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FREQUENCY-DOMAIN MODELING OF TRANSIENTS IN PIPE NETWORKS WITH COMPOUND NODES USING A LAPLACE-DOMAIN ADMITTANCE MATRIX

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Abstract

An alternative to modeling the transient behavior of pipeline systems in the time-domain is to model these systems in the frequency-domain using Laplace transform techniques. A limitation with traditional frequency-domain pipeline models is that they are only able to deal with systems of a limited class of configuration. Despite the development of a number of recent Laplace-domain network models for arbitrarily configured systems, the current formulations are designed for systems comprised only of pipes and simple node types such as reservoirs and junctions. This paper presents a significant generalization of existing network models by proposing a framework that allows not only complete flexibility with regard to the topological structure of a network, but also, encompasses nodes with dynamic components of a more general class (such as air vessels, valves and capacitance elements). This generalization is achieved through a novel decomposition of the nodal dynamics for inclusion into a Laplace-domain network admittance matrix. A symbolic example is given demonstrating the development of the network admittance matrix and numerical examples are given comparing the proposed method to the method of characteristics for 11-pipe and 51-pipe networks.

INTRODUCTION

Networks of interlinked fluid lines occur in many different instances, examples of which are material transport systems (*e.g.* water, gas and petroleum [Fox, 1977; Chaudhry, 1987; Wylie and Streeter, 1993]), control systems (*e.g.* hydraulic and pneumatic [Stecki and Davis, 1986; Barber, 1989]), and biological systems (*e.g.* arterial blood flow [John, 2004]). Given the far reaching nature of these systems, the ability to model the transient response of these networks subjected to boundary perturbations is of broad interest and is fundamental for the purposes of analysis and design.

A particular application of transient network modelling is water hammer analysis in distribution systems. Traditionally, the approach for modelling water hammer within distribution systems is via the use of time-domain approximate discrete methods such as the method of characteristics [Wylie and Streeter, 1993]. However, recently alternative Laplace-domain models have been proposed [Kim, 2007; Zecchin *et al.*, 2009] where classical frequency-domain models [Chaudhry, 1987; Wylie and Streeter, 1993] have been extended to handle arbitrarily configured networks.

The Laplace-domain network admittance formulation from Zecchin *et al.* [2009] was designed for systems comprised of pipes, reservoirs and junctions. Despite the capacity to deal with networks of arbitrary configuration, the formulation is still limited in its application as real world networks contain many other types of hydraulic components such as valves, emitters, surge tanks (accumulators), pumps, and other components. This paper presents a new and extended formulation that is able to deal with, not only arbitrarily configured networks, but also pipe networks containing a general class of hydraulic components. The class of components that can be incorporated into the proposed framework is the extremely general class whose dynamic state equations yield an admittance representation, that is, there exists a definable map from the components state variables to the components connection flows (discussed later). The incorporation of these arbitrary components is achieved by a novel nodal expansion method that enables the inclusion of the nodal dynamics into the network admittance matrix structure. These arbitrary components from hereon are referred to as *compound nodes*, which references the fact that these elements represent dynamic systems in conjunction with the nodal-like property of being a connection point for pipes.

The paper is structured as follows. Firstly, the background is given including current methods for modeling pipe networks and a mathematical formulation of the network equations as well as a brief background to Laplace-domain representations of the fluid network equations. This is followed by a comprehensive mathematical framework for the compound node type. The formulation of the Laplace-domain model for an arbitrarily configured network comprised of arbitrary node types is then presented , where a brief review of the work of Zecchin *et al.*

[2009] is given followed by a staged generalization. This section highlights the concepts and definitions that are required to deal with arbitrary node types within a network setting. Numerical examples are then given for two case studies, a 11-pipe network and a 51-pipe network, followed by the conclusions.

BACKGROUND

Modelling the Transient Behaviour of Pipe Networks

Pipe networks can essentially be viewed as systems comprised of dynamic interacting elements. These networks are comprised of two types of elements, namely distributed elements (*e.g.* pipes) and lumped elements (*e.g.* junctions, air vessels and valves). The fluid variables of each of the hydraulic elements interact with their neighboring components according to the physical laws of conservation of mass, momentum and energy. The network problem involves defining a computable model that can deal with an arbitrary structure of connected hydraulic elements, and determine the value of the associated fluid variables that satisfy the system of underlying equations.

Modeling an arbitrary network in the time-domain has been broadly addressed within the research literature (*e.g.* [Karney, 1984; Chaudhry, 1987; Wylie and Streeter, 1993; Axworthy, 1997; Izquierdo and Iglesias, 2004]), and, within industry, there exist many commercial software packages for the purpose of water hammer analysis within water distribution systems. For these time-domain models, the distributed components are discretized in space and time and modeled using hyperbolic partial differential equation (PDE) solvers [Chaudhry, 1987; Wylie and Streeter, 1993], and the lumped components are modeled by simultaneous equations solvers, which are computed at each time point. The distributed nature of the pipelines means that there is a time delay in the wave propagation of the fluid variables. This time delay means that the network variables are not required to be solved simultaneously, but that, at each time step, the interior points of each fluid line can be computed explicitly in isolation [Wylie and Streeter, 1993], and only the fluid variables at the endpoints of the pipes, incident on common nodes, require simultaneous solving, thus greatly reducing the problem complexity. The fluid variables at the pipes endpoints then serve as the boundary condition to the interior points at the following time step.

Laplace-domain modelling is significantly different to this. The underlying fluid equations of the hydraulic elements (pipelines and lumped components) are first linearised, then transformed using the Laplace transform, and finally solved to yield analytic transfer relationships between the points at which the element connects to other elements within the network [Chaudhry, 1987; Wylie and Streeter, 1993] (*e.g.* pipeline end points). The construction of a full network model from the individual element transfer relationships involves solving the simultaneous set of complex valued equations that arise from the hydraulic elements and their interactions with other elements that are incident to similar nodes. What this means is that as the transformed fluid variables are in the Laplace-domain, the temporal delays are replaced by algebraic operations, and consequently the fluid variables for all components must be solved simultaneously for every frequency point of interest. Interestingly, in this regard the Laplace-domain model is similar to steady-state models for solving the flows and pressures in a water distribution system [Todini and Pilati, 1988], in that there is a direct dependence of one network variable on another.

The classical methods for Laplace-domain modelling of pipe networks are the impedance method [Wylie, 1965; Wylie and Streeter, 1993] and the transfer matrix method [Chaudhry, 1970, 1987]. Both these methods have their origins in Laplace-domain transmission line theory, where pipes are described by their wave propagation characteristics [Brown and Nelson, 1965; Stecki and Davis, 1986] . Within the impedance method, a pipe network is characterised by the distribution of hydraulic impedance throughout the network and impedance relationships are derived to relate the hydraulic impedance values across an element (note that Hydraulic impedance refers to the ratio of transformed pressure to transformed flow). Within the transfer matrix method, each hydraulic element is expressed as a 2×2 transfer matrix relating the upstream and downstream transformed variables of pressure and flow, whereby a network model can be created by an ordered multiplication of these matrices.

