PUBLISHED VERSION

Ma, Yumei; Duan, Fabing; Chapeau-Blondeau, François; Abbott, Derek
[Weak-periodic stochastic resonance in a parallel array of static nonlinearities](#) PLoS One, 2013;
8(3):e58507

http://hdl.handle.net/2440/78322

PLOS ONE

Weak-Periodic Stochastic Resonance in a Parallel Array of Static Nonlinearities

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Abstract

This paper studies the output-input signal-to-noise ratio (SNR) gain of an uncoupled parallel array of static, yet arbitrary, nonlinear elements for transmitting a weak periodic signal in additive white noise. In the small-signal limit, an explicit expression for the SNR gain is derived. It serves to prove that the SNR gain is always a monotonically increasing function of the array size for any given nonlinearity and noisy environment. It also determines the SNR gain maximized by the locally optimal nonlinearity as the upper bound of the SNR gain achieved by an array of static nonlinear elements. With locally optimal nonlinearity, it is demonstrated that stochastic resonance cannot occur, i.e. adding internal noise into the array never improves the SNR gain. However, in an array of suboptimal but easily implemented threshold nonlinearities, we show the feasibility of situations where stochastic resonance occurs, and also the possibility of the SNR gain exceeding unity for a wide range of input noise distributions.

Competing Interests: DA is a PLOS ONE Editorial Board member, and the authors here confirm that this does not alter the authors' adherence to all the PLOS ONE policies on sharing data and materials.

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Introduction

Stochastic resonance (SR) is a nonlinear phenomenon where the transmission of a coherent signal by certain nonlinear systems can be improved by the addition of noise [1–12]. The SR effect was initially observed in a bistable climate model driven by a subthreshold periodic input [1–4]. Then, this phenomenon attracted much attention in physics and biology [2,5,6,8,13,14]. It is reported that SR occurs in peripheral [5,6,8,13–16] and central [17–19] nervous systems, since the nervous systems implement, as a basis, complex dynamics that very often involve nonlinear processes, and commonly have to operate in environments containing noise, either of external or internal origins [8,20]. The SR phenomenon can also be observed at a behavioral level, for instance, feeding paddle-fish [21], human posture stabilization [22,23] and attention control [24]. Currently, the utilization of noise has become an optional and nontrivial strategy for statistical signal processing. It is noted that different static nonlinearities have been employed to exhibit the SR effect, for instance, the threshold nonlinearity [3,4,8,25], the saturation nonlinearity [7], the power-law sensor [26], non-adjustable [27–29] or variable [30] detectors, estimators [30–35] and optimal processors [36]. By including nonlinear elements into an array, the array enhanced SR effect was observed by tuning the array noise level and the coupling strength [37,38]. Moreover, in the generic model of an uncoupled parallel array of static nonlinearities, some significant SR effects, e.g. SR without tuning [39], suprathreshold SR [40] and array SR [26], were subsequently reported. The constructive role of internal noise is adequately reappraised for improving the performance of the array of nonlinearities [31–

34,41–44]. Recently, the SR effect has been further shown with new characteristics in complex network topologies, such as small-world networks and scale-free networks [45–52]. Particularly, the influence of network architectures, as well as the non-zero noise level, on SR is recognized [45–52]. It is interesting to note that these related studies in general also provide evidence that, besides an optimal noise intensity, an optimal network configuration exists, at which the best system response can be obtained [49–52].

There has been considerable interest in the amplification of the signal-to-noise ratio (SNR) of a periodic signal by exploiting the SR effect [2,8,53–57]. From the viewpoint of the gain behavior, i.e., the SNR at the output divided by that at the input, the primary issue of a gain exceeding unity has been found for suprathreshold input signals [54–57]. However, most previous SR studies involved a fixed nonlinearity. When we consider an arbitrary adjustable static nonlinearity, the maximum SNR gain is achieved by a locally optimal nonlinearity for a weak periodic signal in additive white noise [29,58]. Since the SNR gain of a locally optimal nonlinearity is given by the Fisher information of the noise distribution, we demonstrated that the SNR gain of a locally optimal nonlinearity certainly exceeds unity for a weak periodic signal in additive non-Gaussian noise, and SR does not exist in an updated locally optimal nonlinearity [58]. However, the structure of the locally optimal nonlinearity is determined by the noise probability density function (PDF) and also the noise level [58,59]. Then, in some practical signal processing tasks, the locally optimal nonlinearity may be too complex to be implemented, and also can not be established for an unknown noise distribution [59]. Therefore, this provides an opportunity for the suboptimal

nonlinearity to improve the SNR gain by the SR effect [3,4,7,8,25–29,58].

In this paper, we focus on amplifying the output-input SNR gain of an uncoupled parallel array of static nonlinearities for transmitting a weak periodic signal in additive white noise. For an array of arbitrary static nonlinearities, the asymptotic expression of the SNR gain is first developed. Then, for a given nonlinearity and fixed noise levels, we prove that the SNR gain of an array is a monotonically increasing function of the array size. It is shown that the SNR gain maximized by the locally optimal nonlinearity is the upper bound of the performance of an array of static nonlinearities. Furthermore, it is demonstrated that the internal array noise components are incapable of further improving the SNR gain for locally optimal processing. This result extends the study of SR in a single static nonlinearity [58,60] to an array of static nonlinearities. The establishment of a locally optimal nonlinearity needs the complete descriptions of the noise PDF and the noise level. Therefore, when this is not feasible, we propose instead a parallel array of suboptimal but easily implemented threshold nonlinearities for transmitting a weak periodic signal, in order to improve the SNR gain via the SR phenomenon. It is shown that such an array of threshold nonlinearities exhibits the SR effect by increasing the array noise level and the array size. Moreover, with a sufficiently large array size, the fact of the SNR gain exceeding unity is shown for a wide range of underlying noise distributions. These interesting results demonstrate that a parallel array of threshold nonlinearities can be practically exploited, and is useful for nonlinear signal processing.

