All-Pay Auctions with Variable Rewards[*]

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Abstract

We study all-pay auctions with variable rewards under incomplete information. In standard models, a reward depends on a bidder's privately known type; however, in our model it is also a function of his bid. We show that in such models there is a potential for paradoxical behavior where a reduction in the rewards or an increase in costs may increase the expected sum of bids or alternatively the expected highest bid.

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1 Introduction

The corporations Boeing and Lockheed Martin were recently engaged in a competition for the contract to construct the future Joint Strike Fighter. Both corporations incurred large costs associated with the preparation of their designs for a test flyoff. Once the planning and experimentation stage was over, the U.S. government decided to award the project to Lockheed Martin. The other company, Boeing, will receive nothing and yet bear all the costs of preparing its rejected proposal. In the recent Democratic primary race, both Bradley and Gore spent large sums of money on advertisements to increase their chances of winning; however, only Gore enjoyed the benefit of being the nominee.

These scenarios are ubiquitous. They are encountered in R&D competitions, political contests, rent seeking and lobbying activities. Previously, economists have modeled these settings as first-price all-pay auctions: every contestant submits and **pays** a bid for the item being sold, while only the highest bidder receives the item. Hillman and Samet [1987], Hillman and Riley [1989] analyze rent seeking activities; Baye et al. [1993], Becker [1983] and Che and Gale [1998] study lobbying activities; Rosen [1986] investigates job promotion; Snyder [1989] considers political contests; Dasgupta [1986] examines R&D activity and O'Neil [1986] analyzes an arms race. Baye et al. [1996], Clark and Riis [1998] and Barut and Kovenock [1998] provide a more general analysis for the complete information case, while Amann and Leininger [1996], Krishna and Morgan [1997], and Moldovanu and Sela [2000] provide such analysis for the incomplete information case.

All such studies that use the standard first-price all-pay auction capture the important feature that all participants incur costs; however, they miss another crucial aspect of these environments – there is a relationship between the expenditures incurred in the bidding process and the size of the reward collected by the winning contestant. That is, a larger expenditure increases not only the probability of winning the contest but the size of the reward gained by winning.[1]

For instance, in bidding for the Joint Strike Fighter contract, effort put into the proposal can save effort in delivering the final product. Furthermore, the higher quality of the proposal may also increase

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[1]Konrad and Schlesinger [1997] study rent seeking in rent-augmenting games where both the size of the reward and the probability of winning depend upon the effort (bid). The classes of probability functions and their properties that are used in such games are described in Skaperdas [1996]. These do not include an all-pay auction where this probability is either one or zero depending upon whose bid is higher.

the reward, by having more fighters purchased, for which estimates ranged from 200 to 400 billion.[2] The reward for winning the Democratic primary race is participation in the Presidential election of which the value depends upon the chances of winning. These chances have a significant dependence on performance in the primary (both on time of winning and popularity gained). Other examples may include cases where several agents compete for a client and where there is an industry standard result-dependent rate (sports agents, estate agents, lawyers in tort cases, investment managers, etc.) The client will choose the agent with the best proposal. Better initial proposals could mean better results (or at least savings in additional costs), thus larger rewards for the agent.

Under complete information, Kaplan et al. [2000] show that the existence of this relationship may introduce substantial qualitative changes to the behavior of the contestants. In this paper, we study such bid-dependent rewards under incomplete information. By studying an environment where the reward is multiplicatively separable in effort and type, we find two paradoxical changes from all-pay auctions that are only present under incomplete information. Namely, an increase in the rewards for winning the contest or a decrease in the costs for entering a bid may, in fact, reduce the amounts bid. We show that such paradoxes do not exist when the reward is additively separable in effort and type (an environment that generalizes the standard all-pay auction).

These results are surprising. Whenever we use an all-pay auction to analyze a situation, we need to ask how exactly is the reward determined. Does it only depend upon winning or losing? If not, then how exactly does the expenditure affect the reward? We see from our results that both questions are important in deriving implications from analysis.