The advantages of these methods are that they are able to deal with systems comprised of pipes and lumped hydraulic components. The major disadvantage, however, is that such methods are not able to deal with an arbitrary network configuration, but are limited to simple first order looped systems [Fox, 1977] (structural reasons for the transfer matrix method, and practical reasons for the impedance method). Many authors have utilised different methods to achieve a frequency-domain representation of complex networks (*e.g.* Ogawa [1980]; Margolis and Yang [1985]; Boucher and Kitsios [1986]; John [2004]; Kim [2007, 2008a,b]). However, these methods were designed simply for networks with junctions and reservoir node types only, with the exception of Kim [2007, 2008a,b] who included an emitter elements and some surge protection devices in his formulation.

An alternative method for modelling systems comprised of pipes, junctions and reservoirs was proposed in Zecchin *et al.* [2009] in which an admittance matrix expression relating the nodal pressures to the nodal outflows was derived from the basic fluid equations using graph theory concepts. The significance of this is twofold, (i) the network matrix was shown to have an intuitive and simple structure for which analogies with admittance matrices in electrical circuits was made apparent [Desoer and Kuh, 1969], and (ii) it showed that the entire network state was a function of the reduced variable set of nodal pressures and flows. The focus of this paper is the development of a new model to deal with networks than include compound nodes in addition to pipes, junctions and reservoirs.

Problem Definition

The development of a network model not only involves modelling the dynamics of each individual component, but it also involves accounting for the continuity of the fluid variables of the hydraulic elements at their connection points. Before outlining the network equations, some notation is defined below, and the general framework for a node is given.

To facilitate the discussion of the network connectivity equations, it is convenient to describe a network as a connected graph $\mathcal{G}(\mathcal{N}, \Lambda)$ [Diestel, 2000] consisting of the node set $\mathcal{N} = \{1, 2, ..., n_n\}$, and the link set $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_{n_\Lambda}\}$ and where $\lambda_j = (i_{u,j}, i_{d,j})$ where $i_{uj}, i_{dj} \in \mathcal{N}$ are the upstream and downstream nodes of link j respectively. Each node is associated with a lumped hydraulic component that is connected to a number of links, and each link is associated with a distributed pipe element where the directed nature of the link describes the positive flow direction sign convention of the element. There are two link sets associated with each node, these are Λ_{ui} and Λ_{di} which correspond to the set of links directed from and to node i respectively, that is $\Lambda_{ui} = \{(i, k), k \in \mathcal{N} : (i, k) \in \Lambda\}$ and $\Lambda_{di} = \{(k, i), k \in \mathcal{N} : (k, i) \in \Lambda\}$). Note that the first set corresponds to the links whose upstream node is i and the second set correspond to the links whose downstream node is i.

Compound Node Equations

Previous work has presented a methodology that included a specific class of node that describes junctions, demand nodes and reservoirs. This nodal type is called a *simple node* and is defined as a point with an infinitely small volume that has a lossless connection to one or more fluid lines. The infinitely small volume implies that there is no variation of pressure or accumulation of mass, and the lossless connection implies that the pressure at the ends of the fluid line connected to the node are equal. The main contribution of this paper is that it presents a novel way to include a completely general node type into the network equations. The node types considered in this paper are of a much more general class encompassing any hydraulic element whose dynamics can be exactly represented (or adequately approximated) by a passive, time-invariant linear system, as is the case for most hydraulic elements such as valves, emitters, and surge tanks (see Desoer and Vidyasagar [1975] for a discussion on passivity). These nodes are referred to as *compound nodes* and are defined and discussed below.

The equations describing the dynamic behaviour of a compound node are derived from the physical laws of mass and momentum conservation. In the general case, the dynamic behaviour of a compound node is given by the vector equation

$$\boldsymbol{\phi}_i\left(\boldsymbol{p}_i, \boldsymbol{q}_i, \boldsymbol{u}_i, \tilde{\boldsymbol{u}}_i, t\right) = \mathbf{0} \tag{1}$$

where $\boldsymbol{\phi}_i$ is the vector valued operator describing the compound node dynamics, $\boldsymbol{u}_i(t)$ is the set of controlled state variables for the node (*i.e.* valve opening, demand *etc.*), $\tilde{\boldsymbol{u}}_i$ is the set of response state variables (*e.g.* pressure, volume and inflow for a surge tank, or pressure and outflow for an emitter) and $\boldsymbol{p}_i$ and $\boldsymbol{q}_i$ are the sets of pressures and flows of the pipes incident to node i. That is, $\boldsymbol{p}_i$ and $\boldsymbol{q}_i$ are vector organisations of the sets

$$\{p_j(0,\cdot) : \lambda_j \in \Lambda_{ui}\} \cup \{p_j(l_j,\cdot) : \lambda_j \in \Lambda_{di}\}$$
$$\{q_j(0,\cdot) : \lambda_j \in \Lambda_{ui}\} \cup \{q_j(l_j,\cdot) : \lambda_j \in \Lambda_{di}\}$$

respectively, where p_j and q_j are the distributions of pressure and flow along pipe j, and l_j is the length of pipe j. Mathematically, the difference between the controlled states and the response states is that the controlled states are inputs that require specification to compute (1), and the dependent states are outputs that are computed from (1). It is important to note that the compound node framework (1) encompasses basic nodes, such as junctions with emitters, and also nodes of complex configurations involving multiple equations for different elements within the compound node. This is demonstrated in the following example.

