OPTIMISM AND PESSIMISM IN GAMES*

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7th September 2006

Abstract

This paper considers the impact of ambiguity in strategic situations. It extends the earlier literature by allowing for optimistic responses to ambiguity. Ambiguity is modelled by CEU preferences. We study comparative statics of changes in ambiguity-attitude in games with strategic complements or substitutes. This gives a precise statement of the impact of ambiguity on economic behaviour.

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Keywords   Ambiguity in games, support, strategic complementarity, optimism.

JEL Classification   C72, D81.

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1   INTRODUCTION

The main focus of this paper is on ambiguity, which describes situations where individuals cannot or do not assign subjective probabilities to uncertain events. In Eichberger and Kelsey (2002) we studied games of strategic complements or substitutes where players were ambiguity-averse. In particular we showed that in games of strategic complements, the comparative statics of ambiguity were predictable. In games of positive externalities and strategic complements an increase in ambiguity-aversion has the effect of decreasing equilibrium strategies. A possible criticism of these results is that experimental evidence shows individuals are not uniformly ambiguity-averse. While ambiguity-aversion is common, individuals do at times display ambiguity preference. The present paper aims to study the case where individuals may express ambiguity preference.

There is a substantial body of experimental evidence which suggests that people behave differently when probabilities are ambiguous see for instance Camerer and Weber (1992). The importance of the distinction between risk and ambiguity is confirmed by recent research, which shows that different parts of the brain process ambiguity and probabilistic risk, see Camerer, Lowenstein, and Prelec (2004). The majority of individuals respond by behaving cautiously when there is ambiguity. Henceforth we shall refer to such cautious behaviour as *pessimism.*[1] In experiments a minority of individuals behave in the opposite way which we shall refer to as *optimism.* (See for instance Camerer and Weber (1992) or Cohen, Jaffray, and Said (1985).) Moreover the same individual may be pessimistic in one situation and optimistic in another.

Models of the impact of ambiguity on individual decisions can be found in Gilboa and Schmeidler (1989), Sarin and Wakker (1992) or Schmeidler (1989). A theory of games with ambiguity has been proposed by Dow and Werlang (1994) and developed in Eichberger and Kelsey (2000). In Eichberger and Kelsey (2002) we presented some results on the comparative statics of ambiguity in games of strategic complements under the assumption of ambiguity-aversion.[2] Until now, this literature has usually assumed that players are uniformly averse to ambiguity.[3]

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[1] See Wakker (2001) who relates this pessimism to a generalised version of the Allais paradox.

[2] For some alternative approaches to modelling games with ambiguity see Lo (1999) and Marinacci (2000).

[3] One exception is Marinacci (2000), who assumes that players either display global ambiguity-aversion or global ambiguity-preference. However the evidence shows that the same individual can express ambiguity-preference in some situations and ambiguity-aversion in others. We consider such mixed ambiguity attitudes.

The present paper aims to extend the previous literature on ambiguity and strategic interaction by allowing for the possibility of optimism. We find that in games with strategic complements and positive externalities, an increase in optimism has the effect of increasing the equilibrium strategy. If a given player is optimistic, (s)he places more weight on good outcomes than an expected utility maximiser would. In this case, good outcomes would be perceived to be situations where other players use high strategies. Strategic complementarity implies that over-weighting high strategies will increase the given player's incentive to play a higher strategy. In games of strategic substitutes the effect is reversed. An increase in ambiguity-aversion increases the weight placed on low strategy profiles of one's opponents and hence increases the marginal benefit of increasing one's own strategy. Consequently the strategies played in a symmetric equilibrium increase.

Allowing for optimism is a useful extension since it allows us to model phenomena where ambiguity-preference plays an important role in motivating behaviour. This might include setting up businesses, speculative research and development and decisions to enter careers such as acting or rock music where the returns are very uncertain.

If there are positive externalities between the players, then for the usual reasons, too few positive externalities will be produced in Nash equilibrium. As a result the equilibrium will be Pareto inefficient. Thus an increase in equilibrium strategies caused by an increase in ambiguity preference may result in an ex-post Pareto improvement.

One may also consider an ex-ante measure of welfare. Consider the case of ambiguity-preference. In this case the utility increase will be higher, since not only will higher strategies be played for any given state of nature but also ambiguity-preference causes people to place greater weight on more favourable states of nature. However since this effect is a comparison between different preference relations, it is similar to an interpersonal comparison of utility. Thus it is more controversial whether the second effect should only be taken into account in a welfare analysis.

Organisation of the Paper   In section 2 we present our framework and definitions. In section 3 we introduce our solution concept and prove existence of equilibrium. In section 4 we derive the comparative statics of changes of ambiguity-attitude in games of strategic complements or substitutes. Concluding comments can be found in Section 5. Appendix A relates a number of

alternative notions of the support of a capacity and Appendix B contains the proofs of those results not proved in the text.

2 MODELLING AMBIGUITY IN GAMES

This section introduces Choquet Expected Utility (henceforth CEU), which is the main theory of ambiguity we shall use. First we shall present the framework and definitions.

2.1 Games with Aggregate Externalities

This paper focuses on ambiguity in games with aggregate externalities, (defined below). Consider a game $\Gamma = \langle N; (S_i), (u_i) : 1 \leqslant i \leqslant n \rangle$ with finite pure strategy sets S_i for each player and payoff functions $u_i(s_i, s_{-i})$. Player i has a finite strategy set which, for convenience, we identify with an interval of the integers, $S_i = \{\underline{s}_i, \underline{s}_i + 1, ..., \bar{s}_i\}$, for $i = 1, ..., n$.[4] The notation, s_{-i}, indicates a strategy combination for all players except i. The space of all strategy profiles for i's opponents is denoted by S_{-i}. The space of all strategy profiles is denoted by S. Player i has utility function $u_i : S \to \mathbb{R}$, for $i = 1, ..., n$.

Definition 2.1 *A game,* Γ, *has* positive (resp. negative) aggregate externalities *if* $u_i(s_i, s_{-i}) = u_i(s_i, f_i(s_{-i}))$, *for* $1 \leqslant i \leqslant n$, *where* u_i *is increasing (resp. decreasing) in* f_i *and* $f_i : S_{-i} \to \mathbb{R}$ *is increasing in all arguments.*

This is a separability assumption. It says that a player only cares about a one-dimensional aggregate of his/her opponents' strategies. An example would be a situation of team production, in which the utility of a given team member depends on his/her own labour input and the total input supplied by all other members of the team.[5]

A game is symmetric if all players have the same strategy set and pay-off function.

Definition 2.2 *Let,* $\Gamma = \langle N; (S_i), (u_i) : 1 \leqslant i \leqslant n \rangle$ *be a game with aggregate externalities. We say that* Γ *is* symmetric *if* $S_i = S_j, 1 \leqslant i, j \leqslant n; u_i = u, f_i = f$ *for* $i = 1, ..., n$.

Notation 2.1 *Since* S_{-i} *is finite, we may enumerate the possible values of* f_i, $f_i^0 < ... < f_i^M$. *Since* f *is increasing* $f_i^0 = f(0, ..., 0)$ *and* $f_i^M = f(K_1, ..., K_n)$.

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[4] It would be straightforward to extend the results to a multi-dimnesional strategy space.

[5] For a more detailed analysis of the impact of ambiguity on team production see Kelsey and Spanjers (2004).

Marginal benefit without ambiguity is defined in the usual way.

Definition 2.3 *Let* $\Delta_i(s_i, s_{-i}) = u_i(s_i, s_{-i}) - u_i(s_i - 1, s_{-i})$, *i.e.* $\Delta_i(s_i, s_{-i})$ *denotes the* marginal benefit *to individual* i *of increasing his/her action from* $s_i - 1$ *to* s_i *when his/her opponents' strategy profile is* s_{-i}. *If* $f_i(s_{-i}) = f_i^r$, *we shall write* $\Delta_i(s_i, f_i^r)$ *for* $\Delta_i(s_i, s_{-i})$.

Thus in a game of strategic complements (resp. substitutes) a given player's marginal benefit will increase (resp. decrease) when the aggregate of his/her opponents' strategy profile increases.

Definition 2.4 *A game,* Γ, *with aggregate externalities is a game of* strategic substitutes *(resp. complements) if* $\Delta_i(s_i, f_i^r)$ *is a strictly decreasing (resp. increasing) function of* r, *for* $1 \leqslant i \leqslant n$.

The following assumption will be a maintained hypothesis.

Assumption 2.1 *All games,* Γ, *are assumed to be* concave, *by which we mean that for all* $i, u_i(s_i, s_{-i})$ *is a strictly concave function of* s_i.

2.2 Non-Additive Beliefs and Choquet Integrals

First we introduce the CEU model of ambiguity, which represents beliefs as capacities. A capacity assigns non-additive weights to subsets of S_{-i}. Formally, they are defined as follows.

Definition 2.5 *A capacity on* S_{-i} *is a real-valued function* ν *on the subsets of* S_{-i} *such that* $A \subseteq B \Rightarrow \nu(A) \leqslant \nu(B)$ *and* $\nu(\varnothing) = 0$, $\nu(S_{-i}) = 1$.

Thus a capacity is like a subjective probability except that it may be non-additive. The simplest example of a capacity is the complete uncertainty capacity defined below.

Definition 2.6 *The* complete uncertainty capacity, ν_0 *on* S_{-i} *is defined by* $\nu_0(A) = 0$ *for all* $A \subsetneqq S_{-i}$.

Intuitively ν_0 describes a situation where the decision maker knows which states are possible but has no further information about their likelihood.

Definition 2.7 *Let* ν *be a capacity on* S_{-i}. *The dual capacity* $\tilde{\nu}$ *is defined by* $\tilde{\nu}(A) = 1 - \nu(\neg A)$.

The capacity and the dual capacity encode the same information. If beliefs are represented by a capacity ν on S, the expected utility of the payoff obtained from a given act can be found using the Choquet integral, which is defined below.

Notation 2.2 *Let Γ be a symmetric game with* aggregate externalities *we shall use H_r to denote the event $\{s_{-i} \in S_{-i} : f(s_{-i}) \geqslant f_r\}$.*

Definition 2.8 *Let Γ be a symmetric game with positive aggregate externalities. The* Choquet integral *of $u_i(s_i, s_{-i})$ with respect to capacity ν_i on S_{-i} is:*

$$V_i(s_i) = \int u_i(s_i, s_{-i}) \, d\nu_i = u_i(s_i, f_M) \nu_i(H_M) + \sum_{r=0}^{M-1} u_i(s_i, f_r) [\nu_i(H_r) - \nu_i(H_{r+1})].$$

This is a special case of the Choquet integral, which applies in games of positive aggregate externalities. For the general definition see Schmeidler (1989).

