Endogenous debt constraints in collateralized economies with default penalties☆

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Abstract

The objective of the paper is to propose endogenous debt constraints that rule out Ponzi schemes and ensure existence of equilibria in a model with limited commitment and (possible) default. We appropriately modify the definition of finitely effective debt constraints, introduced by Levine and Zame (1996) (see also Levine and Zame (2002)), to encompass models with limited commitment, default penalties and collateral. Along this line, we introduce in the setting of Araujo, Páscoa, and Torres-Martínez (2002), Kubler and Schmedders (2003) and Páscoa and Seghir (2009) the concept of actions with *finite equivalent payoffs*. We show that, independently of the level of default penalties, restricting plans to have finite equivalent payoffs rules out Ponzi schemes and guarantees the existence of an equilibrium that is compatible with the minimal ability to borrow and lend that we expect in our model.

An interesting feature of our debt constraints is that they give rise to budget sets that coincide with the standard budget sets of economies having a collateral structure but no penalties (as defined in Araujo, Páscoa, and Torres-Martínez (2002)). This illustrates the hidden relation between finitely effective debt constraints and collateral requirements.

Key words: Infinite horizon economies, Incomplete markets, Limited commitment, Default, Debt constraints, Collateral, Ponzi schemes

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1. Introduction

One of the main difficulties of extending financial markets economies to an infinite horizon is related to the existence of the so-called Ponzi schemes. In the absence of a terminal date agents would attempt to finance unbounded levels of consumption by renewing their credit at infinite. If such schemes are permitted, the agent's decision problem has no solution. Therefore, without debt constraints that limit the rate at which agents accumulate debt, equilibria fail to exist.

Broadly speaking three approaches have been proposed in the literature to deal with the specification of debt constraints in infinite horizon sequential markets models. The main difference among these lines of research hinges on the specific assumptions made about the enforcement of payments as well as the proposed default punishment.

The first approach, due to Magill and Quinzii (1994), Hernández and Santos (1996) and Levine and Zame (1996) (see also Levine and Zame (2002)), introduces debt constraints in economies where payments are fully enforced and therefore there is no default (even on out of equilibrium paths). Magill and Quinzii (1994) argue in favor of implicit debt constraints that restrict budget sets to include portfolios whose value is a bounded sequence along the event tree. An interesting property of equilibria with implicit debt constraints is that it is always possible to find uniform bounds on the value of short-sales which are non-binding at those equilibria. Moreover, under reasonable assumptions on preferences, equilibria with implicit debt constraints coincide with equilibria with transversality type conditions that are often imposed in macroeconomic models (see Blanchard and Fisher (1989) and Ljungqvist and Sargent (2000)). Hernández and Santos (1996) argue in favor of debt constraints that impose a kind of solvency requirement. Households are allowed to borrow against their current value of future endowment streams. When markets are incomplete, traders may not agree on current value prices. Hernández and Santos (1996) propose a special way of computing current value prices that takes into account the whole set of non-arbitrage

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price systems. Levine and Zame (1996) (see also Levine and Zame (2002)) offer an alternative formulation of the solvency requirement. They formalize debt constraints that induce agents to repay their debt in finite time, that is, the suggested debt constrains are *finitely effective*. Stated differently, finitely effective constraints induce agents to choose plans that are budget compatible with the threat that, at any period, they may be restricted to have access to borrowing only for a finite number of periods. Finitely effective debt constraints provide a general characterization of debt constraints that are compatible with equilibrium. More precisely, Levine and Zame (1996) have shown that any loose and consistent debt constraints that rule out Ponzi schemes and ensure existence of an equilibrium reduce to be finitely effective.[3]

The second approach, due to Kehoe and Levine (1993) (see also Kehoe and Levine (2001)), Zhang (1997) and Alvarez and Jermann (2000), explores debt constraints in economies where commitment is limited and there is a severe punishment for default: if agents do not honor their debts, they are excluded from participating in the asset markets in future periods. In such a setting the authors argue for self-enforcing constraints that are tight enough to prevent default at equilibrium but simultaneously are loose enough to allow for as much risk sharing as possible.

The third and most recent approach to deal with Ponzi schemes also considers models with limited commitment. However, contrary to self-enforcing borrowing constraints (à la Alvarez and Jermann (2000)) that prevent default at equilibrium, this research line addresses the issue of Ponzi schemes in economies where default may be consistent with equilibrium. It is motivated by the empirical observation that modern economies experience a substantial amount of default and bankruptcy.[4] One of the most important and widespread means of securing loans and lowering the level of default in financial markets is collateral.[5] Araujo, Páscoa, and Torres-Martínez (2002) (see also Kubler and Schmedders (2003)) showed that, without imposing any debt constraints or transversality conditions, Ponzi schemes are ruled out in economies where collateral is the only mechanism that enforces agents to (partially) pay their debts. The intuition behind their result is as follows: combining short-sales with the purchase of collateral constitutes a joint operation that yields non-negative returns.[6] By non-arbitrage, at equilibrium, the price of the collateral exceeds the price of the asset, implying that collateral costs exceed the value of loans. Therefore, it becomes impossible to pay a previous debt by issuing new debt.

In most economic systems collateral is not the only mean of securing loans. The default option usually entails additional economic consequences.[7] This explains the fact that even in the midst of the most severe housing downturn on record, many households with negative equity choose to continue meeting their financial obligations (see, e.g., Gerardi, Shapiro, and Willen (2007, 2009)) and Gerardi, Lehnert, Sherland, and Willen (2011)).

One approach to model additional enforcement mechanisms is to introduce linear utility penalties (see Dubey, Geanakoplos, and Shubik (1990), Zame (1993), Dubey, Geanakoplos, and Shubik (2005) and the literature cited therein). These penalties might be interpreted as the consequences (directly assessed in terms of utility) of some third party punishment such as prison terms and pangs of conscience, and/or of some non-modeled economic punishment such as exclusion from credit markets and garnishing of future income.

A surprising result found by Páscoa and Seghir (2009) is that the introduction of default penalties in the model of Araujo, Páscoa, and Torres-Martínez (2002) may induce payments besides the value of the collateral and lead to the reappearance of Ponzi schemes. The intuition is simple: when penalties are severe, agents have incentives to repay more than the value of the depreciated collateral. In this case, the joint operation of combining short-sales with the purchase of collateral no longer yields non-negative returns. Therefore, loans may exceed collateral costs and agents may run Ponzi schemes.

One may think that the reappearance of Ponzi schemes is related to the particular additional enforcement mechanism (linear utility penalties) Páscoa and Seghir (2009) have considered. However, Ferreira and Torres-Martínez (2010) showed that, for sufficiently low collateral requirements, any effective additional enforcement mechanism implies the non-existence of physically feasible optimal plans.[8] That is, any effective additional enforcement mechanism gives rise to Ponzi schemes in infinite horizon collateralized economies. Hence, it is the effectiveness of the mechanism that induces agents to run a Ponzi scheme, not the mechanism per se.

[3]See also Hernández and Santos (1996) for a similar discussion.

[4]Nowadays, there is a vast literature on default that dates back to the seminal contributions of Shubik (1972), Shubik and Wilson (1977) and Dubey and Shubik (1979). Default was introduced in a general equilibrium setting by Dubey, Geanakoplos, and Shubik (1990) and Zame (1993). Modern theoretical contributions on default include among others, Dubey, Geanakoplos, and Zame (1995), Geanakoplos (1997), Geanakoplos and Zame (2002), Araujo, Páscoa, and Torres-Martínez (2002), Kubler and Schmedders (2003), Dubey, Geanakoplos, and Shubik (2005), Fostel and Geanakoplos (2008), Páscoa and Seghir (2009), Ferreira and Torres-Martínez (2010). There are also important contributions on default, collateral and credit constraints in macroeconomics (see Bernanke, Gertler, and Gilchrist (1996), Kiyotaki and Moore (1997) and Caballero and Krishnamurthy (2001)). This literature emphasizes the feedback from the fall in collateral prices to a fall in borrowing capacity. Recently, Chatterjee, Corbae, Nakajima, and Ríos-Rull (2007) and Livshits, MacGee, and Tertilt (2007) have calibrated macroeconomic models with incomplete markets and default and used them to address various policy issues.

[5]Collateral-using activities have expanded rapidly in recent years. Financial institutions extensively employ collateral in lending, in securities trading and derivative markets and in payment and settlement systems. Central banks generally require collateral in their credit operations. Common examples of collateralized lending are home mortgages, margin purchases of securities, overnight repurchase agreements and pawn shop loans.

[6]Since there is no other punishment than the seizure of collateral, borrowers will always deliver the minimum between their promises and the value of the associated collateral requirements.

[7]For instance, if an agent files for bankruptcy under Chapter 7 of the U.S. bankruptcy code, the following things may happen (see Chatterjee, Corbae, Nakajima, and Ríos-Rull (2007)): (1) he is not allowed to save and his existing savings will be completely garnished; (2) he has to pay a proportion of the current income as cost of filling for bankruptcy; (3) a proportion of his current labor income is garnished; (4) his credit history turns bad and he is excluded from the loan market.

[8]An enforcement mechanism is said effective if it entails payments besides the value of the collateral at all nodes of a subtree.

Given the findings of Páscoa and Seghir (2009) and Ferreira and Torres-Martínez (2010) we propose to answer the following question: what kind of borrowing constraints rule out Ponzi schemes and ensure existence of equilibria in models with limited commitment and (possible) default at equilibrium? As a first step to provide an answer to this question it is natural to investigate whether debt constraints that have been proposed in models with full commitment can be compatible with equilibrium existence in models with limited commitment. The paper is an attempt to address this issue. It shows that finitely effective debt constraints, similar to those proposed by Levine and Zame (1996) in environments with full commitment, ensure equilibrium existence in the models of Araujo, Páscoa, and Torres-Martínez (2002), Kubler and Schmedders (2003) and Páscoa and Seghir (2009) where commitment is limited.

A direct adaptation of finitely effective debt constraints à la Levine and Zame (1996) in those environments does not help to control debt along time. The reason is that when commitment is limited, an agent can always satisfy his budget restrictions having access to financial markets for a finite number of periods. He can do this by simply defaulting on his promises. Therefore, requiring finite-time solvency à la Levine and Zame (1996) does not restrict budget sets. In particular, it does not exclude Ponzi schemes. We address this issue by modifying appropriately the definition of finitely effective debt constraints to encompass economies with limited commitment and (possible) default at equilibrium. Working in this direction, we impose debt constraints by introducing in the setting of Araujo, Páscoa, and Torres-Martínez (2002), Kubler and Schmedders (2003) and Páscoa and Seghir (2009) the concept of actions with *finite equivalent payoffs*.

An interesting finding is that there is a close relation between our proposed budget sets and the budget sets of Levine and Zame (1996) as well as the budget sets defined through collateral obligations and no additional punishments (Araujo, Páscoa, and Torres-Martínez (2002) and Kubler and Schmedders (2003)). First, our proposed debt constraints provide a natural formulation of Levine and Zame (1996) solvency requirement in those models. When there is full commitment (and payments are fully enforced) our concept of plans with finitely equivalent payoffs coincides with the concept of plans with finitely effective debt introduced by Levine and Zame (1996). Second and most important, we show that the budget feasible plans in economies with a collateral structure and zero default penalties have finite equivalent payoffs and vice versa. In other words, when there are collateral requirements but no default penalties, our budget set coincides with the standard one defined in Araujo, Páscoa, and Torres-Martínez (2002) and Kubler and Schmedders (2003). This equivalence is valid for any price process (i.e., not only at equilibrium but also on out of equilibrium paths) and illustrates the hidden relation between finitely effective debt constraints and collateral requirements.

