Limits on the observable dynamics of mixed states

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It is shown that the observability of a large class of operations on mixed states is fundamentally limited. We consider trace-preserving, unital operations. This class includes unitary and perfect premeasurement operations. An upper bound on the trace distance between an untransformed state and a state transformed by one of these operations is derived. The bound is dependent only on the purity of the state. In the case of maximal mixedness, the bound implies all operations of this class are unobservable.

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I. INTRODUCTION

Given a particular quantum state that is subjected to a class of operations on the state, does the state change? And if so, how observable is that change? This question is related to the study of the preservation of information subject to quantum processes [1–3] by Nielsen *et al.* Their work focuses on the slightly different question of quantifying changes to quantum states given specific operations. By asking our initial question instead of that raised by Nielsen *et al.* the properties of the state are emphasized. We shall in particular explore the relationship between structural properties of a state (i.e., entanglement) and its dynamics.

There is also a foundational motivation for this. Since the inception of quantum mechanics (QM), there has been an uneasy dichotomy between two points of view: is QM a fundamental description of nature or merely an algorithm to calculate probabilities for outcomes of experiments? The friction between these two viewpoints comes from the manifestly nonclassical phenomena QM predicts, constrains, or allows. This list of phenomena includes interference, the uncertainty principle, nonlocality through entanglement (Bell inequalities) or otherwise [4], quantum teleportation, and no-cloning theorems [5]. The relationships between these phenomena remain unclear [6–9]. Our interest lies in how the structure of the allowed states of a quantum theory constrains its dynamics [8,9]. We suggest that the results of this paper may be extrapolated to probabilistic theories more general than quantum mechanics.

We utilize a symmetry of quantum states called envariance [10] which emerges dynamically due to their entanglement structure. This symmetry is a consequence of the tensor product structure (TPS) of quantum states.[1] In classical deterministic theories, Cartesian products are used to define assemblies of subsystems rather than tensor products, and so a symmetry equivalent to envariance does not exist.

We shall use an operational definition of the observability of the change of a quantum state: the trace distance (defined in Sec. V) between the transformed and untransformed state. An upper bound on this measure of the observability of the dynamics is derived. The bound is only dependent on the purity of the state. Another intriguing aspect of quantum states is the mathematical equivalence of states of subsystems of an entangled system to mixed states representing classical ensembles of quantum states. This connection allows the bound to apply to entangled systems as well.

Envariance is defined in Sec. II. In Sec. III, we describe how the information contained in the subsystems of an entangled bipartite system can be less than the information contained in the whole system. It is also shown how the mixedness of a state constrains knowledge on all nondegenerate observables of that state. These two qualities of quantum states are then used to motivate Sec. IV, where the class of invariant operations on a completely mixed state is considered. This symmetry is then used in Sec. V to derive an upper bound on the trace distance between an untransformed state and a transformed one. Concluding remarks are presented in Sec. VI.

II. ENVARIANCE

Envariance [10,11] is a symmetry of entangled composite systems. We define a composite system, in general, as a state that can be decomposed in terms of eigenstates of two or more mutually commuting observables where subsets of the total set of mutually commuting observables completely describe subsystems. Therefore a particle state with the quantum numbers spin and position can be considered composite, with spin and position describing separate subsystems.

Zurek's [10] original use of envariance was to provide a proof of Born's rule under "very mild" assumptions. We shall be using envariance in a different context and thus assume Born's rule from the outset.

Consider a composite system that can be decomposed into two subsystems, α and β, with the state $|\psi\rangle \in \mathcal{H}_\alpha \otimes \mathcal{H}_\beta$. Now suppose there exist unitary operators U_α and U_β, where

$$U_\alpha := \bar{U}_\alpha \otimes \mathbb{I}_\beta, \tag{1}$$

with $\mathbb{I}_\beta$ being the identity on $\mathcal{H}_\beta$ and $\bar{U}_\alpha : \tilde{\mathcal{H}}_\alpha \to \tilde{\mathcal{H}}_\alpha$, where $\tilde{\mathcal{H}}_\alpha$ is a subspace of $\mathcal{H}_\alpha$, with an analogous definition for U_β. A state $|\psi\rangle$ is said to be envariant under U_α (or U_β) if the following holds:

$$U_\alpha U_\beta|\psi\rangle = |\psi\rangle \quad \text{or} \quad U_\alpha|\psi\rangle = U_\beta^\dagger|\psi\rangle. \tag{2}$$

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[1] It has been shown [8] that TPSs are a generic feature of probabilistic theories with subsystems. Some form of envariance may then exist in such theories as well.

