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The Jacobian for solving water distribution system equations with the Darcy-Weisbach head loss model

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Abstract

The widely used Todini and Pilati method for solving the equations which model water distribution systems was originally developed for pipes in which the head loss is modeled by the Hazen-Williams formula. The friction factors in this formula are independent of flow. Rossman's popular program EPANET implements elements of the Todini and Pilati algorithm, but in the case where the Darcy-Weisbach head loss formula is used, does not take into account the dependence of the friction factors on Reynolds number, and therefore flow, in computing the Jacobian. In this Technical Note we present the correct Jacobian matrix formulae which must be used in order to fully account for the friction factor's dependence on flow when the Todini and Pilati method is applied with Darcy-Weisbach head loss formula. With the correct Jacobian matrix the Todini and Pilati implementation of Newton's method has its normally quadratic convergence restored.

The new formulae are demonstrated with an illustrative example.

INTRODUCTION

This paper considers a new way to deal with the Darcy-Weisbach friction factor in computing the Jacobian matrix when solving the pipe network equations for a water distribution system. In their paper, Todini & Pilati (1988) gave consideration only to the head loss equation of Hazen and Williams, where the Hazen-Williams coefficient is assumed to be independent of flow. Rossman (2000) in his EPANET program, implemented elements of the Todini and Pilati algorithm, but also provided a choice of head loss equations for the solution of flow and pressure in water distribution

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systems. In particular, he incorporated the Darcy-Weisbach formula where the friction factor, f, is dependent on the Reynolds number and the relative roughness. However, while solving the pipe network equations in which the head loss is modeled by the Darcy-Weisbach formula, Rossman (2000) treated the friction factors as independent of flow when computing the Jacobian. In this paper, we give the correct formulae required to take into account this variation of friction factor with flow in the computation of the Jacobian matrix. Using the correct Jacobian for this case restores the normally quadratic convergence of the method.

Results from the application of the new technique to an example network shows a smaller final error for the same number of iterations.

THE DARCY-WEISBACH PIPE HEAD LOSS EQUATION

We consider a water distribution network which has n_p pipes, n_j variable-head nodes and n_f fixed-head nodes. The head loss in all pipes in a network is assumed to be modeled by the Darcy-Weisbach formula so the relation between the heads at two ends of a pipe and the flow is

$$H_i - H_k = \begin{cases} r_j Q_j, & \text{for laminar flow,} \\ r_j Q_j |Q_j|^{n-1}, & \text{for turbulent flow} \end{cases} \tag{1}$$

where Q_j is the flow in pipe p_j, H_i is the HGL or head at node i and $n = 2$. The pipe resistance factors are

$$r_j = \begin{cases} \dfrac{128\nu}{\pi g} \dfrac{L_j}{D_j^4}, & \text{for laminar flow } r_j \text{ is independent of } Q_j, \\ \dfrac{8}{\pi^2 g} \dfrac{L_j f_j}{D_j^5}, & \text{for turbulent flow } f_j \text{ depends on } Q_j. \end{cases} \tag{2}$$

where ν is the kinematic viscosity of water at a given temperature, g is the gravitational acceleration constant, L_j is the pipe length, D_j is the pipe diameter and f_j is the Darcy-Weisbach friction factor.

For turbulent flow the friction factor for each pipe depends on both the *relative roughness*, ϵ_j/D_j, which is the ratio of the pipe roughness ϵ_j to the diameter, and the Reynolds number, $\mathcal{R}$, which is defined as $\mathcal{R} = VD/\nu$, where V is the average fluid velocity. We also define a diagonal matrix, $\boldsymbol{G} = \boldsymbol{G}(\boldsymbol{q}, \boldsymbol{r}) \in \mathbb{R}^{n_p \times n_p}$ by

$$[\boldsymbol{G}]_{jj} = \begin{cases} r_j, & \text{for laminar flow} \\ r_j |Q_j|^{n-1}, & \text{for turbulent flow} \end{cases} , j = 1, 2, \ldots, n_p, \tag{3}$$

where we define the vectors $\boldsymbol{q} = (Q_1, Q_2, \ldots, Q_{n_p})^T$ of the unknown flows and $\boldsymbol{r} = (r_1, r_2, \ldots, r_{n_p})^T$ of the resistance factors.

