standard simplicial set that is homeomorphic to S_1. The subset $\Phi(B)$ of the cube Σ is the cubistic set C_3 depicted below.



The set C_3

The resulting 3-player $2 \times 2 \times 2$ binary game is

$$\begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0,0,0 & 1,1,1 \\ 1,1,1 & 1,1,1 \end{bmatrix} \end{array} \qquad \begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 1,1,1 & 1,1,1 \\ 1,1,1 & 0,0,0 \end{bmatrix} \end{array}$$

where the left-hand matrix reports the payoffs when player 3 chooses A (corresponding to $\sigma_3 = 1$), and the right-hand matrix reports the payoffs when player 3 chooses B (corresponding to $\sigma_3 = 0$). The set C_3 is the success set of this game, and hence a Nash component of the set of Nash equilibria. However, the game has one more Nash equilibrium $\sigma_1 = \sigma_2 = \sigma_3 = \frac{1}{2}$, which forms a singleton Nash component. This shows that the success set of a binary game need not be the entire set of Nash equilibria.

7 Success sets and stability

Let C be the success set of a binary game. Then C is a strict equilibrium set of this game as defined in Balkenborg and Schlag [3]. This means that if $(y_i, x_{-i}) \in C$ and if z_i is a best reply of player i against x_{-i}, then $(z_i, x_{-i}) \in C$. Since all elements of C yield the maximum payoff of 1 to all players, it is clear that C satisfies this condition. Therefore Theorem 6 of Balkenborg and Schlag [3] yields

Theorem 7.1 *The success set C is asymptotically stable and consists of stable restpoints for the replicator dynamics.*

Then Corollary 1 in Demichelis and Ritzberger [6] yields

Proposition 7.2 *Suppose the success set C has non-zero Euler characteristic. Then it contains a strategically stable set in the sense of Mertens [10].*

8 Success sets and index

In this section we provide an alternative for the constructions in Govindan, von Schemde and von Stengel [7], where for every integer $k \in \mathbb{Z}$ a game with a Nash component of index k is presented.

Let $k \in \mathbb{Z}$ be given. We construct a connected polyhedral set of dimension 2 in $\mathbb{R}^3$ with Euler characteristic k. [5]

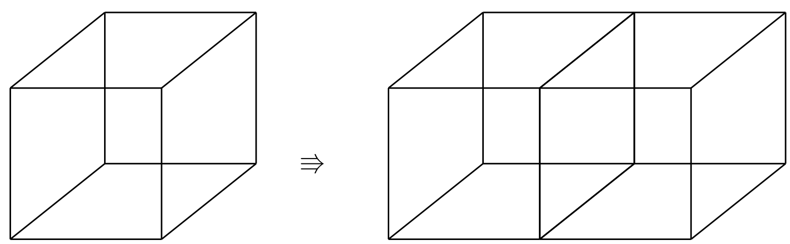
Take a polyhedral complex $\mathcal{D}$ in $\mathbb{R}^3$ with carrier D. In case that the dimension of D is 2 (which means that $\mathcal{D}_3 = \phi$ and $\mathcal{D}_2 \neq \phi$), the formula for the Euler characteristic $\chi(D)$ simplifies to

$$\chi(D) = V - E + F,$$

where $V = |\mathcal{D}_0|$ is the number of vertices in $\mathcal{D}$, $E = |\mathcal{D}_1|$ is the number of edges in $\mathcal{D}$, and $F = |\mathcal{D}_2|$ is the number of faces (2-dimensional polytopes) in $\mathcal{D}$. We use this formula in the remainder of this section to compute the Euler characteristic of 2-dimensionsal polyhedral sets.

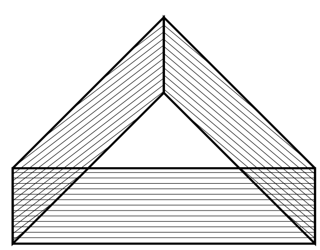
[5] The fact that the polyhedral set we construct is of dimension 2 in $\mathbb{R}^3$ is not essential. We can in general construct a polyhedral set of dimension d in $\mathbb{R}^n$ with Euler characteristic k for any d with $2 \leq d \leq n-1$. Negative integers can be achieved with polyhedral sets of dimension 1 in $\mathbb{R}^2$.

For $k = 2$ we take boundary of the unit cube in $\mathbb{R}^3$, indicated on the left-hand side in the display below, as our polyhedral set. The associated complex consists of 6 squares (the facets of the unit cube), 12 line segments (the edges of the unit cube), and 8 points, the vertices of the unit cube. Its Euler characteristic is therefore $F - E + V = 6 - 12 + 8 = 2$. For $k = 3$, we glue a second unit cube onto one of the sides of the first cube. The result is indicated on the right-hand side of the display below. Its Euler characteristic is $11 - 20 + 12 = 3$.

