A general bound for the limiting distribution of Breitung's statistic*

December 21, 2007

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*We are grateful to Dieter Balkenborg, Nigar Hashimzade, Michael Jansson, Chris Müris, Morten Nielsen, Paolo Paruolo, and Katsuto Tanaka for helpful advice and discussions, and to two referees for constructive comments. Address correspondence to Jan R. Magnus, Department of Econometrics and OR, Tilburg University, PO Box 90153, 5000 LE Tilburg, The Netherlands; e-mail: magnus@uvt.nl.

Proposed running head:
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Abstract:
We consider Breitung's (2002) statistic ξ_n which provides a nonparametric test of the I(1) hypothesis. If ξ denotes the limit in distribution of ξ_n as $n \to \infty$, we prove (Theorem 1) that $0 \leq \xi \leq 1/\pi^2$, a result that holds under any assumption on the underlying random variables. The result is a special case of a more general result (Theorem 3), which we prove using the so-called "cotangent trick" associated with Cauchy's residue theorem.

1 Introduction

Let $x_1, x_2, \ldots, x_n$ represent an arbitrary sequence of random variables, and let $\bar{x}_n$ denote the sample mean. We consider

$$\xi_n := \frac{\sum_{t=1}^{n}\left(\sum_{j=1}^{t}(x_j - \bar{x}_n)\right)^2}{n^2 \sum_{t=1}^{n}(x_t - \bar{x}_n)^2},$$

which is the statistic for Breitung's (2002) nonparametric test of the I(1) hypothesis. Let ξ denote the limit in distribution of ξ_n as $n \to \infty$. We wish to prove the following theorem.

Theorem 1. *ξ is supported on the interval $[0, 1/\pi^2]$.*

This result holds under any assumption on $x_1, \ldots, x_n$ whatsoever. It is trivially true when the process is I(0) because then, as Breitung shows, the distribution is degenerate at 0. In fact, it can be shown that the same holds for a covariance stationary I(d) process when $|d| < 1/2$. Under Breitung's null hypothesis, where x_t is an I(1) process subject to the usual regularity conditions, ξ would correspond to the functional

$$\frac{\int_0^1 \left(\int_0^t W dr - t\int_0^1 W dr\right)^2 dt}{\int_0^1 W^2 dt - \left(\int_0^1 W dt\right)^2},$$

where W denotes standard Brownian motion. However, the theorem also holds when x_t is I(d), for *any* finite d. Some simulations are shown in Figure 1, for cases with $1 \leq d \leq 2$.

FIGURE 1

It will be convenient to write ξ_n in a different form. Let $x := (x_1, x_2, \ldots, x_n)'$ and define the $n \times n$ lower triangular "cumulation" matrix

$$C := \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 1 & 1 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \ldots & 1 \end{pmatrix},$$

as in Tanaka (1996, Equation 1.3). Then,

$$\xi_n = \frac{x'MC'CMx}{n^2 x'Mx},$$

where $M := I_n - (1/n)\imath\imath'$, and $\imath := (1, 1, \dots, 1)'$. Let $A := (1/n)C$ and let Q be an $n \times (n-1)$ matrix containing the eigenvectors associated with the $n-1$ unit eigenvalues of M, so that $M = QQ'$. Letting $y := Q'x$ we have

$$\xi_n = \frac{x'MA'AMx}{x'Mx} = \frac{y'Q'A'AQy}{y'y},$$

so that

$$\xi_n \leq \lambda_{\max}(Q'A'AQ) = \lambda_{\max}(AQQ'A') = \lambda_{\max}(AMA').$$

Hence, the theorem is true if and only if

$$\lambda_{\max}(AMA') \to \frac{1}{\pi^2} \text{ as } n \to \infty.$$

The plan of this paper is as follows. In Section 2 we study the eigenvalues of the matrix AMA', and show that these can be found as the solutions of a particular equation (Theorem 2). In Section 3 we prove a generalization of Theorem 1, which states that, for any fixed j, the j-th largest eigenvalue μ_j of AMA' converges to $1/(j^2\pi^2)$ (Theorem 3). The proof uses the so-called "cotangent trick" associated with Cauchy's residue theorem. In Section 4 we discuss the speed of convergence and the behavior of the whole set of eigenvalues when n is large. Two appendices accompany this paper. In Appendix A we discuss the determinant of the matrix $V - \omega\omega'$ where V is positive semidefinite and ω is a vector. In Appendix B we explain the cotangent trick.