Note that from now on, we shall be considering finite-dimensional Hilbert spaces only.

Suppose we have a state of the form

$$|\psi\rangle = \frac{1}{\sqrt{N}}\sum_{j=1}^{N} e^{i\phi_j}|\alpha_j\rangle|\beta_j\rangle, \quad (3)$$

where $\{|\alpha_j\rangle\}$ and $\{|\beta_j\rangle\}$ form orthonormal bases for $\tilde{\mathcal{H}}_\alpha$ and $\tilde{\mathcal{H}}_\beta$, respectively, $N = \dim(\tilde{\mathcal{H}}_\alpha) = \dim(\tilde{\mathcal{H}}_\beta)$, and ϕ_j are arbitrary phases. These states are envariant under *all* unitary transformations of $\tilde{\mathcal{H}}_\alpha$ (or $\tilde{\mathcal{H}}_\beta$). When $\tilde{\mathcal{H}}_\alpha = \mathcal{H}_\alpha$, the state is maximally entangled (for subsystems α and β), and the group of envariant transformations is the group of all unitary transformations of $\mathcal{H}_\alpha$ alone [i.e., they can be decomposed as in Eq. (1)].

Consider now the case of a state with Schmidt decomposition,

$$|\Omega\rangle = \sum_{j=1}^{n} c_j e^{i\phi_j}|\alpha_j\rangle|\beta_j\rangle, \quad (4)$$

with $c_j \in \mathbb{R}^+$ such that $c_i \neq c_j$ for $i \neq j$, i.e., the coefficients of the Schmidt decomposition have unequal norms. In this case, the group of envariant transformations includes only relative (and overall) phase changes between the components $|\alpha_i\rangle|\beta_i\rangle$, i.e., unitaries of the form

$$U_\alpha := \left(\sum_{j=1}^{n} e^{i\lambda_j}|\alpha_j\rangle\langle\alpha_j| + \sum_{j=n+1}^{N}|\alpha_j\rangle\langle\alpha_j|\right)\otimes \mathbb{I}_\beta, \quad (5)$$

$$U_\beta := \left(\sum_{k=1}^{n} e^{-i\lambda_k}|\beta_k\rangle\langle\beta_k| + \sum_{k=n+1}^{M}|\beta_k\rangle\langle\beta_k|\right)\otimes \mathbb{I}_\alpha, \quad (6)$$

where $\dim(\mathcal{H}_\alpha) = N, \dim(\mathcal{H}_\beta) = M$, and $\lambda_j \in (0,2\pi)\ \forall j$. These unitaries have the desirable property

$$U_\alpha U_\beta|\Omega\rangle = |\Omega\rangle. \quad (7)$$

The most general case is where some of the coefficients c_i are equal and others are not. For the subspaces of $\mathcal{H}_\alpha$ spanned by the components whose coefficients are equal, we have envariance over the entire subspace. For the rest of the space, it is only relative phases of the components with unequal coefficients that can be envariantly transformed.

III. ALLOWED STATES

The emergence of envariance is a reflection of the property of entangled quantum states, where complete knowledge of the entire system (i.e., the state being pure) means incomplete knowledge of the subsystems. This can be understood in several ways.

(1) The reduced density matrices, tracing out α or β ($\mathrm{tr}_\alpha[\rho]$ or $\mathrm{tr}_\beta[\rho]$ for some pure ρ), have nonzero von Neumann entropy, leaving a mixed state partially equivalent to a classical lack of knowledge about the subsystem. However, the number of invariant degrees of freedom is only indirectly related to the amount of entanglement, as mentioned in Ref. [11]. We shall consider this point in more detail later on.

(2) Consider a Bell state,

$$|\nu\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1|\downarrow\rangle_2 + |\downarrow\rangle_1|\uparrow\rangle_2\right), \quad (8)$$

which is maximally entangled and as such has an SU(2) subgroup of envariant transformations which we can parametrize by the Pauli matrices,

$$e^{i\vec{\theta}\cdot\vec{\sigma}_1}|\nu\rangle = e^{i\vec{\theta}\cdot\vec{\sigma}_2}|\nu\rangle, \quad (9)$$

where σ_j^i is the ith Pauli matrix for the jth particle. Thus, rotating the spin of particle 1 is the same as rotating the spin of particle 2 instead. This implies only the *relative* orientations of the rays within the subsystem Hilbert spaces are known. Zurek has cited a similar idea as his motivation for using envariance [12] and calls it the "relativity of quantum observables." The situation can be said to have a kind of Machianity [13], analogous to the situation where the universe consists of point particles and only relative distances between them are known, not global displacement or orientation. The state only contains information about the correlations between the particles.