Results

Model

Consider the observation of a process $x(t) = s(t) + \xi(t)$, where the component $s(t)$ is a weak periodic signal with a maximal amplitude A ($|s(t)| \leq A$) and period T, and zero-mean additive white noise $\xi(t)$, independent of $s(t)$, having a PDF f_ξ and variance $\sigma_\xi^2 = E_\xi[x^2] = \int_{-\infty}^{\infty} x^2 f_\xi(x)dx$. Next, the input $x(t)$ is applied to an uncoupled parallel array of N identical static nonlinearities. In these nonlinearities, the noise terms $\eta_n(t)$, independent of $x(t)$, are the internal noise components for each static nonlinearity g, so as to yield the outputs [26].

$$y_n(t) = g(x(t) + \eta_n(t)), \ n = 1,2,\cdots,N. \tag{1}$$

Here, assume that the derivative $g'(z) = dg(z)/dz$ exists for almost all z, and g has zero mean under f_z, i.e. $E_z[g(x)] = 0$, which is not restrictive since any arbitrary g can always include a constant bias to cancel this average [59]. The internal noise components $\eta_n(t)$ are mutually independent and identically distributed (i.i.d.) with the same PDF f_η and variance σ_η^2. The noise components $\xi(t)$ and $\eta_n(t)$ are all assumed to be stationary random variables. Since $\xi(t)$ and $\eta_n(t)$ are independent, Eq. (1) can be rewritten as $y_n(t) = g(s(t) + z(t))$, where the composite noise components $z(t) = \xi(t) + \eta_n(t)$ are with the same convolved PDF $f_z(x) = \int_{-\infty}^{\infty} f_\xi(x-u)f_\eta(u)du$. Then, the array output $y(t)$ is given by

$$y(t) = \frac{1}{N} \sum_{n=1}^{N} y_n(t). \tag{2}$$

The input SNR for $x(t)$ can be defined as the power contained in the spectral line at $1/T$ divided by the power contained in the noise background in a small frequency bin ΔB around $1/T$, this is [3]

$$R_{in} = \frac{|\langle s(t) \exp(-i2\pi t/T) \rangle|^2}{\sigma_\xi^2 \Delta t \Delta B}, \tag{3}$$

with Δt indicating the time resolution or the sampling period in a discrete-time implementation and the temporal average defined as $\langle \cdots \rangle = \frac{1}{T} \int_0^T \cdots dt$ [3]. Since $s(t)$ is periodic, $y(t)$ is in general a cyclostationary random signal with period T [3]. Similarly, the output SNR at $y(t)$ is expressed as

$$R_{out} = \frac{|\langle E_z[y(t)] \exp(-i2\pi t/T) \rangle|^2}{\langle var[y(t)] \rangle \Delta t \Delta B}, \tag{4}$$

where the nonstationary expectation $E_z[y(t)] = E_z[y_n(t)] = \int_{-\infty}^{\infty} y_n(t)f_z(x)dx$ and nonstationary variance $var[y(t)] = E_z[y^2(t)] - E_z^2[y(t)]$ are also temporal functions of time t [3]. Then, the SNR gain, G_N, is defined as the ratio of the output SNR over the input SNR [3,26,57]

$$G_N = \frac{R_{out}}{R_{in}} = \frac{\sigma_\xi^2 |\langle E_z[y(t)] \exp(-i2\pi t/T) \rangle|^2}{\langle var[y(t)] \rangle |\langle s(t) \exp(-i2\pi t/T) \rangle|^2}, \tag{5}$$

for an array of static nonlinearities with array size N.

SNR Gain of an Array for Weak Signals

For a weak signal $s(t)$ ($A \to 0$ and $|s(t)| \leq A$) and at a fixed time t, we make a Taylor expansion of g around z and have the asymptotic form

$$\begin{aligned} E_z[y(t)] &= E_z[y_n(t)] = E_z[g(z(t)+s(t))] \\ &\approx E_z[g(z)+s(t)g'(z)] = s(t)E_z[g'(z)], \end{aligned} \tag{6}$$

where the outputs y_n are i.i.d. for $n = 1,2,\cdots,N$. The output degree-two moment is given by

$$\begin{aligned} E_z[y^2(t)] &= E_\xi\{E_\eta[y^2(t)]\} = E_\xi\{\frac{1}{N^2} \sum_{n=1}^{N} \sum_{m=1}^{N} E_\eta[y_n y_m]\} \\ &= \frac{1}{N^2} E_\xi\{N E_\eta[y_n^2] + N(N-1)E_\eta[y_n y_m]\}(\forall m \neq n) \\ &= \frac{1}{N} E_\xi\{E_\eta[y_n^2]\} + \frac{N-1}{N} E_\xi\{E_\eta^2[y_n]\} \\ &= \frac{1}{N} E_z[g^2(s+z)] + \frac{N-1}{N} E_\xi\{E_\eta^2[g(s+z)]\}, \end{aligned} \tag{7}$$

where $E_z[\cdot] = E_\xi\{E_\eta[\cdot]\}$ and $E_\eta[y_n] = E_\eta[y_m]$. Therefore, based on Eq. (6), we have

$$var[y(t)] = E_z[y^2(t)] - E_z^2[y(t)]$$

$$\approx \frac{1}{N}E_z[g^2(s+z)] + \frac{N-1}{N}E_\xi\{E_\eta^2[g(s+z)]\}$$

$$- s^2(t)E_z^2[g'(z)] \qquad (8)$$

$$\approx \frac{1}{N}E_z[g^2(z)] + \frac{1}{N}2s(t)E_z[g(z)g'(z)]$$

$$+ \frac{N-1}{N}E_\xi\{E_\eta^2[g(z)]\} + \frac{N-1}{N}2s(t)E_\xi\{E_\eta^2[g'(z)g(z)]\},$$

where the approximations are up to first order in the small signal $s(t)$. Substituting the asymptotic forms of $E_z[y(t)]$ of Eq. (6) and $var[y(t)]$ of Eq. (8) into Eq. (5), we obtain the asymptotic expression of the SNR gain of a parallel array of static nonlinearities as

$$G_N \approx \frac{\sigma_\xi^2 E_z^2[g'(z)]}{\frac{1}{N}E_z[g^2(z)] + \frac{N-1}{N}E_\xi\{E_\eta^2[g(\xi+\eta)]\}}, \qquad (9)$$

where terms $2s(t)E_z[g(z)g'(z)]$ and $2s(t)E_\xi\{E_\eta^2[g'(z)g(z)]\}$, compared with primary terms $E_z[g^2(z)]$ and $E_\xi\{E_\eta^2[g(z)]\}$, are neglected as $A \to 0$ ($|s(t)| \le A$). It is interesting to note that the SNR gain G_N in Eq. (9) is applicable for an arbitrary weak-periodic signal $s(t)$ throughout an array of static (yet arbitrary) nonlinearities.