2 The eigenvalues of AMA'

Before proving the theorem we investigate what can be said about the eigenvalues of AMA'. We know that the matrix $A'A$ is positive definite and that its eigenvalues are given by $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$, where

$$\lambda_j = \frac{1}{4n^2 \sin^2(\alpha_j)}, \qquad \alpha_j := \frac{2j-1}{2n+1} \cdot \frac{\pi}{2}, \qquad j = 1, \dots, n. \tag{1}$$

These eigenvalues were first obtained by Rutherford (1946), see also Tanaka (1996, Equation (1.4)). The eigenvalues used in Dickey and Fuller (1979) are the same, but presented in a different form. Since the sine-function is monotonic on the interval $(0, \pi/2)$, the matrix $A'A$ has no multiple eigenvalues.

The matrix AMA' is positive semidefinite and has rank $n-1$. Its ij-th element is given by

$$(AMA')_{ij} = \frac{1}{n^2}\left(\min(i,j) - \frac{ij}{n}\right),$$

By Theorem 11.11 of Magnus and Neudecker (1988, p. 210) we know that its eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$ satisfy

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n.$$

For fixed j and large n we may use the approximation $\sin(\alpha_j) \approx \alpha_j$, giving the bounds

$$\frac{1}{j^2\pi^2} \cdot \frac{(2j)^2}{(2j+1)^2} \leq \mu_j \leq \frac{1}{j^2\pi^2} \cdot \frac{(2j)^2}{(2j-1)^2}. \tag{2}$$

These bounds are, however, not sharp. For example, when $j = 1$, we find $4/(9\pi^2) \leq \mu_1 \leq 4/\pi^2$ as $n \to \infty$ which is not very precise. On the other hand, the inequality

$$\frac{1}{4n^2} + \frac{j^2\pi^2}{4n^2(2n+1)^2} \leq \mu_{n-j} \leq \frac{1}{4n^2} + \frac{(j+1)^2\pi^2}{4n^2(2n+1)^2} \tag{3}$$

is precise and useful; see Section 4.

Since none of the eigenvalues of AMA' can be eigenvalues of $A'A$, the eigenvalue μ_j of AMA' is found as the unique solution of

$$\imath'(I_n - \mu_j(A'A)^{-1})^{-1}\imath = n, \qquad \lambda_{j+1} \leq \mu_j \leq \lambda_j; \tag{4}$$

see Lemma A3 in the Appendix A.

The eigenvectors of $A'A$ are also known (Dickey and Fuller, 1979). Let s_j be the normalized eigenvector associated with λ_j. The i-th element of s_j is given by

$$s_{ij} = \frac{2}{\sqrt{2n+1}} \cos\frac{\pi(2i-1)(2j-1)}{4n+2}, \qquad i = 1, \ldots, n. \tag{5}$$

If we define $S := (s_1, s_2, \ldots, s_n)$, then $S'A'AS = \Lambda := \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Letting $q := S'\imath$, with components $q_1, \ldots, q_n$, we obtain, after some trigonometric simplifications,

$$\begin{aligned} q_j = \sum_{i=1}^{n} s_{ij} &= \frac{2}{\sqrt{2n+1}} \sum_{i=1}^{n} \cos((2i-1)\alpha_j) \\ &= \frac{1}{\sqrt{2n+1}} (-1)^{j-1} \cot(\alpha_j), \end{aligned}$$