THE NETWORK EQUATIONS

The topological matrices, $\boldsymbol{A}_1 \in \mathbb{R}^{n_p \times n_j}$, the unknown head node incidence matrix, and $\boldsymbol{A}_2 \in \mathbb{R}^{n_p \times n_f}$, the fixed head node incidence matrix are defined by

$$[\boldsymbol{A}_{1,2}]_{ji} = \begin{cases} -1 & \text{if the flow in pipe } j \text{ enters the node } i, \\ 0 & \text{if pipe } j \text{ does not connect to the node } i, \\ 1 & \text{if the flow in pipe } j \text{ leaves the node } i. \end{cases}$$

We also define $\boldsymbol{O}$ as an n_j square, zero matrix and $\boldsymbol{o}$ as an $n_p \times n_j$ zero matrix.

For the nodes we define the vectors:

$\boldsymbol{h} = (H_1, H_2, \ldots, H_{n_j})^T$, the unknown heads; $\boldsymbol{d}_m \in \mathbb{R}^{n_j \times 1}$, the known nodal demands; and $\boldsymbol{e}_\ell \in \mathbb{R}^{n_f \times 1}$, the fixed head elevations.

The energy and continuity equations describing the flows and nodal heads in a water distribution system are, expressed in matrix form,

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} \boldsymbol{G} & -\boldsymbol{A}_1 \\ -\boldsymbol{A}_1^T & \boldsymbol{O} \end{pmatrix} \begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{h} \end{pmatrix} - \begin{pmatrix} \boldsymbol{A}_2 \boldsymbol{e}_\ell \\ \boldsymbol{d}_m \end{pmatrix} = \boldsymbol{o} \tag{4}$$

if we denote by $\boldsymbol{x}$ the $n_p + n_j$ dimensional real vector of unknown flows and heads in the system as $\boldsymbol{x} = (\boldsymbol{q}^T, \boldsymbol{h}^T)^T$.

The matrix in (4) has important structural properties: it is always symmetric, it is sparse whenever the network is large, the (1,1) block is diagonal and the (2,2) block is zero. These properties give an advantage when designing algorithms to solve the system of equations in (4).

The matrix $\boldsymbol{A}_1$ is constant but $\boldsymbol{G}$ depends on the unknown pipe flows in $\boldsymbol{q}$ (except for the case of laminar flows) and this makes the system in (4) non–linear.

Systems of non-linear equations $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$ such as those in (4) are frequently solved by Newton's iterative method

$$\boldsymbol{J}(\boldsymbol{x}^{(m)})(\boldsymbol{x}^{(m+1)} - \boldsymbol{x}^{(m)}) = -\boldsymbol{f}(\boldsymbol{x}^{(m)}),\ \boldsymbol{x}^{(0)} \text{ prescribed },\ m = 0, 1, 2, \ldots \tag{5}$$

where $\boldsymbol{J}$ is the Jacobian of $\boldsymbol{f}$ and m is the iteration number. For $\boldsymbol{f}(\boldsymbol{x})$, the vector function of the vector variable $\boldsymbol{x}$ in (4), we denote by $\nabla_x \boldsymbol{f}$ the gradient of $\boldsymbol{f}$ with respect to $\boldsymbol{x}$. In view of the fact that $\boldsymbol{A}_1$, $\boldsymbol{A}_2\boldsymbol{e}_\ell$ and $\boldsymbol{d}_m$ are constant with respect to $\boldsymbol{q}$ and $\boldsymbol{h}$, the Jacobian for the system (4) is

$$\boldsymbol{J} = \begin{pmatrix} \nabla_q (\boldsymbol{G}\boldsymbol{q}) & -\boldsymbol{A}_1 \\ -\boldsymbol{A}_1^T & \boldsymbol{O} \end{pmatrix}. \tag{6}$$

