AMBIGUITY AND SOCIAL INTERACTION

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Abstract

A decision-maker is said to have an ambiguous belief if it is not precise enough to be represented by a single probability distribution. The pervasive assumption in game theoretic models in economics is that players' beliefs are unambiguous. This paper argues, drawing on examples from economics and politics, that it may be illuminating, in instances, to model players as having ambiguous beliefs. Optimistic and pessimistic responses to ambiguity are formally modelled. We show that pessimism has the effect of increasing (decreasing) equilibrium prices under Cournot (Bertrand) competition. In addition the effects of ambiguity on peace-making are examined. It is shown that ambiguity may select equilibria in coordination games with multiple equilibria. Some comparative statics results are derived for the impact of ambiguity in games with strategic complements.

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1 Introduction

An individual is said to have an ambiguous belief if it is not precise enough to be represented by a single probability distribution. The pervasive assumption in game theoretic models is that players' beliefs are unambiguous. This paper argues, drawing on examples from economics and politics, that it may be illuminating to model players as having ambiguous beliefs. The growing literature on ambiguity in strategic games[1] lacks two important features: First, an elementary framework of ambiguity in games that can be easily understood and applied by non-specialists, and second, the treatment of an optimistic attitude towards ambiguity in addition to ambiguity-aversion or pessimism. In this article we take up both issues.

Ambiguity may be contrasted with risk, where probabilities are known, see Knight (1921). In fact for many economic or political situations, it is not obvious why decision-makers should know probabilities. For instance, it is not easy to assign probabilities to threats from terrorism and rogue states, and the likely impact of new technologies, etc. Nevertheless, for several decades, subjective expected utility (henceforth SEU) by Savage (1954) appeared to have rendered the distinction between risk and Knightian uncertainty obsolete. In this theory, individuals faced with uncertainty behave as if they hold beliefs that can be represented by a subjective probability distribution. Hence, from an analytical point of view, there was little distinction between risk and ambiguity.

However early evidence by Ellsberg (1961) suggests that beliefs cannot be represented by conventional probabilities. Systematic laboratory experiments have confirmed Ellsberg's conjecture, see Camerer and Weber (1992). Despite the experimental evidence, SEU proved to be a successful modelling tool. Important insights were obtained from the distinction between risk preferences and beliefs, which can be made in this approach. The economics of insurance and information could be developed in this context. It is desirable to develop a theory of ambiguity, which is equally suitable for application. The inconsistencies between Savages's theory and empirically observed behaviour have stimulated efforts for alternative theories. We believe one of the most promising of these is Choquet expected utility (henceforth CEU), which involves representing individuals' beliefs by non-additive probabilities (or capacities), see Schmeidler (1989).

[1] See for instance Dow and Werlang (1994), Eichberger and Kelsey (2000), Lo (1996) and Marinacci (2000).

In this theory, individuals maximise the expected value of a utility function with respect to a non-additive belief, and the expectation is expressed as a Choquet integral, Choquet (1953-4). CEU is a generalization of subjective expected utility. It has the advantage that it maintains the separation of beliefs and outcome evaluation, which makes the theory easier to apply in economics and social sciences.

A key element of Knight's work was that people differ in their attitudes to ambiguity. The majority of people tend to avoid ambiguous situations. However a minority of individuals actually appear to seek ambiguity. Laboratory evidence shows a similar pattern. Most experimental subjects behave more cautiously when probabilities are undefined, while a significant minority displays the opposite attitude, (see for instance Camerer and Weber (1992)). Moreover the same individual may be pessimistic (or ambiguity averse) in one situation and optimistic (or ambiguity seeking) in another. The evidence shows that ambiguity attitudes are distinct from risk attitudes, see Cohen, Jaffray, and Said (1985). Individuals may be risk-averse and ambiguity-loving and vice-versa.

Most of the work on CEU has been focused on the case of ambiguity aversion i.e., where the capacity is convex. The generality of arbitrary convex capacities made expositions relatively inaccessible and the focus on pessimism limited the scope of potential applications. Recently, Chateauneuf, Eichberger, and Grant (2008) axiomatised CEU, where preferences are represented as a weighted average of the expected utility, the maximum utility and the minimum utility. While a special case, this representation has the advantage of being simple to apply and easy to interpret. Moreover, it allows for a meaningful notion of both optimism as well as pessimism as weight put on the best and worst outcomes respectively. We take this model further and apply it to strategic games.

We model interaction among players in a game who face ambiguity about their opponents' strategy. A player may react optimistically (resp. pessimistically) towards this ambiguity by over-weighting strategies of the opponent which are good (resp. bad) for the player. We introduce the concept of Equilibrium Under Ambiguity (henceforth EUA), prove existence and relate it to Nash equilibrium without ambiguity. EUA is an equilibrium where beliefs may be non-additive.

We demonstrate the applicability of our framework and present some comparative static results on ambiguity-attitude. Specifically we consider the impact of optimism and pessimism in some familiar models from industrial organisation. In a Cournot oligopoly, the worst (resp. best) outcome would be perceived as a rival producing a large (resp. low) quantity. Provided Cournot oligopoly is a game of strategic substitutes, optimism has the effect of increasing the perceived marginal benefit of producing more and so increases the equilibrium output. This decreases profits but increases consumer surplus. In contrast, in Bertrand competition, a good outcome would be perceived as rival firms charging a high price. In the usual case, where prices are strategic complements, optimism will increase the incentive for any given firm to increase its price and hence also the equilibrium price.

Traditionally, it has been suspected that oligopolies are prone to informal collusive arrangements. Scherer (1970) provides many examples from anti-trust cases. The presumption of regulators that oligopolists collude, suggests that output is, at least sometimes, below the Cournot level without clear evidence of collusion. Ambiguity-aversion may offer an alternative and as yet unexplored explanation for why competition may be less fierce in Cournot-style oligopoly than is predicted by Nash equilibrium.

In Cournot and Bertrand models, there is scope for strategic delegation. Interestingly, in both cases, we show that it is desirable for the owner of a firm to delegate decision-making to a manager who is more optimistic than (s)he is. This result is rather striking since usually comparative results are reversed between the Bertrand and Cournot models. Already Knight (1921) argued in his theory of profit and entrepreneurial activity, that entrepreneurs tend to be individuals who are less ambiguity-averse. Indeed, there is evidence from interviews with new entrepreneurs that their self-assessed chances of success are uncorrelated with objective predictors like education, prior experience, and initial capital, and are on average widely off the mark (**?**). In a different study, new life insurance agents who were optimistic, sold more policies during their first year and were less likely to quit (see **?**).

Ambiguity is not confined to economic situations. For instance, environmental risks are often ambiguous due to limited knowledge of the relevant science and because outcomes will only be seen many decades from now. The effects of global warming and the environmental impact of genetically modified crops are two examples. Other fields characterized by ambiguity are politics, diplomacy and international security as many recent events in world politics dramatically demonstrate. To illustrate the potential of our framework we consider a stylized model of peace-making in section 5. Because no conflict is like another, it is hard for the parties involved for come up with exact probability judgements. Thus ambiguity may play a large role in conflict resolution such as in the Middle East peace-making (see for example **?**) or the conflict in Northern Ireland.

In peace-making, ambiguity leaves room for the optimistic hope that peace indeed will be achieved but pessimistic participants may also respond to ambiguity with distrust in the opponent. We find that in our model ambiguity-attitudes can determine the success or failure of a peace process.

Beside the above mentioned applications, we provide several new general results. For 2-player games with a one-dimensional strategy space, we show that if ambiguity is sufficiently large, then equilibrium under ambiguity is unique. If in addition the game has strategic complements we provide new results on the comparative statics of equilibria with respect to optimism and pessimism. For such games with multiple equilibria we show ambiguity and ambiguity-attitude can select particular equilibria.

Organisation of the Paper Section 2 describes how we model ambiguity in strategic games. Then in section 3 we discuss a solution concept for games, where the behaviour of other players may be perceived to be ambiguous. We show existence and discuss the relationship with Nash equilibrium. In section 4 we demonstrate economic applications by considering the impact of ambiguity in some standard oligopoly models. Non-economic applications are demonstrated by the model of peace-making in section 5. Some general results concerning the comparative statics of ambiguity are presented in section 6 and our conclusions can be found in section 7. The on-line appendix contains proofs of those results not proved in the text.

2 Modelling Ambiguity in Strategic Games

In this section we explain how ambiguity can be modelled in strategic games by non-additive beliefs. We present the concepts of a neo-additive capacity (a class of non-additive beliefs) and of an expectation with respect to a non-additive probability, (the Choquet integral).

2.1 Games

Unless otherwise mentioned, we consider a strategic game $G = \langle (S_i, u_i)_{i=1,2} \rangle$ with two players $i = 1, 2$, where each player's strategy set $S_i \subseteq \mathbb{R}$ is a closed and bounded interval. For most economic applications, it is sufficient to assume that agents choose real-valued variables such as prices or quantities. The pay-off function of player i, $u_i(s_i, s_{-i})$ is assumed to be quasi-concave in his/her strategy and twice continuously differentiable in strategies of both players. The following notational conventions will be maintained throughout. The set of strategy combinations will be denoted by $S = S_1 \times S_2$. A typical strategy combination $s \in S$ can be decomposed into the strategy s_i of player i and the strategy of his/her opponent s_{-i}, hence we may write $s = \langle s_i, s_{-i} \rangle$. The set of strategy combinations of player i's opponent is denoted by S_{-i}.

2.2 Non-Additive Beliefs and Expectations

Consider an economic agent whose profit may depend in part on the actions of rivals. Here, ambiguity concerns the possible play of one's opponent. We shall represent individuals' beliefs by capacities. A capacity plays a similar role to a subjective probability in SEU. In this paper we shall confine attention to neo-additive capacities, defined below.

Definition 2.1 *Let α, δ be real numbers such that $0 \leqslant \delta \leqslant 1, 0 \leqslant \alpha \leqslant 1$, define a neo-additive-capacity ν by $\nu(\emptyset) = 0, \nu(S_{-i}) = 1, \nu(A) = \delta\alpha + (1-\delta)\pi(A), \emptyset \subsetneqq A \subsetneqq S_{-i}$, where π is an additive probability distribution on S_{-i}.*[2]

Let u be a utility function which represents the decision-makers' pay-offs as a function of the acts of his/her opponent. The expectation of u with respect to the neo-additive capacity ν, is given by the Choquet integral (see Chateauneuf, Eichberger, and Grant (2008)).[3]

[2] Neo-additive is an abbreviation for non-extremal outcome additive. Neo-additive capacities are axiomatised in Chateauneuf, Eichberger, and Grant (2008).