which implies that

$$\frac{q_j^2}{n} = \frac{4n^2\lambda_j - 1}{n(2n+1)}. \tag{6}$$

Note that $\imath'(I_n - \mu_j(A'A)^{-1})^{-1}\imath = n$ if and only if $q'(I_n - \mu_j\Lambda^{-1})^{-1}q = n$. The eigenvalues $\mu_1, \dots, \mu_n$ of AMA' are thus found as the n solutions of

$$\sum_{j=1}^{n} \left(\frac{q_j^2}{n} \cdot \frac{\lambda_j}{\lambda_j - \mu} \right) = 1. \tag{7}$$

But since

$$\begin{aligned} \frac{q_j^2}{n} \cdot \frac{\lambda_j}{\lambda_j - \mu} &= \frac{4n^2\lambda_j - 1}{n(2n+1)} \cdot \frac{\lambda_j}{\lambda_j - \mu} \\ &= \frac{4n}{2n+1} \left(\lambda_j + (\mu - \frac{1}{4n^2}) \frac{\lambda_j}{\lambda_j - \mu} \right), \end{aligned}$$

and $\sum_j \lambda_j = \operatorname{tr}(A'A) = (n+1)/(2n)$, we have proved

Theorem 2. *The nonzero (positive) eigenvalues $\mu_1 > \cdots > \mu_{n-1}$ of AMA' are the $n-1$ solutions of the equation*

$$n \sum_{j=1}^{n} \frac{1}{g_n(j) - 1/\mu} = \frac{\mu n^2}{4\mu n^2 - 1},$$

where

$$g_n(j) := 4n^2 \sin^2(\alpha_j).$$

3 Generalization and proof of Theorem 1

To prove Theorem 1 we need to demonstrate that Theorem 2 holds asymptotically for $\mu = 1/\pi^2$, that is, that

$$n \sum_{j=1}^{n} \frac{1}{g_n(j) - \pi^2} \to \frac{1}{4}$$

as $n \to \infty$. We shall prove a more general result, namely that Theorem 2 holds asymptotically for $\mu = 1/(k^2\pi^2)$. This implies that the k-th largest eigenvalue μ_k of AMA' converges to $1/(k^2\pi^2)$ as $n \to \infty$. Theorem 1 is the special case when $k = 1$.

Theorem 3. *Let k be a fixed positive integer, and let*

$$S_k(n) := \sum_{j=1}^{n} \frac{1}{g_n(j) - k^2\pi^2}.$$

Then we have

$$nS_k(n) \to \frac{1}{4}$$

as $n \to \infty$.

Proof. The proof is based on Cauchy's residue theorem (see Appendix B), and consists of applying the so-called cotangent trick. This trick allows us to compute $S_k(n)$ in almost closed form. Let

$$g_n(z) := 4n^2 \sin^2\left(\frac{2z-1}{2n+1}\cdot\frac{\pi}{2}\right), \qquad f(z) := \frac{1}{g_n(z) - k^2\pi^2},$$

where we have suppressed the dependency of f on n. We will apply the theory of Appendix B to the function $f(z)$ on a rectangle Γ_n, with sides parallel to the real and imaginary axes, its vertical sides being intervals on the two lines $L_0 := \{x = 1/2\}$ and $L_n := \{x = n + 1/2\}$, respectively. It turns out that the integral over Γ_n is of the order $o(1/n)$ as $n \to \infty$. Thus the $S_k(n)$ will appear as a (finite) sum of residues plus a term of order $o(1/n)$.

