Are the Treasures of Game Theory Ambiguous?*

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Abstract

Goeree and Holt (2001) observe that, for some parameter values, Nash equilibrium provides good predictions for actual behavior in experiments. For other payoff parameters, however, actual behavior deviates consistently from that predicted by Nash equilibria. They attribute the robust deviations from Nash equilibrium to actual players' considering not only marginal gains and losses but also total payoffs. In this paper, we show that optimistic and pessimistic attitudes towards strategic ambiguity may induce exactly such behavior.

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1 INTRODUCTION

Nash equilibrium (henceforth NE) is the most widely used equilibrium concept in game theory. Though a large and growing number of experimental studies indicate its weaknesses, it has proved difficult to find systematic patterns in the deviations from NE. Given this, we believe that it is worth considering alternative theories.

Goeree and Holt (2001) (henceforth GH) published an article with the provocative title *"Ten Little Treasures of Game Theory and Ten Intuitive Contradictions"* in which they claim that "for each of these ten games there is an experimental treatment in which behavior conforms nicely to predictions of NE" but where "a change in the payoff structure produces a large inconsistency between theoretical predictions and observed behavior".[1] In the present paper we restrict attention to the five one-shot games, which GH studied. We argue that many of the "inconsistencies" in these games can be explained by ambiguity.

Ambiguity describes situations where individuals cannot or do not assign subjective probabilities to uncertain events. This may be because the problem is complex or unfamiliar. There is by now considerable experimental evidence which shows that individuals treat ambiguous decisions differently from risks with known probabilities. The best known example is the Ellsberg paradox, Ellsberg (1961).[2] There is also experimental evidence that behavior in games is affected by ambiguity see Colman and Pulford (2007) or Eichberger, Kelsey, and Schipper (2007). We believe that ambiguity may be present in experimental games since the relevant uncertainty is the strategy choice of one's opponent. Human behavior is not intrinsically easy to predict. We believe that it is plausible that there may be ambiguity in GH's experiments since each game was only played once. Hence subjects did not have time to become familiar with the game or the behavior of their opponents.

We use a model of ambiguity axiomatized by Chateauneuf, Eichberger, and Grant (2008),

[1] Goeree and Holt (2001) p. 1402.

[2] This has been confirmed by the subsequent experimental literature, see for instance Camerer and Weber (1992) and Cohen, Jaffray, and Said (1985).

(henceforth CEG), in which ambiguity is modelled by non-additive beliefs. Applying this theory to games, implies that players will maximize a weighted average of the equilibrium pay-off, the best pay-off and the worst pay-off of any given strategy. Subsection 2.1 will introduce these preferences, provide some intuition for the parameters of this model and point to the experimental literature, which tries to estimate them.

Tversky and Wakker (1995) study the relationship between decision weights and attitudes towards risk and characterize the *possibility* and *certainty effects.* A majority of individuals appear to behave cautiously when there is ambiguity. Following Wakker (2001), who relates such behavior to a generalized version of the Allais paradox, we shall refer to such cautious behavior as *pessimism.* This article contains also a brief survey of the relevant experimental literature. Experimental evidence also shows, that a minority of individuals respond to ambiguity in the opposite way, which we shall refer to as *optimism.*

We model ambiguity in games by postulating that each player views the strategy choice by his/her opponents as ambiguous. Players may react to this ambiguity either in an optimistic way by over-weighting good outcomes or in a pessimistic way by over-weighting bad outcomes. Subsection 2.3 defines an *Equilibrium under Ambiguity (EUA),* the equilibrium notion which will be used in this paper.

Organization of the Paper The next section describes our basic model of ambiguity in games. In section 3 we argue that GH's results on one-shot games can be explained by ambiguity. In Section 4 we discuss competing theories such as Quantal Response Equilibrium and section 5 concludes. The appendix contains proofs of those results not proved in the text.

2 STRATEGIC AMBIGUITY

This section introduces our model of ambiguity and uses it as the basis of a solution concept for normal form games.

2.1 Non-additive beliefs

In the present paper we restrict attention to ambiguity in 2-player games, which requires the following notation. A 2-player game $\Gamma = \langle \{1, 2\}; S_1, S_2, u_1, u_2 \rangle$ consists of players, $i = 1, 2$, finite pure strategy sets S_i and payoff functions $u_i(s_i, s_{-i})$ for each player. The space of all strategy profiles is denoted by S. The notation, s_{-i}, denotes the strategy chosen by i's opponent. The set of all strategies for i's opponent is S_{-i}. We shall adopt the convention that female pronouns (she, her etc.) denote player 1 and male pronouns denote player 2.[3]

Beginning with Schmeidler (1989), ambiguous beliefs have been modelled as *capacities*, which are similar to subjective probabilities except that they are not necessarily additive. We shall use a model of ambiguity from CEG, which has the advantage that it is parsimonious in the number of parameters. This theory represents beliefs by a neo-additive capacity ν defined by:

$$\nu(A|\alpha, \delta, \pi) = \begin{cases} 1 & \text{for} \quad A = S_{-i}, \\ \alpha\delta + (1-\delta)\pi(A) & \text{for} \quad \emptyset \subsetneqq A \subsetneqq S_{-i}, \\ 0 & \text{for} \quad A = \varnothing, \end{cases}$$

where $\alpha, \delta \in [0, 1]$, π is an additive probability distribution π on S_{-i} and $\pi(A) := \sum_{s_{-i} \in S_{-i}} \pi_i(s_{-i})$. They show that preferences may be represented in the form:

$$V_i(s_i; \alpha, \delta, \pi) = \delta \left[\alpha \max_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) + (1-\alpha) \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) \right] + (1-\delta) \cdot \mathbf{E}_\pi u_i(s_i, s_{-i}),$$

where $\mathbf{E}_\pi u_i(s_i, s_{-i})$, denotes a conventional expectation taken with respect to the probability distribution π.[4]

One can interpret π as the decision-maker's belief. However (s)he may not be fully confident in this belief. Thus it is an ambiguous belief. His/her *confidence* is modelled by the weight $(1-\delta)$ given to the expected payoff $\mathbf{E}_\pi u_i(s_i, s_{-i})$. Or equivalently δ can be interpreted as a measure of the ambiguity the decision-maker perceives. The highest (resp. lowest) possible level of ambiguity corresponds to $\delta = 1$, (resp. $\delta = 0$). Ambiguity-attitude is measured by α, which represents the *optimism/pessimism* of the decision maker. Purely optimistic preferences are given by $\alpha = 1$, while the highest level of pessimism occurs when $\alpha = 0$. If $0 < \alpha < 1$, the individual is neither purely ambiguity-averse nor purely ambiguity-loving, since (s)he responds to ambiguity partly in an optimistic way by overweighting good outcomes and partly

[3] Of course this convention is for convenience only and bears no relation to the actual gender of subjects in GH's experiments.

[4] For simplicity, we will write, in slight abuse of notation, $V_i(s_i; \alpha, \delta, \pi)$ instead of $V_i(s_i; \nu(\cdot|\alpha, \delta, \pi))$.

in a pessimistic way by overweighting bad outcomes.

A possible interpretation is that the optimism parameter, α, is a personal characteristic of the decision maker like his/her risk preferences. In contrast, the degree of ambiguity, δ, may depend on the situation. In particular, one would expect there to be more ambiguity when players interact for the first time. Growing familiarity with the game and the behavior of opponents is likely to reduce ambiguity.

CEG also show that these preferences may also be represented in the multiple priors form:[5]

$$V_i\left(s_i;\alpha,\delta,\pi\right)=\alpha\max_{p\in\mathcal{P}}\mathbf{E}_p u_i\left(s_i,s_{-i}\right)+\left(1-\alpha\right)\min_{p\in\mathcal{P}}\mathbf{E}_p u_i\left(s_i,s_{-i}\right), \tag{1}$$

where $\mathcal{P}:=\left\{p\in\Delta\left(S_{-i}\right):p\geqslant\left(1-\delta\right)\pi\right\}$. For the case of one opponent with three pure strategies, Figure 1 shows the set of probability distributions $\mathcal{P}(\delta,\pi)$.

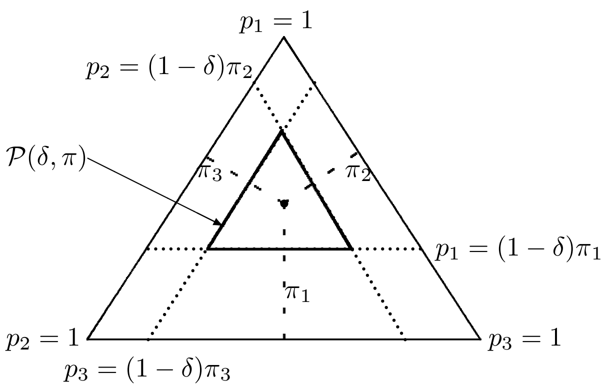

Figure 1. The set $\mathcal{P}(\delta,\pi)$

The multiple priors representation in equation (1) can be interpreted as follows. When an individual perceives a situation as ambiguous (s)he considers more than one probability distribution to be possible. He/she reacts to ambiguity partly in an optimistic way by using the most favorable possible probability and partly in a pessimistic way by using the least favorable distribution.