[3] Gilboa (1987), Schmeidler (1989) and Sarin and Wakker (1992) provide axiomatisations for general CEU preferences. Ghirardato and Marinacci (2002), Wakker (2001) and Epstein (1999) characterise capacities representing

Definition 2.2 (Choquet Integral) *The Choquet expected value of the utility function u_i with respect to the neo-additive capacity $\nu = \delta\alpha + (1-\delta)\pi$ from playing $s_i \in S_i$ is given by:*

$$\int u_i(s_i, s_{-i})\, d\nu = \delta\alpha M_i(s_i) + \delta(1-\alpha)\, m_i(s_i) + (1-\delta)\mathbf{E}_\pi u_i(s_i, s_{-i}), \tag{1}$$

where $\mathbf{E}_\pi u_i(s_i, \cdot)$ denotes the expected utility of u_i with respect to the probability distribution π on S_{-i}, $M_i(s_i) = \max_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})$, and $m_i(s_i) = \min_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})$.

There is experimental evidence that preferences have the form of a weighted average of the minimum, the maximum and the mean pay-offs, see Lopes (1987).[4] Beliefs modelled by neo-additive capacities are easy to interpret. Intuitively, a neo-additive capacity represents an individual whose beliefs are described by the additive probability distribution π. However (s)he lacks confidence in this belief. In part (s)he reacts to this in an optimistic way by over-weighting the best outcome M, as measured by $\delta\alpha$ and in part in a pessimistic way by over-weighting the worst outcome m. Ambiguity-attitude is captured by the parameter α, which we shall refer to as degree of optimism. High values of α correspond to optimistic responses to ambiguity, while low values represent a decision-maker who is generally pessimistic. The amount of perceived ambiguity may be measured by δ, which we shall refer to as the degree of ambiguity.

Definition 2.3 (Optimism/Pessimism) *Let $\nu = \delta a + (1-\delta)\pi$ be a neo-additive capacity. We define the degree of optimism of ν to be α.*[5]

If beliefs are represented by conventional probabilities, it is not possible to model decision-makers who lack confidence in their beliefs. The ability to make this distinction offers opportunities to analyse the impact of ambiguity and optimism/pessimism in economic models. With neo-additive capacities it is relatively easy to study comparative statics with respect to ambiguity-attitude.

Neo-additive capacities reduce the technical complexity of the CEU model. Like a conventional expectation, the Choquet integral is a weighted average of utilities. In particular it is a

ambiguity-averse or pessimistic attitudes of a decision maker. There is also a closely related literature which represents beliefs as sets of conventional probability distributions, see Bewley (2002), Gilboa and Schmeidler (1989), Kelsey (1994).

[4] Such preferences have been axiomatised in the context of risk by Cohen (1992).

[5] Wakker (2001) provides precise definitions of optimism and pessimism in CEU models. Epstein (1999) and Ghirardato and Marinacci (2002) provide alternative concepts of ambiguity aversion.

convex combination of the minimum, the maximum and an average pay-off. More importantly, neo-additive capacities are less mathematically complex than more general classes of capacities. While only $n-1$ parameters will describe a probability distribution on a set with n elements, a general capacity on the same set involves 2^n parameters. In contrast, a neo-additive capacity can be described by just $n+1$ parameters.

The Choquet integral is similar to a conventional expectation since it is a weighted average of utilities and the weights sum to 1. However the weights are not probabilities but decision weights. The best (resp. worst) outcome, M (resp. m) gets weight $\delta\alpha + (1-\delta)\,\pi(M)$ (resp. $\delta(1-\alpha) + (1-\delta)\,\pi(m)$). For any other outcome, x, the decision weight is $(1-\delta)\,\pi(x)$. The Choquet integral is the sum over all outcomes of the act weighted by these decision weights.[6]

Assumption 2.1 *We shall assume that all individuals have CEU preferences and beliefs, which can be represented by a neo-additive capacity.*

Finally, we need to define the support of a capacity.

Definition 2.4 (Support) *The support of the neo-additive capacity,* $\nu(A) = \delta\alpha + (1-\delta)\,\pi(A)$, *is defined by* $\operatorname{supp}(\nu) = \operatorname{supp}(\pi)$.

For the support of a neo-additive capacity we have $\nu(\operatorname{supp}(\nu)) + \nu(S_{-i} \setminus \operatorname{supp}(\nu)) = 1 - \delta + 2\delta\alpha$, which is not in general equal to 1. This can be interpreted as saying that the support of the capacity is itself ambiguous. This could be useful to model a situation where a player believes there is some ambiguity about whether or not his/her opponents will play best responses. If beliefs are represented by a neo-additive capacity then the best and worst outcomes will influence choice in addition to strategies in the support of a player's beliefs.

3 Equilibrium Under Ambiguity

3.1 Definition of Equilibrium

In this section we present an equilibrium concept for strategic games with ambiguity. Suppose player i perceives ambiguity about his/her opponent's choice of strategy. If his/her beliefs are

[6] Sarin and Wakker (1998) provide a detailed discussion of the relationship between decision weights and capacities.

modelled by a neo-additive capacity ν_i on S_{-i}, i.e., with an additive probability distribution π_i on S_{-i}, a degree of optimism α_i and a degree of ambiguity δ_i, then the expected payoff function is the Choquet integral,

$$V_i\left(s_i;\pi_i,\alpha_i,\delta_i\right)=\delta_i\alpha_i M_i\left(s_i\right)+\delta_i\left(1-\alpha_i\right)m_i\left(s_i\right)+\left(1-\delta_i\right)\int u_i(s_i,s_{-i})d\pi_i(s_{-i}). \tag{2}$$

In games, one can determine π_i endogenously as the prediction of the players from the knowledge of the game structure and the preferences of others. In contrast, we treat the degrees of optimism, α_i and ambiguity, δ_i, as exogenous.

Define the best-response correspondence of player i given that his/her beliefs are represented by a neo-additive capacity ν_i by $R_i(\nu_i)=R_i(\pi_i,\alpha_i,\delta_i):=\arg\max_{s_i\in S_i}V_i\left(s_i;\pi_i,\alpha_i,\delta_i\right).$

Definition 3.1 (Equilibrium under Ambiguity) *A pair of neo-additive capacities* (ν_1^*,ν_2^*) *is an Equilibrium Under Ambiguity (EUA) if for* $i=1,2$, $\operatorname{supp}(\nu_i^*)\subseteq R_{-i}(\nu_{-i}^*)$.

In equilibrium, each player assigns strictly positive likelihood to his/her opponent's best responses given the opponent's belief. However, each player lacks confidence in his/her likelihood assessment and responds in an optimistic way by over-weighting the best outcome, or in a pessimistic way by over-weighting the worst outcome. This notion of equilibrium is similar to that suggested by Dow and Werlang (1994) for 2-player games. Neither notion is more general than the other, since we allow for optimistic as well as pessimistic preferences, while the earlier papers allow for arbitrary but convex capacities.

Our notion of Equilibrium under Ambiguity has a close relation to standard Nash equilibrium. Indeed, for no ambiguity, $\delta_i=0$ for all $i=1,2$, this solution concept would coincide with Nash equilibrium of the game $G=\langle(S_i,u_i)_{i=1,2}\rangle$. Recall that Nash equilibrium can be interpreted as an equilibrium in beliefs. Let $\pi^*=(\pi_1^*,\pi_2^*)$ be a (possibly mixed) Nash equilibrium, then π_i^* is player i's additive belief over the opponent's strategy set with the property that any action in the support of π_i^* is a best-response given π_{-i}^*. Above we define an Equilibrium under Ambiguity as an equilibrium in non-additive beliefs (ν_1^*,ν_2^*). Equilibrium strategies are given by the supports of the capacities, which are required to be best-responses. If these are singleton sets, we have a *pure equilibrium.* If there are several strategies, which a player considers as equal best, then any combination of these is possible in equilibrium. For example in Matching Pennies, any combination of "heads " and "tails " will represent equilibrium behaviour as long as both players do not believe that the opponent would favour a particular choice. If there is no ambiguity, then Definition (3.1) specifies a pair of independent additive probability distributions, which is a mixed strategy Nash equilibrium.

There is a second connection between the EUA and Nash equilibrium. Consider the game $\Gamma(\delta, \alpha) = \langle (S_i, V_i(\cdot; \cdot, \delta_i, \alpha_i))_{i=1,2} \rangle$, where $\delta = (\delta_1, \delta_2)$ and $\alpha = (\alpha_1, \alpha_2)$. Hence $\Gamma(\delta, \alpha)$ is the "perturbed" game derived from $G = \langle (S_i, u_i)_{i=1,2} \rangle$ by replacing u_i with the functions, $V_i(s_i, s_{-i}, \delta_i, \alpha_i) = \delta_i \alpha_i M(s_i) + \delta_i (1 - \alpha_i) m(s_i) + (1 - \delta_i) u_i(s_i, s_{-i})$, for $i = 1, 2$. We claim that any pure strategy Nash equilibrium $s^* = (s_1^*, s_2^*)$ of the perturbed game $\Gamma(\delta, \alpha)$ is a pure strategy EUA, $\nu^* = (\nu_1^*, \nu_2^*)$ of the game G with degree of ambiguity δ_i and degree of optimism α_i, for $i = 1, 2$. Indeed, for each player i define a neo-additive capacity $\nu_i^* = \delta_i \alpha_i + (1 - \delta_i) \pi_i^*$ that assigns probability $\pi_i^*(s_{-i}^*) = 1$ to s_{-i}^*. Since (s_1^*, s_2^*) is a Nash equilibrium of $\Gamma(\delta, \alpha)$, we have $\mathrm{supp}(\nu_i^*) = \{s_{-i}^*\} \subseteq R_{-i}(\nu_{-i}^*)$. Thus ν^* is an Equilibrium under Ambiguity. This proves the following proposition:

Proposition 3.1 *For any pure strategy Nash equilibrium $s^* = (s_1^*, s_2^*)$ of $\Gamma(\delta, \alpha) = \langle (S_i, V_i(\cdot; \cdot, \delta_i, \alpha_i))_{i=1,2} \rangle$, there is a pure strategy Equilibrium under Ambiguity (EUA) $\nu^* = (\nu_1^*, \nu_2^*)$ of $G = \langle (S_i, u_i)_{i=1,2} \rangle$, in which player i has degrees of ambiguity δ_i and optimism α_i, and $\pi_i^*(s_{-i}^*) = 1$.*

This observation leads immediately to an existence result. In economic applications, players' strategy sets are mostly continuous variables, such as prices, quantities and investment expenditures. In such situations, pure strategy Nash equilibria exist.[7] To extend this idea to strategic games with ambiguity, we just need to ensure that the best (resp. worst) outcome is "well-behaved".

