UNCERTAINTY, CO-ORDINATION AND PATH DEPENDENCE*

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Abstract

Our objective is to assess whether dynamics hinder or assist co-ordination in a game with strategic complementarities. We study two dynamic aspects: different agents make decisions at different points in time; and extra information about a payoff-relevant state of nature becomes available over time. We find that the dynamic resolution of information matters most for uniqueness of equilibrium. This is demonstrated by showing that the condition for uniqueness is weaker when learning occurs. We also analyse how successfully agents co-ordinate when there is a unique equilibrium. Finally, we show that path dependence occurs: the order in which signals arrive matters, as well as the total amount of information received.

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1 Introduction

Games with strategic complementarities have been the subject of much study in economics, for a number of reasons. They have proved useful for the analysis of a variety of important economic situations, such as currency attacks, financial contagion, bank runs, poverty traps and technology choice with network externalities. In addition, these games provide a tractable environment in which to understand further the role of common knowledge, information and beliefs in Bayesian games. A central concern of this research has been the issue of equilibrium multiplicity. Games with strategic complementarities typically have more than one equilibrium. When the multiple equilibria can be Pareto ranked, this leads naturally to the question of whether the Pareto-preferred equilibrium will in fact be played. Multiple equilibria also present familiar difficulties for an analyst who wishes to predict the outcome of a particular game, or to conduct policy evaluation.

Most applied and theoretical studies have concentrated on one-shot, static games. (Exceptions to this statement are reviewed below.) In this paper, we examine a dynamic game with strategic complementarities. Two features make the game dynamic: first, different agents make decisions at different points in time; secondly, extra information about a payoff-relevant state of nature becomes available over time. Our objective is to assess whether these dynamic features hinder or assist the co-ordination process. In particular, we ask whether equilibrium multiplicity is more or less likely in a dynamic setting; and, when there is a unique equilibrium, how successfully agents co-ordinate in that equilibrium.

We study a two-period game with one risk-less and one risky action. In the fully dynamic version of the game, a mass one of agents acts in each of the periods. Agents who act in the first period receive a noisy signal about an underlying state of nature that affects payoffs. Agents who act in the second period observe not only the first period signal and how many agents chose the risky action in period 1; they observe also a second (noisy) signal about the state. If a sufficient mass of agents across the two periods choose the risky action, then those who chose it receive an additional payoff from successful co-ordination. Hence the two sets of agents face different situations. Agents acting in the first period do so with more limited information. The agents who act in the second period have superior information, but face the irreversible decisions taken by agents in the firs period.

Our results come in two parts. In the first part, we show how the timing of decisions and information resolution affects the outcome of co-ordination. A key measure for the analysis is the *co-ordination effect*—the effect that agents have on the probability of successful co-ordination. The size of the co-ordination effect is critical. If it is large, and if all agents choose the risk-less (risky) action, then there is a low (high) probability that co-ordination will be successful. This sensitivity of the final outcome to agents' decisions leads naturally to multiple equilibria. In contrast, if agents' decisions have little effect on the probability of successful co-ordination, then a unique equilibrium will result. In short, a necessary and sufficient condition for equilibrium uniqueness is that co-ordination effects are small.

As a benchmark, we first consider a static version of the model: agents act simultaneously and all information (signals about the state) are received before decisions are made. An intuitive property holds: co-ordination effects are small when fundamental uncertainty (measured by the variance of posterior beliefs about the state) is high. At extreme values of the underlying state, even co-ordinated action by the agents cannot influence the probability of successful co-ordination—if the state is very low, then there is little prospect of successful co-ordination; the converse holds if the state is high. In these cases, then, agents' payoffs are largely determined by the state of nature, co-ordination is irrelevant, and there is a unique equilibrium. When the posterior variance is high, there is a high probability that the underlying state takes an extreme value, and hence a high probability that co-ordination is irrelevant. Hence high posterior variance (or 'fundamental uncertainty') leads to a unique equilibrium. (A similar result has been pointed out in Morris and Shin (2005).)

This first step concentrates on conditions for equilibrium uniqueness and multiplicity. In this sense, greater uncertainty assists co-ordination by ensuring a unique equilibrium. It also has implications for how successfully agents co-ordinate in equilibrium. We show that when signals about the underlying state are sufficiently weak, greater uncertainty ensures a unique equilibrium with a greater degree of co-ordination (and hence higher payoffs) than the Pareto-preferred outcome when there are multiple equilibria. But when signals are sufficiently strong, the unique equilibrium has lower co-ordination and payoffs than any outcome when there are multiple equilibria. We also show that greater heterogeneity (in agents' idiosyncratic valuations) always assists co-ordination, by ensuring a unique equilibrium with more co-ordination than the Pareto-preferred outcome when there are multiple equilibria.

Next, we show that the co-ordination effect for first-movers is greater than for second movers. This means that if there is a unique equilibrium strategy for early movers, then there is a unique equilibrium overall. Hence we concentrate on the co-ordination effect for early movers.

Finally, we show that what matters most is the dynamic resolution of information. To show this, we consider two more versions of the model. In the second version, agents act sequentially (half of them in the first period, half in the second), but all signals are received before the early movers make their decisions. In the third, fully dynamic version, agents act sequentially and the two signals arrive sequentially at the beginning of each period. In all three versions, there is a unique equilibrium if fundamental uncertainty is sufficiently large. In the first two versions (i.e., the static and partially dynamic cases), the critical value is equal—that is, spreading agents' decisions over time makes no difference to the condition for equilibrium uniqueness in this model. In contrast, the critical value is less in the third, fully dynamic version; and is smaller when the precision of the signals is high. In short, dynamic learning about an unknown state helps to eliminate equilibrium multiplicity.

The importance of learning for equilibrium determination suggests a second set of results, which we label path dependence. As the choices of early movers have an effect on late movers, signals arriving in the first period have a greater effect on the equilibrium outcome. So, two models with the same aggregate information (i.e., the signals in total convey the same information about the state) can have quite different equilibrium paths when the signals arrive in different orders.

We formalize this possibility in several different ways. An increase in the mass of agents moving in the first period shifts the distribution of agents moving in the second period in a first-order stochastic dominance sense. A direct implication of this result is that an early signal has a bigger effect on the equilibrium outcome than a late one. This raises the issue of distinguishing equilibrium multiplicity from path dependence.

An early study of dynamic co-ordination games was provided by Farrell and Saloner (1985). In a complete information model, they find that fully sequential adoption prevents co-ordination failures, ensuring a unique, efficient (sub-game perfect) equilibrium. In the incomplete information version of the game, neither efficiency nor uniqueness of equilibrium can be ensured.[1] The game that we study differs from the two games in Farrell and Saloner in a number of ways. We want to examine the effects of learning; so we include, as well as incomplete information about the types of players, uncertainty about an underlying state of nature that decreases (through the sequential arrival of informative signals) over time. Furthermore, in our model, a large number of agents moves in each period; hence we study co-ordination both within and between groups. In contrast, Farrell and Saloner concentrate on the between-groups problem. The between-group problem is, of course, important; but we argue that within-group co-ordination is equally important, particularly in applications, and merits study.

Two more recent papers that derive unique equilibria in dynamic games with strategic complementarities are Burdzy, Frankel, and Pauzner (2001) and Herrendorf, Valentinyi, and Waldman (2000). Both explicitly introduce heterogeneity among agents and show that sufficient heterogeneity can ensure equilibrium uniqueness. (In Herrendorf, Valentinyi, and Waldman (2000), agents differ in their productivity in the increasing returns-to-scale sector in the Matsuyama (1991) two-sector model. In the same setting, Burdzy, Frankel, and Pauzner (2001) have agents who are different in their ability to revise their strategies.) The focus of both papers is different from ours, and so neither concentrate on the role of sequential actions and signals in ensuring uniqueness of equilibrium.

A paper apparently close to this one is Dasgupta (forthcoming). His objectives are similar to

[1] The potential inefficiency of equilibrium is not surprising. Farrell and Saloner show that there is a unique symmetric equilibrium, but cannot rule out the existence of asymmetric equilibria.

ours: he aims to investigate how separating decisions over time, and the sequential arrival of information, affect a co-ordination game. His set-up is, however, quite different. He bases his model on a global game. Global games are games of incomplete information whose type space is determined by the players each observing a noisy signal of an underlying state; see Carlsson and van Damme (1993), Morris and Shin (1998), and Morris and Shin (2005). Dasgupta looks at the equilibria in monotone pure strategies that emerges as noise in his model becomes small. He considers the uniqueness and efficiency of equilibrium in this limit. A complication of his model is that agents' types are (unconditionally) correlated.[2] As a result, he is not able to eliminate the possibility that there are other, non-monotone equilibria. In contrast, in our model, agents' types are independent; an immediate implication is that any equilibrium must be in monotone pure strategies (see lemma 1). When we find a unique equilibrium in this paper, therefore, we can be sure that no other equilibria exist. We analyse the conditions under which there is a unique equilibrium. We find that we require sufficiently noisy signals of the underlying state. The more homogeneous are the agents (this case being the closest equivalent to Dasgupta's limit case), the greater must be the noise to ensure equilibrium uniqueness.