Definition 2.9 *A capacity is said to be* convex *if $\nu(A \cup B) \geqslant \nu(A) + \nu(B) - \nu(A \cap B)$.*

Schmeidler (1989) argues that convex capacities represent ambiguity-aversion.[6] More recently Wakker (2001) has argued that convexity is implied by a generalised version of the Allais paradox.

Definition 2.10 *Let ν be a capacity on S_{-i}. The* core, $\mathcal{C}(\nu)$, *is defined by*

$$\mathcal{C}(\nu) = \{p \in \Delta(S_{-i}) ; \forall A \subset S_{-i}, p(A) \geqslant \nu(A)\}.$$

The following result shows that for a convex capacity, the Choquet integral for a given act a is equal to the minimum over the core of the expected value over f. Hence convex capacities provide an attractive representation of pessimism. When a decision-maker does not know the true probabilities (s)he considers a set of probabilities to be possible and evaluates any given act by the least favourable of these probabilities.

Proposition 2.1 *If ν is an convex capacity on S_{-i}, then $\int a \, d\nu = \min_{p \in C(\nu)} \mathbf{E}_p a$, where $\mathbf{E}$ denotes the expected value of a with respect to the additive probability p.*

2.3 JP-Capacities

Next we introduce a class of capacities which are useful for representing the impact of ambiguity in games since they are capable of modelling both optimism and pessimism. Jaffray and Philippe (1997) consider the following class of capacities.

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[6]Alternative definitions of ambiguity-aversion have been proposed by Epstein (1999) and Ghirardato and Marinacci (2002).

Definition 2.11 *Say that a capacity ν on S_{-i} is a JP-capacity if there exists a convex capacity μ and $\alpha \in [0,1]$, such that $\nu = \alpha\mu + (1-\alpha)\tilde{\mu}$, where $\tilde{\mu}$ denotes the dual capacity of μ.*

This class of capacities allows preferences to be represented in both the multiple priors and CEU forms. Perceived ambiguity is represented by the capacity μ, while ambiguity attitude is represented by α. Hence JP capacities allow a distinction between ambiguity and ambiguity-attitude, which is formalised in the following definitions.

Definition 2.12 *Let ν and ν' be two capacities on S. We say that ν is* more ambiguity-averse *than ν' if for all $A \subseteq S, \nu(A) \leqslant \nu'(A)$.*[7]

The following result shows that for JP-capacities, an increase in α implies an increase in ambiguity-aversion.

Proposition 2.2 *Suppose that $\hat{\alpha} \geqslant \tilde{\alpha}$ and μ is concave then $\hat{\nu} = \hat{\alpha}\mu + (1-\hat{\alpha})\tilde{\mu}$ is more ambiguity averse than $\tilde{\nu} = \tilde{\alpha}\mu + (1-\tilde{\alpha})\tilde{\mu}$.*

Proof. The result follows from noting that since μ is convex for all $A \subseteq S, \mu(A) \leqslant \tilde{\mu}(A)$. ∎

If $\nu = \alpha\mu + (1-\alpha)\tilde{\mu}$ is a JP capacity then

$$\int f d\nu = \alpha \min_{p \in \mathcal{C}(\mu)} \mathbf{E}_p f + (1-\alpha) \max_{p \in \mathcal{C}(\mu)} \mathbf{E}_p f,$$

where $\mathcal{C}(\mu)$ *is* the core of μ. This equation provides an intuitive representation of behaviour in the presence of ambiguity. When faced with ambiguity, the decision-maker does not know the true probability distribution (if this concept is meaningful). Instead (s)he considers a number of probability distributions to be possible. If $\alpha = 0$ the reaction is pessimistic since (s)he evaluates any given act by the least favourable probability distribution. Similarly if $\alpha = 1$ the reaction to ambiguity is optimistic. In the general case, the decision-maker's reaction to ambiguity is in part pessimistic and in part optimistic.

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[7]Theories of ambiguity-aversion have been proposed by Epstein (1999), Ghirardato and Marinacci (2002) and Kelsey and Nandeibam (1996). Although these papers differ in a number of respects, they agree that this is an appropriate definition of when one capacity is more ambiguity-averse than another. The present paper focuses on the comparative statics of changes in ambiguity attitude. We are interested in when one set of preferences is more ambiguity-averse than another, not the absolute definition of ambiguity-aversion. Controversies concerning the appropriate definition of ambiguity-aversion have been concerned with the absolute definition not the relative definition.

A useful special case is the neo-additive capacity, defined below, which generates CEU preferences that display both optimism and pessimism. They are the simplest class of capacities with this property.

Definition 2.13 *Let α, δ be real numbers such that $0 < \delta < 1, 0 < \alpha < 1$, define a neo-additive-capacity ν by $\nu(A) = \delta(1 - \alpha) + (1 - \delta)\pi(A), \emptyset \subsetneqq A \subsetneqq S.$*[8]

The neo-additive capacity describes a situation where the decision maker's 'beliefs' are represented by π. However (s)he has some doubts about these beliefs. The reaction to these doubts is in part pessimistic and in part optimistic. These preferences maintain a separation between ambiguity and ambiguity-attitude, which are measured by δ and α respectively. The highest possible level of ambiguity corresponds to $\delta = 1$, while $\delta = 0$ corresponding to no ambiguity. Higher levels of α corresponding to more ambiguity-aversion. Purely ambiguity-loving preferences are given by $\alpha = 0$, while the highest level of ambiguity-aversion occurs when $\alpha = 1$.

Chateauneuf, Eichberger, and Grant (2005), show that the Choquet expected value of a real valued function $a : S_{-i} \to \mathbb{R}$ with respect to the neo-additive-capacity ν is given by:

$$\int a d\nu = \delta\alpha \min_{s \in S_{-i}} a(s) + \delta(1 - \alpha) \max_{s \in S_{-i}} a(s) + (1 - \delta) \cdot \mathbf{E}_\pi a.$$

Thus the Choquet integral for a neo-additive capacity is a weighted averaged of the highest payoff, the lowest payoff and the expected payoff. The response to ambiguity is partly optimistic represented by the weight given to the best outcome and partly pessimistic. In Chateauneuf, Eichberger, and Grant (2005) it is shown that CEU preferences with neo-additive capacities can also be represented in the following form:

$$\int a d\nu = (1 - \alpha) \max_{p \in \mathcal{P}} \mathbf{E}_p a + \alpha \min_{p \in \mathcal{P}} \mathbf{E}_p a,$$

where $\nu = \delta(1 - \alpha) + (1 - \delta)\pi$ is a neo-additive capacity, $\mathcal{P} = \{p \in \Delta(S_{-i}) : p \geq (1 - \delta)\pi\}$ and $a \in A(S)$. Thus $\mathcal{P}$ is the set of measures 'centred' around a fixed $\pi \in \Delta(S)$.

Neo-additive capacities are restrictive since they only allow the best and worst outcome to be over-weighted. It is more plausible that ambiguity causes a number of good and bad outcomes

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[8]Neo-additive is an abbreviation for non-extremal outcome additive. Neo-additive capacities are axiomatised in Chateauneuf, Eichberger, and Grant (2005).

to be over-weighted. It does not seem unreasonable to assume that in most cases the worst outcome is death. However people may also be concerned about other bad outcomes such as losing large amounts of money. Similarly optimism will intuitively result in a number of good outcomes being over-weighted.[9] Hence for most of this paper we consider JP-capacities, which represent behaviour in the presence of ambiguity as over-weighting a number of good and bad outcomes.

3 EQUILIBRIUM

In this section we describe our solution concept and prove existence of equilibrium. First we discuss the support of ambiguous beliefs, which is a key concept for defining equilibrium in games. There have been a number of solution concepts for games with ambiguity, see for instance Dow and Werlang (1994), Lo (1996) or Marinacci (2000). In all of these, the support of a player's beliefs is used to represent the set of strategies that (s)he believes his/her opponents will play. An equilibrium is defined to occur when every profile of strategies in the support consists only of best responses. The main difference between the various solution concepts is that they use different support notions. Thus the definition of support is important.

3.1 Support of Ambiguous Beliefs

The support of a capacity plays a crucial role our the definition of equilibrium in a game, (and indeed in most other definitions of equilibrium). As we shall argue it is not possible to apply support definitions from the previous literature unmodified since many of them have implicitly assumed ambiguity-aversion. For *convex* capacities most support notions coincide. (See Appendix A for further discussion of the relation between different support notions.) We restrict attention to preferences which can be represented in both the CEU and multiple priors forms. Below we define the support for a convex capacity.

Definition 3.1 *If μ is a convex capacity on S_{-i}, we define the support of μ, supp μ, by*

$$\operatorname{supp} \mu = \bigcap_{p \in \mathcal{C}(\mu)} \operatorname{supp} p.^{10}$$

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[9]One piece of evidence for this is that lotteries have more than one prize.

[10]This definition is essentially the same as the *inner support* as defined in Ryan (1997).

We define the support of a JP-capacity to be the support of the convex capacity on which it is based.

Definition 3.2 *If* $\nu = \alpha\mu + (1 - \alpha)\tilde{\mu}$ *is a JP-capacity on* S_{-i}, *we define the support of* ν, $\operatorname{supp}\nu$, *by* $\operatorname{supp}\nu = \operatorname{supp}\mu$.

The support always exists but may possibly be empty. In the sequel we shall use capacities to represent the beliefs of players in a game. If these capacities are equilibrium beliefs then we shall interpret the support to be the set of strategies, which may be played in equilibrium. It may seem that the possibility of empty support makes this interpretation difficult. As we shall demonstrate existence of equilibrium beliefs with a non-empty support, this potential problem does not arise in practice. We shall now proceed to relate the support to the decision weights in the Choquet integral.