For the random variable $g(\xi+\eta)$ and the convex function x^2, by the Jensen inequality [61], we have

$$E_\eta[g^2(\xi+\eta)] \ge E_\eta^2[g(\xi+\eta)], \qquad (10)$$

for any fixed variable ξ [61]. Therefore, we have

$$E_z[g^2(z)] = E_\xi\{E_\eta[g^2(\xi+\eta)]\} \ge E_\xi\{E_\eta^2[g(\xi+\eta)]\}. \qquad (11)$$

From Eq. (11) and for any integers $K \ge N \ge 1$, we have

$$(K-N)E_z[g^2(z)] \ge (K-N)E_\xi\{E_\eta^2[g(\xi+\eta)]\}, \qquad (12)$$

$$\frac{1}{N}E_z[g^2(z)] + \frac{N-1}{N}E_\xi\{E_\eta^2[g(\xi+\eta)]\}$$

$$\ge \frac{1}{K}E_z[g^2(z)] + \frac{K-1}{K}E_\xi\{E_\eta^2[g(\xi+\eta)]\}, \qquad (13)$$

$$G_N = \frac{\sigma_\xi^2 E_z^2[g'(z)]}{\frac{1}{N}E_z[g^2(z)] + \frac{N-1}{N}E_\xi\{E_\eta^2[g(\xi+\eta)]\}} \le G_K$$

$$= \frac{\sigma_\xi^2 E_z^2[g'(z)]}{\frac{1}{K}E_z[g^2(z)] + \frac{K-1}{K}E_\xi\{E_\eta^2[g(\xi+\eta)]\}}, \qquad (14)$$

Thus, for the given nonlinearity g and fixed noise components $\xi(t)$ and $\eta_n(t)$, the SNR gain G_N in Eq. (9) is a monotonically increasing function of the array size N. From Eq. (9), we have the minimum.

$$G_1 = \sigma_\xi^2 E_z^2[g'(z)]/E_z[g^2(z)], \qquad (15)$$

for $N = 1$, and the maximum

$$G_\infty = \sigma_\xi^2 E_z^2[g'(z)]/E_\xi\{E_\eta^2[g(\xi+\eta)]\}, \qquad (16)$$

for $N = \infty$.

Naturally, Eq. (12) inspires us to consider the increase of array size N for the further improvement of the SNR gain obtained by a single nonlinearity. We will demonstrate in Eq. (17) that this thought is infeasible for locally optimal processing.

Without the internal noise $\eta_n(t)$, Eq. (9) becomes

$$G_N = \frac{\sigma_\xi^2 E_\xi^2[g'(x)]}{\frac{1}{N}E_\xi[g^2(x)] + \frac{N-1}{N}E_\xi[g^2(x)]} = G_1$$

$$= \frac{\sigma_\xi^2 E_\xi^2[g'(x)]}{E_\xi[g^2(x)]} \le \sigma_\xi^2 E_\xi\left[\frac{f_\xi'^2(x)}{f_\xi^2(x)}\right] = \sigma_\xi^2 I(f_\xi), \qquad (17)$$

where the array size N does not work, and the equality occurs as g becomes a locally optimal nonlinearity

$$g_{opt}(x) \overset{\triangle}{=} Cf'_\xi(x)/f_\xi(x), \qquad (18)$$

for the derivative $f'_\xi(x) = df_\xi(x)/dx$ (without loss of generality $C = -1$) [29,59]. Here, $I(f) = E[f'^2/f^2]$ is the Fisher information of the noise distribution f [61].

We add the extra noise $\eta(t)$ to the observation data X, aiming to improve the performance of g_{opt}. However, based on the Fisher information in inequality $I(f_z) \le \min(I(f_\xi), I(f_\eta))$ [61], we have

$$G_1 = \frac{\sigma_\xi^2 E_z^2[g'(z)]}{E_z^2[g(z)]} \le \sigma_\xi^2 I(f_z) \le \sigma_\xi^2 I(f_\xi), \qquad (19)$$

where the later inequality indicates that the addition of extra noise cannot improve the performance of a single locally optimal nonlinearity g_{opt} [58].

Based on Eq. (12), the SNR gain of an array of arbitrary static nonlinearities attains its maximum G_∞ in Eq. (16). Using the Cauchy-Schwarz inequality, we have

$$G_\infty = \frac{\sigma_\xi^2 E_z^2[g'(z)]}{E_\xi\left\{E_\eta^2[g(\xi+\eta)]\right\}} = \frac{\sigma_\xi^2 E_\xi^2\{dE_\eta[g(\xi+\eta)]/d\xi\}}{E_\xi\left\{E_\eta^2[g(\xi+\eta)]\right\}}$$

$$\le \sigma_\xi^2 E_\xi[\frac{f'^2_\xi(\xi)}{f^2_\xi(\xi)}] = \sigma_\xi^2 I(f_\xi), \qquad (20)$$

where the nonlinearity $E_\eta[g(\xi+\eta)]$ is a function of ξ. The equality occurs as the nonlinearity $E_\eta[g_{opt}(\xi+\eta)] = Cf'_\xi(\xi)/f_\xi(\xi)$, i.e. the nonlinearity $g_{opt}(\xi) = Cf'_\xi(\xi)/f_\xi(\xi)$ and the PDF $f_\eta(x) = \delta(x)$. Here, $\delta(x)$ is the Dirac delta function, and this means there is no internal noise in the nonlinearity. From Eqs. (17), (19) and (20), this result indicates that the upper bound of the SNR gain G_∞ is achieved by g_{opt} in Eq. (18) without the internal noise $\eta_n(t)$. Therefore, the addition of internal noise components $\eta_n(t)$ to the signal is never helpful for improving the SNR gain that is obtained by the locally optimal nonlinearity of Eq. (18).