[5] Gilboa and Schmeidler (1989) axiomatized the multiple priors model, which represents ambiguous beliefs by sets of probability distributions. Multiple priors and non additive beliefs produce related models of ambiguity. However they are not, in general, identical

2.2 Evidence on Individual Decisions

Experiments on decision-making with known probabilities have shown that individuals tend to overweight both high and low probability events. As a result the decision weights assigned to events are an inverse S-shaped function compared to the given probability distribution, (see for instance, Gonzalez and Wu (1999) and Abdellaoui (2000)). This can be explained by insensitivity of perception in the middle of the range. For instance, the change from a probability of 0.55 to 0.60 is not perceived as great as the change from 0 to 0.05.

If probabilities are not known, a similar phenomena has been found (see, Kilka and Weber (2001)). Individuals overweight both highly likely and highly unlikely events. (In this case the likelihood of events is subjective.) This produces a pattern of decision weights like that illustrated in Figure 2. The curved line represents the decision weights of a typical experimental subject and the 45^o line represents SEU beliefs for comparison. This diagram is based on observations that subjects are willing to take courses of action, which yield high outcomes in unlikely events but refuse to accept even a small chance of bad outcomes. The more unfamiliar the source of uncertainty is the lower is the elevation of the curve, i.e. the curve shifts downwards in less familiar situations. This can be interpreted as an effect of ambiguity.

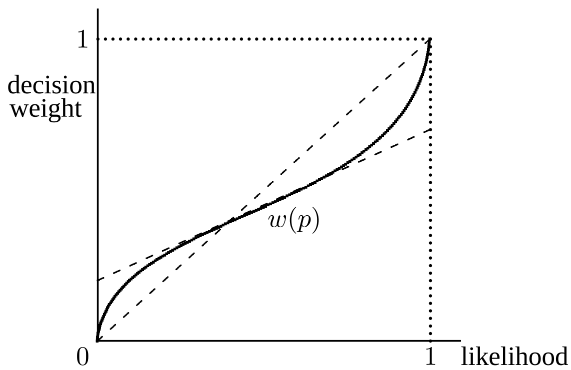

Figure 2. Inverse-S decision weights

Kilka and Weber (2001) report an experimental study of choices in financial markets, which was able to distinguish beliefs from decision weights. They found that decision weights deduced from actual choices were markedly non-additive. Moreover the weighting scheme of a *neo-*

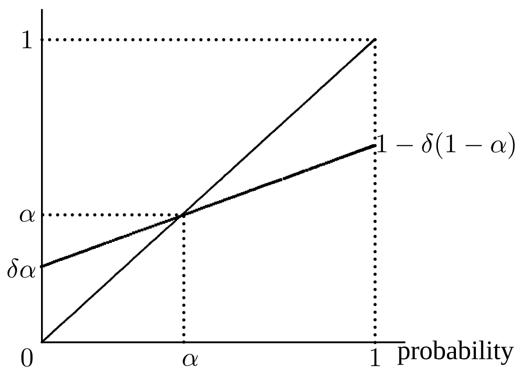

Figure 3. Neo-additive capacity

additive capacity provides a simple version of an inverse-S shaped function relating beliefs, p, to decision weights, $w(p)$.

$$w(p) := \begin{cases} 1 & \text{for} \quad p = 1, \\ \delta\alpha + (1-\delta) \cdot p & \text{for} \quad 0 < p < 1, \\ 0 & \text{for} \quad p = 0. \end{cases}$$

This weighting scheme is illustrated in Figure 3. It can be seen as a piecewise linear approximation to that in Figure 2.

Kilka and Weber (2001) used their data to estimate degrees of optimism and ambiguity separately. In terms of our notation, they report the following values:

	α	δ	$\alpha\delta$
Average	0.5	0.52	0.26
Max.	0.62	0.61	0.34
Min.	0.4	0.41	0.18

The values of both optimism α and ambiguity δ vary around 0.5 with deviations of 0.1.

2.3 Equilibrium under Ambiguity

We shall use a solution concept based on Dow and Werlang (1994).[6] Formally, we assume that each player maximizes his/her expected payoff with respect to a non-additive belief. In equilibrium, beliefs have to be reasonable in the sense that each player "believes" that the opponents play best responses. To model this we require that the *support* of any given player's beliefs contain only best responses of the other players. Denote by $R_i(\nu_i) = \arg\max\{V_i(s_i, \nu_i) \mid s_i \in S_i\}$

[6] Dow and Werlang (1994) assumed ambiguity-aversion. Their solution concept was later generalized to an arbitrary number of players in Eichberger and Kelsey (2000) and extended to include optimistic behavior in Eichberger and Kelsey (2006).

the best response correspondence of player i, given beliefs represented by the capacity ν_i.

Definition 2.1 *A pair of capacities* $\nu^* = \langle \nu_1^*, \nu_2^* \rangle$ *is an* Equilibrium Under Ambiguity (EUA) *if*

$$\varnothing \neq \operatorname{supp} \nu_1^* \subseteq R_1(\nu_2^*) \text{ and } \varnothing \neq \operatorname{supp} \nu_2^* \subseteq R_1(\nu_1^*).$$ [7]

If $s_i^* \in \operatorname{supp} \nu_i^*$ *for* $i = 1, 2$, *we say that* $s^* = \langle s_1^*, s_2^* \rangle$ *is an* equilibrium strategy profile. *If* $\operatorname{supp} \nu_i^*$ *contains a single strategy profile for* $1 = 1, 2$ *we say that it is a* pure *equilibrium, otherwise we say that it is* mixed.[8]

A mixed equilibrium, where the support contains multiple strategy profiles, should be interpreted as an equilibrium in beliefs rather than randomizations.

2.4 Support of Ambiguous Beliefs

Most theories of ambiguity are formulated for single person decisions. To study ambiguity in games it is necessary to extend them to allow for the interactions between different decision-makers. In the absence of ambiguity, each player is assumed to choose a strategy which maximizes his/her expected payoff with respect to beliefs which are compatible with the mixed strategies of their opponents. Most equilibrium notions rest on some degree of consistency between actual behavior and beliefs, since players are likely to adapt their beliefs if they observe behavior which contradicts them. In the presence of ambiguity, perfect consistency is unlikely since there do not exist non-additive randomizing devices, which prevents us from constructing strategies, corresponding precisely to ambiguous beliefs. We consider games where each player believes that the strategy choice of his/her opponents is possibly ambiguous.[9] An equilibrium is a situation where players behave optimally relative to their beliefs.

There is more than one way to extend the notion of a support from probability distributions to capacities. This definition determines how tight the relationship between beliefs and actual behavior is. Definition 2.1 requires the strategies in the support of a given player's equilibrium

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[7] Existence of equilibrium can be proved in a standard way using fixed-point theorems, see Dow and Werlang (1994), Eichberger and Kelsey (2000) and Eichberger, Kelsey, and Schipper (2006).

[8] Our aim is to modify Nash equilibrium by allowing for the possibility that players may view their opponents behavior as ambiguous. If beliefs were additive, then in a 2-player game, Definition 2.1 would coincide with Nash equilibrium. In this sense we have modified Nash equilibrium to allow for ambiguity.

[9] There are other possible modelling choices, for instance, one could consider there is ambiguity about the opponents' type.

belief, be best responses. It is ambiguous whether the opponents play best responses. As result, in addition, the best and worst possible plays by one's opponent are taken into account when evaluating a strategy. Decision-relevant strategies outside the support can be interpreted as events a player views as unlikely but which, due to ambiguity about the behavior of the opponents, cannot be completely ruled out.

Several solution concepts for games with strategic ambiguity have been suggested, (see for instance Marinacci (2000) and Lo (1996)). The main difference between the various solution concepts is that they use different support notions. Thus the definition of support deserves careful consideration.

Definition 2.2 *We define the* support *of the neo-additive capacity* $\nu(\cdot|\alpha, \delta, \pi)$ *by* $\operatorname{supp} \nu = \operatorname{supp} \pi$.