Definition 3.2 (Positive/Negative Externalities) *The game G has positive (resp. negative) externalities if $u_i(s_i, s_{-i})$ is increasing (resp. decreasing) in s_{-i}, for $i = 1, 2$.*

[7] We shall not consider mixed strategies in the present paper. Even for additive beliefs, the interpretation of mixed Nash equilibria is debatable, see Osborne and Rubinstein (1994). If beliefs are strictly non-additive, then behaviour, whether in pure or mixed strategies, cannot coincide with the strategies played, since there are no non-additive randomising devices.

If there are positive (resp. negative) externalities a good outcome will be interpreted as the opponent playing a high (resp. low) strategy. Many games relevant to economics have such a property. Proposition 3.2 shows that for such games EUA can be applied whenever Nash equilibrium can.

Proposition 3.2 (Existence) *If the game has positive or negative externalities, the strategy sets* $S_i \subseteq \mathbb{R}$ *are non-empty, compact and convex for all players* $i = 1, 2$, *and the payoff functions* $u_i(s_i, s_{-i})$ *are continuous in* s *and concave in each player's own strategy* s_i *for any* $s_{-i} \in S_{-i}$, *then there exists an Equilibrium under Ambiguity (EUA) in pure strategies.*

3.2 Example: Entry Deterrence

We illustrate EUA by the following example.[8] There are two players, an incumbent monopolist, I, and an entrant, E. If the entrant chooses not to enter, ne, (s)he will receive payoff 0 and the incumbent will receive the monopoly profits M. If the entrant enters the market, e, the incumbent has the choice of accommodating entry, a or fighting a price war, f. If the incumbent accommodates entry, both firms receive the duopoly profit d. Fighting entry causes both firms to sustain losses $-L$. The interaction between the incumbent and the entrant may be represented as the following normal form game:

		Incumbent	
		a	f
Entrant	e	d, d	$-L, -L$
	ne	$0, M$	$0, M$

where $M > d > 0$ and $L > 0$.

There are two Nash equilibria (without ambiguity), (a, e), and (f, ne). In the first, the incumbent accommodates and the entrant enters, while in the second the incumbent fights and the entrant stays out. Based on standard refinements such as sub-game perfection in an appropriately defined extensive form game, it is common to regard the latter equilibrium as less plausible. Once the entrant is in the industry, the incumbent will make lower profits by fighting

[8] The example also shows that convex sets of strategies are not necessary in Proposition 3.2.

than by accommodating.

Now we shall consider how ambiguity affects this example. We shall assume that agents are purely pessimistic, i.e. $\alpha = 0$. When there is ambiguity, we find a new type of equilibrium. In this, the incumbent does not fight. However entry does not occur because the entrant is pessimistic and perceives considerable ambiguity about the incumbent's behaviour. Consider the following beliefs: $\nu^E(a) = 1 - \delta^E$, $\nu^E(f) = 0$, $1 > \delta^E > \frac{d}{(d+L)}$, $\nu^I(e) = 0$, $\nu^I(ne) = 1 - \delta^I$, $1 \geqslant \delta^I > 0$. These beliefs show a high degree of pessimism for the entrant. In this case, the (Choquet) expected payoff of the incumbent is given by: $V^I(a) = M\left(1 - \delta^I\right) + d\delta^I$, $\quad V^I(f) = M\left(1 - \delta^I\right) - L\delta^I$. Hence, a is a best response for the incumbent. If $\delta^I > 0$, then f is not a best response for the incumbent. Since this holds for all $\delta^I > 0$, even small amounts of ambiguity are capable of eliminating non-credible threats.[9] The (Choquet) expected utility of the entrant is given by,

$$V^E(e) = d\left(1 - \delta^E\right) - L\delta^E, \qquad V^E(ne) = 0. \tag{3}$$

Thus ne is a best response for the entrant if and only if,

$$\delta^E \geq \frac{d}{(d+L)}. \tag{4}$$

We interpret this as saying the entrant will not enter if (s)he feels sufficient ambiguity about the situation. Equation (4) says that entry is more likely, the higher are the profits from successful entry d and the lower are the losses from a price war, L. In Nash equilibrium, entry is independent of these factors provided d and L are both positive. In our opinion, it is not implausible that these variables would affect the outcome.

The case of high ambiguity shows how deviations from Nash equilibrium can arise in EUA. The entrant considers it more likely that the incumbent will accommodate entry and this belief is sustained in equilibrium. It is possible that such a decision might be affected by ambiguity, since a firm will usually have much less information about an industry in which it does not already have a presence. In practice, entry is likely to entail considerable expenditure before

[9] This is true more generally see, Eichberger and Kelsey (2000), Proposition 5.1.

any returns are received. By definition, the entrant is not already in the industry. Thus (s)he may face some considerable ambiguity about relevant variables, in particular the behaviour of the incumbent. It is not implausible that entrants might react by behaving cautiously and not entering even if they do not expect the incumbent to fight a price war.

In standard Nash theory, the assumption that the incumbent will be more likely to play a, implies that it is optimal to play e, which yields the higher payoff of d. This need not be the case in an EUA, if the entrant is sufficiently pessimistic. Clearly, the possibility that the incumbent might fight entry, an event which is not in the support of the belief, influences the equilibrium outcome. Such behaviour is not implausible when players perceive ambiguity. If the entrant thinks that the incumbent will be cautious and accommodate, (s)he may still not be bold enough, to enter, since a misjudgment will earn him/her an outcome of $-L$.

Our intuition suggests that (ne, a) is not an implausible way to behave. We suspect, however, that the degree of ambiguity-aversion depends upon observations. As evidence builds up that the opponent plays a (or f respectively), confidence may grow and choosing e (ne) may become more likely. With a support notion, which insists that strategy combinations outside the support do not affect behaviour, (ne, a) can never be an equilibrium. It seems to us an advantage of EUA, that it opens the possibility to model such testable hypothesis.

3.3 Alternative Notions of Support and Equilibrium

In the literature there have been a number of solution concepts for games with ambiguity, see for instance Lo (1996) and Marinacci (2000). The key difference between the various proposals is that they use different definitions of support.[10] Most of the previous literature has explicitly or implicitly assumed ambiguity-aversion. For ease of comparison, we shall make a similar assumption here. Although there are a number of support notions, most of these coincide for a convex neo-additive capacity.

There are two main possibilities. We can define the support of a capacity to be a minimal set of capacity one (as in Lo (1996)) or we can define it to be a minimal set whose complement has capacity zero (as in Dow and Werlang (1994)). (For an ambiguity-averse neo-additive capacity, this coincides with the definition we have been using in the present paper.) If beliefs

[10]For discussion of alternative support concepts see Ryan (1997) and Eichberger and Kelsey (2006).

are non-additive, the two definitions will often not coincide.

The advantage of Lo's definition is that all strategies relevant to choice are included in the support. For instance, in the present paper, the best and worst outcomes influence choice but the strategies, which give rise to these outcomes, are not necessarily in the support of a player's beliefs. This would not be the case if we had used the other definition. However the Dow-Werlang definition has some advantages. Lo (1996) shows his support notion implies that the resulting equilibrium concept generically coincides with Nash equilibrium. This property makes Lo's solution concept unsuitable for studying deviations from Nash equilibrium due to ambiguity.[11] The reason for this is the Lo-support is not itself an ambiguous set, hence players do not perceive ambiguity about whether their opponents play best responses in his framework.

4 Oligopoly Models

In this section, we shall present some examples of how these techniques can be used to examine the effect of ambiguity on economic behaviour. These examples will illustrate that the consequences of ambiguity can be examined without technical sophistication.

4.1 Cournot Oligopoly

4.1.1 Equilibrium Under Ambiguity

First we consider a symmetric Cournot duopoly with homogenous goods. We show that, in this case, optimism increases competition because it induces more aggressive behaviour. Pessimism will, in general, have the opposite effect.

There are two firms, $i = 1, 2$, which compete in quantities. Assume that firm i faces the linear inverse demand curve $p_i(x_i, x_{-i}) = \max\{1 - x_i - x_{-i}, 0\}$. We shall assume that each firm can produce at constant marginal cost equal to c. Firm i chooses the quantity it wants to supply, x_i, from the interval $[0, 1]$. If beliefs are represented by neo-additive capacities, a firm over-weights the best and worst outcomes. We assume that firm i perceives the worst scenario to be a situation, where its rival dumps a large quantity on the market, driving the price down to zero. The firm's perceived best outcome is assumed to be where the rival produces zero

[11] In fairness this was not the stated aim of Lo's paper.

output and the firm is a monopolist. Under these assumptions firm i's (Choquet) expected profit is:

$$V_i(x_i, x_{-i}) = \delta_i \alpha_i x_i (1 - x_i) + (1 - \delta_i) x_i [1 - x_{-i} - x_i] - c x_i. \tag{5}$$

A possible criticism of this model is that the choice of the best and the worst outcome is arbitrary. However our results remain true provided the best (resp. worst) outcome is below (resp. above) the Nash equilibrium output.[12]

The first order condition for maximising firm 1's profit is,

$$\frac{\partial V_1}{\partial x_1} = \delta_1 \alpha_1 (1 - 2x_1) + (1 - \delta_1)(1 - 2x_1 - x_2) - c = 0.$$

Hence the reaction function of firm 1 is given by,

$$R^1(x_2) = \frac{\delta_1 \alpha_1 + (1 - \delta_1)(1 - x_2) - c}{2(1 - \delta_1 + \delta_1 \alpha_1)}. \tag{6}$$

Having obtained the reaction function, one can easily derive the equilibrium values of price and quantity. Here, we illustrate this claim for the case of a symmetric equilibrium[13].