Thus, Eq. (17) extends our previous result of the incapability of SR in the SNR gain improvement of a single locally optimal

nonlinearity [58] to the configuration of the array of static nonlinearities. For instance, consider the Gaussian noise components $\xi(t)$ and $\eta_n(t)$ with PDFs $f_\xi(x) = \exp(-x^2/(2\sigma_\xi^2))/\sqrt{2\pi\sigma_\xi^2}$ and $f_\eta(x) = \exp(-x^2/(2\sigma_\eta^2))/\sqrt{2\pi\sigma_\eta^2}$, respectively. Then, the composite noise $z(t)$ is also Gaussian distributed with PDF $f_z(x) = \exp(-x^2/(2\sigma_z^2))/\sqrt{2\pi\sigma_z^2}$ and variance $\sigma_z^2 = \sigma_\xi^2 + \sigma_\eta^2$. The Gaussian distribution corresponds to the locally optimal nonlinearity $g_{opt}(x) = x$ [58]. Substituting g_{opt} into Eq. (9), we have

$$G_N = \frac{\sigma_\xi^2 E_z^2[g'_{opt}(z)]}{\frac{1}{N}E_z[g_{opt}^2(z)] + \frac{N-1}{N}E_\xi\{E_\eta^2[g_{opt}(\xi+\eta)]\}}$$
$$= \frac{\sigma_\xi^2}{\frac{1}{N}\sigma_z^2 + \frac{N-1}{N}\sigma_\xi^2} = \frac{\sigma_\xi^2}{\sigma_\xi^2 + \frac{1}{N}\sigma_\eta^2} \qquad (21)$$
$$\leq G_\infty = \sigma_\xi^2 I(f_\xi) = 1,$$

where $E_\xi\{E_\eta^2[g_{opt}(\xi+\eta)]\} = \sigma_\xi^2$ and the Fisher information of f_ξ is $I(f_\xi) = 1/\sigma_\xi^2$ [59]. In Eq. (21), it is seen that the increase of noise variance σ_η^2 only degrades the SNR gain. The upper bound of unity can be only achieved for the infinite array size $N = \infty$ or $\sigma_\eta^2 = 0$ by the locally optimal nonlinearity of Eq. (18).

Noise-enhanced Signal Transmission in Arrays

It is seen in Eq. (17) that, for transmitting a weak periodic signal in additive white noise, the addition of internal array noise to an uncoupled parallel array of nonlinearities is incapable of improving the SNR gain of the locally optimal nonlinearity g_{opt}. However, the structure of g_{opt} in Eq. (18) depends on the complete description of the noise PDF and the noise level, and in practice it may be difficult to obtain an explicit analytical expression of g_{opt} in the unknown noisy environment [59]. Moreover, the presence of internal noise $\eta_n(t)$ is unavoidable in some practical signal processing cases [5,6,38–41,59]. Thus, we place suboptimal but easily implemented nonlinearities in a parallel array to transmit a weak periodic signal, and then show the feasibility of the SR phenomenon [1,3,26].

In the observation model of Eq. (1), the external noise $\xi(t)$ is considered as zero-mean generalized Gaussian noise, which is a flexible family containing some common important cases (e.g. Gaussian noise and Laplacian noise) [27,59,61]. The generalized Gaussian noise $\xi(t)$ has PDF

$$f_\xi(x) = \frac{c_1}{\sigma_\xi} \exp(-c_2 |\frac{x}{\sigma_\xi}|^\alpha), \qquad (22)$$

where $c_1 = \frac{\alpha}{2}\Gamma^{\frac{1}{2}}(3\alpha^{-1})/\Gamma^{\frac{3}{2}}(\alpha^{-1})$ and $c_2 = [\Gamma(3\alpha^{-1})\Gamma(\alpha^{-1})]^{\alpha/2}$ for a decay exponent $\alpha > 0$ [59]. The array noise terms $\eta_n(t)$ are assumed to be i.i.d. uniform noise with the same PDF

$$f_\eta(x) = 1/(2b), \qquad (23)$$

for $-b \leq x \leq b$ ($b = \sqrt{3}\sigma_\eta > 0$) and zero otherwise. When the exponent $\alpha = 2$, Eq. (22) represents the PDF of Gaussian noise $\xi(t)$. In this case, the signal $s(t)$ is buried in the composite noise $z(t) = \xi(t) + \eta_n(t)$. Then, the corresponding locally optimum nonlinearity g_{opt} of Eq. (18) needs to be updated as

$$g_{opt}(x) = -\frac{f'_z(x)}{f_z(x)}$$
$$= \frac{\exp\left(-\frac{(x-\sqrt{3}\sigma_\eta)^2}{2\sigma_\xi^2}\right) - \exp\left(-\frac{(x+\sqrt{3}\sigma_\eta)^2}{2\sigma_\xi^2}\right)}{\sqrt{2\pi}\sigma_\xi[Q(\frac{x-\sqrt{3}\sigma_\eta}{\sigma_\xi}) - Q(\frac{x+\sqrt{3}\sigma_\eta}{\sigma_\xi})]}, \qquad (24)$$

with $Q(x) = \int_x^\infty \exp(-t^2/2)/\sqrt{2\pi}dt$. It is seen in Eq. (24) that the structure of g_{opt} is rather complicated and depends closely on the noise root-mean-square (RMS) amplitudes σ_ξ and σ_η. An illustrative plot of the locally optimum nonlinearity is shown in Fig. 1 for $\sigma_\xi = \sigma_\eta = 1$.

A suboptimal but easily implemented nonlinearity that we consider is the three-level threshold nonlinearity [3,55]

$$g_{th}(x) = \frac{1}{2}[\text{sign}(x-\theta) + \text{sign}(x+\theta)], \qquad (25)$$

with the sign or signum function $\text{sign}(\cdot)$ and the response threshold θ. Furthermore, the SNR gain G_N of a parallel array of threshold elements is plotted in Fig. 2 as a function of the RMS amplitude σ_η of the array noise $\eta_n(t)$ and the array size N. Here, the Gaussian noise $\xi(t)$ is with RMS amplitude $\sigma_\xi = 1$, and the response threshold of g_{th} takes $\theta = 1$. From the bottom up, the SNR gain G_N is shown for $N = 1,2,5,100,500$ and ∞ in Fig. 2 (solid lines). It is seen in Fig. 2 that, for an isolated static nonlinearity g_{th} ($N=1$), the SR effect does not appear, and the SNR gain decreases monotonically as σ_η increases. The SNR gain G_1^{opt} of a single locally optimum nonlinearity g_{opt} is also plotted in Fig. 2 (dashed line). It is seen in Fig. 2 that the SNR gain G_1^{opt} of g_{opt} is always better than G_1 of a single threshold nonlinearity g_{th} ($N=1$). However, as the array size $N \geq 2$ and the array noise RMS amplitude σ_η increases, G_N of the array of threshold nonlinearities gradually catches up, and finally exceeds G_1^{opt} of the isolated locally optimum nonlinearity g_{opt}, as shown in Fig. 2. Additionally, upon increasing the array size N, the bell-shape behavior of G_N of a parallel array of threshold elements versus σ_η and N is clearly visible, this is the array SR effect. It is also noted in Fig. 2 that, for a sufficiently large array size $N > 500$, the SNR gain G_N tends to its upper limit of G_∞ for $N = \infty$. Of course, based on Eq. (17), the upper limit of G_∞ is less than the quantity of $\sigma_\xi^2 I(f_\xi) = 1$ achieved by the locally optimal nonlinearity g_{opt} in Eq. (18) (without the internal noise $\eta_n(t)$), as shown in Fig. 2.