As explained above a neo-additive capacity is intended to represent a situation where the decision-maker's belief is represented by the additive probability distribution π but (s)he is not fully confident in this belief. Given this it is plausible that the support of ν should coincide with that of π. Eichberger and Kelsey (2006) show that, for a neo-additive capacity $\nu(\cdot|\alpha, \delta, \pi)$,

$$\operatorname{supp} \nu = \bigcap_{p \in \mathcal{P}} \operatorname{supp} p,$$ [10]

where $\mathcal{P}$ is the set of probability distributions defined in equation (1).[11]

3 EXPERIMENTAL GAMES

Goeree and Holt (2001) present evidence that NE is a good predictor in some games but not in others. In particular they consider five one-shot games, in which there is evidence in favour of NE. However, in each case, a seemingly irrelevant parameter change produces the opposite result. In this section we argue that this evidence can be explained by ambiguity. For

[10] This definition of support essentially coincides with the inner support notion in Ryan (1997).

[11] Much of the existing literature on ambiguity in games has explicitly or implicitly restricted attention to the case of pessimistic players. In the present paper, an important part of our explanation of behavior in experimental games relies to a large extent on optimistic responses to ambiguity. It is, therefore, necessary to reconsider the support notions put forth in the previous literature. For a more detailed discussion of the relation of our proposal to earlier support notions see Eichberger and Kelsey (2006).

expository reasons we shall discuss the experiments in a different order to GH. To avoid undue repetition, we shall discuss the first example in detail and present an outline of the argument for the remaining games.

3.1 The Kreps Game

The impact of ambiguity is illustrated by the Kreps game, which is an asymmetric coordination game with a safe strategy for Player 2, NN.[12] The normal form of the game is described in the following table:

Game A

		Player 2			
		L (26%)	M (8%)	NN (68%)	R (0%)
Player 1	T (68%)	$200, 50$	$0,\ 45$	$10,\ 30$	$20, -250$
	B (32%)	$0,\ -250$	$10,\ -100$	$30,\ 30$	$50, 40$

The numbers in brackets denote the number of subjects playing the respective strategies in GH's experiment. The only two Nash equilibria in pure strategies are $\langle T, L\rangle$ and $\langle B, R\rangle$. There is also a mixed strategy equilibrium, in which Player 2 chooses M and L each with positive probability. The only strategy which will not be played in any NE is NN. In stark contrast, almost two thirds of subjects chose NN. Interestingly, this game shows another behavioral feature not mentioned in GH. Given the strong incentive of Player 2 to choose NN, Player 1 could be expected to play the best reply B. This is, however, not the case for subjects in GH's experiment.

We claim that these results can be explained by ambiguity. For player 2, strategy NN gives a certain pay-off of 30, even with ambiguity. All the other strategies can potentially give him a negative pay-off. Thus pessimistic responses to ambiguity can motivate him to choose NN . Suppose that he has an ambiguous belief that player 1 will play s_1, where s_1 can either take the values T or B, then the Choquet expected utility of his other strategies is given by:[13]

$$\begin{aligned} V_2(L) &\leqslant \delta \cdot [\alpha \cdot 50 - (1-\alpha) \cdot 250] + (1-\delta) \cdot \max\{u_2(T,L), u_2(B,L)\} \\ &= 50 - \delta \cdot (1-\alpha) \cdot 300 \leqslant 30; \end{aligned}$$

[12] The name comes from Kreps (1995) who discusses the possibility that the level of payoffs, rather than their relative values, may affect players' behavior. The payoffs have been modified to allow the game to be run experimentally. These modifications do not affect the set of equilibria.

[13] For convenience we are suppressing the arguments α, δ and π.

$$V_2(M) \leqslant \delta[\alpha.45-(1-\alpha)\cdot 100]+(1-\delta)\cdot\max\{u_2(T,M),u_2(B,M)\}$$
$$= 45-\delta(1-\alpha)\cdot 145 \leqslant 30;$$

$$V_2(R) \leqslant \delta\cdot[\alpha\cdot 40-(1-\alpha)\cdot 250]+(1-\delta)\cdot\max\{u_2(T,R),u_2(B,R)\}$$
$$= \delta\cdot[\alpha\cdot 40-(1-\alpha)\cdot 250]+(1-\delta)\cdot 40 = 40-\delta\cdot(1-\alpha)\cdot 290 \leqslant 30;$$

provided $\delta(1-\alpha) \geqslant \max\{\frac{50-30}{300},\frac{45-30}{145},\frac{40-30}{290}\} = \frac{3}{29} = 0.103$. From which it follows that NN is a best response. Most estimates for α and δ exceed this value by far. Hence in EUA, Player 2 will choose NN, for any beliefs compatible with the estimated values of α and δ.

The observed behavior of more than two thirds of Player 1's choosing T, can be obtained as an equilibrium under ambiguity, $\pi_1^*(T) = \pi_2^*(NN) = 1$, but never as a NE. Assuming 2 is believed to play NN, the CEU value of payoffs for Player 1 are:

$$V_1(s_1,NN) = \delta\cdot\left[\alpha\cdot\max_{s_2\in S_2}u_1(s_1,s_2)+(1-\alpha)\cdot\min_{s_2\in S_2}u_1(s_1,s_2)\right]+(1-\delta)\cdot p_1(s_1,NN)$$
$$= \begin{cases}\delta\cdot\alpha\cdot 200+(1-\delta)\cdot 10) & \text{for} \quad s_1 = T,\\ \delta\cdot\alpha\cdot 50+(1-\delta)\cdot 30 & \text{for} \quad s_1 = B.\end{cases}$$

Thus T, is preferred to B if and only if $150\delta\alpha-(1-\delta)20 \geqslant 0$, which would be positive for $\delta\alpha \approx 0.25$ and $\delta \approx 0.5$. Hence, with the experimentally observed parameter values for α and δ, (T,NN) are equilibrium strategies.

3.2 The Traveller's Dilemma

In the Traveller's Dilemma, each player makes a claim n_i for a payment between 180 and 300 cents, i.e., $n_i \in S := \{180,181,182,...,298,299,300\}$.[14] Given two claims (n_1,n_2), both players obtain the minimum $\min\{n_1,n_2\}$, but, if the claims are not equal, the player with the higher claim pays $R > 1$ to the other, yielding the payoff function:

$$u_i(n_1,n_2) = \min\{n_1,n_2\}+R\cdot\text{sign}(n_j-n_i),$$

with $i,j \in \{1,2\}$ and $i \neq j$.

It is easy to see that for $R > 1$ each player has an incentive to undercut the opponent's claim by one unit. The following diagram shows the best-reply of Player 1. Hence, for any $R > 1$,

[14] The story which motivates the game can be found in Basu (1994), where this game was introduced.

claiming the minimum amount, $(n_1^*, n_2^*) = (180, 180)$, is the unique NE. In fact, $n_i^* = 180$ is the only rationalizable strategy for each player, since it is the only strategy which cannot be undercut by the opponent.

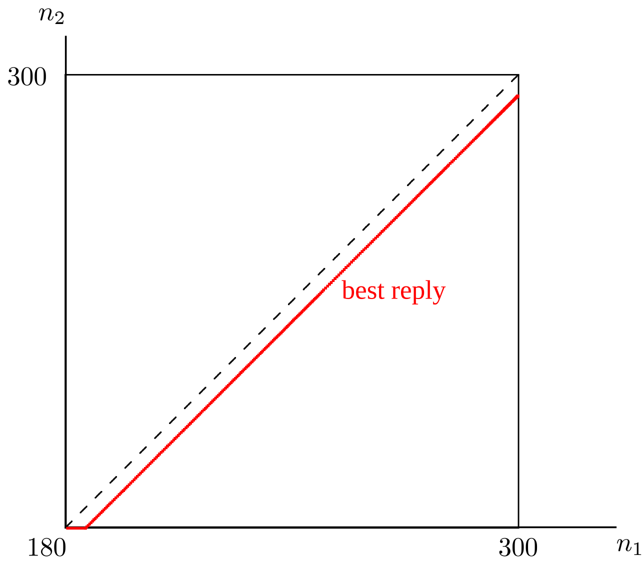


Figure 4. Best response of Player 1 without ambiguity

GH show, however, that the experimental results, depend on R. For large R, players claim 180, or close to this amount, as predicted by the NE. For $R = 180$, almost 80 percent of the subjects chose $n_i \leqslant 185$. In contrast to the NE predictions, for small R, players make claims close to 300, i.e., for $R = 5$, almost 80 percent of the players chose $n \geqslant 295$.

The evidence can be explained by ambiguity as follows. In the Traveller's Dilemma, payoffs are high if players coordinate on a high claim. As a result there are two possible best responses to an action by one's opponent. Either one can undercut by one unit or alternatively one can choose 299, which yields the highest coordination gain and maintains at least the chance to avoid the penalty. For $R = 180$ however, the penalty for being the highest bidder is extreme, wiping out any possible gain from coordination. Hence, even a small amount of pessimism in response to ambiguity will deter players from making a high claim and the only possible equilibrium is where both claim 180.