Our approach to debt constraints is certainly not the only one possible. Instead of adapting the restrictions proposed by Levine and Zame (1996), one may follow another route by considering restrictions in the spirit of Magill and Quinzii (1994) or Hernández and Santos (1996). However, it is not clear whether those borrowing constraints would be innocuous in models with collateral requirements and zero default penalties as it is the case for the constraints we propose. In that respect, we believe that modifying the approach of Levine and Zame (1996) to control debt is more suitable for models with limited commitment and collateral requirements.

Proposing any kind of debt constraints raises an equally important issue: how difficult is to implement those constraints in anonymous and competitive markets. In the context of full commitment, Magill and Quinzii (1994) give two possible interpretations of their implicit debt constraints: a subjective (self-monitoring) interpretation where agents restrict themselves to satisfy these constraints and an objective (market based) one where an external agent (an agency) has the ability to restrict agents to choose plans satisfying the borrowing constraints. In our context of limited commitment, restricting plans to have finite equivalent payoffs can be given a similar interpretation. This is due to the fact that, under mild conditions on primitives, equilbria with finite equivalent payoffs are equilibria with implicit (or explicit and non-binding) constraints on short-selling.[9] In particular, one can show that there exists a threshold bound related only to primitives (aggregate resources) of the economy such that any posted bound greater than this threshold will be non-binding at equilibrium.

The paper is structured as follows. In Section 2 we set out the model, introduce notation, assumptions and the equilibrium concept in the absence of borrowing constraints. In Section 3 we present and discuss the new debt constraints we impose on budget feasible plans. We also introduce an equilibrium concept associated with those constraints and highlight its relation with the equilibrium concepts introduced by Levine and Zame (1996) and Araujo, Páscoa, and Torres-Martínez (2002). Section 4 proves the existence of what we term *equilibrium with finite equivalent payoffs* under a mild condition on default penalties. In Section 5 we discuss equilibrium refinement and highlight a problem that has been overlooked by the literature. Section 6 concludes.

2. The Model

The model is essentially the one developed in Araujo, Páscoa, and Torres-Martínez (2002) and extended by Páscoa and Seghir (2009) to allow for the possibility of linear default penalties.

2.1. Uncertainty and time

Let $\mathcal{T} \equiv \{0, 1, \ldots, t, \ldots\}$ denote the set of time periods and let S be a (infinite) set of states of nature. The available information at period $t \in \mathcal{T}$ is the same for each agent and is described by a finite partition P_t of S. Information is revealed along time, i.e., the partition P_{t+1} is finer than P_t for every t. Every pair (t, σ) where σ is a set in P_t is called a node. The set

[9] Our bounds (implicit or explicit) are different than those imposed by Magill and Quinzii (1994). Our bounds restrict short sales while theirs restrict the real value of debt.

of all nodes is denoted by D and is called the event tree. We assume that there is no information at $t = 0$ and we denote by $\xi_0 = (0, S)$ the initial node. If $\xi = (t, \sigma)$ belongs to the event tree, then t is denoted by $t(\xi)$. We say that $\xi' = (t', \sigma')$ is a successor of $\xi = (t, \sigma)$ if $t' \geqslant t$ and $\sigma' \subset \sigma$; we use the notation $\xi' \geqslant \xi$. We denote by ξ^+ the set of immediate successors defined by

$$\xi^+ \equiv \{\xi' \in D : \ t(\xi') = t(\xi) + 1\}.$$

Because P_t is finer than P_{t-1} for every $t > 0$, for a given node $\xi \neq \xi_0$, there is a unique node ξ^- in D such that ξ is an immediate successor of ξ^-. Given a period $t \in \mathcal{T}$ we let $D_t \equiv \{\xi \in D : \ t(\xi) = t\}$ denote the set of nodes at period t. The set of nodes up to period t is denoted $D^t \equiv \{\xi \in D : \ t(\xi) \leqslant t\}$.

2.2. Agents and commodities

There exists a finite set L of commodities available for trade at every node $\xi \in D$. We interpret $x(\xi) \in \mathbb{R}^L_+$ as a claim to consumption at node ξ. We also write $\mathbf{1}_{\{\ell\}} \in \mathbb{R}^L_+$ for the commodity bundle consisting of one unit of commodity $\ell \in L$ and nothing else. We depart from the usual intertemporal models by allowing for some commodities to be *non-perishable*, that is, we allow for storable and durable goods as well as for commodities that may serve as physical assets (i.e., Lucas trees). Transformation of commodities is represented by a family $(Y(\xi))_{\xi \in D}$ of linear functionals $Y(\xi)$ from $\mathbb{R}^L_+$ to $\mathbb{R}^L_+$. The bundle $Y(\xi)z(\xi^-)$ represents what is obtained at node ξ if the bundle $z(\xi^-) \in \mathbb{R}^L_+$ is purchased at node ξ^-. We say that the commodity ℓ is perishable at node ξ^- if $Y(\xi)\mathbf{1}_{\{\ell\}}$ is the zero vector in $\mathbb{R}^L_+$, and non-perishable otherwise. At each node there are spot markets for trading every commodity. We let $p = (p(\xi))_{\xi \in D}$ be the spot price process where $p(\xi) = (p(\xi, \ell))_{\ell \in L} \in \mathbb{R}^L_+$ is the price vector at node ξ.

There is a finite set I of infinitely lived agents. Each agent $i \in I$ is characterized by an endowment process $\omega^i = (\omega^i(\xi))_{\xi \in D}$ where $\omega^i(\xi) = (\omega^i(\xi, \ell))_{\ell \in L}$ is a vector in $\mathbb{R}^L_+$ representing the endowment available at node ξ. Each agent chooses a consumption process $x = (x(\xi))_{\xi \in D}$ where $x(\xi) \in \mathbb{R}^L_+$. We denote by X the set of consumption processes. The utility function $U^i : X \longrightarrow [0, +\infty]$ is assumed to be additively separable, i.e.,

$$U^i(x) \equiv \sum_{\xi \in D} u^i(\xi, x(\xi))$$

where $u^i(\xi, \cdot) : \mathbb{R}^L_+ \longrightarrow [0, \infty)$.

2.3. Assets and collateral

There is a finite set J of short-lived real financial assets available for trade at each node. For each asset j, the bundle yielded at node ξ is denoted by $A(\xi, j) \in \mathbb{R}^L_+$. We let $q = (q(\xi))_{\xi \in D}$ be the asset price process where $q(\xi) = (q(\xi, j))_{j \in J} \in \mathbb{R}^J_+$ represents the asset price vector at node ξ. We denote by $\theta^i(\xi) \in \mathbb{R}^J_+$ the vector of purchases and by $\varphi^i(\xi) \in \mathbb{R}^J_+$ the vector of short-sales at each node ξ.

Following the seminal contribution of Geanakoplos (1997) and Geanakoplos and Zame (2002) for finite horizon models, and Araujo, Páscoa, and Torres-Martínez (2002) together with Páscoa and Seghir (2009) for infinite horizon models, assets are collateralized in the sense that for every unit of asset j sold at a node ξ, agents should buy a collateral bundle $C(\xi, j) \in \mathbb{R}^L_+$ that protects lenders in case of default. We assume that payments can be enforced through the seizure of the collateral. At a node ξ, agent i should deliver the promise $V(p, \xi)\varphi^i(\xi^-)$ where

$$V(p, \xi) = (V(p, \xi, j))_{j \in J} \quad \text{and} \quad V(p, \xi, j) \equiv p(\xi)A(\xi, j).$$

However, agent i may decide to default and choose a delivery $d^i(\xi, j)$ in units of account. Since the collateral can be seized, this delivery must satisfy

$$d^i(\xi, j) \geqslant D(p, \xi, j)\varphi^i(\xi^-, j)$$

where

$$D(p, \xi, j) \equiv \min\{p(\xi)A(\xi, j), p(\xi)Y(\xi)C(\xi^-, j)\}.$$

Remark 2.1. Kubler and Schmedders (2003) propose a model where the collateral requirements are imposed in terms of physical assets. We show hereafter that a simplified version of their model can be seen as a particular case of the model proposed by Araujo, Páscoa, and Torres-Martínez (2002). In that respect whenever we are referring to the model proposed by Araujo, Páscoa, and Torres-Martínez (2002) we are also referring to the one proposed by Kubler and Schmedders (2003).

If there is a specific commodity $g \in L$ satisfying the following properties, then this commodity can be interpreted as a physical asset or a Lucas tree.

(i) At initial node ξ_0, each agent i has an initial endowment $\omega^i(\xi_0, g) \geqslant 0$ of commodity g which represents his share of the tree. At subsequent nodes $\xi > \xi_0$, agent i has no initial endowment in commodity g.

(ii) One unit of commodity g purchased at node ξ delivers at node $\mu \in \xi^+$ the bundle

$$y(\mu) \equiv Y(\mu)\mathbf{1}_{\{g\}} \in \mathbb{R}^L_+.$$

The g-th coordinate $y(\mu, g)$ is equal to 1, i.e., the physical asset is long lived.

(iii) Each agent i is indifferent with respect to commodity g, i.e., for each agent $i \in I$, for each node $\xi \in D$, for each consumption bundle $c \in \mathbb{R}^L_+$, we have

$$u^i(\xi, c + \mathbf{1}_{\{g\}}) = u^i(\xi, c).$$

(iv) In every successor node $\mu \in \xi^+$, the transformed bundle of one unit of commodity g purchased at any node ξ, is a desirable bundle, i.e., $y(\mu)$ is a bundle in $\mathbb{R}^L_+$ such that for each consumption bundle $c \in \mathbb{R}^L_+$, we have[10]

$$u^i(\mu, c + y(\mu)) > u^i(\mu, c).$$

[10]Since each agent i is indifferent with respect to commodity g, the bundle delivered by the tree must satisfy $y(\mu, \ell) > 0$ for at least one commodity $\ell \neq g$.

If at every node $\xi \in D$, the collateral bundle $C(\xi, j)$ is only in terms of commodity g, then the collateral structure of our model (and the one in Araujo, Páscoa, and Torres-Martínez (2002) and Páscoa and Seghir (2009)) reduces to the one considered by Kubler and Schmedders (2003).

Following Dubey, Geanakoplos, and Shubik (1990) (and Dubey, Geanakoplos, and Shubik (2005)), we assume that agent i feels a disutility $\lambda^i(\xi, j) \in [0, +\infty]$ from defaulting.[11] More precisely, if an agent defaults at node ξ, then he suffers at $t = 0$, the disutility

$$\sum_{j\in J} \lambda^i(\xi, j) \frac{\left[V(p, \xi, j)\varphi^i(\xi^-, j) - d^i(\xi, j)\right]^+}{p(\xi)v(\xi)}$$

where $(v(\xi))_{\xi\in D}$ is an exogenously specified process in $\mathbb{R}^L_{++}$ that is uniformly bounded away from 0.[12] In that case, agent i may have an incentive to deliver more than the minimum between his debt and the depreciated value of his collateral, i.e., we may have $d^i(\xi, j) > D(p, \xi, j)\varphi^i(\xi^-, j)$.

As in Dubey, Geanakoplos, and Shubik (2005) assets are thought as pools. At each node ξ the sales $\varphi^i(\xi, j)$ are pooled at the market for asset j. The deliveries $d^i(\xi, j)$ on asset j are also pooled and the buyers of pool j receive a pro rata share of all its different sellers' deliveries. We assume that lenders rationally anticipate that every borrower delivers at least $D(p, \xi, j)$ on each unit of asset j sold at node ξ^-. Therefore, agents anticipate that each share of pool j delivers a fraction $V(\kappa, p, \xi, j)$ of its promise $V(p, \xi, j)$ defined by

$$V(\kappa, p, \xi, j) = \kappa(\xi, j)V(p, \xi, j) + (1 - \kappa(\xi, j))D(p, \xi, j)$$

where $\kappa(\xi, j) \in [0, 1]$ will be determined at equilibrium such that deliveries match payments.[13] The buyer of asset j does not need to know the identities of the sellers or the quantities of their sales. All that matters to him is the price $q(\xi, j)$ and the anticipated delivery rates $(\kappa(\mu, j))_{\mu\in\xi^+}$.

