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THE HAUSDORFF DIMENSION OF THE VISIBLE SETS OF CONNECTED COMPACT SETS

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Abstract. For a compact set $\Gamma \subset \mathbb{R}^2$ and a point x, we define the visible part of Γ from x to be the set

$$\Gamma_x = \{u \in \Gamma : [x, u] \cap \Gamma = \{u\}\}.$$

(Here $[x, u]$ denotes the closed line segment joining x to u.)

In this paper, we use energies to show that if Γ is a compact connected set of Hausdorff dimension larger than one, then for (Lebesgue) almost every point $x \in \mathbb{R}^2$, the Hausdorff dimension of Γ_x is strictly less than the Hausdorff dimension of Γ. In fact, for almost every x,

$$\dim_H(\Gamma_x) \leq \frac{1}{2} + \sqrt{\dim_H(\Gamma) - \frac{3}{4}}.$$

We also give an estimate of the Hausdorff dimension of those points where the visible set has dimension larger than $\sigma + \frac{1}{2} + \sqrt{\dim_H(\Gamma) - \frac{3}{4}}$ for $\sigma > 0$.

1. Introduction

Given a subset E of the plane, Urysohn [11, 12] defined the notion of linear accessibility for a point $p \in E$: p is *linearly accessible* if there is a non-degenerate line segment L that only meets E at the point p. In a sequence of papers, Nikodym [7, 8, 9] investigated the relationship between the set theoretic complexity of E and the set of linearly accessible points.

In this paper, we consider those points of a compact connected set Γ set that are linearly accessible from a given fixed point x and investigate the relationship between the (Hausdorff) dimensions of the compact set and its linearly accessible part from x for Lebsgue almost all $x \in \mathbb{R}^2 \setminus \Gamma$. Denoting Γ_x to be the points of Γ that are linearly accessible from x, it is clear that $\dim_H(\Gamma_x) \leq \dim_H(\Gamma)$ for all $x \in \mathbb{R}^2 \setminus \Gamma$. What is perhaps surprising though is that for most points there is a drop in dimension.

Proceeding more formally, if for a compact set in the plane, K, and $x \in \mathbb{R}^2$ we define the visible part of K from x by

$$K_x = \{u \in K : [x, u] \cap K = \{u\}\},$$

where $[x, u]$ denotes the closed line segment joining x to u, then our results may be summarised as follows.

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Theorem 1.1. *If $\Gamma \subset \mathbb{R}^2$ is a compact connected set with $\dim_H(\Gamma) > 1$, then for (Lebesgue) almost all $x \in \mathbb{R}^2$,*

$$\dim_H(\Gamma_x) \leq \frac{1}{2} + \sqrt{\dim_H(\Gamma) - \frac{3}{4}}.$$

This follows directly from the theorem that we prove in this paper.

Theorem 1.2. *Let $\Gamma \subset \mathbb{R}^2$ be a compact connected set with $\dim_H(\Gamma) > 1$. Then for $\frac{1}{2} + \sqrt{\dim_H(\Gamma) - \frac{3}{4}} < s \leq \dim_H(\Gamma)$,*

$$\dim_H\{x \in \mathbb{R}^2 : \dim_H(\Gamma_x) > s\} \leq \frac{\dim_H(\Gamma) - s}{s - 1}.$$

In an earlier paper [3], it was shown that for a particular class of compact connected sets (namely quasicircles), whenever x lies outside the set, $\dim_H(\Gamma_x) = 1$. Since quasicircles can have dimension arbitrarily close to 2, and for connected sets of positive dimension, $\dim_H(\Gamma_x) \geq 1$ whenever $x \notin \Gamma$, it follows that, unless the optimal upper bound for $\dim_H(\Gamma_x)$ is one, there is no general result concerning the lower bound of $\dim_H(\Gamma_x)$ beyond the trivial estimate.

There are many possible directions for future work. Despite the fact that the upper bound given in Theorem 1.1 is the golden-ratio for $\dim_H(\Gamma) = 2$, there is no good reason to believe that this bound is optimal, since the proof we give in this paper uses at least one sub-optimal estimate. It would be interesting to know the correct upper bound. Our method of proving Theorem 1.2 relies in an essential way on the properties of connected sets in the plane, and it is unclear whether a similar result could hold in higher dimensions. Whether a dimension drop will occur for totally disconnected sets is also unclear: in [3], it is shown that, for the cross-product of a Cantor set with itself in the plane, there is a dimension drop (to 1), provided that the original Cantor set has Hausdorff dimension sufficiently close to 1.

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2. Background results and preliminary estimates

In this section we summarise the main definitions and results that we use.

Most of the time we shall be working in the plane, $\mathbb{R}^2$, endowed with the usual norm, $|\cdot|$ and inner product $\langle \cdot\,,\,\cdot \rangle$. We let e_1 and e_2 denote the usual basis vectors in $\mathbb{R}^2$ and set $x^\wedge = x/|x|$ for $x \neq 0$, and $x^\perp = \langle x,\, e_1\rangle e_2 - \langle x,\, e_2\rangle e_1$ for $x \in \mathbb{R}^2$. For $x \in \mathbb{R}^2$ and $A \subseteq \mathbb{R}^2$, define $\text{arc-diam}_x(A)$ to be the angle (in radians) subtended by the smallest arc in the circle $\{u : |x-u| = 1\}$ that contains the radial projection of A onto this circle. (If $x \in A$, then $\text{arc-diam}_x(A) = 2\pi$.)

For a subset A of the plane and $r > 0$, let

$$B(A, r) = \{y \in \mathbb{R}^2 : \text{There is } x \in A \text{ with } |y - x| \leq r\}$$

and, in a slight abuse of notation, let $B(x, r) = B(\{x\}, r)$, the usual closed ball of centre x and radius r.

Let X be a Polish space. (That is, X is a complete, separable, metrisable topological space.) A sub-additive, non-negative set function μ on X is a Radon

measure if it is a Borel measure (all Borel sets are μ-measurable) for which all compact sets have finite measure and both

$$\mu(U) = \sup\{\mu(K) : K \subset U,\ K \text{ is compact}\}, \text{ for open sets } U$$

and

$$\mu(A) = \inf\{\mu(U) : A \subset U,\ U \text{ is open}\}, \text{ for } A \subseteq X.$$

We denote the set of Radon measures on X by $\mathcal{M}(X)$.

We let $\sigma(\mathbf{A}(X))$ denote the σ-algebra generated by the analytic subsets of X, we suppress mention of X when this is clear from the context. If μ is a Radon measure on this space then all sets in $\sigma(\mathbf{A})$ are μ-measurable. See [4, (21.10)].

For $s \in \mathbb{R}$ and $A \subseteq X$, we define

$$\begin{aligned}&\mathcal{M}^s(A)\\ &= \{\nu \in \mathcal{M}(X) : \nu(A) > 0 \text{ and } \nu(B(x,r)) \le r^s \text{ for } x \in X,\ 0 < r \le 1\}.\end{aligned}$$

If μ is a Radon measure on the plane and $s \in \mathbb{R}$, then $I_s(\mu)$ denotes the s-energy of μ given by

$$I_s(\mu) = \iint |x-y|^{-s}\, d\mu(x)\, d\mu(y).$$

The Hausdorff dimension of a set is defined in the usual way via Hausdorff measures, see [1, 2, 6, 10]. The following theorem summarises some useful equivalent ways of finding the Hausdorff dimension of a set.

Theorem 2.1. *Let A be an analytic subset of a Euclidean space, $\mathbb{R}^n$. Then*

$$\begin{aligned}\dim_H(A) &= \sup\{s \in \mathbb{R} : \mathcal{M}^s(A) \neq \emptyset\}\\ &= \sup\{s \in \mathbb{R} : \textit{There is } \mu \in \mathcal{M}(\mathbb{R}^n) \textit{ with } \nu(A) > 0 \textit{ and } I_s(\mu) < \infty\}\\ &= \sup\{\dim_H(K) : K \subseteq A \textit{ and } K \textit{ is compact}\}.\end{aligned}$$

Proof. See [1, Theorem 6.4] together with [2, 2.10.48] or [10, Theorem 57]. □

We record some simple geometric estimates for future use. For $x \in \mathbb{R}^2$, $d_-, d_+ \in \mathbb{R}^+$, let $A(x, d_-, d_+) = B(x, d_+) \setminus B(x, d_-)$, a half-open annulus.

Lemma 2.1. *Let $0 < d_- \le d_+$ with $d_- \le 1$ and let $a \in \mathbb{R}^2 \setminus \{0\}$ and $E \subseteq A(0, d_-, d_+)$ be compact. Suppose that $|a| \le \frac{1}{2} d_-$ and let $\alpha = \min\{|\langle p,\, a^\perp\rangle / \langle p,\, a\rangle| : p \in E\}$. If $\alpha \le 1$, then for all $p \in E$*

$$\frac{1}{2} \le \frac{\langle p-a,\, p+a\rangle}{|p-a||p+a|} \le 1 - \frac{9}{17 d_+^2}(|a|\alpha)^2.$$

Proof. For $p \in E$,

$$\langle p-a,\, p+a\rangle = |p|^2 - |a|^2$$

and

$$|p-a|^2|p+a|^2 = (|p|^2 + |a|^2)^2 - 4\langle p,\, a\rangle^2.$$

If $A = \langle p,\, a^\perp\rangle / \langle p,\, a\rangle$, then $1 + A^2 = \frac{|p|^2|a|^2}{\langle p,\, a\rangle^2}$, and so

$$|p-a|^2|p+a|^2 = \frac{(|p|^2 - |a|^2)^2}{1+A^2}\left(1 + \left(\frac{|p|^2+|a|^2}{|p|^2-|a|^2}\right)^2 A^2\right).$$

(If $|A| = +\infty$, then read the formula as $|p-a|^2|p+a|^2 = (|p|^2+|a|^2)^2$.) Hence

$$(*) \qquad \frac{\langle p-a,\, p+a\rangle}{|p-a||p+a|} = \sqrt{\frac{1+A^2}{1+(1+\mu)^2A^2}} = \sqrt{1-\frac{\mu(2+\mu)A^2}{1+(1+\mu)^2A^2}},$$

where

$$2\frac{|a|^2}{d_+^2} \le 2\left(\frac{|a|}{|p|}\right)^2 \le \mu = 2\frac{(|a|/|p|)^2}{1-(|a|/|p|)^2} \le \frac{8}{3}\left(\frac{|a|}{|p|}\right)^2 \le \frac{2}{3}.$$

It is easy to see that for $p \in E$, (*) is maximised when $A = |\langle p,\, a^\perp\rangle/\langle p,\, a\rangle| = \alpha$. However

$$(1-x)^{\frac{1}{2}} \le 1-\tfrac{1}{2}x, \text{ for } 0 \le x \le 1,$$

and so, since $\frac{\mu(2+\mu)\alpha^2}{1+(1+\mu)^2\alpha^2} = 1-\frac{1+\alpha^2}{1+(1+\mu)^2\alpha^2} \le 1$, and since $\mu \le \frac{2}{3}$,

$$\frac{\langle p-a,\, p+a\rangle}{|p-a||p+a|} \le 1-\frac{1}{2}\left(\frac{\mu(2+\mu)\alpha^2}{1+(1+\mu)^2\alpha^2}\right) \le 1-\tfrac{9}{34}\mu\alpha^2 \le 1-\tfrac{9}{17}\left(\frac{|a|\alpha}{d_+}\right)^2.$$

The lower bound follows from recognising that (*) is minimised when $p = d_-a^\perp/|a|$. □

For $x \in \mathbb{R}^2$, $u \in \mathbb{R}^2 \setminus \{0\}$ and $\sigma > 0$, let

$$V(x,u,\sigma) = \{y \in \mathbb{R}^2 : |\langle y-x,\, u^\perp\rangle| < \sigma\langle y-x,\, u\rangle\},$$

the open cone with vertex x, direction u and opening σ. The next lemma gives a lower bound on the distance of a point in a particular subregion of a cone from the vertex.

Lemma 2.2. *Let $p \in \mathbb{R}^2 \setminus \{0\}$ and $\sigma, \tau > 0$. If*

$$u \in V(0,p,\sigma) \setminus V(p,-p,\tau),$$

then

$$\langle u-p,\, p^\wedge\rangle \ge -\frac{\sigma}{\sigma+\tau}|p|.$$

Proof. Suppose that $u \in V(0,p,\sigma) \setminus V(p,-p,\tau)$, then

$$\langle u-p,\, p^\wedge\rangle \ge \langle q-p,\, p^\wedge\rangle$$

where

$$q = \mu(p+\sigma p^\perp) = p+\lambda(-p+\tau p^\perp),$$

for some $\mu, \lambda > 0$. Calculating $\langle q,\, p^\perp\rangle$ gives

$$\mu = \lambda\frac{\tau}{\sigma}$$

and substituting for μ in $\langle q,\, p\rangle$ gives

$$\lambda = \frac{\sigma}{\sigma+\tau}.$$

Hence

$$\langle q-p,\, p^\wedge\rangle \ge -\frac{\sigma}{\sigma+\tau}|p|,$$

as required. □

2.1. **Elementary measure estimates.** We now prove some estimates concerning the geometric distribution of mass for Radon measures in the plane.

We start by recording a simple mass estimate.

Lemma 2.3. *Fix $s > 0$ and $0 < d_- \leq \frac{1}{2}d_+$. Let ν be a Radon measure such that for all $u \in \mathbb{R}^2$ and $r > 0$, $\nu(B(u,r)) \leq r^s$. Suppose that $x \in \mathbb{R}^2$ and $V \subseteq \mathbb{R}^2$, then*

$$\nu(V \cap A(x, d_-, d_+)) \leq c\,\text{arc-diam}_x(V \cap A(x, d_-, d_+))^{s-1},$$

for some fixed positive constant c depending only on d_-, d_+ and s.