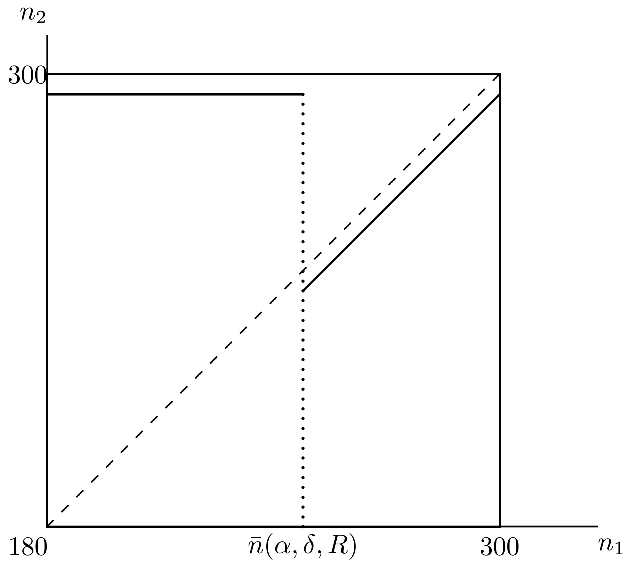


Figure 5. Best reply of Player 1 with ambiguity

In contrast, for $R = 5$, the penalty is so low that a little ambiguity and optimism, $\frac{\alpha\delta}{1-\delta} \geqslant 0.1$, suffices to make it worthwhile to claim 299. Figure 5 illustrates the equilibrium best-reply correspondence. There is a mixed equilibrium with two best responses 299 and $\bar{n}$. For example, if $\alpha = 0.4$ and $\delta = 0.6$, the ambiguous beliefs that one's opponent would choose $[\overline{n}] = 285$ would be 0.18 and the belief for 299 would equal 0.82. The observation, that 80 percent of subjects chose a claim higher than 295, seems to be not obviously incompatible with EUA for these values of α and δ. Further details are given in Proposition A.2 in the appendix. The table below gives the values of $\bar{n}$ for $R = 5$ and plausible parameter values.

δ \ α	0.4	$0, 5$	0.62
0.41	268	274	280
0.52	279	283	286
0.61	285	288	291

3.3 Matching Pennies

In experiments on the Matching Pennies game, GH discover that subjects tend to conform with NE predictions if the game is symmetric, but deviate systematically if the payoffs are asymmetric. They study the following two versions of matching pennies.[15] The ratios to the left

[15] GH also consider a third version of matching pennies. This can be analyzed in a similar way.

of the strategies indicate the unique NE mixed strategies and the bold numbers in brackets to the right of the strategies show the percentage of players choosing the respective strategy in the experiments.

Game **B**

			Player 2	
			0.5 *L* **(48%)**	0.5 *R* **(52%)**
Player 1	0.5 *T* **(48%)**		**80**, 40	40, 80
	0.5 *B* **(52%)**		40, 80	80, 40

Game **C**

			Player 2	
			0.12 *L* **(16%)**	0.88 *R* **(84%)**
Player 1	0.5 *T* **(96%)**		**320**, 40	40, 80
	0.5 *B* **(4%)**		40, 80	80, 40

The games differ only in the payoff of Player 1 for $\langle T, L\rangle$, which is indicated by a boldface number. In mixed-strategy NE, the probabilities which a given player uses to randomize are chosen to make his/her opponent indifferent between all of his/her equilibrium strategies. Hence, a change in player 1's payoff is predicted to leave her own behavior unchanged, while causing a change in the behavior of player 2.

Actual play reveals, however, a quite different pattern. While the relative frequency of strategy choices in Game B correspond to the NE prediction, they deviate dramatically from the predictions in the asymmetric game. In Game C, Player 1's choose almost exclusively strategy T, the payoff which has been increased. As a result they make their behavior predictable, which is exploited by the subjects in the role of Player 2. It is surprising that Player 1 does not appear to foresee this shift in the behavior of her opponent. It appears as if Player 2 understands the change in Player 1's incentives better than she does herself. One interpretation of Player 1's behavior may be a shift in decision weights to extremely attractive low-probability events.

Ambiguity makes little difference to game B. Symmetry implies that the only equilibrium is where each player believes that his/her opponent is equally likely to use either strategy. In game C, optimistic responses to ambiguity cause player 1 to overweight unlikely events which yield the high payoff 320. This causes her to choose strategy T almost exclusively. From Player 2's point of view, the two strategies are symmetric. However 1 has a bias in favour of T. Hence

R is a best response for 2. Thus there is an equilibrium with ambiguity where the equilibrium strategy combination is $\langle T, L\rangle$. There is no NE which describes such behavior. There is a unique NE, where Player 2 plays $\pi(L) = \frac{1}{8}$ and Player 1 plays $\pi(T) = \frac{1}{2}$. Such randomizations are incompatible with the observed choices.

3.4 A Coordination Game with a Secure Option

GH study a coordination game, which is modified by giving Player 2 an extra secure option.

Game D

Player 1 \ Player 2	L (?)	H (84)	S(?)
L (4)	$90, 90$	$0,\ 0$	$\mathbf{0}, 40$
H (96)	$0,\ 0$	$180,\ 180$	$0, 40$

Game E

Player 1 \ Player 2	L (?)	H (76)	S(?)
L (36)	$90, 90$	$0,\ 0$	$\mathbf{400}, 40$
H (64)	$0,\ 0$	$180,\ 180$	$0, 40$

The two versions of the game are distinguished by the payoff of Player 1 when the secure strategy, S, is played, (shown in bold). This strategy is strictly dominated by a mixture of L and H, hence it will never be played in a NE.

The prediction of NE is the same for the two games. In each case there are three NEs, one where both play H, one where both play L and a mixed strategy equilibrium where both players randomize between H and L. GH note however, that 96 percent of subjects in the role of Player 1 and 84 percent of subjects in the role of Player 2 chose the strategy of the Pareto-dominant equilibrium $\langle H, H\rangle$ in Game D. Increasing the payoff of Player 1 for the strategy combination $\langle S, L\rangle$ in Game E made it substantially more likely that Player 1 would choose L. This is even more surprising since there was no corresponding change in the behavior of Player 2.

If players view their opponents behavior as ambiguous then Player 1's perceives the worst pay-off of both strategies to be the same. Thus pessimism will not influence her decision. In game D, optimism increases the incentive for Player 1 to choose H and thus have some chance of achieving the maximum pay-off, 180.

As discussed previously, experimental evidence suggests that $\delta \approx 0.5$ and $\alpha\delta \approx 0.25$ are plausible parameter values. For these values there remain multiple EUA leading either to coordinated behavior on $\langle H, H \rangle$ or $\langle L, L \rangle$. However it is not implausible that there is more ambiguity in coordination games, since the multiplicity of equilibria makes prediction of opponents' behavior harder. If ambiguity is sufficiently high, $\delta > \frac{5}{7}$, even a low degree of optimism $\alpha = 0.4$ will induce Player 1 to choose strategy H, no matter what belief π_2 she holds regarding the opponent. In this case, the only EUA would be $\pi_1^*(H) = \pi_2^*(H) = 1$, which corresponds fairly closely to GH's observations for Game D.

In Game E, optimistic responses to ambiguity may cause Player 1 to switch to strategy L, even if she believes that Player 2 chooses H. Thus, there is a tendency to deviate from coordinated behavior on H. In this case, any ambiguous beliefs compatible with the empirically observed values of $\alpha\delta \approx 0.25$ and $\delta \approx 0.5$ will induce Player 1 to choose L. For Player 2 the CEU payoff difference remains unchanged. Hence, for ambiguity and optimism represented by the parameter values $\alpha\delta \approx 0.25$ and $\delta \approx 0.5$, there is a unique pure equilibrium in which both players use strategy L.

The experimental results reported by GH (p.1408) show that coordination on the Pareto-dominant NE, $\langle H, H \rangle$, is high (80 percent) in Game D, whereas coordination was low, 32 percent on $\langle H, H \rangle$ and 16 percent on $\langle L, L \rangle$, in Game E with the outside option yielding a payoff of 400 for Player 1. This increase in "uncoordinated" behavior indicates that Player 2 found it more difficult to adjust to the incentive created by the potential payoff of 400 for optimistic opponents.