Next, an interesting question is, for transmitting a weak periodic signal, whether the SNR gain G_N of an array of threshold nonlinearities can exceed unity or not. This possibility, for the case of SNR gain exceeding unity, is shown for Laplacian noise $\xi(t)$ with $\alpha = 1$ in Eq. (22). In this case, when the array noise components $\eta_n(t)$ are i.i.d. uniform random variables, the locally optimum nonlinearity should be updated as

$$g_{opt}(x) = -\frac{f'_z(x)}{f_z(x)} = -\frac{f_\xi(x+\sqrt{3}\sigma_\eta) - f_\xi(x-\sqrt{3}\sigma_\eta)}{F_\xi(x+\sqrt{3}\sigma_\eta) - F_\xi(x-\sqrt{3}\sigma_\eta)}, \qquad (26)$$

where $F_\xi(x) = [1 + \text{sign}(x)(1 - \exp(-\sqrt{2}|x|)/\sigma_\xi)]/2$ is the cumulative distribution function of Laplacian noise $\xi(t)$. For the noise RMS amplitudes $\sigma_\xi = \sigma_\eta = 1$, an illustrative example of the structure of g_{opt} is plotted in Fig. 3. Furthermore, when we fix $\sigma_\xi = 1$ and tune σ_η, the SNR gains G_N of an array of threshold
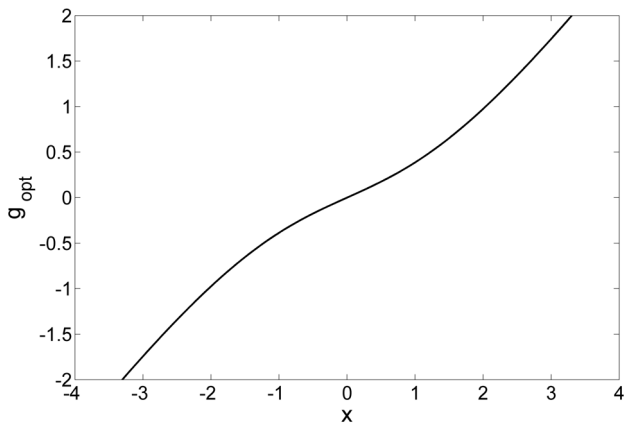
Figure 1. The locally optimum nonlinearity g_{opt}**.** The locally optimum nonlinearity g_{opt} in Eq. (24). The internal uniform noise components $\eta_n(t)$ have the RMS amplitude $\sigma_\eta = 1$. The external noise $\xi(t)$ is with the RMS amplitude $\sigma_\xi = 1$ and the decay parameter $\alpha = 2$ (Gaussian noise).
doi:10.1371/journal.pone.0058507.g001
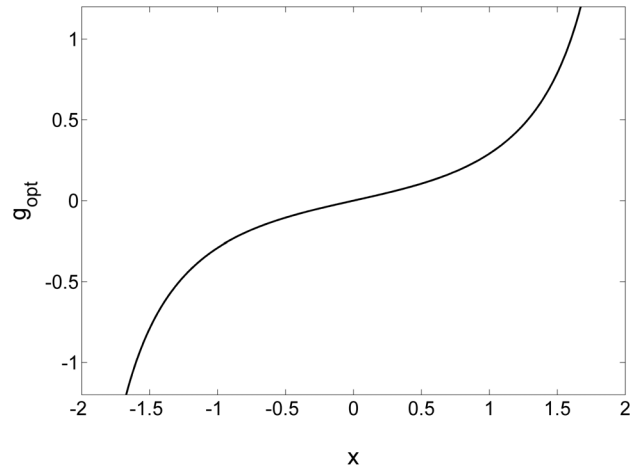
Figure 3. The locally optimum nonlinearity g_{opt}**.** The locally optimum nonlinearity g_{opt} in Eq. (26). The internal noise terms $\eta_n(t)$ are uniform noises with the RMS amplitude $\sigma_\eta = 1$. The external noise $\xi(t)$ is with the RMS amplitude $\sigma_\xi = 1$ and the decay parameter $\alpha = 1$ (Laplacian noise).
doi:10.1371/journal.pone.0058507.g003

elements with the threshold $\theta = 1$ are presented in Fig. 4. It is seen in Fig. 4 that, for a sufficiently larger array size $N \geq 5$, the fact of the SNR gain G_N exceeding unity is clearly demonstrated in this case. It is also interesting to note in Fig. 4 that SR effect survives for a single threshold nonlinearity g_{th} with $N = 1$.

In Eq. (12), it is known that the performance of an array of nonlinearities increases monotonically with the array size. As indicated in Figs. 2 and 4, we advocate the significance of a parallel array of nonlinearities with large array size N: The region of the noise level that improves the SNR gain of an array is gradually expanded as the array size N increases. Thus, increasing the array size N provides a simple alternative means of improving the performance of nonlinearities, especially when the optimal

noise associated with a single nonlinearity [28,30] is not known or accessible.