Proof. We may suppose that $x = 0$. Let $\theta = \text{arc-diam}_0(V)$. If $\theta \leq 1/2$, then $\theta d_+ \leq d_+ - d_-$ and so $V \cap A(0, d_-, d_+)$ may be covered by $1 + \frac{d_+ - d_-}{d_+\theta}$ boxes of side $d_+\theta$. Hence a simple estimate of mass gives

$$\nu(V \cap A(0, d_-, d_+)) \leq 2^{\frac{1}{2}s}(d_+\theta + d_+ - d_-)(d_+\theta)^{s-1} \leq 3d_+^s 2^{\frac{1}{2}s-1}\theta^{s-1},$$

and the lemma follows for $\theta \leq 1/2$. If $\theta \geq 1/2$, then we use the estimate that $\nu(A(0, d_-, d_+)) \leq d_+^s$. □

We now prove a lemma on the distribution of mass for an arbitrary measure in semi-infinite tubes. To do this we define for $x \in \mathbb{R}^2$ and $r > 0$,

$$T^+(x,r) = \{z \in \mathbb{R}^2 : |p_1(z) - p_1(x)| < r \text{ and } p_2(z) > p_2(x)\}$$

and

$$T^-(x,r) = \{z \in \mathbb{R}^2 : |p_1(z) - p_1(x)| < r \text{ and } p_2(z) < p_2(x)\},$$

where p_1 and p_2 denote orthogonal projection onto the x- and y-axis, respectively. Thus $T^+(x,r)$ is an open vertical tube of width $2r$ extending upwards from x and $T^-(x,r)$ is an open vertical tube of width $2r$ extending downwards from x.

Proposition 2.1. *Suppose ν is a compactly supported Radon measure in the plane. Then for $\xi > 0$ and ν-a.e. x*

$$\liminf_{r\to 0} \frac{\nu(T^+(x,r))}{r^{1+\xi}} = \liminf_{r\to 0} \frac{\nu(T^-(x,r))}{r^{1+\xi}} = +\infty.$$

Proof. We give the proof for T^+; the proof for T^- is similar. Without loss of generality we assume that $\operatorname{spt}\nu$ lies in the unit square $[0,1] \times [0,1]$ and let

$$E_\infty = \left\{x : \liminf_{r\to 0} \frac{\nu(T^+(x,r))}{r^{1+\xi}} = +\infty\right\}.$$

Since $\nu(\mathbb{R}^2 \setminus \operatorname{spt}\nu) = 0$, it is enough to show that $\nu(\operatorname{spt}\nu \setminus E_\infty) = 0$.

For M and $j \in \mathbb{N} \cup \{0\}$, let

$$E^{M,j} = \{x \in \operatorname{spt}\nu : \nu(T^+(x,r)) < Mr^{1+\xi} \text{ for some } 0 < r \leq 2^{-j}\}.$$

Then

$$\operatorname{spt}\nu = E_\infty \cup \bigcup_{M\in\mathbb{N}} \bigcap_{j\in\mathbb{N}\cup\{0\}} E^{M,j},$$

and

$$E^{M,j} \subset \bigcup_{k\geq j} \underbrace{\{x \in \operatorname{spt}\nu : \nu(T^+(x, 2^{-k})) < 2^{1+\xi}M2^{-k(1+\xi)}\}}_{E^{M,j,k},\text{ say}}.$$

We now estimate the ν measure of $E^{M,j,k}$ for $k \geq j \in \mathbb{N}$. Choose $F \subseteq E^{M,j,k}$ compact such that

$$\nu(F) \geq \nu(E^{M,j,k})/2.$$

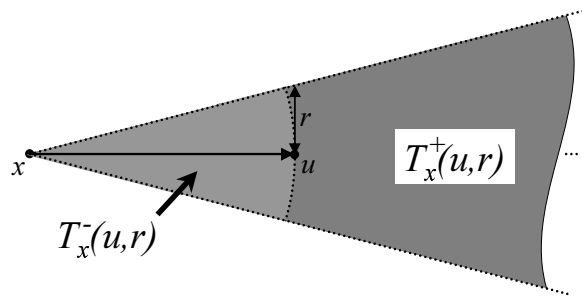

FIGURE 1. The radial tubes $T_x^-(u,r)$ and $T_x^+(u,r)$.

We consider the 2^{k+2} columns $C_i = [i2^{-(k+2)}, (i+1)2^{-(k+2)}] \times \mathbb{R}$, $i = 0, \ldots, 2^{k+2}-1$. For each i with $C_i \cap F \neq \emptyset$, we choose $x_i \in C_i \cap F$ to have minimum possible height above the x-axis, ie

$$\text{dist}\,(C_i \cap F, \mathbb{R} \times \{0\}) = \text{dist}\,(x_i, \mathbb{R} \times \{0\}).$$

For such an i,

$$\nu(F \cap T^+(x_i, 2^{-k}) \leq \nu(T^+(x_i, 2^{-k})) < 2^{1+\xi} M 2^{-k(1+\xi)}.$$

Clearly

$$F \subseteq \bigcup_{i: F \cap C_i \neq \emptyset} F \cap C_i \subseteq \bigcup_{i: F \cap C_i \neq \emptyset} F \cap T^+(x_i, 2^{-k}).$$

And so

$$\begin{aligned} \nu(F) &\leq \sum_{i: F \cap C_i \neq \emptyset} \nu(F \cap T^+(x_i, 2^{-k})) \\ &< 2^{k+2} \times 2^{1+\xi} M 2^{-k(1+\xi)} \\ &= 2^{3+\xi} M 2^{-k\xi}. \end{aligned}$$

Hence

$$\nu(E^{M,j,k}) < 2^{4+\xi} M 2^{-k\xi}$$

and so

$$\nu(E^{M,j}) \leq \sum_{k=j}^{\infty} \nu(E^{M,j,k}) < \frac{2^{4+\xi} M}{1 - 2^{-\xi}} 2^{-j\xi}.$$

Thus,

$$\nu\left(\bigcup_{M \in \mathbb{N}} \bigcap_{j \in \mathbb{N}} E^{M,j} \right) = 0$$

and the lemma follows. □

For $x \neq u \in \mathbb{R}^2$ and $r > 0$, define radial tubes $T_x^+(u,r)$ and $T_x^-(u,r)$ by

$$T_x^+(u,r) = V(x, u - x, r/d(x,u)) \cap \{z \in \mathbb{R}^2 : d(x,z) > d(x,u)\}$$

and

$$T_x^-(u,r) = V(x, u - x, r/d(x,u)) \cap \{z \in \mathbb{R}^2 : d(x,z) < d(x,u)\},$$

see Figure 1.

It is easy to use a bi-Lipschitz transformation to transform our lemma about parallel tubes to one about radial tubes.

Lemma 2.4. *Let ν be a compactly supported Radon measure in the plane and $x \notin \operatorname{spt}\nu$. Then for $\xi > 0$ and for ν-a.e. u*

$$\liminf_{r\to 0} \frac{\nu(T_x^+(u,r))}{r^{1+\xi}} = \liminf_{r\to 0} \frac{\nu(T_x^-(u,r))}{r^{1+\xi}} = +\infty.$$

Proof. Since $x \notin \operatorname{spt}\nu$, there is $\rho > 0$ with $B(x,\rho) \cap \operatorname{spt}\nu = \emptyset$. Since $\operatorname{spt}\nu$ is compact, we can find some $R > \rho$ for which $\operatorname{spt}\nu \subset B(x,R)$. Moreover, by restricting and translating ν suitably, we may suppose that $\operatorname{spt}\nu$ is a subset of a quadrant of the plane with corner at x, $Q(x)$, say, intersected with the annulus $A(x,\rho/2,R)$. It is now straightforward to find a transformation (namely, $re^{i\theta} \mapsto (r,\theta)$) which transforms radial lines segments through x and intersecting this region to half-lines parallel to the y-axis. This transformation is bi-Lipschitz when restricted to $Q(x) \cap A(x,\rho/2,R)$. This gives us the situation described in Proposition 2.1 and the claim follows. □

This lemma allows us to show that measures with dimension larger than one have mass far from the origin of these radial tubes for typical points:

Lemma 2.5. *Let $s > 1$, $0 < r_1 \le r_0 \le 1$ and $\xi, M, d_-, c > 0$, and $x \in \mathbb{R}^2$. Suppose that ν is a compactly supported Radon measure on the plane and $F \subseteq E$ are compact sets in the plane satisfying:*

(1) *for all $u \in E$, $|u-x| \ge d_-$;*
(2) *for all $u \in E$ and $0 < r \le r_0$,*
$$\nu B(u,r) \le cr^s;$$
(3) *for all $u \in F$ and $0 < r \le r_1$,*
$$\nu(E \cap T_x^{\pm}(u,r)) > Mr^{1+\xi}.$$

Then there are constants $r_2 \in (0, r_1/\sqrt{2}]$ and $d_0 > 0$ such that for $u \in F$ and

$$0 < r \le r_2,$$
$$\nu(E \cap T_x^{\pm}(u,r) \cap (\mathbb{R}^2 \setminus A(x, |u-x| - d_0 r^{2+\xi-s}, |u-x| + d_0 r^{2+\xi-s}))) > 0.$$

Proof. Let

$$d_0 = \tfrac{M}{12} 2^{-s/2} \text{ and } r_2 = \min\{r_1/\sqrt{2}, \tfrac{1}{2}\sqrt{3}d_-, (d_-/d_0)^{\frac{1}{2+\xi-s}}, d_0^{\frac{1}{s-1-\xi}}\}.$$

We give the proof for $T_x^+(u,r)$; the proof for $T_x^-(u,r)$ is similar. By rotating and translating, we may assume that $x = 0$ and the line segment $[x,u]$ is on the positive x-axis. Let $\Delta = |u-x| \ge d_-$.

Elementary geometry shows, since $r \le \frac{1}{2}\sqrt{3}d_- \le \frac{1}{2}\sqrt{3}\Delta$ and so $(1+(r/\Delta)^2)^{-\frac{1}{2}} \ge 1 - \frac{1}{2}(r/\Delta)^2$, that

$$\begin{aligned} &T_x^+(u,r) \cap A(x, \Delta - R, \Delta + R) \subseteq T_x^+(u,r) \cap B(x, \Delta + R) \\ &\subseteq \left([\Delta - \tfrac{1}{2}r^2/\Delta, \Delta] \times [-r,r]\right) \cup \left([\Delta, \Delta + R] \times [-r(1+R/\Delta), r(1+R/\Delta)]\right), \end{aligned}$$

for any $R \ge 0$. We choose $R = d_0 r^{2+\xi-s}$.

We estimate that $T_x^+(u,r) \cap (A, \Delta - R, \Delta + R)$ can be covered by

$$2 + (1 + 2R/r)(r(1+R/\Delta) + 1)$$

closed squares of side r, since $\frac{1}{2}r^2/\Delta \le \frac{1}{2}\frac{r_2}{d_-}r < r$. We find that

$$2 + (1+2R/r)(r(1+R/\Delta)+1) \le 2R/r + (3R/r)(2r+1) \le 11R/r,$$

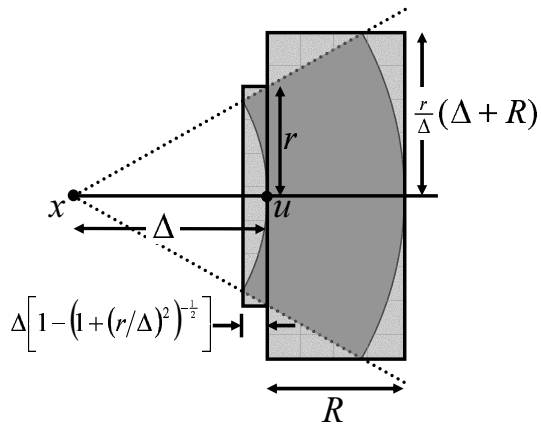

FIGURE 2. Estimating the mass of $T_x^+(u,r)$.

since $r \le R \le \Delta$ and $r \le 1$. Hence we require at most $11R/r$ balls of radius $\sqrt{2}r$ to cover $E \cap T_x^+(u,r) \cap (A, \Delta - R, \Delta + R)$.

So, since $\sqrt{2}r \le r_1 \le r_0$, we estimate that

$$\begin{aligned}\nu(E \cap T_x^+(u,r) \cap (A, \Delta - R, \Delta + R)) &\le (11R/r) \times 2^{s/2} r^s \\ &= 11 \cdot 2^{\frac{s}{2}} d_0 r^{1+\xi} \\ &< M r^{1+\xi},\end{aligned}$$

proving the lemma. □

2.2. **A 'two measures' estimate.** In this subsection, we investigate the interaction of two measures of large dimension when they are supported on different visible sets of Γ. The result that we prove in this section is the crux of our method. It shows that if two measures of large dimension are supported in different visible sets, then they will be 'disjoint' in the sense that balls containing points from both visible sets will have small mass for both measures. The remainder of the paper consists mainly of trying to place ourselves in a position to use this observation.

In the following proposition, $\mathcal{T}(x,y,p)$ denotes the closed triangle with vertices x, y and p, and $H(x,y;u)$ denotes the closed upper-half plane that has the line segment $[x,y]$ in its boundary and u lying in its interior.