Consider political races. If society wants to reduce advertisements in political races, increasing the costs of advertising may, in fact, increase advertisements. Even though the rewards for being governor are higher than that of being mayor, higher campaign expenses in the mayoral race may not be an indicator of that competition being closer. In addition, the government may want to reduce the socially wasteful efforts towards competing for government procurement contracts. One may recommend to reduce the size of the reward by purchasing more on the free-market; however, this

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[2] The initial contract specifies initial orders only. A higher quality design could lead to additional orders from the US as well as orders from US allies. Furthermore, it may delay any possible introduction of the next generation fighter (Holson [2001]).

may, to the converse, increase overall effort.

Patent races are another situation that is amenable to our analysis. A firm choosing a time t of innovation replaces a contestant choosing a bid x. The time (until innovation) should decrease in x with $x = 0$ corresponding to $t = \infty$ (for instance if $t = 1/x$). Clearly, the reward should depend upon the time of innovation (usually decreasing).

Again, in this situation we may get counter-intuitive results. Decreasing rewards to innovations (such as shortening the patent) may in fact speed up innovation time. An intermediate innovation that reduces costs (to all participants) may delay the innovation time.

The paper proceeds as follows. In Section 2 we present the model. In Section 3 we analyze an additively separable environment and show that paradoxical behavior cannot occur in such settings. In Section 4 we study a multiplicatively separable environment and obtain paradoxical behavior for a range of possible cost and reward structures. In Section 5 we discuss the results and offer an explanation for the paradox. In Section 6, we conclude.

2 The model

We now describe the environment of an all-pay auction with bid-dependent rewards. The set of contestants is $N = \{1, 2, ..., n\}$. Each contestant i in N submits a bid x_i and, by doing so, incurs a disutility (or cost) denoted by $c(x_i)$, where $c : R_+ \to R_+$ is an increasing function with $c(0) = 0$. The contestant j that chooses the highest bid wins a reward $R(\theta_j, x_j)$, where $R(\theta_j, x_j) : R_+^2 \to R_+$, and individual type θ_j is independently drawn from the interval $[0, 1]$ according to the distribution function F. While F is common knowledge, each contestant is privately informed about his own type.

Note that in the standard all-pay auction model $R(\theta_j, x) = R(\theta_j)$, namely the reward of each contestant is a function of his own type only.

2.1 Equilibrium

A contestant's strategy is a function $b(\theta)$ that indicates a bid to submit for each realization of θ. In a Bayesian equilibrium, the bidding function (strategy) chosen by each contestant maximizes his expected utility given the bidding functions chosen by the other contestants. Hence, a symmetric

equilibrium consists of a bidding function $b(\theta)$ (assumed to be monotonic increasing and differentiable) where for each θ the bid $b(\theta)$ solves the following maximization problem:

$$\max_x F^{n-1}(b^{-1}(x)) \cdot R(\theta, x) - c(x) \tag{1}$$

3   Additively separable auctions

We say that an environment is additively separable if $R(\theta, x)$ is separable into $g(\theta) + h(x)$ where g is strictly increasing. Note that when $h(x)$ is constant, the environment is the same as a regular all-pay auction, and when $h(x) = -x$ and $c(x) = 0$, the environment is the same as a first-price auction. In this section, we assume that conditions are such that there exists an equilibrium described in the previous section where the equilibrium bid function is monotonic increasing.[3]

Proposition 1 *In an additively separable environment, a reduction in the rewards or an increase in the cost of bidding decrease the expected sum of bids or alternatively the expected highest bid.*

Proof. We show that if there are two additively separable environments (c^a, h^a) and (c^b, h^b) where $h^b - c^b \geq h^a - c^a$ it must be that the equilibrium bids satisfy $x_b \geq x_a$. We show these properties by using the Revenue Equivalence Theorem as stated in Klemperer [1999].[4] First, the bidder with the highest θ always wins the auction. A bidder with $\theta = 0$ has zero expected payoff. This environment is equivalent to a mechanism (c, h) where the seller takes $c(x(\theta))$ from a buyer reporting θ and pays $h(x(\theta))$ to a buyer with the highest θ (and value $g(\theta)$) while giving him the object. By revenue equivalence, any such mechanism should give the same expected surplus to a buyer of type θ. Therefore, if there are two mechanisms: (c^a, h^a) and (c^b, h^b) with $h^b - c^b \geq h^a - c^a$, then the equilibrium bidding functions $x_a(\theta)$ and $x_b(\theta)$ should satisfy:

$$F(\theta) \cdot [g(\theta) + h^a(x_a(\theta))] - c^a(x_a(\theta)) = F(\theta) \cdot [g(\theta) + h^b(x_b(\theta))] - c^b(x_b(\theta)) \tag{2}$$

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[3] This is a standard result for a large class of auctions (see Reny and Zamir [2000]).