(3) For maximally entangled subsystems, the probabilities for fine-grained (nondegenerate) measurement outcomes of a subsystem whose reduced density matrix has its maximum von Neumann entropy become equal *in any basis*. This can be seen with the use of envariance, which is equivalent to a basis ambiguity of the subsystems. For instance, with the Bell state the probabilities for a particular particle to be spin up or down in the z direction are the same, while the probabilities for the particle to be spin left or right in the x or y direction are also the same. This is in contrast to an unentangled spin-1/2 particle where there always exists a direction where the spin is definitely known.

This last example is a special case of a phenomenon that does not apply to classical physics. The uncertainty principle is usually applied to pure states, but the situation changes for mixed states, such that the bounds on the uncertainties for incompatible observables become more strict. To show this, we utilize the concavity of the expression $-x\ln x$ (with $x \in \mathbb{R}^+$) [14] such that for a density matrix ρ and a fine-grained basis $\{|i\rangle\}$, $\langle i|j\rangle = \delta_{ij}$, $\sum_i |i\rangle\langle i| = \mathbb{I}$, the von Neumann entropy $S(\rho)$ has the property

$$S(\rho) := -\mathrm{tr}[\rho\ln\rho] = \sum_i \langle i| - \rho\ln\rho|i\rangle$$
$$\leqslant -\sum_i \langle i|\rho|i\rangle\ln\langle i|\rho|i\rangle. \quad (10)$$

Choosing $\{|i\rangle\}$ to be the eigenstates of a fine-grained observable, then $\langle i|\rho|i\rangle$ is the probability to measure outcome "i" such that the Shannon entropy of said observable (call it O) is given by

$$H_O(\rho) = -\sum_i \langle i|\rho|i\rangle\ln\langle i|\rho|i\rangle. \quad (11)$$

Thus,

$$S(\rho) \leqslant H_O(\rho). \quad (12)$$

For an alternative proof see Ref. [15].

This applies to all fine-grained observables of the system described by ρ. In the case of the Bell state, the reduced density matrix obtained by tracing out one of the particles has maximum von Neumann entropy such that all nontrivial observables of the subsystem also have maximum Shannon entropy. Thus if a subsystem contains quantum correlations with another, the information we have about the subsystem is more constrained than in the case of classical physics where Shannon entropies of "incompatible" observables are allowed to be independent.

IV. OBSERVABLE DYNAMICS OF COMPLETELY MIXED STATES

Intuitively, when one lacks knowledge of a system, one expects our ability to distinguish the dynamics of the system to be lessened. We have seen that in the case of mixed quantum states, our knowledge of the system is less than allowed classically.

We begin quantifying the distinguishability of dynamics of mixed states by extending envariance of completely mixed states to nonunitary operations. In this regard we choose to describe a quantum process in an operator-sum representation which maps density matrices to density matrices. A general physical operation on α can be described by a set [16] of operation elements $E_{k\alpha} \in \mathcal{H}_\alpha \otimes \bar{\mathcal{H}}_\alpha$, where $\bar{\mathcal{H}}_\alpha$ is the dual to $\mathcal{H}$. The operation is then given by

$$\mathcal{E}_\alpha(\rho) = \sum_{k=1}^{K} E_{k\alpha}\rho E_{k\alpha}^\dagger. \tag{13}$$

We shall be concerned with operations $\mathcal{E}_\alpha$, whose elements are trace preserving ($\sum_k E_{k\alpha}^\dagger E_{k\alpha} = \mathbb{I}_\alpha$) and also unital:

$$\sum_{k=1}^{K} E_{k\alpha}E_{k\alpha}^\dagger = \mathbb{I}_\alpha. \tag{14}$$

Let ρ_α be a completely mixed state of a system α, purified by system β,

$$\rho_\alpha := \mathrm{tr}_\beta[|\psi\rangle\langle\psi|] = \tfrac{1}{N}\mathbb{I}_\alpha. \tag{15}$$