NUMERICAL CONSIDERATIONS

The calculations in this paper have been performed using two programs: one written by the authors in Matlab (Mathworks 2008), and EPANET V2.00.12 written by L. Rossman. Both codes use IEEE standard double precision arithmetic with precision, measured by machine epsilon (Forsythe & Moler 1967) $\epsilon_{mach} \approx 2 \times 10^{-16}$. The EPANET program was modified in only three ways:

(a) by replacing those constants in the code which are given to a smaller number of decimal places by the same constants to full 20 decimal digit accuracy, as shown below,

```
#define A1      3.14159265358979323850e+03  /* 1000*PI */
#define A2      1.57079632679489661930e+03  /* 500*PI  */
#define A3      5.02654824574366918160e+01  /* 16*PI   */
#define A4      6.28318530717958647700e+00  /* 2*PI    */
#define A8      4.61841319859066668690e+00  /* 5.74*(PI/4)^.9 */
#define A9     -8.68588963806503655300e-01  /* -2/ln(10)      */
#define AA     -1.56346013485170657950e+00  /* -2*.9*2/ln(10) */
#define AB      3.28895476345399058690e-03  /* 5.74/(4000^.9) */
#define AC     -5.14214965799093883760e-03  /* AA*AB */
```

(b) by outputting certain intermediate quantities which are normally not available to the user.

(c) by allowing stopping tolerances smaller than 10^{-5} (the current built–in minimum stopping tolerance) to be set by the user. This was done to allow better comparisons of the accuracy between the authors' Matlab code and EPANET.

To verify our implementation, we configured our Matlab code to mirror the convergence criterion and the Jacobian calculation used in EPANET. The Matlab code took the same number of iterations (± 1) as EPANET in these tests. In addition, the solutions obtained by the Matlab and EPANET codes agreed in norm to better than $10\delta_{stop}$ in all cases and the residuals were all better that δ_{stop}. The Matlab code, when configured to mirror the EPANET calculations produced results at each step which agree with those of EPANET to better than 13 decimals of accuracy.

The iterations were terminated using the same criterion as is applied in EPANET when there are no pumps or valves (the case considered in this paper): terminate execution when the relative change in flow

$$\phi\left(\boldsymbol{q}^{(m+1)}\right) = \sum_{k=1}^{n_p} \left| Q_k^{(m+1)} - Q_k^{(m)} \right| \Big/ \sum_{k=1}^{n_p} \left| Q_k^{(m+1)} \right| \leq \delta_{stop}, \tag{7}$$

a predetermined stopping tolerance.

THE JACOBIAN FOR THE DARCY-WEISBACH FORMULATION

We now consider the resistance factors for the Darcy-Weisbach head loss model in more detail.

The Reynolds number $\mathcal{R}$ may be expressed in terms of discharge as $\mathcal{R} = 4|Q|/(\pi\nu D)$ and falls into the three ranges of interest shown in the first column of Table 1 (although many would consider a Reynolds number of 2300 as the limit of the laminar flow region, this choice was made to maintain consistency with EPANET). For these three ranges of $\mathcal{R}$ we have the following formulae for friction factors, resistance factors and terms on the diagonal of the matrix $\boldsymbol{G}$:

Case 1: Laminar flow $\mathcal{R} \leq 2000$ For this range of $\mathcal{R}$ the Hagen-Poiseuille formula (Bhave 1991) $f = 16\pi\nu D/|Q|$ is applicable and so the term on the diagonal of the matrix $\boldsymbol{G}$ in (3) is just $[\boldsymbol{G}]_{jj} = r_j$. Importantly, this term does not depend on the pipe flow.

Case 2: Transitional flow $2000 < \mathcal{R} < 4000$ We use Dunlop's interpolating cubic splines (Dunlop 1991) (expressed in a slightly different form) in order to ensure a smooth transition of the friction factors from laminar to turbulent flow in this Reynolds number range. The following representation gives exactly the same Dunlop cubic spline approximation as that used in EPANET and which is discussed on pages 189-190 in the EPANET User's Manual (Rossman 2000): $f = \sum_{k=0}^{3} (\alpha_k + \beta_k/\theta)\,\eta^k$ where α_k, β_k are defined in Table 2, and where we have introduced the new variables $\eta = \mathcal{R}/2000$,