[4] The revenue equivalence theorem in Klemperer [1999] is: "Assume each of a given number of risk-neutral potential buyers of an object has a privately-known signal drawn from a common, strictly increasing, atomless distribution. Then any auction mechanism in which (i) the object always goes to the buyer with the highest signal, and (ii) any bidder with the lowest-feasible signal expects zero surplus, yields the same expected revenue and results in each bidder making the same expected payment as a function of her signal."

If there were a $\widehat{\theta}$ where $x_b(\widehat{\theta}) < x_a(\widehat{\theta})$, then by revealed preferences (prefers $x_b(\widehat{\theta})$ to $x_a(\widehat{\theta})$ when $(c, h) = (c^b, h^b)$)

$$F(\widehat{\theta}) \cdot [g(\widehat{\theta}) + h^b(x_b(\widehat{\theta}))] - c^b(x_b(\widehat{\theta})) \geq F(x_b^{-1}(x_a(\widehat{\theta}))) \cdot [g(\widehat{\theta}) + h^b(x_a(\widehat{\theta}))] - c^b(x_a(\widehat{\theta}))$$

Since x_b is monotonic, $x_b^{-1}(x_a(\widehat{\theta})) > \widehat{\theta}$. Therefore,

$$F(\widehat{\theta}) \cdot [g(\widehat{\theta}) + h^b(x_b(\widehat{\theta}))] - c^b(x_b(\widehat{\theta})) > F(\widehat{\theta}) \cdot [g(\widehat{\theta}) + h^b(x_a(\widehat{\theta}))] - c^b(x_a(\widehat{\theta}))$$

Using equation (2) (derived from revenue equivalence) to substitute for the LHS,

$$F(\widehat{\theta}) \cdot h^a(x_a(\widehat{\theta})) - c^a(x_a(\widehat{\theta})) > F(\widehat{\theta}) \cdot h^b(x_a(\widehat{\theta})) - c^b(x_a(\widehat{\theta}))$$

Which can only happen if

$$F(\widehat{\theta}) \cdot [h^a(x_a(\widehat{\theta})) - h^b(x_a(\widehat{\theta}))] > c^a(x_a(\widehat{\theta})) - c^b(x_a(\widehat{\theta}))$$

If this inequality holds for $1 > F(\widehat{\theta}) > 0$, then

$$h^a(x_a(\widehat{\theta})) - h^b(x_a(\widehat{\theta})) > c^a(x_a(\widehat{\theta})) - c^b(x_a(\widehat{\theta}))$$

This contradicts that $h^b - c^b \geq h^a - c^a$. $\square$

So, in an additively separable environment, an increase in rewards (or a decrease in costs) causes, as can be expected, an increase in the bids submitted.[5] In the next section we examine another environment with radically different results.

4 Multiplicatively separable auctions

We say that an environment is multiplicatively separable if $R(\theta, x) = \theta \cdot R(x)$, where $R(\cdot)$ is strictly increasing and $R(0) > 0$. Here, θ is the ability of a contestant where the higher θ, the more he is able to exploit the reward. For instance, the reward may represent the profit that a firm makes for a given contract. The amount bid (representing quality) affects both probability of winning and, in addition,

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[5]As mentioned, the standard all-pay auction is a specific case of our additively separable environment. Gavrious, Moldovanu and Sela [2001] find that adding bid caps improves the revenue of an all-pay auction. This is puzzling since adding a cap is the equivalent of increasing the cost function (albeit to one with infinite slope at the cap). However, the bid function is not strictly monotonic and thus there is a chance that the bidder with the highest θ will lose. We conjecture that conditions such as continuity of costs will be sufficient to eliminate such behavior.

the quantity ordered once a contractor has been chosen. Different abilities represent different per unit profit levels.