All unital operations leave ρ_α invariant (see Appendix A), i.e.,

$$\mathcal{E}_\alpha(\rho_\alpha) = \rho_\alpha. \tag{16}$$

Some operations satisfying these conditions include (1) unitary $\mathcal{E}_U(\rho) = U\rho U^\dagger$, (2) perfect premeasurements $\mathcal{E}_P(\rho) = \sum_{k=1}^{K} P_k\rho P_k$, where P_k are projectors of a complete basis, and (3) combinations of unitary and perfect premeasurement operations, e.g., $\mathcal{E}_{UP}(\rho) = \mathcal{E}_U \circ \mathcal{E}_P(\rho)$. Interestingly, generalized measurements [16] where the outcome is unknown do not necessarily satisfy these conditions, e.g., for measurement operators of a two-level system $M_1 = |0\rangle\langle 0|$ and $M_2 = |0\rangle\langle 1|$, the left-hand side of Eq. (14) with $E_{i\alpha} = M_i$ does not equal unity; $M_1M_1^\dagger + M_2M_2^\dagger \neq \mathbb{I}$.

V. UPPER BOUND FOR GENERAL MIXED STATES

Let us now consider a general mixed state of α,

$$\tilde{\rho}_\alpha = \sum_{j=1}^{n} |c_j|^2 |\alpha_j\rangle\langle\alpha_j|. \tag{17}$$

A purification of α by β is given by the Schmidt decomposition (4), where now c_i and c_j may be equal for $i \neq j$. The group of envariant operations on $|\Omega\rangle$ is, in general, greatly reduced compared to the maximally entangled state. Thus the set of all operations that can be shown to leave $\tilde{\rho}_\alpha$ invariant by the use of envariance is also reduced. This limits the previous proof of the unobservability of the dynamics of α for cases where the state is not completely mixed.

Our proposal is that even with a large reduction in the set of symmetries, the original set may apply in a partial sense. The motivation is that a large reduction in the symmetry can occur with only a very small reduction in the von Neumann entropy of α [11]. Mixed states with less than maximum von Neumann entropy may still have some form of limitations on their observable dynamics for the full set of trace-preserving operations satisfying Eq. (14). This turns out to be the case.

To see this, we initially rewrite Ω. Let us extend the sum over j from 1 to N and define $c_j = 0$ for $n+1 \leqslant j \leqslant N$.[2] For $M < N$, we enlarge $\mathcal{H}_\beta$ until the dimensionalities are equal. We then decompose Ω into two parts, one that is maximally symmetric over $\mathcal{H}_\alpha$ and the rest of the state;

$$c_j = \frac{1}{\sqrt{N}} + d_j, \tag{18}$$

where $d_j := c_j - 1/\sqrt{N}$, such that

$$|\Omega\rangle = \frac{1}{\sqrt{N}}\sum_{j=1}^{N} e^{i\phi_j}|\alpha_j\rangle|\beta_j\rangle + \sum_{j=1}^{N} d_j e^{i\phi_j}|\alpha_j\rangle|\beta_j\rangle. \tag{19}$$

Define

$$|\Omega_1\rangle = \frac{1}{\sqrt{N}}\sum_{j=1}^{N} e^{i\phi_j}|\alpha_j\rangle|\beta_j\rangle, \tag{20}$$

$$Q|\Omega_2\rangle = \sum_{j=1}^{N} d_j e^{i\phi_j}|\alpha_j\rangle|\beta_j\rangle, \tag{21}$$

where the constant $Q = \sqrt{\sum_j d_j^2}$ is chosen such that Ω_2 is normalized to 1.

Our measure of the purity of α is given by Q. It is not equal to the usual measure of purity, which is $\mathrm{tr}[\tilde{\rho}_\alpha^2]$. One can consider Q^2 as the χ^2 value between the distribution of amplitudes $\{c_j\}$ and the constant distribution $1/\sqrt{N}$. It follows from $\sum_j c_j^2 = 1$ and $0 \leqslant c_j^2 \leqslant c_j$ that Q is bounded:

$$0 \leqslant Q \leqslant \sqrt{2 - 2/\sqrt{N}}. \tag{22}$$

The maximal value occurs for pure states of α, while $Q = 0$ corresponds to completely mixed states (all c_j equal).

We now utilize a measure of the distinguishability of quantum states, the trace distance, defined as

$$D(\rho,\sigma) := \tfrac{1}{2}\mathrm{tr}|\rho - \sigma|, \tag{23}$$

where ρ and σ are density matrices and $|X| := \sqrt{X^\dagger X}$ is the positive square root of $X^\dagger X$ (defined by taking a spectral decomposition $X^\dagger X = \sum_i e_i|x_i\rangle\langle x_i|$ and taking the positive square roots of the eigenvalues $\sqrt{X^\dagger X} = \sum_i \sqrt{e_i}|x_i\rangle\langle x_i|$).