$$\theta = \frac{\epsilon}{3.7D} + \frac{5.74}{\mathcal{R}^{9/10}} = \frac{b\epsilon}{D} + c\left|\frac{D}{Q}\right|^{9/10}, \quad \hat{\theta} = \frac{b\epsilon}{D} + \frac{5.74}{4000^{9/10}}. \tag{8}$$

where $b = 1/3.7$ and $c = 5.74\,(\pi\nu/4)^{9/10}$. Note that η, θ, α_k and β_k all depend on ϵ and D and so are different for each pipe. With this representation the term on the diagonal of the matrix $\boldsymbol{G}$ in (3) is that given in the third column of Table 3 for this case.

Case 3: Turbulent flow $\mathcal{R} \geq 4000$ The Darcy-Weisbach friction factor can be estimated by the Swamee and Jain approximation (Swamee & Jain 1976) to the Colebrook–White formula:

$$f = \frac{0.25}{\left[\log_{10}\left(\epsilon/3.7D + 5.74/\mathcal{R}^{0.9}\right)\right]^2} = \frac{\ln^2 10}{4\ln^2\theta} \tag{9}$$

with θ defined in (8). The term on the diagonal of the matrix $\boldsymbol{G}$ in (3) is therefore that given in the third column of Table 3 for this case.

In both Cases 2 and 3, r_j depends on the flow. Table 3 summarizes the formulae for the diagonal elements of the matrix $\boldsymbol{G}$ for Cases 1,2 and 3.

INCLUDING THE FLOW DEPENDENCE OF FRICTION FACTORS

Todini & Pilati (1988) use the Hazen-Williams head loss formulae and so in the formulation of the Newton method to solve (4), the resistance factors r_j are treated as constant with respect to the flows Q_j. EPANET, (Rossman 2000) which implements the Todini-Pilati method for the Hazen-Williams head loss model allows also for the Darcy-Weisbach head loss formula. As noted earlier the Darcy-Weisbach friction formula for turbulent flow is dependent on the fluid Reynolds number which itself depends on the unknown flow. When using the Darcy-Weisbach head loss formula, EPANET appears to use a Jacobian with a (1,1) block which is $n\boldsymbol{G}$. Using this value for the (1,1) block in the Jacobian does not correctly account for the dependence of the friction factors on flow for this case and this means it is no longer a true Newton method. Nevertheless, the iteration can still give the correct solution, if it converges, but at a linear convergence rate rather than the quadratic rate that is in general characteristic of the Newton method (Isaacson & Keller 1966). In fact, ignoring the dependence of the friction factors on flow in computing the Jacobian is roughly equivalent to using a variation of the Newton method called the *chord method* (Isaacson & Keller 1966) where an approximation to the true Jacobian is used in place of the exact form.

To implement a true Newton method either as it stands or as set out by Todini and Pilati but with the Darcy-Weisbach head loss formula for turbulent flow one needs to add terms to the (1,1) block, $n\boldsymbol{G}$, of the Jacobian which account for the dependence of the resistance factors on the flows.

We now derive the correct terms for the Jacobian when head loss is modeled by the Darcy-Weisbach formula and demonstrate the quadratic convergence which is restored when these terms are included.

In computing the Jacobian of $\boldsymbol{f}(\boldsymbol{x})$ in (4) we need the differential of the quantity $\boldsymbol{Gq}$. From (3) we see that the vector $\boldsymbol{Gq}$ has elements given by $[\boldsymbol{Gq}]_j = r_j Q_j$ for laminar flow and $[\boldsymbol{Gq}]_j = r_j Q_j |Q_j|$ for turbulent flow. Thus, (dropping subscripts where there is no ambiguity and remembering that the resistance factor r for laminar flow is independent of flow) the differential, with respect to Q, of an element of $\boldsymbol{Gq}$ is

$$\frac{\partial[\boldsymbol{Gq}]_j}{\partial Q} = \begin{cases} \frac{\partial}{\partial Q} rQ = r & \text{laminar flow} \\ \frac{\partial}{\partial Q} rQ|Q| = 2r|Q| + \frac{\partial r}{\partial Q} Q|Q| & \text{turbulent flow} \end{cases}.$$