Example 1. *Consider the compound node configuration in Figure 1(a) consisting of a closed branch and a controlled demand bounded by valves A and B. Pipe [a] is incident to valve A and pipes [b] and [c] are incident to valve B. The nodal states can be taken as the internal pressure ψ_o, the capacitive inflow into the closed branch θ_o, and the controlled flow injection (or demand) θ_d. Labeling this node with i, the nodal vectors are*

$$\boldsymbol{p}_i(t) = \begin{bmatrix} p_a(l_a,t) \\ p_b(l_b,t) \\ p_c(0,t) \end{bmatrix}, \ \boldsymbol{q}_i(t) = \begin{bmatrix} q_a(l_a,t) \\ q_b(l_b,t) \\ q_c(0,t) \end{bmatrix}, \ \boldsymbol{u}_i(t) = \theta_d(t), \ \tilde{\boldsymbol{u}}_i(t) = \begin{bmatrix} \psi_o(t) \\ \theta_o(t) \end{bmatrix}. \tag{2}$$

The vector equation $\boldsymbol{\phi}_i$ for compound node is

$$\boldsymbol{\phi}_i(\boldsymbol{p}_i, \boldsymbol{q}_i, \boldsymbol{u}_i, \tilde{\boldsymbol{u}}_i, t) = \left\{ \begin{array}{rll} p_b(l_b,t) - \psi_o(t) - f_B\left(q_b(l_b,t) - q_c(0,t)\right) & = 0 & \textit{pressure change across valve } B \\ q_a(l_a,t) + q_b(l_b,t) - q_c(0,t) - \theta_o(t) + \theta_d(t) & = 0 & \textit{continuity within node} \\ p_b(l_b,t) - p_c(0,t) & = 0 & \textit{connectivity of links b and c} \\ \dfrac{V_o}{K_e}\dfrac{d\psi_o(t)}{dt} - \theta_o(t) & = 0 & \textit{capacitance equation for branch} \\ p_a(l_a,t) - \psi_o(t) - f_A\left(q_a(l_a,t)\right) & = 0 & \textit{pressure change across valve } A \end{array} \right. \tag{3}$$

where V_o and K_e are volume and effective modulus of the branch, and

$$f_X(q) = \frac{\rho}{2}\frac{|q|q}{\left(C_{dX}A_{vX}\right)^2}$$

is the valve pressure change where C_{dX} and A_{vX} are the valve coefficient and valve cross-sectional area for valve X for subscripts $X = A, B$.

Network Equations

With the given notation, a compound node network can be defined as the triple

$$(\mathcal{G}(\mathcal{N}, \Lambda), \mathcal{P}, \mathcal{C})$$

where $\mathcal{G}(\mathcal{N}, \Lambda)$ is the network graph of nodes $\mathcal{N}$ and links Λ, $\mathcal{P} = \{\mathcal{P}_j : \lambda_j \in \Lambda\}$ is the set of pipeline properties where $\mathcal{P}_j$ are the properties for pipe j (*i.e.* l_j, diameter, roughness *etc.*), and $\mathcal{C} = \{\boldsymbol{\phi}_i : i \in \mathcal{N}_c\}$ is the set of compound node functions for the set of compound nodes $\mathcal{N}_c \subset \mathcal{N}$. The state space of the network $(\mathcal{G}(\mathcal{N}, \Lambda), \mathcal{P}, \mathcal{C})$ is given by the distributions of pressure and flow along each line of the network, and the compound node response states, which are given by

$$\boldsymbol{p}\left(\boldsymbol{x}, t\right) = \begin{bmatrix} p_1\left(x_1, t\right) \\ \vdots \\ p_{n_\Lambda}\left(x_{n_\Lambda}, t\right) \end{bmatrix}, \quad \boldsymbol{q}\left(\boldsymbol{x}, t\right) = \begin{bmatrix} q_1\left(x_1, t\right) \\ \vdots \\ q_{n_\Lambda}\left(x_{n_\Lambda}, t\right) \end{bmatrix}, \quad \tilde{\boldsymbol{u}}(t) = \begin{bmatrix} \tilde{\boldsymbol{u}}_1(t) \\ \hdashline \vdots \\ \hdashline \tilde{\boldsymbol{u}}_{n_c}(t) \end{bmatrix}, \tag{4}$$

respectively, where $\boldsymbol{x} = [x_1 \cdots x_{n_\Lambda}]^T$ is the vector of spatial coordinates, (*i.e.* $\boldsymbol{x} \in \mathcal{X} = \mathcal{X}_1 \times \cdots \mathcal{X}_{n_\Lambda}$ where $\mathcal{X}_j = [0, l_j]$), $t \in \mathbb{R}$ is time, n_Λ is the number of links, and n_c is the number of compound nodes.

For a given network $(\mathcal{G}(\mathcal{N}, \Lambda), \mathcal{P}, \mathcal{C})$, the network modeller is interested in the transient response of the states (4) for a specific hydraulic scenario, where hydraulic scenario is defined by a set of specified initial and boundary conditions. In addition to the compound node controls $\boldsymbol{u}_i, i \in \mathcal{N}_c$, each simple node either has controlled nodal pressure (as in the case of a reservoir) or a controlled nodal flow (as in the case of a demand node or a junction). Therefore, partitioning the set of simple nodes $\mathcal{N}_s = \mathcal{N}/\mathcal{N}_c$ as $\mathcal{N}_s = \mathcal{N}_J \cup \mathcal{N}_d \cup \mathcal{N}_r$ where $\mathcal{N}_J$ is the set of junctions, $\mathcal{N}_d$ is the set of demand nodes, and $\mathcal{N}_r$ is the set of reservoir nodes, the system of dynamic equations governing the network states (4) is

$$\frac{\partial q_j}{\partial t}+\frac{A_j}{\rho}\frac{\partial p_j}{\partial x}+\frac{q_j}{A_j}\frac{\partial q_j}{\partial x}+\frac{\pi D_j}{\rho}\tau_j=0,\quad x\in\mathcal{X}_j,\,\lambda_j\in\Lambda, \tag{5}$$

$$\frac{\partial p_j}{\partial t}+\frac{\rho c_j^2}{A_j}\frac{\partial q_j}{\partial x}+\frac{q_j}{A_j}\frac{\partial p_j}{\partial x}=0,\quad x\in\mathcal{X}_j,\,\lambda_j\in\Lambda, \tag{6}$$

$$p_j(\varphi_{ji},t)-p_k(\varphi_{ki},t)=0,\quad \lambda_j,\lambda_k\in\Lambda_i,\,i\in\mathcal{N}_J\cup\mathcal{N}_d \tag{7}$$

$$p_j(\varphi_{ji},t)-\psi_{ri}(t)=0,\quad \lambda_j\in\Lambda_i,\,i\in\mathcal{N}_r, \tag{8}$$

$$\sum_{\lambda_j\in\Lambda_{di}}q_j(l_j,t)-\sum_{\lambda_j\in\Lambda_{ui}}q_j(0,t)=0,\quad i\in\mathcal{N}_J \tag{9}$$

$$\theta_{di}(t)+\sum_{\lambda_j\in\Lambda_{di}}q_j(l_j,t)-\sum_{\lambda_j\in\Lambda_{ui}}q_j(0,t)=0,\quad i\in\mathcal{N}_d \tag{10}$$

$$\boldsymbol{\phi}_i\left(\boldsymbol{p}_i,\boldsymbol{q}_i,\boldsymbol{u}_i,\tilde{\boldsymbol{u}}_i,t\right)=\mathbf{0},\quad i\in\mathcal{N}_c \tag{11}$$

$$p_j(x,0)=p_j^0(x),\,q_j(x,0)=q_j^0(x),\quad x\in\mathcal{X}_j,\lambda_j\in\Lambda \tag{12}$$