2.4. Budget set without debt constraints

We let A be the space of adapted processes $a = (a(\xi))_{\xi\in D}$ with[14]

$$a(\xi) = (x(\xi), \theta(\xi), \varphi(\xi), d(\xi)) \in \mathbb{R}^L_+ \times \mathbb{R}^J_+ \times \mathbb{R}^J_+ \times \mathbb{R}^J_+.$$

Given a process (p, q, κ) of commodity prices, asset prices and delivery rates, agent i's choice $a^i = (x^i, \theta^i, \varphi^i, d^i) \in A$ must satisfy, in each decision node $\xi \in D$, the following constraints:

(a) solvency constraint:

$$p(\xi)x^i(\xi) + \sum_{j\in J} d^i(\xi, j) + q(\xi)\theta^i(\xi) \leqslant p(\xi)\omega^i(\xi)$$
$$+p(\xi)Y(\xi)x^i(\xi^-) + V(\kappa, p, \xi)\theta^i(\xi^-) + q(\xi)\varphi^i(\xi); \quad (2.1)$$

(b) collateral requirement:

$$C(\xi)\varphi^i(\xi) \leqslant x^i(\xi); \quad (2.2)$$

(c) minimum delivery:

$$\forall j \in J, \quad D(p, \xi, j)\varphi^i(\xi^-, j) \leqslant d^i(\xi, j). \quad (2.3)$$

The set of plans $a = (x, \theta, \varphi, d) \in A$ satisfying constraints (2.1), (2.2) and (2.3) is called the (unconstrained) budget set and is denoted by $B^i(p, q, \kappa)$.

2.5. The payoff function

Consider that agent i has chosen the plan $a = (x, \theta, \varphi, d)$ under a process of prices and delivery rates $\pi = (p, q, \kappa)$. He enjoys the utility

$$U^i(x) = \sum_{\xi\in D} u^i(\xi, x(\xi)) \in [0, \infty]$$

but he suffers the disutility $W^i(p, a) \in [0, \infty]$ defined by

$$W^i(p, a) \equiv \sum_{\xi>\xi_0} \sum_{j\in J} \lambda^i(\xi, j) \frac{[V(p, \xi, j)\varphi(\xi^-, j) - d(\xi, j)]^+}{p(\xi)v(\xi)}.$$

We would like to define the payoff $\Pi^i(p, a)$ of the plan a as the following difference

$$\Pi^i(p, a) = U^i(x) - W^i(p, a).$$

Unfortunately, $\Pi^i(p, a)$ may not be well defined if both $U^i(x)$ and $W^i(p, a)$ are infinite. We propose to consider the binary relation $\succ_{i,p}$ defined on A by $\widetilde{a} \succ_{i,p} a$ when

$$\exists \varepsilon > 0, \quad \exists T \in \mathbb{N}, \quad \forall t \geqslant T, \quad \Pi^{i,t}(p, \widetilde{a}) \geqslant \Pi^{i,t}(p, a) + \varepsilon$$

where

$$\Pi^{i,t}(p, a) \equiv U^{i,t}(x) - W^{i,t}(p, a), \quad U^{i,t}(x) \equiv \sum_{\xi\in D^t} u^i(\xi, x(\xi))$$

and

$$W^{i,t}(p, a) \equiv \sum_{\xi\in D^t\setminus\{\xi_0\}} \sum_{j\in J} \lambda^i(\xi, j) \frac{[V(p, \xi, j)\varphi(\xi^-, j) - d(\xi, j)]^+}{p(\xi)v(\xi)}.$$

According to this definition, a plan $\widetilde{a}$ is strictly preferred to a if the difference of payoffs $\Pi^{i,t}(p, \widetilde{a}) - \Pi^{i,t}(p, a)$ between the two plans is uniformly strictly positive for every period t large enough.[15]

Observe that if $\Pi^i(p, \widetilde{a})$ and $\Pi^i(p, a)$ are finite then $\widetilde{a} \succ_{i,p} a$ if and only $\Pi^i(p, \widetilde{a}) > \Pi^i(p, a)$. We denote by $\mathrm{Pref}^i(p, a)$ the set of plans strictly preferred to plan a by agent i.

[11] Models with non-pecuniary penalties for default also include Diamond (1984), Rea (1984), who considers contracts involving "arm-breaking", Zame (1993), Araujo, Monteiro, and Páscoa (1998), Bisin and Gottardi (1999), Santos and Scheinkman (2001), Lacker (2001) and Páscoa and Seghir (2009).

[12] More precisely, we assume that there exists $\underline{v} > 0$ such that for every node $\xi \in D$ and every commodity $\ell \in L$, we have $v(\xi, \ell) \geqslant \underline{v}$.

[13] If all the sellers of asset j at node ξ^- fully deliver on their promises at the successor node ξ then $\kappa(\xi, j) = 1$, while if all sellers fully default on their promises then $\kappa(\xi, j) = 0$.

[14] By convention we pose $a(\xi_0^-) = (x(\xi_0^-), \theta(\xi_0^-), \varphi(\xi_0^-), d(\xi_0^-)) = (0, 0, 0, 0)$.

[15] The sequence of differences $(\Pi^{i,t}(p, \widetilde{a}) - \Pi^{i,t}(p, a))_{t\geqslant 1}$ need not be converging.

2.6. Assumptions

For each agent i, we denote by Ω^i the process of accumulated endowments, defined recursively by $\Omega^i(\xi) = Y(\xi)\Omega^i(\xi^-) + \omega^i(\xi)$ where $\Omega^i(\xi_0) = \omega^i(\xi_0)$. The process $\sum_{i\in I}\Omega^i$ of accumulated aggregate endowments is denoted by Ω. The following assumptions on the characteristics of the economy are standard in the literature of infinite horizon models with collateral requirements.

Assumption 2.1 (Agents)**.** For every agent i,

(H.1) the process of accumulated endowments is strictly positive and uniformly bounded from above, i.e.,

$$\exists \overline{\Omega}^i \in \mathbb{R}^L_{++}, \quad \forall \xi \in D, \quad \Omega^i(\xi) \in \mathbb{R}^L_{++} \quad \text{and} \quad \Omega^i(\xi) \leqslant \overline{\Omega}^i;$$

(H.2) for every node ξ, the utility function $u^i(\xi,\cdot)$ is concave, continuous and strictly increasing,[16] with $u^i(\xi, 0) = 0$;

(H.3) the infinite sum $U^i(\Omega)$ is finite.

Assumption 2.2 (Financial assets)**.** For every asset j and node ξ, the collateral $C(\xi, j)$ is not zero.

It should be clear that these assumptions always hold throughout the paper.

2.7. Equilibrium without debt constraints

We denote by Ξ the set of prices and delivery rates (p, q, κ) normalized as follows: for every node ξ, we have $p(\xi) \in \mathbb{R}_{++}$, $\kappa(\xi) \in [0, 1]^J$ and $(p(\xi), q(\xi))$ belongs to the simplex $\Delta(L\times J)$.[17] Given a process (p, q, κ) of commodity prices, asset prices and delivery rates, we denote by $d^i(p, q, \kappa)$ the demand set defined by

$$d^i(p, q, \kappa) \equiv \{a \in B^i(p, q, \kappa) \ : \ \mathrm{Pref}^i(p, a) \cap B^i(p, q, \kappa) = \emptyset\}.$$

Definition 2.1. A *competitive equilibrium* for the economy $\mathcal{E}$ is a family of prices and delivery rates $(p, q, \kappa) \in \Xi$ and an allocation $\boldsymbol{a} = (a^i)_{i\in I}$ with $a^i \in A$ such that

(a) for every agent i, the plan a^i is optimal, i.e.,

$$a^i \in d^i(p, q, \kappa);$$

(b) commodity markets clear at every node, i.e.,

$$\sum_{i\in I} x^i(\xi_0) = \sum_{i\in I} \omega^i(\xi_0) \tag{2.4}$$

and for all $\xi \neq \xi_0$,

$$\sum_{i\in I} x^i(\xi) = \sum_{i\in I} \left[\omega^i(\xi) + Y(\xi)x^i(\xi^-)\right]; \tag{2.5}$$

(c) asset markets clear at every node, i.e., for all $\xi \in D$,

$$\sum_{i\in I} \theta^i(\xi) = \sum_{i\in I} \varphi^i(\xi); \tag{2.6}$$

(d) deliveries match at every node, i.e., for all $\xi \neq \xi_0$ and all $j \in J$,

$$\sum_{i\in I} V(\kappa, p, \xi, j)\theta^i(\xi^-, j) = \sum_{i\in I} d^i(\xi, j). \tag{2.7}$$

The set of allocations $\boldsymbol{a} = (a^i)_{i\in I}$ in A satisfying the market clearing conditions (2.4), (2.5) and (2.6) is denoted by F. Each allocation in F is called physically feasible. A plan $a^i \in A$ is called physically feasible if there exists a physically feasible allocation $\boldsymbol{b}$ such that $a^i = b^i$. The set of physically feasible plans is denoted by F^i. We denote by $\mathrm{Eq}(\mathcal{E})$ the set of competitive equilibria for the economy $\mathcal{E}$.

3. Debt constraints

In this section, we show how to adapt the *finitely effective* debt constraints proposed by Levine and Zame (1996) to infinite horizon models with limited commitment and default penalties. While keeping the minimal ability to borrow and lend that we expect in our model, we prove that the proposed constraints are compatible with equilibrium (precluding agents to run Ponzi schemes). Moreover, our constraints appear to have an additional appealing feature: we show that the budget sets associated with those constraints coincide with the standard budget sets of economies having a collateral structure but no penalties (as defined in Araujo, Páscoa, and Torres-Martínez (2002) and Kubler and Schmedders (2003)).

3.1. Infinite default penalties

When default penalties are infinite and the collateral requirements are zero, our model reduces to the one studied by Magill and Quinzii (1994) and Levine and Zame (1996). In the absence of debt constraints, an equilibrium may not exist: all traders would attempt to finance unbounded levels of consumption by unbounded levels of borrowing. To rule out Ponzi schemes, Levine and Zame (1996) (see also Levine and Zame (2002)) formalize the concept of plans with *finitely effective* debt by requiring agents' actions to be budget compatible with the threat that, at any period, agents may be restricted to have access to borrowing for only a finite number of periods. In other words, an agent's debt is finitely effective if at any period, the debt is repayable within a finite horizon. More formally, we consider the following definition due to Levine and Zame (1996).

Definition 3.1. A plan $a \in B^i(p, q, \kappa)$ is said to have *finitely effective debt*, if for each period $t \geqslant 0$, there exists a period $T > t$ and a plan $\widehat{a}$ also in the budget set $B^i(p, q, \kappa)$ such that

(i) up to period t both plans coincide, i.e.,

$$\forall \xi \in D^t, \quad \widehat{a}(\xi) = a(\xi);$$

[16] Assuming that the function $u^i(\xi,\cdot)$ is strictly increasing is not compatible with the interpretation of a commodity as a Lucas tree. This assumption was made only for expositional purposes and can be weakened as follows: for every ξ the function $u^i(\xi,\cdot)$ is non-decreasing and there exists a commodity ℓ that is strictly desirable in the sense that for every pair x, y in $\mathbb{R}^L_+$, we have $u^i(\xi, x+y) > u^i(\xi, x)$ provided that $y(\ell) > 0$.