Thus, for turbulent flow, a term $\frac{\partial r}{\partial Q} Q|Q|$ must be added to the diagonal element of the (1,1) block, $n\boldsymbol{G}$, of the Hazen-Williams Jacobian. The Darcy-Weisbach Jacobian matrix (6) now has $\nabla_q(\boldsymbol{Gq}) = \boldsymbol{F}$ where the n_p diagonal terms in $\boldsymbol{F}$ include the correction term. The linear system which must be solved at each iteration of the Newton method is now seen to be

$$\begin{pmatrix} \boldsymbol{F} & -\boldsymbol{A}_1 \\ -\boldsymbol{A}_1^T & \boldsymbol{O} \end{pmatrix} \begin{pmatrix} \boldsymbol{q}^{(m+1)} \\ \boldsymbol{h}^{(m+1)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{F}-\boldsymbol{G} & \boldsymbol{o} \\ \boldsymbol{o}^T & \boldsymbol{O} \end{pmatrix} \begin{pmatrix} \boldsymbol{q}^{(m)} \\ \boldsymbol{h}^{(m)} \end{pmatrix} + \begin{pmatrix} \boldsymbol{A}_2\boldsymbol{e}_\ell \\ \boldsymbol{d}_m \end{pmatrix} \tag{10}$$

and the block equations of this system are

$$\boldsymbol{Fq}^{(m+1)} - \boldsymbol{A}_1\boldsymbol{h}^{(m+1)} = (\boldsymbol{F}-\boldsymbol{G})\boldsymbol{q}^{(m)} + \boldsymbol{A}_2\boldsymbol{e}_\ell, \tag{11}$$

$$-\boldsymbol{A}_1^T\boldsymbol{q}^{(m+1)} = \boldsymbol{d}_m. \tag{12}$$

We now derive the terms for the elements of $\boldsymbol{F}$ for the three ranges of Reynolds numbers and display them in Table 4.

Case 1: Laminar flow $\mathcal{R} \le 2000$ For this range of $\mathcal{R}$ we have $\frac{\partial r}{\partial Q} = 0$ and so $[\boldsymbol{F}]_{jj} = [\boldsymbol{G}]_{jj}$.

Case 2: Transitional flow $2000 < \mathcal{R} < 4000$ Denote $t_k = (\alpha_k + \beta_k/\theta)\,\eta^k$. The differential of t_k with respect to Q is

$$\frac{\partial t_k}{\partial Q} = \frac{1}{Q}\left\{\frac{9c}{10}\frac{\beta_k}{\theta^2}\left|\frac{D}{Q}\right|^{9/10} + k\,(\alpha_k + \beta_k/\theta)\right\}\eta^k$$

and so

$$\frac{\partial r}{\partial Q}Q|Q| = \left(\frac{8}{\pi^2 g}\right)\frac{L}{D^5}|Q|\sum_{k=0}^{3}\left\{\frac{9c}{10}\frac{\beta_k}{\theta^2}\left|\frac{D}{Q}\right|^{9/10} + k\,(\alpha_k + \beta_k/\theta)\right\}\eta^k \tag{13}$$

from which the term shown in Table 4 follows.

Case 3: Turbulent flow $\mathcal{R} \ge 4000$ Using (9) and (2) we get

$$\frac{\partial r}{\partial Q} = \left(\frac{18c\ln^2 10}{5\pi^2 g}\right)\frac{L}{D^5}\frac{1}{Q}\frac{1}{\theta\ln^3\theta}\left|\frac{D}{Q}\right|^{9/10}$$

and again the term shown in Table 4 follows.