Proposition 4.1 *In a symmetric equilibrium of the Cournot model, where $\delta_1 = \delta_2 = \delta$ and $\alpha_1 = \alpha_2 = \alpha$, the equilibrium output, price and profit are given by:*

$$\begin{aligned} \bar{x} &= \frac{\delta(\alpha - 1) + 1 - c}{3(1 - \delta) + 2\delta\alpha}, \quad \bar{p} = \frac{1 - \delta + 2c}{3(1 - \delta) + 2\delta\alpha}, \\ \bar{\pi} &= (\bar{p} - c)\bar{x} = \frac{[(1-\delta)(1-c) + 2\delta(1-\alpha)c][\delta(\alpha - 1) + 1 - c]}{[3(1-\delta) + 2\delta\alpha]^2}. \end{aligned}$$

Proof. From the reaction curve, $\bar{x} = \frac{\delta\alpha + (1-\delta)(1-\bar{x}) - c}{2(1-\delta+\delta\alpha)}$. Thus $\bar{x}\frac{2(1-\delta+\delta\alpha)+1-\delta}{2(1-\delta+\delta\alpha)} = \frac{\delta\alpha+1-\delta-c}{2(1-\delta+\delta\alpha)}$, which implies $\bar{x} = \frac{\delta\alpha+1-\delta-c}{3(1-\delta)+2\delta\alpha}$. The equilibrium price is given by, $\bar{p} = 1 - 2\bar{x} = \frac{3(1-\delta)+2\delta\alpha-2\delta(\alpha-1)-2+2c}{3(1-\delta)+2\delta\alpha} = \frac{1-\delta+2c}{3(1-\delta)+2\delta\alpha}$. ■

Consider the following special cases of Proposition 4.1. For $\delta = 0$, the firms experience no ambiguity. Hence, the degree of optimism does not affect their behaviour and we obtain the well-known symmetric Nash equilibrium $\bar{x}^c = \frac{1-c}{3}$, $\bar{p}^c = \frac{1+2c}{3}$, and $\bar{\pi}^c = \frac{(1-c)^2}{9}$. In contrast, for

[12] See Eichberger and Kelsey (2002) Proposition 3.1, for a related result, which does not depend on such assumptions.

[13] The asymmetric case where some firms are optimistic and others are pessimistic has been studied in **?**.

complete ambiguity, $\delta = 0$, one obtains:

$$\bar{x} = \frac{\alpha - c}{2\alpha}, \quad \bar{p} = \frac{c}{\alpha}, \; \overline{\pi} = \frac{(1-\alpha)(\alpha - c)\, c}{2\alpha^2}.$$

Firms will only supply if they are sufficiently optimistic, $\alpha > c$. With extreme optimism $\alpha = 1$ and strong ambiguity about the behaviour of the opponent $\delta = 1$, both firms behave like a monopolist $\bar{x} = \frac{1-c}{2}$, which drives down their profit to zero.[14]

The next result studies the comparative statics of a change in ambiguity-attitude on equilibrium prices and quantities. An increase in optimism will increase the weight the firm puts on its rivals producing a low output. This increases the marginal benefit of producing more and hence results in an increase in equilibrium output.

Proposition 4.2 *Suppose firms face some ambiguity about the behaviour of the opponent,* $\delta > 0$. *In Cournot equilibrium an increase in optimism,* α :

1. *increases output and decreases prices;*

2. *decreases equilibrium profits, provided* $\bar{x} > \frac{1}{4}(1-c)$.

Proof. By inspection, $\bar{p}$ is an decreasing function of α. Since $\bar{x} = \frac{1}{2}(1 - \bar{p})$, it follows that $\bar{x}$ is an increasing function of α.

Symmetric equilibrium profits are given by, $V = (p(\bar{x}(\alpha)) - c)\,\bar{x}(\alpha)$, hence $\frac{\partial V}{\partial \alpha} = \frac{\partial \bar{x}}{\partial \alpha}\left[p(\bar{x}(\alpha)) - c + \frac{dp}{d\bar{x}}\bar{x}(\alpha)\right] = \frac{\partial \bar{x}}{\partial \alpha}\left[1 - c - 4\bar{x}(\alpha)\right]$. Thus provided $\bar{x} > \frac{1}{4}(1-c)$ an increase in α decreases profit. ■

Intuitively, more optimism causes a firm to place more weight on the possibility that its rival will produce a low output. This increases the marginal profitability of extra output. Thus the given firm will produce more. This reasoning is not restricted to the specific demand and cost functions but will apply whenever Cournot oligopoly is a game of strategic substitutes. The condition $\bar{x} > \frac{1}{4}(1-c)$ says that the effects of ambiguity are relatively small, in the sense that they do not induce firms to produce less than the collusive output. We would view this as the normal case.

[14] It is an artifact of our numerical example that the price would equal marginal cost in this case.

Comparative static effects with respect to ambiguity naturally depend on the degree of optimism α. The following result characterises this effect.

Proposition 4.3 *In Cournot equilibrium an increase in ambiguity, δ :*

1. *increases output and decreases prices if optimism is sufficiently high, $\alpha > \frac{3c}{1+2c}$;*

2. *decreases output and increases prices, if optimism is low, $\alpha < \frac{3c}{1+2c}$.*

Proof. Differentiating $\overline{x}$ with respect to δ yields $\frac{\partial \overline{x}}{\partial \delta} = \frac{\alpha - 3c + 2\alpha c}{(3(1-\delta)+2\delta\alpha)^2}$. Hence, $\frac{\partial \overline{x}}{\partial \delta} \gtrless 0$ iff $\alpha \gtrless \frac{3c}{1+2c}$. The price effect follows because the inverse demand function is decreasing in quantity. ■

To illustrate this result consider the case where there is no optimism $\alpha = 0$. By equation (6), the symmetric equilibrium is characterised by $\bar{x} = \frac{(1-\delta)(1-\bar{x})-c}{2(1-\delta)}$ or

$$\frac{1}{2} - \frac{3}{2}\bar{x} = \frac{c}{2(1-\delta)}. \tag{7}$$

Assume that there is an increase in ambiguity, i.e. δ rises. Then the rhs. of equation (7) increases. Since the lhs. of equation (7) is decreasing in $\bar{x}(\delta)$, $\bar{x}$ must be a decreasing function of δ. An increase in ambiguity will decrease the quantities in a symmetric Cournot equilibrium, as depicted in Figure 1. As firms are symmetric, EUA are intersections of the best response function with the 45-degree line.

With increasing ambiguity, pessimism reduces the amount brought to market. Intuitively, ambiguity makes a decision-maker cautious about the behaviour of the opponent. By dumping output onto the market, the rival can drive down the price. If firms become more concerned about this possibility, they will reduce output in order to avoid the potential losses.

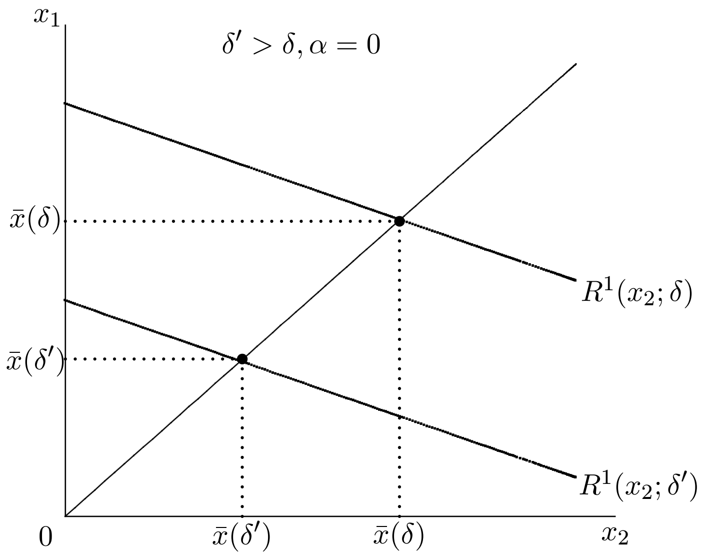

Figure 1: Cournot equilibrium with pessimistic firms

4.1.2 Strategic Delegation

In this section we show that it may be profitable to delegate decision-making to a manager who is more optimistic than the owner of the firm. This allows the owner to commit to producing a larger output, which is advantageous in a game of strategic substitutes. Hiring an optimistic manager has a similar effect to giving the manager an incentive to maximise a weighted sum of profits and revenue, as discussed in Vickers (1985).

Assume that firm 1 has a profit maximising owner who is ambiguity neutral, i.e., has additive beliefs. The owner hires a manager to operate the firm on his/her behalf. The owner pays him a wage, which is fraction θ of firm 1's profit. The manager has CEU preferences and has beliefs represented by a neo-additive capacity. The owner chooses the manager to maximise his/her profit. Firm 2 is a conventional profit maximising firm. The following result finds the degrees of optimism, α_1 and ambiguity, δ_1, which are optimal for the owner of firm 1.

Proposition 4.4 *The profit maximising levels of α_1 and δ_1 satisfy:*

$$\alpha_1 = \frac{1 - c - \delta_1 + 5c\delta_1}{4c\delta_1}. \tag{8}$$

Proof. Profit is maximised where the equilibrium output of firm 1 is equal to that of a Stackelberg leader, which is $\frac{1}{2}(1-c)$. Thus by Lemma A.1,[15] $\frac{1-\delta_1+2\delta_1\alpha_1-(1+\delta_1)c}{3(1-\delta_1)+4\delta_1\alpha_1} = \frac{1}{2}(1-c)$. Cross multiplying and simplifying we obtain: $4\delta_1\alpha_1 c - 2c - 2\delta_1 c = 1 - \delta_1 - 3c + 3c\delta_1$, from which the result follows. ■

To understand this result, consider the special case of a manager who feels complete ambiguity, $\delta_1 = 1$. In this case, the manager should also be extremely optimistic, $\alpha_1 = 1$. Being a complete optimist, the manager will assign weight one to the possibility that his/her opponent will produce zero output and will himself produce the monopoly output. This is desirable, since in this example, the monopoly output coincides with that of a Stackelberg leader, which is the most profitable output. In general, for managers who have a lower degree of ambiguity δ_1, less optimism according to equation (8) will be optimal. It is not very likely that a manager would assign the decision-weight one to the possibility that the opponent will produce zero output.