[17] In the sense that $p(\xi) \in \mathbb{R}^L_+$, $q(\xi) \in \mathbb{R}^J_+$ and $\sum_{\ell\in L} p(\xi, \ell) + \sum_{j\in J} q(\xi, j) = 1$.

(ii) at every node after period T, there is solvency without borrowing, i.e.,

$$\forall \xi \in D, \quad t(\xi) \geqslant T \Longrightarrow \widehat{\varphi}(\xi) = 0.$$

The intuition behind Definition 3.1 can be better understood if we think about the role of those restrictions in the finite horizon framework. No short selling at the terminal date implicitly imposes a solvency requirement at earlier dates. That is, at any node agents should hold an amount of debt that they will be able to repay by the end of the terminal date. In the absence of a terminal date, it is necessary to impose explicitly or implicitly that solvency requirement.

Remark 3.1. Consider the following notation. For each period t, we denote by A^t the set of plans $a \in A$ where $a(\xi) = (0,0,0,0)$ for each ξ such that $t(\xi) > t$. If a is a plan in A and t is a period, we denote by $a\mathbf{1}_{[0,t]}$ the plan in A^t which coincides with a for every node $\xi \in D^t$.[18] Following this notation, a plan a has a finitely effective debt if for each period $t \geqslant 0$, there exists a subsequent period $T > t$ and a plan $\widehat{a}$ such that

$$\widehat{a} \in B^i(p,q,\kappa) \cap C^T \quad \text{and} \quad a\mathbf{1}_{[0,t]} = \widehat{a}\mathbf{1}_{[0,t]}$$

where C^T is the set of plans a in A without borrowing after period T in the sense that

$$\forall \xi \in D, \quad t(\xi) \geqslant T \Longrightarrow \widehat{\varphi}(\xi) = 0.$$

Instead of restricting plans to be finitely effective, one may consider the following alternative restriction.

Definition 3.2. A budget feasible plan $a \in B^i(p,q,\kappa)$ is said to have *finite equivalent utility* when for every period $t \geqslant 0$ and every $\varepsilon > 0$ there exists a subsequent period $T > t$ and a plan $\widehat{a}$ such that

(i) the plans a and $\widehat{a}$ coincide up to period t, i.e., $a\mathbf{1}_{[0,t]} = \widehat{a}\mathbf{1}_{[0,t]}$;

(ii) the plan $\widehat{a}$ is budget feasible and there is no borrowing after period T, i.e., $\widehat{a} \in B^i(p,q,\kappa) \cap C^T$;

(iii) the utility of the plan $\widehat{a}$ may be lower than the payoff of a but not more than ε, i.e.,

$$\inf_{\tau \geqslant T} \left[U^{i,\tau}(p,\widehat{a}) - U^{i,\tau}(p,a) \right] \geqslant -\varepsilon.$$

In other words, a budget feasible plan a has finite equivalent utility if in case where at some period t the agent is restricted to have access to borrowing for finitely many periods, then he can find an alternative plan $\widehat{a}$ doing the job, i.e., satisfying (i) and (ii); but at the same time the utility loss can be made as small as desired.

The following proposition shows the equivalence between plans with finitely effective debt and plans having finite equivalent utility. This alternative characterization will be proven particularly useful in the process of modifying finitely effective constraints to encompass models with limited commitment.

[18] The plan $a\mathbf{1}_{[0,t]}$ can be interpreted as a "truncation" of a up to period t.

Proposition 3.1. Assume that the default penalty is infinite and consider a budget feasible plan $a \in B^i(p,q,\kappa)$ with a finite utility $U^i(x) < \infty$. The plan a has finitely effective debt, if and only if, it has finite equivalent utility.

Proof of Proposition 3.1. Let $a \in B^i(p,q,\kappa)$ be a budget feasible plan with a finite utility $U^i(x) < \infty$. It is obvious that if a has finite equivalent utility, then it has a finitely effective debt. The converse deserves more attention. Assume that the plan a has a finitely effective debt. Fix a period $t \geqslant 0$ and $\varepsilon > 0$. If we apply the definition to the period t, we get the existence of a period $T > t$ and a plan $\widehat{a}$ such that

$$\widehat{a} \in B^i(p,q,\kappa) \cap C^T \quad \text{and} \quad a\mathbf{1}_{[0,t]} = \widehat{a}\mathbf{1}_{[0,t]}.$$

Unfortunately, we do not know if $U^{i,T}(\widehat{x}) \geqslant U^{i,T}(x) - \varepsilon$. However, we know that the utility $U^i(x)$ is finite. Therefore, there exists $t' > t$ such that

$$\sum_{s>t'} \sum_{\xi \in D_s} u^i(\xi, x(\xi)) \leqslant \varepsilon. \tag{3.1}$$

Now, applying the definition of *finitely effective debt* for the period t', there exists a period $T > t'$ and a plan $\widehat{a}$ such that

$$\widehat{a} \in B^i(p,q,\kappa) \cap C^T \quad \text{and} \quad a\mathbf{1}_{[0,t']} = \widehat{a}\mathbf{1}_{[0,t']}.$$

Now fix $\tau \geqslant T$. Since $T > t'$, we have

$$\begin{aligned} U^{i,\tau}(\widehat{x}) \geqslant U^{i,t'}(\widehat{x}) &= U^{i,t'}(x) \\ &\geqslant U^{i,\tau}(x) - \sum_{t'<s\leqslant\tau} \sum_{\xi \in D_s} u^i(\xi, x(\xi)). \end{aligned}$$

It follows from (3.1) that $U^{i,\tau}(\widehat{x}) \geqslant U^{i,\tau}(x) - \varepsilon$. □

3.2. Finite default penalties

The concept of finitely effective debt constraints makes perfect sense in models with full enforcement and perfect commitment (i.e., no default). However, with limited commitment, imposing finitely effective debt constraints does not help to control debt along time. We provide an explanation below. Let $a = (x, \theta, \varphi, d)$ be a plan in $B^i(p,q,\kappa)$ and t be any period. Consider the plan $\widehat{a}$ defined by

$$\widehat{a}(\xi) = \begin{cases} a(\xi) & \text{if} \quad t(\xi) \leqslant t \\ (\omega^i(\xi), 0, 0, D(p,\xi)\varphi(\xi^-)) & \text{if} \quad t(\xi) = t+1 \\ (\omega^i(\xi), 0, 0, 0) & \text{if} \quad t(\xi) > t+1. \end{cases}$$

This plan belongs to the set $B^i(p,q,\kappa) \cap C^{t+1}$ and coincides with a on every node up to period t. That is, under limited commitment, any plan $a \in B^i(p,q,\kappa)$ has finitely effective debt according to Definition 3.1. Agents can always default up to the minimum value between their debt and the depreciated value of their collateral. Therefore, there is no hope to bound debt along time.

We introduce hereafter an endogenous restriction on trades that allows to encompass models with limited commitment and finite default penalties. The point of our departure is Proposition 3.1 where it is shown that, when default penalties are infinite, restricting plans to have finitely effective debt is equivalent to restricting plans to have finite equivalent utility. This equivalence breaks down in the presence of finite default penalties. In this case, we proceed by replacing "utility" by "payoff" and we introduce the concept of plans with *finite equivalent payoffs*. We claim that requiring plans to have finite equivalent payoffs provides an appropriate adaptation of finitely effective debt constraints to models with limited commitment and finite default penalties. The formal definition is as follows.

Definition 3.3. A plan a in the budget set $B^i(p,q,\kappa)$ has *finite equivalent payoffs* if for every period $t \geqslant 0$ and every $\varepsilon > 0$ there exists a subsequent period $T > t$ and a plan $\widehat{a}$ such that

(i) the plans a and $\widehat{a}$ coincide up to period t, i.e., $a\mathbf{1}_{[0,t]} = \widehat{a}\mathbf{1}_{[0,t]}$;

(ii) the plan $\widehat{a}$ is budget feasible and there is no borrowing after period T, i.e. $\widehat{a} \in B^i(p,q,\kappa) \cap C^T$;

(iii') the payoff of the plan $\widehat{a}$ may be lower than the payoff of the initial plan a but not more than ε, i.e.,

$$\inf_{\tau \geqslant T} \left[\Pi^{i,\tau}(p,\widehat{a}) - \Pi^{i,\tau}(p,a)\right] \geqslant -\varepsilon.$$

The interpretation of a plan with finite equivalent payoff is similar to the one of a plan with finite equivalent utility. The only difference is that we replace "utility" by "payoff". This is very intuitive since agents may suffer a loss in utility when defaulting.

3.3. Equilibrium with finite equivalent payoffs

We denote by $B^i_\star(p,q,\kappa)$ the set of all plans in $B^i(p,q,\kappa)$ having finite equivalent payoffs and we let $d^i_\star(p,q,\kappa)$ be the associated demand set.[19]

Definition 3.4. A *competitive equilibrium with finite equivalent payoffs* for the economy $\mathcal{E}$ is a family of prices and delivery rates $(p,q,\kappa) \in \Xi$ together with an allocation $\boldsymbol{a} = (a^i)_{i\in I}$ with $a^i \in A$ such that the conditions of market clearing (b), (c) and (d) in Definition 2.1 are satisfied and the unconstrained optimality condition (a) is replaced by

(a') for every agent i, the plan a^i has finite equivalent payoffs and is optimal among all budget feasible plans with finite equivalent payoffs, i.e., $a^i \in d^i_\star(p,q,\kappa)$.

We denote by $\mathrm{Eq}_\star(\mathcal{E})$ the set of competitive equilibria with finite equivalent payoffs for the economy $\mathcal{E}$. We prove in Section 4.3 that the set $\mathrm{Eq}_\star(\mathcal{E})$ is non-empty under a mild condition on default penalties. Before addressing the existence issue, we explore hereafter the relation between the equilibrium concept that we have just introduced with the one found in Araujo, Páscoa, and Torres-Martínez (2002).

3.4. No default penalty

We consider the case where collateral repossession is the only enforcement mechanism and that default penalties are equal to zero as in Araujo, Páscoa, and Torres-Martínez (2002) and Kubler and Schmedders (2003). One may expect $B^i_\star(p,q,\kappa)$ to be a strict subset of $B^i(p,q,\kappa)$. However, as the following proposition shows, the two sets coincide. In fact, in the model proposed by Araujo, Páscoa, and Torres-Martínez (2002), any budget feasible allocation with a finite utility has finite equivalent payoffs. This is a consequence of the absence of default penalties or explicit economic punishments.

Proposition 3.2. Assume that there is no default penalty and let $a = (x,\theta,\varphi,d)$ be a plan in the budget set $B^i(p,q,\kappa)$. If $U^i(x)$ is finite then a has finite equivalent payoffs, i.e., a belongs to $B^i_\star(p,q,\kappa)$.