Definition 3.3 *If* ν *is a capacity on* S_{-i}, *define*

$$\mathcal{B}(\nu) = \{s \in S_{-i} : \forall A \subsetneqq S_{-i}, s \notin A; \nu(A \cup s) > \nu(A)\}.$$

The set $\mathcal{B}(\nu)$ consists of those states which always get positive weight in the Choquet integral, no matter which act is being evaluated. To see, this, recall that the Choquet expected utility of a given act, a, is a weighted sum of utilities. The weight assigned to strategy profile $\tilde{s}$ is $\nu(\{s : a(s) \succ a(\tilde{s})\} \cup \{\tilde{s}\}) - \nu(s : a(s) \succ a(\tilde{s}))$. These weights depend on the way in which the act ranks the states. Since there are $n!$ ways the states can be ranked, in general there are $n!$ profiles of decision weights used in evaluating the Choquet integral with respect to a given capacity. If $\forall A \subsetneqq S, s \notin A, \nu(A \cup \tilde{s}) > \nu(A)$ then the decision-weight on state $\tilde{s}$ is positive no matter how state $\tilde{s}$ is ranked by act a. The state space can be partitioned into three sets, those states which are given positive weight by all of the decision weights, those states given positive weight by some sets of decision weights but not others and those states which are given weight zero by all decision weights. Sarin and Wakker (1998) argue that the decision-maker's beliefs may be deduced from these decision weights. With this interpretation, $\mathcal{B}(\nu)$ is the set of states in which the decision-maker 'believes' in the strong sense that they always get positive weight in the Choquet integral. Proposition 3.1 shows that the support consists of states in which the decision-maker believes in this sense. Similarly the states which always get zero weight are those

which the decision-maker believes to be impossible. The remaining states can be interpreted as those which the decision-maker believes to be ambiguous. The weight they get in the Choquet integral may or may not be positive depending on the context.[11]

Proposition 3.1 *Let ν be a JP-capacity then $\operatorname{supp}\nu \subseteq \mathcal{B}(\nu)$.*

Proof. Since ν is a JP-capacity we may write $\nu = \alpha\mu + (1-\alpha)\tilde{\mu}$ for some convex capacity μ. Take $\hat{s} \in \operatorname{supp}\nu$ and $A \subsetneq S$, $\hat{s} \notin A$. Let $\hat{p} = \operatorname{argmin}_{p\in\mathcal{C}(\mu)} p(A \cup \hat{s})$. Because μ is convex $\mu(A \cup \hat{s}) - \mu(A) = \hat{p}(A \cup \hat{s}) - \mu(A) = \hat{p}(A) - \mu(A) + \hat{p}(\hat{s}) > 0$, since $\hat{p}(A) \geqslant \mu(A)$ and $\hat{p}(\hat{s}) > 0$ by Proposition A.1.

Let $\tilde{p} = \operatorname{argmin}_{p\in\mathcal{C}(\mu)} p(S\backslash A)$. Then $\tilde{\mu}(A \cup \hat{s}) - \tilde{\mu}(A) = \mu(S\backslash A) - \mu(S\backslash(A \cup \hat{s})) \geqslant \tilde{p}(S\backslash A) - \tilde{p}(S\backslash(A \cup \hat{s})) = 1 - \tilde{p}(A) - [1 - \tilde{p}(A \cup \hat{s})] = \tilde{p}(A) + \tilde{p}(\hat{s}) - \tilde{p}(A) > 0$, since $\tilde{p}(\hat{s}) > 0$. Hence for $A \subsetneq S, s \notin A, \nu(A \cup \tilde{s}) > \nu(A)$. ∎

The following example demonstrates that it is not possible to prove a converse to Proposition 3.1.

Example 1 *Assume there are three states of nature, $S = \{s_1, s_2, s_3\}$. Consider capacities ν, μ defined on S in the table below:*

	$\varnothing$	s_1	s_2	s_3	s_1s_2	s_1s_3	s_2s_3	S
μ	0	β	0	0	$\frac{1}{2}$	β	0	1
$\tilde{\mu}$	0	1	$1-\beta$	$\frac{1}{2}$	1	1	$1-\beta$	1
ν	0	$1-\alpha(1-\beta)$	$(1-\alpha)(1-\beta)$	$\frac{1}{2}(1-\alpha)$	$1-\frac{1}{2}\alpha$	$1-\alpha(1-\beta)$	$(1-\alpha)(1-\beta)$	1

where $\nu = \alpha\mu + (1-\alpha)\tilde{\mu}$ and $0 < \alpha < 1$, $0 < \beta < \frac{1}{2}\alpha$. Then $\operatorname{supp}\nu = \{s_1\}$ and $\mathcal{B}(\nu) = \{s_1, s_2\}, \operatorname{supp}\mu = \{s_1\}, \mathcal{B}(\mu) = \{s_1\}$.

Proof. Since μ is convex, it follows from Proposition A.1, that $\operatorname{supp}\mu = \{s_1\}$. By definition $\operatorname{supp}\nu = \operatorname{supp}\mu$.

By Proposition 3.1, $s_1 \in \mathcal{B}(\nu)$. We can show $s_2 \in \mathcal{B}(\nu)$, since $\nu(\varnothing \cup s_2) = 1 - \alpha > \nu(\varnothing)$, $\nu(s_1 \cup s_2) = 1 - \frac{1}{2}\alpha > \nu(s_1) = 1 + \alpha\beta - \alpha$, $\nu(s_3 \cup s_2) = (1-\beta)(1-\alpha) > \nu(s_3) = \frac{1}{2}(1-\alpha)$, $\nu(s_1s_3 \cup s_2) = 1 > \nu(s_1s_3) = 1 + \alpha\beta - \alpha$. However $s_3 \notin \mathcal{B}(\nu)$, since $\nu(s_1s_3) = \nu(s_1)$. ∎

Since μ is a belief function, ν is a JP capacity. Consider the capacity $\nu_\alpha = \alpha\mu + (1-\alpha)\tilde{\mu}$, where μ is a convex capacity. Intuitively as α changes, ambiguity-attitude changes, while beliefs and perceived ambiguity remain constant. If the $\mathcal{B}(\nu)$ is taken to represent beliefs, one has to argue that the decision-maker's beliefs change as his/her ambiguity-attitude changes. However $\operatorname{supp}\nu_\alpha$ is independent of α. This suggests to us that $\operatorname{supp}\nu$ is a superior notion of support to $\mathcal{B}(\nu)$, since it makes a clear distinction between ambiguity-attitude and beliefs. Indeed the fact that $\operatorname{supp}\nu$ is the intersection of support of the probabilities in the core is quite intuitive.

3.2 Definition of Equilibrium

We define an equilibrium to be a situation where each player maximises his/her (Choquet) expected utility given his/her beliefs. In addition beliefs have to be reasonable in the sense that each player believes that his/her opponents play best responses. We interpret this require-ment as implying the support of any given player's beliefs should consist of best responses for the other players. Let $R_i(\nu_i) = \arg\max_{s_i \in S_i}\{\int u_i(s_i, s_{-i})\,d\nu_i(s_{-i})\}$ denote the best response correspondence of player i given beliefs ν_i.

Definition 3.4 *An n-tuple of capacities $\nu^* = \langle \nu_1^*, ...\nu_n^* \rangle$ is an* equilibrium under ambiguity *if:*

$$\forall i, \varnothing \neq \operatorname{supp}\nu_i^* \subseteq \underset{j \neq i}{\times} R_j(\nu_j^*).$$

If $\forall i, k_i^* \in \operatorname{supp}\nu_i^*$, we say that $k^* = \langle k_1^*, ...k_n^* \rangle$ is an equilibrium strategy profile.

In equilibrium, the beliefs of player i are represented by a capacity ν_i^*, whose support consists of strategies that are best responses for his/her opponents. A player's evaluation of a particular strategy may, in part, depend on strategies of his/her opponents which do not lie in the support. We interpret these as events a player views as unlikely but which cannot be ruled out. This may reflect some doubts (s)he may have about the rationality of the opponents or whether (s)he correctly understands the structure of the game.

Definition 3.5 *Let Γ be a game and let $\hat{\nu} = \langle \hat{\nu}_1, ..., \hat{\nu}_n \rangle$ be an equilibrium of Γ, if $\forall i, \operatorname{supp}\hat{\nu}_i$ contains a single strategy profile we say that it is* pure, *otherwise we say that it is* mixed.

Players choose pure strategies and do not randomise. Hence we are not able to interpret a mixed equilibrium as a randomisation. In a mixed equilibrium some player i say, will have two

or more best responses. The support of other players' beliefs about i's play, will contain some or all of them. Thus an equilibrium, where the support contains multiple strategy profiles, is an equilibrium in beliefs rather than randomisations. If, in addition, it is required that beliefs are additive and then a pure equilibrium is a Nash equilibrium.

3.3   Existence of Equilibrium

We shall confine attention to situations where beliefs may be represented by JP-capacities. The reason for this is that this is a large class of capacities in which it is possible to specify ambiguity and the ambiguity-attitude exogenously. This enables us to study the comparative statics of changing ambiguity-attitude in games. The class of capacities, defined below, are a set of candidate equilibria for games, which allow us to vary ambiguity and ambiguity-attitude exogenously.

Definition 3.6 (Constant Contamination CC) *A capacity on S_{-i} is said to display constant contamination if it may be written in the form $\nu_i = \nu_i(\pi_i, \chi_i, \delta_i, \alpha_i) = \delta_i \pi_i(A) + (1 - \delta_i)[\alpha_i \chi_i(A) + (1 - \alpha_i)\tilde{\chi}_i(A)]$, where π_i is an additive probability distribution and χ_i is a convex capacity with $\operatorname{supp}\chi_i = \emptyset$. If $\pi_i^{\tilde{s}}$ denotes the probability distribution on S_{-i}, which assigns probability 1 to $\tilde{s}_{-i}$ we shall write $\nu_i^{\tilde{s}} = \nu_i(\pi_i^{\tilde{s}}, \chi_i, \delta_i, \alpha_i)$. We shall suppress the arguments $(\alpha_i, \delta_i, \chi_i)$ when it is convenient.*

Thus $\nu_i(\pi_i, \chi_i, \delta_i, \alpha_i)$ describes a situation where player i 'believes' that his/her opponents will play the mixed or pure strategy profile described by π_i but lacks confidence in this belief. The CC-capacity has a separation between beliefs represented by π, ambiguity represented by χ and δ and ambiguity attitude represented by α. The situation where i believes that his/her opponents will play the pure strategy profile $\tilde{s}$ is described by the capacity $\nu_i^{\tilde{s}}$. The parameter δ determines the weight the individual gives to ambiguity. Lower values of δ correspond to more ambiguity hence δ may be interpreted as a measure of confidence in the decision-maker's probabilistic belief π. The capacity χ determines which strategy profiles the player regards as ambiguous. CC-capacities are a special case of JP-capacities. The following result finds the support of a CC capacity.