Recall that $2i\sin(z) = e^{iz} - e^{-iz}$, so that $|\sin(x+iy)| \approx 2e^{|y|}$ for $|y|$ large. Taking orientation into account, it then follows from Remark B1 in Appendix B that we can shift the horizontal sides of our rectangle to ∞. Defining $F(z) := \pi\cot(\pi z) f(z)$, this gives

$$\int_{\Gamma_n} F(z)\,dz = \int_{L_n} F(z)\,dz - \int_{L_0} F(z)\,dz.$$

On L_n we have $z = n + 1/2 + iy$, so that $\cot(\pi z) = -\tan(i\pi y)$, which is bounded by one in absolute value. Also,

$$\begin{aligned} g_n(n+1/2+iy) &= 4n^2\sin^2\left(\frac{2n+2iy}{2n+1}\cdot\frac{\pi}{2}\right) \\ &= 4n^2\cos^2\left(\frac{-1+2iy}{2n+1}\cdot\frac{\pi}{2}\right). \end{aligned} \tag{8}$$

We now show that $n\int_{L_n} F(z) \to 0$ as $n \to \infty$. Using the fact that $\cot(\pi z) = -\tan(i\pi y)$ and (8), we substitute $y = ns/\pi$ and obtain

$$\begin{aligned} n\int_{L_n} F(z)\,dz &= n\int_{-\infty}^{\infty} \frac{-\pi\tan(i\pi y)}{4n^2\cos^2\left(\left(\frac{-1}{2n+1} + \frac{2iy}{2n+1}\right)\frac{\pi}{2}\right) - k^2\pi^2}\,i\,dy \\ &= \int_{-\infty}^{\infty} \frac{-\tan(ins)}{4\cos^2\left(\frac{-\pi}{2(2n+1)} + \frac{ins}{2n+1}\right) - \left(\frac{k\pi}{n}\right)^2}\,i\,ds. \end{aligned}$$

Lebesgue's dominated convergence theorem applies so that we may interchange limit and integral. Thus,

$$\lim_{n\to\infty} n \int_{L_n} F(z)\, dz = \int_{-\infty}^{\infty} \frac{\mathrm{sgn}(s)}{4\cos^2(is/2)}\, ds = 0,$$

using the fact that $\lim_{n\to\infty} -i\tan(ins) = \mathrm{sgn}(s)$, the sign of s. Similarly, the integral over L_0 is $o(1/n)$.

It remains to compute the residues of F at the singular points that come from f, that is, at the zeros of $g_n(z) - k^2\pi^2$ that are located inside Γ_n. We write

$$g_n(z) - k^2\pi^2 = \left(2n\sin\left(\frac{2z-1}{2n+1}\cdot\frac{\pi}{2}\right) + k\pi\right)\left(2n\sin\left(\frac{2z-1}{2n+1}\cdot\frac{\pi}{2}\right) - k\pi\right).$$

The zeros of this equation are solutions of

$$\frac{2z-1}{2n+1}\cdot\frac{\pi}{2} = \pm\arcsin(k\pi/2n),$$

which we rewrite as

$$z = \frac{1}{2} \pm \frac{2n+1}{\pi}\arcsin(k\pi/2n) \approx \frac{1}{2} \pm \frac{k(2n+1)}{2n} \mod (2n+1).$$

Of these solutions only the one close to $1/2 + k$ is inside Γ_n. We compute the residue at the solution $z_1 \approx 1/2 + k$, using Remark B2, l'Hôpital's rule, and the fact that $2n\sin\left(\frac{2z_1-1}{2n+1}\cdot\frac{\pi}{2}\right) = k\pi$. This gives

$$\begin{aligned}
\mathrm{Res}(F, z_1) &= \lim_{z\to z_1} \frac{z - z_1}{2n\sin\left(\frac{2z-1}{2n+1}\cdot\frac{\pi}{2}\right) - k\pi} \cdot \frac{\pi\cot(\pi z)}{2n\sin\left(\frac{2z-1}{2n+1}\cdot\frac{\pi}{2}\right) + k\pi} \\
&= \frac{1}{2n\frac{\pi}{2n+1}\cos\left(\frac{2z_1-1}{2n+1}\cdot\frac{\pi}{2}\right)} \cdot \frac{\pi\cot(\pi z_1)}{2n\sin\left(\frac{2z_1-1}{2n+1}\cdot\frac{\pi}{2}\right) + k\pi} \\
&= \frac{1}{\frac{2n\pi}{2n+1}\sqrt{1 - \left(\frac{k\pi}{2n}\right)^2}} \cdot \frac{\pi\cot(\pi z_1)}{2k\pi}.
\end{aligned}$$