Proposition 2.2. *Let Γ be a non-empty compact connected subset of $\mathbb{R}^2$. Suppose that $s > 1$, $0 < \xi < s-1$, $0 < r_1 \le r_0 \le 1$, $0 < d_- \le d_+$ with $d_- \le 1$ and $M > 0$ are given. Let $x, y \in \mathbb{R}^2 \setminus \Gamma$ satisfy*

$$0 < 2|x-y| < d_- \le \min\{d(x,\Gamma),\, d(y,\Gamma)\} \le \max\{d(x,\Gamma),\, d(y,\Gamma)\} + |\Gamma| \le d_+.$$

Let ν_x and ν_y be Radon measures supported in Γ_x and Γ_y respectively and let

$$F_x \subseteq E_x \subseteq \Gamma_x \text{ and } F_y \subseteq E_y \subseteq \Gamma_y$$

be compact sets. Suppose that:

(1) *for all $u \in E_x$, $v \in E_y$ and $0 < r \le r_0$ both*

$$\nu_x(B(u,r) \le r^s \text{ and } \nu_y(B(v,r)) \le r^s;$$

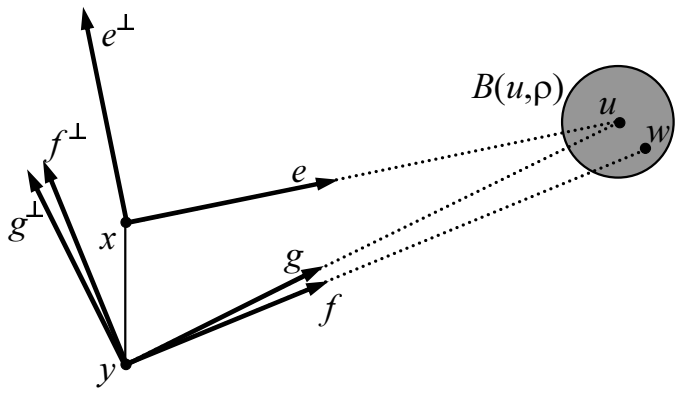

FIGURE 3. The vectors e, f and g.

(2) *for all* $u \in F_x$, $v \in F_y$ *and* $0 < r \le r_1$ *both*

$$\nu_x(T_x^{\pm}(u,r) \cap E_x) \ge Mr^{1+\xi} \text{ and } \nu_y(T_y^{\pm}(v,r) \cap E_y) \ge Mr^{1+\xi};$$

(3) *there is* $\psi \in (0, 1/2)$ *such that for* $u \in F_x \cup F_y$,

$$\langle (u-x)^\wedge, (u-y)^\wedge \rangle \in [\tfrac{1}{2}, 1-\psi].$$

Then there are constants $\alpha_0, d_1, c_1 > 0$ *such that for* $u \in F_x$, *if* $0 < \rho \le d_1 \psi^{\frac{1}{2}\frac{1}{s-1-\xi}}$, *then*

$$\nu_y(B(u,\rho) \cap F_y) \le c_1 \psi^{-\frac{1}{2}\left(\frac{s-1}{2+\xi-s}\right)} \rho^{\frac{1+\xi}{2+\xi-s}}. \tag{2.1}$$

Furthermore, if $v \in F_y \cap B(u,\rho)$, *then there is*

$$p \in [\tfrac{1}{2}(x+y), u] \cap B(u, \alpha_0 \psi^{-1/2} \rho)$$

such that $\mathcal{T}(x,y,p) \cap \Gamma = \emptyset$ *and*

$$V(p, \tfrac{1}{2}(x+y) - u, \tfrac{2}{5}\psi^{\frac{1}{2}}) \cap \Gamma \cap H(x,y;u) = \emptyset.$$

Notice that the symmetry of the hypotheses in this proposition imply that a version of (2.1) holds for $u \in F_y$ with ν_y replaced by ν_x and F_y replaced by F_x.

Here $\alpha_0 = 60 d_+/d_-$, $d_1 = \min\{(r_2/\alpha_1)^{2+\xi-s}, d_-/\alpha_0\}$ and $c_1 = 2^{5+s/2}\alpha_1^{s-1}(d_+/d_-)$, where $\alpha_1^{2+\xi-s} = (\alpha_0+1)/d_0$ and d_0, and r_2 are the constants determined in Lemma 2.5.

Proof. Suppose the conditions of the proposition are satisfied. Fix

$$0 < \rho \le d_1 \psi^{\frac{1}{2}\frac{1}{s-1-\xi}},$$

we must show that

$$\nu_y(F_y \cap B(u,\rho)) \le c_1 \psi^{-\frac{1}{2}\left(\frac{s-1}{2+\xi-s}\right)} \rho^{\frac{1+\xi}{2+\xi-s}}.$$

If $F_y \cap B(u,\rho) = \emptyset$, then there is nothing to prove. So suppose $w \in F_y \cap B(u,\rho)$ and set

$$e = (u-x)^\wedge, \quad f = (w-y)^\wedge \quad \text{and} \quad g = (u-y)^\wedge.$$

Notice that the third hypothesis of the proposition states

$$\tfrac{1}{2} \le \langle e, g \rangle \le 1 - \psi \tag{2.2}$$

and since

$$\langle e^\perp,\, g\rangle^2 = 1 - \langle e,\, g\rangle^2 \geq 1 - (1-\psi)^2 = \psi(2-\psi) \geq \psi,$$

it follows that

$$|\langle e^\perp,\, g\rangle| \geq \psi^{\frac{1}{2}}. \tag{2.3}$$

In order to prove the theorem, we make a sequence of geometric observations. In the first observation, we make some further estimates relating the angles between various of the vectors $e, e^\perp, f, f^\perp, g$ and $g^\perp$.

Observation 1. *If* $0 < \rho < \frac{1}{4}d_-$, *then*

$$|\langle f,\, g^\perp\rangle| \leq \rho/d_- \tag{2.4}$$

$$\langle f,\, g\rangle \geq 1 - 2\rho/d_- \tag{2.5}$$

$$\langle e,\, f\rangle > \tfrac{1}{4}d_-/d_+. \tag{2.6}$$

Proof. For inequality (2.4), we use $g^\perp = \langle g,\, f\rangle f^\perp - \langle g,\, f^\perp\rangle f$ and calculate

$$\langle f,\, g^\perp\rangle = 0 - \langle g,\, f^\perp\rangle = -\frac{1}{|u-y|}\langle u-y,\, f^\perp\rangle = -\frac{\langle u-w,\, f^\perp\rangle}{|u-y|}.$$

Hence $|\langle f,\, g^\perp\rangle| \leq \rho/d_-$.

For inequality (2.5), on noting

$$\langle f,\, g\rangle|w-y||u-y| = \langle w-y,\, u-y\rangle = \langle w-u,\, u-y\rangle + |u-y|^2,$$

we find

$$\langle f,\, g\rangle = \frac{1}{|w-y|}\langle w-u,\, g\rangle + \frac{|u-y|}{|w-y|} \geq -\frac{\rho}{d_-} + \left(1 - \frac{|u-w|}{|w-y|}\right) \geq 1 - 2\rho/d_-.$$

To verify inequality (2.6), note that $w = y + |w-y|f \in B(u,\rho)$, and so $w = y + (x-y) + |u-x|e + z$ for some $|z| \leq \rho$. Hence $|w-y|f = (x-y) + |u-x|e + z$ and

$$|w-y|\langle f,\, e\rangle = \langle x-y,\, e\rangle + |u-x| + \langle z,\, e\rangle.$$

Now

$$|\langle x-y,\, e\rangle| \leq \tfrac{1}{2}d_- \leq \tfrac{1}{2}|u-x| \quad \text{and} \quad |\langle z,\, e\rangle| \leq \rho \leq \tfrac{1}{4}d_- \leq \tfrac{1}{4}|u-x|.$$

Thus

$$|w-y|\langle e,\, f\rangle \geq \tfrac{1}{4}|u-x| \geq \tfrac{1}{4}d_-$$

and so $\langle e,\, f\rangle \geq \frac{1}{4}d_-/d_+$, as required. □

We now note that if $z \in T_y(w,r)$, then it is also in $T_y(u,r')$ for r' not too much bigger than r.

Observation 2. *If* $0 < \rho \leq \frac{1}{4}d_-$, *then*

$$V\left(y, f, \tfrac{\rho}{d_-}\right) \subseteq V\left(y, g, 4\tfrac{\rho}{d_-}\right).$$

Proof. If $z \in V(y, f, \rho/d_-)$, then

$$|\langle z-y,\, f^\perp\rangle| < \frac{\rho}{d_-}\langle z-y,\, f\rangle. \tag{2.7}$$

Since

$$z - y = \langle z-y,\, f\rangle f + \langle z-y,\, f^\perp\rangle f^\perp,$$

we find

$$\langle z-y,\, g^{\perp}\rangle = \langle z-y,\, f\rangle\langle f,\, g^{\perp}\rangle + \langle z-y,\, f^{\perp}\rangle\langle f^{\perp},\, g^{\perp}\rangle.$$

Hence (2.4) implies

$$|\langle z-y,\, g^{\perp}\rangle| \le \frac{\rho}{d_-}|\langle z-y,\, f\rangle| + |\langle z-y,\, f^{\perp}\rangle|.$$

Thus (2.7) gives

$$\begin{aligned} |\langle z-y,\, g^{\perp}\rangle| &\le \frac{\rho}{d_-}|\langle z-y,\, f\rangle| + \frac{\rho}{d_-}\langle z-y,\, f\rangle \\ &= 2(\rho/d_-)\langle z-y,\, f\rangle \end{aligned} \tag{2.8}$$

It only remains to estimate $\langle z-y,\, f\rangle$ in terms of $\langle z-y,\, g\rangle$. As $f = \langle f,\, g\rangle g + \langle f,\, g^{\perp}\rangle g^{\perp}$,

$$0 < \langle z-y,\, f\rangle \le \langle z-y,\, g\rangle\langle f,\, g\rangle + \langle z-y,\, g^{\perp}\rangle\langle f,\, g^{\perp}\rangle,$$

which, on using (2.8) and (2.4), gives

$$0 < \langle z-y,\, f\rangle \le \langle z-y,\, g\rangle\langle f,\, g\rangle + \frac{2\rho}{d_-} \times \frac{\rho}{d_-}\langle z-y,\, f\rangle.$$

Rearranging and using $0 < \langle f,\, g\rangle \le 1$, we find

$$\langle z-y,\, f\rangle[1 - 2(\rho/d_-)^2] \le \langle z-y,\, g\rangle.$$

Substituting back into (2.8), then gives

$$|\langle z-y,\, g^{\perp}\rangle| \le 2(\rho/d_-)[1 - 2(\rho/d_-)^2]^{-1}\langle z-y,\, g\rangle$$

which, as $\rho \le d_-/2$, proves the claim. □

Observation 3. *If* $0 < \rho \le \frac{1}{20}d_-\psi^{1/2}$, *then*

$$V(x, e, \rho/d_-) \cap V(y, f, \rho/d_-) \subseteq B(u, \alpha_0\psi^{-1/2}\rho),$$

where $\alpha_0 = 60\frac{d_+}{d_-}$.

Proof. Fix $z \in V(y, f, \rho/d_-) \cap V(x, e, \rho/d_-)$. Since $0 < \rho \le \frac{1}{20}d_-\psi^{1/2} \le d_-/4$, observation 2 implies $z \in V(y, g, 4\rho/d_-)$. Hence there are $\lambda, \mu > 0$ for which

$$z = y + \lambda(g - bg^{\perp}) = x + \mu(e + ae^{\perp})$$

where $|b| \le 4\rho/d_-$ and $|a| \le \rho/d_-$. We wish to find an upper bound for $|z-u|$. Now

$$\langle z-x,\, e\rangle = \mu \quad \text{and} \quad \langle z-y,\, g\rangle = \lambda.$$

Notice that

$$|z-u|^2 = \langle z-u,\, g\rangle^2 + \langle z-u,\, g^{\perp}\rangle^2 = (\lambda - |y-u|)^2 + b^2\lambda^2, \tag{2.9}$$

and so upper estimates for $(\lambda - |y-u|)^2$ and λ^2 give an upper estimate for $|z-u|$.

Now

$$\langle z-u,\, e\rangle = \langle y-u,\, e\rangle + \lambda(\langle g,\, e\rangle - b\langle g^{\perp},\, e\rangle) = \langle x-u,\, e\rangle + \mu$$

and so

$$\mu = |x-u| - |u-y|\langle g,\, e\rangle + \lambda(\langle g,\, e\rangle - b\langle g^{\perp},\, e\rangle).$$

Also

$$\langle z-u,\, e^{\perp}\rangle = \langle y-u,\, e^{\perp}\rangle + \lambda(\langle g,\, e^{\perp}\rangle - b\langle g^{\perp},\, e^{\perp}\rangle) = a\mu$$

and so

$$\begin{aligned}&-|u-y|\langle g,\, e^{\perp}\rangle + \lambda(\langle g,\, e^{\perp}\rangle - b\langle g^{\perp},\, e^{\perp}\rangle)\\ &= a|u-x| - a|u-y|\langle g,\, e\rangle + a\lambda(\langle g,\, e\rangle - b\langle g^{\perp},\, e\rangle).\end{aligned}$$

This rearranges to give

$$\lambda\gamma = a|u-x| + |u-y|(\langle g,\, e^{\perp}\rangle - a\langle e,\, g\rangle),$$

where

$$\gamma = (1-ab)\langle g,\, e^{\perp}\rangle - (a+b)\langle e,\, g\rangle.$$

Thus

$$\begin{aligned}\lambda - |u-y| &= \gamma^{-1}\left[a|u-x| + |u-y|(\langle g,\, e^{\perp}\rangle - a\langle e,\, g\rangle - \gamma)\right]\\ &= \gamma^{-1}\left[a|u-x| + |u-y|(ab\langle g,\, e^{\perp}\rangle + b\langle e,\, g\rangle)\right]\\ &= \gamma^{-1}\left[a|u-x| + b|u-y|(a\langle g,\, e^{\perp}\rangle + \langle e,\, g\rangle)\right].\end{aligned}$$

Since $|a| \le \rho/d_-$ and $|b| \le 4\rho/d_-$, it follows that $|ab| \le 1/2$ and $|a+b| \le 5\rho/d_-$. From equation (2.3) we know $|\langle e^{\perp},\, g\rangle| \ge \psi^{1/2}$, and so

$$|\gamma| \ge \tfrac{1}{2}\psi^{1/2} - 5(\rho/d_-) \ge \tfrac{1}{4}\psi^{1/2},$$

since $\rho \le \frac{1}{20}d_-\psi^{1/2}$.