Proof of Proposition 3.2. Fix an agent i and consider a budget feasible plan $a \in B^i(p,q,\kappa)$ with a finite utility. Fix a period $t \geqslant 0$ and $\varepsilon > 0$. Since $U^i(x)$ is finite, there exists $T \geqslant t+1$ such that

$$\sum_{\tau \geqslant T} \sum_{\xi \in D_\tau} u^i(\xi, x(\xi)) \leqslant \varepsilon.$$

Consider now the plan $\widehat{a}$ defined by

$$\widehat{a}(\xi) = \begin{cases} a(\xi) & \text{if} \quad t(\xi) < T \\ (\omega^i(\xi), 0, 0, \widehat{d}(\xi)) & \text{if} \quad t(\xi) = T \\ (\omega^i(\xi), 0, 0, 0) & \text{if} \quad t(\xi) > T \end{cases}$$

where

$$\forall \xi \in D_T, \quad \forall j \in J, \quad \widehat{d}(\xi, j) = D(p,\xi,j)\varphi(\xi^-, j).$$

Observe that the plan $\widehat{a}$ is budget feasible, belongs to C^T and satisfies

$$\widehat{a}\mathbf{1}_{[0,T-1]} = a\mathbf{1}_{[0,T-1]}.$$

Fix $\tau \geqslant T$. Since $T-1 \geqslant t$, in order to prove that the plan a has finite equivalent payoffs, we need to compare $U^{i,\tau}(\widehat{x})$ and $U^{i,\tau}(x)$. Observe that

$$\begin{aligned} U^{i,\tau}(\widehat{x}) &= U^{i,T-1}(x) + \sum_{T \leqslant s \leqslant \tau} \sum_{\xi \in D_s} u^i(\xi, \omega^i(\xi)) \\ &\geqslant U^{i,T-1}(x) \\ &\geqslant U^{i,\tau}(x) - \sum_{T \leqslant s \leqslant \tau} \sum_{\xi \in D_s} u^i(\xi, x(\xi)) \\ &\geqslant U^{i,\tau}(x) - \varepsilon. \end{aligned}$$

We have thus proved that the plan a has finite equivalent payoffs.

□

[19] That is, $d^i_\star(p,q,\kappa) \equiv \{a \in B^i_\star(p,q,\kappa) : \ \mathrm{Pref}^i(p,a) \cap B^i_\star(p,q,\kappa) = \emptyset\}$.

A direct implication of the last proposition is that, when there is no loss of utility in case of default, the sets $\mathrm{Eq}(\mathcal{E})$ and $\mathrm{Eq}_\star(\mathcal{E})$ coincide. This observation allows us to obtain the existence result of Araujo, Páscoa, and Torres-Martínez (2002) as a direct corollary of our equilibrium existence result (see Section 4).

Proposition 3.3. If there is no default penalty then $(\pi, \boldsymbol{a})$ is a competitive equilibrium, if and only if, it is a competitive equilibrium with finite equivalent payoffs, i.e., the sets $\mathrm{Eq}(\mathcal{E})$ and $\mathrm{Eq}_\star(\mathcal{E})$ coincide.

Proof of Proposition 3.3. Let $(\pi, \boldsymbol{a}) \in \mathrm{Eq}(\mathcal{E})$ be a competitive equilibrium. Fix an agent $i \in I$. In order to prove that a^i belongs to the demand $d^i_\star(\pi)$, it is sufficient to prove that a^i has finite equivalent payoffs. Since $\boldsymbol{a}$ is feasible we have $x^i(\xi) \leqslant \Omega(\xi)$. From (H.3), we get that $U^i(x^i)$ is finite. The desired result follows from Proposition 3.2.

Now let $(\pi, \boldsymbol{a}) \in \mathrm{Eq}_\star(\mathcal{E})$ be a competitive equilibrium with finite equivalent payoffs. We only have to prove that a^i belongs to $d^i(\pi)$ for each agent i. Fix an agent i and assume by contradiction that there exists a plan a in $B^i(\pi)$ such that $U^i(x) > U^i(x^i)$. If $U^i(x)$ is finite then, applying Proposition 3.2, we get that $a \in B^i_\star(\pi)$: contradiction. Therefore, we must have $U^i(x) = \infty$, implying that there exists $T \geqslant 1$ such that

$$U^{i,T}(x) > U^i(x^i).$$

Consider the plan $\widehat{a}$ defined by

$$\widehat{a}(\xi) = \begin{cases} a(\xi) & \text{if} \quad t(\xi) \leqslant T \\ (\omega^i(\xi), 0, 0, \widehat{d}(\xi)) & \text{if} \quad t(\xi) = T+1 \\ (\omega^i(\xi), 0, 0, 0) & \text{if} \quad t(\xi) > T+1 \end{cases}$$

where

$$\forall \xi \in D_{T+1}, \quad \forall j \in J, \quad \widehat{d}(\xi, j) = D(p, \xi, j)\varphi(\xi^-, j).$$

Since the plan $\widehat{a} \in B^i(p, q, \kappa)$ and $U^i(\widehat{x}) < \infty$, Proposition 3.2 implies that it has finite equivalent payoffs, i.e., $\widehat{a} \in B^i_\star(p, q, \kappa)$. Moreover we have

$$U^i(\widehat{x}) = U^{i,T}(x) + \sum_{\xi \in D \setminus D^T} u^i(\xi, \omega^i(\xi)) > U^i(x^i).$$

This contradicts the optimality of a^i in $B^i_\star(p, q, \kappa)$. □

4. Precluding Ponzi schemes

Levine and Zame (1996) proved that finitely effective debt constraints are compatible with equilibrium when the default penalty is infinite and no collateral is required. We argued in the previous section that a reasonable adaptation of those endogenous borrowing constraints to models with limited commitment is to restrict plans to have finite equivalent payoffs. We formally defined the concept of equilibrium with finite equivalent payoffs and we have shown its relation with respect to the equilibrium concepts found in the papers of Araujo, Páscoa, and Torres-Martínez (2002) and Kubler and Schmedders (2003). In this section, we are concerned with the issue of existence of such equilibria. We show that if agents are myopic with respect to default penalties, restricting actions to have finite equivalent payoffs allows to rule out Ponzi schemes and guarantees the existence of an equilibrium. Myopia in our setting refers to the time preference of default: the disutility of defaulting today is greater than the disutility of defaulting in the distant future and vanishes in the long run. In other words, myopia implies a reasonable restriction on the asymptotic behavior of default penalties. We exhibit below a large class of "standard" economies for which agents are myopic with respect to default penalties.

4.1. Myopia with respect to default penalties

Before introducing the formal definition of myopic agents with respect to default penalties, we need to introduce some notations. For each asset j and node ξ, we denote by $M(\xi, j)$ the real number

$$\min_{\ell \in L} \frac{\Omega(\xi, \ell)}{C(\xi, j, \ell)}$$

which corresponds to the maximum amount of short-sales in asset j at node ξ that is consistent with the equilibrium condition of market clearing. Observe that under Assumption 2.2, we have $M(\xi, j) < \infty$. Finally, for every node $\xi \neq \xi_0$ we let[20]

$$H(\xi, j) = \sup_{p \in \Delta(L)} \frac{[pA(\xi, j) - pY(\xi)C(\xi^-, j)]^+}{pv(\xi)}.$$

The quantity $H(\xi, j)$ is the maximum amount in real terms that an agent may default on every unit of asset j he sold short at the preceding node ξ^-. Indeed, it is straightforward to verify that if $a = (x, \theta, \varphi, d)$ in A is a physically feasible plan and (p, q, κ) in Π is a process of prices and delivery rates, then for each node ξ and each asset j, we have $\varphi(\xi, j) \leqslant M(\xi, j)$ and

$$\frac{[V(p, \xi, j)\varphi(\xi^-, j) - d(\xi, j)]^+}{pv(\xi)} \leqslant M(\xi^-, j)H(\xi, j).$$

Definition 4.1. Agent i is said to be *myopic* with respect to default penalties if the disutility suffered at the initial period from defaulting in the long run is negligible, i.e.,

$$\liminf_{T \to \infty} \sum_{\xi \in D_T} \sum_{j \in J} \lambda^i(\xi, j) H(\xi, j) M(\xi^-, j) = 0.$$

Agent i is said to be *uniformly myopic* with respect to default penalties when

$$\liminf_{T \to \infty} \sum_{\xi \in D_T} \sum_{j \in J} \lambda^i(\xi, j) H(\xi, j) = 0.$$

Assuming that agents are myopic with respect to default penalties is a very mild assumption since it is automatically satisfied for every standard economy as defined below (see e.g. Araujo and Sandroni (1999)).

[20] The set $\Delta(L)$ is the simplex in $\mathbb{R}^L_+$, i.e., $\Delta(L) = \{p \in \mathbb{R}^L_+ : \ \sum_{\ell \in L} p(\ell) = 1\}$.

Definition 4.2. The economy $\mathcal{E}$ is said *standard* if Assumptions 2.1 and 2.2 are satisfied and if for each agent i there exist

(S.1) a discount factor $\beta_i \in (0,1)$;

(S.2) a sequence $(P^i_t)_{t\geqslant 1}$ of beliefs about nodes at period t represented by a probability $P^i_t \in \mathrm{Prob}(D_t)$;

(S.3) a Bernoulli function $v^i : D \times \mathbb{R}^L_+ \to [0,\infty)$ where $v^i(\xi,\cdot)$ is the cardinal felicity function at node ξ;

(S.4) a current default penalty $\mu^i(\xi, j) \in (0,\infty)$ for each node $\xi > \xi_0$;

such that for each node $\xi \in D$,

$$u^i(\xi,\cdot) = [\beta_i]^{t(\xi)} P^i_{t(\xi)}(\xi) v^i(\xi,\cdot)$$

for each $j \in J$,

$$\lambda^i(\xi, j) = [\beta_i]^{t(\xi)} P^i_{t(\xi)}(\xi)\mu^i(\xi, j)$$

and the processes $(A(\xi, j))_{\xi>\xi_0}$, $(\mu^i(\xi, j))_{\xi>\xi_0}$ and $(G(\xi, j))_{\xi\in D}$ are uniformly bounded from above, where

$$G(\xi, j) = \frac{1}{\max_{\ell\in L} C(\xi, j, \ell)}.$$

Remark 4.1. In a standard economy, one may have that current default penalties are time and state independent, i.e., $\mu^i(\xi, j) = \overline{\mu}(j)$. In that case, assuming that agents are myopic with respect to default penalties does not impose any restriction on $\overline{\mu}(j)$: it can be as large as desired.

Remark 4.2. If every process $M(j) \equiv (M(\xi, j))_{\xi\in D}$ is uniformly bounded away from 0 then myopia implies uniform myopia. In particular, this is the case if we strengthen Assumptions 2.1 and 2.2 by assuming the following properties:

(A.1) The process Ω is uniformly bounded away from 0, i.e., there exists $\underline{\Omega} \in \mathbb{R}^L_{++}$ such that $\Omega(\xi) \geqslant \underline{\Omega}$ for every ξ.

(A.2) For every asset j, the process $C(j) \equiv (C(\xi, j))_{\xi\in D}$ is uniformly bounded from above, i.e., there exists $\overline{C}(j) \in \mathbb{R}^L_+$ such that $C(\xi, j) \leqslant \overline{C}(j)$ for every ξ.

On the other hand, if every process $M(j)$ is uniformly bounded from above then uniform myopia implies myopia. This is in particular the case if we impose the following additional assumption.

(A.3) For every asset j, the process $C(j)$ of collateral requirements does not eventually vanishes in the sense that there exists $\underline{C}(j) > 0$ such that

$$\forall \xi \in D, \quad \max_{\ell\in L} C(\xi, j, \ell) \geqslant \underline{C}(j).$$

Finally, if the process $M(j)$ is uniformly bounded from above and away from 0, then the concepts of uniform myopia and myopia coincide. Uniform myopia is useful when we discuss issues regarding the implementation of our equilibrium concept (See Section 4.2).

When agents are myopic with respect to default penalties, any budget and physically feasible plan $a \in B^i(p,q,\kappa) \cap \mathrm{F}^i$ has actually finite equivalent payoffs. This result will turn out to be crucial in the process of proving the existence of an equilibrium with finite equivalent payoffs.