Hence the limit becomes

$$\begin{aligned}
\lim_{n\to\infty} n\,\mathrm{Res}(F, z_1) &= \lim_{n\to\infty} n\frac{\cot(\pi z_1)}{2k\pi} \\
&= \lim_{n\to\infty} \frac{-n\tan((2n+1)\arcsin(k\pi/(2n)))}{2k\pi} = -1/4.
\end{aligned}$$

Then, using (13), we conclude that $\lim_{n\to\infty} nS_k(n) = 1/4$, and this completes the proof of Theorem 3. ∎

4 Further discussion

In fact we can say a little more. Theorem 3 tells us that the j-th largest eigenvalue μ_j of AMA' converges to $1/(j^2\pi^2)$ as $n \to \infty$. But it does not tell us how fast this convergence takes place, nor does it tell us what happens to the whole set of eigenvalues. Further analysis, not provided here, shows that, for $n \geq 4$,

$$\mu_j = \frac{1}{j^2\pi^2} + \frac{c_j}{n^2\pi^2} \qquad (j = 1, \ldots, n-1), \tag{9}$$

where, for each given n, c_j is a slowly increasing function of j bounded by $\underline{c} \leq c_j \leq \overline{c}$, with

$$\underline{c} := \lim_{n\to\infty} n^2 \left(\pi^2\mu_1 - 1\right) = \frac{\pi^2}{12} \approx 0.8225$$

and

$$\overline{c} := \lim_{n\to\infty} n^2\pi^2\mu_{n-1} - 1 = \frac{\pi^2}{4} - 1 \approx 1.4674,$$

and satisfying $(1/n)\sum_j c_j \to 1$. In particular,

$$\max_{1\leq j\leq n-1} \left|\mu_j - \frac{1}{j^2\pi^2}\right| \leq \frac{1}{n^2} \cdot \frac{\pi^2 - 4}{4\pi^2} \approx \frac{0.1487}{n^2}.$$

Hence we may say that "the set of eigenvalues $\{\mu_j\}$ of AMA' converges to the set $\{\mu_j^*\}$," where $\mu_j^* = 1/(j^2\pi^2)$ for $j = 1, \ldots, n-1$ and $\mu_n^* = 0$. Some caution is however required in interpreting this phrase. For fixed j it means that μ_j approaches $1/(j^2\pi^2)$. But for j dependent on n, it only means that the difference approaches zero. Thus, $\mu_{n-j} \approx 1/(4n^2)$ (see (3)), while $\mu_{n-j}^* = 1/((n-j)^2\pi^2) \approx 1/(n^2\pi^2)$. Their difference is of order $O(1/n^2)$, but μ_{n-j} is better approximated by $1/(4n^2)$ than by μ_{n-j}^*.

We note that our findings agree with the facts that

$$\sum_{j=1}^{n} \mu_j = \operatorname{tr}(AMA') = \frac{n^2-1}{6n^2} = \frac{1}{6} + O(\frac{1}{n^2})$$

and

$$\sum_{j=1}^{n} \mu_j^2 = \operatorname{tr}(AMA')^2 = \frac{(n^2-1)(2n^2+7)}{180n^4} = \frac{1}{90} + \frac{1}{36n^2} + O(\frac{1}{n^4}),$$

while we also know that

$$\sum_{j=1}^{n} \frac{1}{j^2} = \frac{\pi^2}{6} - \frac{1}{n} + O(\frac{1}{n^2}), \qquad \sum_{j=1}^{n} \frac{1}{j^4} = \frac{\pi^4}{90} + O(\frac{1}{n^3}).$$