Proposition 3.2 *Let $\nu = \delta\pi(A) + (1 - \delta)[\alpha\chi(A) + (1 - \alpha)\hat{\chi}(A)]$ be a CC capacity. Then $\operatorname{supp}\nu = \operatorname{supp}\pi$.*

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Proof. Let $\mu = \delta\pi(A) + (1-\delta)\chi(A)$. Then by Lemma B.2, $\nu = \alpha\mu + (1-\alpha)\hat{\mu}$ is a JP capacity. By definition $\operatorname{supp}\nu = \operatorname{supp}\mu$. By Lemma B.2, $\operatorname{supp}\mu = \operatorname{supp}\pi$. ∎

Remark 1 *If ν_0 denotes the complete uncertainty capacity, then a neo-additive capacity may be written in the form $\nu = \delta\pi(A) + (1-\delta)[\alpha\nu_0(A) + (1-\alpha)\tilde{\nu}_0(A)]$. Proposition B.2 implies that $\operatorname{supp}\nu = \operatorname{supp}\pi$. Recall we interpret a neo additive capacity as describing a situation where the decision-maker's beliefs are represented by the additive probability distribution π, however (s)he may lack confidence in this belief. Given this, it seems intuitive that the support of the neo-additive capacity should coincide with the support of π. This provides an argument in favour of our definition of support, since it gives the intuitively correct result for neo-additive capacities.*

In game theory it is common to assume that each player believes that his/her opponents act independently. This can be achieved if we assume that the capacity μ is an independent product.[12]

We commence the analysis of equilibria with ambiguity by defining the marginal benefit under ambiguity.

Definition 3.7 *Marginal Benefit*   *Suppose that player i has beliefs described by a capacity ν_i on S_{-i}. Define*

$$\mathrm{MB}_i(s_i, \nu) = \int u_i(s_i, s_{-i}) \, d\nu_i(s_{-i}) - \int u_i(s_i - 1, s_{-i}) \, d\nu_i(s_{-i}).$$

We interpret $\mathrm{MB}_i(s_i, \nu)$ as player i's perceived marginal benefit from increasing his/her strategy from $s_i - 1$ to s_i, given that (s)he has beliefs represented by the capacity ν. In general this will be different to the marginal benefit without ambiguity from Definition 2.3.

The following result demonstrates existence of equilibrium. We take the ambiguity represented by χ_i and δ_i, and ambiguity-attitude represented by α_i as given. In games of aggregate externalities with strategic complementarity we are able to demonstrate the existence of pure equilibria.

Theorem 3.1 *Let Γ be a game of positive aggregate externalities with strategic complements. Then for any given n-tuples $\alpha = \langle \alpha_1, ..., \alpha_n \rangle$, $\delta = \langle \delta_1, ..., \delta_n \rangle$ and $\chi = \langle \chi_1, ..., \chi_n \rangle$, Γ has a pure equilibrium in CC capacities.*

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[12]Technically we need to assume that μ is a Möbius independent product of belief functions defined on the marginals. For a definition of the Möbius independent product and further discussion see Ghirardato (1997).

3.4 Symmetric Games

In symmetric games we can strengthen the previous result by proving existence of symmetric equilibrium. For this, we require that the strategy sets and payoff functions be symmetric. Moreover we need to ensure that players perceive ambiguity in a symmetric way. This is captured by the following definition.

Notation 3.1 *Let* $\psi : \{1, ..., n\} \setminus i \rightarrow \{1, ..., n\} \setminus i$ *be a permutation. Then* ψ *induces a map* $\tilde{\psi} : S_{-i} \rightarrow S_{-i}$ *by* $\tilde{\psi}(\langle s_1, ..., s_n \rangle) = \langle s_{\psi(1)}, ..., s_{\psi(n)} \rangle.$

Definition 3.8 *A capacity* ζ *on* S_{-i} *is said to be* anonymous *if for any permutation* $\psi :$ $\{1, ..., n\} \setminus i \rightarrow \{1, ..., n\} \setminus i$ *and all* $A \subseteq S_{-i}, \zeta\left(\tilde{\psi}(A)\right) = \zeta(A).$[13]

Definition 3.9 *Define* $\xi : \{1, ..., n\} \setminus i \rightarrow \{1, ..., n\} \setminus j$ *by* $\xi(j) = i$ *and* $\xi(k) = k, k \neq, j.$ *Capacities* χ_i *on* S_{-i} *and on* S_{-j} *are said to be* similar *if for all* $A \subseteq S_{-i}, \chi_j(\xi(A)) = \chi_i(A).$

Definition 3.10 *Let* Γ *be a game and let* $\nu^* = \langle \nu_1^*, ..., \nu_n^* \rangle$ *be an equilibrium of* $\Gamma.$ *The profile of capacities,* ν^* *is said to be a* symmetric equilibrium, *if for all* i, j, ν_i^* *is similar to* $\nu_j^*.$

In a symmetric equilibrium, all players have the same best responses and have correspondingly similar beliefs about what their opponents will do. The next result demonstrates the existence of symmetric equilibria in symmetric games. Compared to Theorem 3.1 it relaxes the assumption of strategic complementarity. However the cost of this is we are only able to demonstrate the existence of mixed equilibria.

Theorem 3.2 (Existence of symmetric equilibrium) *Let* Γ *be a symmetric game, then for any given numbers* $\alpha, \delta, 0 \leqslant \alpha, \delta \leqslant 1$ *and any anonymous capacity* χ *with* $\text{supp}\,\chi = 0,$ Γ *has a symmetric equilibrium* $\nu_i = \nu(\pi, \chi, \delta, \alpha)$ *in CC capacities.*

4 COMPARATIVE STATICS

In this section we investigate the comparative statics of changes in ambiguity-attitude on equilibrium.

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[13]We refer to such capacities as *anonymous* rather than *symmetric*, since the term symmetric is commonly used to denote capacities which satisfy $\nu(A) + \nu(\neg A) = 1,$ see Gilboa (1989).

4.1 Strategic Complements

For games of positive aggregate externalities with strategic complements, an increase in ambiguity-aversion increases equilibrium strategies. Intuitively if a given player becomes more ambiguity-averse (s)he will place more weight on outcomes which are perceived as bad. If there are positive externalities, a bad outcome would be when an opponent plays a low strategy. In the presence of strategic complementarity if a given player increases the decision weight on low strategies of his/her opponents this will reduce his/her incentive to play a high strategy.

The following theorem is our comparative result on games with strategic complementarity. It shows that an increase in pessimism will reduce the equilibrium strategies in games with positive aggregate externalities and strategic complements. If there are multiple equilibria, the strategies played in the highest and lowest equilibria will decrease. For this result we assume that the ambiguity-attitude of one player changes, the ambiguity-attitudes of other players and the perceived ambiguity measured by δ and χ are held constant.

Theorem 4.1 *Let Γ be a game of positive aggregate externalities with* strategic complements. *Assume that players' beliefs may be represented by CC capacities $\nu_i(\pi_i, \chi_i, \delta_i, \alpha_i)$. Let $\underline{k}'$ (resp. $\bar{k}'$) denote the lowest (resp. highest) equilibrium strategy profile when the degree of ambiguity-aversion is $\alpha^* = \langle \alpha_1^*, ..., \alpha_n^* \rangle$. Let $\underline{k}^*$ (resp. $\bar{k}^*$) denote the lowest (resp. highest) equilibrium strategy profile when the degree of ambiguity-aversion is $\alpha' = \langle \alpha_1', ..., \alpha_n' \rangle$. If $\alpha^* > \alpha'$, then $\underline{k}' \geqslant \underline{k}^*$ and $\bar{k}' \geqslant \bar{k}^*$.*

Proof. Assume first that α^* and α' only differ in one component, i.e. $\exists i, \forall j \neq i, \alpha_j^* = \alpha_j'$. By Lemma B.4, $\forall i, \mathrm{MB}_i\left(s_i, \nu_i^{\tilde{s}}(\alpha_i', \delta_i, \chi_i)\right) < 0$ for $\bar{k}' < \tilde{s} \leqslant m^*$. By Lemma B.5, $\forall i, \mathrm{MB}_i\left(s_i, \nu_i^{\tilde{s}}(\alpha_i', \delta_i, \chi_i)\right) \geqslant \mathrm{MB}_i\left(s_i, \nu_i^{\tilde{s}}(\alpha_i^*, \delta_i, \chi_i)\right)$. Hence $\forall i, \mathrm{MB}_i\left(s_i, \nu_i^{\tilde{s}}(\alpha_i^*, \delta_i, \chi_i)\right) < 0$, for $\bar{k}' < \tilde{s} \leqslant m^*$. However, $\mathrm{MB}_i\left(s_i, \nu_i^{\tilde{s}}(\alpha_i^*, \delta_i, \chi_i)\right) \geqslant 0$ is a necessary condition for there to be a equilibrium in which s is the equilibrium strategy. This implies $\bar{k}^* \leqslant \bar{k}'$. A similar argument applies to the smallest equilibrium.

The general result follows by repeated application of the result where α^* and α' only differ in one component. ■

Remark 2 *The comparative statics are reversed in games of negative aggregate externalities, for further details see Eichberger and Kelsey (2002).*

4.2 Strategic Complementarity and Multiple Equilibria

Strategic Complementarity can give rise to multiple Nash equilibria. Under some assumptions we can show if there are multiple equilibria without ambiguity, if there is sufficient optimism (resp. pessimism) equilibrium will be unique and will correspond to the highest (resp. lowest) equilibrium without ambiguity. Thus ambiguity has a double effect. Below we propose a measure of how ambiguous a capacity is.

Definition 4.1 *The* minimal degree of ambiguity *of capacity μ is defined by:*

$$\lambda\left(\mu\right) = 1 - \min_{A \subseteq S}\left(\mu\left(A\right) + \mu\left(\neg A\right)\right).$$

The next result shows that if players are sufficiently optimistic, equilibrium is unique and is higher than the highest equilibrium without ambiguity. To prove this we need the following assumption.

Assumption 4.1 *For $1 \leqslant i \leqslant n, u_i\left(s_i, \bar{s}_{-i}\right)$ has a unique maximiser, i.e.*
$\left|\mathrm{argmax}_{s_i \in S_i}\, u_i\left(s_i, \bar{s}_{-i}\right)\right| = 1$.

This assumption is required for technical reasons. If the strategy space were continuous it would be implied by our other assumptions. It says that the gaps in the discrete strategy space do not fall in the wrong place. The following result is a corollary of Theorem 4.1 and Lemma B.7.