3.5 A Minimum-Effort Coordination Game

In the minimum effort coordination game (also know as the weakest link model of public goods, see Cornes and Sandler (1986)) two players have to choose effort levels from the set $E = \{110, ..., 170\}$ at a marginal cost of $c < 1$ yielding payoffs

$$u_i(e_1, e_2) := \min\{e_1, e_2\} - c \cdot e_i,$$

for $i = 1, 2$. GH played this experiment with the marginal cost parameters 0.1 and 0.9 and observed the following distributions of play:[16]

[16] Note that GH have grouped the data for five successive integers.

Game F	$e = 115$	$e = 125$	$e = 135$	$e = 145$	$e = 155$	$e = 165$
$c = 0.1$	0.1	0.02	0.1	0.1	0.08	0.6
$c = 0.9$	0.5	0.18	0.05	0.07	0.05	0.15

The observations show a clear concentration of play on high effort levels in the case of low costs, $c = 0.1$, and on the low ones for high costs, $c = 0.9$. Coordinating on any of the six possible effort levels is a Nash equilibrium for either possible value of costs i.e., the set of NE's is $\{(e_1^*, e_2^*) \in E^2 | \, e_1^* = e_2^*\}$. Thus Nash equilibrium is unable to explain why an increase in the cost parameter changes behavior. Since the experiments were one-shot games and there were many possible equilibria, coordination is not very likely.

We shall argue that such observations can be explained by ambiguity. In this game, the best outcome is that your opponent plays the highest possible strategy. Suppose there is an equilibrium with ambiguity in which both players coordinate on an effort level other than the highest. If Player 1 increases her effort by one unit, the perceived marginal benefit is $\delta\alpha,$ which is the weight on the highest outcome. The marginal cost of increasing effort is $c.$ Thus if $\delta\alpha > c$ it is in her interest to increase her contribution. Player 2 will think similarly. Hence under the assumption $\delta\alpha > c$ the only possible equilibrium is where both players make the highest contribution.

Suppose there is an equilibrium with ambiguity in which both players use an effort level other than the lowest (i.e. 115). If Player 1 decreases her contribution by one unit the perceived marginal reduction in benefit is $\delta\alpha + (1 - \delta)$. The marginal cost saving is $c.$ Thus if $c > \delta\alpha + (1 - \delta)$ it is worth decreasing effort, which implies that the only possible equilibrium is where both players coordinate on the lowest effort level.

With the average values of $\alpha\delta \approx 0.25$ and $\delta \approx 0.5$ reported by Kilka and Weber (2001), one has $\delta \cdot \alpha - c > 0$ for $c = 0.1$ and $c > [\delta \cdot \alpha + (1 - \delta)]$ for $c = 0.9$. For marginal costs of $0.1,$ equilibrium with ambiguity predicts that players would try to coordinate on the highest effort level, while for $c = 0.9$ they should coordinate on the lowest effort level. The observed behavior seems to correspond well with this prediction. In particular, taking into consideration that $\alpha\delta \approx 0.25$ and $\delta \approx 0.5$ are average values of ambiguity and optimism and that the multiplicity of equilibria may create a high degree of ambiguity in a one-shot game, it is possible that individual players have degrees of optimism and ambiguity satisfying

$$[\delta \cdot \alpha + (1 - \delta)] > c > \delta \cdot \alpha. \tag{2}$$

In this case, there would be multiple EUA with equilibrium contributions between the extremes.

4 COMPETING EXPLANATIONS

The above discussion of the "Treasures" has shown the potential of EUA to explain observations in many experiments. GH survey evolutionary and learning models as descriptions of behavior in games and suggest several alternatives, which appear to provide good explanations of experimental data. In particular, they find that evolutionary models with random shocks may describe observed behavior better than NE. The logistic quantal response function of McKelvey and Palfrey (1995) also offers better predictions of behavior in experiments on one-shot games. This may be interpreted either as dynamic noisy learning rule or as an iterated noisy introspection process. They used this as the basis of an alternative equilibrium concept, *Quantal Response Equilibrium (QRE)*, which proved to be a successful alternative in explaining the results of experimental games. For example, Boylan and Grant (2006) show that behavior in the asymmetric Matching-Pennies game can be explained by QRE, a fact also noted by Goeree, Holt, and Palfrey (2004).

For two players[17] with finite strategy sets S_1 and S_2 and payoff functions $u_1(s_1, s_2)$ and $u_2(s_1, s_2)$, a *Quantal Response Equilibrium (QRE)* is a mixed strategy combination (π_1, π_2) such that

$$\pi_i(s_i) = f_i(s_i, \pi_{-i}) := \frac{\exp\left(\lambda_i \cdot \mathcal{E}_{\pi_{-i}} u_i(s_i, \cdot)\right)}{\sum\limits_{s_i \in S_i} \exp\left(\lambda_i \cdot \mathcal{E}_{\pi_{-i}} u_i(s, \cdot)\right)}$$

for all $s_i \in S_i$. The parameter λ_i measures the responsiveness of a player. For $\lambda_i = 0$ one has a uniform distribution over one's strategies, independent of the expected payoff. For $\lambda_i \to \infty$ all the weight is shifted to the strategy with the highest expected payoff. The parameter λ_i can be adjusted such that the best-possible fit of the equilibrium probabilities π_i with the relative

[17] A generalisation to games with many players is straightforward.

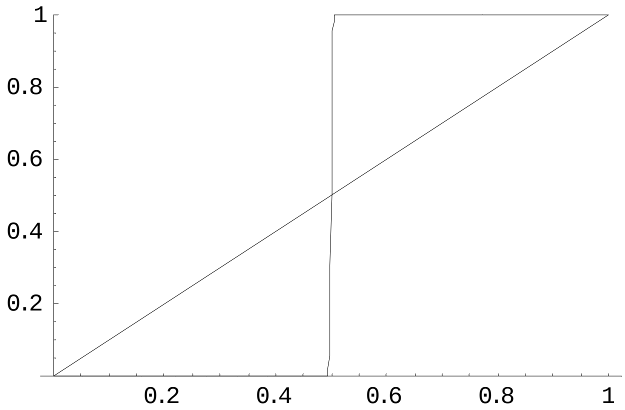


Figure 6.

frequencies of actual behavior is achieved.

In the matching-pennies games of Section 3.3 we have only two pure strategies. Thus, we can identify the mixed strategy of each player with the probability q_i of playing the strategies T or L, respectively.

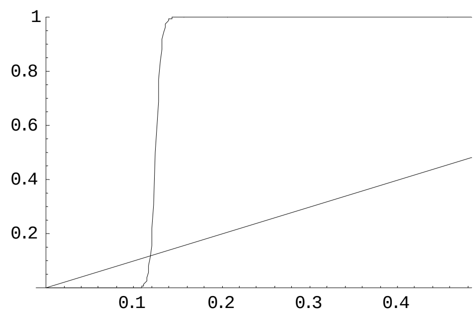
$$\begin{aligned} q_1 = \quad f_1(T, q_2) \quad &:= \frac{\exp(\lambda_1 \cdot [80q_2 + 40(1-q_2)])}{\exp(\lambda_1 \cdot [80q_2 + 40(1-q_2)]) + \exp(\lambda_1 \cdot [40q_2 + 80(1-q_2)])}, \\ q_2 = \quad f_2(L, q_1) \quad &:= \frac{\exp(\lambda_2 \cdot [40q_1 + 80(1-q_1)])}{\exp(\lambda_2 \cdot [80q_1 + 40(1-q_1)]) + \exp(\lambda_2 \cdot [40q_1 + 80(1-q_1)])}. \end{aligned}$$

For $\lambda = 10$, the quantal response function $f_1(T, q_2)$ looks almost like the best-reply correspondence of the symmetric matching-pennies Game A.

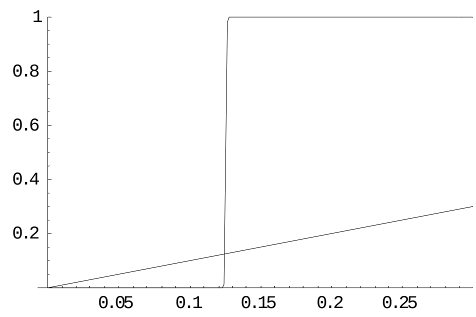
On the other hand, if the payoff of Player 1 from (T, L) is increased to 320 as in Game B, the following equilibrium conditions obtain:

$$\begin{aligned} q_1 = \quad f_1(T, q_2) \quad &:= \frac{\exp(\lambda_1 \cdot [320q_2 + 40(1-q_2)])}{\exp(\lambda_1 \cdot [320q_2 + 40(1-q_2)]) + \exp(\lambda_1 \cdot [40q_2 + 80(1-q_2)])}, \\ q_2 = \quad f_2(L, q_1) \quad &:= \frac{\exp(\lambda_2 \cdot [40q_1 + 80(1-q_1)])}{\exp(\lambda_2 \cdot [80q_1 + 40(1-q_1)]) + \exp(\lambda_2 \cdot [40q_1 + 80(1-q_1)])}. \end{aligned}$$

The figure below shows the quantal response function $f_1(T, q_2)$, again for the parameter values $\lambda = 1$ and $\lambda = 10$. The quantal response function becomes now almost vertical at $q_2 = 0.125$.