Hence, as $|a| \le \rho/d_- \le 1$,

$$|\lambda| \le 4\psi^{-1/2}\left[|a||u-x| + |u-y|(|\langle g,\, e^{\perp}\rangle| + |a||\langle e,\, g\rangle|)\right] \le 12d_+\psi^{-1/2}$$

and, as $|b| \le 4\rho/d_-$,

$$\begin{aligned}&|\lambda - |u-y||\\ &\le 4\psi^{-1/2}\left[|a||u-x| + |b||u-y|(|a||\langle g,\, e^{\perp}\rangle| + |\langle e,\, g\rangle|)\right]\\ &\le 36(d_+/d_-)\psi^{-1/2}\rho.\end{aligned}$$

Thus estimating λ in (2.9) gives

$$|z-u|^2 \le (36(d_+/d_-)\psi^{-1/2}\rho)^2 + (48(d_+/d_-)\psi^{-1/2}\rho)^2,$$

and so

$$|z-u| \le 60(d_+/d_-)\psi^{-1/2}\rho,$$

as required. □

We now observe that there is a 'large' triangle that is disjoint from Γ and with a vertex close to u (and hence w).

Observation 4. *If*

$$0 < \rho \le d_1\psi^{\frac{1}{2}\left(\frac{1}{s-1-\xi}\right)},$$

and $r = \alpha_1(\psi^{-\frac{1}{2}}\rho)^{\frac{1}{2+\xi-s}}$, *then there is*

$$z \in V(x, e, r/d_-) \cap V(y, f, r/d_-)$$

with

$$\mathcal{T}(x, y, z) \cap \Gamma = \emptyset$$

and

$$z \in B(u, \alpha_0\psi^{-1/2}\rho).$$

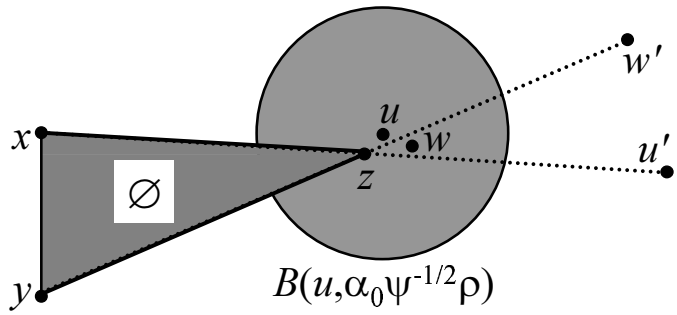

FIGURE 4. $\emptyset \neq [x, u'] \cap [y, w'] \subseteq B(u, \alpha_0 \psi^{-1/2} \rho)$.

Proof. We aim to find a point z which is visible from both x and y. Recall that

$$\alpha_1^{2+\xi-s} = (\alpha_0 + 1)/d_0 \text{ and } d_1 = \min\{(r_2/\alpha_1)^{2+\xi-s},\, d_-/\alpha_0\}.$$

(The constant α_0 is given in observation 3, and r_2 and d_0 are given in Lemma 2.5.)

Since $w \in F_y$ and

$$r = \alpha_1 (\psi^{-\frac{1}{2}} \rho)^{\frac{1}{2+\xi-s}} \leq \alpha_1 d_1^{\frac{1}{2+\xi-s}} \leq r_2,$$

we may use Lemma 2.5 applied to ν_y and w to find $w' \in E_y \cap T_y^+(w, r)$, and in particular lying in $V(y, w - y, r/d_-)$, for which

$$|w - w'| > d_0 r^{2+\xi-s} = d_0 \alpha_1^{2+\xi-s} \psi^{-\frac{1}{2}} \rho.$$

Hence, as $|w - u| \leq \rho$,

$$|w' - u| > d_0 \alpha_1^{2+\xi-s} \psi^{-\frac{1}{2}} \rho - \rho = (d_0 \alpha_1^{2+\xi-s} \psi^{-\frac{1}{2}} - 1)\rho \geq \alpha_0 \psi^{-1/2} \rho.$$

But

$$\rho \leq d_1 \psi^{\frac{1}{2}\left(\frac{1}{s-1-\xi}\right)} \leq d_-/\alpha_0 < d_-/4$$

and

$$r = \alpha_1 (\psi^{-\frac{1}{2}} \rho)^{\frac{1}{2+\xi-s}} \rho^{-1} \rho = \alpha_1 (\psi^{-\frac{1}{2}} \rho^{s-1-\xi})^{\frac{1}{2+\xi-s}} \rho \leq \alpha_1 d_1^{\frac{1}{2+\xi-s}} \rho \leq \rho.$$

Hence, by observation 2, $w' \in V(y, g, 4\rho/d_-)$.

Similarly, there is $u' \in E_x \cap T_x^+(u, r)$ for which

$$|u' - u| \geq \alpha_0 \psi^{-1/2} \rho$$

and, clearly, $u' \in V(x, u - x, r/d_-)$.

Now both $|u - x|$ and $|u - y|$ are no less than d_- and

$$\alpha_0 \psi^{-1/2} \rho \leq \alpha_0 d_1 \psi^{\frac{1}{2}\left(\frac{2+\xi-s}{s-1-\xi}\right)} < \alpha_0 d_1 \leq d_-,$$

hence

$$\min\{|u - x|,\, |u - y|\} > \alpha_0 \psi^{-1/2} \rho.$$

Moreover

$$r \leq \rho \leq d_1 \psi^{\frac{1}{2}\left(\frac{1}{s-1-\xi}\right)} \leq d_1 \psi^{\frac{1}{2}} < \tfrac{1}{20} d_- \psi^{1/2},$$

and so it follows from observation 3 that

$$\emptyset \neq [x, u'] \cap [y, w'] \subseteq B(u, \alpha_0 \psi^{-1/2} \rho).$$

Let z denote this intersection point. Then

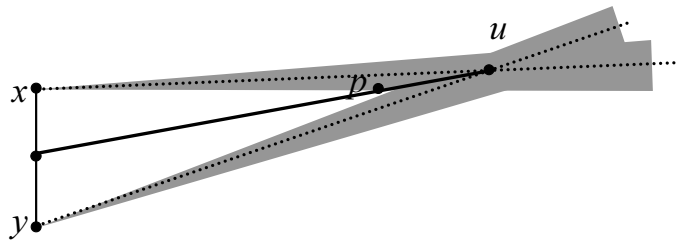

FIGURE 5. $p \in [\frac{1}{2}(x+y), u] \cap (V(x, e, 4\rho/d_-) \cup V(y, g, 4\rho/d_-))$.

$$([x, z] \cup [x, y] \cup [y, z]) \cap \Gamma = \emptyset,$$

since Γ is connected, u' is visible from x and w' is visible from y. The observation follows. □

We now use this observation to find an empty cone with base point near to u.

Observation 5. *If* $0 < \rho \leq d_1 \psi^{\frac{1}{2}\left(\frac{1}{s-1-\xi}\right)}$, *then there is* $p \in [\frac{1}{2}(x+y), u] \cap B(u, \alpha_0 \psi^{-\frac{1}{2}} \rho)$ *for which*

$$\mathcal{T}(x, y, p) \cap \Gamma = \emptyset$$

and

$$V(p, \tfrac{1}{2}(x+y) - u, \tfrac{2}{5}\psi^{1/2}) \cap H[x, y; u] \cap \Gamma = \emptyset.$$

Proof. Let

$$p \in [\tfrac{1}{2}(x+y), u] \cap (V(x, e, 4\rho/d_-) \cup V(y, g, 4\rho/d_-))$$

be chosen to be at the minimum possible distance from $\frac{1}{2}(x+y)$, see Figure 5. Then there is $\lambda > 0$ such that

$$p = u - \lambda \frac{|u-x|e + |u-y|g}{||u-x|e + |u-y|g|}$$

Suppose (without loss of generality) that $p \in V(y, g, 4\rho/d_-)$, then there is $\mu > 0$ and $\sigma \in \{+1, -1\}$ such that

$$p = y + \mu(g + 4\rho d_-^{-1} \sigma g^{\perp}).$$

Hence, if we set $h = |u-x|e + |u-y|g$, then

$$u - \lambda h/|h| = y + \mu(g + 4\rho d_-^{-1} \sigma g^{\perp}),$$

which, as $u - y = |u-y|g$, rearranges to give

$$|u-y|g - \lambda h/|h| = \mu(g + 4\rho d_-^{-1} \sigma g^{\perp}). \tag{2.10}$$

So taking the inner product of (2.10) with $g^{\perp}$ gives

$$-\lambda \frac{|u-x|}{|h|} \langle e,\, g^{\perp} \rangle = 4 \frac{\rho}{d_-} \sigma \mu \tag{2.11}$$

and taking the inner product of (2.10) with g and rearranging gives

$$|u-y| - \lambda \frac{|u-x| \langle e,\, g \rangle + |u-y|}{|h|} = \mu.$$

Substituting for μ from (2.11) gives

$$|u-y| - \lambda\frac{|u-x|\langle e,\, g\rangle + |u-y|}{|h|} = -\frac{d_-}{4\rho\sigma}\cdot\frac{|u-x|\langle e,\, g^\perp\rangle}{|h|}\lambda$$

and this rearranges to give

$$\lambda\left[\left(\frac{d_-}{4\rho\sigma}\langle e,\, g^\perp\rangle - \langle e,\, g\rangle\right)|u-x| - |u-y|\right] = -|u-y||h|$$

and so, substituting for h,

$$\lambda\left[(d_-\langle e,\, g^\perp\rangle - 4\rho\sigma\langle e,\, g\rangle) - 4\rho\sigma\frac{|u-y|}{|u-x|}\right] = -4\rho\sigma|u-y|\left|e + \frac{|u-y|}{|u-x|}g\right|$$

As $|x-y| \le d_-/2$, it easily follows that

$$\frac{2}{3} \le \frac{|u-x|}{|u-y|},\ \frac{|u-y|}{|u-x|} \le \frac{3}{2},$$

and so

$$|\lambda| \le \frac{4\rho}{d_-}\times\left(1+\tfrac{3}{2}\right)\frac{|u-y|}{|\sigma\langle e,\, g^\perp\rangle - 4\rho d_-^{-1}(\langle e,\, g\rangle + |u-y|/|u-x|)|}.$$

Now $|\langle e,\, g\rangle + |u-y|/|u-x|| \le 5/2$ and by (2.3),

$$|\langle e,\, g^\perp\rangle| = |\langle e^\perp,\, g\rangle| \ge \sqrt{\psi}.$$

Thus

$$|\sigma\langle e,\, g^\perp\rangle - 4\rho d_-^{-1}(\langle e,\, g\rangle + |u-y|/|u-x|)| \ge \sqrt{\psi} - 10\rho/d_- \ge \tfrac{1}{2}\sqrt{\psi},$$

since $\rho \le d_1\psi^{\frac{1}{2}\left(\frac{1}{s-1-\xi}\right)} \le d_1\psi^{\frac{1}{2}} \le (d_-/\alpha_0)\psi^{\frac{1}{2}} < \frac{1}{20}d_-\sqrt{\psi}$. Hence

$$|\lambda| \le 10\rho d_-^{-1}|u-y| \times 2\psi^{-1/2} \le 20(d_+/d_-)\psi^{-1/2}\rho \le \alpha_0\psi^{-\frac{1}{2}}\rho,$$

and $p \in B(u, \alpha_0\psi^{-1/2}\rho)$, as claimed.

Since the hypotheses of observation 4 are satisfied, there is a point z satisfying its conclusions, and we note that $p \in \mathcal{T}(x,y,z)$. Hence $\mathcal{T}(x,y,p)\cap\Gamma = \emptyset$.

To show that

$$V(p,\, \tfrac{1}{2}(x+y)-u,\, \tfrac{2}{5}\psi^{1/2}) \cap H[x,y;u] \cap \Gamma = \emptyset$$

we recall that $h = |u-x|e + |u-y|g$ and so

$$h^\perp = e^\perp|u-x| + g^\perp|u-y|.$$

We shall estimate $|\langle x-u,\, h^\perp\rangle|, |\langle y-u,\, h^\perp\rangle|, |\langle x-u,\, h\rangle|$ and $|\langle y-u,\, h\rangle|$ for if $q \in H(x,y;u)$ satisfies

$$|\langle q-p,\, h^\perp\rangle| \le m\langle q-p,\, h\rangle$$

where $m = \min\{|\langle x-u,\, h^\perp\rangle/\langle x-u,\, h\rangle|,\, |\langle y-u,\, h^\perp\rangle/\langle y-u,\, h\rangle|\}$, then $q \in \mathcal{T}(x,y,p)$. Now

$$\begin{aligned}
\langle x-u,\, h^\perp\rangle &= -|x-u||y-u|\langle e,\, g^\perp\rangle\\
\langle x-u,\, h\rangle &= -|x-u|^2 - |x-u||y-u|\langle e,\, g\rangle\\
\langle y-u,\, h^\perp\rangle &= -|x-u||y-u|\langle g,\, e^\perp\rangle\\
\langle y-u,\, h\rangle &= -|y-u|^2 - |x-u||y-u|\langle e,\, g\rangle,
\end{aligned}$$

and so

$$\left|\frac{\langle x-u,\, h^{\perp}\rangle}{\langle x-u,\, h\rangle}\right| = \left|\frac{\langle e,\, g^{\perp}\rangle}{\langle e,\, g\rangle + |x-u|/|y-u|}\right|$$

and

$$\left|\frac{\langle y-u,\, h^{\perp}\rangle}{\langle y-u,\, h\rangle}\right| = \left|\frac{\langle e,\, g^{\perp}\rangle}{\langle e,\, g\rangle + |y-u|/|x-u|}\right|.$$

Hence $m \geq \frac{2}{5}\psi^{1/2}$ and the observation follows. □

We now reach the main part of the proof of the proposition. The existence of a large empty cone near to u and w forces all other points of $F_y \cap B(u,\rho)$ to lie in a narrow strip in direction $w-y$.

Observation 6. *Let* $0 < \rho \leq d_1\psi^{\frac{1}{2}\left(\frac{1}{s-1-\xi}\right)}$ *and* $r = \alpha_1(\psi^{-\frac{1}{2}}\rho)^{\frac{1}{2+\xi-s}}$. *If* $v, w \in F_y \cap B(u,\rho)$, *then* $v \in V(y, w-y, 3r/d_-)$.