We also emphasize that the fact of the SNR gain exceeding unity is not exceptive. Here, we employ an array of g_{th} threshold elements with the threshold $\theta = 0$ and the array size $N = 100$. The external generalized Gaussian noise $\xi(t)$ is with the RMS amplitude $\sigma_\xi = 1$. The array noise is uniform noise with its RMS amplitude $\sigma_\eta = 1$. It is shown in Fig. 5 that, for a sufficiently large array size $N = 100$, the SNR gain G_{100} (red line) can be larger than
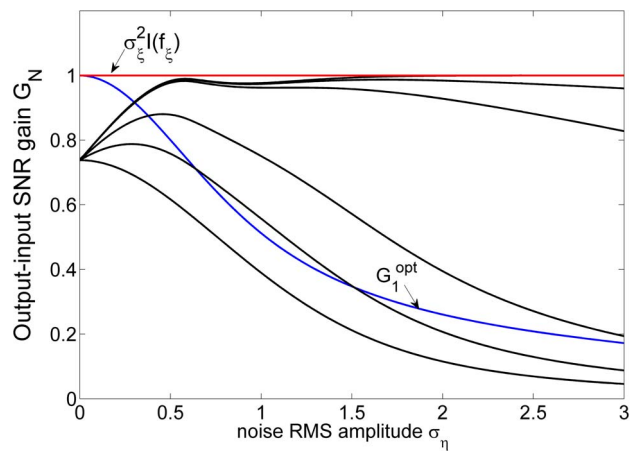
Figure 2. Output-input SNR gain G_N**.** Output-input SNR gain G_N as a function of the RMS amplitude σ_η of the array uniform noise terms $\eta_n(t)$ in the array of threshold nonlinearities of Eq. (25). The external noise $\xi(t)$ is with the RMS amplitude $\sigma_\xi = 1$ and the decay parameter $\alpha = 2$ (Gaussian noise). The threshold of g_{th} takes $\theta = 1$. The SNR gain G_N of Eq. (9) is plotted by black lines for $N = 1,2,5,100,500$ and ∞ (from the bottom up). For comparison, the SNR gains G_1^{opt} (blue line) of the locally optimum nonlinearities $g_{opt}(x)$ in Eq. (24) and the quantity of $\sigma_\xi^2 I(f_\xi) = 1$ (red line) achieved by the locally optimal nonlinearity g_{opt} in Eq. (18) (without the internal noise $\eta_n(t)$) are also illustrated.
doi:10.1371/journal.pone.0058507.g002
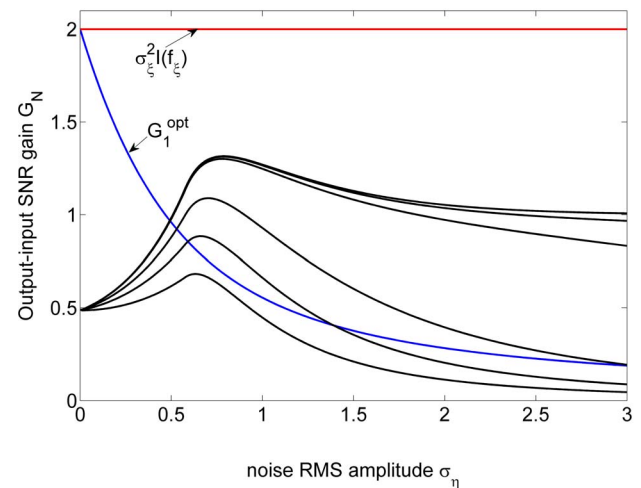
Figure 4. Output-input SNR gain G_N**.** Output-input SNR gain G_N as a function of the RMS amplitude σ_η of the array uniform noises $\eta_n(t)$ in the array of threshold nonlinearities of Eq. (25). The external noise $\xi(t)$ is with the RMS amplitude $\sigma_\xi = 1$ and the decay parameter $\alpha = 1$ (Laplacian noise). The threshold of g_{th} takes $\theta = 1$. The SNR gain G_N of Eq. (9) is plotted by black lines for $N = 1,2,5,100,500$ and ∞ (from the bottom up). For comparison, the SNR gains G_1^{opt} (blue line) of the locally optimum nonlinearities g_{opt} in Eq. (26) and the quantity of $\sigma_\xi^2 I(f_\xi) = 2$ (red line) achieved by the locally optimal nonlinearity g_{opt} in Eq. (18) (without the internal noise $\eta_n(t)$) are also illustrated.
doi:10.1371/journal.pone.0058507.g004
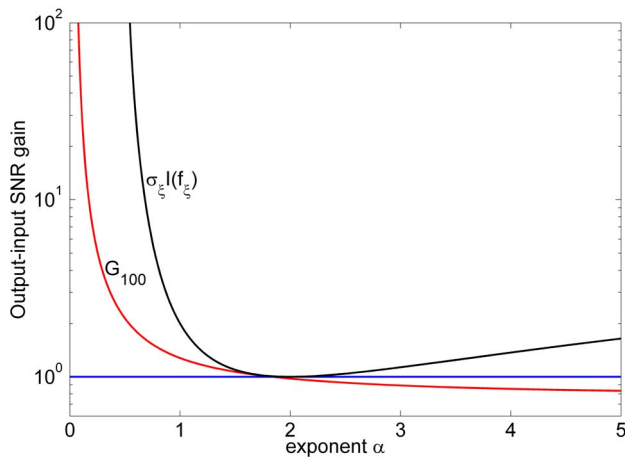
Figure 5. Output-input SNR gain G_{100}. Output-input SNR gain G_{100} (red line) of an array of threshold nonlinearities with array size $N = 100$ as a function of the decay exponent α for the response threshold $\theta = 0$. The external noise $\xi(t)$ is the generalized Gaussian noise with PDF in Eq. (22) and RMS amplitude $\sigma_\xi = 1$. The uniform array noise components $\eta_n(t)$ are with the RMS amplitude $\sigma_\eta = 1$. The black line indicates the SNR gain $\sigma_\xi^2 I(f_\xi) = \alpha^2 \Gamma(3\alpha^{-1})\Gamma(2-\alpha^{-1})/\Gamma^2(\alpha^{-1})$ achieved by the locally optimal nonlinearity g_{opt} in Eq. (18) (without the internal noise $\eta_n(t)$), which is also the upper limit that the SNR gain G_∞ cannot exceed. The blue line is the benchmark of unity.
doi:10.1371/journal.pone.0058507.g005

unity for the decay exponent $\alpha < 1.83$, which represents a wide range of generalized Gaussian noise distributions. Here, the black line indicates the upper limit of $\sigma_\xi^2 I(f_\xi) = \alpha^2 \Gamma(3\alpha^{-1})\Gamma(2-\alpha^{-1})/\Gamma^2(\alpha^{-1})$ [58,59] that the SNR gain G_∞ cannot exceed, as Eq. (20) indicated.