Proposition 4.1 *Consider a game of* positive *aggregate externalities with strategic complements which satisfies Assumption 4.1. There exist $\bar{\alpha}$ and $\bar{\lambda}$ such that if the minimal degree of ambiguity is $\lambda\left(\mu_i\right) \geqslant \bar{\lambda}$ and $\alpha_i \geqslant \bar{\alpha}$, (resp. $\leqslant \underline{\alpha}$) for $1 \leqslant i \leqslant n$, equilibrium is unique and is larger (resp. smaller) than the largest (resp. smallest) equilibrium without ambiguity.*

In a game with strategic complements with multiple Nash equilibria, increasing ambiguity causes the multiplicity of equilibria to disappear while increasing ambiguity-aversion causes the equilibrium strategies to decrease. Thus ambiguity and ambiguity-attitude have distinct effects. Combined with sufficient optimism, ambiguity can cause the economy to move to a higher level equilibrium.

4.3 Strategic Substitutes

Next we study games with strategic substitutes. For this section, we shall assume that all games are symmetric, since there are no general comparative statics results for non-symmetric games with strategic substitutes. However it is possible to obtain a result for symmetric equilibria of symmetric games. The next result shows that in games with strategic substitutes and positive externalities, increasing ambiguity-aversion raises the strategy played in symmetric equilibrium.

Proposition 4.2 *Consider a game of* positive *aggregate externalities with* strategic substitutes. *Let* $\bar{\nu} = \nu\left(\pi, \mu, \delta, \bar{\alpha}\right)$ *(resp.* $\hat{\nu} = \nu\left(\pi, \mu, \delta, \hat{\alpha}\right)$*) be a symmetric equilibrium in CC-capacities in which* $\bar{k}$ *(resp.* $\hat{k}$*) is the highest strategy played and* $\bar{\ell}$ *(resp.* $\hat{\ell}$*) is the lowest strategy played. Then* $\hat{\alpha} > \bar{\alpha}$ *implies* $\hat{k} \geqslant \bar{k}$ *and* $\hat{\ell} \geqslant \bar{\ell}$.[14]

With positive externalities, an increase in ambiguity-aversion increases the weight on lower strategies of the opponent. In a game of strategic substitutes this will increase the perceived marginal benefit of a given player's own action and hence (s)he will have a higher best response to any given strategy profile of his/her opponents. As a result the strategy played in a symmetric equilibrium will increase.

5   CONCLUSION

Compared to our previous work, e.g. Eichberger and Kelsey (2002), this paper makes a number of innovations. One of the most important are that we allow for optimistic responses to ambiguity. Much of the innovation in the present paper involves developing techniques for modelling optimistic responses to ambiguity. New definitions of support and measures of ambiguity-attitude were needed. This has enabled us to make a clear distinction between ambiguity and ambiguity attitude. In our previous research Eichberger and Kelsey (2002) both ambiguity and ambiguity-attitude were varied simultaneously in our comparative static exercises. (It is hard to avoid this in a model which assumes ambiguity-aversion.) Moreover we have extended the previous results to a larger class of games since Eichberger and Kelsey (2002) restricted attention to symmetric equilibria of symmetric games, these assumptions have been relaxed apart from in the section on strategic substitutes. Possible applications of these results would include

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[14]Since we are not able to prove existence of a pure equilibrium, the equilibrium strategies are not necessarily unique hence we need to consider the highest and lowest equilibrium strategies.

oligopoly models, public goods and environmental models. Applications of our comparative statics results can be found in Eichberger and Kelsey (2002) and Eichberger, Kelsey, and Schipper (2005).

It is possible, in principle, to extend our results to extensive form games. However this will pose some new technical problems. Since players will receive new information during the course of play, it will be necessary to model how information is updated. Some possible updating rules are discussed in Eichberger, Grant, and Kelsey (2005).

APPENDIX

A   ALTERNATIVE NOTIONS OF SUPPORT

In this appendix we discuss some alternative notions of support which have been proposed. We show that most of them coincide for the important case of convex capacities. Dow and Werlang (1994) define the support of a capacity to be a minimal set whose complement has capacity 0.

Definition A.1 *The* DW-support *of capacity ν, $\operatorname{supp}_{DW} \nu$ is a set $E \subseteq S_{-i}$, such that $\nu \left(S_{-i} \backslash E \right) = 0$ and $\nu \left(F \right) > 0$, for all F such that $S_{-i} \backslash E \subsetneq F$.*

This definition has the advantage that $\operatorname{supp}_{DW} \nu$ always exists, however it may not be unique. In contrast, Marinacci (2000) defines the support of a capacity ν to be the set of states with positive capacity.

Definition A.2 *The* M-support *of capacity ν, is defined by $\operatorname{supp}_M \nu = \left\{ s \in S_{-i} : \nu \left(s \right) > 0 \right\}.$*

When it exists $\operatorname{supp}_M \nu$ is always unique. However there are capacities for which it does not exist, for example the *complete uncertainty capacity*, (see definition 2.6).

It is difficult to apply Definitions A.1 and A.2 when decision makers may be ambiguity-loving. For capacities which are not necessarily ambiguity-averse, it is quite possible that *all* states will get positive capacity. As an example consider a neo-additive capacity $\nu = \delta \left(1 - \alpha \right) + \left(1 - \delta \right) \pi$. Then if $\alpha > 0$, $\operatorname{supp}_{DW} \nu = \operatorname{supp}_M \nu = S$.

The following results formally demonstrate the relationships between the various support notions.

Lemma A.1 *If ν is a capacity then $\operatorname{supp}_M \nu \subset \operatorname{supp}_{DW} \nu$.*

Proof. Let $\tilde{s} \in \operatorname{supp}_M \nu$ and suppose, if possible, $\tilde{s} \notin \operatorname{supp}_{DW} \nu$. Then $\nu (S \backslash \operatorname{supp}_{DW} \nu) \geqslant \nu(\tilde{s}) > 0$, which is a contradiction. ■

The following lemma shows that our definition of support coincides with the M-support for a convex capacity.

Proposition A.1 *If ν is an convex capacity with support $\operatorname{supp} \nu$, then $\operatorname{supp} \nu = \operatorname{supp}_M \nu = \mathcal{B}(\nu)$.*

Proof. Let $\tilde{s} \in \operatorname{supp}_M \nu$ and let $\pi \in \mathcal{C}(\nu)$.[15] Then $\pi(\tilde{s}) \geq \nu(\{\tilde{s}\}) > 0$. Hence, $\operatorname{supp}_M \nu \subseteq \bigcap_{\pi \in \mathcal{C}(\nu)} \operatorname{supp} \pi = \operatorname{supp} \nu$. On the other hand, suppose $s \in \bigcap_{\pi \in \mathcal{C}(\nu)} \operatorname{supp} \pi$. Since ν is convex, $\nu(s) = \min_{\pi \in \mathcal{C}(\nu)} \pi(s) > 0$.[16] Hence $\bigcap_{\pi \in \mathcal{C}(\nu)} \operatorname{supp} \pi \subseteq \operatorname{supp}_M \nu$.

Suppose $s \in \operatorname{supp}_M \nu$. Then $\nu(s) > 0$. For any $A \subset S$, $s \notin A$, by convexity of ν, $\nu(A \cup s) \geq \nu(A) + \nu(s) > \nu(A)$. Hence, $s \in \mathcal{B}(\nu)$. Conversely suppose $s \in \mathcal{B}(\nu)$, then $\nu(s) = \nu(\varnothing \cup s) > \nu(\varnothing) = 0$. Hence, $s \in \operatorname{supp}_M \nu$. Thus $\operatorname{supp}_M \nu = \mathcal{B}(\nu)$. The result follows. ■

Even for a convex capacity, our support notion does not coincide with that of Dow and Werlang. An example of this is the complete uncertainty capacity, ν^0, (see definition 2.6), which has empty support but does have a non-unique DW-support.[17]

Lemma A.2 *Let ν be a capacity on S_{-i} then $\operatorname{supp}_{DW} \nu$ is unique if and only if $\operatorname{supp}_M \nu$ is a DW-support.*

Proof. Suppose that $\operatorname{supp}_{DW} \nu$ is unique. Let E be a DW-support. By Lemma A.1, $\operatorname{supp}_M \nu \subseteq E$. Suppose, if possible, there exists $\hat{s} \in E \backslash \operatorname{supp}_M \nu$, then $\nu(\hat{s}) = 0$. Hence, $F := S \backslash \{\hat{s}\}$ satisfies $\nu(S \backslash F) = 0$. Let G be a minimal set such that $G \subseteq F$ and $\nu(S \backslash G) = 0$. Then, $G \neq E$ is another DW-support which contradicts uniqueness. Hence $\operatorname{supp}_M \nu = \operatorname{supp}_{DW} \nu$.

Now suppose $\operatorname{supp}_M \nu$ is a DW-support. Let F be an arbitrary DW-support. By Lemma A.1, $\operatorname{supp}_M \nu \subset F$. Thus by the minimality part of the definition of a DW-support, we must have $F = \operatorname{supp}_M \nu$, which gives uniqueness. The result follows. ■

Thus for a convex capacity if the DW-support is unique, the various support notions coincide. The following proposition summarises the above analysis.

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[15] Recall $\mathcal{C}(\nu)$ denotes the core of the capacity ν, see definition 2.10.

[16] Although $\mathcal{C}(\nu)$ is an infinite set, the minimum must occur at one of the extremal points. The set of extremal points of a core is finite. Thus the minimum must be positive.

[17] If $s \in S$, then $\{s\}$ is a DW-support of ν^0.

Proposition A.2 *For a* convex *capacity* ν

    1. if supp_{DW} *is unique, then* $\mathcal{B}(\nu) = \operatorname{supp}_M \nu = \operatorname{supp} \nu = \operatorname{supp}_{DW} \nu,$

    2. otherwise, $\operatorname{supp}_M \nu = \mathcal{B}(\nu) = \operatorname{supp} \nu \subset \operatorname{supp}_{DW} \nu.$

Proof. Part 1 follows from Proposition A.1 and Lemma A.2. Part 2 follows from Proposition A.1 and Lemma A.1. ∎

    Alternatively the support of a capacity could be defined to be the complement of the set of states which are always given zero weight in the Choquet integral. Solution concepts for games based on this notion of support have been studied by Dow and Werlang (1991) and Lo (1996). They show that this definition of support does not result in a solution concept which is significantly different to Nash equilibrium. These conclusions seem incompatible with our objective of modelling deviations from Nash equilibrium due to ambiguity.