[15] See the online appendix.

Even so profit can be raised by delegating to a manager who is more optimistic than the owner. From the point of view of the owner, there is an additional advantage of hiring an optimistic manager. The more optimistic the manager the lower the fraction of the profit, θ, needs to be paid to the manager to induce him/her to work. This second effect also implies that it is advantageous to hire an optimistic manager.

4.2 Bertrand Oligopoly

4.2.1 Equilibrium Under Ambiguity

We shall now consider price (Bertrand) competition. Consider 2 firms producing heterogeneous goods, which are close (but not perfect) substitutes. Firm i can produce at constant marginal and average cost, $k > 0$. Firm i charges price p_i for its output. We assume that firm i faces a linear demand curve: $D_i(p_i, p_{-i}) = \max\{0, a + bp_{-i} - cp_i\}$, $a, b, c > 0$, $a > k$, $2c > b$.

Each firm perceives its rival's behaviour as ambiguous and has beliefs represented by neo-additive capacities. Suppose that each firm perceives the worst case to be where its rival reduces price to marginal cost. The best case is perceived to be where the rival firm sets a high price K, which is above the Nash equilibrium level, i.e., $K > \frac{(a+ck)}{(2c-b)}$. We require $a + bk - cK > 0$ to ensure that demand is positive at all quantities in the firms' strategy sets. Hence we assume, a firm's strategy set is the interval $[k, K] \subseteq \mathbb{R}$ for some sufficiently high K.

With these assumptions the (Choquet) expected profit of firm i becomes:

$$V_i = (1-\delta_i)(p_i - k)(a + bp_{-i} - cp_i) + \delta_i (1-\alpha_i)(p_i - k)(a + bk - cp_i) + \delta_i\alpha_i (p_i - k)(a + bK - cp_i),$$

which can be simplified to

$$V_i = (p_i - k)(a - cp_i) + (p_i - k)\, b\left[(1-\delta_i) p_{-i} + \delta_i (1-\alpha_i) k + \delta_i\alpha_i K\right].$$

Using the first-order condition for profit maximisation, $\frac{\partial V_i}{\partial p_i} = a - cp_i - c(p_i - k)$ $+ b\left[(1-\delta_i) p_{-i} + \delta_i (1-\alpha_i) k + \delta_i\alpha_i K\right] = 0$, one obtains firm i's reaction function,

$$\rho^i(p_{-i}) = \frac{a + b\left[(1-\delta_i) p_{-i} + \delta_i (1-\alpha_i) k + \delta_i\alpha_i K\right] + ck}{2c}. \tag{9}$$

Equation (9) defines a non-singular system of linear equations, hence the Bertrand equilibrium is unique. Since $K \geqslant p_{-i} \geqslant k$, an increase in α_i will shift firm i's reaction curve up and hence increase the equilibrium price. The price of the rival firm will also increase, since reaction curves slope upwards. Consider firm 1. An increase in optimism causes it to place more weight on good outcomes. In this context, a good outcome would be firm 2 charging a high price. Since the model exhibits strategic complementarity, this gives firm 1 an incentive to increase its price. This discussion is summarised in the following proposition.

Proposition 4.5 *If firm i perceives its opponent's price policy to be ambiguous, $\delta > 0$, then in Bertrand oligopoly an increase in optimism, α_i, of firm i causes both firms to set higher prices in equilibrium.*

In contrast to the Cournot case, this result does not depend on symmetry of the firms. On the other hand, the comparative static effect of an increase in ambiguity δ depends on the degree of optimism of both firms in equilibrium. Assuming the firms have similar attitudes to ambiguity, $\alpha_1 = \alpha_2 =: \alpha$, and feel the same degree of ambiguity, $\delta_1 = \delta_2 =: \delta$, it is straightforward to compute the EUA of the Bertrand game as:

$$\overline{p} = \frac{a + ck + \delta b\left[\alpha K + (1-\alpha)k\right]}{2c - (1-\delta)b}. \tag{10}$$

Notice that for $\delta = 0$, one obtains the usual Nash equilibrium of Bertrand oligopoly, $\overline{p}^B = \frac{a+ck}{2c-b}$. In order to investigate the comparative static effect of an increase in ambiguity, it is useful to rewrite equation (10) as follows:

$$\overline{p} = \left(\frac{2c-b}{2c-b+\delta b}\right)\overline{p}^B + \left(\frac{\delta b}{2c-b+\delta b}\right)\left[\alpha K + (1-\alpha)k\right].$$

The equilibrium price of the symmetric EUA is a weighted average of the usual Bertrand equilibrium price $\overline{p}^B$, and the best and worst price policy of the opponent $\left[\alpha K + (1-\alpha)k\right]$. The less ambiguity a firm perceives about the opponent the closer is its behaviour to the Nash equilibrium. On the other hand, the more ambiguous the firm is, the more its price will be determined by its optimistic or pessimistic view of the opponent's behaviour. The weight $\frac{2c-b}{2c-b+\delta b}$ is clearly decreasing in δ. Hence an increase in ambiguity will shift more weight to the ambiguity attitude $[\alpha K + (1-\alpha)k]$. Whether such a shift will increase or decrease the equilibrium price will depend upon whether $\overline{p}^B$ is larger or smaller than $[\alpha K + (1-\alpha)k]$.

Proposition 4.6 *In a symmetric EUA of the Bertrand game, an increase in ambiguity δ about the other firm's price policy will increase (resp. decrease) the equilibrium price $\overline{p}$ if $\alpha > \bar{\alpha}$ (resp. $\alpha < \bar{\alpha}$), where $\bar{\alpha} = \frac{\overline{p}^B - k}{K-k}$.*

Figure 2 illustrates the result of Proposition 4.6 for the case of pure pessimism, $\alpha = 0$. An increase in ambiguity causes the best-response function to shift down and the slope to decrease. Firms have their own markets in which to react to the other's price. Uncertainty about the other price is equivalent to uncertainty about a firm's own demand. The lower a given firm sets the price, the smaller the market the opponents will face. Firms' concern about low demand in their respective market, provides an incentive for charging lower prices than in a conventional (Bertrand) equilibrium. Hence, pessimism tends to increase the competitiveness of Bertrand markets. For firms which are sufficiently optimistic, the comparative statics are reversed, as illustrated in figure 3.

In Bertrand oligopoly there is also a strategic advantage from delegating decisions to an optimistic manager. This can be seen from equation (9), which shows that an increase in α_i will increase the equilibrium prices of both firms. In Bertrand oligopoly, a given firm can gain a strategic advantage by committing to price above the equilibrium level, see Fershtman and Judd (1987). This causes rivals to raise their prices, which gives the first firm an indirect benefit since its profits are higher the greater the prices of its rivals. Appointing an optimistic manager would be one way to commit to a high price and hence increase profits.

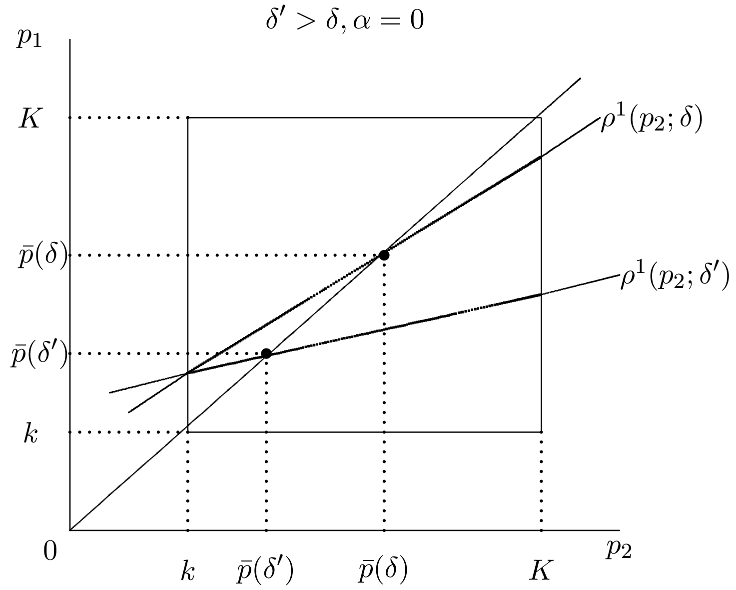

Figure 2: Bertrand equilibrium with pessimistic firms

5 Peace-Making

We believe that throughout social sciences there are many natural applications for our formal framework of ambiguity, since game theory can be applied in many non-economic contexts. To illustrate this point, we next consider a model of peace-making. This model also allows us to study the impact of ambiguity in a game with multiple equilibria.[16]

5.1 A Peace-Making Game

Consider two players $i = 1, 2$ interpreted as the parties involved in a conflict. Each player i chooses a strategy $s_i \in S_i = [0, 1]$. We interpret $s_i = 0$ as no effort, whereas $s_i = 1$ corresponds to individual i making the highest possible effort to establish peace. Higher values of s_i correspond to greater peace-making efforts by individual i. We assume that benefits from peace-making have the following form:

$$b(s_1, s_2) = \begin{cases} 1 & \text{if} \quad s_1 = s_2 = 1, \\ \varepsilon (s_1 + s_2)^2 & \quad \text{otherwise,} \end{cases} \tag{11}$$

where $0 < \varepsilon < \frac{1}{4}$. The benefits from peace making are increasing in the efforts of both parties, hence there are positive externalities. An effort towards peace by one party brings benefits to both. The benefit function is convex, which implies there is strategic complementarity in peace-making. The more effort is supplied by one party, the greater the marginal benefit of peace-making for the other. The discontinuity at $\langle 1, 1 \rangle$ indicates that there is a qualitative difference between peace and a war of very low intensity.

Peace-making can be costly. For example, decision makers may face political pressure and/or threats from extremists in their own camp. For simplicity we assume the costs are linear cs_i, $c > 0$. The payoff function u of either party $i = 1, 2$ is written,

$$u_i(s_i, s_j) = b(s_i, s_j) - cs_i. \tag{12}$$

[16] This example also shows that continuous payoff functions are not necessary in Proposition 3.2.