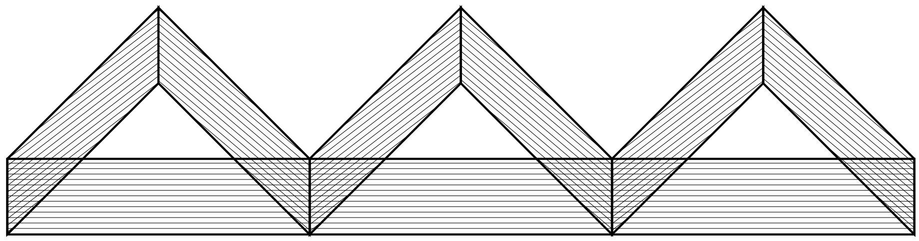


In general, adding a next cube adds 5 new squares, 8 new edges, and 4 new vertices, so that the Euler characteristic increases by $5-8+4 = 1$. Hence, a sequence of k cubes glued together in a row has Euler characteristic $k + 1$.

For $k = 0$, we take the bracelet (see below) which is a polyhedral set consisting of three rectangles glued together in a tubular shape. The Euler characteristic is $3 - 9 + 6 = 0$.



Gluing bracelets together yields negative Euler characteristics. For example, for $k = -2$ we need three bracelets as indicated below.



The Euler characteristic of the three adjoined bracelets is $9 - 25 + 14 = -2$. The Euler characteristic of two adjoined bracelets is $6 - 17 + 10 = -1$. In general, addition of an extra bracelet adds 3 squares, 8 edges, and 4 vertices. Thus the Euler characteristic changes by $3 - 8 + 4 = -1$, so a net reduction of 1. Hence, a sequence of $m + 1$ bracelets attached along an edge yields Euler characteristic $-m$.

Thus, for every integer k there exists a polyhedral set with Euler characteristic k.

Theorem 8.1 *The Euler characteristic of a success set equals its index. Moreover, for every integer k, there exists a binary game with a connected success set C and $\chi(C) = k$.*

Proof. Note that the success set of a binary game is a polyhedral set so its Euler characteristic is defined. From Theorem 7.1 above, together with Theorem 1 of Demichelis and Ritzberger [6] it follows that the Euler characterisic of the success set equals its index.

In order to prove the second claim, take the polyhedral set constructed above with Euler characteristic k. Since a polyhedral set is semi-algebraic, we know that it is also a simplicial set. [6] Then, since the Euler characteristic is a topological invariant, the success set associated with this polyhedral set has Euler characteristic k as well. ■

9 Concluding remarks and open questions

In our construction the number of strategies per player is as small as possible. However, our construction needs as many players as there are vertices in the initial simplicial complex. For example, any polyhedral set that is homeomorphic to the torus has at least 7 vertices [7]. So, our construction uses at least 7 players, and hence embeds the torus into the hypercube in $\mathbb{R}^7$,

[6]See Lemma 4 of Vermeulen and Jansen [13] for a direct construction.

[7]We thank Bernhard von Stengel for drawing our attention to this result.

whereas the torus can be embedded in $\mathbb{R}^3$. The question remains how far our construction is from the minimum dimension implementation.

Bubelis [4] provides a construction which identifies the set of Nash equilibria for an n-player ($n \geq 3$) game with the set of Nash equilibria in a 3-player game. This suggests that the collection of Nash equilibrium components in 3-player games is universal for the collection of compact and connected semi-algebraic sets. This question, or the equivalent question for 2-player games, is currently open.

We showed that every cubistic set is a finite union of Nash equilibrium components of a binary game. However, as we already noted in the example above, typically the game constructed will have additional Nash equilibria. Since the sum of the indices over all components equals one, this is necessarily the case if the Euler characteristic of the success set is not 1. It is an open question however whether in our construction the success set equals the set of Nash equilibria when the Euler characteristic (and hence the index) of the success set equals 1. More general, if a polyhedral set has Euler characteristic 1, can we construct a game such that the polyhedral set is homeomorphic to the set of *all* Nash equilibria in this game?

We considered here only the topological structure. One might more generally ask whether any compact connected semi-algebraic set is diffeomorphic or even algebraically equivalent to a Nash equilibrium component. Finally, another open question is whether similar results can be achieved for the sets of perfect or proper equilibria instead of the set of Nash equilibria.

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