B   GAMES WITH AMBIGUITY

This appendix contains the proofs of our main results and some supporting results.

B.1   Existence

If player i has beliefs represented by the capacity $\nu_i^{\tilde{s}} = \nu_i^{\tilde{s}}(\alpha_i, \delta_i, \mu_i)$ then intuitively (s)he believes his/her opponents will play strategy profile $\tilde{s}$, however (s)he lacks confidence in this belief. The following lemma says that if a given player believes (in this sense) his/her opponents will play a higher strategy profile then this will raise his/her marginal benefit of increasing his/her own strategy.

Lemma B.1 *Let* Γ *be a game with positive aggregate externalities and strategic complements, then if* $\hat{s} \geqslant \tilde{s}, \forall s_i \in S_i, \operatorname{MB}_i\left(s_i, \nu^{\hat{s}}\right) \geqslant \operatorname{MB}_i\left(s_i, \nu^{\tilde{s}}\right).$

Proof. By Lemma B.3, $\operatorname{MB}_i\left(s_i, \nu^{\hat{s}}\right) - \operatorname{MB}_i\left(s_i, \nu^{\tilde{s}}\right) = \delta\left[\Delta_i(s_i, \hat{s}) - \Delta_i(s_i, \tilde{s})\right] > 0$, by strategic complementarity. ∎

Lemma B.2 *Let* χ *be a convex capacity on* S *with* $\operatorname{supp} \chi = \varnothing$. *Define a capacity* ξ *on* S *by* $\xi = \delta\pi + (1 - \delta)\chi$ *then:*

1. *Core of $\xi = \{\delta\pi + (1-\delta)\,p : p \in \mathcal{C}\,(\chi)\}$;*

2. $\operatorname{supp}\xi = \operatorname{supp}\pi$;

3. $\tilde{\xi} = \delta\pi\,(A) + (1-\delta)\,\tilde{\chi}\,(A)\,.$

Proof. If $p \in \mathcal{C}\,(\chi)$ then for all $A \subseteq S$, $p\,(A) \geqslant \chi\,(A)\,,$ hence $\delta\pi\,(A) + (1-\delta)\,p\,(A) \geqslant \delta\pi\,(A) + (1-\delta)\,\chi\,(A)\,,$ which implies $\delta\pi + (1-\delta)\,p \in \mathcal{C}\,(\chi)\,.$ Conversely suppose $q \in \mathcal{C}\,(\chi)\,.$ Then for all $A \subseteq S, q \geqslant \delta\pi\,(A) + (1-\delta)\,\chi\,(A)$ hence $\frac{1}{(1-\delta)}\,(q - \delta\pi\,(A)) \geqslant \chi\,(A)\,,$ which implies $\frac{1}{(1-\delta)}\,(q - \delta\pi\,(A)) \in \mathcal{C}\,(\chi)\,.$ Hence $\mathcal{C}\,(\xi) = \{\delta\pi + (1-\delta)\,p : p \in \mathcal{C}\,(\chi)\}\,.$

It is clear that ξ is convex since a linear combination of convex capacities is convex.

Suppose that $\hat{s} \in \operatorname{supp}\pi$, then $\delta\pi\,(\hat{s}) + (1-\delta)\,p\,(\hat{s}) > 0$ for all $p \in C\,(\chi)\,,$ which implies $\hat{s} \in \operatorname{supp}\xi$. Conversely suppose that $\tilde{s} \in \operatorname{supp}\xi$. Then since $\operatorname{supp}\chi = \varnothing$, there exists $q \in \mathcal{C}\,(\chi)$ such that $q\,(\tilde{s}) = 0$. Since $\tilde{s} \in \operatorname{supp}\xi, \delta\pi\,(\tilde{s}) + (1-\delta)\,q\,(\tilde{s}) > 0$, which implies $\pi\,(\tilde{s}) > 0$ hence $\tilde{s} \in \operatorname{supp}\pi$. Thus $\operatorname{supp}\xi = \operatorname{supp}\pi$.

Part 3 follows directly from the definitions. ∎

Proof of Theorem 3.1   The first order conditions for a pure equilibrium in which $\hat{s} = \langle\hat{s}_1, ..., \hat{s}_n\rangle$ is the equilibrium strategy may be expressed as $\mathrm{MB}_i\,(\hat{s}_i, \nu^{\hat{s}}) \geqslant 0, \mathrm{MB}_i\,(\hat{s}_i + 1, \nu^{\hat{s}}) \leqslant 0$, or $\mathrm{MB}_i\,(\hat{s}_i, \nu^{\hat{s}}) < 0$, and $\hat{s}_i = 0.$[18]

Let $\mathcal{T} = \{s \in S : \forall i,\ \text{either } \mathrm{MB}_i\,(s_i, \nu^{\hat{s}}) \geqslant 0 \text{ or } s_i = 0\}\,.$ Note that $\mathcal{T}$ is non-empty, since $\mathbf{0} \in \mathcal{T}$. Let $\tilde{s}$ be a maximal element of $\mathcal{T}$. Then

$$\forall i, \mathrm{MB}_i\,(\tilde{s}_i, \nu^{\tilde{s}}) \geqslant 0 \text{ or } \tilde{s}_i = 0. \tag{1}$$

Let $\mathcal{I} = \{i : \tilde{s}_i \neq K_i\}\,,$ i.e. $\mathcal{I}$ denotes the set of individuals who are not playing their highest strategy. If $i \in \mathcal{I}$, define $\tilde{s}^{+i} = \langle\tilde{s}_{-i}, \tilde{s}_i + 1\rangle$. Then since $\tilde{s}$ is maximal, $\exists j, \mathrm{MB}_j\,(\tilde{s}^{+i}, \nu^{\tilde{s}^{+i}}) < 0$ and $\tilde{s}_j^{+i} \neq 0$. However by Lemma B.1, $\tilde{s}^{+i} \geqslant \tilde{s}$ implies that if $j \neq i$ and $\tilde{s}_j \neq 0, \mathrm{MB}_j\,(\tilde{s}_j, \nu^{\tilde{s}^{+i}}) \geqslant 0$. Hence it is that case that $\forall i \in \mathcal{I}, \mathrm{MB}_i\,(\tilde{s}_i + 1, \nu^{\tilde{s}^{+i}}) < 0$. By Lemma B.1,

$$\forall i \in \mathcal{I}, \mathrm{MB}_i\,(\tilde{s}_i + 1, \nu^{\tilde{s}}) < 0. \tag{2}$$

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[18]Recall $\nu_i^{\tilde{s}} = \nu_i\,(\pi_i^{\tilde{s}}, \chi_i, \delta_i, \alpha_i)\,,$ where $\pi_i^{\tilde{s}}$ denotes the probability distribution on S_{-i}, which assigns probability 1 to $\tilde{s}_{-i}$.

Equations (1) and (2) imply $\nu^{\tilde{s}}$ is a pure equilibrium, in which all play $\tilde{s}$ and beliefs are represented by CC capacities with degrees of ambiguity-aversion $\langle \alpha_1, ..., \alpha_n \rangle$.[19] ∎

Theorem 3.2 Let Γ be a symmetric game, then for any given numbers $\alpha, \delta, 0 \leqslant \alpha, \delta \leqslant 1$ and any anonymous capacity χ, Γ has a symmetric equilibrium $\nu_i = \nu_i (\pi_i, \chi_i, \delta_i, \alpha_i)$ in CC capacities.

Proof of Theorem 3.2 Define $\Psi (s_i, s_{-i}) = u (s_i, s_{-i}) + (1 - \delta) \alpha \int u (s_i, s_{-i}) d\chi$

$+ (1 - \delta) (1 - \alpha) \int u (s_i, s_{-i}) d\tilde{\chi}$, for $1 \leqslant i \leqslant n$. Consider the symmetric game (without ambiguity) $\Gamma' = \langle N, (S_i) (u_i) : 1 \leqslant i \leqslant n \rangle$, where the players have strategy sets $\{0, 1, ..., m^*\}$ and player i's utility function is Ψ. Since the sum of concave functions is concave, Ψ is concave in s_i. Hence the game Γ' has a symmetric Nash equilibrium, in which players independently choose a strategy according to the probability distribution π_i^* on S_i, (see Moulin (1986) p. 115). Let π_{-i}^* denote probability distribution on S_{-i}, which is the independent product of the marginals, $\pi_j^*, j \neq i$.

Define by $\nu^* (A) = \delta \pi_{-i}^* (A) + (1 - \delta) [\alpha \chi (A) + (1 - \alpha) \tilde{\chi} (A)]$. We assert that the profile of CC capacities $\nu^* = \langle \nu_1^*, ..., \nu_n^* \rangle$ is a symmetric equilibrium of Γ. Consider player i. Suppose that strategy $\hat{s}_i$ is in the support of ν^*. Then by Lemma 3.2, $\pi^* (\hat{s}_i) > 0$. Thus strategy $\hat{s}_i$ is given positive probability in the Nash equilibrium of the game Γ'. This implies, $\mathbf{E}_{\pi^*} \Psi (\hat{s}_i, \hat{s}_{-i}) \geqslant \mathbf{E}_{\pi^*} \Psi (\hat{s}_i, \hat{s}_{-i})$, for $\hat{s}_i \in S_i$, where the expectation is over $\hat{s}_{-i}$.

Expanding, $\int \Psi^i (\hat{s}_i, s_{-i}) d\pi^* (s_{-i}) = \delta \int u (\hat{s}_i, s_{-i}) d\pi^* (s_{-i}) + (1 - \delta) \alpha \int u (\hat{s}_i, s_{-i}) d\chi$

$+ (1 - \delta) (1 - \alpha) \int u (\hat{s}_i, s_{-i}) d\tilde{\chi} \geqslant \delta \int u (s_i, s_{-i}) d\pi^* (s_{-i}) + (1 - \delta) \alpha \int u (s_i, s_{-i}) d\chi$

$+ (1 - \delta) (1 - \alpha) \int u (s_i, s_{-i}) d\tilde{\chi}$, for $s_i \in S_i$. This is equivalent to $\int u (\hat{s}_i, s_{-i}) d\nu^* (s_{-i}^* s_{-i}) \geqslant$ $\int u (s_i, s_{-i}) d\nu^* (s_{-i})$, which establishes that $\hat{s}_i$ is a best response for player i, given that his/her beliefs can be represented by capacity ν^*. It follows that ν^* is a symmetric equilibrium with ambiguity. ∎

B.2 Strategic Complements

Notation B.1 *For given* $\alpha_i, \delta_i, \chi_i$ *and* π_i, *define* $\varsigma_r (\alpha_i) = \alpha_i [\chi_i (H_r) - \chi_i (H_{r+1})] + (1 - \alpha_i) \chi_i (\neg H_{r+1})$ $- (1 - \alpha_i) \chi_i (\neg H_r)$, *for* $1 \leqslant r \leqslant M - 1$; $\varsigma_M (\alpha_i) = (1 - \alpha_i) + \alpha_i \chi (H_M) - (1 - \alpha_i) \chi_i (\neg H_M)$, $c_r (\alpha_i) = \delta_i [\pi_i (H_r) - \pi_i (H_{r+1})] + (1 - \delta_i) \varsigma_r (\alpha)$, *for* $1 \leqslant r \leqslant M - 1$ *and* $c_M (\alpha_i) = \delta_i \pi (H_M) +$ $(1 - \delta_i) \varsigma_M (\alpha_i)$.