$\lambda_i = 1$



$\lambda_i = 10$

The equilibrium value of q_1 must be close to this value and therefore, QRE describes actual behavior better than NE.

There is a deeper reason why EUA, like QRE allows a better description of behavior. In both cases, players consider absolute as well as relative payoffs. In the QRE this is achieved by the assumption that the probability that a pure strategy will be played depends on its expected payoff. In an EUA, we consider still best replies, but due to optimism and ambiguity the best and worst outcomes are over-weighted. Depending on the degree of ambiguity these effects may be stronger or weaker, just as the parameter λ increases or reduces the impact of the expected payoffs.

Both approaches are more flexible in describing actual behavior than NE, however we believe that EUA may be given a more attractive interpretation. Optimism can be viewed as a personal characteristic, which we may take as given, like a player's preferences. Ambiguity, on the other hand, is more situation-dependent. In particular, one would predict that a one-shot game is by nature more ambiguous than a repeated game. Hence, there are testable hypotheses regarding EUA which have no counter-part in QRE. Moreover, there is substantial evidence on individual behavior, which allows one to predict the attitude of a decision-maker towards ambiguity. Such evidence can help to restrict behavior in EUA, which makes the theory more powerful. It is a particular strength of EUA that it can explain the diverging behavior in many games with the same set of ambiguity and optimism parameters.

Indeed, a recent paper by Haile, Hortascu, and Kosenok (2003) shows that "without a priori distributional assumptions, a QRE can match any distribution of behavior by each player in any normal form game".

Boylan and Grant (2006) also find that the Quantal Response Equilibrium of McKelvey and Palfrey (1995) explains behavior in the asymmetric matching pennies experiments of GH better than NE. Moreover, they show that fairness-based payoff transformations as suggested in Fehr and Schmidt (1999) and Rabin (1993) do not predict the observed behavior.

5 CONCLUSION

In this paper we have shown that many of the treasures of game theory from GH can be explained as responses to ambiguity. We have only analyzed those treasures based on normal form games. The other experiments concern dynamic games some of which also have incomplete information. To study the impact of ambiguity in these cases it would be necessary to develop new solution concepts for such games. This is beyond the scope of a short article. Nevertheless we believe that explanations based on ambiguity could be found for many of these games as well. For instance, the treasure from GH entitled ‘Should you believe a threat that is not credible’ is very similar to the model of frivolous lawsuits in Eichberger and Kelsey (2004).

The preferences we use have the effect of over weighting the best and worst outcomes. The worst outcome may often be death. However it is likely that ambiguity-aversion would cause other bad outcomes to be over weighted such as losing large sums of money. Similarly optimism might have the effect that a number of good outcomes are overweighted rather than just the best outcome. While this objection may have some merits in general, the games studied in this paper typically have salient best and worst outcomes. It does not seem implausible that these should be subjectively overweighted. In Eichberger and Kelsey (2006) we show that much of our analysis can be extended to the more general case where a number of good and bad outcomes are over weighted.

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Appendix A. TECHNICAL APPENDIX

This appendix is not intended for publication. It is included to enable the referee and editor to check the proofs. Each section gives details of the model and proves the claims made in the corresponding section of the text.

A.1 The Traveller's Dilemma

Consider Player 1. Suppose she holds beliefs which are represented by a *neo-additive capacity:* $\nu(A|\alpha,\delta,\pi)=\alpha\delta+(1-\delta)\,\pi(A)$. Given these beliefs, Player 1's Choquet expected utility from the choice of n_1 is

$$\begin{aligned} V(n_1;\alpha,\delta,\pi) &= \delta\left[\alpha\max_{n_2\in S}u_1\left(n_1,n_2\right)+(1-\alpha)\min_{n_2\in S}u_1\left(n_1,n_2\right)\right] \\ &\quad +(1-\delta)\cdot\sum_{n_2\in S}u_1\left(n_1,n_2\right)\cdot\pi(n_2). \end{aligned}$$

Lemma A.1 *Suppose Player 1's beliefs are given by a neo-additive capacity $\nu(\cdot|\alpha,\delta,n_2)$ defined by $\nu(\varnothing|\alpha,\delta,n_2)=0$, $\nu(S_2|\alpha,\delta,n_2)=1$, $\nu(A|\alpha,\delta,n_2)=\delta\alpha$ if $n_2\notin A$, $\nu(A|\alpha,\delta,n_2)=\delta\alpha+(1-\delta)$, if $n_2\in A$. Note that $\nu(\cdot|\alpha,\delta,n_2)$ has degree of pessimism α, degree of ambiguity δ. If α and δ satisfy $\alpha<0.5,\delta<0.5$ and $0.1\leqslant\frac{\alpha\delta}{1-\delta}\leqslant0.9$, then the best-reply correspondence is*

$$R_1(\nu(\cdot|\alpha,\delta,n_2))=\begin{cases}299 & \textit{for}\quad n_2<\overline{n}(\alpha,\delta,R)\\ n_2-1 & \textit{otherwise}\end{cases}$$

with

$$\overline{n}(\alpha,\delta,R):=300-\frac{1-\delta}{\alpha\delta}\left[2R-1\right].$$

Proof. First note that $R>1$, $n_1=299$ weakly dominates $n_1=300$, moreover, for $\alpha,\delta>0$, $n_1=300$ is strictly dominated. Since the highest pay-off for $n_1=299$ is greater than that for $n_1=300$, and under our assumptions on α and δ, the highest payoff gets positive weight in the Choquet integral. Thus we may eliminate the possibility that either player plays strategy 300.

Consider $n_2=180$. The CEU of a pure strategy n_1 is easily computed as[18]

$$\begin{aligned} &V(n_1;\alpha,\delta,\pi^{180}):=\int u_1\left(n_1,\cdot\right)d\nu_1(\cdot|\alpha,\delta,\pi^{180}) \\ &= \delta\left[\alpha\max_{n_2\in S}u_1\left(n_1,n_2\right)+(1-\alpha)\min_{n_2\in S}u_1\left(n_1,n_2\right)\right]+(1-\delta)\cdot u_1\left(n_1,180\right) \end{aligned}$$

[18] Here π^{n_2} denotes the Dirac measure with support n_2.

$$= \begin{cases} \delta\left[\alpha\left(180+R\right)+(1-\alpha)180\right]+(1-\delta)\cdot 180 & \text{for} \quad n_1 = 180, \\ \delta\left[\alpha\left(n_1+R\right)+(1-\alpha)\left(180-R\right)\right]+(1-\delta)\cdot[180-R] & \text{for} \quad 180 < n_1 < 300, \end{cases}$$

Consider next $n_2 \in (180, 300)$. The CEU of a pure strategy combination (n_1, n_2) is,

$$V(n_1; \alpha, \delta, \pi^{n_2}) := \int u_1\left(n_1, \cdot\right) d\nu_1(\cdot|\alpha, \delta, \pi^{n_2})$$

$$= \begin{cases} \delta\left[\alpha\left(180+R\right)+(1-\alpha)180\right]+(1-\delta)\cdot[180+R] & \text{for} \quad n_1 = 180; \\ \delta\left[\alpha\left(n_1+R\right)+(1-\alpha)\left(180-R\right)\right]+(1-\delta)\cdot[n_1+R] & \text{for} \quad 180 < n_1 < n_2; \\ \delta\left[\alpha\left(n_1+R\right)+(1-\alpha)\left(180-R\right)\right]+(1-\delta)\cdot n_1 & \text{for} \quad n_1 = n_2; \\ \delta\left[\alpha\left(n_1+R\right)+(1-\alpha)\left(180-R\right)\right]+(1-\delta)\cdot[n_2-R] & \text{for} \quad n_2 < n_1 < 300. \end{cases}$$

(ii) For $n_2 = 180$, $V(n_1; \alpha, \delta, \pi^{180})$ is strictly increasing in n_1 for $n_1 > 180$. Hence,

$$\begin{aligned} V(299; \alpha, \delta, \pi^{180}) - V(180; \alpha, \delta, \pi^{180}) &= \alpha\delta\left(299-180\right)-(1-\alpha\delta)R \\ &= 119\alpha\delta-(1-\alpha\delta)R. \end{aligned}$$

Thus

$$R_1(\nu(\cdot|\alpha, \delta, 180)) = \begin{cases} 299 & \text{for} \quad \alpha\delta > \frac{R}{119+R}, \\ 180 & \text{otherwise.} \end{cases}$$

Notice, for $R = 5$, $\frac{R}{119+R} = \frac{5}{124} \approx 0.041 \leqslant 0.1$. Hence, $R_1(\nu(\cdot|\alpha, \delta, 180)) = 299$. For $R = 180$, we have $\frac{R}{119+R} = \frac{180}{299} \approx 0.6$ and depending on the values of α and δ the best reply may be $R_1(\nu(\cdot|\alpha, \delta, 180)) = 180$.