We now present a paradoxical behavior in multiplicatively separable auctions that differs from behavior in standard all-pay auctions.

Proposition 2 *In a multiplicatively separable all-pay auction, for a large family of possible reward and cost functions (costs can be linear, concave or convex), a reduction in the rewards may increase the expected sum of equilibrium bids or alternatively the expected highest bid.*

Proof. Assume that θ is uniformly distributed on $[0,1]$, i.e., $F(\theta) = \theta$. In order to characterize the equilibrium bid functions we show first the following lemma.

Lemma 3 *In a multiplicatively separable all-pay auction with n bidders, if either $c'(x) > R'(x)$ for all x or $c'(x)/R'(x)$ is a strictly increasing function of x for all x, then $b(\theta) = u^{-1}(\theta^n)$ is a symmetric equilibrium bidding strategy where, $u(x) = R(x)^{-\frac{n}{n-1}}[\int_0^x \frac{n}{n-1} c'(t) R(t)^{\frac{1}{n-1}} dt]$.*

Proof. See in the Appendix. $\square$

Now, we show two cases a and b such that the reward function in case a is smaller than the reward function in case b (the cost function is identical in both cases) and nevertheless the expected bid of every bidder as well as the expected highest bid in case a is larger than in case b.

Consider the following cases when $n = 2$. Case a' : the reward function is $R^{a'}(x) = x^{1/4}$ and the cost function is $c(x) = x$. Case b : the reward function is $R^b(x) = 5^{1/3}/2$ and the cost function is $c(x) = x$.

By Lemma 3, the bid functions $b^{a'}(\theta) = (5/8)^{4/3} \cdot \theta^{8/3}$ and $b^b(\theta) = 5^{1/3}/4 \cdot \theta^2$ respectively form equilibria for cases a' and b.

Now, let Case a be as follows: the reward function is $R^a(x) = Min\{x^{1/4}, (5/8)^{1/3}\}$ and the cost function is $c(x) = x$.

By comparing case a and case b, we will show the paradoxical behavior that if the reward goes down, both the average bid and the average highest bid may go up. First notice that the reward function over the range of equilibrium bids in case a' is less than that in case b. The maximum bid for case a' is

$b^{a'}(1) = (5/8)^{4/3}$. Thus, the highest possible reward for case a' is $R^{a'}((5/8)^{4/3}) = (5/8)^{1/3}$ where the reward for case b is $(5/8)^{1/3}$ for all x. Second notice that $b^{a'}$ also forms an equilibrium with reward function $R^a(x)$. (The points not chosen with $R^{a'}$ will yield less reward with R^a and thus still not be chosen.) Thus, $R^a(x) \leq R^b(x)$ for all $x \geq 0$.

The expected average bid in case a is $E^a = \int_0^1 (5/8)^{4/3} \cdot \theta^{8/3} d\theta = 0.14574$, while the expected average bid in case b is $E^b = \int_0^1 5^{1/3}/4 \cdot \theta^2 d\theta = 0.1425$. That is, the expected average bid in case a is larger than in case b. In addition, the highest expected bids in cases a and b are $\left(\int_0^1 b^b(\theta) \cdot 2\theta \cdot d\theta = \right)$ 0.22901 and 0.21374, respectively.[6]

Notice that the costs in the example above are linear. We can use the following lemma to show that these results apply to a large family of cost and reward functions.

Lemma 4 *If in an environment with cost $c(x)$ and reward $R(x)$ there is an equilibrium bid function $b(\theta)$, then for any continuous, strictly-increasing transformation $y = g(x)$ (where $g(0) = 0$), the environment with $\widehat{c}(y) = c(g^{-1}(y))$ and $\widehat{R}(y) = R(g^{-1}(y))$ has an equilibrium bid function $\widehat{b}(\theta) = g(b(\theta))$.*

Proof. See in the Appendix. $\square$

Notice that if $R^1(x) \geq R^2(x)$ for all x, then $\widehat{R}^1(y) \geq \widehat{R}^2(y)$ for all y, likewise for costs. Also notice that for any θ where $b^1(\theta) > b^2(\theta)$, then $\widehat{b}^1(\theta) > \widehat{b}^2(\theta)$ as well. This implies that applying any transformation to our example will maintain the relationship between the reward functions while also maintaining the paradoxical relationship for the same range of θ. What happens with the average bids and expected highest bids is less clear.