Path dependence arises also in the literature on herding. Since Banerjee (1992) and Bikhchandani, Hirshleifer, and Welch (1992), numerous papers have shown that models in which there is sequential learning from other agents' action choices produce an extreme form of path dependence. In these models, the action choices of agents who make decisions after observing others' choices may depend entirely on those earlier choices i.e., later movers ignore entirely their own private information. The present paper identifies a different form of path dependence, which, unlike the informational externality in the herding literature, arises due to the complementarity among choices. Since there is no (relevant) private information in our model, there is no issue of information aggregation—the crucial ingredient for path dependence in the herding literature. The path dependence in the present paper follows because earlier choices change the payoff from the risky action choice.

The paper is organized as follows. In section 2, a simple two-period model is developed in which agents' payoffs are affected by the possibility of co-ordinated action with other agents, and an unknown state of nature. (The model can be interpreted in terms of investors facing a risky project; or a firm selling a durable good with uncertain demand.) Section 3 starts with a preliminary analysis where the necessary and sufficient condition for equilibrium uniqueness is obtained in a simplified, static model; following subsections extend the analysis to the dynamic case, showing that the dynamic setting provides a mechanism which facilitates co-ordination. Section 4 analyzes the dependence of the equilibrium outcome on the order in which information about the state arrives.

[2]Each agent receives a noisy signal of the unknown state. Hence the agents' types in the Bayesian game are correlated by the common state on which their signals are based; although conditioned on the state, their types are independent.

The final section concludes. The appendix contains the proof of lemma 2 and an extension to the basic model to test the robustness of our conclusions.

2 Model

Let the state of the world be denoted $\theta \in \mathbf{R}$ which is not observed by agents. All agents have a common prior on θ which is normally distributed with mean μ_0 and variance σ_0^2. In each period $t \in \{1, 2\}$, a signal X_t is drawn and observed by all agents.[3] The signal X_t in period t is determined by $X_t = \theta + \epsilon_t$; $\{\epsilon_t\}_{t\in\{1,2\}}$ are drawn independently from the same normal distribution with zero mean and variance σ_ϵ^2.

For convenience we collect the following standard results in Bayesian updating for normal distributions. After a signal draw $X_1 = x_1$, the agents' posterior is normally distributed with mean $\mu_1 = (\sigma_\epsilon^2\mu_0 + \sigma_0^2 x_1)/(\sigma_0^2 + \sigma_\epsilon^2)$ and variance $\sigma_1^2 = \sigma_0^2\sigma_\epsilon^2/(\sigma_0^2 + \sigma_\epsilon^2)$. After signal draws $X_1 = x_1$ and $X_2 = x_2$, the agents' posterior is normally distributed with mean $\mu_2 = (\sigma_\epsilon^2\mu_0 + \sigma_0^2(x_1 + x_2))/(\sigma_0^2 + 2\sigma_\epsilon^2)$ and variance $\sigma_2^2 = \sigma_0^2\sigma_\epsilon^2/(\sigma_0^2 + 2\sigma_\epsilon^2)$.

In each period, there is a continuum of agents of total mass 1, represented by the unit interval $[0, 1]$, making the total population 2. Agents must choose an action, y, from a binary action space, $\{0, 1\}$. Choosing action 0 guarantees the agent zero payoff. On the other hand, the utility from choosing action 1 consists of *i)* ζ, which is an idiosyncratic parameter in the agent's utility, *ii)* the cost of choosing the action, which is normalized to 1, and *iii)* γ, which is the extra payoff in the event of co-ordination success. The first component, ζ, represents the heterogeneity among agents and is assumed to be uniformly distributed on the interval $[0, \beta]$, $\beta > 0$, throughout the population of agents each period. The third component, $\gamma > 0$, represents the benefit of successful co-ordination. The agent receives it if the sum of the random state, θ, and the numbers of agents choosing action 1 in periods $t \in \{1, 2\}$, is greater than some threshold; the size of the threshold matters little in the subsequent analysis, so we set it equal to 1. Otherwise the agent receives zero from this component. Hence choosing action $y = 1$ yields the following utility:

$$U_\zeta(y = 1|\, \theta, \alpha_1, \alpha_2) = \begin{cases} \zeta - 1 + \gamma & \text{if} \quad \theta + \alpha_1 + \alpha_2 \geq 1, \\ \zeta - 1 & \text{if} \quad \theta + \alpha_1 + \alpha_2 < 1 \end{cases} \tag{1}$$

where α_t is the mass of agents who choose action 1 in period t.

Given α_t representing the strategy choices of agents in period $t \in \{1, 2\}$, the expected utility of

[3] We use the notational convention that a Roman alphabet denoting a random variable is written in upper-case and its realization is written in lower-case.

an agent who chooses action y in period t on receiving signal X_t is:

$$\mathbb{E}[U_\zeta(y|\,\theta,\alpha_1,\alpha_2)|X_t] = [\zeta - 1 + \gamma\,\Pr(\theta + \alpha_1 + \alpha_2 \geq 1|X_t)]y. \tag{2}$$

The timing of the game in each period is that all agents observe the same random signal X_t, and then choose an action simultaneously. The game is repeated twice with the signal and the choices of the first period revealed to the agents before the second period decision. The information sets Ω_t for all agents at time t are, therefore, $\Omega_1 = \{X_1\}$ and $\Omega_2 = \{X_1, X_2, \alpha_1\}$. Notice that α_2 and X_2 are both random variables in period 1. In period 2, the signal X_2 as well as the first period signal, X_1, and fraction of agents choosing action 1 in the first period, α_1, are revealed to the agents.

This model can be generated by a number of stories. For example, a firm operates with an existing debt of D which has to be serviced out of the firm's profit at the end. If the profit is less than debt service requirement, then the firm goes bankrupt. Profit is earned from selling to consumers; in addition, random shocks affect the firms' profit. Each consumer has unit demand, and gains additional utility if the firm is not bankrupt at the end (this utility may come from e.g., continued availability of parts after purchase in the case of durable goods).[4] Alternatively, consider a group of investors deciding whether to invest in a safe project (action 0) or a risky project (action 1). The risky project succeeds only if a critical mass of investors backs it and the underlying state is favourable. Information about the state arrives over time and so investors have the chance of learning more about the probability of success before acting.

2.1 Complete information benchmark

As a benchmark, first consider the complete information game in which all agents observe θ (which is therefore common knowledge), and choose an action simultaneously. A (pure) strategy for an agent is mapping from its type ζ to an action. For extreme values of θ, there is clearly a unique equilibrium. When $\theta < -1$, an agent chooses action 0 iff its type $\zeta < 1$, otherwise it chooses action 1. (Obviously, if $\beta < 1$, then no agent chooses action 1 in this case.) In fact, this equilibrium exists for all values of $\theta \leq (2-\beta)/\beta$. When $\theta > 1$, an agent chooses action 0 iff its type $\zeta < 1-\gamma$, otherwise it chooses action 1. (Obviously, if $\gamma > 1$, then no agent chooses action 0 in this case.) In fact, this equilibrium exists for all values of $\theta \geq (2-\beta-2\gamma)/\beta$. We illustrate these equilibria in figure 1, in which we assume that $\beta > 1 > \gamma$. In the figure, ζ^* refers to the type that is indifferent between actions 0 and 1.

[4]The model can be applied to various settings other than the case of durable goods. For instance, a factor supplier might be concerned about the financial viability of a firm and willing to invest in the relationship only if the firm is likely to survive. The investment decision of the factor supplier influences the firm's production cost which in turn affects the financial viability of the firm.

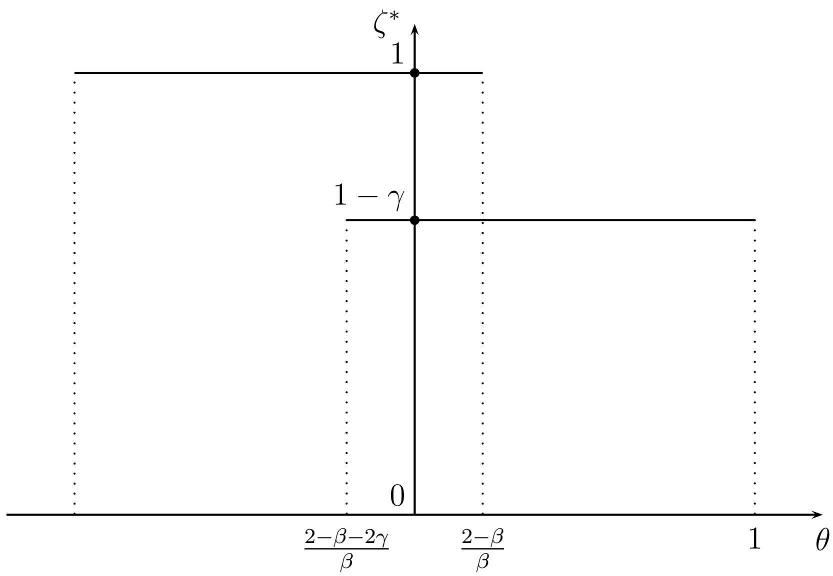

Figure 1: Complete information equilibria

As the figure demonstrates, there are multiple equilibria in the complete information game when θ is in the interval $[(2-\beta-2\gamma)/\beta, (2-\beta)/\beta]$, if $\beta > 1$ and $\gamma < 1$. In the remainder of the paper, we shall assume that $\beta > 1 > \gamma$. We note, however, that none of our arguments rely particularly on these assumptions. They simplify the analysis, by reducing the number of cases that has to be considered. With these parametric restrictions, there are positive masses of agents for whom it is strictly dominant to choose action 0 and action 1. But our general results do not rely on this simplification, and apply also when there are equilibria in which all agents choose either action 0 or 1. (We expand on this remark in section 3.1.)