$$\boldsymbol{u}_i(0)=\boldsymbol{u}_i^0,\quad i\in\mathcal{N}_c \tag{13}$$

where the symbols are defined as follows: for the fluid lines ρ is the fluid density, c_j, A_j, D_j and $\tau_j=\tau_j(q_j)$ are the fluid line wavespeed, the cross-sectional area, the diameter and the cross sectional shear stress for pipe j respectively; for the nodes ψ_{ri} is the controlled temporally varying reservoir pressure for the reservoir nodes in the reservoir node set $\mathcal{N}_r$, θ_{di} is the controlled temporally varying nodal demand for the demand nodes in the demand node set $\mathcal{N}_d$; p_j^0 and q_j^0 are the initial distribution of pressure and flow in each pipe $\lambda_j\in\Lambda$; $\boldsymbol{u}_i^0$ are the initial values for the compound node response states; and $\varphi_{ji}=l_j$ if $\lambda_j\in\Lambda_{di}$ and 0 otherwise.

The network equations (5)-(13) can be divided into five groups: (5) and (6) are the unsteady equations of motion and mass continuity for each fluid line; (7) and (8) are the nodal equations of equal pressures in pipe ends connected to the same node for junctions (nodes for which the inline pressure is the free variable) and reservoirs (nodes for which the nodal flow is the free variable) respectively; (9) and (10) are the nodal equations of mass conservation for junctions and demand nodes; (11) is the vector equation governing the behaviour of the compound nodes; (12)-(13) are the initial conditions for the link states and node states.

Laplace-Domain Representation of Fluid Equations

The Laplace transform is a useful tool in dealing with partial differential equations as it removes the time-dependency of the variables and yields a simpler ordinary differential equation [Kreyszig, 1999]. Via the Laplace transform, the real variable $p(x,t)$ becomes the complex Laplace-domain variable $P(x,s)$, where $s\in\mathbb{C}$, and all differential operations involving t become algebraic operations involving s. An important property of the Laplace transform is that the frequency-domain behaviour of a variable is given by the value of the Laplace-domain variable restricted to the positive imaginary axis, that is $s = \mathrm{i}\omega$ where i is the imaginary unit, and ω is the radial frequency [Franklin *et al.*, 2001].

The Laplace-domain representation of the network equations (5)-(13) requires (i) the linearisation of (5)-(10) in p_j, q_j, and $\tilde{\boldsymbol{u}}_i$ and (ii) the assumption of homogeneous initial conditions. The standard approach to satisfy both these requirements is to linearise the system (5)-(11) about the initial conditions (12) and consider the transient fluctuations in p_j, q_j and $\tilde{\boldsymbol{u}}_i$ about these values [Chaudhry, 1987; Wylie and Streeter, 1993]. For the application of transient modelling within water distribution systems, it is common to take the initial conditions as the steady state conditions. The assumption with this approach is that the unsteadiness of the flow during normal operation is negligible and the system is approximately at steady-state conditions. This assumption is typically adequate as the transient events to be simulated are orders of magnitude greater than background transients caused by the mild unsteadiness within the system. For the nodal conditions (7)-(10), no approximation is required, as these equations are linear, but linearisation is required for the unsteady fluid equations (5) and (6) and the compound node equation (11).

For small Mach number flows the convective terms in (5) and (6) maybe neglected, and the Laplace transform of these PDEs becomes

$$\frac{\partial P_j}{\partial x} = -\frac{\rho}{A_j}\left[s + \frac{\pi D_j}{\rho}\mathcal{L}\left\{\bar{\tau}_j\right\}(s)\right]Q_j, \tag{14}$$

$$\frac{\partial Q_j}{\partial x} = -s\frac{A_j}{\rho c_j^2}P_j, \tag{15}$$

on $x \in \mathcal{X}_j, s \in \mathbb{C}$ where P_j and Q_j are the transformed pressure and flow, the operator $\bar{\tau}_j = \bar{\tau}_j(q_j)$ is a linear approximation of the nonlinear operator τ_j, and $\mathcal{L}\{f\}$ denotes the Laplace transform of the function f. For turbulent flow $\tau_j(q_j) = \bar{\tau}_j(q_j) + O\left\{\left(q_j - q_j^0\right)^2\right\}$ but for unsteady formulations with terms in τ_j involving convolution operations on q_j (*e.g.* Zielke [1968]; Vardy and Brown [2003, 2004]), $\bar{\tau}_j$ exactly captures the unsteady effects as a convolution is a linear operation (see Stecki and Davis [1986] for more detail). Note that in the case of nonhomogeneous initial conditions, P_j and Q_j are taken as the Laplace transform of the transient fluctuations about these initial conditions.

The solution to the linearised equations (5) and (6) is given by the standard solution to a constant coefficient first order ordinary differential equation [Kreyszig, 1999], that is

$$\begin{bmatrix} P_j(x,s) \\ Q_j(x,s) \end{bmatrix} = B_j^-(s)e^{-\tilde{\Gamma}_j(s)x} \begin{bmatrix} 1 \\ Z_{c,j}^{-1}(s) \end{bmatrix} + B_j^+(s)e^{\tilde{\Gamma}_j(s)x} \begin{bmatrix} 1 \\ -Z_{c,j}^{-1}(s) \end{bmatrix}, \tag{16}$$

on $x \in \mathcal{X}_j, s \in \mathbb{C}$ where B_j^- and B_j^+ are arbitrary functions dependent on the boundary conditions of pipe j, and $\tilde{\Gamma}_j$ and $Z_{c,j}$ are the propagation operator and the characteristic impedance, respectively, for pipe j and are given by

$$\tilde{\Gamma}_j(s) = \frac{s}{c_j}\sqrt{1 + \frac{\pi D_j}{\rho}\frac{\mathcal{L}\{\bar{\tau}_j\}(s)}{s}}, \quad Z_{c,j}(s) = \frac{c_j \rho}{A_j}\sqrt{1 + \frac{\pi D_j}{\rho}\frac{\mathcal{L}\{\bar{\tau}_j\}(s)}{s}}.$$

The propagation operator $\tilde{\Gamma}_j$ describes the rate of attenuation and phase change experienced by a propagating wave within a pipeline, and the series impedance $Z_{c,j}$ describes the amplitude and phase coupling between a pressure wave and its associated flow perturbation.