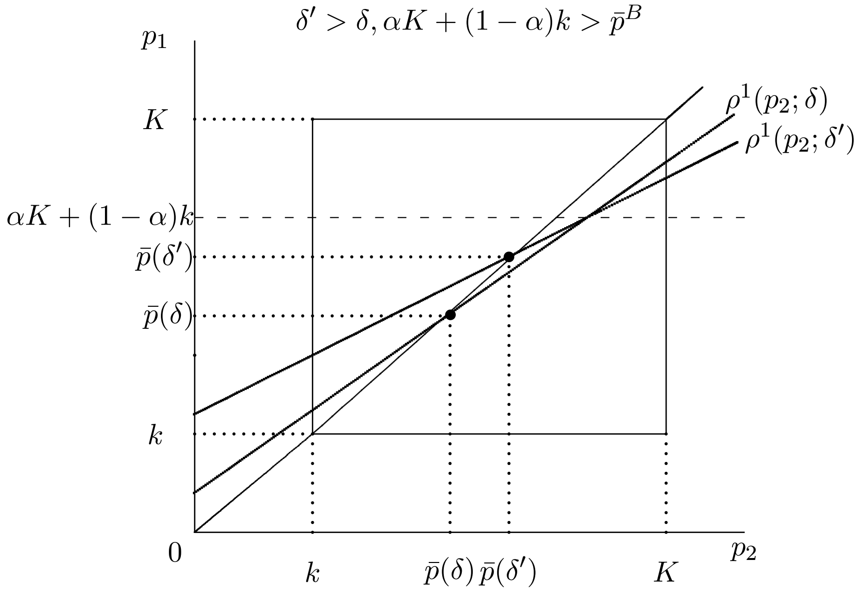

Figure 3: Bertrand equilibrium with optimistic firms

This game is denoted by $G^P = \langle (S_i, u_i)_{i=1,2} \rangle$. Peace is viewed as a public good produced with increasing returns to scale. The following result characterises the Nash equilibria of the peace game. This is the benchmark without ambiguity.

Proposition 5.1 *Solutions without ambiguity of the peace-making game G^P, are characterised as follows:*

1. *if $\varepsilon > c$, then full peace-making effort is the strictly dominant strategy for $i = 1, 2$;*

2. *if $1 - \varepsilon \geqslant c \geqslant \varepsilon$ then there exist two Nash equilibria in pure strategies one with full peace-making effort and one where no effort is supplied by either party;*[17]

3. *if $c > 1 - \varepsilon$, then no effort is the strictly dominant strategy for $i = 1, 2$.*

Case (3) describes a situation where each side views the benefits of peace as being less than the costs of peace-making, regardless of what the other party does. Consequently peace is not established. Such a situation would arise if benefits from peace are small compared to costs of peace-making efforts. This does not seem a realistic representation of situations such as in the Middle East or Northern Ireland, where it seems most people perceive peace as worth achieving if possible.

Case (1) is the non-problematic case. Benefits from efforts toward peace are always strictly larger than the costs. Hence both parties provide full effort and peace is established. Again this does not appear to be a reasonable model of a conflict situation in which peace is difficult to achieve.

Case (2) is the interesting intermediate case. There are substantial benefits from peace but benefits from intermediate peace-making efforts are not enough to justify the costs. This seems to fit the circumstances in many peace processes. Two Nash equilibria in pure strategies arise, one in which the conflict ends because both parties engage in full peace-making efforts and one in which no effort is made and hence peace is not achieved.

[17] There is also a Nash equilibrium in mixed strategies where parties mix between zero and full peace-making effort.

5.2 Peace-Making with Ambiguity

Now we study the impact of ambiguity on peace-making. The following proposition shows more optimism makes a successful peace process more likely. In the case where Nash equilibrium is not unique, ambiguity can play a role in equilibrium selection. If the degree of optimism is sufficiently high, there is a unique equilibrium in which the peace process succeeds. Pessimism has the opposite effect. If there is enough pessimism, peace will not be established.

Proposition 5.2 *The impact of ambiguity in the peace-making game, G^P, is as follows:*

1. *if $\varepsilon \geqslant c$, then any in equilibrium under ambiguity both parties make full effort towards establishing peace, i.e., $s_i = 1$, $i = 1, 2$;*

2. *if $1 - \varepsilon > c > \varepsilon$, then any equilibrium under ambiguity involves only strategies $s_i = 1$ or $s_i = 0$, $i = 1, 2$. Moreover, there exists $\bar{\lambda}$ (resp. $\bar{\gamma}$) such that if $\delta\alpha \geqslant \bar{\lambda}$ (resp. $\delta(1-\alpha) \geqslant \bar{\gamma}$) then $s_i = 1$ (resp. $s_i = 0$) is the unique equilibrium strategy for $i = 1, 2$;*

3. *If $c \geqslant 1 - \varepsilon$, then in any equilibrium under ambiguity neither party puts any effort into peace-making, i.e., $s_i = 0$, $i = 1, 2$.*

We believe that case (2) is the relevant one when peace-making poses a serious political problem. It is in this case, that ambiguity makes a difference. If the degree of ambiguity, δ, is sufficiently high and agents are pessimistic (i.e. α is low) peace-making efforts will break down. On the other hand, high ambiguity and optimism (or more confidence in the actions of the other side) can cause the peace-process to be successful.

6 General Results

In this section we present some more general results for 2-player games with strategic complements. We consider a game with 2-players, $i = 1, 2$. In particular let $S_i = [\underline{s}_i, \bar{s}_i] \subseteq \mathbb{R}$ for $i = 1, 2$ and $S = S_1 \times S_2$. Player i has a concave utility function $u^i(s_1, s_2)$ and has beliefs on S_{-i} represented by a neo-additive capacity $\nu_i = \delta_i \alpha_i + (1 - \delta_i)\pi_i$. The following assumption is maintained throughout this section.

Assumption 6.1 (Strict Concavity) *For all $s_1, s_2 \in S$, $u^i_{11}(s_1, s_2) < 0$, $i = 1, 2$.*

Proposition 6.1 (Uniqueness) *Under Assumption 6.1 and positive or negative externalities, if the amount of ambiguity perceived by player i, δ_i is sufficiently large then the equilibrium under ambiguity is unique.*

This may be explained as follows. If a given player believes his/her opponents' behaviour to be more ambiguous, then that player's behaviour becomes less responsive to changes in their strategies. Thus the best-response functions become flatter, which results in a unique equilibrium. Next we investigate the comparative statics of changing ambiguity attitudes. To get unambiguous results we assume strategic complementarity.

Assumption 6.2 (Strategic Complementarity) *The game G has strategic complements, if $u^i_{12}(s_1, s_2) > 0$, for $i = 1, 2$.*[18]

Strategic complementarity says that if a given player increases his/her strategy this raises the marginal benefit to his/her opponents of increasing their own strategies. If a given player becomes more optimistic (s)he will place higher weight on good outcomes. If there are positive externalities a good outcome will be interpreted as the other player playing a high strategy. In the presence of strategic complementarity the given player has an incentive to increase his/her strategy. If equilibrium is unique, an increase in optimism will increase equilibrium strategies of both players. If equilibrium is not unique we get a similar result. The *set of equilibria* increases, in the sense that the strategies played in the highest and lowest equilibria increase.

Proposition 6.2 (Comparative Statics) *Under Assumptions 6.2 and positive externalities the strategies of both players in the highest and lowest equilibria are increasing functions of α_1 and α_2.*

We obtain following corollary from Propositions 6.1 and 6.2:

Corollary 6.1 (Equilibrium Selection) *Under Assumptions 6.2 and positive externalities, if $\delta_i\alpha_i$ is sufficiently large, then equilibrium is unique and is larger than the largest equilibrium without ambiguity. If $\delta_i(1-\alpha_i)$ is sufficiently large, then equilibrium is unique and is smaller than the smallest equilibrium without ambiguity.*

[18] As usual u^i_{12} denotes $\frac{\partial^2 u^i}{\partial s_1 \partial s_2}$, etc.

Thus ambiguity can act to select equilibria. Under positive externalities, if there is a high degree of ambiguity and agents are sufficiently optimistic, all will focus on an equilibrium in which high strategies are played. The assumption of positive externalities and strategic complementarity implies that the highest equilibrium is Pareto superior. In this case, optimism would select the equilibrium with the highest level of economic activity. As usual, pessimism has the opposite effect. If there are negative externalities the comparative statics are reversed. For instance, an increase in optimism would tend to reduce equilibrium strategies.

Without further assumptions, Proposition 6.2 can not directly be extended to games with strategic substitutes, i.e., games for which $u^i_{12}(s_1, s_2) < 0$, even in the case of two players. Although one can show that for positive externalities, best-response strategies decrease in the degree of optimism, the change of equilibrium depends on the relative shift of those best-responses correspondences.

It is worth noting that ambiguity and ambiguity-attitude have distinct comparative static effects. Increases in ambiguity, as measured by δ, cause multiple equilibria to collapse into a single equilibrium, while ambiguity-attitude, as measured by α, causes the equilibrium strategies to rise or fall.

7 Concluding Discussion

We introduce a simple model of ambiguity in strategic games and show how it can be applied to many situations of interest in economics and social sciences. New results are derived for both, optimistic as well as pessimistic attitudes towards ambiguity. The applications here were chosen to represent cases of strategic substitutes (Cournot equilibrium), strategic complements with a unique equilibrium (Bertrand equilibrium) and strategic complements with multiple equilibria (peace-making).[19]

Some related results can be found in Eichberger and Kelsey (2002). The present paper extends those results since they apply to optimistic as well as pessimistic attitudes to ambiguity. This has enabled us to establish comparative static results demonstrating the effects of varying ambiguity and ambiguity-attitude independently. In addition the present paper has continu-

[19] Extensions to more than two players are possible but introduce technical complications concerning the product capacity, see Eichberger and Kelsey (2000) and Eichberger and Kelsey (2002).

ous rather than discrete strategy spaces. Eichberger and Kelsey (2002) confined attention to symmetric equilibria of symmetric games, assumptions not used in our paper. Moreover, they established uniqueness of equilibrium with a high degree of ambiguity only when a restrictive assumption was satisfied (Assumption 3.2 of Eichberger and Kelsey (2002)). The results in Eichberger and Kelsey (2002) are proved for general pessimistic CEU preferences. They provide some further applications. For example, in a model of voluntary contributions to public goods it is shown that ambiguity increases the provision of public goods.