2.2 Incomplete information game

When choosing an optimal strategy in the incomplete information game, agents rely on a posterior obtained from the prior and the signal according to Bayes' rule. We employ the concept of perfect Bayesian equilibrium where each agent's strategy maximizes its expected utility given the strategies of all other agents and its Bayesian posterior.

Definition 1 *The agents' choices $y(\zeta, X_t)$ and the beliefs on the strategy choice constitute a perfect Bayesian equilibrium of the game if*

1. *for* $\zeta \in [0, \beta]$, $y(\zeta, \Omega_1) = \arg\max_y E[U_\zeta(y|\theta, \alpha_1, \alpha_2)|\Omega_1]$, *and*

$y(\zeta, \Omega_2) = \arg\max_y E[U_\zeta(y|\theta, \alpha_1, \alpha_2)|\Omega_2]$,

2. $\alpha_1 = \int_0^\beta y(\zeta, \Omega_1)\frac{1}{\beta}\, d\zeta$ *and* $\alpha_2 = \int_0^\beta y(\zeta, \Omega_2)\frac{1}{\beta}\, d\zeta$.

Note that for given $X_t = x_t$, α_1 and α_2, all agents with ζ satisfying $1 - \gamma \Pr(\theta + \alpha_1 + \alpha_2 \geq 1|\Omega_t) \leq \zeta \leq \beta$ have non-negative expected utility from action 1 and so choose it.[5] This observation implies that

Lemma 1 *The best response of an agent to any strategy profile played by its opponents is a monotone pure strategy of the form:*

$$y = \begin{cases} 0 & \text{if } \zeta \leq \zeta^*, \\ 1 & \text{if } \zeta > \zeta^*. \end{cases}$$

Consequently, $\alpha_1(x_1)$ as a function of the first period signal x_1 is determined by

$$\alpha_1(x_1) = \begin{cases} 0 & \text{if } 1 - \gamma \Pr(\theta + \alpha_2 \geq 1|x_1) \geq \beta, \\ 1 & \text{if } 1 - \gamma \Pr(\theta + 1 + \alpha_2 \geq 1|x_1) \leq 0, \\ \frac{\beta - 1 + \gamma \Pr(\theta + \alpha_1 + \alpha_2 \geq 1|x_1)}{\beta} \in (0,1) & \text{otherwise} \end{cases}$$

and similarly for $\alpha_2(x_2)$. Hence we can identify the equilibrium with a function $\alpha_t(x)$ which maps from the signal space to the unit interval $[0, 1]$ according to these three cases. Note, however, that since we assume that $\beta > 1 > \gamma$, we can rule out the equilibria corresponding to $\alpha = 0$ and $\alpha = 1$. Hence equilibrium is given by the (implicit) solution to

$$\alpha_1(x_1) = \frac{\beta - 1 + \gamma \Pr(\theta + \alpha_1 + \alpha_2 \geq 1|x_1)}{\beta} \in (0, 1). \tag{3}$$

Multiple equilibria exist if, for a realization of signal, there are multiple roots to the implicit equation (3).

3 Uncertainty and Co-ordination

In the following we take three different information structures/choice orders to highlight the interaction of the three factors for equilibrium determination.

[5] This follows from the fact that optimization requires choosing action 1 if and only if its payoff is positive.

3.1 Simultaneous Information and Choice

We first analyze the benchmark case where two signals are received simultaneously and a mass 2 of agents with idiosyncratic valuations ζ distributed on $[0, \beta]$ choose simultaneously. Hence the information set of the agents $\Omega_1 = \Omega_2 = \{X_1, X_2\}$. This case reduces the fully dynamic model, which we shall study in section 3.3, to a static one. This benchmark will help us to identify the separate effects of sequential choice and learning.

Given two independently drawn signals x_1 and x_2, the agents' common posterior on the state is normally distributed with mean $\mu_2 = (\sigma_\epsilon^2 \mu_0 + \sigma_0^2(x_1 + x_2))/(\sigma_0^2 + 2\sigma_\epsilon^2)$ and variance $\sigma_2^2 = \sigma_0^2 \sigma^2/(\sigma_0^2 + 2\sigma_\epsilon^2)$. The expected utility of a type-ζ agent from choosing action 1 is therefore

$$\mathbb{E}[U_\zeta(\theta, \alpha)] = \zeta - 1 + \gamma \Pr[\theta + 2\alpha \geq 1 | X_1, X_2].$$

(For comparison with later calculations, α is the fraction of a unit mass of agents choosing action 1 in each period, so that in total a mass 2α chooses action 1.) Since θ is normally distributed with mean μ_2 and variance σ_2^2,

$$\mathbb{E}[U_\zeta(\theta, \alpha)] = \zeta - 1 + \gamma \left(1 - \Phi\left(\frac{1 - 2\alpha - \mu_2}{\sigma_2}\right)\right),$$

where $\Phi(\cdot)$ is the standard normal distribution function.

Agents choose action 1 if and only if the expected net utility from doing so is greater than zero. In equilibrium, therefore, α is determined by

$$\alpha = \frac{\beta - 1 + \gamma\left(1 - \Phi\left(\frac{1-2\alpha-\mu_2}{\sigma_2}\right)\right)}{\beta} \in (0, 1) \tag{4}$$

(recalling that $\beta > 1 > \gamma$ implies that there is no equilibrium with $\alpha = 0$ or $\alpha = 1$).

Proposition 1 *There is a unique equilibrium in the simultaneous information/choice case if and only if* $\sigma_2 \geq \frac{\gamma}{\beta}\sqrt{\frac{2}{\pi}}$.

Proof. Rearrange equation (4) to define the function

$$f(\alpha | x) \equiv \alpha\beta - \beta + 1 - \gamma\left(1 - \Phi\left(\frac{1 - 2\alpha - \mu_2}{\sigma_2}\right)\right). \tag{5}$$

An equilibrium is given by $f(\alpha|x) = 0$. A necessary and sufficient condition for there to be a unique solution to this equation is that the function $f(\alpha|x)$ is single-upward crossing in α. We show in this proof that the necessary and sufficient condition is in fact that the function $f(\alpha|x)$ is non-decreasing in α for all $\alpha \in [0, 1]$.

We start with the sufficient condition. For $f(\alpha|x)$ to be single-upward crossing, it is clearly sufficient that $f(\alpha|x)$ is non-decreasing in α. Differentiation of the expression in equation (5) gives

$$\frac{df(\alpha|x)}{d\alpha} = \beta - \frac{2\gamma}{\sigma_2}\phi\left(\frac{1-2\alpha-\mu_2}{\sigma_2}\right).$$

Since $\phi(\cdot) \leq 1/\sqrt{2\pi}$, $f(\alpha|x)$ is non-decreasing if

$$\beta - \frac{2\gamma}{\sigma_2}\frac{1}{\sqrt{2\pi}} \geq 0 \tag{6}$$

which establishes the first part of the proposition.

To establish necessity, we find a particular signal realization for which multiple equilibria exist if the condition of the proposition is violated. Consider the signal realization $\hat{x}$ defined by

$$\mu(\hat{x}) \triangleq 1 - \frac{2(\beta-1)+\gamma}{\beta}.$$

At this signal realization, equation (4) shows that

$$\alpha = \frac{\beta - 1 + \gamma/2}{\beta}$$

is an equilibrium. We now show that other equilibria exist when the condition in the proposition is violated.

Clearly, the function $f(\alpha|x)$ is continuous in α. Given that $f(0|\hat{x}) < 0$ and $f(1|\hat{x}) > 0$, if the derivative of $f(\alpha|\hat{x}) = 0$ with respect to α is negative around $\alpha = \frac{\beta-1+\gamma/2}{\beta}$, then the equation $f(\alpha|\hat{x}) = 0$ has at least two more roots since $f(\alpha|\hat{x})$ is continuous in α and $f(0|\hat{x}) < 0$ and $f(1|\hat{x}) > 0$. Taking the derivative of $f(\alpha|\hat{x})$ with respect to α, we get

$$\frac{df(\alpha|\,\hat{x})}{d\alpha} = \beta - \frac{2\gamma}{\sigma_2}\phi(0) = \beta - \frac{2\gamma}{\sigma_2\sqrt{2\pi}}.$$

It follows that if the condition in the proposition is not satisfied i.e.,

$$\beta < \frac{2\gamma}{\sigma_2}\frac{1}{\sqrt{2\pi}},$$

then there are multiple equilibria. The proof is complete. ■

The basic working of proposition 1 is illustrated in figure 2. The dotted line represents a case in which there are multiple equilibria, marked by points at which the function crosses zero (in this illustration, there are three equilibria). The solid line shows a case in which there is a unique equilibrium. The two cases are distinguished by the slope of the function $f(\alpha|x)$. In the former case, $f(\alpha|x)$ is a non-monotonic function of α, and has multiple crossing points. In the latter, $f(\alpha|x)$ is single-upward crossing in α.