Proposition 4.1. If agent i is myopic with respect to default penalties, then every budget and physically feasible plan has finite equivalent payoffs. In other words, we have

$$B^i(p,q,\kappa) \bigcap \mathrm{F}^i \subset B^i_\star(p,q,\kappa).$$

Proof of Proposition 4.1. Fix an agent i and consider a plan a that is budget and physically feasible, i.e., $a \in B^i(p,q,\kappa) \cap \mathrm{F}^i$. Fix a period $t \geqslant 0$ and $\varepsilon > 0$. Since the allocation a is physically feasible, we have $x(\xi) \leqslant \Omega(\xi)$, implying that

$$\sum_{\xi\in D} u^i(\xi, x(\xi)) < \infty.$$

Therefore there exists $T^0 \geqslant 1$ such that

$$\sum_{T\geqslant T^0}\sum_{\xi\in D_T} u^i(\xi, x(\xi)) \leqslant \frac{\varepsilon}{2}.$$

Since agent i is myopic with respect to default penalties, there exists $T > \max\{t, T^0\}$ such that

$$\sum_{\xi\in D_T}\sum_{j\in J} \lambda^i(\xi, j)H(\xi, j) \leqslant \frac{\varepsilon}{2}.$$

Consider now the plan $\widehat{a}$ defined by

$$\widehat{a}(\xi) = \begin{cases} a(\xi) & \text{if} \quad t(\xi) < T \\ (\omega^i(\xi), 0, 0, \widehat{d}(\xi)) & \text{if} \quad t(\xi) = T \\ (\omega^i(\xi), 0, 0, 0) & \text{if} \quad t(\xi) > T \end{cases}$$

where

$$\forall \xi \in D_T, \quad \forall j \in J, \quad \widehat{d}(\xi, j) = D(p, \xi, j)\varphi(\xi^-, j).$$

Observe that the plan $\widehat{a}$ satisfies

$$\widehat{a} \in B^i(p,q,\kappa) \cap C^T \quad \text{and} \quad \widehat{a}\mathbf{1}_{[0,T-1]} = a\mathbf{1}_{[0,T-1]}.$$

Moreover, for every $\tau \geqslant T$ we have

$$\begin{aligned}
\Pi^{i,T}(p,\widehat{a}) &\geqslant \Pi^{i,T-1}(p,\widehat{a}) + \sum_{\xi\in D_T} u^i(\xi, \omega^i(\xi)) \\
&\qquad - \sum_{\xi\in D_T}\sum_{j\in J} \lambda^i(\xi, j)H(\xi, j)M(\xi^-, j) \\
&\geqslant \Pi^{i,T-1}(p,a) - \sum_{\xi\in D_T}\sum_{j\in J} \lambda^i(\xi, j)H(\xi, j)M(\xi^-, j) \\
&\geqslant \Pi^{i,T-1}(p,a) - \frac{\varepsilon}{2} \\
&\geqslant \Pi^{i,\tau}(p,a) - \frac{\varepsilon}{2} - \sum_{T\leqslant s\leqslant\tau}\sum_{\xi\in D_s} u^i(\xi, x(\xi)) \\
&\geqslant \Pi^{i,\tau}(p,a) - \varepsilon.
\end{aligned}$$

It follows that for every $\tau \geqslant T$

$$\Pi^{i,\tau}(p,\widehat{a}) = \Pi^{i,T}(p,\widehat{a}) + \sum_{\xi \in D^\tau \setminus D^T} u^i(\xi, \omega^i(\xi)) > \Pi^{i,\tau}(p,a) - \varepsilon.$$

Since $T-1 \geqslant t$, this implies that the plan a has finite equivalent payoffs. □

Remark 4.3. Given Proposition 4.1 one may wonder whether restricting plans to have finite equivalent payoffs is relevant to the issue of existence. Since myopia implies that budget and physically feasible plans have finite equivalent payoffs, why one should impose any kind of debt constraints on available plans to ensure existence? The answer to this question lies on the fact that in decentralized economies agents do not take into account feasibility restrictions when they solve their maximization problem. Only budgetary restrictions are relevant for them. But if this is the case, in the absence of borrowing constraints, agents can run a Ponzi scheme and equilibria may fail to exist.[21]

4.2. Implementation of equilibria with finite equivalent payoffs

The introduction of debt constraints raises issues related to the implementation of those constraints in decentralized anonymous markets. When there is no default penalty implementation is not an issue. Indeed, Proposition 3.3 states that in this case our constraints on plans is innocuous.[22] The case where default penalties are non-zero requires more elaboration.

When there is full commitment, Magill and Quinzii (1994) rule out Ponzi schemes by imposing implicit or explicit and non-binding bounds on the real value of debt. They subsequently argue for either a subjective (self-monitoring) interpretation of implicit bounds or an objective (market-based) interpretation when bounds are explicit. We propose to show that in our context of limited commitment the same kind of interpretation applies. More precisely, we show that equilibria with finite equivalent payoffs are equilibria with implicit (or explicit and non-binding) bounds on short sales.

To prove our claim, we introduce first some notations. Given a number $m > 0$, we let $B^i_m(p,q,\kappa)$ be the subset of all plans $a = (x,\theta,\varphi,d)$ in the unconstrained budget set $B^i(p,q,\kappa)$ such that the process φ is uniformly bounded by m. The set $\cup_{m>0} B^i_m(p,q,\kappa)$ is denoted by $B^i_\infty(p,q,\kappa)$ and corresponds to all plans $a = (x,\theta,\varphi,d)$ in the unconstrained budget set $B^i(p,q,\kappa)$ such that the process φ is uniformly bounded, i.e., $\varphi \in \ell^\infty(D)$. The set $B^i_\infty(p,q,\kappa)$ is called the budget set with implicit bounds on short-selling and $B^i_m(p,q,\kappa)$ is called the budget set with explicit bound m on short-selling.

Definition 4.3. A competitive equilibrium with explicit bounds (implicit bounds) on short-selling is a family of prices and delivery rates (p,q,κ) and a allocation $a = (a^i)_{i\in I}$ where each a^i is optimal in the constrained budget set $B^i_m(p,q,\kappa)$ (resp. $B^i_\infty(p,q,\kappa)$) and all markets clear. We denote by $\mathrm{Eq}_m(\mathcal{E})$ ($\mathrm{Eq}_\infty(\mathcal{E})$) the set of all competitive equilibria with explicit bounds m (resp. implicit bounds) on short-selling.

The following proposition shows that, under uniform myopia with respect to default penalties, plans with implicit constraints on short-sales have finite equivalent payoffs.[23]

Proposition 4.2. If agent i is uniformly myopic with respect to default penalties then every plan with implicit bounds on short-selling has finite equivalent payoffs, i.e., for every price process (p,q,κ) we have $B^i_\infty(p,q,\kappa) \subset B^i_\star(p,q,\kappa)$.

We can now provide a formal proof of our claim: equilibria with finite equivalent payoffs are indeed equilibria with implicit (or explicit and non-binding) bounds on short sales.

Corollary 4.1. *Consider an economy where every agent is uniformly myopic with respect to default penalties. Assume that for every asset j, there exists a lower bound $\underline{C}(j) > 0$ on collateral requirements, i.e.,*

$$\forall \xi \in D, \quad \max_{\ell \in L} C(\xi, j, \ell) \geqslant \underline{C}(j). \tag{4.1}$$

Then every competitive equilibrium with finite equivalent payoffs is

(1) *an equilibrium with implicit bounds on short-selling;*

(2) *an equilibrium with explicit bounds $m \geqslant \overline{m}$ on short-selling where*

$$\overline{m} \equiv \max_{j \in J} \frac{\sum_{i\in I} \overline{\Omega}^i}{\underline{C}(j)}.$$

Proof of Corollary 4.1. Let $((p,q,\kappa), \boldsymbol{a})$ be a competitive equilibrium with finite equivalent payoffs, i.e., $((p,q,\kappa),\boldsymbol{a}) \in \mathrm{Eq}_\star(\mathcal{E})$. Given Proposition 4.2, to prove the corollary we only have to show that for every agent i, the plan a^i belongs to $B^i_m(p,q,\kappa)$ for every $m \geqslant \overline{m}$. Since the allocation $\boldsymbol{a}$ is physically feasible, we get that the process φ of short-sales satisfies

$$\forall \xi \in D, \quad \forall j \in J, \quad \varphi^i(\xi, j) \leqslant M(\xi, j).$$

The desired result follows from Assumption 2.1 and condition (4.1). □

Remark 4.4. It follows from the above arguments that if the bound m is such that $m > \overline{m}$ then this bound is never binding at equilibrium.

4.3. Existence

The main contribution of this paper is the following existence result.

Theorem 4.1. *If every agent is myopic with respect to default penalties then a competitive equilibrium with finite equivalent payoffs exists, i.e.,* $\mathrm{Eq}_\star(\mathcal{E}) \neq \emptyset$.

[21] Páscoa and Seghir (2009) provide an example of an economy with myopic agents and no borrowing constraints in which equilibria fail to exist.

[22] In particular, we recover the existence result in Araujo, Páscoa, and Torres-Martínez (2002). See Corollary 4.2.

[23] The proof follows almost verbatim the arguments of the proof of Proposition 4.1 and therefore is omitted.

We propose a simple proof based on the standard "truncation argument".

Proof of Theorem 4.1. For each $\tau \in \mathcal{T}$, we denote by $\mathcal{E}^\tau$ the truncation of the economy for which the final period is τ. Following the arguments in Páscoa and Seghir (2009),[24] it is possible to prove that under our set of assumptions, there exist a process of prices and delivery rates $\pi^\tau = (p^\tau, q^\tau, \kappa^\tau)$ and a process of plans $\boldsymbol{a}^\tau = (a^{i,\tau})_{i\in I}$ such that $(\pi^\tau, \boldsymbol{a}^\tau)$ is a competitive equilibrium for the truncated economy $\mathcal{E}^\tau$ with $\|p^\tau(\xi)\| \geqslant m(\xi) > 0$ for some $m(\xi)$ that depends only on the primitives of the economy $\mathcal{E}$ (and is independent of the truncation size τ).[25]

We denote by $\mathrm{cl}\,\Xi$ the closure of Ξ under the weak topology.[26] Each process π^τ belongs to $\mathrm{cl}\,\Xi$ which is weakly compact as a product of compact sets. Passing to a subsequence if necessary, we can assume that the sequence $(\pi^\tau)_{\tau\in\mathcal{T}}$ converges to a process $\pi = (p, q, \kappa)$ in $\mathrm{cl}\,\Xi$. Observe that, for each node $\xi \in D$, we have $\|p(\xi)\| \geqslant m(\xi) > 0$. In particular, for each period t and every plan $a \in A$, the payoff $\Pi^{i,t}(p, a)$ is well defined.

By feasibility at each node ξ, we get for each j

$$x^{i,\tau}(\xi) \leqslant \Omega(\xi), \quad \varphi^{i,\tau}(\xi, j) \leqslant M(\xi, j) \quad \text{and} \quad \theta^{i,\tau}(\xi, j) \leqslant M(\xi, j).$$

This implies that the sequence $(x^{i,\tau}(\xi), \varphi^{i,\tau}(\xi), \theta^{i,\tau}(\xi))_{\tau\in\mathcal{T}}$ is uniformly bounded. By optimality, the delivery $d^{i,\tau}(\xi, j)$ is always lower than the promise $V(p^\tau, \xi, j)\varphi^{i,\tau}(\xi^-, j)$ and therefore the sequence $(d^{i,\tau}(\xi))_{\tau\in\mathcal{T}}$ is uniformly bounded. Passing to a subsequence if necessary, we can assume that for each i, the sequence $(a^{i,\tau})_{\tau\in\mathcal{T}}$ converges to a process $a^i \in A$.