Proof. Suppose that $\langle v-y,\, (w-y)^{\perp}\rangle > 0$. (If not, then interchange v and w — note that $\langle v-y,\, (w-y)^{\perp}\rangle \neq 0$ since v and w are both visible from y.) By observation 4, there is

$$z \in V(x, e, r/d_-) \cap V(y, f, r/d_-) \cap B(u, \alpha_0\psi^{-1/2}\rho)$$

for which $\Gamma \cap \mathcal{T}(x,y,z) = \emptyset$.

By lemma 2.5, we can find $v' \in E_y \cap T_y^-(v,r)$ for which

$$|v'-u| \geq |v'-v| - |u-v| > d_0 r^{2+\xi-s} - \rho = (d_0\alpha_1^{2+\xi-s}\psi^{-\frac{1}{2}} - 1)\rho \geq \alpha_0\psi^{-1/2}\rho. \tag{2.12}$$

We show that if

$$\langle v-y,\, (w-y)^{\perp}\rangle \geq 3(r/d_-)\langle v-y,\, w-y\rangle, \tag{2.13}$$

then $v' \in \mathcal{T}(x,y,z)$, which is impossible. To do this, it is enough to show

$$\langle v'-x,\, e^{\perp}\rangle < -(r/d_-)\langle v'-x,\, e\rangle \tag{2.14}$$

and

$$\langle v'-y,\, (w-y)^{\perp}\rangle > (r/d_-)\langle v'-y,\, w-y\rangle, \tag{2.15}$$

since $z \in V(x, e, r/d_-) \cap V(y, f, r/d_-)$.

For (2.14): If

$$v' \in V(y, v-y, r/d_-) \cap V(x, e, r/d_-),$$

then observation 3 applied to v implies that $|v'-u| < \alpha_0\psi^{-1/2}\rho$ contradicting (2.12). Hence, as $v' \in V(y, v-y, r/d_-)$, we deduce

$$|\langle v'-x,\, e^{\perp}\rangle| > (r/d_-)\langle v'-x,\, e\rangle$$

and it only remains to show that $\langle v'-x,\, e^{\perp}\rangle < 0$. If $\langle v'-x,\, e^{\perp}\rangle \geq 0$, then $\langle v'-x,\, e^{\perp}\rangle > (r/d_-)\langle v'-x,\, e\rangle$. But, by observation 2 applied to v, $|\langle v'-y,\, g^{\perp}\rangle| \leq 4(\rho/d_-)\langle v'-y,\, g\rangle$ and so, as $|v'-y| < |v-y|$, we find

$$\begin{aligned} |\langle v'-y,\, g^{\perp}\rangle| < 4\rho/d_-|v-y| &\leq 4\rho/d_-(|u-y|+\rho) \\ &\leq (5\rho/d_-)|u-y|, \text{ as } \rho \leq d_-/4 \\ &\leq 5(d_+/d_-)\rho \end{aligned}$$

and $\langle v'-u,\, g^{\perp}\rangle = \langle v'-y,\, g^{\perp}\rangle$. So

$$|\langle v'-u,\, g^{\perp}\rangle| \leq 5(d_+/d_-)\rho.$$

Now

$$\langle v'-u,\, g\rangle = \langle v'-y,\, g\rangle - |u-y| \le |v-y| - |u-y| \le \rho.$$

Let q be the point of intersection of $[x,u]$ with $[y, y+|y-u|(g+4(\rho/d_-)g^\perp)]$. Then since $v' \in V(y,g,4(\rho/d_-))$ and $\langle v'-x,\, e^\perp\rangle > (r/d_-)\langle v'-x,\, e\rangle$, it follows that $\langle v'-u,\, g\rangle \ge \langle q-u,\, g\rangle$. Hence it is enough to find a lower bound for $\langle q-u,\, g\rangle$. There is $0<\lambda<|u-y|$ and $0<\mu<|x-u|$ such that

$$q = y + \lambda(g+4(\rho/d_-)g^\perp) = x + \mu e,$$

and so

$$-|u-y|g + \lambda(g+4(\rho/d_-)g^\perp) = (\mu - |u-x|)e.$$

Taking inner products of this expression with g and $g^\perp$, and solving for λ gives

$$\lambda = |u-y|\left(1+4\left(\frac{\rho}{d_-}\right)\frac{\langle e,\, g\rangle}{\langle e^\perp,\, g\rangle}\right)^{-1}.$$

Hence

$$\lambda \ge |u-y|(1+4(\rho/d_-)\psi^{-1/2})^{-1} \ge |u-y|(1-4(\rho/d_-)\psi^{-1/2}).$$

Thus

$$\langle q-u,\, g\rangle \ge -4(\rho/d_-)\psi^{-1/2}|u-y|$$

and so

$$\langle v'-u,\, g\rangle \ge -4(\rho/d_-)\psi^{-1/2}|u-y| \ge -4(d_+/d_-)\psi^{-\frac{1}{2}}\rho.$$

Hence

$$|v'-u| \le 5\rho\tfrac{d_+}{d_-} + 4\rho\psi^{-1/2}\tfrac{d_+}{d_-} \le 9(d_+/d_-)\psi^{-1/2}\rho < \alpha_0\psi^{-1/2}\rho,$$

a contradiction.

For (2.15), notice that $v'-y = \alpha(v-y)+\beta(v-y)^\perp$ for some $0<\alpha<1$ and $|\beta|<\alpha(r/d_-)$. Thus, using (2.13),

$$\begin{aligned}\langle v'-y,\, (w-y)^\perp\rangle &= \alpha\langle v-y,\, (w-y)^\perp\rangle + \beta\langle (v-y)^\perp,\, (w-y)^\perp\rangle\\ &\ge 3\alpha(r/d_-)\langle v-y,\, w-y\rangle - |\beta||\langle v-y,\, w-y\rangle|\\ &\ge 2\alpha r/d_-\langle v-y,\, w-y\rangle\end{aligned}$$

and

$$\langle v'-y,\, w-y\rangle = \alpha\langle v-y,\, w-y\rangle + \beta\langle (v-y)^\perp,\, w-y\rangle.$$

But

$$\langle (v-y)^\perp,\, w-y\rangle = \langle (v-y)^\perp,\, w-v\rangle$$

and so $|\langle (v-y)^\perp,\, w-y\rangle| \le 2\rho|v-y|$, and

$$\langle v-y,\, w-y\rangle = \langle v-y,\, w-v\rangle + |v-y|^2$$

and so

$$|\langle v-y,\, w-y\rangle| \ge |v-y|(|v-y|-2\rho) \ge (d_-/2)|v-y|,$$

since $|v-y|\ge d_-$ and $\rho \le d_-/4$. Thus

$$|\langle (v-y)^\perp,\, w-y\rangle| \le 2\rho|v-y| \le 4(\rho/d_-)\langle v-y,\, w-y\rangle.$$

So

$$\begin{aligned}\langle v'-y,\, w-y\rangle &\le (\alpha+4(\rho/d_-)|\beta|)\langle v-y,\, w-y\rangle\\ &\le (1+4(\rho/d_-)(r/d_-))\alpha\langle v-y,\, w-y\rangle\\ &< (1+4(\rho/d_-)(r/d_-))(d_-/(2r))\langle v'-y,\, (w-y)^\perp\rangle\end{aligned}$$

and rearranging gives

$$\langle v'-y,\, (w-y)^{\perp}\rangle > (2r/d_-)(1+4(\rho/d_-)(r/d_-))^{-1}\langle v'-y,\, w-y\rangle$$

which, since $4(\rho/d_-)(r/d_-) \le r/d_- \le r_2/d_- \le 1$, implies (2.15). □

We can now finish the proof of the proposition.

Let $0<\rho\le d_1\psi^{\frac{1}{2}\left(\frac{1}{s-1-\xi}\right)}$ and $r=\alpha_1(\psi^{-\frac{1}{2}}\rho)^{\frac{1}{2+\xi-s}}$. Suppose $w\in B(u,\rho)\cap F_y$, then

$$F_y\cap B(u,\rho)\subset V(y,(w-y),3r/d_-).$$

Thus $F_y\cap B(u,\rho)$ is contained in a rectangle of height 2ρ and width $6rd_+/d_-$ which can be covered by $(2+2\rho/r)(2+6d_+/d_-)$ boxes of side r. Since

$$(2+2\rho/r)(2+6d_+/d_-)\le 32(\rho/r)d_+/d_-,$$

and $\sqrt{2}r\le\sqrt{2}\alpha_1\left(d_1\psi^{\frac{1}{2}\left(\frac{2+\xi-s}{s-1-\xi}\right)}\right)^{\frac{1}{2+\xi-s}}\le\sqrt{2}\alpha_1 d_1^{\frac{1}{2+\xi-s}}\le\sqrt{2}r_2\le r_0$, we estimate that

$$\nu_y(F_y\cap B(u,\rho))\le 2^{5+s/2}(d_+/d_-)\rho r^{s-1}=2^{5+s/2}\alpha_1^{s-1}(d_+/d_-)\psi^{-\frac{1}{2}\left(\frac{s-1}{2+\xi-s}\right)}\rho^{\frac{1+\xi}{2+\xi-s}},$$

as required. The remainder of the proposition follows from observation 5. □

2.3. **Mass estimate proposition.** The main utility of Proposition 2.2 lies in its use in proving the following proposition.

Proposition 2.3. *Let Γ be a non-empty compact connected subset of $\mathbb{R}^2$ and let A and B be compact subsets of $\mathbb{R}^2$. Suppose that $s>1$, $0<\xi<s-1$, $0<r_1\le r_0\le 1$, $0<d_-\le d_+$ with $d_-\le 1$ and $M>0$ are given. Let $x,y\in\mathbb{R}^2\setminus\Gamma$ satisfy*

$$0<2|x-y|<d_-\le\min\{d(x,\Gamma),\, d(y,\Gamma)\}\le\max\{d(x,\Gamma),\, d(y,\Gamma)\}+|\Gamma|\le d_+.$$

Let ν_x and ν_y be Radon measures supported in Γ_x and Γ_y respectively and let

$$F_x\subseteq E_x\subseteq\Gamma_x \text{ and } F_y\subseteq E_y\subseteq\Gamma_y$$

be compact sets. Suppose that:

(1) *for all $u\in E_x$, $v\in E_y$ and $0<r\le r_0$ both*

$$\nu_x(B(u,r)\le r^s \text{ and } \nu_y(B(v,r))\le r^s;$$

(2) *for all $u\in F_x$, $v\in F_y$ and $0<r\le r_1$ both*

$$\nu_x(T_x^{\pm}(u,r)\cap E_x)\ge Mr^{1+\xi} \text{ and } \nu_y(T_y^{\pm}(v,r)\cap E_y)\ge Mr^{1+\xi};$$

(3) *there is $\psi\in(0,1/2)$ such that for $u\in(F_x\cap A)\cup(F_y\cap B)$,*

$$\langle (u-x)^{\wedge},\, (u-y)^{\wedge}\rangle\in[1/2,1-\psi].$$

Then there are constants $c_2,d_2>0$ such that if $0<\rho\le d_2\psi^{\frac{1}{2}\frac{1}{s-1-\xi}}$, then

$$(\nu_x\otimes\nu_y)\left((F_x\times F_y)\cap(A\times B)\cap\{(u,v):|u-v|\le\rho\}\right)$$
$$\le c_2\,\text{arc-diam}_{\frac{1}{2}(x+y)}(A\cap F_x\cap B(F_y\cap B,\rho))(\psi^{-\frac{1}{2}}\rho)^{\frac{s+\xi}{2+\xi-s}}.$$

Proof. Let $d_2 = \frac{5}{2^{3/2}\alpha_0} d_1$ and observe that, as $d_2 \le d_1$, Proposition 2.2 implies

$$\begin{aligned}
&(\nu_x \otimes \nu_y)\left((F_x \times F_y) \cap (A \times B) \cap \{(u,v) : |u-v| \le \rho\}\right) \\
&= \int_{B(F_y \cap B, \rho)} \nu_y|_{F_y \cap B}\left(\{v : |u-v| \le \rho\}\right) d\nu_x|_{F_x \cap A}(u) \\
&= \int_{B(F_y \cap B, \rho)} \nu_y\left(F_y \cap B \cap B(u,\rho)\right) d\nu_x|_{F_x \cap A}(u) \\
&\le c_1 \psi^{-\frac{s-1}{2(2+\xi-s)}} \rho^{\frac{1+\xi}{2+\xi-s}} \nu_x\left(F_x \cap A \cap B(F_y \cap B, \rho)\right).
\end{aligned}$$

It remains to estimate

$$\nu_x\left(F_x \cap A \cap B(F_y \cap B, \rho)\right).$$

We begin by noticing that for each

$$u \in F_x \cap A \cap B(F_y \cap B, \rho),$$

Proposition 2.2 guarantees the existence of $p_u \in [\frac{1}{2}(x+y), u] \cap B(u, \alpha_0 \psi^{-1/2}\rho)$ such that $\mathcal{T}(x,y,p_u) \cap \Gamma = \emptyset$ and

$$V(p_u, \tfrac{1}{2}(x+y) - u, \tfrac{2}{5}\psi^{\frac{1}{2}}) \cap \Gamma \cap H(x,y;u) = \emptyset.$$

Let $\sigma = \frac{2}{5}\frac{\alpha_0}{d_+}\rho$ and fix $v \in F_x \cap A \cap B(F_y \cap B, \rho)$. Then Lemma 2.2 guarantees that if

$$w \in V(\tfrac{1}{2}(x+y), p_v - \tfrac{1}{2}(x+y), \sigma) \setminus V(p_v, -(p_v - \tfrac{1}{2}(x+y)), \tfrac{2}{5}\psi^{\frac{1}{2}}),$$

then

$$\begin{aligned}
\langle w - p_v,\, (p_v - \tfrac{1}{2}(x+y))^\wedge \rangle &\ge -\frac{\sigma}{\sigma + \frac{2}{5}\psi^{\frac{1}{2}}} |p_v - \tfrac{1}{2}(x+y)| \\
&\ge -\tfrac{5}{2} d_+ \left(\frac{2}{5}\frac{\alpha_0}{d_+}\right) \psi^{-\frac{1}{2}} \rho = -\alpha_0 \psi^{-\frac{1}{2}} \rho.
\end{aligned}$$