(iii) Consider now $n_2 \in (180, 300)$.

For $n_1 \in (180, n_2) \cup (n_2, 300)$ the CEU value is strictly increasing in n_1. Hence, only $n_1 = 180,\ n_1 = n_2 - 1,\ n_1 = n_2$, or $n_1 = 299$ can be best responses.

(a) Comparing $n_1 = n_2 - 1$ and $n_1 = n_2$, we observe that

$$\begin{aligned} & V(n_2; \alpha, \delta, \pi^{n_2}) - V(n_2-1; \alpha, \delta, \pi^{n_2}) \\ = \;& \left\{\delta\left[\alpha\left(n_2+R\right)+(1-\alpha)\left(180-R\right)\right]+(1-\delta)\cdot n_2\right\} \\ & -\left\{\delta\left[\alpha\left(n_2-1+R\right)+(1-\alpha)\left(180-R\right)\right]+(1-\delta)\cdot[n_2-1+R]\right\} \\ = \;& \delta\alpha+(1-\delta)\cdot[1-R] = [\delta\alpha+\delta(R-1)]-(R-1) < 0 \end{aligned}$$

holds, for $R > 1 + \alpha$. For $R = 5$ and $R = 180$, this condition is satisfied. Hence, $n_1 = n_2$ cannot be a best reply.

(b) Comparing $n_1 = 180$ and $n_1 = 299$.

(b1) Suppose $181 \leqslant n_2 \leqslant 298$, then we observe that

$$V(299; \alpha, \delta, \pi^{n_2}) - V(180; \alpha, \delta, \pi^{n_2}) = [\delta [\alpha (299 + R) + (1 - \alpha) (180 - R)] + (1 - \delta) \cdot [n_2 - R]]$$
$$- [\delta [\alpha (180 + R) + (1 - \alpha) 180] + (1 - \delta) \cdot [180 + R]]$$
$$= 119\delta\alpha + (1 - \delta) \cdot [n_2 - 180 - 2R] \gtreqless 0 \Leftrightarrow n_2 \gtreqless 180 + 2R - \frac{\delta\alpha}{1-\delta} 119.$$

By the assumption $0.1 \leqslant \frac{\alpha\delta}{1-\delta} \leqslant 0.9$, we obtain for $R = 5$, $180 > 180 + 2R - 0.1 \cdot 119 \geqslant 180 + 2R - \frac{\delta\alpha}{1-\delta} 119$. Hence, for $R = 5$, $V(299; \alpha, \delta, \pi^{n_2}) > V(180; \alpha, \delta, \pi^{n_2})$ for all $n_2 > 180$. Moreover, for $R = 180$, we find that $180 + 2R - \frac{\delta\alpha}{1-\delta} 119 > 180 + 2R - 0.9 \cdot 119 > 300$. Thus, $V(180; \alpha, \delta, \pi^{n_2}) > V(299; \alpha, \delta, \pi^{n_2})$ in this case.

(b2) For $n_2 = 299$

$$\begin{aligned} & V(299; \alpha, \delta, \pi^{299}) - V(180; \alpha, \delta, \pi^{299}) \\ = \; & \{\delta [\alpha (299 + R) + (1 - \alpha) (180 - R)] + (1 - \delta) \cdot 299\} \\ & - \{\delta [\alpha (180 + R) + (1 - \alpha) 180] + (1 - \delta) \cdot [180 + R]\} \\ = \; & 119\delta\alpha + (1 - \delta) \cdot [119 - R] \gtreqless 0 \iff 119 \left(1 + \frac{\alpha\delta}{1-\delta}\right) \gtreqless R. \end{aligned}$$

For $R = 5$, this condition is satisfied, hence, $n_1 = 299$ is the best reply to a belief concentrated on $n_2 = 299$. For $R = 180$, however, $n_1 = 180$ may be the best reply.

(c) Comparing $n_1 = n_2 - 1$ with $n_1 = 299$.

(c1) For $n_2 = 299$, we obtain

$$\begin{aligned} & V(298; \alpha, \delta, \pi^{299}) - V(299; \alpha, \delta, \pi^{299}) \\ = \; & \{\delta [\alpha (298 + R) + (1 - \alpha) (180 - R)] + (1 - \delta) \cdot (298 + R)\} \\ & - \{\delta [\alpha (299 + R) + (1 - \alpha) (180 - R)] + (1 - \delta) \cdot 299\} \\ = \; & \delta\alpha (-1) + (1 - \delta) \cdot (R - 1) \gtreqless 0 \end{aligned}$$

for $R \gtreqless 1 + \frac{\alpha\delta}{1-\delta}$. From $\delta < 0.5$ we have $R > 1 + \frac{\alpha\delta}{1-\delta}$ for $R \geqslant 2$. Thus, $n_1 = 298$ is the best reply to 299.

(c2) For $n_2 < 299$, we obtain

$$\begin{aligned} & V(n_2 - 1; \alpha, \delta, \pi^{n_2}) - V(299; \alpha, \delta, \pi^{n_2}) \\ = & \{\delta [\alpha (n_2 - 1 + R) + (1 - \alpha)(180 - R)] + (1 - \delta) \cdot (n_2 - 1 + R)\} \\ & - \{\delta [\alpha (299 + R) + (1 - \alpha)(180 - R)] + (1 - \delta) \cdot (n_2 - R)\} \\ = & \delta\alpha (n_2 - 300) + (1 - \delta) \cdot (2R - 1), \end{aligned}$$

thus

$$n_2 \gtreqless 300 - \frac{1-\delta}{\alpha\delta} [2R - 1] =: \overline{n}(\alpha, \delta, R).$$

Hence,

$$R_1(\nu(\cdot|\alpha, \delta, n_2)) = \begin{cases} 299 & \text{for} \quad n_2 < \overline{n}(\alpha, \delta, R), \\ n_2 - 1 & \text{otherwise.} \end{cases}$$

Notice, for $R = 5$, $\overline{n}(\alpha, \delta, R)$ can range between 210 and 290. For $R = 180$, we have $180 > \overline{n}(\alpha, \delta, R)$. Thus, $n_1 = n_2 - 1$ is the best response for all $n_2 > 180$. ■

The following proposition yields the symmetric equilibrium under ambiguity of this game. The notation $[x]$ refers to the smallest integer larger or equal to x. For ease of notation, we will suppress the arguments of the function $\overline{n}(\alpha, \delta, R)$ and will write $\overline{n}$ for its value.

Proposition A.2 (EUA of the traveller's dilemma) *Suppose the conditions of Lemma A.1 are satisfied.*

1. *For* $R = 180$, *in the* unique symmetric EUA *both players have beliefs:*
$$\pi^*(180) = 1.$$
In response, both players choose $n_1^* = n_2^* = 180$.
2. *For* $R = 5$, *in the* unique symmetric EUA *both players have beliefs:*
$$\begin{aligned} \pi^*([\overline{n}]) &= \left[1 + \frac{\alpha\delta}{1-\delta}\right] \cdot \frac{299 - [\overline{n}]}{299 - [\overline{n}] + R}, \\ \pi^*(299) &= \frac{R}{(299 - [\overline{n}] + R)} - \frac{\alpha\delta}{1-\delta} \cdot \frac{(299 - [\overline{n}])}{(299 - [\overline{n}] + R)}, \end{aligned}$$
where
$$\overline{n}(\alpha, \delta, R) := 300 - \frac{1-\delta}{\alpha\delta} [2R - 1].$$
In response, both players choose $n_1^*, n_2^* \in \{[\overline{n}], 299\}$.