With respect to the hydraulic component, to be able to apply the Laplace transform, the equations (1) must be approximated by a linear, time-invariant system. The method of constructing this approximation is dependent on the nature of the nonlinearities in the node equation $\boldsymbol{\phi}$. A standard property of the nonlinearities within many hydraulic components is that they are memoryless, that is the integrodifferential and delay terms are linear in the nodal variables. For these circumstances, the linear time-invariant approximation is constructed by taking only the linear terms in a Taylor series approximation about a selected operating point. Performing such a linearization about a selected operating point and taking the Laplace-transform of (1) leads to the following expression for the dynamics of the i-th hydraulic component

$$\boldsymbol{\Phi}_i(s) \begin{bmatrix} \boldsymbol{P}_i(s) \\ \hdashline \boldsymbol{Q}_i(s) \\ \hdashline \boldsymbol{U}_i(s) \\ \hdashline \widetilde{\boldsymbol{U}}_i(s) \end{bmatrix} = \boldsymbol{0} \tag{17}$$

where $\boldsymbol{\Phi}_j(s)$ is the matrix Laplace-transform of the linear approximation of the vector function $\boldsymbol{\phi}_j$, and $\boldsymbol{P}_i$, $\boldsymbol{Q}_i$, $\boldsymbol{U}_i$ and $\tilde{\boldsymbol{U}}_i$ are the Laplace transforms of their lower case counterparts (note that in the case were $\boldsymbol{p}_i$, $\boldsymbol{q}_i$, $\boldsymbol{u}_i$ and $\tilde{\boldsymbol{u}}_i$ have nonhomogeneous initial conditions, the Laplace variables are taken as the transient fluctuations about the initial values). Consider the following example.

Example 2. *Revisiting the compound node from Figure 1(a) in Example 1. Linearising the valve pressure loss functions of* (3)*, as in Wylie and Streeter [1993], and taking the Laplace transform leads to the* (17)*-type repre-*

sentation where

$$\mathbf{\Phi}_i(s) = \left[\begin{array}{ccc:ccc:c:cc} 0 & 1 & 0 & 0 & -c_B & c_B & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c(s) & -1 \\ 1 & 0 & 0 & -c_A & 0 & 0 & 0 & -1 & 0 \end{array}\right] \tag{18}$$

where $c(s) = V_0 s/K_e$, $c_X = 2\rho|q_{oX}|/(C_{dX}A_{vX})^2, X = A, B$ *where the* q_{oX} *are the operating points for the linearisation of the valve headloss functions and* $\boldsymbol{P}_i$, $\boldsymbol{Q}_i$, $\boldsymbol{U}_i$, *and,* $\widetilde{\boldsymbol{U}}_i$, *are the Laplace transforms of the transient fluctuation of the variables* (2) *about the initial values. The partitions of* (18) *correspond to the matrix sections that act on the node states* $\boldsymbol{P}_i$, $\boldsymbol{Q}_i$, $\boldsymbol{U}_i$, *and,* $\widetilde{\boldsymbol{U}}_i$, *respectively.*

MATHEMATICAL FRAMEWORK FOR COMPOUND NODE

For a compound node element to be incorporated within a network model, a special representation of the compound node equation (17) must be determined. In a network context, a compound node is comprised of a hydraulic *component* and a number of *connection* points (*i.e.* junctions between the compound node component and the incident pipes). The hydraulic component is the physical structure of the compound node that governs the dynamic behaviour of the node, and the connections are the junctions through which the compound node interacts with the network. Formally, a connection for a compound node is defined as an interface between one or more links and the compound node's component, within which there is no accumulation of fluid or change in pressure. The significance of a connection is that as the link end pressures and flows are uncoupled, the component experiences the aggregated effect of all links incident to a connection and does not differentiate between the contributions to the connection flow from individual links. Consider the following example.

Example 3. *The compound node in Figure 1(a) used in Examples 1 and 2 has two connections, each just exterior to the valves* A *and* B*, and the component of the compound node includes the valves and everything in between the valves. Figure 1(b) demonstrates the connectivity of the compound node with the link [a] incident to connection* A*, and links [b] and [c] both incident to the connection at* B*. Figure 1(c) demonstrates the fluid states of pressure and flow at the connections, where the pressures* P_{ciA} *and* P_{ciB} *are the pressures at the end point of the links, and the inflows* Q_{ciA} *and* Q_{ciB} *are the aggregated link flows into the component.*

In general, for a compound node i with n_{si} connections, the connection states of pressure and flow are given by the vectors

$$\boldsymbol{P_c}_i(s) = \begin{bmatrix} P_{ci1}(s) \\ \vdots \\ P_{cin_{si}}(s) \end{bmatrix}, \quad \boldsymbol{Q_c}_i(s) = \begin{bmatrix} Q_{ci1}(s) \\ \vdots \\ Q_{cin_{si}}(s) \end{bmatrix}, \tag{19}$$

where P_{cik} is the common pressure shared at all link ends incident to the k-th connection of compound node i, and Q_{cik} is the aggregated flow from the links incident to the k-th connection of compound node i into the component. With this notation, the desired admittance representation of the compound node dynamics from (17) is given by

$$\boldsymbol{Y_c}_i(s)\boldsymbol{P_c}_i(s) - \boldsymbol{Y_u}_i(s)\boldsymbol{U}_i(s) = \boldsymbol{Q_c}_i(s) \tag{20}$$

where $\boldsymbol{Y_c}_i$ and $\boldsymbol{Y_u}_i$ are stable transfer matrices of size $n_{si} \times n_{si}$ and $n_{si} \times n_{ui}$ respectively. Technically, a stable transfer function is one for which all poles (singularities) are located in the right hand plane of the complex domain [Franklin *et al.*, 2001]. Practically, stability is a property of most physical systems, and is observed as the temporal decay in the system response when subject to transient inputs. This canonical representation of the node dynamics is interpreted as a hydraulic admittance as $\boldsymbol{Y_c}_i(s)$ is the admittance transfer matrix from the connection pressures to the connection flows, and $\boldsymbol{Y_u}_i(s)$ is the admittance transfer matrix from the controlled nodal states to the connection flows.

The derivation of (20) from (17) involves three steps: (i) the expression of the nodal equations in terms of the compound node variables $\boldsymbol{U}_i$ and $\tilde{\boldsymbol{U}}_i$, and the connection variables (19), (ii) the decoupling of the nodal equations from $\widetilde{\boldsymbol{U}}_i$, and (iii) the extraction of the form (20). Details of these steps are omitted here, but are given in the Appendix , where the criteria for the existence of such a representation is also given. Continuing on from Example 6 in the Appendix , the following example gives the form of (20) for the compound node in Figure 1(a).