So suppose $u, v \in F_x \cap A$ with $u \in V(\frac{1}{2}(x+y), v, \sigma)$ and assume, without loss of generality, that

$$|u - \tfrac{1}{2}(x+y)| \le |v - \tfrac{1}{2}(x+y)|.$$

We wish to estimate $\langle u - v,\, (v - \frac{1}{2}(x+y))^\wedge \rangle$ from below. (An easy upper bound is given by zero.) From the preceding we know that

$$\langle u - p_v,\, (v - \tfrac{1}{2}(x+y))^\wedge \rangle = \langle u - p_v,\, (p_v - \tfrac{1}{2}(x+y))^\wedge \rangle \ge -\alpha_0 \psi^{-\frac{1}{2}} \rho.$$

Hence

$$\begin{aligned}
\langle u - v,\, (v - \tfrac{1}{2}(x+y))^\wedge \rangle &= \langle u - p_v,\, (v - \tfrac{1}{2}(x+y))^\wedge \rangle + \langle p_v - v,\, (v - \tfrac{1}{2}(x+y))^\wedge \rangle \\
&\ge -\alpha_0 \psi^{-\frac{1}{2}} \rho - \alpha_0 \psi^{-\frac{1}{2}} \rho \\
&\ge -2\alpha_0 \psi^{-\frac{1}{2}} \rho.
\end{aligned}$$

Thus

$$V(\tfrac{1}{2}(x+y), v - \tfrac{1}{2}(x+y), \sigma) \cap (F_x \cap A \cap B(F_y \cap B, \rho) \cap B(\tfrac{1}{2}(x+y), |v - \tfrac{1}{2}(x+y)|))$$

can be covered by

$$\frac{2\alpha_0 \psi^{-1/2}\rho}{\frac{2}{5}\alpha_0 \rho} = 5\psi^{-\frac{1}{2}}$$

boxes of side $\frac{4}{5}\alpha_0\rho$. Hence, by using the mass estimate in Proposition 2.2 (for ν_x rather than ν_y), since $2\sqrt{2}\frac{2}{5}\alpha_0\rho \leq d_1\psi^{\frac{1}{2}\frac{1}{s-1-\xi}}$,

$$\nu_x\left(V(\tfrac{1}{2}(x+y), v-\tfrac{1}{2}(x+y), \sigma) \cap F_x \cap A \cap B(F_y \cap B, \rho) \cap B(\tfrac{1}{2}(x+y), |v-\tfrac{1}{2}(x+y)|)\right)$$

is at most

$$5\psi^{-\frac{1}{2}} \times c_1\psi^{-\frac{1}{2}\frac{s-1}{2+\xi-s}}\left(\frac{4\sqrt{2}}{5}\alpha_0\rho\right)^{\frac{1+\xi}{2+\xi-s}} = 5c_1\left(\frac{4\sqrt{2}}{5}\alpha_0\right)^{\frac{1+\xi}{2+\xi-s}}(\psi^{-\frac{1}{2}}\rho)^{\frac{1+\xi}{2+\xi-s}}.$$

By choosing v to be as far from $\frac{1}{2}(x+y)$ as possible and counting the number of such cones needed to cover $F_x \cap A$, we obtain

$$\begin{aligned}
&\nu_x(F_x \cap A \cap B(F_y \cap B, \rho)) \\
&\leq 2\,\text{arc-diam}_{\frac{1}{2}(x+y)}(F_x \cap A \cap B(F_y \cap B, \rho))\sigma^{-1} \times 5c_1\left(\frac{4\sqrt{2}}{5}\alpha_0\right)^{\frac{1+\xi}{2+\xi-s}}(\psi^{-\frac{1}{2}}\rho)^{\frac{1+\xi}{2+\xi-s}} \\
&= c_2\,\text{arc-diam}_{\frac{1}{2}(x+y)}(F_x \cap A \cap B(F_y \cap B, \rho))(\psi^{-\frac{1}{2}}\rho)^{\frac{1+\xi}{2+\xi-s}}\rho^{-1}
\end{aligned}$$

for $c_2 = 25c_1d_+(4\sqrt{2}/5)^{\frac{1+\xi}{2+\xi-s}}\alpha_0^{\frac{s-1}{2+\xi-s}}$, and this implies the claim. □

3. Measurability results

In this section we prove the measurability of various maps that we use in the proof of Theorem 1.2. In particular, we show that if B is a compact set that is disjoint from Γ, then there is a universally-measurable map that assigns to each point $x \in B$ for which Γ_x has large dimension, a Radon measure of large dimension that is 'supported' on Γ_x.

Let B be a compact subset of the plane disjoint from the non-empty compact connected set Γ. Letting S^1 denote the unit circle, we define $K \subseteq B \times S^1 \times \mathbb{R}^+$ by

$$K = \{(x, \theta, t) \in B \times S^1 \times \mathbb{R}^+ : x + t\hat{\theta} \in \Gamma\}$$

where $\hat{\theta} = (\cos\theta, \sin\theta)$. Notice that K is compact and a lifting of Γ.

For $x \in B$ and $\theta \in S^1$, define $\gamma\colon B \times S^1 \to \mathbb{R}^+ \cup \{\infty\}$ by

$$\gamma(x,\theta) = \begin{cases} \inf\{t > 0 : x + t\hat{\theta} \in \Gamma\} & \text{if } (x + \mathbb{R}\hat{\theta}) \cap \Gamma \neq \emptyset \\ \infty & \text{otherwise.} \end{cases}.$$

Observe that $x + \gamma(x,\theta)\hat{\theta} \in \Gamma$ for any $x \in B$ and $\theta \in S^1$ for which $(x + \mathbb{R}\hat{\theta}) \cap \Gamma \neq \emptyset$.

Let

$$\text{gr}\,(\gamma) = \{(x, \theta, \gamma(x,\theta)) : (x,\theta) \in B \times S^1, \gamma(x,\theta) < \infty\},$$

then

$$\text{gr}\,(\gamma) \subseteq K \subseteq B \times S^1 \times I,$$

where $I = [0, \text{diam}\,(B \cup \Gamma)]$.

Lemma 3.1. *The function γ is lower semi-continuous. In particular,* $\text{gr}\,(\gamma)$ *is a G_δ-subset of K.*

Proof. That γ is lower semi-continuous follows readily from the observation that its graph is the lower envelope of the compact set K.

The fact that $\text{gr}\,(\gamma)$ is G_δ is a standard result concerning functions of Baire class 1, see for example, [5, Ch II,§31 VII, Theorem 1]. □

For $C \subseteq K$ and $x \in B$, let C_x be given by

$$C_x = C \cap \left(\{x\} \times S^1 \times I\right),$$

the slice of C through x. For ease, we let $\mathrm{gr}_x(\gamma)$ denote $(\mathrm{gr}(\gamma))_x$.

Recall that $\mathcal{M}(K)$ denotes the Radon measures supported in K. The set $\mathcal{M}(K)$ can be given the topology of weak convergence by using as a base, sets of the form

$$\left\{\mu \in \mathcal{M}(K) : \int f \, d\mu < a\right\},$$

where $a \in \mathbb{R}$ and $f \in C(K)$, the set of real-valued continuous functions on K. It turns out that $\mathcal{M}(K)$ with this topology is a Polish space, see [6, §14.15] and [4, II.17].

Lemma 3.2. *Let E be a Borel subset of K. Then the functions $F_E \colon \mathcal{M}(K) \to \mathbb{R}$ and $G_E \colon B \times \mathcal{M}(K) \to \mathbb{R}$ given by*

$$F(\nu) = \nu(E) \text{ and } G_E(x, \nu) = \nu(E_x)$$

are Borel.

In particular,

$$\{(x, \nu) \in B \times \mathcal{M}(K) : \nu(E_x) > 0\}$$

is a Borel set.

Proof. Let $E \subseteq K$ be a Borel set. We show that G_E is Borel; the proof that F_E is Borel is similar.

Suppose first that E is a compact subset of K. Then for $x \in B$, E_x is also compact, and for $\mu \in \mathcal{M}(K)$,

$$G_E(x, \nu) = \nu(E_x) < c \text{ if and only if}$$
$$\text{there is } f \in C^+(K) \text{ such that } f > 1 \text{ on } E_x \text{ and } \int f \, d\nu < c.$$

(Here $C^+(K)$ denotes the set of non-negative real-valued continuous functions on K.) For a given $f \in C^+(K)$, the sets

$$B_f = \{x \in B : f > 1 \text{ on } C_x\}$$

and

$$M_f = \{\nu \in \mathcal{M}(K) : \int f \, d\nu < c\}$$

are open subsets of B and $\mathcal{M}(K)$, respectively. Hence

$$\{(x, \nu) : \nu(E_x) < c\} = \bigcup_{f \in C^+(K)} B_f \times M_f$$

is an open set, and so G_E is upper semi-continuous, and in particular, Borel.

If $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$ is an increasing sequence of compact sets, and $G_1, G_2, G_3, \ldots$, the associated sequence of maps, then

$$G_{\cup_{i \in \mathbb{N}} E_i} = \lim_{i \to \infty} G_i$$

is a Borel map. Similarly, if $E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$ is a decreasing sequence of compact sets, and $G_1, G_2, G_3, \ldots$, the associated sequence of maps, then

$$G_{\cap_{i \in \mathbb{N}} E_i} = \lim_{i \to \infty} G_i$$

is also a Borel map. It follows that for a general Borel set E, the map G_E is Borel, as required. □

Lemma 3.3. *Let E be a Borel subset of K. The set*

$$\{(x,\nu) \in B \times \mathcal{M}(K) : \nu(K \setminus (\mathrm{gr}_x(\gamma) \cap E)) = 0\}$$

is Borel in $B \times \mathcal{M}(K)$.

Proof. Observe that

$$\begin{aligned}
&\{(x,\nu) \in B \times \mathcal{M}(K) : \nu(K \setminus (\mathrm{gr}_x(\gamma) \cap E)) = 0\} \\
&= \{(x,\nu) : \nu((\mathrm{gr}_x(\gamma) \cap E)) = \nu(K)\} \\
&= \{(x,\nu) : \nu((\mathrm{gr}\,(\gamma) \cap E)_x) - \nu(K) = 0\} \\
&= \{(x,\nu) : G_{\mathrm{gr}\,(\gamma)\cap E}(x,\nu) - F_K(\nu) = 0\}.
\end{aligned}$$

Hence, since lemma 3.2 implies F_K and $G_{\mathrm{gr}\,(\gamma)\cap E}$ are Borel functions, this set is Borel. □

Define $\Pi\colon B \times S^1 \times I \to \mathbb{R}^2$ by

$$\Pi(x,\theta,t) = x + t\hat{\theta}$$

and observe that Π is continuous.

Lemma 3.4. *For $x \in B$, if $A \subseteq (\{x\} \times S^1 \times I) \cap K$, then*

$$\dim_H(\Pi(A)) = \dim_H(A).$$

Proof. This follows from the fact that Π is bi-Lipschitz when restricted to $\{x\} \times S^1 \times I$. □

In particular, since

$$\Gamma_x = \Pi((\{x\} \times S^2 \times I) \cap \mathrm{gr}\,(\gamma)) = \Pi(\mathrm{gr}_x(\gamma)),$$

it follows that

$$\dim_H(\Gamma_x) = \dim_H(\mathrm{gr}_x(\gamma)),$$

for each $x \in B$. Recall that for $A \subseteq K$ and $s \in \mathbb{R}$,

$$\begin{aligned}
&\mathcal{M}^s(A) \\
&= \{\nu \in \mathcal{M}(K) : \nu(A) > 0 \text{ and } \nu(B(\zeta,r)) \le r^s, \text{ for } \zeta \in K,\, r \in (0,1]\}.
\end{aligned}$$

It is an easy calculation, which we omit, to check that $\mathcal{M}^s(K)$ is a Borel set. Since $\mathrm{gr}_x(\gamma)$ is a Borel set,

$$\dim_H(\mathrm{gr}_x(\gamma)) = \sup\{\sigma : \mathcal{M}^\sigma(\mathrm{gr}_x(\gamma)) \neq \emptyset\}.$$

Proposition 3.1. *Let C be a Borel subset of the plane. Then for $t \ge 0$,*

$$\{x \in B : \dim_H(\Gamma_x \cap C) > t\}$$

is an analytic set.

Proof. Let $E \subseteq K$ be given by $E = \Pi^{-1}(C) \cap K$, and observe that $\mathrm{gr}\,(\gamma) \cap E$ is a Borel subset of K. For $t \geq 0$

$$\begin{aligned}\{x : \dim_H(\Gamma_x \cap C) > t\} &= \{x : \dim_H((\mathrm{gr}\,(\gamma) \cap E)_x) > t\} \\ &= \{x : \mathcal{M}^\tau((\mathrm{gr}\,(\gamma) \cap E)_x) \neq \emptyset \text{ for some } \tau > t\} \\ &= \bigcup_{p \in \mathbb{Q}^+} \{x : \mathcal{M}^{t+p}((\mathrm{gr}\,(\gamma) \cap E)_x) \neq \emptyset\}.\end{aligned}$$

However, if $\pi_B \colon B \times \mathcal{M}(K) \to B$ denotes coordinate projection onto B, then

$$\begin{aligned}&\{x : \mathcal{M}^{t+p}((\mathrm{gr}\,(\gamma) \cap E)_x) \neq \emptyset\} \\ &= \pi_B(\{(x,\nu) : x \in B,\, \nu \in \mathcal{M}^{t+p}((\mathrm{gr}\,(\gamma) \cap E)_x)\}) \\ &= \pi_B\left(\{(x,\nu) \in B \times \mathcal{M}^{t+p}(K) : \nu((\mathrm{gr}\,(\gamma) \cap E)_x) > 0\}\right).\end{aligned}$$

Hence lemmas 3.1 and 3.2 together imply that $\{x : \mathcal{M}^{t+p}((\mathrm{gr}\,(\gamma) \cap E)_x) \neq \emptyset\}$ is the coordinate-wise projection of a Borel set from a product of Polish spaces, and so it is analytic, see [4, Chapter III]. Hence $\{x : \dim_H(\Gamma_x \cap C) > t\}$ is also analytic. □

Our last result in this section is a selection theorem and allows us to choose, in a measurable way, an element of $\mathcal{M}^t(\mathrm{gr}_x(\gamma))$ whenever $x \in B$ is such that $\dim_H(\Gamma_x) > t$.