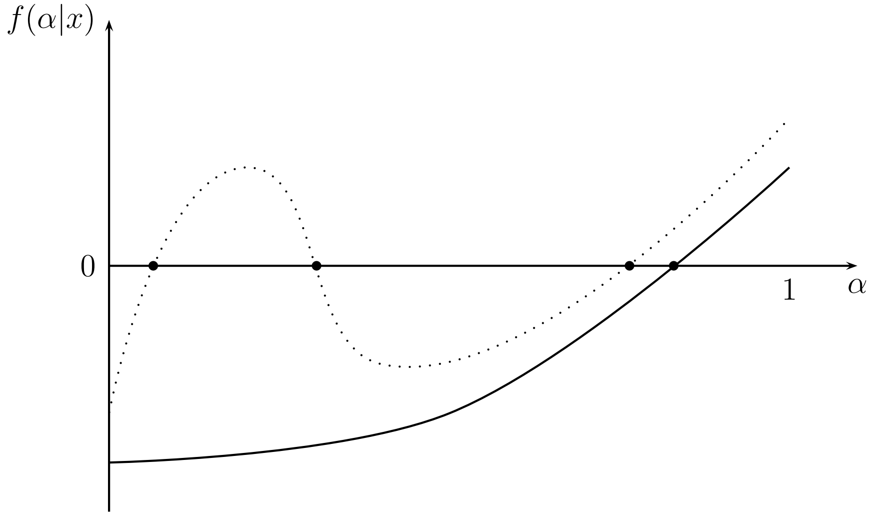

Figure 2: Illustration of proposition 1

This observation also shows why the assumption that $\beta > 1 > \gamma$ does not sacrifice generality. With these parameter values, any equilibrium must be interior i.e., have $\alpha \in (0, 1)$. If we allow other parameter values, then there can be equilibria in which $\alpha = 0$ or $\alpha = 1$. The former occurs if $f(\alpha = 0|x) \geq \beta$; the latter if $f(\alpha = 1|x) \leq 0$. Suppose that these equilibria are possible. It is still the case that there is a unique equilibrium (which might involve either $\alpha = 0$ or $\alpha = 1$) iff $f(\alpha|x)$ is non-decreasing in α.[6]

The mechanism which determines the equilibrium level of α is central to the understanding of the result. Given the signal realization, the strategy choice of agents is represented by the proportion of agents who choose action 1, i.e., α. The relationship between the equilibrium strategy and the equilibrium proportion is the consistency condition that given the α implied by the strategy, agents do not want to deviate from the strategy. The implicit function in the last line of equation (3) represents this consistency condition. There is a unique equilibrium when, for any signal realization, there is a unique value of α which satisfies the consistency condition.

Technically, we show that the implicit function in α that defines equilibrium is a contraction

[6] To see why this statement is true, consider first sufficiency. If $f(\alpha|x)$ is non-decreasing in α, then only three cases are possible: (i) $f(0|x) > 0$ and so $f(\alpha|x) > 0$ for all $\alpha \in [0, 1]$; (ii) $f(1|x) <)$ and so $f(\alpha|x) < 0$ for all $\alpha \in [0, 1]$; (iii) $f(0|x) < 0$ and $f(1|x) > 0$, so that there is a unique $\hat{\alpha} \in (0, 1)$ such that $f(\hat{\alpha}|x) = 0$. (With $\beta > 1 > \gamma$, only case (iii) is possible.) In all three cases, there is a unique equilibrium, given by $\alpha = 0$, $\alpha = 1$ and $\alpha = \hat{\alpha}$ respectively. Necessity follows from observing (as in the proof of proposition 1) that if $f(a|x)$ is non-monotonic, then there are signal realisations such that there are multiple equilibria.

mapping. Usually, this gives only a sufficient condition for a unique solution. But since the function must be a contraction for all possible signal realizations, the contraction condition is both necessary and sufficient. Intuitively, the contraction condition ensures that the *co-ordination effect* is small. We define the co-ordination effect as

$$\mathrm{CE} \equiv \frac{\partial \Pr[\theta + 2\alpha \geq 1 | x_1, x_2]}{\partial \alpha}$$

that is, the effect that agents have on the probability of successful co-ordination. If the co-ordination effect is large, then if all agents choose action 0 (1), then there is a low (high) probability that co-ordination will be successful i.e., $\theta + 2\alpha < (\geq) 1$. This sensitivity of the final outcome to agents' decisions leads naturally to multiple equilibria. In contrast, if agents' decisions have little effect on the probability of successful co-ordination, then a unique equilibrium will result.

The proposition also shows that the co-ordination effect is small when fundamental uncertainty (measured by σ_2, the variance of the posterior) is high. This is quite intuitive; a similar observation has been made by Morris and Shin (2005).[7] At the extreme values of the underlying state, even co-ordinated action by the agents cannot influence the probability of successful co-ordination. If θ is very low, then there is little prospect of successful co-ordination; and conversely if θ is high. In these cases, then, co-ordination is irrelevant and there is a unique equilibrium. When the posterior variance is high, there is a high probability that the underlying state θ takes an extreme value. The proposition makes this statement precise, showing that when the posterior variance exceeds a critical value, the co-ordination effect is small and therefore equilibrium is unique.

The critical value of σ_2 depends on the degree of heterogeneity in agents' idiosyncratic payoffs, measured by β, and the payoff from successful co-ordination, γ. The smaller is the degree of agent heterogeneity, the greater is the critical level of σ_2 required for a unique equilibrium. In the limit, as β tends to zero, the heterogeneity between agents becomes negligible and payoffs are determined almost entirely by co-ordination. Clearly in this case, the condition for equilibrium uniqueness becomes very demanding.[8] A similar intuition applies to the comparative static of the critical value with respect to the co-ordination parameter. When γ is very large, (successful) co-ordination is critical for agents' payoffs. In the limit, as γ tends to infinity, payoffs are determined entirely by co-ordination; and again, the condition for equilibrium uniqueness becomes very demanding. In fact, due to the functional form of agents' payoffs, only the ratio of the two payoff parameters β and

[7]Morris and Shin deal with interaction games, which include global games. The result here is a special case of their setting, in which there is zero correlation between agents' types.

[8]This observation highlights the difference between our model and a global game, as used by e.g., Dasgupta (forthcoming). In a global game, a unique equilibrium is selected in the limit as agents become homogeneous (i.e., as their private types become very highly, but not perfectly, correlated). In our model, the condition for a unique equilibrium becomes harder to satisfy as agents' types become homogeneous.

γ matters for the analysis, as the condition in Proposition 1 shows.

3.1.1 Hindering or assisting co-ordination?

Proposition 1 determines when there is a unique equilibrium in the co-ordination game. When $\sigma_2 < \frac{\gamma}{\beta}\sqrt{\frac{2}{\pi}}$, there are multiple equilibria for certain signal realisations. Given those signals realisations, an increase in σ_2 (or equivalently a decrease in γ/β), so that $\sigma_2 \geq \frac{\gamma}{\beta}\sqrt{\frac{2}{\pi}}$, leads to a unique equilibrium. Uncertainty, in this sense, assists co-ordination. But does uncertainty make co-ordination more successful, in the sense of increasing the proportion of agents choosing action 1 in equilibrium? Since agents' actions are strategic complements, multiple equilibria can be Pareto-ranked; the equilibrium that involves the largest α (the smallest critical type ζ^* that is indifferent between actions 0 and 1) is Pareto-preferred. When there is a unique equilibrium, how does it relate to the Pareto-preferred outcome with multiple equilibria?

Consider first a decrease in the ratio γ/β. It is straightforward to show that the value of the function $f(\alpha|x)$ is decreasing in this ratio. This is illustrated in figure 2: the non-monotonic function with multiple equilibria (the dotted line) lies everywhere above the non-decreasing function with a unique equilibrium (the solid line). Consequently, the unique equilibrium involves an α which is greater than the α in the Pareto-preferred outcome with multiple equilibria. The intuition for this is straightforward. The ratio γ/β is low when there are large benefits from successful co-ordination, or when the heterogeneity in agents' idiosyncratic payoffs. In both cases, the mass of agents that chooses 1 in the unique equilibrium is necessarily larger.