We claim that $(\pi, \boldsymbol{a})$ is a competitive equilibrium with finite equivalent payoffs for the economy $\mathcal{E}$. It is straightforward to check that each plan a^i belongs to the budget set $B^i(p, q, \kappa)$ and that the feasibility conditions (2.4), (2.5), (2.6) and (2.7) are satisfied. Applying Proposition 4.1, we get that the plan a^i has finite equivalent payoffs. We propose now to prove that a^i is optimal among plans with finite equivalent payoffs, i.e., the set $\mathrm{Pref}^i(p, a^i) \cap B^i_\star(p, q, \kappa)$ is empty. Assume by way of contradiction that there exist $\varepsilon > 0$, a plan $\overline{a}$ in the budget set $B^i_\star(p, q, \kappa)$ and $T^1 \in \mathbb{N}$ satisfying

$$\forall T \geqslant T^1, \quad \Pi^{i,T}(p, \overline{a}) > \Pi^{i,T}(p, a^i) + \varepsilon. \tag{4.2}$$

Since a^i is physically feasible, we have $x^i(\xi) \leqslant \Omega(\xi)$ for every node $\xi \in D$. It follows from Assumptions (H.2) and (H.3) that $U^i(x^i) \leqslant U^i(\Omega) < +\infty$. This implies that the limit

$$\Pi^i(p, a^i) \equiv \lim_{T\to\infty} \Pi^{i,T}(p, a^i)$$

exists in $[-\infty, \infty)$. In particular, there exists $T^2 \geqslant T^1$ such that

$$\forall T \geqslant T^2, \quad \Pi^{i,T}(p, a^i) + \frac{\varepsilon}{2} > \Pi^i(p, a^i). \tag{4.3}$$

Since the plan $\overline{a}$ has finite equivalent payoffs, there exists $T > T^2$ and $\widetilde{a}$ in the set $B^i(p, q, \kappa) \cap C^T$ such that

$$\widetilde{a}\mathbf{1}_{[0,T^2]} = \overline{a}\mathbf{1}_{[0,T^2]} \quad \text{and} \quad \inf_{\tau\geqslant T}\left[\Pi^{i,\tau}(p, \widetilde{a}) - \Pi^{i,\tau}(p, \overline{a})\right] \geqslant -\frac{\varepsilon}{4}.$$

We denote by $\widehat{a}$ the plan defined by

$$\forall \xi \in D, \quad \widehat{a}(\xi) = \begin{cases} \widetilde{a}(\xi) & \text{if} \quad t(\xi) \leqslant T \\ (0, 0, 0, 0) & \text{if} \quad t(\xi) > T. \end{cases}$$

Observe that $\widehat{a}$ belongs to the truncated budget set $B^i(p, q, \kappa) \cap B^T$ and satisfies

$$\widehat{a}\mathbf{1}_{[0,T^2]} = \overline{a}\mathbf{1}_{[0,T^2]} \quad \text{and} \quad \Pi^{i,T}(p, \widehat{a}) \geqslant \Pi^{i,T}(p, \overline{a}) - \frac{\varepsilon}{4}. \tag{4.4}$$

Combining (4.2), (4.3) and (4.4) we get

$$\Pi^{i,T}(p, \widehat{a}) > \Pi^i(p, a^i) + \frac{\varepsilon}{4}.$$

We let ψ^i be the correspondence from A to A^T defined by

$$\forall a \in A, \quad \psi^i(a) = \left\{b \in B^T \;:\; \Pi^{i,T}(p, b) > \Pi^i(p, a) + \frac{\varepsilon}{4}\right\}.$$

Let F^i be the correspondence from $\Xi \times A$ to A^T defined by

$$\forall (\pi', a') \in \Xi \times A, \quad F^i(\pi', a') = B^{i,T}(\pi') \cap \psi^i(a').$$

Observe that $\widehat{a} \in F^i(\pi, a^i)$. Moreover, we proved that there exists a strictly increasing sequence $(T_n)_{n\in\mathbb{N}}$ with $T_n \in \mathbb{N}$ such that

$$\lim_{n\to\infty}((p_n, q_n, \kappa_n), a^i_n) = ((p, q, \kappa), a^i)$$

where

$$(p_n, q_n, \kappa_n) = (p^{T_n}, q^{T_n}, \kappa^{T_n}) \quad \text{and} \quad a^i_n = a^{i,T_n}.$$

Since F^i is lower semi-continuous on $\Xi \times A$ for product topologies,[27] we can conclude that there exists $\nu \in \mathbb{N}$ large enough such that $T_\nu \geqslant T$ and the set $F^i((p_\nu, q_\nu, \kappa_\nu), a^i_\nu)$ is non-empty. Let $\widehat{a}_\nu$ be an element of that set. This means that

$$\widehat{a}_\nu \in B^{i,T}(p_\nu, q_\nu, \kappa_\nu) \quad \text{and} \quad \Pi^{i,T}(p_\nu, \widehat{a}_\nu) \geqslant \Pi^i(p_\nu, a^i_\nu) + \frac{\varepsilon}{4}.$$

Since $T_\nu \geqslant T$, we have

$$B^{i,T}(p_\nu, q_\nu, \kappa_\nu) \subset B^{i,T_\nu}(p_\nu, q_\nu, \kappa_\nu)$$

and

$$\Pi^{i,T_\nu}(p_\nu, \widehat{a}_\nu) \geqslant \Pi^{i,T}(p_\nu, \widehat{a}_\nu).$$

It follows that

$$\Pi^{i,T_\nu}(p_\nu, \widehat{a}_\nu) > \Pi^i(p_\nu, a^i_\nu) = \Pi^{i,T_\nu}(p_\nu, a^i_\nu)$$

[24] We can also adapt the arguments of the proof of Theorem 1 in Araujo, Páscoa, and Torres-Martínez (2002).

[25] We refer to the appendix for the precise definition of the truncated economy $\mathcal{E}^\tau$ and the associated (finite-horizon) equilibrium concept.

[26] The process (p, q, κ) belongs to $\mathrm{cl}\,\Xi$ if the condition "$p(\xi) \in \mathbb{R}^L_{++}$" in (**??**) is replaced by "$p(\xi) \in \mathbb{R}^L_+$".

[27] See Páscoa and Seghir (2009) for detailed arguments.

contradicting the optimality of a^i_ν in the truncated economy $\mathcal{E}^{T_\nu}$ under the price process $(p_\nu, q_\nu, \kappa_\nu)$.

We have thus proved that for each i, the plan a^i has finite equivalent payoffs and satisfies

$$\mathrm{Pref}^i(p, a^i) \cap B^i_\star(p, q, \kappa) = \emptyset.$$

This means that a^i belongs to the demand set $d^i_\star(\pi)$. We already proved that all markets clear. This means that $(\pi, \boldsymbol{a})$ is a competitive equilibrium with finite equivalent payoffs. □

Given Proposition 3.3, we can obtain the main existence result in Araujo, Páscoa, and Torres-Martínez (2002, Theorem 2) as a direct corollary of Theorem 4.1.

Corollary 4.2 (Araujo, Páscoa, and Torres-Martínez (2002))**.** *If there is no default penalty then there exists a competitive equilibrium, i.e.,* $\mathrm{Eq}(\mathcal{E}) \neq \emptyset$.

Remark 4.5. The proof of the above result proposed by Araujo, Páscoa, and Torres-Martínez (2002) is different than ours. They also consider a sequence of equilibria for truncated economies and pass to the limit. However, to prove that the limit allocation is optimal, they follow a rather involved argument showing that the sequence of marginal utilities of income associated to the sequence of truncated equilibria is uniformly bounded.

5. Equilibrium refinement

In this section we address an issue related to the indeterminacy of delivery rates in the definition of a competitive equilibrium. In a companion paper (see Martins-da-Rocha and Vailakis (2011)) we have shown that for the two examples proposed in Páscoa and Seghir (2009), no-trade is a possible equilibrium outcome. This is due to the fact that the standard equilibrium concept leaves room for spurious inactivity on asset markets due to unduly pessimistic expectations on asset deliveries. In the definition of a competitive equilibrium, the market clearing equation defining the delivery rate expected by lenders leaves its value undeterminate when there is no-trade at equilibrium. A similar issue was already pointed out by Dubey, Geanakoplos, and Shubik (2005). However, in their model it is easy to support equilibria with no-trade in the asset markets on account of absurdly pessimistic expectations about repayment rates. Indeed, if lenders expect assets to deliver nothing, then we can support any pure-spot equilibria by choosing the asset prices to be equal to zero. One may think that this problem does not arise anymore in the presence of collateral requirements since lenders rationally expect deliveries to be at least larger than the minimum between the promise and the value of the depreciated collateral. It is true that we cannot support pure-spot equilibria in a trivial manner as it is the case in Dubey, Geanakoplos, and Shubik (2005). However, spurious inactivity on asset markets due to overpessimistic expectations is still a problem even in the presence of collateral requirements. We propose to clarify this issue and explain how the refinement proposed by Dubey, Geanakoplos, and Shubik (2005) can be adapted to our setting.

Let $(\pi, (a^i)_{i\in I})$ be a competitive equilibrium with finite equivalent payoffs where $\pi = (p, q, \kappa) \in \Xi$ and $a^i = (x^i, \theta^i, \varphi^i, d^i)$. Fix a node $\xi > \xi_0$. Since agent i delivers in node ξ at least the amount $D(p, \xi, j)\varphi^i(\xi^-, j)$, we let $\sigma^i(\xi, j) \in [0, 1]$ be the individual delivery rate defined by the equation

$$d^i(\xi, j) = \left[\sigma^i(\xi, j)\{V(p, \xi, j) - D(p, \xi, j)\} + D(p, \xi, j)\right]\varphi^i(\xi^-, j)$$

if agent i has a short position $\varphi^i(\xi^-, j) > 0$, and we pose $\sigma^i(\xi, j) = 0$ elsewhere. If there is trade in node ξ^-, i.e., $\varphi^i(\xi^-, j) > 0$ for some agent i, then equation (2.7) in the definition of a competitive equilibrium can be restated as follows

$$\kappa(\xi, j)\sum_{i\in I}\varphi^i(\xi^-, j) = \sum_{i\in I}\sigma^i(\xi, j)\varphi^i(\xi^-, j) \tag{5.1}$$

and $\kappa(\xi, j)$ can be interpreted as the average delivery rate (per unit of asset sold) above the minimum delivery $D(p, \xi, j)$. If there is no-trade in asset j in node ξ^- then the delivery rate $\kappa(\xi, j)$ is left undeterminate. That is, when the asset is not traded, our equilibrium concept makes no assumption about the expected delivery rate. We have shown in Martins-da-Rocha and Vailakis (2011) that pessimistic expectations about deliveries (i.e., low values of $\kappa(\xi, j)$) may by itself render the asset market inactive if default penalties are large enough. This finding shares some similarities with the issue of trivial equilibria pointed out by Dubey, Geanakoplos, and Shubik (2005). To explain this link we recall some notations. In Dubey, Geanakoplos, and Shubik (2005) assets are not collateralized. The repayment rate, denoted by $K(\xi, j)$, is defined by the equation

$$K(\xi, j)V(p, \xi, j)\sum_{i\in I}\varphi^i(\xi^-, j) = \sum_{i\in I}d^i(\xi, j).$$

As explained in Páscoa and Seghir (2009), when assets are collateralized agents deliver at least $D(p, \xi, j)$ per unit of asset sold. In this case, if $D(p, \xi, j)$ and $V(p, \xi, j)$ are not zero, rational agents expect $K(\xi, j)$ to be greater than the ratio $D(p, \xi, j)/V(p, \xi, j)$, and in particular it must be non-null.[28] In other words, when there is trade in node ξ^- we have the relation

$$K(\xi, j)V(p\xi, j) = \kappa(\xi, j)\{V(p\xi, j) - D(p\xi, j)\} + D(p\xi, j).$$

In Dubey, Geanakoplos, and Shubik (2005) it is easy to support equilibria with no-trade in the asset on account of absurdly pessimistic expectations about repayment rates. However, in a model with collateral requirements, it is not clear whether such equilibria can be supported.[29] In Martins-da-Rocha and Vailakis (2011) we show that although agents expect per unit repayments $K(\xi, j)$ to be strictly positive (actually above or equal

[28] This is the reason why in our model we have chosen to parameterize agents' expectations about delivery by the average delivery rate above the minimum delivery, denoted by $\kappa(\xi, j)$.