Proposition 3.2. *Let $t \geq 0$ and C be a Borel subset of the plane. There is a map*

$$\begin{aligned}\omega \colon \{x \in B : \dim_H(\Gamma_x \cap C) > t\} &\to \mathcal{M}(K) \\ x &\mapsto \omega_x\end{aligned}$$

such that:

1. *ω is $\sigma(\mathbf{A})$-measurable,*
2. *$\omega_x \in \mathcal{M}^t(\mathrm{gr}_x(\gamma) \cap \Pi^{-1}(C))$ for each x, and*
3. *$\omega_x(K \setminus (\mathrm{gr}_x(\gamma) \cup \Pi^{-1}(C))) = 0$ for each x.*

(Here $\sigma(\mathbf{A})$ denotes the σ-algebra generated by the analytic sets in B.)

In particular, ω is μ-measurable for every Radon measure μ on B.

Proof. Let $E = \Pi^{-1}(C) \cap K$, a Borel set. Since

$$(B \times \sigma(K)) \cap \{(x,\nu) : \nu((\mathrm{gr}\,(\gamma) \cap E)_x) > 0\} \cap \{(x,\nu) : \nu(K \setminus (\mathrm{gr}\,(\gamma) \cap E)_x) = 0\}$$

is Borel in $B \times \mathcal{M}(K)$, claims 1,2, and 3 follow readily from the Jankov-von Neumann Uniformisation Theorem. (See [4, Theorem 18.1] for a statement of this theorem.)

See [4, Theorem 21.10] for a proof of Lusin's Theorem that analytic sets are universally measurable, from which it follows that sets in the σ-algebra generated by analytic sets are also universally measurable. □

4. Proof of Theorem 1.2

We now draw our preparatory work together and prove Theorem 1.2.

Let Γ be a compact connected subset of the plane for which $1 < \dim_H(\Gamma) \leq 2$. If $\dim_H(\Gamma) = 2$, then let $d = 2$, otherwise choose $\dim_H(\Gamma) < d < 2$. Notice that in both cases this implies that whenever ν is a non-zero Radon measure supported in Γ, then

$$I_d(\nu) = +\infty.$$

(If $d = 2$, then, since $\mathcal{H}^2(\Gamma) < \infty$, Theorem 8.7 of [6] implies $I_2(\nu) = +\infty$.)

4.1. **Measure theoretic decomposition.** Fix $d > s > 1$ and let $\emptyset \neq B^{(0)} \subseteq \mathbb{R}^2 \backslash \Gamma$ be a compact set for which $\operatorname{diam}(B^{(0)}) \leq \frac{1}{100} \operatorname{dist}(B^{(0)}, \Gamma)$. It is enough for us to show that

$$\dim_H(\{x \in B^{(0)} : \dim_H(\Gamma_x) > s\}) < \tfrac{1}{2} + \sqrt{d - \tfrac{3}{4}}.$$

Since Γ is compact, we can find finitely many open sets U_1, U_2,..., U_N that intersect Γ such that $\Gamma \subseteq \bigcup_{i=1}^N U_i$ and $\operatorname{diam}(U_i) \leq \frac{1}{100} \operatorname{dist}(B^{(0)}, \Gamma)$ for each i.

It follows that

$$E = \bigcup_{i=1}^N \{x \in B^{(0)} : \dim_H(\Gamma_x \cap U_i) > s\} = \bigcup_{i=1}^N E_i, \text{ say.}$$

Clearly each E_i satisfies

$$\operatorname{diam}(E_i) \leq \tfrac{1}{100} \operatorname{dist}(E_i, \Gamma) \leq \tfrac{1}{100} \operatorname{dist}(B^{(0)}, \Gamma).$$

From Proposition 3.1, we see that each E_i is an analytic set.

Moreover, for $t > 0$, if

$$\dim_H \Big(\{x \in B^{(0)} : \dim_H(\Gamma_x) > s\} \Big) > t,$$

then we can find an i such that

$$\dim_H(E_i) > t. \tag{4.1}$$

So suppose $t > 0$ and i are such that $\dim_H(E_i) > t$. Our objective is to find an upper bound for the size of t in terms of d and s.

By Theorem 2.1, there is a nonzero Radon measure μ with compact support $B^{(1)} \subseteq A_i \subseteq B^{(0)}$ such that whenever $x \in \mathbb{R}^2$ and $r > 0$, then

$$\mu(B(x, r)) \leq r^t. \tag{4.2}$$

Proposition 3.2 enables us to find a $\sigma(\mathbf{A})$-measurable function

$$\omega \colon B^{(1)} \to \mathcal{M}^s(K)$$
$$x \mapsto \omega_x,$$

(where $K = \{(x, \theta, t) \in B^{(1)} \times S^1 \times \mathbb{R}^+ : x + t\hat{\theta} \in \Gamma\}$) such that

- $\omega_x(\operatorname{gr}_x(\gamma) \cap \Pi^{-1}(U_i)) > 0$,
- $\omega_x(K \setminus (\operatorname{gr}_x(\gamma) \cap \Pi^{-1}(U_i))) = 0$, for each $x \in B^{(1)}$.

Moreover, there is a constant C such that $\omega_x(K) \leq C$ for all x. By Lusin's Theorem [2, 2.3.5], there is a compact set $B^{(2)} \subseteq B^{(1)}$ such that

- $\mu(B^{(2)}) > 0$, and
- $\omega|_{B^{(2)}}$ is a continuous map.

Let $\mu^{(2)} = \mu|_{B^{(2)}}$ and for Borel $E \subseteq K$ define

$$m^*(E) = \int \omega_x(E) \, d\mu^{(2)}(x),$$

and extend m^* to arbitrary $A \subseteq K$ by setting $m^*(A) = \inf\{m^*(E) : A \subseteq E \text{ and } E \text{ is Borel}\}$. We omit the routine verification that m^* is a Radon measure on K.

For $x \in B^{(2)}$ define a Radon measure ν_x on Γ by

$$\nu_x(A) = \omega_x(\Pi^{-1}(A)) = \omega_x(\Pi^{-1}(A) \cap (\{x\} \times S^1 \times I)), \text{ for } A \subseteq \mathbb{R}^2,$$

and observe that the continuity of the map ω implies that $x \mapsto \nu_x$ is a Borel measurable function. Also notice:

- for $x \in B^{(2)}$, $\nu_x(\mathbb{R}^2 \setminus \Gamma_x) = 0$ and $0 < \nu_x(\mathbb{R}^2) \leq C$,
- for $x \in B^{(2)}$, $u \in \mathbb{R}^2$ and $0 < r \leq 1$, $\nu_x(B(u,r)) \leq r^s$.

We now analyse the geometry of the measures ν_x.

Fix $0 < \xi < s - 1$. Then for all $x \in B^{(2)}$, Lemma 2.4 implies that for ν_x-a.e. $u \in \Gamma_x$,

$$\min\left\{\liminf_{r\searrow 0} \frac{\nu_x(T_x^+(u,r))}{r^{1+\xi}}, \liminf_{r\searrow 0} \frac{\nu_x(T_x^-(u,r))}{r^{1+\xi}}\right\} = +\infty. \tag{4.3}$$

That is, for all $x \in B^{(2)}$ and ω_x-a.e. $\zeta \in K$,

$$\liminf_{r\searrow 0} \frac{\omega_x(\Pi^{-1}(T_x^+(\Pi(\zeta),r)))}{r^{1+\xi}} = \liminf_{r\searrow 0} \frac{\omega_x(\Pi^{-1}(T_x^-(\Pi(\zeta),r)))}{r^{1+\xi}} = +\infty.$$

It is easy to verify if $K^{(2)} = K \cap (B^{(2)} \times S^1 \times I)$, then $f\colon K^{(2)} \to \mathbb{R} \cup \{+\infty\}$ given by

$$f(x,\theta,t) = \min\left\{\liminf_{r\searrow 0} \frac{\nu_x(T_x^+(x+t\hat{\theta},r))}{r^{1+\xi}}, \liminf_{r\searrow 0} \frac{\nu_x(T_x^-(x+t\hat{\theta},r))}{r^{1+\xi}}\right\}$$

is a Borel function and so

$$K_\infty^{(2)} = \{\zeta \in K^{(2)} : f(\zeta) = +\infty\}$$

is a Borel set with $\omega_x(K^{(2)} \setminus K_\infty^{(2)}) = 0$ for all $x \in B^{(2)}$. Hence, $m^*(K^{(2)} \setminus K_\infty^{(2)}) = 0$. Now

$$K_\infty^{(2)} = \bigcap_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} K_{m,n}^{(2)}$$

where

$$K_{m,n}^{(2)} = \Big\{(x,\theta,t) \in K^{(2)} : \text{if } r \in (0, \tfrac{1}{n}], \text{ then } \min\left\{\nu_x(T^+(x+t\hat{\theta},r)), \nu_x(T^-(x+t\hat{\theta},r))\right\} > mr^{1+\xi}\Big\}.$$

Thus we can find $m, n \in \mathbb{N}$ such that $m^*(K_{m,n}^{(2)}) > 0$ and so we can choose a compact set $K^{(3)} \subseteq K_{m,n}^{(2)}$ with $m^*(K^{(3)}) > 0$. It follows that we can find a compact set $B^{(3)} \subseteq \pi_B(K^{(3)}) \subseteq B^{(2)}$ and $p > 0$ such that $\mu^{(2)}(B^{(3)}) > 0$ and for all $x \in B^{(3)}$, we have $\omega_x(K^{(3)}) > p$. For $x \in B^{(3)}$, let

$$F_x = \Pi(K^{(3)} \cap (\{x\} \times S^1 \times I)) \subseteq \Gamma_x \cap \Pi^{-1}(U_i)$$

and notice F_x is a compact set with $\nu_x(F_x) = \omega_x(K^{(3)}) > p$. Thus, summarising, we have

- $x \mapsto \nu_x$ is a Borel measurable function on $B^{(3)}$,
- $B^{(3)} \subseteq \pi_B(K^{(3)})$ is compact with $\mu^{(2)}(B^{(4)}) > 0$,
- for $x \in B^{(4)}$, $\nu_x(F_x) > p$,
- for $x \in B^{(4)}$, $0 < r \leq 1/n$ and $u \in F_x \subseteq \Gamma_x$,
$$\min\left\{\nu_x(T^+(u,r)), \nu_x(T^-(u,r))\right\} > mr^{1+\xi}.$$

Thus we have found a compact set $B^{(3)} \subseteq E_i \subseteq B^{(0)}$, a compact set $\bar{U}_i \cap \Gamma \subseteq \Gamma$, a non-zero Radon measure μ and a constant $c > 0$ such that:

(1) for $x \in B^{(3)}$ and $u \in \bar{U}_i \cap \Gamma$, $|u - x| \geq \frac{99}{100} d(B,\Gamma) \geq 99\,\mathrm{diam}\,(B)$;
(2) $\mu^{(2)}(B^{(3)}) > 0$;

(3) for $x \in B^{(3)}$ and $0 < r \leq 1$,
$$\mu B(x,r) \leq r^t;$$
(4) for $x \in B^{(3)}$, there is a Radon measure ν_x and a compact set $F_x \subseteq U_i \cap \Gamma_x \cap B_i$ such that
  (a) $\nu_x(F_x) > p$;
  (b) for $0 < r \leq 1$ and $u \in \Gamma$,
$$\nu_x B(u,r) \leq r^s;$$
  (c) for $0 < r \leq n^{-1}$ and $u \in F_x$,
$$\min\{\nu_x(T_x^+(u,r)),\, \nu_x(T_x^-(u,r))\} > mr^{1+\xi}.$$

If $x, y \in B^{(3)}$, then $|x-y| \leq \frac{1}{100}\operatorname{dist}(B,\Gamma)$. So let $d_- = (1/50)\operatorname{dist}(B^{(0)},\Gamma)$ and $d_+ = \operatorname{diam} B^{(0)} + \operatorname{diam}\Gamma + \operatorname{dist}(B^{(0)},\Gamma)$. By rescaling if necessary, we can assume that $d_+ \leq 1$.