Discussion

In this paper, for a weak periodic signal in additive white noise, we study the characteristics of the SNR gain of an uncoupled parallel array of arbitrary static nonlinearities. Under the assumption of weak signal, an explicit expression of the SNR gain of an array is developed. Then, it is proven that, for a given nonlinearity and fixed noise levels, the SNR gain of an array is a monotonically increasing function of the array size. Furthermore, it is demonstrated that the internal array noise components are incapable of further improving the SNR gain of locally optimal processing. However, since the locally optimal nonlinearity requires a complete knowledge of the underlying noise statics,

the structure of the locally optimal nonlinearity may have no analytical expression or be intractable. Therefore, a parallel array of suboptimal but easily implemented threshold nonlinearities becomes an optional approach. It is shown that such an array of threshold nonlinearities can exhibit the SR effect by increasing the array noise level and the array size. For a sufficiently large array size, we also show that the SNR gain of an array of threshold nonlinearities can exceed unity for a wide range of noise distributions, e.g. the exponent $\alpha \leq 1.83$ in Fig. 5.

Some interesting open questions arise. For example, we only considered the array of threshold nonlinearities for processing a weak noisy signal. Therefore, can other tractable nonlinearities be connected in parallel for achieving improved output-input SNR gain via the array SR effect? As indicated in Fig. 2 and 4, we can operate an array of nonlinearities with large array size at a feasible level of noise. Therefore, given an acceptance criterion of the performance of nonlinearities, how large the array size is and which level the noise takes are interesting questions. These questions will be of interest for further studies of nonlinear signal processing in the context of array SR, especially in the ensemble of neurons. Often quite a number of neurons have similar properties and respond to the same stimuli [5,6,15], thus the condition of all neurons in parallel having the same pattern of input and output connections will be considered. It is of interest to explore how the external (internal) noise components assist the information transfer through the neural network. For the static nonlinearity considered in Eq. (1), Eq. (20) provides the upper bound of the performance of the array of static nonlinearities. While many neuron models, such as the leaky integrate-and-fire model and the Hodgkin-Huxley model [6,8], represent the neurodynamics with the time evolution nonlinear process (not a static nonlinearity), then whether the transmission efficiency of a parallel array of neurons has an upper bound for the neural signal propagate or not deserves to be studied.

Methods

Under the assumption of weak signals, the Taylor expansion of the noise PDF is utilized in Eqs. (6), (7), (8) and (9). The Jensen inequality is applied to Eq. (11). The Cauchy-Schwarz inequality is extensively used in Eqs. (17), (20) and (21).

Author Contributions

Proofreading: FCB DA. Performed the experiments: YM FD. Analyzed the data: FCB DA. Contributed reagents/materials/analysis tools: FCB DA. Wrote the paper: YM FD.

References

1. Benzi R, Sutera A, Vulpiani A (1981) The mechanism of stochastic resonance. Journal of Physics A: Mathematical and General 14: L453–L457.
2. Jung P, Hänggi P (1991) Amplication of small signals via stochastic resonance. Physical Review A 44: 8032–8042.
3. Chapeau-Blondeau F, Godivier X (1997) Theory of stochastic resonance in signal transimission by static nonlinear systems. Physical Review E 55: 1478–1495.
4. Godivier X, Chapeau-Blondeau F (1996) Noise-assisted signal transmission in a nonlinear electronic comparator: Experiment and theory. Signal Processing 56: 293–303.
5. Collins JJ, Chow CC, Imhoff TT (1995) Stochastic resonance in neuron models. Physical Review E 52: 3321–3324.
6. Lindner B, García J, Neiman A, Schimansky-Geier L (2004) Effects of noise in excitable systems. Physics Reports 392: 321–424.
7. Rousseau D, Rojas-Varela J, Chapeau-Blondeau F (2003) Stochastic resonance for nonlinear sensors with saturation. Physical Review E 67: no. 021102.
8. Gammaitoni L, Hänggi P, Jung P, Marchesoni F (1998) Stochastic resonance. Reviews of Modern Physics 70: 233–287.
9. Perc M, Gosak M, Kralj S (2008) Stochastic resonance in soft matter systems: combined effects of static and dynamic disorder. Soft Matter 4: 1861–1870.
10. Li J, Xu B (2006) Parameter-induced stochastic resonance with a periodic signal. Chinese Physics 15: 2867–2871.
11. Gosak M, Perc M, Kralj S (2012) The impact of static disorder on vibrational resonance in a ferroelectric liquid crystal. Molecular Crystals & Liquid Crystals 553: 13–20.
12. Ward LM, MacLean SE, Kirschner A (2010) Stochastic resonance modulates neural synchronization within and between cortical sources. PLoS ONE 5: no. e14371.
13. Wiesenfeld K, Moss F (1995) Stochastic resonance and the benefits of noise: from ice ages to crayfish and squids. Nature 373: 33–36.
14. Hänggi P (2002) Stochastic resonance in biology: how noise can enhance detection of weak signals and help improve biological information processing. ChemPhysChem 3: 285–290.
15. Moss F, Pei X (1995) Neurons in parallel. Nature 376: 211–212.
16. Bezrukov SM, Vodyanoy I (1995) Noise-induced enhancement of signal transduction across voltagedependent ion channels. Nature 378: 362–364.