Example 4. *Consider the compound node from Figure 1(a). Example 2 showed that the controlled state is* $\boldsymbol{U}_i = \Theta_d$*, and Example 3 demonstrated that the connection states of pressure and flow are*

$$\boldsymbol{P_c}_i(s) = \begin{bmatrix} P_{ciA}(s) \\ P_{ciB}(s) \end{bmatrix}, \quad \boldsymbol{Q_c}_i(s) = \begin{bmatrix} Q_{ciA}(s) \\ Q_{ciB}(s) \end{bmatrix}. \tag{21}$$

Following the three step process outlined in Appendix , the matrices of the admittance form for this compound *node that relates $\boldsymbol{Q}_{\boldsymbol{c}i}$ to $\boldsymbol{P}_{\boldsymbol{c}i}$ and $\boldsymbol{U}_i$, is determined from* (18) *as*

$$\boldsymbol{Y}_{\boldsymbol{c}i}(s) = \frac{1}{c_A + c_B + c_A c_B c s}\begin{bmatrix} 1 + c_B c s & -1 \\ -1 & 1 + c_A c s \end{bmatrix},$$
$$\boldsymbol{Y}_{\boldsymbol{u}i}(s) = -\frac{1}{c_A + c_B + c_A c_B c s}\begin{bmatrix} c_B \\ c_A \end{bmatrix}.$$

The details of each step are given in Example 6 in Appendix .

NETWORK FORMULATION

The derivation of the network admittance matrix for hydraulic networks comprised of pipelines and compound nodes is presented in the following sections. Firstly, as background for this work, the network admittance matrix for a simple node network is derived [Zecchin *et al.*, 2009]. Based on this, a staged generalisation is presented. Firstly , the special case of pressure dependent nodal outflows compound node is considered. This case highlights the majority of the necessary steps for the inclusion of compound nodes into the network admittance matrix form. Secondly, the general admittance form of the compound node dynamics are incorporated into the network matrix structure. This formulation represents a full treatment of the network equations (5)-(13). Finally, a computable input-output network transfer matrix model is presented.

Review of Network Matrix for a simple link simple node network

The case of a simple node network comprised of only pipes, junctions and reservoirs serves as the basis for dealing with the more complex case of a compound node network. A simple node network is defined as the pair $(\mathcal{G}(\mathcal{N},\Lambda),\mathcal{P})$ where, similarly to the compound node network, $\mathcal{G}(\mathcal{N},\Lambda)$ is the underlying network graph structure comprised of n_Λ links and n_n simple nodes, and $\mathcal{P}$ is the set of link data. The nodal states for such a network are the nodal pressures and flows

$$\boldsymbol{\Psi} = [\Psi_1 \cdots \Psi_{n_n}]^T, \quad \boldsymbol{\Theta} = [\Theta_1 \cdots \Theta_{n_n}]^T.$$

The equations governing the behaviour of a simple node network are (5)-(13) without the compound node equations (11) and (13). Based on these equations, Zecchin *et al.* [2009] derived the form of the network admittance matrix mapping from the nodal pressures $\boldsymbol{\Psi}$ to the nodal flows $\boldsymbol{\Theta}$. The main results are briefly reviewed below.

The Laplace-domain description of transient fluid lines (16) can be organised into the end-to-end transfer matrix admittance form [Goodson and Leonard, 1972]

$$\begin{bmatrix} Q_j(s,0) \\ -Q_j(s,l_j) \end{bmatrix} = \frac{1}{Z_{cj}(s)} \begin{bmatrix} \coth\Gamma_j(s) & -\text{csch}\Gamma_j(s) \\ -\text{csch}\Gamma_j(s) & \coth\Gamma_j(s) \end{bmatrix} \begin{bmatrix} P_j(s,0) \\ P_j(s,l_j) \end{bmatrix},$$

where $\Gamma_j = l_j \tilde{\Gamma}_j$. The admittance matrix functions for each link $\lambda \in \Lambda$ can be organised into the matrix form

$$\begin{bmatrix} \boldsymbol{Q}(s,\boldsymbol{0}) \\ \hdashline -\boldsymbol{Q}(s,\boldsymbol{l}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{Z_c}^{-1}(s)\coth\boldsymbol{\Gamma}(s) & -\boldsymbol{Z_c}^{-1}(s)\text{csch}\boldsymbol{\Gamma}(s) \\ \hdashline -\boldsymbol{Z_c}^{-1}(s)\text{csch}\boldsymbol{\Gamma}(s) & \boldsymbol{Z_c}^{-1}(s)\coth\boldsymbol{\Gamma}(s) \end{bmatrix} \begin{bmatrix} \boldsymbol{P}(s,\boldsymbol{0}) \\ \hdashline \boldsymbol{P}(s,\boldsymbol{l}) \end{bmatrix}$$

with link state vectors

$$\boldsymbol{P}(s,\boldsymbol{x}) = \left[P_1(s,x_1),\ldots,P_{n_\Lambda}(s,x_{n_\Lambda})\right]^T, \quad \boldsymbol{Q}(s,\boldsymbol{x}) = \left[Q_1(s,x_1),\ldots,Q_{n_\Lambda}(s,x_{n_\Lambda})\right]^T,$$

and the diagonal link function matrices

$$\boldsymbol{\Gamma}(s) = \text{diag}\ \left\{\Gamma_1(s),\ldots,\Gamma_{n_\Lambda}(s)\right\}, \boldsymbol{Z_c}(s) = \text{diag}\ \left\{Z_{c1}(s),\ldots,Z_{cn_\Lambda}(s)\right\},$$

where $\boldsymbol{x} = [x_1,\ldots,x_{n_\Lambda}]^T$ is the vector of spatial coordinates, and $\boldsymbol{x} = \boldsymbol{0}$ ($\boldsymbol{x} = \boldsymbol{l}$) corresponds to all coordinates set at their start (end) points. Defining the upstream and downstream node incidence matrices as

$$\{\boldsymbol{N_u}\}_{i,j} = \begin{cases} 1 & \text{if} \quad \lambda_j \in \Lambda_{u,i} \\ 0 & \text{otherwise} \end{cases}, \qquad \{\boldsymbol{N_d}\}_{i,j} = \begin{cases} 1 & \text{if} \quad \lambda_j \in \Lambda_{d,i} \\ 0 & \text{otherwise} \end{cases},$$

the upstream and downstream pressure and flow link variables can be related to the pressure and flow nodal variables by the matrix equations

$$\begin{bmatrix} \boldsymbol{P}(s,\boldsymbol{0}) \\ \hdashline \boldsymbol{P}(s,\boldsymbol{l}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{N_u} & \vdots & \boldsymbol{N_d} \end{bmatrix}^T \boldsymbol{\Psi}(s), \quad \begin{bmatrix} \boldsymbol{N_u} & \vdots & \boldsymbol{N_d} \end{bmatrix} \begin{bmatrix} \boldsymbol{Q}(s,\boldsymbol{0}) \\ \hdashline -\boldsymbol{Q}(s,\boldsymbol{l}) \end{bmatrix} = \boldsymbol{\Theta}(s),$$