[2]However, the choice of the size of the extension may be chosen to be smaller depending on whether $\mathcal{E}$ leaves certain subspaces of $\mathcal{H}_\alpha$ invariant.

It can be shown that [16]

$$D(\rho,\sigma) = \max_P \operatorname{tr}[P(\rho - \sigma)], \tag{24}$$

where P is a projector and the maximization is taken over all possible projectors. This gives a clear physical interpretation of the trace distance. If experimentalists wanted to distinguish whether they had state ρ or σ, the trace distance gives the maximum possible difference in probabilities for a projective measurement outcome for the two states. For instance, if for two states $D = 1$, it is, in principle, possible to do a projective measurement where the probability of getting a confirmatory result for one state is 1 while the other is 0, and hence only one measurement is ever needed to distinguish the states.

We are now in a position to derive an upper bound on $D_\alpha := D(\mathcal{E}_\alpha(\tilde{\rho}_\alpha),\tilde{\rho}_\alpha)$. Let the state $\rho_\Omega = |\Omega\rangle\langle\Omega|$ be acted on by $\mathcal{E}_\alpha$ as defined in Sec. IV. In the case where $Q = 0$, $|\Omega\rangle = |\Omega_1\rangle$ and

$$\mathcal{E}_\alpha(\rho_\Omega|_{Q=0}) = \mathcal{E}_\beta(\rho_\Omega|_{Q=0}), \tag{25}$$

where $\mathcal{E}_\beta(\rho) = \sum_{k=1}^K E_{k\beta}\rho E_{k\beta}^\dagger$ (cf. Appendix A).[3] As these two states are equal, they are indistinguishable. For the general case where Q may not be zero, a measure for the distinguishability of the two states can be given by the trace distance:

$$D_{\alpha\beta} := D(\mathcal{E}_\alpha(\rho_\Omega),\mathcal{E}_\beta(\rho_\Omega)). \tag{26}$$

In Appendix B, we show that $D_{\alpha\beta}$ satisfies the following bound:

$$D_{\alpha\beta} \leqslant 2\sqrt{1 - \left|1 - Q^2 + \tfrac{1}{4}Q^4\right|}. \tag{27}$$

This is related to D_α in the following way. If β is an ancilla subsystem used to purify α or the experimentalist does not have access to subsystem β, then we can ask about our ability to tell whether $\mathcal{E}_\alpha$ has happened at all. This can be quantified by

$$D(\operatorname{tr}_\beta[\mathcal{E}_\alpha(\rho_\Omega)],\operatorname{tr}_\beta[\mathcal{E}_\beta(\rho_\Omega)]) = D(\mathcal{E}_\alpha(\operatorname{tr}_\beta[\rho_\Omega]),\operatorname{tr}_\beta[\rho_\Omega]) = D_\alpha. \tag{28}$$

The partial trace over β is trace preserving, so D_α is bounded by $D_{\alpha\beta}$:

$$D_\alpha \leqslant D_{\alpha\beta} \leqslant 2\sqrt{1 - \left|1 - Q^2 + \tfrac{1}{4}Q^4\right|}. \tag{29}$$

For values of $Q < \sqrt{2-\sqrt{3}} \approx 0.5$, the right-hand side of (29) becomes less than 1 and hence bounds D_α. For $Q = 0$, (29) gives $D_\alpha = 0$, which is the same result achieved in Sec. IV. The upper bound on D_α given by (29) is our central result.

The nontrivial nature of this bound can be seen by considering cases where D_α is not bounded because Q is larger than $\sqrt{2-\sqrt{3}}$.

(1) Consider two bases for α, $\{|\alpha_k\rangle\}$ and $\{|\tilde{\alpha}_k\rangle\}$, such that $\langle\alpha_m|\tilde{\alpha}_m\rangle = 0$ for some m. Take a pure state $\sigma = |\alpha_m\rangle\langle\alpha_m|$. The purity as given by Q is then

$$Q = \sqrt{2 - 2/\sqrt{N}} > \sqrt{2-\sqrt{3}} \tag{30}$$

for $N \geqslant 2$. One can see that if α experiences a unitary transformation

$$U_\alpha = \sum_k |\tilde{\alpha}_k\rangle\langle\alpha_k|, \tag{31}$$

then the states have zero overlap:

$$\langle\alpha_m|U_\alpha\sigma U_\alpha^\dagger|\alpha_m\rangle \quad \text{or} \quad D(U_\alpha\sigma U_\alpha^\dagger,\sigma) = 1. \tag{32}$$

Thus the two states are, in principle, easily distinguishable.