17. Levin JE, Miller JP (1996) Broadband neural encoding in the cricket cercal sensory system enhanced by stochastic resonance. Nature 380: 165–168.
18. Mori T, Kai S (2002) Noise-induced entrainment and stochastic resonance in human brain waves. Physical Review Letters 88: no. 218101.
19. Stacey WC, Durand DM (2001) Synaptic noise improves detection of subthreshold signals in hippocampal CA1 neurons. Journal of Neurophysiology 86: 1104–1112.
20. Chapeau-Blondeau F, Duan F, Abbott D (2010) Synaptic signal transduction aided by noise in a dynamical saturating model. Physical Review E 81: no. 021124.
21. Russell DF, Wilkens LA, Moss F (1999) Use of behavioural stochastic resonance by paddle fish for feeding. Nature 401: 291–294.
22. Collins JJ, Priplata AA, Gravelle DD, Niemi J, Harry J, et al. (2003) Noise-enhanced human sensorimotor function. IEEE Engineering in Medicine and Biology 22: 76–83.
23. Priplata AA, Patritti BL, Niemi JB, Hughes R (2006) Noise-enhanced balance control in patients with diabetes and patients with stroke. Annals of Neurology 59: 4–12.
24. Kitajo K, Yamanaka K, Ward LM (2006) Stochastic resonance in attention control. Europhysics Letters 76: 1029.
25. Greenwood PE, Müller UU, Ward LM (2004) Soft threshold stochastic resonance. Physical Review E 70: no. 051110.
26. Chapeau-Blondeau F, Rousseau D (2004) Enhancement by noise in parallel arrays of sensors with power-law characteristics. Physical Review E 70: no. 060101(R).
27. Kay S (2000) Can detectabilty be improved by adding noise? IEEE Signal Processing Letters 7: 8–10.
28. Chen H, Varshney PK, Michels JH, Kay SM (2007) Theory of the stochastic resonance effect in signal detection: Part I–fixed detectors. IEEE Transactions on Signal Processing 55: 3172–3184.
29. Zozor S, Amblard PO (2003) Stochastic resonance in locally optimal detectors. IEEE Transactions on Signal Processing 51: 3177–3181.
30. Chen H, Varshney PK (2008) Theory of the stochastic resonance effect in signal detection: Part II–variable detectors. IEEE Transactions on Signal Processing 56: 5031–5041.
31. Chapeau-Blondeau F, Blanchard S, Rousseau D (2008) Fisher information and noise-aided power estimation from one-bit quantizers. Digital Signal Processing 18: 434–443.
32. McDonnell MD, Stocks NG, Pearce CEM, Abbott D (2008) Stochastic resonance: from suprathreshold stochastic resonance to stochastic signal quantization. Cambridge: Cambridge University Press.
33. Rousseau D, Chapeau-Blondeau F (2007) Noise-improved Bayesian estimation with arrays of onebit quantizers. IEEE Transactions on Instrumentation and Measurement 56: 2658–2662.
34. Patel A, Kosko B (2010) Optimal mean-square noise benefits in quantizer-array linear estimation. IEEE Signal Processing Letters 17: 1005–1009.
35. Chen H, Varshney PK, Michels JH (2008) Noise enhanced parameter estimation. IEEE Transactions on Signal Processing 56: 5074–5081.
36. Rousseau D, Chapeau-Blondeau F (2005) Stochastic resonance and improvement by noise in optimal detection strategies. Digital Signal Processing 15: 19–32.
37. Lindner JF, Meadows BK, Ditto WL, Inchiosa ME, Bulsara AR (1995) Array enhanced stochastic resonance and spatiotemporal synchronization. Physical Review Letters 75: 3–6.
38. Inchiosa ME, Bulsara AR (1996) Signal detection statistics of stochastic resonators. Physical Review E 53: R2021–R2024.
39. Collins JJ, Chow CC, Imhoff TT (1995) Stochastic resonance without tuning. Nature 376: 236–238.
40. Stocks NG (2000) Suprathreshold stochastic resonance in multilevel threshold systems. Physical Review Letters 84: 2310–2313.
41. Gailey PC, Neiman A, Collins JJ, Moss F (1997) Stochastic resonance in ensembles of nondynamical elements: the role of internal noise. Physical Review E 79: 4701–4704.
42. Stocks NG (2001) Information transmission in parallel threshold arrays: Suprathreshold stochastic resonance. Physical Review E 63: no. 041114.
43. Duan F, Chapeau-Blondeau F, Abbott D (2008) Stochastic resonance in a parallel array of nonlinear dynamical elements. Physics Letters A 372: 2159–2166.
44. Patel A, Kosko B (2011) Noise benefits in quantizer-array correlation detection and watermark decoding. IEEE Transactions on Signal Processing 59: 488–505.
45. Perc M (2007) Stochastic resonance on excitable small-world networks via a pacemaker. Physical Review E 76: no. 066203.
46. Benzi R (2009) Stochastic resonance in complex systems. Journal of Statistical Mechanics: Theory and Experiment 1: no. P01052.
47. Perc M (2008) Stochastic resonance on weakly paced scale-free networks. Physical Review E 78: no. 036105.
48. Wang Q, Perc M, Duan Z, Chen G (2009) Delay-induced multiple stochastic resonances on scalefree neuronal networks. Chaos 19: no. 023112.
49. Gan C, Perc M, Wang Q (2010) Delay-aided stochastic multiresonances on scale-free FitzHugh-Nagumo neuronal networks. Chinese Physical B 19: no. 040508.
50. Hütt MT, Jain MK, Hilgetag CC, Lesne A (2012) Stochastic resonance in discrete excitable dynamics on graphs. Chaos, Solitons & Fractals 45: 611–618.
51. Gosak M, Perc M, Kralj S (2011) Stochastic resonance in a locally excited system of bistable oscillators. The European Physical Journal B 80: 519–528.
52. Teramae J, Tsubo Y, Fukai T (2012) Optimal spike-based communication in excitable networks with strong-sparse and weak-dense links. Scientific Reports 2: no. 00485.
53. Dykman MI, McClintock PVE (1998) What can stochastic resonance do? Nature 391: 344.
54. Duan F, Chapeau-Blondeau F, Abbott D (2006) Noise-enhanced SNR gain in parallel array of bistable oscillators. Electronics Letters 42: 1008–1009.
55. Chapeau-Blondeau F, Rousseau D (2006) Nonlinear devices acting as SNR amplifiers for a harmonic signal in noise. Circuits, Systems and Signal Processing 25: 431–446.
56. Rousseau D, Chapeau-Blondeau F (2004) Suprathreshold stochastic resonance and signal-to-noise ratio improvement in arrays of comparators. Physics Letters A 321: 280–290.
57. Hänggi P, Inchiosa ME, Fogliatti D, Bulsara AR (2000) Nonlinear stochastic resonance: The saga of anomalous output-input gain. Physical Review E 62: 6155–6163.
58. Duan F, Chapeau-Blondeau F, Abbott D (2011) Fisher-information condition for enhanced signal detection via stochastic resonance. Physical Review E 84: no. 051107.
59. Kassam SA (1988) Signal Detection in Non-Gaussian Noise. New York: Springer-Verlag.
60. Duan F, Chapeau-Blondeau F, Abbott D (2012) Fisher information as a metric of locally optimal processing and stochastic resonance. PLoS ONE 7: no. e34282.
61. Cover TM, Thomas JA (1991) Elements of Information Theory. New York: Wiley.