Now consider a mean-preserving spread in posteriors, by increasing σ_2 while keeping μ_2 constant.[9] The sign of the change in the value of the function $f(\alpha|x)$ is equal to the sign of $-(1 - 2\alpha - \mu_2)$. So, for very strong signals (high x_1 and x_2, such that $\mu_2 > 1$), the value of $f(\alpha|x)$ increases with an increase in σ_2. For very low signals (such that $\mu_2 < -1$), it decreases. Referring again to figure 2, we see that the unique equilibrium in the first case involves an α which is lower than any equilibrium α with multiple equilibria. In the second case, the α in a unique equilibrium is greater than any α with multiple equilibria. So, greater uncertainty increases the degree of equilibrium co-ordination when signals indicate a low value of θ, but decreases co-ordination when signals indicate a high θ. This observation is a direct consequence of the effect of a mean-preserving spread on the probability $1 - \Phi((1 - 2\alpha - \mu_2)/\sigma_2)$ of successful co-ordination. An increase in σ_2 places more mass in the tails of the posterior distribution of beliefs about θ. With low signals, this increases the probability of successful co-ordination; with high signals, it decreases it.

[9] In order to increase σ_2, either σ_0 or σ_ϵ must be increased. But note that this also increases μ_2. In this discussion, we assume that any increase in σ_2 is done in such a way that μ_2 is kept constant.

We summarise this discussion in the following proposition.

Proposition 2 *Consider signal realisations such that there are multiple equilibria when $\sigma_2 < \frac{\gamma}{\beta}\sqrt{\frac{2}{\pi}}$. Let $\underline{\alpha}_M$ be the lowest $\alpha \in (0,1)$ occurring among the multiple equilibria; let $\overline{\alpha}_M$ be the highest equilibrium $\alpha \in (0,1)$.*

1. *Hold σ_2 constant, and decrease γ/β so that $\sigma_2 \geq \frac{\gamma}{\beta}\sqrt{\frac{2}{\pi}}$. In the unique equilibrium that results, $\alpha > \overline{\alpha}$.*
2. *Hold γ/β constant, and increase σ_2 so that $\sigma_2 \geq \frac{\gamma}{\beta}\sqrt{\frac{2}{\pi}}$ (but μ_2 is constant). In the unique equilibrium that results, if $\mu_2 > 1$, then $\alpha < \underline{\alpha}$; if $\mu_2 < -1$, then $\alpha > \overline{\alpha}_M$.*

Greater uncertainty can, therefore, both hinder and assist co-ordination. It always assists co-ordination, in the sense that with sufficient uncertainty (σ_2 sufficiently large), there is a unique equilibrium. When signals are weak, it assists also by ensuring an equilibrium with a greater degree of co-ordination (and hence higher payoffs) than the Pareto-preferred outcome when there are multiple equilibria. But when signals are strong, the unique equilibrium has lower payoffs than any outcome when there are multiple equilibria. Greater heterogeneity, in the sense of an increase in the ratio γ/β, always assists co-ordination, in both senses.

3.2 Simultaneous Information and Sequential Choice

The next benchmark examined, before turning to the 'fully' sequential problem, is the one in which a unit mass of agents choose in each of the two periods, with the same information: signals X_1 and X_2 drawn at the beginning of period 1 and observed by both sets of agents. Hence $\Omega_1 = \{X_1, X_2\}$ and $\Omega_2 = \{X_1, X_2, \alpha_1\}$ and the posterior on the state for both periods is normally distributed with mean μ_2 and variance σ_2^2.

We consider this case in order to uncouple the timing of decisions among agents from the timing of resolution of information. This allows us to assess whether the importance of co-ordination is reduced when agents move sequentially. The result in this subsection indicates that this is not the case, since the necessary and sufficient condition for unique determination of equilibrium is identical to the static case of the previous subsection. In fact we obtain a stronger result than the identical necessary and sufficient condition; the next proposition proves that the equilibrium under simultaneous information and sequential choice is identical to that under simultaneous information and choice.

Proposition 3 *The equilibrium in the simultaneous information/sequential choice case is identical to that in the simultaneous information and choice case.*

Proof. We first show that the equilibrium under this case is symmetric in the sense that the equilibrium value of α_1 and α_2 are identical. Note that the equilibrium condition for α_1 and α_2 are given as:

$$\alpha_1 = \frac{\beta - 1 + \gamma\left(1 - \Phi\left(\frac{1-\alpha_1-\alpha_2(\alpha_1)-\mu_2}{\sigma_2}\right)\right)}{\beta}. \tag{7}$$

and

$$\alpha_2 = \frac{\beta - 1 + \gamma\left(1 - \Phi\left(\frac{1-\overline{\alpha}_1-\alpha_2-\mu_2}{\sigma_2}\right)\right)}{\beta}. \tag{8}$$

Suppose that α_1 and α_2 are not identical. However the right hand sides of equations (7) and (8) evaluated for given α_1 and α_2 are identical. Hence they cannot be equal to different α_1 and α_2.

The symmetry of the equilibrium α's imply that the equilibrium condition is reduced to

$$\alpha = \frac{\beta - 1 + \gamma\left(1 - \Phi\left(\frac{1-2\alpha-\mu_2}{\sigma_2}\right)\right)}{\beta}. \tag{9}$$

which is the equilibrium condition for simultaneous information and choice. The proof is complete.
■

The proof relies on the fact that the response functions for α_1 and α_2 are identical. The symmetry of the equilibrium is more general than in the present model as demonstrated in Lee and Lee (2005), which shows that the symmetry of equilibrium for games with symmetric and non-decreasing response functions.

The proposition indicates that the case of simultaneous information and sequential choice is essentially identical to that of simultaneous information and choice. Hence the necessary and sufficient condition for a unique equilibrium is also identical, which is stated in the following corollary without proof.

Corollary 1 *The sufficient and necessary condition for a unique equilibrium in the simultaneous information/sequential choice case is the same as that in the simultaneous information and choice case.*

Those agents who move later observe the decision of early-movers, and so their co-ordination effect is smaller (i.e., $\mathrm{CE}_1 \geq \mathrm{CE}_2$). Intuition based on this observation would suggest that co-ordination should be easier with sequential action choices. But early movers, anticipating the choices of late movers, face the same co-ordination difficulty since the effect of their decision is subsequently amplified by those who move in the later stage. Hence sequential action choice alone does not make co-ordination easier in this model. This result appears to be quite robust. For example, we have verified that it holds when there are different masses of agents moving in the two periods.[10]

The current case can be contrasted to the complete information model in Farrell and Saloner (1985). In the latter, there is a unique subgame perfect equilibrium. In contrast, our analysis indicates that sequential decision making does not help, relative to the static (i.e., simultaneous information/choice) case, in ensuring a unique equilibrium. The difference can be understood by the fact that our model has a continuum of agents; hence there is still a co-ordination problem in each period.

3.3 Sequential Information and Choice

Finally, we analyze the fully dynamic case in which signals are revealed and choices are made sequentially. We assume that each period, $t \in \{1, 2\}$, a signal is revealed to the agents who subsequently choose their strategy conditional on the information. In the second period agents also observe the decisions made in the first period before making their own choice. Hence $\Omega_1 = \{X_1\}$ and $\Omega_2 = \{X_1, X_2, \alpha_1\}$.

The agents who move in the second period have the same information as the previous case of simultaneous information/sequential choice. The first period problem is different, however. Previously, the fraction α_2 of agents choosing action 1 in the second period was not subject to uncertainty from a further signal draw (even if indeterminate due to multiplicity). Now, α_2 is a random variable from the perspective of period-1 agents since it will be determined conditional on the realization of the signal X_2.

In the first period, the agents' common posterior on θ is normally distributed with mean $\mu_1 = (\sigma_\epsilon^2 \mu_0 + \sigma_0^2 x_1)/(\sigma_0^2 + \sigma_\epsilon^2)$ and variance $\sigma_1^2 = \sigma_0^2 \sigma_\epsilon^2/(\sigma_0^2 + \sigma_\epsilon^2)$. In the second period, the agents' common posterior on θ is determined as in the previous subsection: it is normally distributed with mean $\mu_2 = (\sigma_\epsilon^2 \mu_0 + \sigma_0^2(x_1 + x_2))/(\sigma_0^2 + 2\sigma_\epsilon^2)$ and variance $\sigma_2^2 = \sigma_0^2 \sigma_\epsilon^2/(\sigma_0^2 + 2\sigma_\epsilon^2)$. In addition we need information on how the second period signal, X_2, and the fundamental, θ, are correlated: their covariance is given by σ_1^2 so that the correlation coefficient is computed as $\rho = \frac{\sigma_1^2}{\sigma_1 \sqrt{\sigma_1^2 + \sigma_\epsilon^2}}$.[11]

Proposition 4 *Define*

$$\lambda \equiv \inf_\tau \left(\Pr(|Z| \leq \tau) + \exp\left[-\frac{\rho^2 \tau^2}{2(1-\rho^2)} \right] \Pr(|Z| > \tau) \right)$$

where Z is a standard normal random variable and ρ is the correlation coefficient between θ and

[10] The proof of this statement is available on request.

[11] Since the random variable, X_2, is a noisy signal of the fundamental θ, the correlation coefficient ρ takes values only between 0 and $\frac{\sqrt{2}}{2}$. We will provide more discussion on this observation.