[29] The intuition behind the existence of trivial equilibria in Dubey, Geanakoplos, and Shubik (2005) is as follows. Consider a sequence of pure spot markets and an associated equilibrium. Introduce next an asset j in node ξ^-. Choose the repayment rate $K(\xi, j)$ of the asset equal to zero and the price $q(\xi^-, j)$ equal to zero. Then no agent would have an incentive to trade in node ξ^-. In a model with collateralized obligations this argument breaks down since $K(\xi, j)$ must be larger that $D(p, \xi, j)/V(p, \xi, j)$. In case the asset's promise is larger than the

to the minimum $D(p,\xi,j)/V(p,\xi,j)$) there is still room for unduly pessimistic expectations that sustain equilibrium with no-trade.

This raises an interesting issue. The equilibrium concept should be refined in order to rule out such pathological no-trade equilibria. We show below that the refinement procedure proposed by Dubey, Geanakoplos, and Shubik (2005) can be easily adapted to our framework.

Following Dubey, Geanakoplos, and Shubik (2005) we propose an equilibrium refinement in which the government intervenes to sell infinitesimal quantities $\varepsilon > 0$ of each asset at each node and fully delivers on its promises. Since the government does not default, it does not need to constitute collateral bundles. However, since it delivers fully $\varepsilon V(p,\xi,j)$ but it gets delivered only $\varepsilon V(\kappa,p,\xi,j)$, on net the government injects the vector of commodities $\varepsilon b(\kappa,p,\xi,j)v(\xi)$ where $b(\kappa,p,\xi,j) \geqslant 0$ is defined by the equation

$$b(\kappa,p,\xi,j)p(\xi)v(\xi) = V(p,\xi,j) - V(\kappa,p,\xi,j).$$

This touch of honesty banishes whimsical pessimism and rules out spurious inactivity on asset markets. We adapt the definition of a competitive equilibrium with the government intervention proposed by Dubey, Geanakoplos, and Shubik (2005) to our framework.

Definition 5.1. An ε-equilibrium is a family $\pi = (p,q,\kappa) \in \Xi$ of prices and delivery rates and an allocation $(a^i)_{i\in I}$ such that: (1) as in the standard competitive equilibrium with finite equivalent payoffs, for every agent i the plan a^i is optimal among the budget feasible plans and the asset market clears at every period; (2) different to the standard competitive equilibrium, commodity markets ε-clear, i.e., for every $\xi \in D$,[30]

$$\sum_{i\in I} x^i(\xi) = \sum_{i\in I}[\omega^i(\xi) + Y(\xi)x^i(\xi^-)] + \varepsilon b(\kappa,p,\xi,j)v(\xi)$$

and delivery rates are boosted by the external agent, i.e., for every $\xi > \xi_0$,

$$V(\kappa,p,\xi,j)\left[\varepsilon + \sum_{i\in I}\theta^i(\xi^-,j)\right] = \varepsilon V(p,\xi,j) + \sum_{i\in I} d^i(\xi,j). \quad (5.2)$$

Equation (5.2) defining the delivery rate $\kappa(\xi,j)$ can be restated as follows

$$\kappa(\xi,j)\left[\varepsilon + \sum_{i\in I}\theta^i(\xi^-,j)\right] = \varepsilon + \sum_{i\in I}\sigma^i(\xi,j)\varphi^i(\xi^-,j)$$

where $\sigma^i(\xi,j)$ is agent i's individual delivery rate as defined by (5.1). The delivery rate $\kappa(\xi,j)$ is the weighted average of individual rates and is boosted due to the fact that the government delivers fully on its promises. As the government intervention disappears, i.e., ε tends to 0, this boost disappears for periods where the asset is positively traded in the limit.

Definition 5.2. A competitive equilibrium $(\pi,(a^i)_{i\in I})$ with finite equivalent payoffs is called a *refined equilibrium* if for every $\varepsilon > 0$ small enough there exists an ε-equilibrium $(\pi(\varepsilon),(a^i(\varepsilon))_{i\in I})$ such that

$$\lim_{\varepsilon\to 0}(\pi(\varepsilon),(a^i(\varepsilon))_{i\in I}) = (\pi,(a^i)_{i\in I}).$$

It is straightforward to adapt our arguments to get existence (under standard assumptions) of an ε-equilibrium with finite equivalent payoffs. In order to prove that the limit $(\pi,(a^i)_{i\in I})$ is an equilibrium, the only difficulty is to show that the plan a^i is optimal in the budget set defined by the price process π. The arguments follow almost verbatim those in the proof of Theorem 4.1 and are based on the lower semi-continuity of the correspondence F^i.[31]

6. Conclusion

What makes general equilibrium models with collateral requirements (Araujo, Páscoa, and Torres-Martínez (2002) and Kubler and Schmedders (2003)) very appealing is that collateral constraints not only do exist in actual markets but seem to be an efficient mechanism to preclude Ponzi schemes without imposing any *ad-hoc* constraint on debt. The recent contributions of Páscoa and Seghir (2009) and Ferreira and Torres-Martínez (2010) show that the positive results in Araujo, Páscoa, and Torres-Martínez (2002) may not be robust: the effectiveness of collateral requirements to bound debt may not be valid anymore in the natural case where there are other mechanisms leading agents to overpay, that is, to repay more than the collateral when the value of their debt actually exceeds the collateral value.

To formally close the model and restore equilibrium, we need to impose borrowing constraints. Among the different approaches already existing in the literature with full commitment, we argue in favor of the endogenous debt constraints à la Levine and Zame (1996). We introduce in the setting of Araujo, Páscoa, and Torres-Martínez (2002), Kubler and Schmedders (2003) and Páscoa and Seghir (2009) the concept of plans with *finite equivalent payoffs*. When payments are fully enforced, our concept of plans with finite equivalent payoffs coincides

depreciated value of the collateral, i.e., $D(p\xi,j) = p(\xi)Y(\xi,j)C(\xi^-,j)$, one may try to implement no-trade by choosing $\kappa(\xi,j) = 0$ (or equivalently $K(\xi,j) = D(p,\xi,j)/V(p,\xi,j)$) and fixing the asset price $q(\xi^-,j) = p(\xi^-)C(\xi^-,j)$. No agent would have incentives to invest. Indeed, it would be better to buy the bundle $C(\xi^-,j)$ instead of one unit of the asset because of the utility obtained from consuming the collateral. However, it is not clear whether agents would have no incentives to sell the asset. It depends on whether the gain from consuming the collateral in node ξ can compensate the future penalty suffered in case of default or the loss in consumption due to the repayment of debt besides the value of the depreciated collateral.

[30] By convention we let $a_{-1} = (x_{-1},\theta_{-1},\varphi_{-1},d_{-1}) = (0,0,0,0)$ and $b(\kappa,p,\xi_0,j) = 0$.

[31] Here we will have

$$\lim_{\varepsilon\to 0}((p(\varepsilon),q(\varepsilon),\kappa(\varepsilon)),a^i(\varepsilon)) = ((p,q,\kappa),a^i)$$

instead of

$$\lim_{n\to\mathbb{N}}((p_n,q_n,\kappa_n),a^i_n) = ((p,q,\kappa),a^i).$$

Moreover, we will obtain a contradiction with respect to the optimality of $a^i(\varepsilon)$ in the boosted ε-economy (for ε small enough) instead of the truncated economy $\mathcal{E}^T$ for T large enough.

with the concept of plans with finitely effective debt introduced by Levine and Zame (1996). When there are collateral requirements but no default penalties, any budget feasible plan has automatically finite equivalent payoffs. In particular, our budget set coincides with the standard one defined in Araujo, Páscoa, and Torres-Martínez (2002) and Kubler and Schmedders (2003). Assuming a mild assumption on default penalties, namely that agents are myopic with respect to default penalties, we show that restricting actions to have finite equivalent payoffs rules out Ponzi schemes and guarantees equilibrium existence while keeping the minimal ability to borrow and lend that we expect in our model. The proof is very simple and intuitive. In particular, the main existence result in Araujo, Páscoa, and Torres-Martínez (2002) is a direct corollary of our existence result.

A. Appendix: Truncated economy

Fix $\tau \in \mathcal{T}$ with $\tau > 0$. Recall that A^τ denotes the set of all plans $a \in A$ such that

$$\forall \xi \in D, \quad t(\xi) > \tau \Longrightarrow a(\xi) = 0.$$

We let B^τ be the set of plans $a \in A^\tau$ satisfying the additional condition

$$\forall \xi \in D, \quad t(\xi) = \tau \Longrightarrow \varphi(\xi) = 0.$$

Given a process $(p, q, \kappa) \in \Xi$, we denote by $B^{i,\tau}(p, q, \kappa)$ the set defined by

$$B^{i,\tau}(p, q, \kappa) \equiv B^i(p, q, \kappa) \cap B^\tau.$$

A competitive equilibrium for the *truncated* economy $\mathcal{E}^\tau$ is a family of prices and delivery rates $\pi = (p, q, \kappa) \in \Xi$ and an allocation $\boldsymbol{a} = (a^i)_{i \in I}$ with $a^i \in B^\tau$ such that

(a) for every agent i, the plan a^i is optimal, i.e.,[32]

$$a^i \in d^{i,\tau}(p, q, \kappa); \tag{A.1}$$

(b) commodity markets clear at every node up to period τ, i.e.,

$$\sum_{i \in I} x^i(\xi_0) = \sum_{i \in I} \omega^i(\xi_0) \tag{A.2}$$

and for all $\xi \in D^\tau \setminus \{\xi_0\}$,

$$\sum_{i \in I} x^i(\xi) = \sum_{i \in I} \left[\omega^i(\xi) + Y(\xi) x^i(\xi^-)\right]; \tag{A.3}$$

(c) asset markets clear at every node up to period $\tau - 1$, i.e., for all $\xi \in D^{\tau-1}$,

$$\sum_{i \in I} \theta^i(\xi) = \sum_{i \in I} \varphi^i(\xi); \tag{A.4}$$

(d) deliveries match up to period τ, i.e., for all $\xi \in D^\tau \setminus \{\xi_0\}$ and all $j \in J$,

$$\sum_{i \in I} V(\kappa, p, \xi, j) \theta^i(\xi^-, j) = \sum_{i \in I} d^i(\xi, j). \tag{A.5}$$

Observe that if a plan a belongs to B^τ, then $\Pi^{i,\tau}(p, a)$ and $\Pi^i(p, a)$ coincide for every price process p. Moreover, if $(\pi, \boldsymbol{a})$ is a competitive equilibrium for the truncated economy $\mathcal{E}^\tau$, then without any loss of generality, we can assume that $q(\xi) = 0$ and $\theta(\xi) = 0$ for every terminal node $\xi \in D_\tau$.

[32] The demand set is defined by

$$d^{i,\tau}(p, q, \kappa) \equiv \operatorname{argmax}\{\Pi^{i,\tau}(p, a) : \ a \in B^{i,\tau}(p, q, \kappa)\}.$$

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