In our example above, both equilibrium bid functions are increasing in θ, while the bid function for case a is higher than the bid function for case b after a certain point. This implies that any convex transformation, will increase the disparity of the average bid and expected highest bid between the cases. Furthermore, any concave transformation $y = x^\alpha$ where $\alpha = 1 - \varepsilon$ will lead to only a slight deviation in the equilibrium bid functions. Thus, for small enough ε, we will have the relationship between the average and expected highest bids maintained. Therefore, there exist paradoxical reward examples with both concave and convex costs. $\square$

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[6] The disparity for the highest expected bid is always greater than the disparity for the expected bid.

A set of closely related results to our findings here is present in the study of optimal auctions. Myerson [1981] finds that revenue in a first-price auction with a (suitable) minimum bid is higher than in one without. A minimum bid can also be represented by a bid dependent reward function. Formally, a minimum bid of m with reward R is equivalent to a variable reward $\widehat{R}(x) = R$ when $x > m$ and 0 otherwise. This reward function is less than $R^*(x) = R$ for all x. Thus, although $\widehat{R}(x)$ is not continuous and hence does not satisfy the requirements imposed on the reward functions in our model, a result similar to ours is obtained, mainly a decrease in rewards may increase revenues.

We now present another paradoxical behavior that differs from behavior in standard all-pay auctions.

Proposition 5 *In a multiplicatively separable all-pay auction, and for a large family of possible reward and cost functions, an increase in the costs may increase the expected sum of equilibrium bids or alternatively the expected highest bid.*

Proof. Assume again that θ is uniformly distributed on $[0, 1]$, i.e., $F(\theta) = \theta$. In the following we show two cases a and b such that the cost function in case a is higher than the cost function in case b and nevertheless, in equilibrium, the average bids in case a are higher than the average bids in case b.

Consider the following cases for $n = 2$. Case $a' : R(x) = x^{1/18}$ and $c^{a'}(x) = 3/4 \cdot x^{1/9}$. Case $b :$ $R(x) = x^{1/18}$ and $c^b(x) = 3/4 \cdot x^{1/6}$.

Using the solution offered by Lemma 3, the equilibrium bid functions are, $b^{a'}(\theta) = \theta^{36}$ and $b^b(\theta) = (8/9)^9 \cdot \theta^{18}$.

Now, let case a be as follows: $R(x) = x^{1/18}$ and $c^a(x) = max\{c^{a'}(x), c^b(x)\}$.

For the same reasoning as with different reward functions, the bid function $b^{a'}$ will be an equilibrium for a cost function $c^a(x) = max\{c^{a'}(x), c^b(x)\}$. Thus, costs in case a (with cost function c^a) are higher than costs in case b. As with the reward example at $\theta = 1$, the equilibrium bid is higher in case a, as we see that $b^a(1) = 1$ while $b^b(1) = (8/9)^9$.

Now, the expected average bid in case a is $E^a = \int_0^1 \theta^{36} d\theta = 0.027$, while the expected average bid in case b is $E^b = \int_0^1 (8/9)^9 \cdot \theta^{18} d\theta = 0.018$. That is, the expected average bid in case a is higher than in case b. In addition, the highest expected bids in cases a and b are approximately 0.053 and 0.035 respectively.

As before using Lemma 4 implies that this result holds for a large family of cost and reward functions. $\square$

5 Discussion

As we showed, in the multiplicative environment there exist examples of paradoxical behavior. However, there are simple conditions that yield "regular" behavior. For example, behavior will be regular if the change in rewards or costs is multiplicative (by a factor of $\alpha > 1$). Namely, if $R^2(x) = \alpha \cdot R^1(x)$ or $c^1(x) = \alpha \cdot c^2(x)$, then $b^2(\theta) \geq b^1(\theta)$ for all θ. To see that note that by the solution derived in Lemma 3 (see equation 5 in the appendix), we can easily arrive at $u^1(x) = \alpha \cdot u^2(x)$. Since $b(\theta) = u^{-1}(\theta^n)$, we have $b^2(\theta) \geq b^1(\theta)$. Hence we see that for paradoxical behavior to occur it is necessary to change more than just the magnitude.