We believe that our approach is simple and intuitive in order to be applicable to various problems in economics and social sciences. The applications, presented in this paper, can only serve to illustrate the type of results which one can expect to obtain by including ambiguity in economic analysis. Our intuition suggests that the conclusions obtained are not unreasonable. So far, there exists experimental evidence mainly for the impact of ambiguity-aversion on individual decision making, see for instance Kilka and Weber (2001). However experimental evidence that ambiguity does affect behaviour in games can be found in **?** and Eichberger, Kelsey, and Schipper (2007). The present paper provides some testable hypotheses. For instance we show that ambiguity has the opposite effect in games of strategic complements and substitutes. Eichberger, Kelsey, and Schipper (2007) find some support for this hypothesis.

Supplementary Material

Supplementary material (the Appendix) is available on-line at the OUP website.

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A Appendix

APPENDIX TO AMBIGUITY AND SOCIAL INTERACTION

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A.1 Existence

Proof of Proposition 3.2 By Proposition 3.1, it suffices to show that a pure strategy Nash equilibrium of the game $\Gamma(\delta, \alpha)$ exists. By standard arguments, i.e., Glicksberg (1952), $\Gamma(\delta, \alpha)$ has a pure strategy Nash equilibrium if for $i = 1, 2$, S_i is a non-empty compact convex subset of an Euclidian space and V_i is continuous in s and quasi-concave in s_i. We need to show that V_i is quasi-concave in s_i, all other properties are straightforward.

Since G has positive externalities, u_i is continuous in s and $S_{-i} \subseteq \mathbb{R}$ is non-empty and compact, we have for any $s_i'' \in S_i$ that $\max S_{-i} \in \arg\max_{s_{-i} \in S_{-i}} u_i(s_i'', s_{-i})$ and $\min S_{-i} \in \arg\min_{s_{-i} \in S_{-i}} u_i(s_i'', s_{-i})$. Together with u_i being concave in s_i for each $s_{-i} \in S_{-i}$, it follows that M_i and m_i are concave in s_i. Finally, note that for any δ_i, α_i with $0 \leqslant \alpha_i \leqslant 1$ and $0 \leqslant \delta_i \leqslant 1$, V_i is a positively weighted sum of functions concave in s_i, which implies that V_i itself is concave in s_i. The proof for negative externalities is similar. ■

A.2 Oligopoly

Lemma A.1 *Assume that the manager of firm 2 perceives no ambiguity (i.e., $\delta_2 = 0$), while the manager of firm 1 is not necessarily ambiguity neutral. Then under Cournot quantity competition, the equilibrium output of firm 1 is given by:* $\bar{x}_1 = \frac{1-\delta_1+2\delta_1\alpha_1-(1+\delta_1)c}{3(1-\delta_1)+4\delta_1\alpha_1}$.

Proof. From equation (6), firm 1's (resp. 2's) reaction function is given by, $R^1(x_2) = \frac{\delta_1\alpha_1+(1-\delta_1)(1-x_2)-c}{2(1-\delta_1+\delta_1\alpha_1)}$, (resp. $R^2(x_1) = \frac{1-c-x_1}{2}$). Solving for equilibrium in the usual way we obtain: $x_1 = \frac{2\delta_1\alpha_1+(1-\delta_1)(1+c+x_1)-2c}{4(1-\delta_1+\delta_1\alpha_1)}$ or $\left[\frac{4(1-\delta_1+\delta_1\alpha_1)-(1-\delta_1)}{4(1-\delta_1+\delta_1\alpha_1)}\right] x_1 = \frac{1-\delta_1+2\delta_1\alpha_1-(1+\delta_1)c}{4(1-\delta_1+\delta_1\alpha_1)}$, from which the result follows. ■

Lemma A.2 *Provided $K \geqslant \frac{10+9k}{14}$, in Example 2 the optimal value of δ_1 is*

$$\delta_1 = \frac{10 - 5k}{56K - 38 - 37k} \geqslant 0. \tag{13}$$

Proof. By equation (9), firm 2's reaction function is given by $p_2 = \frac{2+p_1+2k}{4}$. Firm 1's reaction function is given by $p_1 = \frac{2+2k+\delta_1K+(1-\delta_1)p_2}{4}$ (after substituting $\alpha_1 = 1$). Solving for equilibrium, $p_1 = \frac{8+8k+4\delta_1K+(1-\delta_1)(2+2k)+(1-\delta_1)p_1}{16}$. Hence $p_1 = \frac{8+8k+4\delta_1K+(1-\delta_1)(2+2k)}{16-(1-\delta_1)}$.

Profit is maximised when the equilibrium price is equal to the price which would be chosen by a price leader without ambiguity. Hence $\frac{8+8k+4\delta_1K+(1-\delta_1)(2+2k)}{16-(1-\delta_1)} = \frac{10+9k}{14}$. Solving for δ_1 : $112 + 112k + 56\delta_1K + 28(1-\delta_1)(1+k) = 160 - 10(1-\delta_1) + 144k - 9k(1-\delta_1)$ simplifying, $56\delta_1K - \delta_1 38 - \delta_1 37k = 10 - 5k$, from which the result follows.

Note that $56K - 38 - 37k \geqslant 56\left(\frac{10+9k}{14}\right) - 38 - 37k = 2 - k \geqslant 0$, since $2 = a > k$. (Recall $\frac{10+9k}{14}$ is the output a price leader would choose.) ■

A.3 Peace-making

Proof of Proposition 5.1 Since b is convex, any party's best response is either $s_i = 0$ or $s_i = 1$.

Case (1) If $s_2 \neq 1$, $u(1, s_2) - u(s_1, s_2) = \varepsilon (1 + s_2)^2 - c - \left[\varepsilon (s_1 + s_2)^2 - cs_1\right]$ $= (1 - s_1)[\varepsilon (1 + s_1 + 2s_2) - c] > 0$, since, by assumption, $\varepsilon > c$. The case $s_2 = 1$ can be covered as follows: $u(1,1) - u(s_1, 1) = 1 - c - \left[\varepsilon (s_1 + 1)^2 - cs_1\right] = (1 - 4\varepsilon) + 2\varepsilon (1 - s_1) + \varepsilon \left(1 - s_1^2\right) - c (1 - s_1) = (1 - 4\varepsilon) + (1 - s_1)[2\varepsilon + \varepsilon (s_1 + 1) - c] > 0$ since $\varepsilon > c$.

Case (2) To show that $s_1 = s_2 = 1$ is a Nash equilibrium, by convexity of b it is enough to show $u(1, 1) \geqslant u(0, 1)$. This holds since, $u(1, 1) - u(0, 1) = 1 - \varepsilon - c \geqslant 0$. Now $u(0, 0) = 0$, $u(1, 0) = \varepsilon - c \leqslant 0$, by assumption, which implies that $s_1 = s_2 = 0$ is also a Nash equilibrium.

Case (3) If $s_1 \neq 1$, $u(0, s_2) - u(s_1, s_2) = cs_1 - \varepsilon \left(2s_1s_2 + s_1^2\right) \geqslant c - 3\varepsilon > 0$, since $c > 1 - \varepsilon$ implies $c > \frac{3}{4} \geqslant 3\varepsilon$. The remaining case follows since $u(1, 1) - u(0, 1) = 1 - \varepsilon - c < 0$. ■

Proof of Proposition 5.2 Cases (1) and (3) follow from Proposition 5.1 and the observation that CEU preferences respect strict dominance.

Case (2) By convexity the only best responses can be 0 or 1. Without loss of generality consider player 1. Assume that his/her beliefs are represented by a neo-additive-capacity $\nu = \delta\alpha + (1 - \delta)\pi$. Let $V(1)$ (resp. $V(0)$) denote his/her (Choquet) expected utility if (s)he chooses 1 (resp. 0). Then, $V(1) = \delta\alpha (1 - c) + \delta (1 - \alpha)(\varepsilon - c) + (1 - \delta)\left[\varepsilon \mathbf{E}_\pi (1 + s_2)^2 - c\right]$, and $V(0) = \delta\alpha\varepsilon + (1 - \delta)\varepsilon \mathbf{E}_\pi s_2^2$.

Thus $V(1) - V(0) = \delta\alpha (1 - c - \varepsilon) + \delta (1 - \alpha)(\varepsilon - c) + (1 - \delta)[\varepsilon \mathbf{E}_\pi (1 + 2s_2) - c]$. By assumption $1 - \varepsilon - c > 0$ and $\varepsilon - c < 0$, hence if $\delta\alpha$ (resp. $\delta (1 - \alpha)$) is sufficiently large $V(1) > V(0)$ (resp. $V(1) < V(0)$), from which the result follows. ■

A.4 General Results

Lemma A.3 *Under Assumption 6.1 and positive or negative externalities, the slope of the best-response functions is given by:*

$$R^{1\prime}(s_2) = \frac{-(1 - \delta_1) u_{12}^1 \left(R^1(s_2), s_2\right)}{\delta_1\alpha_1 M_{11}^1 \left(R^1(s_2)\right) + \delta_1 (1 - \alpha_1) m_{11}^1 \left(R^1(s_2)\right) + (1 - \delta_1) u_{11}^1 \left(R^1(s_2), s_2\right)},$$

$$R^{2\prime}(s_1) = \frac{-(1 - \delta_2) u_{12}^2 \left(s_1, R^2(s_1)\right)}{\delta_2\alpha_2 M_{22}^2 \left(R^2(s_1)\right) + \delta_2 (1 - \alpha_2) m_{22}^2 \left(R^2(s_1)\right) + (1 - \delta_2) u_{22}^2 \left(s_1, R^2(s_1)\right)}.$$

Proof. Let R^i denote the best-response function of player i. By positive externalities, we can set $M^i(s_i) \equiv u_i(s_i, \bar{s}_{-i})$ and $m^i(s_i) \equiv u_i(s_i, \underline{s}_{-i})$ for any $s_i \in S_i$. Consider player 1, his/her Choquet expected utility is given by: $\delta_1\alpha_1 M^1(s_1) + \delta_1 (1 - \alpha_1) m^1(s_1) + (1 - \delta_1) u^1(s_1, s_2)$. By