Let us denote the n_j–square matrix (*Schur complement*) $\boldsymbol{V} = \boldsymbol{A}_1^T\boldsymbol{F}^{-1}\boldsymbol{A}_1$. Multiplying (11) on the left by $-\boldsymbol{A}_1^T\boldsymbol{F}^{-1}$, using (12) and rearranging gives

$$\boxed{\boldsymbol{h}^{(m+1)} = \boldsymbol{V}^{-1}\left[-\boldsymbol{d}_m + \boldsymbol{A}_1^T\boldsymbol{F}^{-1}\left((\boldsymbol{G}-\boldsymbol{F})\boldsymbol{q}^{(m)} - \boldsymbol{A}_2\boldsymbol{e}_\ell\right)\right]} \tag{14}$$

which leads, using (11) again, to

$$\boxed{\boldsymbol{q}^{(m+1)} = \boldsymbol{q}^{(m)} + \boldsymbol{F}^{-1}\boldsymbol{A}_1\boldsymbol{h}^{(m+1)} - \boldsymbol{F}^{-1}\left(\boldsymbol{G}\boldsymbol{q}^{(m)} - \boldsymbol{A}_2\boldsymbol{e}_\ell\right).} \tag{15}$$

These equations, with $\boldsymbol{F}$ as defined in Table 4, are the Todini-Pilati implementation of the true Newton method with the correct Jacobian where the Darcy-Weisbach head loss model is used. We now demonstrate the effect of including the correction terms on an illustrative example.

Example 1 *The network shown in Figure 1 has pipe and node parameters as shown in Table 5. In addition, columns five and nine of Table 5 show the steady state flows as determined by the Todini-Pilati method. All pipes have head loss modeled by the Darcy-Weisbach formula.*

In Table 6 we show the convergence data for two runs of the Todini-Pilati method as described by equations (14) and (15): the first using EPANET with the (uncorrected) Jacobian with $n\boldsymbol{G}$ *as the (1,1) block and updating of the* r *factors after each iteration, and the second using the full derivative terms shown Table 4 on the diagonal of the (corrected) Jacobian.*

Column two of Table 6 shows the EPANET error measure $\phi_u(\boldsymbol{q})$*, the relative flow measured by (7), for the uncorrected Jacobian and the third column shows error measure* $\phi_c(\boldsymbol{q})$*, for the corrected Jacobian.*

With a stopping tolerance of $\delta_{stop} = 10^{-6}$ *EPANET terminates after six iterations with an accuracy of* 3.6×10^{-8} *while using the corrected Jacobian obtains an accuracy of* 1.3×10^{-14}.

Quadratic convergence is often characterized by an approximate (asymptotic) doubling of the number of correct decimals as the solution is approached. We see that the reduction in errors shown in Table 6 is consistent with linear convergence for the case of flow independent friction factors (column two) and is consistent with quadratic convergence for the case of flow dependent friction factors (column three).

CONCLUSIONS

A method for the computation of the Jacobian matrix in the case where the head loss is modeled by the Darcy-Weisbach formula is proposed. The new method is based on taking full account of the variation of the Darcy-Weisbach friction factor with flow when computing the Jacobian elements.

The method is demonstrated on an example network and shows an improvement over the accuracy obtained when not fully accounting for the dependence of the friction factor on flow in the computation of the Jacobian.

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NOMENCLATURE

$\boldsymbol{A}_1$ = unknown head node incidence matrix

$\boldsymbol{A}_2$ = fixed head node incidence matrix

$b = 1/3.7$

$c = 5.74\,(\pi\nu/4)^{9/10}$

D_j = diameter of pipe j

$\boldsymbol{d}_m$ = vector of nodal demands

$\boldsymbol{e}_\ell$ = vector of fixed head elevations

$\boldsymbol{F}$ = diagonal matrix with elements defined by Table 4

f_j = Darcy-Weisbach friction factor for pipe j

$\boldsymbol{f}(\boldsymbol{x})$ = function for the energy and continuity equations