Hence we obtain

$$n\left(\sum_{j=1}^{n}\mu_j-\sum_{j=1}^{n}\frac{1}{j^2\pi^2}\right)=\frac{1}{\pi^2}+O(\frac{1}{n})$$

and

$$n^2\left(\sum_{j=1}^{n}\mu_j^2-\sum_{j=1}^{n}\frac{1}{j^4\pi^4}\right)=\frac{1}{36}+O(\frac{1}{n}).$$

We conclude by noting that Theorem 1 also provides an asymptotic bound for various statistics related to ξ_n. Suppose first that M is replaced by M_Z, the projection matrix orthogonal to some collection of nonstochastic regressors Z, including the intercept. The obvious example is the inclusion of the linear trend. Since $M_Z = MM_ZM$, Theorem 11.11 of Magnus and Neudecker (1988) may be redeployed to show that the ordered eigenvalues of AM_ZA' are bounded by the corresponding eigenvalues of AMA'. On the other hand, if M is replaced by I so that we consider the version of ξ_n obtained from non-centred data, the exact bound for the ratio $x'A'Ax/x'x$ is provided, directly from (1), by the case $j = 1$ of the limits

$$\lambda_j \to \frac{1}{\pi^2(j-\frac{1}{2})^2}.$$

As Breitung (2002) points out, ξ_n corresponds to n^{-1} times the so-called KPSS statistic of Kwiatkowski *et al.* (1992), except that the kernel estimator of the long-run variance in the denominator of the latter statistic is replaced by the simple variance of the sample. A recently proposed variant of the KPSS test is the V/S test of Giraitis *et al.* (2003), in which the the numerator of the ratio is expressed in mean deviation form. The corresponding modification of Breitung's statistic takes the form $\xi_n^* = x'MA'MAMx/x'Mx$. If $\xi_n^* \to_d \xi^*$, observe that when, in particular, x is an I(1) vector,

$$\xi^*=\frac{\int_0^1\left(\int_0^t W\,dr-t\int_0^1 W\,dr\right)^2 dt-\left[\int_0^1\left(\int_0^t W\,dr-t\int_0^1 W\,dr\right)\,dt\right]^2}{\int_0^1 W^2\,dt-\left(\int_0^1 W\,dt\right)^2}.$$

In view of the general inequality $0 \le \xi_n^* \le \xi_n$ we are able to say that Theorem 1 also applies to ξ^*.

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Appendix A: The matrix $V - \omega\omega'$

We consider a positive semidefinite $n \times n$ matrix V, and an $n \times 1$ vector ω.

Lemma A1. *We have*

$$|V - \omega\omega'| = \begin{cases} |V|(1 - \omega' V^{-1}\omega) & \textit{if } \operatorname{rk}(V) = n; \\ -(\omega' x)^2 \cdot p(V) & \textit{if } \operatorname{rk}(V) = n-1, Vx = 0, x'x = 1; \\ 0 & \textit{if } \operatorname{rk}(V) \leq n-2, \end{cases}$$

where $p(V)$ denotes the product of the nonzero eigenvalues of V.

Proof. We prove the lemma first for a diagonal $n \times n$ matrix Λ with nonnegative diagonal elements, and an $n \times 1$ vector a.
(i) If all diagonal elements of Λ are nonzero (hence positive), then

$$|\Lambda - aa'| = |\Lambda^{1/2}(I_n - \Lambda^{-1/2}aa'\Lambda^{-1/2})\Lambda^{1/2}| = |\Lambda|(1 - a'\Lambda^{-1}a).$$

(ii) If one of the diagonal elements of Λ is zero (say, the n-th), then we partition

$$\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

If $a_2 \neq 0$, then

$$|\Lambda - aa'| = \begin{vmatrix} \Lambda_1 - a_1a_1' & -a_2a_1 \\ -a_2a_1' & -a_2^2 \end{vmatrix} = -a_2^2|\Lambda_1|,$$

see Exercise 5.30(b) in Abadir and Magnus (2005). The result remains true when $a_2 = 0$, because both sides of the equality are then zero.
(iii) If two or more diagonal elements of Λ are zero, then

$$\mathrm{rk}(\Lambda - aa') \leq \mathrm{rk}(\Lambda) + \mathrm{rk}(aa') \leq (n-2) + 1 = n - 1,$$

and hence $|\Lambda - aa'| = 0$.