Assumption 6.1, his/her best-response function is defined by,

$$\delta_1 \alpha_1 M_1^1 \left(R^1(s_2)\right) + \delta_1 (1-\alpha_1) m_1^1 \left(R^1(s_2)\right) + (1-\delta_1) u_1^1 \left(R^1(s_2), s_2\right) = 0. \qquad (14)$$

Differentiating (14) with respect to s_2 we obtain:

$$R^{1\prime}(s_2) = \frac{-(1-\delta_1) u_{12}^1 \left(R^1(s_2), s_2\right)}{\delta_1 \alpha_1 M_{11}^1 (R^1(s_2)) + \delta_1 (1-\alpha_1) m_{11}^1 (R^1(s_2)) + (1-\delta_1) u_{11}^1 (R^1(s_2), s_2)}.$$

From which the result follows. The slope of R^2 can be derived by analogous reasoning. ■

Proof of Proposition 6.1 Consider the function, $g : S^1 \times S^2 \to S^1 \times S^2$, defined by $g(s_1, s_2) = \left\langle R^1(s_2) - s_1, R^2(s_1) - s_2 \right\rangle$. The partial derivatives of g are $\frac{\partial g^1}{\partial s_1} = -1, \frac{\partial g^1}{\partial s_2} = R^{1\prime}(s_2), \frac{\partial g^2}{\partial s_1} = R^{2\prime}(s_1)$ and $\frac{\partial g^2}{\partial s_2} = -1$. Let J denote the Jacobian matrix of g. Then $J = \begin{pmatrix} -1 & R^{1\prime}(s_2) \\ R^{2\prime}(s_1) & -1 \end{pmatrix}$. The trace of J is -2. Thus if the determinant of J is positive, both eigenvalues must be negative and hence J is negative definite. The determinant of J is $1 - R^{1\prime}(s_2) R^{2\prime}(s_1) \geqslant 1 - \frac{(1-\delta_2)(1-\delta_1)Q^2}{\eta^2}$, since

$$R^{1\prime}(s_2) = \frac{-(1-\delta_1) u_{12}^1 \left(R^1(s_2), s_2\right)}{\delta_1 \alpha_1 M_{11}^1 (R^1(s_2)) + \delta_1 (1-\alpha_1) m_{11}^1 (R^1(s_2)) + (1-\delta_1) u_{11}^1 (R^1(s_2), s_2)},$$

$\left|R^{1\prime}(s_2)\right| \leqslant \frac{(1-\delta_1)Q}{\eta}$, where $Q = \max_{\langle s_1, s_2\rangle \in S} \left|u_{12}^1(s_1, s_2)\right|$ and $\eta = \min_{\langle s_1, s_2\rangle \in S} \left|u_{11}^1(s_1, s_2)\right|$. Since the strategy space is compact, the maximum and minimum exist. It follows that J is negative definite if δ_1 is sufficiently large. By Theorem 4.3 of Eichberger (1993), this implies that equilibrium is unique. ■

The next result characterises extremal equilibria in terms of the slope of the best-response functions.

Lemma A.4 *If the highest and lowest equilibria are interior equilibria, then $R^{1\prime}(s_2) R^{2\prime}(s_1) \leqslant 1$ at these equilibria.*

Proof. Define $\rho : [\underline{s}_1, \bar{s}_1] \to [\underline{s}_1, \bar{s}_1]$ by $\rho(s_1) = R^1\left(R^2(s_1)\right)$. By assumption there are no corner equilibria, hence $\rho(\underline{s}_1) > \underline{s}_1$ and $\rho(\bar{s}_1) < \bar{s}_1$. Let $\langle \hat{s}_1, \hat{s}_2 \rangle$ be an equilibrium such that $R^{1\prime}(s_2) R^{2\prime}(s_1) > 1$. Then for all sufficiently small $\gamma > 0$, $\rho(\hat{s}_1 + \gamma) > \hat{s}_1 + \gamma$. Let $\phi(s_1) = \rho(s_1) - s_1$. Then $\phi(\hat{s}_1 + \gamma) > 0$ and $\phi(\bar{s}_1) < 0$. By the intermediate value theorem, there exists $s' \in (\hat{s}_1 + \gamma, \bar{s}_1)$ such that $\phi(s_1') = s_1'$. Therefore $\langle \hat{s}_1, \hat{s}_2 \rangle$ is not the highest equilibrium. Similar arguments apply to the lowest equilibrium. ■

Proof of Proposition 6.2 Let $\langle \hat{s}_1, \hat{s}_2 \rangle$ denote the highest equilibrium. Assume first that $\langle \hat{s}_1, \hat{s}_2 \rangle$ is an interior equilibrium. Since $\langle \hat{s}_1, \hat{s}_2 \rangle$ is an interior equilibrium, it satisfies the first order conditions for best responses:

$$\delta_1 \alpha_1 M_1^1(s_1) + \delta_1(1-\alpha_1) m_1^1(s_1) + (1-\delta_1) u_1^1(s_1,s_2) = 0, \tag{15}$$
$$\delta_1 \alpha_2 M_2^2(s_2) + \delta_2(1-\alpha_2) m_2^2(s_2) + (1-\delta_2) u_2^2(s_1,s_2) = 0. \tag{16}$$

Differentiating (16) with respect to α_1 we obtain: $(1-\delta_2) u_{12}^2(s_1,s_2) \frac{\partial s_1}{\partial \alpha_1}$ $+ \left[\delta_1 \alpha_2 M_{22}^2(s_2) + \delta_2(1-\alpha_2) m_{22}^2(s_2) + (1-\delta_2) u_{22}^2(s_1,s_2)\right] \frac{\partial s_2}{\partial \alpha_1} = 0$. Hence

$$\frac{\partial s_2}{\partial \alpha_1} = R^{2\prime}(s_1) \frac{\partial s_1}{\partial \alpha_1}. \tag{17}$$

Differentiating (15) with respect to α_1 we obtain, $(1-\delta_1) u_{12}^1(s_1,s_2) \frac{\partial s_2}{\partial \alpha_1}$ $+ \left[\delta_1 \alpha_1 M_{11}^1(s_1) + \delta_1(1-\alpha_1) m_{11}^1(s_1) + (1-\delta_1) u_{11}^1(s_1,s_2)\right] \frac{\partial s_1}{\partial \alpha_1} = m_1^1(s_1) - M_1^1(s_1)$. Hence

$$\begin{aligned}&\frac{\partial s_1}{\partial \alpha_1} + \frac{(1-\delta_1) u_{12}^1(s_1,s_2)}{\delta_1 \alpha_1 M_{11}^1(s_1) + \delta_1(1-\alpha_1) m_{11}^1(s_1) + (1-\delta_1) u_{11}^1(s_1,s_2)} \frac{\partial s_2}{\partial \alpha_1} \\ = \; & \frac{m_1^1(s_1) - M_1^1(s_1)}{\delta_1 \alpha_1 M_{11}^1(s_1) + \delta_1(1-\alpha_1) m_{11}^1(s_1) + (1-\delta_1) u_{11}^1(s_1,s_2)}.\end{aligned}$$

Substituting from (17), $\frac{\partial s_1}{\partial \alpha_1} = \frac{m_1^1(s_1) - M_1^1(s_1)}{\delta_1 \alpha_1 M_{11}^1(s_1) + \delta_1(1-\alpha_1) m_{11}^1(s_1) + (1-\delta_1) u_{11}^1(s_1,s_2)} \left[1 - R^{1\prime}(s_2) R^{2\prime}(s_1)\right]^{-1}$. If the game has positive externalities and strategic complements, then $m_1^1(s_1) - M_1^1(s_1) = u_1^1(s_1, \underline{s}_2) - u_1^1(s_1, \bar{s}_2) < 0$. By Lemma A.4, $1 - R^{1\prime}(s_2) R^{2\prime}(s_1) \geqslant 0$ and by concavity in own pay-off, $\delta_1 \alpha_1 M_{11}^1(s_1) + \delta_1(1-\alpha_1) m_{11}^1(s_1) + (1-\delta_1) u_{11}^1(s_1,s_2) < 0$. Hence, $\frac{\partial s_1}{\partial \alpha_1} \geqslant 0$, and by equation (17), $\frac{\partial s_2}{\partial \alpha_1} \geqslant 0$.

Now consider the case where the highest equilibrium is on the boundary of the strategy set and the game has positive externalities. In particular suppose that when $\alpha_1 = \tilde{\alpha}_1$ that the highest equilibrium is $\langle \bar{s}_1, \bar{s}_2 \rangle$. Firstly it is trivially true that a decrease in α_1 must (weakly) decrease the equilibrium strategies of both players. Now suppose α_1 increases from $\tilde{\alpha}_1$ to $\hat{\alpha}_1$. The equilibrium at $\langle \bar{s}_1, \bar{s}_2 \rangle$ satisfies the Kuhn-Tucker conditions:

$$\delta_1 \tilde{\alpha}_1 M_1^1(\bar{s}_1) + \delta_1(1-\tilde{\alpha}_1) m_1^1(\bar{s}_1) + (1-\delta_1) u_1^1(\bar{s}_1,\bar{s}_2) \geqslant 0, \tag{18}$$
$$\delta_2 \alpha_2 M_1^2(\bar{s}_2) + \delta_2(1-\alpha_2) m_1^2(\bar{s}_2) + (1-\delta_2) u_1^2(\bar{s}_1,\bar{s}_2) \geqslant 0. \tag{19}$$

Since $\delta_1 \hat{\alpha}_1 M_1^1(\bar{s}_1) + \delta_1(1-\hat{\alpha}_1) m_1^1(\bar{s}_1) + (1-\delta_1) u_1^1(\bar{s}_1,\bar{s}_2) \geqslant \delta_1 \tilde{\alpha}_1 M_1^1(\bar{s}_1) + \delta_1(1-\tilde{\alpha}_1) m_1^1(\bar{s}_1) + (1-\delta_1) u_1^1(\bar{s}_1,\bar{s}_2)$, the Kuhn-Tucker conditions are still satisfied when $\alpha_1 = \hat{\alpha}_1$. By concavity, these conditions are sufficient, hence $\langle \bar{s}_1, \bar{s}_2 \rangle$ remains the highest equilibrium when $\alpha_1 = \hat{\alpha}_1$.

Analogous reasoning applies to the lowest equilibrium, other parameter changes and negative externalities. ■

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