An *initial* economic intuition one may have to explain this is as follows. Increasing one's bid changes not only the probability of receiving a reward, but affects the reward itself $(d(P(x)R(x)) = P'(x)R(x) + P(x)R'(x))$. Increasing the slope of the reward function could provide enough incentive to increase one's bid even if there is a reduction in the rewards. However, this cannot fully explain the paradox – if this explains the paradox in the mulitiplicatively-separable case, it should also imply the existence of a paradox in additively-separable case, which we proved does not exist. Furthermore, Kaplan et. al [2000] showed that the paradox does not exist with complete information.

So what is the full (theoretical) explanation? In environments with asymmetric information the expected payoff of a bidder depends on his type (the value of the realized θ). In our setup the expected payoff of a bidder depends positively on his type and only bidders with the lowest type receive a zero expected payoff. The higher payoffs associated with the higher types are referred to as informational rents (see Macho-Stadler & Perez-Castrillo [1997] and Salanie [1997]). Under (symmetric) complete information, there are no informational rents (all information is public) and all bidders have zero expected profit. Therefore, any change in rewards or costs must have a corresponding change in bids to leave profit the same: an increase in rewards must imply an increase in bids in order to leave the bidder with zero expected profit and likewise for a decrease in costs.

This logic still holds in the additively separable case. As we used the revenue equivalence to assist

in the proof, again the main intuition is that the informational rent of each bidder is not affected by changes in rewards or costs. Therefore, by simple accounting no paradox can occur. Things change for the multiplicatively separable case. The expected payoff (informational rent) of a bidder of type θ in equibrium, denoted by $\pi(\theta)$, is given by $\pi(\theta) = F(\theta)^{n-1}R(x(\theta))\theta - c(x(\theta))$. Using the envelope theorem we see the informational rents are given by the following.[7]

$$F(\theta)^{n-1}R(x(\theta))\theta - c(x(\theta)) = \int_0^\theta F(\hat{\theta})^{n-1}R(x(\hat{\theta}))d\hat{\theta}$$

From this equation we can see why an increase in the rewards may decrease bids. Take for example an increase in the reward function R only for low values of x. Keeping bids constant, this will increase the RHS of the equation for all θ. However, again keeping the bids constant the LHS of the equation does not increase for higher values of θ. For the equation to return to equality, the LHS must increase. This occurs when bids are lowered.

This also explains why the paradox does not occur when R is multiplied by a positive constant. Take for instance a change whereby R is doubled. When bids stay fixed the RHS is doubled, but the LHS is more than doubled. Hence, bids in the new equilibrium must increase. Notice that the preceeding explanation also applies to the case where R is a constant.

One may have noticed that the RHS is not directly affected by a change in costs. How then can a decrease in costs influence the rents? The answer is that decreasing the costs for low values of θ increases the LHS for those values. This then means that the bids must be increased for those values. This then indirectly affects the RHS for not only those values, but higher values as well and we have the same story as before.

While we now see that such a paradox can occur in theory, we should ask if there is any common policy that may result in such a paradox against the wishes of its creators. For instance, changing the duration of a patent alters the rewards. Can this result in a paradox? If the market size were constant over time, then reducing the length L to $\gamma \cdot L$ would be equivalent to multiplying the rewards

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[7] The bidder's payoff is given by: $\pi(\theta) = Max_x F^{n-1}(b^{-1}(x)) \cdot \theta \cdot R(x) - c(x)$. Invoking the envelope theorem we get $\frac{d\pi(\theta)}{d\theta} = F^{n-1}(b^{-1}(x(\theta))) \cdot R(x(\theta))$. Since equilibrium strategies are assumed to be symmetric, $b^{-1}(x(\theta)) = \theta$ and we get $\frac{d\pi(\theta)}{d\theta} = F^{n-1}(\theta) \cdot R(x(\theta))$. Integrating this expression we get the integral representation of informational rent as: $\pi(\theta) = \int_0^\theta F(\hat{\theta})^{n-1}R(x(\hat{\theta}))d\hat{\theta}$. Note that a similar formula can be derived for general $R(\theta, x)$. In this case the envelope theorem yields the derivative $F(\theta)^{n-1}R_\theta(\theta, x)$. It can also be derived for ability-dependent costs where the derivative is $F(\theta)^{n-1}R_\theta(\theta, x) - c_\theta(\theta, x)$.