---

[19] The strategy of proof is similar to that of Tarski's fixed point theorem, see Tarski (1955).

Thus the c_r's are the decision weights in the Choquet integral with respect to a CC capacity.

Lemma B.3 *Let* Γ *be a game with* positive aggregate externalities *and let the beliefs of player* i *be represented by the CC capacity* $\nu_i = \delta_i \pi_i + (1 - \delta_i) [\alpha_i \chi + (1 - \alpha_i) \tilde{\chi}_i]$ *on* S_{-i}, *then*

$$\mathrm{MB}_i(s_i, \nu_i) = \sum_{r=0}^{M} \Delta_i (s_i, f_i^r) \, c_r (\alpha_i)$$

$$= \delta_i \pi (H_M) \Delta_i (s_i, f_M) + \sum_{r=0}^{M} \Delta_i (s_i, f_i^r) \delta_i [\pi_i(H_r) - \pi_i(H_{r+1})] + (1 - \delta_i) \sum_{r=0}^{M} \Delta_i (s_i, f_i^r) \varsigma_r (\alpha_i).$$

$$(3)$$

Proof.   Consider player i. Then

$$\int u_i (s_i, s_{-i}) \, d\nu_i(s_{-i}) = u_i (s_i, f_M) \nu_i (H_M) + \sum_{r=0}^{M-1} u_i (s_i, f_r) [\nu_i(H_r) - \nu_i(H_{r+1})].$$

Similarly

$$\int u_i (s_i - 1, s_{-i}) \, d\nu_i(s_{-i}) = u_i (s_i - 1, f_M) \nu_i (H_M) + \sum_{r=0}^{M-1} u_i (s_i - 1, f_r) [\nu_i(H_r) - \nu_i(H_{r+1})].$$

By taking the difference of these two expressions we obtain: $\mathrm{MB}_i (s_i, \nu_i) = \Delta_i (s_i, f_M) \nu_i (H_M) + \sum_{r=0}^{M-1} \Delta_i (s_i, f_r) [\nu_i(H_r) - \nu_i(H_{r+1})]$. If we substitute $\nu_i(H_r) = \delta_i \pi_i(H_r) + (1 - \delta_i) \alpha_i \chi_i(H_r) + (1 - \delta_i) (1 - \alpha_i) \tilde{\chi}_i(H_r)$, we obtain:

$$\mathrm{MB}_i (s_i, \nu_i) = \Delta_i (s_i, f_M) \delta_i \pi_i (H_M) + \Delta_i (s_i, f_M) (1 - \delta_i) \{[\alpha_i \chi_i (H_M) + (1 - \alpha_i) \tilde{\chi}_i (H_M)]\}$$

$$+ \sum_{r=0}^{M-1} \Delta_i (s_i, f_r) \delta_i [\pi_i(H_r) - \pi_i(H_{r+1})] + \sum_{r=0}^{M-1} \Delta_i (s_i, f_r) (1 - \delta_i) [\alpha_i \chi_i(H_r) + (1 - \alpha_i) \tilde{\chi}_i(H_r)]$$

$$- \sum_{r=0}^{M-1} \Delta_i (s_i, f_r) (1 - \delta_i) [\alpha_{i_i} \chi_i(H_{r+1}) + (1 - \alpha_i) \tilde{\chi}_i(H_{r+1})]$$

$$= \Delta_i (s_i, f_M) \{\delta_i \pi_i (H_M) + (1 - \delta_i) [\alpha_i \chi_i (H_M) + (1 - \alpha_i) - (1 - \alpha_i) \chi_i (\neg H_M)]\}$$

$$+ \sum_{r=0}^{M-1} \Delta_i (s_i, f_r) \{\delta_i [\pi_i(H_r) - \pi_i(H_{r+1})] + (1 - \delta_i) [\alpha_i \chi_i(H_r) + (1 - \alpha_i) [1 - \chi_i (\neg H_r)]]$$

$$- (1 - \delta_i) [\alpha_i \chi_i(H_{r+1}) + (1 - \alpha_i) [1 - \chi_i (\neg H_{r+1})]]\}.$$

Hence

$$
\begin{aligned}
\mathrm{MB}_i\left(s_i, \nu_i\right) \;=\; & \Delta_i\left(s_i, f_M\right)\left\{\delta_i \pi_i\left(H_M\right)+\left(1-\delta_i\right) \varsigma_M\left(\alpha_i\right)\right\} \\
& +\sum_{r=0}^{M-1} \Delta_i\left(s_i, f_r\right)\left\{\delta_i\left[\pi\left(H_r\right)-\pi\left(H_{r+1}\right)\right]+\left(1-\delta_i\right) \varsigma_r\left(\alpha_i\right)\right\}
\end{aligned}
$$

from which, the result follows. ∎

Lemma B.4 *Let* Γ *be a game with* positive aggregate externalities. *Then for any given n-tuples* α, δ *and* χ, *suppose that* $\mathrm{MB}_i\left(\tilde{s}_i, \nu_i^{\tilde{s}}\right) \geqslant 0$ *for* $1 \leqslant i \leqslant n$.[20] *Then there exists* $\hat{s} \geqslant \bar{s}$ *such that there is a pure equilibrium in CC capacities* $\nu^* = \nu^{\tilde{s}} = \nu\left(\pi^{\tilde{s}}, \chi, \delta, \alpha\right)$ *in which* $\hat{s}$ *is the equilibrium strategy.*

Proof. Suppose first that $\mathrm{MB}_i\left(s_i', \nu_i^{s'}\right) \geqslant 0$, for all $s' \geqslant \bar{s}$, then there is a corner equilibrium, in which player i plays K_i, for $1 \leqslant i \leqslant n$. Otherwise, let $\mathcal{K} = \left\{s \in S : s \geqslant \bar{s}, \forall i, \mathrm{MB}_i\left(s_i', \nu_i^{s_i'}\right) \geqslant 0\right\}$. Let $\tilde{k}$ be a maximal element of $\mathcal{K}$, (maximal elements exist since the space of all strategy profiles is finite).

Then we may show that $\tilde{k}$ is an equilibrium strategy profile when beliefs are $\nu^{\hat{k}}$. The proof can be completed in a similar way to the last part of the proof of Theorem 3.1. ∎

The next result shows that if individual i's ambiguity-aversion increases then his/her perceived marginal benefit falls. This is the key step for establishing our comparative static results.

Lemma B.5 *Let* Γ *be a game with positive aggregate externalities and strategic complements. Assume* $\alpha_i^* \geqslant \alpha_i'$, *then* $\mathrm{MB}_i\left(s_i, \nu_i^{\tilde{s}}\left(\alpha_i', \delta_i, \chi_i\right)\right) > \mathrm{MB}_i\left(s_i, \nu_i^{\tilde{s}}\left(\alpha_i^*, \delta_i, \chi_i\right)\right).$

Proof. Note that $\sum_{r=t}^{M} \varsigma_r\left(\alpha\right) = \left[\alpha \chi_i\left(H_M\right)-\left(1-\alpha\right) \chi_i\left(\neg H_M\right)\right]+\sum_{r=t}^{M-1}\left[\alpha \chi_i\left(H_r\right)-\alpha \chi_i\left(H_{r+1}\right)\right]$ $+\sum_{r=t}^{M-1}\left(1-\alpha\right)\left[\chi_i\left(\neg H_{r+1}\right)-\chi_i\left(\neg H_r\right)\right] = \left(1-\alpha\right)+\alpha \chi_i\left(H_t\right)-\left(1-\alpha\right) \chi_i\left(\neg H_t\right).$ Hence

$$
\sum_{r=t}^{M} \varsigma_r\left(\alpha\right) = \alpha\left[\chi_i\left(H_t\right)+\chi_i\left(\neg H_t\right)-1\right]+1-\chi_i\left(\neg H_t\right). \tag{4}
$$

---

[20] Recall if, $\tilde{s} \in S$ is a given strategy profile $\pi_i^{\tilde{s}}$ is the probability distribution on S_{-i}, which assigns probability 1 to $\tilde{s}_{-i}$ then $\nu_i^{\tilde{s}} = \nu_i^{\tilde{s}}\left(\alpha_i, \delta_i, \chi_i\right) = \delta_i \pi_i^{\tilde{s}}+\left(1-\delta_i\right)\left[\alpha_i \chi_i+\left(1-\alpha_i\right) \tilde{\chi}_i\right].$

Since χ_i is strictly sub-additive, $\sum_{r=t}^{M} \varsigma_r(\alpha_i') > \sum_{r=t}^{M} \varsigma_r(\alpha_i^*)$, hence the $\varsigma_r(\alpha_i')$'s first order stochastically dominate the $\varsigma_r(\alpha_i^*)$'s. Since $\Delta_i(s, f_r)$ is strictly increasing in r,

$$\sum_{r=0}^{M} \varsigma_r(\alpha_i') \Delta_i(s, f_i^r) < \sum_{r=0}^{M} \varsigma_r(\alpha_i^*) \Delta_i(s, f_i^r). \tag{5}$$

The result follows from equation (3). ∎

Lemma B.6 *Let Γ be a game with positive aggregate externalities. Consider a given player, i say. Let $s' > s'' > s'''$, be three possible strategies for player i. Suppose that with beliefs over S_{-i} given by ν, s' is indifferent to s''', then s'' is strictly preferred to s'.*