Let $A, B \subseteq \mathbb{R}^2$ be compact and suppose that $\psi \in (0,1/2)$ is such that for $u \in (A \cap F_x) \cup (B \cap F_y)$,
$$|\langle (u-x)^\wedge,\, (u-y)^\wedge\rangle| \in [1/2, 1-\psi].$$
Then all the hypotheses of Propositions 2.2 and 2.3 are satisfied (after suitable relabelling) and so, for $u \in A \cap F_x$, $v \in B \cap F_y$ and $0 < \rho \leq d_1 \psi^{\frac{1}{2}\frac{1}{s-1-\xi}}$, we find that
$$\nu_y(A \cap F_y \cap B(u,\rho)) \leq c_1 \psi^{-\frac{1}{2}\frac{s-1}{2+\xi-s}} \rho^{\frac{1+\xi}{2+\xi-s}}$$
and for $0 < \rho \leq d_2 \psi^{\frac{1}{2}\frac{1}{s-1-\xi}}$

(4.4)
$$\begin{aligned}(\nu_x \otimes \nu_y)\left((F_x \times F_y) \cap (A \times B) \cap \{(u,v) : |u-v| \leq \rho\}\right)\\ \leq c_2 \operatorname{arc-diam}_{\frac{1}{2}(x+y)}(A \cap F_x \cap B(F_y \cap B, \rho))(\psi^{-\frac{1}{2}}\rho)^{\frac{s+\xi}{2+\xi-s}}.\end{aligned}$$

4.2. **Energy estimate.** We now pull all our estimates together and explicitly calculate the d-energy of the measure ν given by
$$\nu(E) = \int \nu_x|_{F_x}(E)\, d\mu|_{B^{(3)}}(x) \text{ for Borel } E \subseteq \mathbb{R}^2$$
and
$$\nu(A) = \inf\{\nu(E) : A \subseteq E \text{ and } E \text{ is Borel}\}, \text{ for non-Borel } A.$$

On noting that for Borel sets E,
$$\nu(E) = \int \omega_x(\Pi^{-1}(E) \cap K^{(3)})\, d\mu|_{B^{(3)}}(x),$$
it is straightforward to verify that ν is a Radon measure. Note that for $\tau > 0$
$$\int |u-v|^{-\tau} d(\nu \times \nu)(u,v) = \int_{B^{(3)} \times B^{(3)}} \int_{F_x \times F_y} |u-v|^{-\tau} d(\nu_x \times \nu_y)(u,v) d(\mu \times \mu)(x,y).$$

Hence, as our choice of d guarantees that $I_d = \int |u-v|^{-d} d(\nu \times \nu)(u,v) = +\infty$,
$$\int_{B^{(3)} \times B^{(3)}} \int_{F_x \times F_y} |u-v|^{-d}\, d(\nu_x \otimes \nu_y)(u,v) d(\mu \otimes \mu)(x,y) = +\infty. \tag{4.5}$$

Fix $x \neq y \in B^{(3)}$. In order to reduce writing, we translate so that $\frac{1}{2}(x+y) = 0$ and let $a = y$, so $|x-y| = 2|a|$.

Using Fubini's theorem, we find

$$\begin{aligned}
&\int_{F_x\times F_y} |u-v|^{-d}\, d(\nu_x\otimes\nu_y)(u,v)\\
&=\int_0^\infty (\nu_x\otimes\nu_y)\left(\{(u,v)\in F_x\times F_y : |u-v|^{-d}\geq r\}\right)\, dr\\
&=d\int_0^\infty \rho^{-d-1}(\nu_x\otimes\nu_y)\left(\{(u,v)\in F_x\times F_y : |u-v|\leq \rho\}\right)\, d\rho\\
&=d\int_{F_x}\int_0^\infty \rho^{-d-1}\nu_y(F_y\cap B(u,\rho))\, d\rho\, d\nu_x(u).
\end{aligned}$$

Let

$$A_0^+=\{w\in A(0,d_-,d_+) : \langle w,\, a^\perp\rangle \geq |\langle w,\, a\rangle|\},$$

$$A_0^-=\{w\in A(0,d_-,d_+) : \langle w,\, a^\perp\rangle \leq -|\langle w,\, a\rangle|\},$$

and for $m,n\in\{0,1\}$ and $i\in\mathbb{N}$, set

$$\begin{aligned}
A_i^{mn}=&\{w\in A(0,d_-,d_+) : |\langle w,\, a^\perp\rangle/\langle w,\, a\rangle|\in[2^{-i},2^{1-i}],\\
&\qquad (-1)^n\langle w,\, a^\perp\rangle>0 \text{ and } (-1)^m\langle w,\, a\rangle>0\},
\end{aligned}$$

noting that

$$\text{arc-diam}_0(A_i^{mn})\preceq 2^{-i}.$$

Observe that if w is in A_i^{mn}, then, by lemma 2.1,

$$\frac{1}{2}\leq\frac{\langle w-a,\, w+a\rangle}{|w-a||w+a|}\leq 1-\frac{9}{17d_+^2}(|a|2^{-i})^2,$$

and if $w\in A_0^+\cup A_0^-$, then

$$\frac{1}{2}\leq\frac{\langle w-a,\, w+a\rangle}{|w-a||w+a|}\leq 1-\frac{9}{17d_+^2}|a|^2.$$

For $i\in\mathbb{N}\cup\{0\}$, set $\psi_i=\left(\frac{2}{5}\frac{d_-}{d_2}|a|2^{-i}\right)^2$ and let $\rho_i=d_2\psi_i^{\frac{1}{2}\frac{1}{s-1-\xi}}$. Observe that, since $\rho_i\leq\frac{2}{5}d_-2^{-i}$, if $u\in A_i^{mn}$ and $|u-v|\leq\rho_i$ with $v\in A(0,d_-,d_+)$, then $v\in A_{i-1}^{mn}\cup A_i^{mn}\cup A_{i+1}^{mn}$. Similarly, if $u\in A_0^+\cup A_0^-$, and $v\in A(0,d_-,d_+)$ with $|u-v|\leq\rho_0$, then $v\in A_1^{mn}$ for some choice of m and n.

Writing $f(\rho)=\rho^{-d-1}\nu_y(F_y\cap B(u,\rho))$, we let $I_0^\pm=\int_{F_x\cap A_0^\pm}\int_0^\infty f(\rho)\, d\rho d\nu_x(u)$ and $I_i^{mn}=\int_{F_x\cap A_i^{mn}}\int_0^\infty f(\rho)d\rho\, d\nu_x(u)$.

We must estimate

$$\begin{aligned}
&\int_{F_x}\int_0^\infty \rho^{-d-1}\nu_y(F_y\cap B(u,\rho))\, d\rho\, d\nu_x(u)\\
&=\left(\int_{F_x\cap A_0^+}+\int_{F_x\cap A_0^-}+\sum_{m,n=0}^{1}\sum_{i=1}^{\infty}\int_{F_x\cap A_i^{mn}}\right)\int_0^\infty f(\rho)\, d\rho\, d\nu_x(u)\\
&=I_0^++I_0^-+\sum_{m,n=0}^{1}\sum_{i=1}^{\infty}I_i^{mn}.
\end{aligned}$$

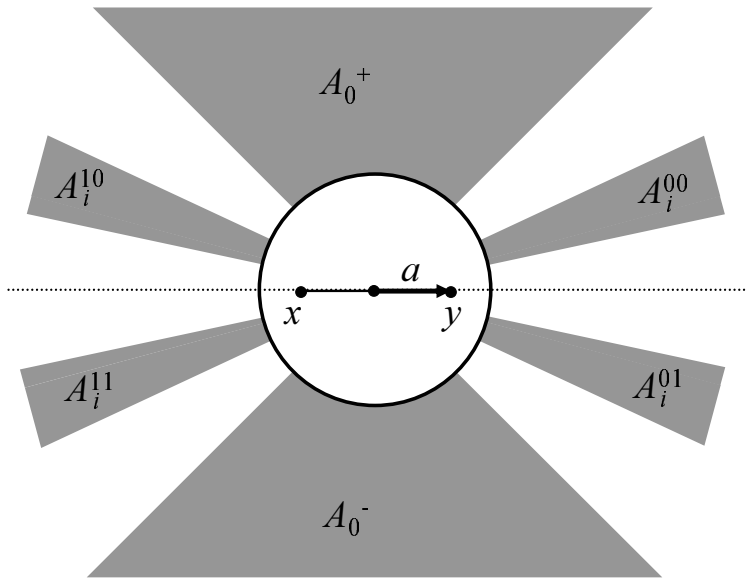

FIGURE 6. A_i^{mn}.

We can write

$$\begin{aligned} I_i^{mn} &= \int_{F_x \cap A_i^{mn}} \left(\int_0^{\rho_i} + \int_{\rho_i}^{\infty} \right) f(\rho)\, d\rho\, d\nu_x(u) \\ &= \int_{F_x \cap A_i^{mn} \cap B(F_y \cap B(A_i^{mn}, \rho_i), \rho_i)} \int_0^{\rho_i} f(\rho)\, d\rho\, d\nu_x(u) + \int_{F_x \cap A_i^{mn}} \int_{\rho_i}^{\infty} f(\rho)\, d\rho\, d\nu_x(u) \\ &= I_{i,1}^{mn} + I_{i,2}^{mn}, \text{ say.} \end{aligned}$$

Lemma 4.1. *Suppose that $V \subseteq A(0, d_-, d_+)$ and $0 < r < 1$. Then*

$$\int_{F_x \cap V} \int_r^{\infty} f(\rho)\, d\rho \le c r^{s-d} \operatorname{arc-diam}_0(F_x \cap V)^{s-1},$$

where c is a positive constant that depends only on d_-, d_+, s and d.

In the proof of the lemma, and subsequently, we let $\preceq$ denote inequality up to a finite constant independent of x and y.

Proof. Using the crude estimate that for $u \in F_x \cap V$, $\nu_y(F_y \cap B(u, \rho)) \le \min\{1, 2^s \rho^s\}$ together with Lemma 2.3, we find

$$\begin{aligned} &\int_{F_x \cap V} \int_r^{\infty} f(\rho)\, d\rho \\ &\le 2^s \left(\frac{1}{d-s} r^{s-d} + \frac{1}{d} \right) \nu_x(F_x \cap V) \\ &\preceq r^{s-d} \operatorname{arc-diam}_0(F_x \cap V)^{s-1}. \end{aligned}$$

□

In particular, Lemma 4.1 implies that

$$I_{i,2}^{mn} \preceq \rho_i^{s-d}\,\text{arc-diam}_0(F_x \cap A_i^{mn})^{s-1} \preceq |a|^{\frac{s-d}{s-1-\xi}} 2^{-\left(\frac{s-d}{s-1-\xi}+s-1\right)i}.$$

In order to estimate $I_{i,1}^{mn}$, we use equation (4.4) (of section 4.1), Fubini's theorem and the fact that if $u \in F_x \cap A_i^{mn}$ and $v \in B(u,\rho_i) \cap F_y$, then $v \in A_{i-1}^{mn} \cup A_i^{mn} \cup A_{i+1}^{mn}$, to calculate that, provided $\frac{s+\xi}{2+\xi-s} - d > 0$,

$$\begin{aligned}
&I_{i,1}^{mn} \\
&= \int_0^{\rho_i} \rho^{-d-1}(\nu_x \otimes \nu_y)\{(u,v) \in (F_x \cap A_i^{mn}) \times (F_y \cap B(A_i^{mn},\rho)) : |u-v| \le \rho\}\, d\rho \\
&\le c_2\,\text{arc-diam}_0(A_i^{mn} \cap F_x) \int_0^{\rho_i} \rho^{-d-1}(\psi_{i+1}^{-\frac{1}{2}}\rho)^{\frac{s+\xi}{2+\xi-s}}\, d\rho \\
&\preceq \text{arc-diam}_0(A_i^{mn} \cap F_x)\psi_{i+1}^{-\frac{1}{2}\frac{s+\xi}{2+\xi-s}} \rho_i^{\frac{s+\xi}{2+\xi-s}-d} \\
&\preceq \text{arc-diam}_0(A_i^{mn} \cap F_x)\psi_i^{\frac{1}{2}\left(\frac{s+\xi-d}{s-1-\xi}\right)} \\
&\preceq |a|^{\frac{s+\xi-d}{s-1-\xi}} 2^{-i\left(\frac{2s-1-d}{s-1-\xi}\right)}.
\end{aligned}$$

Combining these estimates for $I_{i,1}^{mn}$ and $I_{i,2}^{mn}$, we deduce that, provided $\frac{s+\xi}{2+\xi-s} - d > 0$,

$$\begin{aligned}
I_i^{mn} &\preceq |a|^{\frac{s+\xi-d}{s-1-\xi}} 2^{-i\left(\frac{2s-1-d}{s-1-\xi}\right)} + |a|^{\frac{s-d}{s-1-\xi}} (2^{-i})^{\frac{s-d}{s-1-\xi}+s-1} \\
&= |a|^{\frac{s-d}{s-1-\xi}} \left((2^{-i})^{\frac{2s-1-d}{s-1-\xi}} + (2^{-i})^{\frac{s-d}{s-1-\xi}+s-1}\right).
\end{aligned}$$

Hence

$$\sum_{m,n=0}^{1} \sum_{i=1}^{\infty} I_i^{mn} \preceq |a|^{\frac{s-d}{s-1-\xi}},$$

provided that $\min\left\{\frac{s+\xi}{2+\xi-s} - d,\ \frac{2s-1-d}{s-1-\xi},\ \frac{s-d}{s-1-\xi} + s - 1\right\} > 0$. Estimating I_0^+ and I_0^- in a similar way, we find

$$I_0^+ + I_0^- + \sum_{m,n=0}^{1} \sum_{i=1}^{\infty} I_i^{mn} \preceq |a|^{\frac{s-d}{s-1-\xi}},$$

provided that $\min\left\{\frac{s+\xi}{2+\xi-s} - d,\ \frac{2s-1-d}{s-1-\xi},\ \frac{s-d}{s-1-\xi} + s - 1\right\} > 0$. Hence, provided that $\min\left\{\frac{s+\xi}{2+\xi-s} - d,\ \frac{2s-1-d}{s-1-\xi},\ \frac{s-d}{s-1-\xi} + s - 1\right\} > 0$,

$$\int_{F_x \times F_y} |u-v|^{-d}\, d(\nu_x \otimes \nu_y)(u,v) \preceq |x-y|^{-\frac{d-s}{s-1-\xi}}.$$

Thus, if $\min\left\{\frac{s+\xi}{2+\xi-s} - d,\ \frac{2s-1-d}{s-1-\xi},\ \frac{s-d}{s-1-\xi} + s - 1\right\} > 0$, then

$$+\infty = I_d(\nu) \preceq I_{\frac{d-s}{s-1-\xi}}(\mu)$$

and this gives a contradiction if $\frac{d-s}{s-1-\xi} < t$, the dimension of μ. Since $0 < \xi < s-1$ is arbitrary, it follows that if $s > \max\left\{\frac{1}{2}(d+1), \frac{2d}{d+1}, \frac{1}{2} + \sqrt{d - \frac{3}{4}}\right\} = \frac{1}{2} + \sqrt{d - \frac{3}{4}}$, then $t \le \frac{d-s}{s-1}$ and Theorem 1.2 (and hence Theorem 1.1) follows.

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