In the general case, we diagonalize V as $S'VS = \Lambda$ and define $a := S'\omega$. The results (i) and (iii) follow immediately. For (ii) we partition $S = (S_1, x)$, so that $\omega'x = a'S'x = a_1'S_1'x + a_2x'x = a_2$. ■

Lemma A2. $|V - \omega\omega'| = 0$ *if and only if*

$$\begin{cases} \mathrm{rk}(V) = n, \ \omega'V^{-1}\omega = 1; \\ \mathrm{rk}(V) = n - 1, \ \omega'x = 0, \ Vx = 0; \\ \mathrm{rk}(V) \leq n - 2. \end{cases}$$

Proof. This follows directly from Lemma A1. ■

Lemma A3. *Let C be a nonsingular $n \times n$ matrix, and let $M := I_n - (1/n)\imath\imath'$. Then, μ is an eigenvalue of CMC' if and only if*

$$\begin{cases} \mu \textit{ is not an eigenvalue of } CC', \ \imath'(I_n - \mu(C'C)^{-1})^{-1}\imath = n; \\ \mu \textit{ is a simple eigenvalue of } CC' \textit{ with associated eigenvector } x, \ \imath'C'x = 0; \\ \mu \textit{ is a multiple eigenvalue of } CC'. \end{cases}$$

Proof. The eigenvalues of CMC' are given by

$$\left| CC' - \mu I_n - \frac{1}{n}C\imath\imath'C' \right| = 0.$$

Let $V =: CC' - \mu I_n$ and $\omega := C\imath/\sqrt{n}$, and apply Lemma A2. ■

Appendix B: Cauchy's residue theorem and the cotangent trick

In this appendix we state some results from complex function theory that are required in the proof of Theorem 3; see Conway (1978). Our proof depends heavily on the "cotangent trick." Although this trick is "well-known," it is not easy to find a good reference for it. (Conway (1978) has an exercise on page 122.) Hence it will prove useful to state this trick explicitly. Our starting point is

Theorem B1 (Cauchy's residue theorem). *Let f be a function defined on a domain D and its boundary Γ. Assume that f is holomorphic (that is, complex differentiable) except for a finite set of singular points $a_1, \dots, a_k$ in D. Then,*

$$\frac{1}{2\pi i}\int_\Gamma f(z)\,dz = \sum_{h=1}^{k} \operatorname{Res}(f, a_h), \tag{10}$$

where $\operatorname{Res}(f,a)$ *denotes the residue of f at an isolated singularity a, and is defined as*

$$\operatorname{Res}(f,a) := \lim_{\epsilon\to 0} \frac{1}{2\pi i}\int_{|z-a|=\epsilon} f(z)\,dz.$$

We note that, for ϵ sufficiently small, the integral above becomes independent of ϵ. Moreover, one can show that f may be developed into a convergent series near a:

$$f(z) = \sum_{l=-\infty}^{\infty} c_l (z-a)^l, \tag{11}$$

so that $\operatorname{Res}(f,a) = c_{-1}$.

Remark B1. If there are no singularities inside the boundary Γ, then the integral in (10) equals zero. This allows for changing the path of integration without altering the value of the integral as long as we do not "cross" any singularities.

Remark B2. It follows from (11) that if f has a singular point at a and if $L := \lim_{z\to a} f(z)(z-a)$ exists, then $\operatorname{Res}(f,a) = L$.