Proof. Playing strategy s_i yields (Choquet) expected utility: $V_i(s_i) = \int u(s_i, s_{-i}) d\nu(s_{-i})$. There exists λ such that $s'' = \lambda s' + (1-\lambda)s'''$. Since s' and s''' are indifferent: $V_i(s') = \lambda V_i(s') + (1-\lambda)V_i(s''') = \lambda \int u_i(s', s_{-i}) d\nu(s_{-i}) + (1-\lambda) \int u_i(s''', s_{-i}) d\nu(s_{-i})$

$= \int [\lambda u_i(s', s_{-i}) + (1-\lambda) u_i(s''', s_{-i})] d\nu(s_{-i})^{21} < \int u_i(s'', s_{-i}) d\nu(s_{-i}) = V(s'')$, since u_i is strictly concave in s_i. The result follows. ∎

Lemma B.7 *Consider a game of* positive *aggregate externalities with strategic complements. There exist $\bar{\alpha}$ and $\bar{\lambda}$ such that if the minimal degree of ambiguity is $\lambda(\mu_i) \geqslant \bar{\lambda}$ and $\alpha_i \geqslant \bar{\alpha}$, (resp. $\leqslant \underline{\alpha}$) then in any equilibrium $\nu_i = \alpha_i \mu_i + \tilde{\alpha}_i \tilde{\mu}_i, \operatorname{supp} \nu_i \subseteq \operatorname{argmax}_{s_i \in S_i} u_i(s_i, \bar{s}_{-i})$, (resp. $\operatorname{supp} \nu_i \subseteq \operatorname{argmax}_{s_i \in S_i} u_i(s_i, \underline{s}_{-i}))$.*

Proof. Let $s_i'' \in \operatorname{argmax}_{s_i \in S_i} u_i(s_i, \underline{s}_{-i}))$ and let $\epsilon = u_i(s_i'', \bar{s}_{-i}) - \max_{s_i \notin \operatorname{argmax}_{s_i \in S_i} u_i(s_i, \underline{s}_{-i})} u_i(s_i, \underline{s}_{-i})$. By construction $\epsilon > 0$. Let $\hat{\nu}_i = \alpha_i \mu_i + \tilde{\alpha}_i \tilde{\mu}_i$ denote i's equilibrium beliefs. If individual i plays strategy s_i'' (s)he receives utility: $u_i(s_i'', f_M) \nu_i(H_M) + \sum_{r=0}^{M-1} u_i(s_i'', f_r) [\nu_i(H_r) - \nu_i(H_{r+1})]$

$= u_i(s_i'', f_M) (\alpha_i \mu_i(H_M) + \tilde{\alpha}_i \tilde{\mu}_i(H_M)) + \sum_{r=0}^{M-1} u_i(s_i'', f_r) \alpha_i(\mu_i(H_r) - \mu_i(H_{r+1}))$

$+ \sum_{r=0}^{M-1} u_i(s_i'', f_r) \tilde{\alpha}_i [\tilde{\mu}_i(H_r) - \tilde{\mu}_i(H_{r+1})]$. Since $\tilde{\alpha}_i \to 0$ as $\alpha_i \to 1$, the terms involving $\tilde{\alpha}_i$ can be neglected, hence this is approximately,

$$u_i(s_i'', f_M) \alpha_i \mu_i(H_M) + \alpha_i \sum_{r=0}^{M-1} u_i(s_i'', f_r) (\mu_i(H_r) - \mu_i(H_{r+1}))$$

$$= u_i(s_i'', f_M) \alpha_i \mu_i(H_M) + \sum_{r=0}^{M-1} u_i(s_i'', f_r) \alpha_i (\mu_i(H_r) - \mu_i(H_{r+1})).$$

---

[21] This step is valid since, if there are positive aggregate externalities $u(s', s_{-i})$ and $u(s''', s_{-i})$ are comonotonic.

Similarly, if individual i plays strategy $s'_i \notin \mathrm{argmax}_{s_i \in S_i} u_i(s_i, \bar{s}_{-i})$ his her utility is approximately: $V(s'_i) = u_i(s'_i, f_M) \alpha_i \mu_i(H_M) + \sum_{r=0}^{M-1} u_i(s'_i, f_r) \alpha_i(\mu_i(H_r) - \mu_i(H_{r+1}))$.

If we define $d_r = u_i(s''_i, f_r) - u_i(s'_i, f_r)$, the extra utility i gets from playing strategy s''_i rather than s'_i, is approximately: $V(s''_i) - V(s'_i) = d_M \alpha_i \mu_i(H_M) + \sum_{r=1}^{M-1} d_r \alpha_i(\mu_i(H_r) - \mu_i(H_{r+1}))$

$= d_0(\mu_i(H_0) - \mu_i(H_1)) + \sum_{r=1}^{M-1} d_r \alpha_i(\mu_i(H_r) - \mu_i(H_{r+1})) + d_M \alpha \mu_i(H_M)$

$= d_0(1 - \mu_i(H_1)) + \sum_{r=1}^{M-1} d_r \alpha_i(\mu_i(H_r) - \mu_i(H_{r+1})) + d_M \alpha_i \mu_i(H_M)$

$\geqslant d_0(1 - \mu_i(H_1)) + d_M \mu_i(H_1) \geqslant \bar{\lambda} d_0 + d_M(1 - \bar{\lambda}) \geqslant \bar{\lambda} \epsilon + d_M(1 - \bar{\lambda}) > 0$, provided $\bar{\lambda}$ is sufficiently large.

The first inequality follows since strategic complementarity implies that d_r is increasing in r. It follows that for sufficiently large minimal degrees of ambiguity and large α_i, individual i will play a strategy from $\mathrm{argmax}_{s_i \in S_i} u_i(s_i, \underline{s}_{-i})$ in equilibrium. ∎

B.3 Strategic Substitutes

Notation B.2 *If* $1 \leqslant k \leqslant m^*$, *we shall use* $r(k)$ *to denote that value of* r *which satisfies* $f_{r(k)} = f(k, ..., k)$.

Proposition 4.2 Consider a game of *positive* aggregate externalities with *strategic substitutes*. Let $\bar{\nu} = \nu(\pi, \chi, \delta, \bar{\alpha})$ (resp. $\hat{\nu} = \nu(\pi, \chi, \delta, \hat{\alpha})$) be a symmetric equilibrium in CC-capacities in which $\bar{k}$ (resp. $\hat{k}$) is the highest strategy played and $\bar{\ell}$ (resp. $\hat{\ell}$) is the lowest strategy played. Then $\hat{\alpha} > \bar{\alpha}$ implies $\hat{k} \geqslant \bar{k}$ and $\hat{\ell} \leqslant \bar{\ell}$.

Proof of Proposition 4.2 Suppose if possible $\hat{k} < \bar{k}$, we shall show that this leads to a contradiction.

Claim If we assume $\hat{k} < \bar{k}$, then $\bar{\pi}(H_t) \geqslant \hat{\pi}(H_t)$ for $1 \leqslant t \leqslant M$.

By Lemma B.6 the set of equilibrium strategies must be either $\{\hat{k}\}$ or $\{\hat{k}, \hat{k} - 1\}$. This implies that $\hat{\pi}(H_t) = 0$, if $t > r(\hat{k})$, $= 1$ if $r(\hat{k} - 1) \leqslant t$. Similarly $\bar{\pi}(H_t) = 0$, if $t > r(\bar{k})$ and $\bar{\pi}(H_t) = 1$ if $r(\bar{k} - 1) \geqslant t$. Since $\hat{k} < \bar{k}, \bar{k} - 1 \geqslant \hat{k}$. Thus for $t > r(\hat{k})$, $\bar{\pi}(H_t) \geqslant \hat{\pi}(H_t) = 0$; if $r(\bar{k} - 1) \geqslant t$, $\bar{\pi}(H_t) = 1 \geqslant \hat{\pi}(H_t)$; which establishes the claim.

By Lemma B.3, we may write $\mathrm{MB}(\bar{\nu}, k) = \sum_{r=0}^{M} \Delta(s_i, f_r) c_r(\bar{\alpha})$ and $\mathrm{MB}(\hat{\nu}, k)$ $= \sum_{r=0}^{M} \Delta(s_i, f_r) c_r(\hat{\alpha})$. Since χ is convex, $\chi(H_t) + \chi(\neg H_t) \leqslant 1$, hence as $\hat{\alpha} > \bar{\alpha}$,

$\hat{\alpha}\left[\chi(H_t) + \chi\left(\neg H_t\right) - 1\right] \leqslant \bar{\alpha}\left[\chi(H_t) + \chi\left(\neg H_t\right) - 1\right].$ By equation (4),

$$
\begin{aligned}
\sum_{r=t}^{M} c_r\left(\alpha\right) &= \delta\pi\left(H_M\right) + \sum_{r=t}^{M-1} \delta\left[\pi(H_r) - \pi(H_{r+1})\right] + (1-\delta)\sum_{r=t}^{M} \varsigma_r\left(\alpha\right) \\
&= \delta\pi(H_t) + (1-\delta)\left\{\alpha\left[\chi_i\left(H_t\right) + \chi_i\left(\neg H_t\right) - 1\right] + 1 - \chi_i\left(\neg H_t\right)\right\}.
\end{aligned}
$$

Hence the $c_r\left(\bar{\alpha}\right)$'s first order stochastically dominate the $c_r\left(\hat{\alpha}\right)$'s. Since $\Delta\left(s_i, f_r\right)$ is strictly decreasing in r, first order stochastic dominance implies $\mathrm{MB}(\bar{\nu}, k) < \mathrm{MB}(\hat{\nu}, k)$ for all k.

A necessary condition for $\hat{k}$ to be an equilibrium action with beliefs $\hat{\nu}$ is $0 \geqslant \mathrm{MB}\left(\hat{k}-1, \hat{\nu}\right)$. Hence by concavity $0 \geqslant \mathrm{MB}\left(\bar{k}, \hat{\nu}\right) > \mathrm{MB}\left(\bar{k}, \bar{\nu}\right)$. However, a necessary condition for $\bar{k}$ (resp. $\bar{k}, \bar{k}+1$) to be an equilibrium contribution level (resp. levels) with beliefs $\bar{\nu}$, is $\mathrm{MB}\left(\bar{k}, \bar{\nu}\right) \geqslant 0$, (resp. $\mathrm{MB}\left(\bar{k}, \bar{\nu}\right) = \mathrm{MB}\left(\bar{k}+1, \bar{\nu}\right) = 0$). This contradiction establishes that $\hat{k} \leqslant \bar{k}$. A similar argument applies to the lowest equilibrium strategy. ∎

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