$\boldsymbol{G}$ = diagonal matrix with elements r_j or $r_j|Q_j|^{n-1}$

g = gravitational acceleration constant

H_i = head at node i

$\boldsymbol{h} = (H_1, H_2, \ldots, H_{n_j})^T$ = vector of heads

$\boldsymbol{J}$ = Jacobian matrix

k = counter variable

L_j = length of pipe j

n = head loss equation exponent

n_f = number of fixed–head nodes

n_j = number of variable–head nodes

n_p = number of pipes

$\boldsymbol{O}$ = n_j–square zero matrix

$\boldsymbol{o} = n_p \times n_j$ zero matrix

p_j = pipe j

Q_j = flow in pipe j

$\boldsymbol{q} = (Q_1, Q_2, \ldots, Q_{n_p})^T$ = vector of flows

$\mathcal{R}$ = Reynolds number for pipe j, $\mathcal{R} = V_j D_j/\nu$

$\boldsymbol{r} = (r_1, r_2, \ldots, r_{n_p})^T$ = vector of resistance factors

r_j = resistance factor for pipe j

t_k = quantity defined as $(\alpha_k + \beta_k/\theta)\,\eta^k$

$\boldsymbol{V} = \boldsymbol{A}_1^T \boldsymbol{F}^{-1} \boldsymbol{A}_1$ = matrix for Jacobian correction method

V_j = average fluid velocity for pipe j

$\boldsymbol{x} = \begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{h} \end{pmatrix}$

α_{jk} = interpolating spline coefficient

β_{jk} = interpolating spline coefficient

δ_{stop} = EPANET stopping tolerance

ϵ_j = roughness height of pipe j

ϵ_{mach} = machine epsilon

$\eta = \mathcal{R}/2000$

ν = kinematic viscosity of water

θ = parameter defined in (8)

$\hat{\theta}$ = parameter defined in (8)

ϕ = EPANET error measure, defined in (7)

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TABLES AND FIGURES

Table 1: The three ranges of Reynolds numbers of interest, their corresponding resistance factor formulae and sources.

Case	Range of $\mathcal{R}$	Resistance factor r	Formula source
1	$\mathcal{R} \leq 2000$ Laminar flow	$\frac{128\nu}{\pi g} \frac{L}{D^4}$	Hagen-Poiseuille
2	$2000 < \mathcal{R} < 4000$ Transitional flow	$\left(\frac{8}{\pi^2 g}\right) \frac{L}{D^5} \sum_{k=0}^{3} (\alpha_k + \beta_k/\theta)\, \eta^k$	Dunlop α_k, β_k as in Table 2
3	$\mathcal{R} \geq 4000$ Turbulent flow	$\left(\frac{2 \ln^2 10}{\pi^2 g}\right) \frac{L}{D^5} \frac{1}{\ln^2 \theta}$	Swamee-Jain (see (9))

Table 2: Coefficients of the cubic interpolating spline defining the friction factor for $2000 < \mathcal{R} < 4000$. The constants are $\tau = 0.00514215$ and $\xi = -0.86859$.

k	α_k	β_k
0	$5/(\xi^2 \ln^2 \hat{\theta})$	$\tau/(\xi^3 \ln^3 \hat{\theta})$
1	$0.128 - 12/(\xi^2 \ln^2 \hat{\theta})$	$-5\tau/(2\xi^3 \ln^3 \hat{\theta})$
2	$-0.128 + 9/(\xi^2 \ln^2 \hat{\theta})$	$2\tau/(\xi^3 \ln^3 \hat{\theta})$
3	$0.032 - 2/(\xi^2 \ln^2 \hat{\theta})$	$-\tau/(2\xi^3 \ln^3 \hat{\theta})$

Table 3: The terms on the diagonal of the matrix $\boldsymbol{G}$ for Darcy-Weisbach head loss model.

Case	Range of $\mathcal{R}$	Terms on the diagonal of $\boldsymbol{G}$
1	$\mathcal{R} \leq 2000$	$\left(\frac{128\nu}{\pi g}\right) \frac{L}{D^4}$
2	$2000 < \mathcal{R} < 4000$	$\lvert Q\rvert \left(\frac{8}{\pi^2 g}\right) \frac{L}{D^5} \sum_{k=0}^{3} (\alpha_k + \beta_k/\theta)\, \eta^k$
3	$\mathcal{R} \geq 4000$	$\lvert Q\rvert \left(\frac{2 \ln^2 10}{\pi^2 g}\right) \frac{L}{D^5} \frac{1}{\ln^2 \theta}$

Table 4: The diagonal terms of the matrix $\boldsymbol{F}$, the Jacobian, for Darcy-Weisbach head loss model