by α where $\alpha > 1$ if $\gamma > 1$ and $\alpha < 1$ if $\gamma < 1$.[8] Thus, as shown before, this has been shown to lead to standard results. However, if the market size is not constant the paradox might occur.

The example we presented has an increase in the rewards function with a greater increase for low-bid winners than for high-bid winners. How does this correspond to a change in patent life? This occurs when the market is growing fast enough (at least temporarily) such that the present value is also growing. This is common for (although not limited to) goods associated with network externalities. The Internet was growing so fast that the present value of each year's market was growing and thus a year of a patent (for example, PriceLine's auction) is worth more towards the end of its life. Likewise, if the number of births in a country is growing at 10 percent per year, then an additional year for a patent of a baby product would be worth more in present value the later it is (assuming the discount rate is less than 10 percent). We also see certain products that weren't successful because the market/technological conditions weren't right and years later a similar product was a hit. A noticable example of this is the palm computer. The Palm Pilot was a hit in the late 90's, but its predecessors (GO or Apple Newton) of the early 90's were failures.[9] In these examples, an increased patent life would benefit low-bid winners (late innovators) more than high-bid (early inventors).[10]

6   Conclusions

We study an all-pay auction with bid-dependent rewards under incomplete information. We find results that differ from both a regular all-pay auction under incomplete information and an all-pay auction with bid-dependent rewards and complete information. In particular, we find the paradoxical behavior that a reduction in the rewards may increase effort, while a decrease in the costs to bidding may decrease effort. An important insight implied by these findings is that it is possible to achieve an increase in effort (bid) by only lowering the reward for each winning and by doing so modifying the

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[8] The value of holding a patent from time t until time $t + \gamma L$ is $\int_t^{t+\gamma L} e^{-rx} R dx = (1 - e^{-r\gamma L}) R e^{-rt}/r$. Notice that this expression is increasing in γ.

[9] See Kaplan [1996] for a description of the early competition in this market.

[10] One may find that the ability parameter on the rewards is unpalatable for such a patent example; however, one can just as easily put the ability parameter on the cost function and have the same examples if one divides by θ. The equilibrium bid functions would not change, and in this environment rewards are the same for all firms while costs depend on the firm's type.

shape of the reward function. Such paradoxical behavior does not exist for all environments, as we showed for any additively-separable environment; however, we mention in our discussion indications as to why the paradox should exist for a variety of situations.

Our findings cast suspicions on policy recommendations such as subsidies to R&D or patent protection as a means to increase innovative activity. It may also cause doubts on conclusions in many other areas such as government procurement and congressional lobbying. Thus, we hope that this and further work in turn should help improve policy recommendations regarding regulation of such aforementioned contests.

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8 Appendix

Proof of Lemma 3.

We prove the Lemma in the following manner. First, we show that the above bid function solves the differential equation derived from the bidder's maximization problem. Second, we show that either of the two conditions on $c(x)$ and $R(x)$ mentioned is sufficient for the proposed bid function to be strictly monotonic and differentiable.

Before examining the first-order condition, we use the above assumptions on $R(\theta, x)$ and F to reduce (1) to

$$\max_x \theta \cdot (b^{-1}(x))^{n-1} R(x) - c(x) \tag{3}$$

Differentiating with respect to x we obtain the first order condition:

$$\theta(n-1)(b^{-1}(x))^{n-2}(b^{-1})'(x)R(x) + \theta(b^{-1}(x))^{n-1}R'(x) - c'(x) = 0$$