X_2. There is a unique equilibrium in the sequential information/choice case if

$$\sigma_2 \geq \frac{\gamma}{\beta}\frac{(1+\lambda)}{\sqrt{2\pi}}$$

where $0 < \lambda < 1$.

Proof. We start with the second period problem, which is identical to that for the simultaneous information/sequential choice case considered in the previous subsection. We collect the main results here for reference: the implicit equation defining α_2 involves a non-decreasing function of α_2 if and only if

$$\beta \geq \frac{\gamma}{\sigma_2}\phi\left(\frac{1-\overline{\alpha}_1-\alpha_2-\mu_2}{\sigma_2}\right) \tag{10}$$

where α_2 is determined by

$$\alpha_2 = \frac{\beta - 1 + \gamma\left(1-\Phi\left(\frac{1-\overline{\alpha}_1-\alpha_2-\mu_2}{\sigma_2}\right)\right)}{\beta}. \tag{11}$$

Hence the necessary and sufficient condition for unique equilibrium in the second period is $\beta \geq \frac{\gamma}{\sqrt{2\pi}}\frac{1}{\sigma_2}$.

The expected utility of a type-ζ agent in the first period is

$$\zeta - 1 + \gamma \Pr\left[\theta + \alpha_1 + \tilde{\alpha}_2 \geq 1 | x_1\right],$$

where the notation $\tilde{\alpha}_2$ emphasizes that it is a random variable. Those agents who make a decision in the first period must compute the probability of an event which depends on the sum of two random variables, θ and α_2. Since the second period's decision is made conditional on X_2, α_2 is a function of the random variable X_2.

Consider

$$\frac{d\Pr[\theta + \tilde{\alpha}_2 \geq 1 - \alpha_1 | x_1]}{d\alpha_1}$$

where the left hand side of the inequality inside the probability contains only random variables while the right hand side contains only parameters. First observe that θ and $X_2 = \theta + \epsilon_2$ are bivariate-normally distributed random variables. Conditional on the observation of x_1, θ and X_2 have the same mean μ_1 and they have variances σ_1^2 and $\sigma_1^2 + \sigma_\epsilon^2$ while their covariance is given by σ_1^2 so that the correlation coefficient $\rho = \frac{\sigma_1^2}{\sigma_1\sqrt{\sigma_1^2+\sigma_\epsilon^2}}$. Write

$$\begin{aligned}\Pr[\theta + \tilde{\alpha}_2(X_2|\ \alpha_1) \geq 1 - \alpha_1 | x_1] &= \int_{-\infty}^{\infty}\int_{1-\alpha_1-\tilde{\alpha}_2(X_2|\ \alpha_1)}^{\infty} \phi_{x_1}(\theta, X_2)\, d\theta\, dX_2 \\ &= \int_{-\infty}^{\infty}\int_{\frac{1-\alpha_1-\tilde{\alpha}_2(\hat{X}_2|\ \alpha_1)-\mu_1}{\sigma_1}}^{\infty} \phi(\hat{\theta}, \hat{X}_2)\, d\hat{\theta}\, d\hat{X}_2\end{aligned} \tag{12}$$

where $\phi_{x_1}(\theta, X_2)$ on the first line is the bivariate normal distribution of θ and X_2 conditional on the observation of x_1, while $\hat{\theta} = \frac{\theta - \mu_1}{\sigma_1}$ and $\hat{X}_2 = \frac{X_2 - \mu_1}{\sqrt{\sigma_1^2 + \sigma_\epsilon^2}}$ so that $\phi(\hat{\theta}, \hat{X}_2)$ on the second line is the standard bivariate normal distribution.

Using equation (12), the derivative of the probability with respect to α_1 can be computed as follows:

$$\frac{d}{d\alpha_1} \Pr[\theta + \tilde{\alpha}_2(X_2|\alpha_1) \geq 1 - \alpha_1 | x_1] = \int_{-\infty}^{\infty} \frac{d}{d\alpha_1} \int_{\frac{1-\alpha_1 - \tilde{\alpha}_2(\hat{X}_2) - \mu_1}{\sigma_1}}^{\infty} \phi(\hat{\theta}, \hat{X}_2)\, d\hat{\theta}\, d\hat{X}_2 \tag{13}$$

Applying Leibniz's rule, the derivative inside the outer integral is computed:

$$\frac{d}{d\alpha_1} \int_{\frac{1-\alpha_1 - \tilde{\alpha}_2(\hat{X}_2|\alpha_1) - \mu_1}{\sigma_1}}^{\infty} \phi(\hat{\theta}, \hat{X}_2)\, d\hat{\theta}$$
$$= \frac{1}{\sigma_1} \left(1 + \frac{d\tilde{\alpha}_2(\hat{X}_2|\alpha_1)}{d\alpha_1} \right) \phi(\frac{1 - \alpha_1 - \tilde{\alpha}_2(\hat{X}_2|\alpha_1) - \mu_1}{\sigma_1}, \hat{X}_2).$$

Since α_2 is determined from the implicit equation (11), we can totally differentiate it to obtain

$$\frac{d\tilde{\alpha}_2(\hat{X}_2|\alpha_1)}{d\alpha_1} = \frac{\gamma\phi(.)}{\beta\sigma_2 - \gamma\phi(.)} \leq \frac{\gamma\frac{1}{\sqrt{2\pi}}}{\beta\sigma_2 - \gamma\frac{1}{\sqrt{2\pi}}}$$

where the inequality follows from the fact that $\phi(.) \leq \frac{1}{\sqrt{2\pi}}$.

Collecting these results and substituting them into (13) yields

$$\frac{d}{d\alpha_1} \Pr\left[\theta + \tilde{\alpha}_2(\hat{X}_2|\ \alpha_1) \geq 1 - \alpha_1 | x_1\right]$$
$$\leq \frac{1}{\sigma_1} \left(\frac{\beta\sigma_2}{\beta\sigma_2 - \gamma\frac{1}{\sqrt{2\pi}}} \right) \int_{-\infty}^{\infty} \phi\left(\frac{1 - \alpha_1 - \tilde{\alpha}_2(\hat{X}_2) - \mu_1}{\sigma_1}, \hat{X}_2 \right) d\hat{X}_2. \tag{14}$$

The integral on the last line of equation (14) is bounded above by $\frac{\lambda}{\sqrt{2\pi}\sqrt{1-\rho^2}}$ for $\lambda < 1$ as shown in the following lemma whose proof is provided in the appendix.

Lemma 2 *Define*

$$\lambda \equiv \inf_{\tau} \left(\Pr(|Z| \leq \tau) + \exp\left[-\frac{\rho^2\tau^2}{2(1-\rho^2)} \right] \Pr(|Z| > \tau) \right)$$

where Z is a standard normal random variable. For such a λ,

$$\int_{-\infty}^{\infty} \phi\left(\frac{1 - \alpha_1 - \tilde{\alpha}_2(\hat{X}_2)}{\sigma_1}, \hat{X}_2 \right) d\hat{X}_2 \leq \frac{\lambda}{\sqrt{2\pi}\sqrt{1-\rho^2}}.$$

Lemma 2 implies that

$$\frac{d}{d\alpha_1}\Pr\left[\theta+\tilde{\alpha}_2(\hat{X}_2|\ \alpha_1)\geq 1-\alpha_1|x_1\right]$$
$$\leq (\frac{1}{\sigma_1}\frac{\beta\sigma_2}{\beta\sigma_2-\gamma\frac{1}{\sqrt{2\pi}}})\frac{\lambda}{\sqrt{2\pi}\sqrt{1-\rho^2}}=(\frac{1}{\sigma_2}\frac{\beta\sigma_2}{\beta\sigma_2-\gamma\frac{1}{\sqrt{2\pi}}})\frac{\lambda}{\sqrt{2\pi}}.$$

Therefore, there is a unique equilibrium if

$$\beta\geq\gamma(\frac{1}{\sigma_2}\frac{\beta\sigma_2}{\beta\sigma_2-\gamma\frac{1}{\sqrt{2\pi}}})\frac{\lambda}{\sqrt{2\pi}}$$

which can be rewritten as

$$\sigma_2\geq\frac{\gamma}{\beta}\frac{(1+\lambda)}{\sqrt{2\pi}}.$$

The proof is complete. ■

There is a step in the proof that deserves further explanation. The agents who move in the first period have to deal with two random variables: the second period signal, and the proportion of agents who will choose action 1 in the second period. The former is a normal random variable whose distribution is given. However the distribution of the second random variable is endogenously determined as a function of the second period signal. We apply a transformation of random variable technique to find a bound on the derivative of α_2 with respect to α_1. This difficulty prevents us from obtaining a necessary condition for uniqueness in this case; the proposition provides only a sufficient condition.

The proposition indicates that there is a unique equilibrium in the sequential information/choice case for a weaker condition on the three parameters than either of the previous two cases analysed. The following corollary formalizes this observation. We omit the formal proof since it follows from the fact that $\lambda\leq 1$.