Case	Range of $\mathcal{R}$	The diagonal terms in $\boldsymbol{F}$
1*	$\mathcal{R} \leq 2000$	$\left(\frac{128\nu}{\pi g}\right)\frac{L}{D^4}$
2	$2000 < \mathcal{R} < 4000$	$\left(\frac{8}{\pi^2 g}\right)\frac{L}{D^5}\lvert Q\rvert\sum_{k=0}^{3}\left\{\frac{9c}{10}\frac{\beta_k}{\theta^2}\left\lvert\frac{D}{Q}\right\rvert^{9/10} + (2+k)\left(\alpha_k + \beta_k/\theta\right)\right\}\eta^k$
3	$\mathcal{R} \geq 4000$	$\left(\frac{2\ln^2 10}{\pi^2 g}\right)\frac{L}{D^5}\frac{\lvert Q\rvert}{\ln^2\theta}\left(2 + \left(\frac{9c}{5\theta\ln\theta}\right)\left\lvert\frac{D}{Q}\right\rvert^{9/10}\right)$

* Note that for Case 1 the diagonal term in $\boldsymbol{F}$ is constant and is independent of Q

Table 5: Pipe and node parameters and the steady state solution for the network shown in Figure 1.

Pipe ID	L(m)	D(mm)	ϵ(mm)	Flow (m^3/s)	Node ID	$\boldsymbol{d}_m$(m^3/s)	$\boldsymbol{e}_\ell$(m)	Head (m)
Pipe 1	400	200	0.30	0.12665	Tank 1	-	240	240
Pipe 2	100	300	0.30	0.49335	Node 2	0.050	-	203.403
Pipe 3	500	200	0.30	0.03123	Node 3	0.030	-	200.509
Pipe 4	700	300	0.30	0.04542	Node 4	0.020	-	223.588
Pipe 5	700	200	0.30	0.00123	Node 5	0.030	-	202.360
Pipe 6	400	300	0.30	0.27959	Node 6	0	-	200.499
Pipe 7	400	250	0.30	0.19377	Node 7	0.080	-	197.048
Pipe 8	100	300	0.30	0.16472	Node 8	0.090	-	191.818
Pipe 9	900	300	0.30	0.13029	Node 9	0.090	-	191.058
Pipe 10	500	300	0.30	0.16594	Node 10	0.090	-	118.384
Pipe 11	900	300	0.30	0.09123	Node 11	0.080	-	139.416
Pipe 12	700	100	0.30	0.02254	Node 12	0.060	-	188.456
Pipe 12	700	100	0.30	0.02254				
Pipe 13	100	200	0.30	0.03590				
Pipe 14	1000	200	0.30	0.09562				
Pipe 15	300	300	0.30	0.11185				
Pipe 16	800	200	0.30	-0.06746				
Pipe 17	700	150	0.30	-0.05185				

Table 6: Convergence data for the application to the network of Figure 1 of the Todini-Pilati method with uncorrected (u) Jacobian (from EPANET) and from the authors' Matlab code with corrected Jacobian (c).

i	$\phi_u(\boldsymbol{q}^{(i)})$	$\phi_c(\boldsymbol{q}^{(i)})$
1	9.0e-001	9.0e-001
2	8.1e-002	8.2e-002
3	4.6e-003	5.5e-003
4	7.9e-005	2.3e-005
5	1.5e-006	1.0e-009
6	3.6e-008	1.3e-014

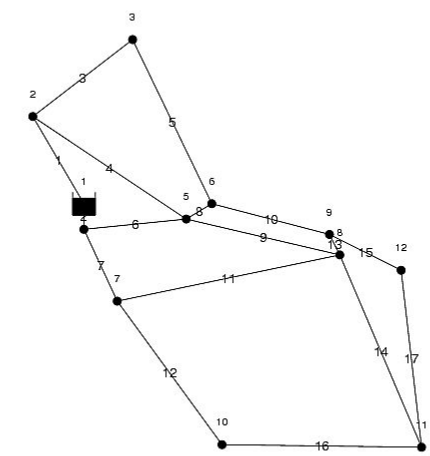

Figure 1: The network discussed in Example 1. It has $n_p = 17$, $n_j = 11$ and $n_f = 1$.