(2) Suppose the system α, still given by the pure state σ, experiences a perfect premeasurement such that

$$\sigma \to \sum_{i=1}^N P_i\sigma P_i, \tag{33}$$

where $P_i = |A_i\rangle\langle A_i|$ and $\{|A_i\rangle\}$ forms a complete orthonormal basis for $\mathcal{H}_\alpha$ such that $|\alpha_m\rangle = \sum_{i=n}^N (1/\sqrt{N})|A_i\rangle$. It is convenient to use the definition of fidelity [16] for density matrices ρ and τ,

$$F(\rho,\tau) := \operatorname{tr}[\sqrt{\rho^{1/2}\tau\rho^{1/2}}], \tag{34}$$

to obtain

$$\left[F\left(\sigma,\sum_i P_i\sigma P_i\right)\right]^2 = \langle\alpha_m|\sum_i P_i\sigma P_i|\alpha_m\rangle = \frac{1}{N}. \tag{35}$$

In this case, the fidelity bounds the trace distance

$$1 - \left[F\left(\sigma,\sum_i P_i\sigma P_i\right)\right]^2 = 1 - \frac{1}{N} \leqslant D\left(\sigma,\sum_i P_i\sigma P_i\right). \tag{36}$$

Thus, the observability of the process $D(\sigma,\sum_i P_i\sigma P_i)$ tends to 1 as N tends to ∞.

Finally, we note that the bound may be extended to mixed states of a composite α and β system. Let

$$\bar{\rho} = \sum_m r_m\rho_m, \tag{37}$$

where $\sum_m r_m = 1$ and ρ_m is a pure density matrix of the composite system $\forall\, m$. Define Q_m as the Q measure of the purity of the $\operatorname{tr}_\beta(\rho_m)$ state and define $\mathcal{E}_\alpha$ and $\mathcal{E}_\beta$ in the usual way. Then, using the convexity of the trace distance,

$$\begin{aligned} D(\mathcal{E}_\alpha(\bar{\rho}),\mathcal{E}_\beta(\bar{\rho})) &\leqslant \sum_m r_m D(\mathcal{E}_\alpha(\rho_m),\mathcal{E}_\beta(\rho_m)) \\ &\leqslant 2\sum_m r_m\sqrt{1 - \left|1 - Q_m^2 + \frac{1}{4}Q_m^4\right|}. \end{aligned} \tag{38}$$

Thus, the distinguishability of the dynamics of α in a mixed composite state is bounded by the average of the bounds of the pure states ρ_m.

VI. REMARKS

We have shown that given a trace-preserving unital operation, the trace distance between the transformed state and its original is bounded by (29) given that the purity is small enough ($Q \lesssim 0.5$). For maximally mixed states where $Q = 0$, the bound implies the operation must be unobservable.

[3]For cases where $\mathcal{H}_\beta$ has to be enlarged and β is considered a real subsystem, $\mathcal{E}_\beta$ may not strictly be physical.

The bound (29) is motivated in Secs. III and IV on the intuition that lack of knowledge of a state leads to lack of an ability to distinguish the dynamics. We note that trace-preserving, unital operations cannot decrease the von Neumann entropy [17]. This leads us to ask whether the class of trace-preserving, unital operations is the largest such class where (29) or a stronger bound holds that depends only on the purity of the input state.

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APPENDIX A: SYMMETRIES OF COMPLETELY MIXED STATES

The proof of the invariance of ρ_α under unital operations is trivial. Here we provide an alternative proof which gives the tools needed for Sec. V. The first step is to extend the symmetry of the second version of (2) for pure state (3) to $\mu_\alpha|\psi\rangle = \mu_\beta|\psi\rangle$, where $\mu_\alpha := \bar{\mu}_\alpha \otimes \mathbb{I}_\beta$ is a general linear operation on pure states. The operations $\bar{\mu}_\alpha$ could, for instance, be a projector onto a subspace of $\mathcal{H}_\alpha$. Also, suppose $|\psi\rangle$ is maximally entangled with respect to subsystems α and β. Since μ_α acts identically on subsystem β, it follows that

$$\mu_\alpha|\psi\rangle = \frac{1}{\sqrt{N}}\sum_{j=1}^{N} e^{i\phi_j}(\bar{\mu}_\alpha|\alpha_j\rangle)|\beta_j\rangle. \tag{A1}$$