which are expressions of the pressure preservation, and mass conservation of a simple node, respectively (*i.e.* matrix versions of equations (7) and (9) respectively). Combining these link and node relationship expressions with the link functions (22) yields an admittance matrix expression for the network dynamics

$$\boldsymbol{Y}(s)\boldsymbol{\Psi}(s) = \boldsymbol{\Theta}(s)$$

where $\boldsymbol{Y}(s)$ is the symetric $n_n \times n_n$ admittance matrix given by

$$\boldsymbol{Y}(s) = \left[\begin{array}{c|c} \boldsymbol{N_u} & \boldsymbol{N_d} \end{array}\right] \left[\begin{array}{c:c} \boldsymbol{Z_c}^{-1}(s)\coth\boldsymbol{\Gamma}(s) & -\boldsymbol{Z_c}^{-1}(s)\mathrm{csch}\boldsymbol{\Gamma}(s) \\ \hdashline -\boldsymbol{Z_c}^{-1}(s)\mathrm{csch}\boldsymbol{\Gamma}(s) & \boldsymbol{Z_c}^{-1}(s)\coth\boldsymbol{\Gamma}(s) \end{array}\right] \left[\begin{array}{c|c} \boldsymbol{N_u} & \boldsymbol{N_d} \end{array}\right]^T \tag{22}$$

which posesses the elementwise representation

$$\{\boldsymbol{Y}(s)\}_{i,k} = \begin{cases} \displaystyle\sum_{\lambda_j \in \Lambda_i} \frac{\coth\Gamma_j(s)}{Z_{cj}(s)} & \text{if } k = i \\ \displaystyle -\frac{\mathrm{csch}\Gamma_j(s)}{Z_{cj}(s)} & \text{if } \lambda_j \in \Lambda_i \cap \Lambda_k \\ 0 & \text{otherwise} \end{cases}. \tag{23}$$

Despite the fact that (23) represents a solution for a simple node network only, it provides the basis and framework within which to include the compound node dynamics.

Network Matrix for a Network with Pressure Dependent Nodal Flows

The first extension to the work outlined above is the consideration of the case of compound nodes consisting of only one connection, that is, compound nodes consisting of a hydraulic component connected to a single junction. Examples of such components are emitters, scour valves, surge tanks or pressure relief valves. The flow into the hydraulic component is clearly pressure dependent, but to generalise slightly further, it is assumed to be influenced by a control action U_i (*e.g.* time varying valve opening, or fluctuating air volume). For a network with such node types, a general expression for the flow into the compound node's component is

$$Q_{ci}(s) = Y_{ci}(s)P_{ci}(s) - Y_{ui}(s)U_i(s) \tag{24}$$

where the first term on the right side of (24) represents the pressure dependent flow with admittance function Y_{ci} and connection pressure P_{ci}, and the second term represents the controlled flow with admittance function Y_{ui} and control U_i. Note that (24) is simple a scalar version of (20).

Consider a network $(\mathcal{G}(\mathcal{N}, \Lambda), \mathcal{C}, \mathcal{P})$ with n_c such compound nodes collected into the set $\mathcal{N}_c$, with $\mathcal{N}_s$ as the set of remaining simple nodes ($\mathcal{N} = \mathcal{N}_s \cup \mathcal{N}_c$). Ordering the nodal states with the $\mathcal{N}_c$ nodes first, a network admittance expression can be derived

$$\boldsymbol{Y}(s)\left[\begin{array}{c}\boldsymbol{\Psi_c}(s)\\ \hdashline \boldsymbol{\Psi_s}(s)\end{array}\right]=\left[\begin{array}{c}\boldsymbol{\Theta_c}(s)\\ \hdashline \boldsymbol{\Theta_s}(s)\end{array}\right] \tag{25}$$

where $\boldsymbol{Y}$ is the admittance matrix (23) for the simple node network given by $(\mathcal{G}(\mathcal{N},\Lambda),\mathcal{P})$, $\boldsymbol{\Psi_c}$ and $\boldsymbol{\Psi_s}$ are the nodal pressures at the compound junction and simple nodes respectively, and $\boldsymbol{\Theta_c}$ and $\boldsymbol{\Theta_s}$ are the nodal flows at the compound junction and simple nodes respectively. In the expression (25), $\boldsymbol{\Theta_c}$ corresponds to the flow that *enters* the network $(\mathcal{G}(\mathcal{N},\Lambda),\mathcal{P})$ from the compound nodes component, which is external to the network $(\mathcal{G}(\mathcal{N},\Lambda),\mathcal{P})$, and as such, the component dynamics are not directly incorporated in (25). To incorporate the component dynamics the relationship between each Θ_{ci} and Q_{ci} from (24) must be used. Given that Θ_{ci} is the flow at the junction into the network, and Q_{ci} is the flow at the junction into the component, for continuity it is required that $\Theta_{ci} + Q_{ci} = 0$. Therefore, infact

$$\begin{aligned}\boldsymbol{\Theta_c}(s) = &\ -\boldsymbol{Q_c}(s)\\ = &\ -\text{diag}\ \{Y_{c1}(s),\ldots,Y_{cn_c}(s)\}\,\boldsymbol{\Psi_c}(s)+\text{diag}\ \{Y_{u1}(s),\ldots,Y_{un_c}(s)\}\,\boldsymbol{U}(s)\end{aligned} \tag{26}$$

where $\boldsymbol{Q_c}$ and $\boldsymbol{U}$ are vector organisations of the compound node connection flows and controlled states. Combining (26) with (25) yields the admittance form for the compound node network as

$$\left(\boldsymbol{Y}(s)+\left[\begin{array}{c:c}\text{diag}\ \{Y_{c1}(s),\ldots,Y_{cn_c}(s)\} & \boldsymbol{0}\\ \hdashline \boldsymbol{0} & \boldsymbol{0}\end{array}\right]\right)\left[\begin{array}{c}\boldsymbol{\Psi_c}(s)\\ \hdashline \boldsymbol{\Psi_s}(s)\end{array}\right]=\left[\begin{array}{c:c}\text{diag}\ \{Y_{u1}(s),\ldots,Y_{un_c}(s)\} & \boldsymbol{0}\\ \hdashline \boldsymbol{0} & \boldsymbol{I}\end{array}\right]\left[\begin{array}{c}\boldsymbol{U}(s)\\ \hdashline \boldsymbol{\Theta_s}(s)\end{array}\right] \tag{27}$$

where, using the identity for $\boldsymbol{Y}$ from (23) for the simple node network $(\mathcal{G}(\mathcal{N},\Lambda),\mathcal{P})$, the elementwise expression for (27) is,

$$\{\boldsymbol{Y}(s)+\text{diag}\ \{\text{diag}\ \{Y_{c1}(s),\ldots,Y_{cn_c}(s)\}\,,\boldsymbol{0}\}\}_{i,k} = \begin{cases}\displaystyle\sum_{j\in\Lambda_i}\frac{\coth\Gamma_j(s)}{Z_j(s)} & \text{if } k=i\in\mathcal{N}_s\\ \displaystyle\sum_{j\in\Lambda_i}\frac{\coth\Gamma_j(s)}{Z_j(s)}+Y_{ci}(s) & \text{if } k=i\in\mathcal{N}_c\\ \displaystyle-\frac{\text{csch}\Gamma_j(s)}{Z_j(s)} & \text{if } \lambda_j=\Lambda_i\cap\Lambda_j\\ 0 & \text{otherwise}\end{cases}. \tag{28}$$

where the diagonalisation refers to a block matrix organisation. Note that in (28), only the diagonal terms in the upper left block of the original network matrix are altered, that is, the terms that correspond to a nodal's pressure influence on its nodal flow.