The cotangent trick is based on the following expansion:

$$\pi\cot(\pi z) = \sum_{j=-n}^{n} \frac{1}{z-j} + g_n(z) = \sum_{j=-\infty}^{\infty} \frac{1}{z-j}.$$

Here g_n is a holomorphic function on $\mathbb{C}$ except for singularities at $\pm(n+1), \pm(n+2), \ldots$. (The last equality should be treated with care, because the sum is only conditionally convergent.)

Now, if f is a holomorphic function with singularities at $a_1, \ldots, a_k$, say all less than n and none of which are integers, and if we let $F(z) := \pi \cot(\pi z) f(z)$, then by Theorem B1,

$$\frac{1}{2\pi i} \int_{\Gamma_n} F(z)\, dz = \sum_{a \in A} \operatorname{Res}(F, a), \tag{12}$$

where Γ_n is a square in $\mathbb{C}$ with vertices $(\pm(n+1/2), \pm(n+1/2)i)$, and A is the set of singularities of F inside the square Γ_n. Since the residue $\operatorname{Res}(F, j)$ at an integer j equals $f(j)$, we can rewrite (12) as

$$\sum_{j=-n}^{n} f(j) = \frac{1}{2\pi i} \int_{\Gamma_n} F(z)\, dz - \sum_{h=1}^{k} \operatorname{Res}(F, a_h). \tag{13}$$

This is the cotangent trick. Observing that the cotangent is uniformly bounded on the squares Γ_n, one can often show that the integral in (13) tends to zero as $n \to \infty$. For suitable f with finitely many singularities one can then compute the limit of the left hand sum in (13).

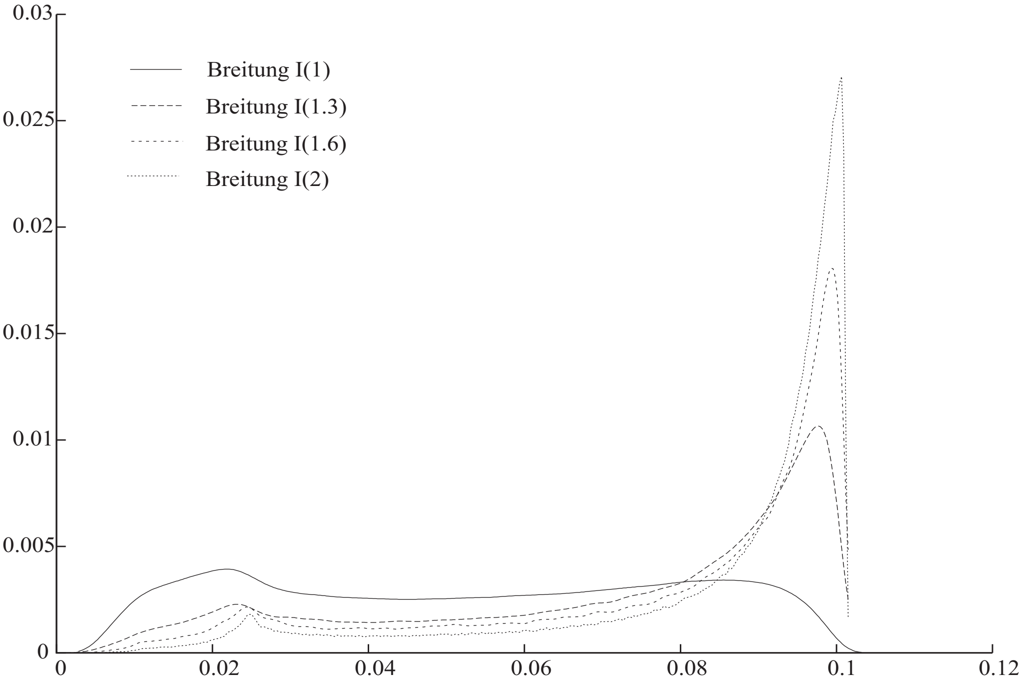

Figure 1. Kernel density plots of Breitung's statistic computed from one million replications of I(d) series ($n = 1000$).