Define $\langle\alpha_i|\bar{\mu}_\alpha|\alpha_j\rangle := \mu_{ij}$ such that

$$\begin{aligned}\mu_\alpha|\psi\rangle &= \frac{1}{\sqrt{N}}\sum_{j=1}^{N} e^{i\phi_j}\left(\sum_{i=1}^{N}\mu_{ij}|\alpha_i\rangle\right)|\beta_j\rangle \\ &= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}|\alpha_i\rangle\left(\sum_{j=1}^{N} e^{i\phi_j}\mu_{ij}|\beta_j\rangle\right) \\ &= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}|\alpha_i\rangle|\tilde{\beta}_i\rangle,\end{aligned} \tag{A2}$$

where $|\tilde{\beta}_i\rangle := \sum_{j=1}^{N} e^{i\phi_j}\mu_{ij}|\beta_j\rangle$. For nonunitary μ_α, $\{|\tilde{\beta}_j\rangle\}$ need not be orthonormal.[4]

Define

$$\mu_\beta := \mathbb{I}_\alpha \otimes \sum_{k=1}^{N} e^{i\phi_k}|\tilde{\beta}_k\rangle\langle\beta_k| := \mathbb{I}_\alpha \otimes \bar{\mu}_\beta, \tag{A3}$$

$$\therefore \mu_\alpha|\psi\rangle = \mu_\beta|\psi\rangle. \tag{A4}$$

This symmetry is not equivalent to envariance as μ_β may not be invertible, and hence, in general, there does *not* exist a μ_β^{-1} such that $\mu_\alpha\mu_\beta^{-1}|\psi\rangle = |\psi\rangle$. With Eq. (A4), we can consider symmetry properties of completely mixed states. Let the completely mixed state be a state of system α;

$$\rho_\alpha := \mathrm{tr}_\beta[|\psi\rangle\langle\psi|] = \frac{1}{N}\mathbb{I}_\alpha. \tag{A5}$$

Suppose we have a quantum operation $\mathcal{E}_\alpha(\rho) = \rho'$ that is given in an operator-sum representation;

$$\mathcal{E}_\alpha(\rho) = \sum_{k=1}^{K} E_{k\alpha}\rho E_{k\alpha}^\dagger, \tag{A6}$$

where $E_{k\alpha}$ are linear maps $E_{k\alpha} : \mathcal{H}_\alpha \to \mathcal{H}_\alpha$. The effect of $\mathcal{E}_\alpha$ upon ρ_α is then

$$\begin{aligned}\mathcal{E}_\alpha(\rho_\alpha) &= \sum_{k=1}^{K} E_{k\alpha}\mathrm{tr}_\beta\,[|\psi\rangle\langle\psi|]\,E_{k\alpha}^\dagger \\ &= \mathrm{tr}_\beta\left[\sum_{k=1}^{K} E_{k\alpha}|\psi\rangle\langle\psi|E_{k\alpha}^\dagger\right].\end{aligned} \tag{A7}$$

Since $E_{k\alpha}$ is of the form $\bar{\mu}_\alpha$, there exists an $E_{k\beta}$ such that $E_{k\alpha}|\psi\rangle = E_{k\beta}|\psi\rangle$ and, similarly, $\langle\psi|E_{k\alpha}^\dagger = \langle\psi|E_{k\beta}^\dagger$. Then

$$\begin{aligned}\mathcal{E}_\alpha(\rho_\alpha) &= \mathrm{tr}_\beta\left[\sum_{k=1}^{K} E_{k\beta}|\psi\rangle\langle\psi|E_{k\beta}^\dagger\right] \\ &= \mathrm{tr}_\beta\left[\sum_{k=1}^{K} E_{k\beta}^\dagger E_{k\beta}|\psi\rangle\langle\psi|\right].\end{aligned} \tag{A8}$$