In a symmetric equilibrium it must be that $b^{-1}(x) = \theta$, hence the above reduces to:

$$(n-1)(b^{-1}(x))^{n-1}(b^{-1})'(x)R(x) + (b^{-1}(x))^n R'(x) - c'(x) = 0$$

We now let $u(\cdot)$ equal $(b^{-1}(\cdot))^n$. The first order condition is then:

$$u' + u\frac{nR'(x)}{(n-1)R(x)} = \frac{nc'(x)}{(n-1)R(x)} \tag{4}$$

The solution of this differential equation and boundary condition $u(0) = 0$ is given by:[11]

$$u(x) = R(x)^{-\frac{n}{n-1}} \int_0^x \frac{n}{n-1} c'(t) R(t)^{\frac{1}{n-1}} \, dt \tag{5}$$

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[11] The solution of the differential equation $u' + f(x)u = g(x)$ is given by $u(x) = e^{-\int_0^x f(t)dt}[c + \int_0^x g(t)e^{\int_0^t f(y)dy} dt]$

Thus, the proposed bid function is $b(\theta) = u^{-1}(\theta^n)$. However, this bid function must be strictly increasing to form a symmetric Bayesian equilibrium. The bid function $b(\cdot)$ is strictly increasing if and only if the function $u(x)$ is strictly increasing as well. Following, we find sufficient conditions for $u(\cdot)$ to be a strictly increasing function. Let us first show that $c'(x) > R'(x)$ is sufficient for u to be strictly increasing. The derivative $u'(x)$ equals

$$\frac{n}{n-1}\left(\frac{c'(x)}{R(x)} - \frac{R'(x)u(x)}{R(x)}\right)$$

Since $u(x) \leq 1$ for all possible x, we obtain:

$$u'(x) \geq \frac{n}{n-1}\left(\frac{c'(x)}{R(x)} - \frac{R'(x)}{R(x)}\right)$$

Consequently, $c'(x) > R'(x)$ implies that $u(x)$ is strictly increasing. That is, $c'(x) > R'(x)$ is a sufficient condition for the existence of symmetric Bayesian equilibrium.

Second, we show that if $c'(x)/R'(x)$ is a strictly increasing function, then $u'(x) > 0$. Notice that from (4), the sign of $u'(x)$ is the same as $c'(x) - u(x)R'(x)$. Through algebraic manipulation we have:

$$
\begin{aligned}
c'(x) - u(x)R'(x) &= c'(x) - R'(x)\left[R(x)^{-\frac{n}{n-1}}\int_0^x \frac{n}{n-1}c'(t)R(t)^{\frac{1}{n-1}}dt\right] \\
&= c'(x) - \frac{\int_0^x \frac{n}{n-1}R(t)^{\frac{1}{n-1}}R'(t)\frac{c'(t)}{R'(t)}dt}{R(x)^{\frac{n}{n-1}}}R'(x) \\
&= c'(x) - \frac{R(x)^{\frac{n}{n-1}}\frac{c'(x)}{R'(x)} - \int_0^x R(t)^{\frac{n}{n-1}}\left(\frac{c'(t)}{R'(t)}\right)'dt}{R(x)^{\frac{n}{n-1}}}R'(x) \\
&= \frac{R'(x)}{R(x)^{\frac{n}{n-1}}}\int_0^x R(t)^{\frac{n}{n-1}}\left(\frac{c'(t)}{R'(t)}\right)'dt
\end{aligned}
$$

Thus, we see that if $c'(x)/R'(x)$ is an increasing function, $u(x)$ is increasing as well and, thereby, there exists a symmetric Bayesian equilibrium. $\square$

Proof of Lemma 4.

Since $b(\theta)$ is an equilibrium bid function, $x = b(\theta)$ must solve

$$\max_x \theta \cdot (b^{-1}(x))^{n-1} \cdot R(x) - c(x)$$

However, by transformation of variables $y = g(x) = g(b(\theta))$ must solve

$$\max_y \theta \cdot (b^{-1}(g^{-1}(y)))^{n-1} \cdot R(g^{-1}(y)) - c(g^{-1}(y))$$

Thus, $y = \widehat{b}(\theta)$ must solve $\max_y \theta \cdot (\widehat{b}^{-1}(y))^{n-1} \cdot \widehat{R}(y) - \widehat{c}(y)$. $\square$

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