Corollary 2 *For any β and γ, there is a set non-empty Σ_2 of values of σ_2 (i.e., σ_0 and σ_ϵ) for which a unique equilibrium exists under sequential information/choice while there are multiple equilibria under simultaneous information/choice or simultaneous information/sequential choice. The set is defined by*

$$\Sigma_2\equiv\{\sigma_2\mid\frac{\gamma}{\beta}\frac{(1+\lambda)}{\sqrt{2\pi}}\leq\sigma_2\leq\frac{\gamma}{\beta}\frac{2}{\sqrt{2\pi}}\}.$$

The sufficient condition in proposition 4 relies on the parameter, λ. A close inspection of λ reveals that it is the best two-point approximation of the normal distribution.[12] The approximation assigns 1 to the normal density at the centre of the support, while any points at a distance greater

[12] λ is the minimum among all two-point approximations and in that sense it is the best approximation.

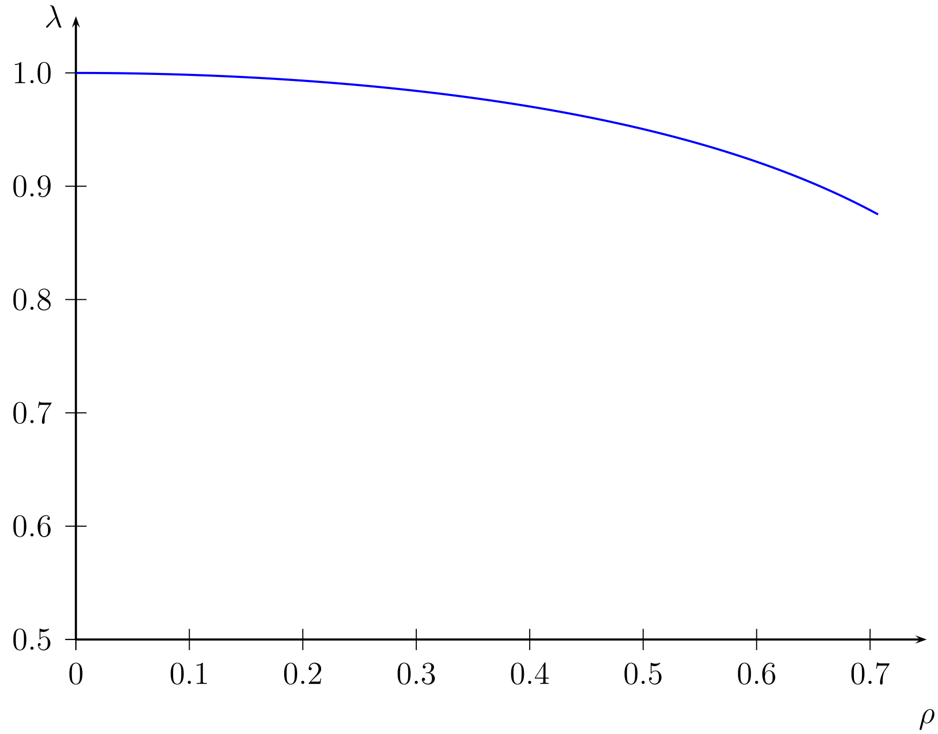

Figure 3: λ and ρ in sequential information/choice case

than τ from the centre is given the value of the density at τ. Since the standard normal density is symmetric around 0 and monotone decreasing as the distance increases, this approximation provides an upper bound on the density. In the computation of the upper bound λ, the correlation coefficient ρ plays a crucial role.

Since a closed form expression of λ cannot be obtained, we compute it using a numerical method to visualize the relation between ρ and λ. Figure 3[13] shows that λ is a monotone decreasing and concave function of ρ. Notice that if λ is close to 1, then the sufficient condition indicates little relaxation in the uniqueness condition compared to the previous cases in which the signals arrive simultaneously. On the other hand, smaller λ implies that the uniqueness condition for the sequential model is substantially weaker than that of the previous two cases. Hence when θ and X_2 are independent, the sufficient condition is almost identical to the simultaneous information models; while when they are correlated, the bound for the sufficient condition is weaker than that for the other two cases.

When θ and X_2 are independent, the second period signal is not informative about the fundamental and hence the second choice is likely to be similar to the first period choice. To put it differently, an X_2 which is independent of θ means that agents have similar information regard-

[13]Since $\rho \leq \frac{\sqrt{2}}{2}$, the figure is valid only up to $\frac{\sqrt{2}}{2}$ on the horizontal axis.

less of when they move; and hence the informational environment is similar to the simultaneous information/sequential choice model. We have found that the simultaneous information/sequential choice model has an identical condition to the simultaneous information/choice model for uniqueness. Hence the sufficient condition for the sequential information/choice model is similar to the simultaneous information/choice model.

On the other hand, when θ and X_2 are strongly correlated (i.e., ρ is close to $\frac{\sqrt{2}}{2}$), the second period signal is strongly informative of the fundamental. To see this, define $\eta \equiv \sigma_0^2/\sigma_\epsilon^2$; η measures the relative precision of period 1 and period 2 signals, since

$$\frac{\sigma_1^2}{\sigma_2^2} = \frac{\sigma_o^2 + 2\sigma_\epsilon^2}{\sigma_o^2 + \sigma_\epsilon^2} = \frac{1 + 2\eta}{1 + \eta}.$$

In the limit when $\eta = 0$, the precisions of the signals in the two periods are equal; conversely, as $\eta \to \infty$, the period 2 signal is much more precise than the period 1 signal. Notice also that

$$\rho = \sqrt{\frac{\eta}{2\eta + 1}}.$$

When $\eta = 0$, $\rho = 0$; and when $\eta \to \infty$, $\rho \to \frac{\sqrt{2}}{2}$. Now consider the limit $\sigma_0 \to \infty$ or $\sigma_\epsilon^2 \to 0$, in which $\eta \to \infty$ and $\rho \to \frac{\sqrt{2}}{2}$, the (relative) precision of the period 2 signal is much larger than the precision of the period 1 signal.[14] This in turn implies that the second period choice is not (necessarily) similar to the first period choice. Hence the agents who move in the first period have less of a co-ordination effect and the sufficient condition is weaker than for the simultaneous information models.

In summary: in the simultaneous information/choice and simultaneous information/seq-uential choice cases, the conditions for equilibrium uniqueness are identical. In the simultaneous information/sequential choice model, a simple intuition (e.g., from a two-player game) suggests that the co-ordination problem will be less, because half of the agents move after observing the choice of the other half. However those agents who move in the first period fully anticipate the consequence of their choice for those agents who move in the second period. As a result, the co-ordination effect for period-1 agents is unchanged; and so is the necessary and sufficient condition for equilibrium uniqueness. In contrast, the sequential information/choice case shows that gradual revelation of information can have a significant effect on co-ordination, particularly when the precision of the second period signal is high.

[14]Note that at the limit $\sigma_\epsilon = 0$, the signals X_1 and X_2 are perfectly precise about the value of θ. At the limit, therefore, there is no difference between the signals arriving simultaneously or sequentially. Along the path to the limit, however, the period 2 signal is relatively much more precise than the period 1 signal.

4 Path Dependence

The mechanism behind equilibrium determination in the sequential information/choice case has an interesting implication for the dynamic behavior of the model. The agents who move in the first period have to make their choice with less information than those in the second period. On the other hand, the first-period choices of agents are irreversible for later movers in the second period. Multiple equilibria occur when agents have too much effect on the probability of successful coordination. This observation implies that the first period agents effectively 'select' one of multiple equilibria. Consequently, different sequences of signals lead to different equilibrium paths. Due to the irreversibility of first period choices, the signal that arrives in the first period has a bigger effect on the determination of the equilibrium path. We refer to this effect as path dependence. In this section, we make this argument precise by establishing how the equilibrium α_t, $t \in \{1, 2\}$, are affected by signal realisations.

We start by showing that our model exhibits a strong form of stochastic dominance on the equilibrium α_t. In the next proposition, we consider a fixed amount of information, and show that both α_1 and α_2 are increasing functions of the first period signal.

Proposition 5 *Given $\tilde{x}$, consider signal draws such that $x_1 + x_2 = \tilde{x}$. The equilibrium values of α_1 and α_2 are increasing functions of x_1.*

Proof. Consider two realizations of signal draws, $x = (x_1, x_2)$ and $x' = (x_1', x_2')$ for which $x_1 + x_2 = x_1' + x_2' = \tilde{x}$. Moreover assume that $x_1 \geq x_1'$. The first period agents' choice satisfies $\alpha_1|_{x_1} > \alpha_1|_{x_1'}$ since the posterior distribution in the first period has a higher mean when conditioned on x_1 than on x_1'.

Recall that the second period equilibrium is determined from

$$\alpha_2 = \frac{\beta - 1 - \gamma\Phi(\frac{1-\alpha_1-\alpha_2-\mu_2}{\sigma_2})}{\beta}.$$

Notice that both sequences of signals produce the same mean since both sequences have the same value for the sum of the signals. Moreover $\alpha_1|_{x_1} > \alpha_1|_{x_2}$ from above. We know that α_2 as a function of α_1 is increasing in α_1. Since α_2 depends on μ_2, σ_2, and α_1 where the first two are identical for both sequences and only α_1 differs in the two sequences, it follows that $\alpha_2|_x > \alpha_2|_{x'}$. ■

The implication of the proposition is seen clearly in the following example.