If $\sum_{k=1}^{K} E_{k\beta}^\dagger E_{k\beta} = \mathbb{I}_\beta$, then

$$\mathcal{E}_\alpha(\rho_\alpha) = \rho_\alpha. \tag{A9}$$

The completely mixed state is then invariant under $\mathcal{E}_\alpha$ if $\mathcal{E}_\alpha$ satisfies the condition $\sum_{k=1}^{K} E_{k\beta}^\dagger E_{k\beta} = \mathbb{I}_\beta$, which is equivalent to the unital condition

$$\sum_{k=1}^{K} E_{k\alpha}E_{k\alpha}^\dagger = \mathbb{I}_\alpha. \tag{A10}$$

APPENDIX B: PROOF OF UPPER BOUND OF TRACE DISTANCE

To prove Eq. (27), we shall need a few identities. Using the definition of d_i and taking Ω to be normalized, we find

$$\begin{aligned}\sum_{i=1}^{N} c_i^2 = 1 &= \sum_{i=1}^{N}\left(\frac{1}{\sqrt{N}} + d_i\right)^2 \\ &= 1 + \sum_{i=1}^{N} d_i^2 + \frac{2}{\sqrt{N}}\sum_{i=1}^{N} d_i, \\ \therefore \sum_{i=1}^{N} d_i^2 &= -\frac{2}{\sqrt{N}}\sum_{i=1}^{N} d_i.\end{aligned} \tag{B1}$$

We also compute the overlap

$$\langle\Omega_1|\Omega_2\rangle = \frac{1}{Q\sqrt{N}}\sum_{i=1}^{N} d_i = -\frac{1}{2}Q, \tag{B2}$$

using Eq. (B1). Defining

$$\begin{aligned}\rho_{11} &:= |\Omega_1\rangle\langle\Omega_1|, \quad \rho_{12} := |\Omega_1\rangle\langle\Omega_2|, \\ \rho_{21} &:= |\Omega_2\rangle\langle\Omega_1|, \quad \rho_{22} := |\Omega_2\rangle\langle\Omega_2|,\end{aligned} \tag{B3}$$

[4] Many $|\tilde{\beta}_j\rangle$ may even be the null state.

we have

$$\rho = |\Omega\rangle\langle\Omega| = \rho_{11} + Q(\rho_{12} + \rho_{21}) + Q^2\rho_{22}. \tag{B4}$$

The ρ's have some useful relationships, namely,

$$\rho_{11}^2 = \rho_{11}, \quad \rho_{22}^2 = \rho_{22}, \quad \rho_{12}\rho_{21} = \rho_{11},$$
$$\rho_{12}\rho_{12} = -\frac{Q}{2}\rho_{12}, \quad \rho_{21}\rho_{12} = \rho_{22}, \rho_{21}\rho_{21} = -\frac{Q}{2}\rho_{21}. \tag{B5}$$

The trace distance has particular properties which shall also be used. Thus,

$$\begin{aligned} D_{\alpha\beta} &:= D(\mathcal{E}_\alpha(\rho),\mathcal{E}_\beta(\rho)) \\ &\leqslant D(\mathcal{E}_\alpha(\rho),\mathcal{E}_\alpha(\rho_{11})) + D(\mathcal{E}_\alpha(\rho_{11}),\mathcal{E}_\beta(\rho)) \\ &= D(\mathcal{E}_\alpha(\rho),\mathcal{E}_\alpha(\rho_{11})) + D(\mathcal{E}_\beta(\rho_{11}),\mathcal{E}_\beta(\rho)) \\ &\leqslant D(\rho,\rho_{11}) + D(\rho_{11},\rho) = 2D(\rho_{11},\rho). \end{aligned} \tag{B6}$$

Because ρ and ρ_{11} are pure,

$$D(\rho_{11},\rho) = \sqrt{1 - [F(\rho_{11},\rho)]^2}, \tag{B7}$$

where F is the fidelity;

$$\begin{aligned} F(\rho_{11},\rho) &= \mathrm{tr}\big[\big(\rho_{11}^{1/2}\rho\rho_{11}^{1/2}\big)^{1/2}\big] = \mathrm{tr}[(\rho_{11}\rho\rho_{11})^{1/2}] \\ &= \mathrm{tr}\big[(\rho_{11}[\rho_{11} + Q(\rho_{12}+\rho_{21}) + Q^2\rho_{22}]\rho_{11})^{1/2}\big] \\ &= \mathrm{tr}\big[\big(\rho_{11} - \tfrac{1}{2}Q^2\rho_{11} - \tfrac{1}{2}Q^2\rho_{11} + \tfrac{1}{4}Q^4\rho_{11}\big)^{1/2}\big] \\ &= \sqrt{1 - Q^2 + \tfrac{1}{4}Q^4}. \end{aligned} \tag{B8}$$

Thus,

$$D_{\alpha\beta} \leqslant 2\sqrt{1 - \left|1 - Q^2 + \tfrac{1}{4}Q^4\right|}. \tag{B9}$$

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