Proof. The equilibrium beliefs π^* of an EUA must make players indifferent between the claims of $[\overline{n}]$ and 299. Clearly, all strategies which are not best responses will be played with probability zero. Hence, we can set $\pi^*(n) = 0$ for all $n \notin \{[\overline{n}], 299\}$. For notational convenience, let $\pi^*([\overline{n}]) = \beta$ and $\pi^*(299) = 1 - \beta$. An EUA is defined by the equation,

$$V([\overline{n}]; \alpha, \delta, \pi^*) - V(299; \alpha, \delta, \pi^*) = 0.$$

Equivalently, one has

$$\begin{aligned} & \delta\left[\alpha\left([\overline{n}]+R\right)+(1-\alpha)\left[180-R\right]\right]+(1-\delta)\cdot\{[\overline{n}]\cdot\pi^*([\overline{n}])+([\overline{n}]+R)\cdot\pi^*(299)\} \\ & -\delta\left[\alpha\left(299+R\right)+(1-\alpha)\left[180-R\right]\right]+(1-\delta)\cdot\{([\overline{n}]-R)\cdot\pi([\overline{n}])+(299)\cdot\pi(299)\} \\ = \; & \delta\alpha\left([\overline{n}]-299\right)+(1-\delta)\cdot\{R\cdot\beta+([\overline{n}]-299+R)\cdot(1-\beta)\}=0. \end{aligned}$$

Solving for β, we obtain

$$\beta = \left[1 + \frac{\alpha\delta}{1-\delta}\right] \cdot \frac{299 - [\overline{n}]}{299 - [\overline{n}] + R}.$$

Hence,

$$\begin{aligned} \pi^*([\overline{n}]) &= \left[1 + \frac{\alpha\delta}{1-\delta}\right] \cdot \frac{299 - [\overline{n}]}{299 - [\overline{n}] + R} \\ \pi^*(299) &= \frac{R}{(299 - [\overline{n}] + R)} - \frac{\alpha\delta}{1-\delta} \cdot \frac{(299 - [\overline{n}])}{(299 - [\overline{n}] + R)}. \end{aligned}$$

■

A.2 Matching Pennies

Let $\mathbf{x}$ be the payoff of Player 1 at the strategy combination (T, L). Consider first Player 1's payoffs:[19]

$$\begin{aligned} V_1(T) - V_1(B) &= \delta\left[\alpha\mathbf{x} + (1-\alpha)\,40\right] + (1-\delta)\left[\mathbf{x}\cdot\pi(L) + 40\cdot\pi(R)\right] \\ &\quad -\delta\left[\alpha 80 + (1-\alpha)\,40\right] - (1-\delta)\left[40\cdot\pi(L) + 80\cdot\pi(R)\right] \\ &= \delta\alpha\left(\mathbf{x} - 80\right) + (1-\delta)\left[\mathbf{x}\cdot\pi(L) - 40\right]. \end{aligned}$$

[19] For convenience we suppress α, δ and π.

Similarly, for Player 2 we obtain

$$\begin{aligned} V_2(L) - V_2(R) &= \delta\left[\alpha 80 + (1-\alpha)\,40\right] + (1-\delta)\left[40 \cdot \pi(T) + 80 \cdot \pi(B)\right] \\ &\quad -\delta\left[\alpha 80 + (1-\alpha)\,40\right] - (1-\delta)\left[80 \cdot \pi(T) + 40 \cdot \pi(B)\right] \\ &= (1-\delta)40 \cdot \left[\pi(B) - \pi(T)\right] = (1-\delta)80\left[\frac{1}{2} - \pi(T)\right]. \end{aligned}$$

In game B, where $\mathbf{x} = \mathbf{80}$,

$$\begin{aligned} \pi^*(L) &= \pi^*(R) = \frac{1}{2}, \\ \pi^*(T) &= \pi^*(B) = \frac{1}{2}, \end{aligned}$$

is the only EUA for any degree of optimism α and any degree of ambiguity δ.

Straightforward computations show that when $\mathbf{x} = 320$,

$$V_1(T) - V_1(B) = 240\delta a + (1-\delta)320\left[\pi(L) - \frac{1}{8}\right] \geqslant 240\delta a - (1-\delta)40.$$

For values of $\alpha\delta \approx 0.25$ and $\delta \approx 0.5$, which are approximately the average values found in Kilka and Weber (2001), T will be preferred to B for any beliefs about player 2's behavior. Consistent beliefs of Player 2 are $\pi(T) = 1$ for which value

$$V_2(L;\alpha,\delta,\pi) - V_2(R;\alpha,\delta,\pi) = (1-\delta)80\left[\frac{1}{2} - \pi(T)\right] < 0$$

follows. Hence,

$$\begin{aligned} \pi^*(L) &= 0, \;\; \pi^*(R) = 1, \\ \pi^*(T) &= 1, \;\; \pi^*(B) = 0, \end{aligned}$$

will be equilibrium beliefs in the unique EUA for optimism and ambiguity parameters close to $\alpha\delta \approx 0.25$ and $\delta \approx 0.5$.

A.3 A Coordination Game with a Secure Option

Consider players who view their opponents behavior as ambiguous, one obtains the following evaluations of their actions. For Game $\mathbf{D}$, one obtains the CEU-payoff differences:

$$V_1(H;\alpha,\delta,\pi_2) - V_1(L;\alpha,\delta,\pi_2) = 90\delta\alpha + 180(1-\delta)\left[\pi_2(H) - \frac{1}{2}\pi_2(L)\right]$$

and $V_2(H;\alpha,\delta,\pi_1) - V_2(L;\alpha,\delta,\pi_1) = 90\delta\alpha + 180(1-\delta)\left[\pi_1(H) - \frac{1}{2}\pi_1(L)\right]$.

With the level of ambiguity, $\delta \approx 0.5$, and ambiguity-attitudes, $\alpha\delta \approx 0.25$, there remain multiple EUA leading either to coordinated behavior on $\langle H, H\rangle$ or $\langle L, L\rangle$. If ambiguity is sufficiently high, $\delta > \frac{5}{7}$, even a low degree of optimism $\alpha = 0.4$ will induce Player 1 to choose the strategy H, no matter what belief π_2 she holds regarding the opponent. In this case, the only EUA would be $\pi_1^*(H) = \pi_2^*(H) = 1$.

In Game **E**, we obtain the following CEU payoff:

$$V_1(H;\alpha,\delta,\pi_2) - V_1(L;\alpha,\delta,\pi_2) = -220\delta\alpha + (1-\delta)\cdot[580\cdot\pi_2(H) + 310\cdot\pi_2(L) - 400]$$
$$\leqslant -220\delta\alpha + 180(1-\delta).$$

In this case, the empirically observed values of $\alpha\delta \approx 0.25$ and $\delta \approx 0.5$ are high enough to induce Player 1 to choose strategy L, no matter what beliefs π_2^* she holds over her opponent's behavior. For Player 2 the CEU payoff difference remains unchanged. Hence, for ambiguity and optimism represented by the parameter values $\alpha\delta \approx 0.25$ and $\delta \approx 0.5$, the unique equilibrium under ambiguity has beliefs $\pi_1^*(L) = \pi_2^*(L) = 1$.

A.4 A Minimum-Effort Coordination Game

Consider player 1. Her (Choquet) expected pay-off is given by:

$$\begin{aligned}
V_1(e_1;\alpha,\delta,\pi^{e_2}) &= \delta\cdot\left[\alpha\cdot\max_{e_2\in E_2}u_1(e_1,e_2) + (1-\alpha)\cdot\min_{e_2\in E_2}u_1(e_1,e_2)\right] + (1-\delta)\cdot u_1(e_1,e_2)\\
&= \delta\cdot[\alpha\cdot(e_1 - c\cdot e_1) + (1-\alpha)\cdot(110 - c\cdot e_1)] + (1-\delta)\cdot(\min\{e_1,e_2\} - c\cdot e_1)\\
&= \delta\cdot(1-\alpha)\cdot 110 + [\delta\cdot\alpha\cdot e_1 + (1-\delta)\cdot\min\{e_1,e_2\}] - c\cdot e_1.
\end{aligned}$$

For any level of effort e_2, which the opponent may choose the payoff from increasing effort by one unit is:

$$\begin{cases} [\delta\cdot\alpha + (1-\delta)] - c & \text{if } e_1 < e_2,\\ \delta\cdot\alpha - c & \text{if } e_1 \geqslant e_2.\end{cases}$$

Hence, choosing the highest effort level is always optimal if $\delta\cdot\alpha - c > 0$. The unique equilibrium beliefs of EUA are in this case $\pi_1^*(170) = \pi_2^*(170) = 1$. Conversely, for $c > [\delta\cdot\alpha + (1-\delta)]$ the lowest level of effort is optimal, yielding an EUA with beliefs $\pi_1^*(110) = \pi_2^*(110) = 1$. With the average values of $\alpha\delta \approx 0.25$ and $\delta \approx 0.5$, one has $\delta\cdot\alpha - c > 0$ for $c = 0.1$ and $c > [\delta\cdot\alpha + (1-\delta)]$ for $c = 0.9$. For marginal costs of 0.1 one would expect players to try to coordinate on the highest effort level, while for $c = 0.9$ they should coordinate on the lowest effort level.