Example 1 *Consider two realizations of signals of the same total information content but with reversed orders: $x = (x_1, x_2)$ and $x' = (x_2, x_1)$ where $x_1 > x_2$. Then $\alpha_1|_x > \alpha_1|_{x'}$ and $\alpha_2|_x > \alpha_2|_{x'}$.*

In the example, there are two sets of second-period agents, both with the same aggregate information (and hence belief about the state of nature), but distinguished by observing different first-period signals. Of the set that receives the higher first period signal, a greater proportion chooses action 1. In short, first-period signals have not only an informational effect, but also a real effect on the equilibrium path.

In the next proposition, we explore further properties of the dynamic path to show the relative importance of the first-period signal for the determination of the equilibrium outcome.

Proposition 6 *Suppose that $\alpha_2(x_2) = \alpha_2(x_2')$ where $x_2 > x_2'$. Then $x_1 < x_1'$ and thus $\alpha_1(x_1) < \alpha_1(x_1')$. Moreover $x_1 + x_2 > x_1' + x_2'$.*

Proof. In order for $\alpha_2(x_2) = \alpha_2(x_2')$ for $x_2 \geq x_2'$, the equilibrium condition

$$\alpha_2 = \frac{\beta - 1 + (1 - \Phi(\frac{1-\alpha_1-\alpha_2-\mu(x_1,x_2)}{\sigma_2}))}{\beta},$$

implies that $\alpha_1(x_1) + \mu(x_1, x_2) = \alpha_1(x_1') + \mu(x_1', x_2')$. It is easy to see that $x_1 < x_1'$ since otherwise $\alpha_1(x_1) > \alpha_1(x_1')$ and $\mu(x_1, x_2) > \mu(x_1', x_2')$ so that $\alpha_1(x_1) + \mu(x_1, x_2) > \alpha_1(x_1') + \mu(x_1', x_2')$.

Hence suppose that $x_1 < x_1'$. Aiming at contradiction, suppose that $x_1 + x_2 \leq x_1' + x_2'$. However under this condition, $\alpha_1(x_1) < \alpha_1(x_1')$ and $\mu(x_1, x_2) < \mu(x_1', x_2')$ so that $\alpha_1(x_1) + \mu(x_1, x_2) < \alpha_1(x_1') + \mu(x_1', x_2')$. Since this contradicts the hypothesis that $\alpha_2(x_2) = \alpha_2(x_2')$, we have proved that $x_1 + x_2 > x_1' + x_2'$. ■

Suppose that a signal which is indicative of a low fundamental is received in the first period. The proposition implies that to offset this early shock, it takes a larger shock in the opposite direction. Hence the early signal has a bigger effect on the equilibrium outcome than the late one. One implication of this observation is that a form of equilibrium multiplicity can occur, even if the sufficient condition in proposition 4 is satisfied. The same overall information can lead to different equilibrium outcomes, distinguished by the order in which signals arrive. In cases where it is possible to observe only an aggregate of information, and not the full sequence of signals, the effect is similar to equilibrium multiplicity. The cause, however, is not the problem of co-ordination (that is eliminated by proposition 4)—it is instead informational.

5 Summary and Conclusions

In this paper, we have investigated the importance of dynamics for equilibrium determination in games with strategic complementarities. We have shown that the timing of information is crucial—in our model, more important than the timing of actions. The importance of learning is seen in three ways: first, in the effect that it has on the conditions for equilibrium uniqueness; secondly, in the implications for equilibrium payoffs; and finally, in the way that equilibrium outcomes can display path dependence.

While we have shown these points in a very simple model, we think that they are robust to a number of generalizations. For example, we use throughout the normal distribution to model uncertainty. This allows us to have very tractable expressions for posteriors, and single parameters to measure variance. But it seems unlikely that the normality assumption is driving any of the main results; and so more general distributions could be used, if the additional complexity is warranted. We assume that an equal mass of agents move in each period; it is easy to show that this feature could be replaced with arbitrary masses with no change in the results.

The results of the paper underscore the importance of uncertainty for the determination of equilibrium in economic problems in which co-ordination matters. For instance, the stability of a market mechanism which is subject to uncertainty resolution and co-ordination externality may be affected by how much information is released to the agents who move over time. The result has an implication for the policy suggestion by David (2001) that to avoid the danger of making a wrong decision, agents should delay their decisions. According to the conditions for equilibrium uniqueness, a delay in the decisions by agents may instead lead to equilibrium multiplicity. On the other hand, there seems no way to rule out the realization of bad outcomes due to path dependence if the decision is made with less than comprehensive information. Our results also imply that any empirical assessment of an economic event that involves co-ordination should pay attention to the dynamic nature of the environment.

Appendix

Proof of Lemma 2

Proof. Since $\phi(\hat{\theta}, \hat{X}_2)$ is the standard bivariate normal distribution with covariance ρ, it holds that

$$\begin{aligned}
\phi(\hat{\theta}, \hat{X}_2) &= \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}(\hat{\theta}^2 - 2\rho\hat{\theta}\hat{X}_2 + \hat{X}_2{}^2)\right] \\
&= \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}(\rho\hat{X}_2 - \hat{\theta})^2\right] \times \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\hat{X}_2^2\right] \\
&= \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}(\rho\hat{X}_2 - \hat{\theta})^2\right] \phi(\hat{X}_2).
\end{aligned}$$

Then,

$$\begin{aligned}
&\int_{-\infty}^{\infty} \phi(\frac{1-\alpha_1-\tilde{\alpha}_2(\hat{X}_2)-\mu_1}{\sigma_1}, \hat{X}_2)\, d\hat{X}_2 \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\rho\hat{X}_2 - \frac{1-\alpha_1-\tilde{\alpha}_2(\hat{X}_2)-\mu_1}{\sigma_1}\right)^2\right] \phi(\hat{X}_2)\, d\hat{X}_2. \quad (15)
\end{aligned}$$

Define

$$g(\hat{X}_2) = \rho \hat{X}_2 - \frac{1-\alpha_1-\tilde{\alpha}_2(\hat{X}_2)-\mu_1}{\sigma_1}$$

and define $\hat{X}_2^*$ such that $g(\hat{X}_2^*) = 0$. Note that $\tilde{\alpha}_2(\hat{X}_2)$ is monotone increasing in $\hat{X}_2$ and hence $g(\hat{X}_2)$ is monotone increasing and there exists a unique $\hat{X}_2^*$.

Fix $\tau > 0$ and define

$$f(\hat{X}_2) = \begin{cases} 1 & \text{if} \quad \hat{X}_2^* - \tau < \hat{X}_2 < \hat{X}_2^* + \tau, \\ \exp\left[-\frac{\rho^2\tau^2}{2(1-\rho^2)}\right] & \text{if} \quad \hat{X}_2 \le \hat{X}_2^* - \tau \text{ or } \hat{X}_2 \ge \hat{X}_2^* + \tau. \end{cases}$$

Then for all $\hat{X}_2$,

$$\exp\left[-\frac{1}{2(1-\rho^2)}\left(\rho \hat{X}_2 - \frac{1-\alpha_1-\tilde{\alpha}_2(X_2)-\mu_1}{\sigma_1}\right)^2\right] \le f(\hat{X}_2)$$

since given $\tau > 0$, $|g(\hat{X}_2)| \ge \rho\tau$ for $\hat{X}_2$ such that $|\hat{X}_2 - \hat{X}_2^*| \ge \tau$ while $\exp\left[-\frac{\rho^2\tau^2}{2(1-\rho^2)}\right] \le 1$ for X_2 such that $|\hat{X}_2 - \hat{X}_2^*| < \tau$. It follows that

$$\begin{aligned} &\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\rho \hat{X}_2 - \frac{1-\alpha_1-\tilde{\alpha}_2(\hat{X}_2)-\mu_1}{\sigma_1}\right)^2\right] \phi(\hat{X}_2)\, d\hat{X}_2 \\ &\le \int_{-\infty}^{\infty} f(\hat{X}_2)\phi(\hat{X}_2) d\hat{X}_2 \\ &= \Pr(|Z| \le \tau) + \exp\left[-\frac{\rho^2\tau^2}{2(1-\rho^2)}\right] \Pr(|Z| > \tau) \\ &< 1 \end{aligned} \tag{16}$$

where Z in the third line of the equation is s standard normal random variable.

Define λ as

$$\lambda \equiv \inf_{\tau} \Pr(|Z| \le \tau) + \exp\left[-\frac{\rho^2\tau^2}{2(1-\rho^2)}\right] \Pr(|Z| > \tau).$$

Then $0 < \lambda < 1$ and

$$\int_{-\infty}^{\infty} \phi(\frac{1-\alpha_1-\tilde{\alpha}_2(\hat{X}_2)-\mu_1}{\sigma_1}, \hat{X}_2) d\hat{X}_2 \le \frac{\lambda}{\sqrt{2\pi}\sqrt{1-\rho^2}}.$$

The proof is complete. ■

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