Admittance Matrix for a General Compound Node Network

In this section, the admittance matrix for a compound node network $(\mathcal{G}(\mathcal{N},\Lambda),\mathcal{P},\mathcal{C})$ comprised of compound nodes of a general type is derived. Before this can be done, an important preliminary concept must be introduced. Given a compound node network $(\mathcal{G}(\mathcal{N},\Lambda),\mathcal{P},\mathcal{C})$ with compound nodes $\mathcal{N}_c$ and simple nodes $\mathcal{N}_s$, the associated *simple node expanded network* is given by the simple node network $(\mathcal{G}(\mathcal{N}_o,\Lambda_o),\mathcal{P}_o)$ where the node set is given as the union of the simple node set $\mathcal{N}_s$ and the compound node connection sets $\mathcal{N}_i$, that is

$$\mathcal{N}_o = \mathcal{N}_s \cup \bigcup_{i\in\mathcal{N}_c} \mathcal{N}_i,$$

where $\mathcal{N}_i$ is the set of connections for compound node i (as defined in Appendix), the link set Λ_o is given by a relabelling of the original link set Λ to the nodes in $\mathcal{N}_o$, which is given by

$$\Lambda_o = \{\langle\lambda\rangle_o : \lambda\in\Lambda\}$$

where the function $\langle\lambda\rangle_o : \mathcal{N}\times\mathcal{N}\mapsto\mathcal{N}_o\times\mathcal{N}_o$ is the relabelling function given by

$$\langle(i,j)\rangle_o = \begin{cases} (i,j) & \text{if } i,j\in\mathcal{N}_s \\ (i,l) & \text{if } i\in\mathcal{N}_s \text{ and } (i,j)\in\Lambda_{djl}, l\in\mathcal{N}_j, j\in\mathcal{N}_c \\ (k,j) & \text{if } (i,j)\in\Lambda_{uik}, k\in\mathcal{N}_i, i\in\mathcal{N}_c \text{ and } j\in\mathcal{N}_s \\ (k,l) & \text{if } (i,j)=\Lambda_{uik}\cup\Lambda_{djl}, k\in\mathcal{N}_i, l\in\mathcal{N}_j, i,j\in\mathcal{N}_c \\ \emptyset & \text{otherwise} \end{cases}, \tag{29}$$

and $\mathcal{P}_o$ is the link data set $\mathcal{P}$ for the relabeled links [note that, as defined in Appendix , $\Lambda_{ujk}\subset\Lambda_{uj}$ ($\Lambda_{djk}\subset\Lambda_{dj}$) in (29) are defined as the set of links whose upstream (downstream) node is the k-th connection of compound node j]. This concept of a simple node expanded network is fundamental to the developments within this section as it provides the basic framework within which to include compound nodes. An example of the expanded simple connection network for a given compound node network in Figure 2(a) is given in Figure 2(b), this is studied in greater depth later.

The nodal states for an expanded simple node network are given as

$$\boldsymbol{\Psi}(s) = \begin{bmatrix} \boldsymbol{\Psi}_1(s) \\ \hdashline \vdots \\ \hdashline \boldsymbol{\Psi}_{n_c}(s) \\ \hdashline \boldsymbol{\Psi}_s(s) \end{bmatrix}, \quad \boldsymbol{\Theta}(s) = \begin{bmatrix} \boldsymbol{\Theta}_1(s) \\ \hdashline \vdots \\ \hdashline \boldsymbol{\Theta}_{n_c}(s) \\ \hdashline \boldsymbol{\Theta}_s(s) \end{bmatrix}, \tag{30}$$

where $\boldsymbol{\Psi}_i(s), \boldsymbol{\Theta}_i(s)$ are the pressures and flows associated with the simple connections in $\mathcal{N}_i$ for each $i \in \mathcal{N}_c$ and $\boldsymbol{\Psi}_s(s)$, and $\boldsymbol{\Theta}_s(s)$ are the pressures and flows associated with the simple nodes $\mathcal{N}_s$. As in the previous section, it is important to explain the meaning of the $\boldsymbol{\Theta}_i, i \in \mathcal{N}_c$. These variables correspond to nodal flow injections that enter the network $(\mathcal{G}(\mathcal{N}_o, \Lambda_o), \mathcal{P}_o)$ through the connections from a compound node's component. The primary motivation for the construction of the simple node expanded network is that the connection states (30) can be related by the admittance relationship

$$\boldsymbol{Y_o}(s) \begin{bmatrix} \boldsymbol{\Psi}_1(s) \\ \hdashline \vdots \\ \hdashline \boldsymbol{\Psi}_{n_c}(s) \\ \hdashline \boldsymbol{\Psi}_s(s) \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Theta}_1(s) \\ \hdashline \vdots \\ \hdashline \boldsymbol{\Theta}_{n_c}(s) \\ \hdashline \boldsymbol{\Theta}_s(s) \end{bmatrix}, \tag{31}$$

where $\boldsymbol{Y_o}$ is the network admittance matrix (23) for the simple node network $(\mathcal{G}(\mathcal{N}_o, \Lambda_o), \mathcal{P}_o)$. As with (25) for the special case of pressure dependent flows, (31) deals only with the flow into the network $(\mathcal{G}(\mathcal{N}_o, \Lambda_o), \mathcal{P}_o)$ and does not directly incorporate the dynamics of the compound node's component. To incorporate the component dynamics, the flows into the network $\boldsymbol{\Theta}_i$ are related to the compound node connection flows $\boldsymbol{Q_{c}}_i$ by applying continuity at the connections. This yields $\boldsymbol{\Theta}_i + \boldsymbol{Q_{c}}_i = \mathbf{0}$ (as explained in the previous section for the special case of compound nodes with only a single connection). Given this relationship, by (20), the following relationship between the nodal flows and the admittance form of the compound node can be derived

$$\begin{aligned} \begin{bmatrix} \boldsymbol{\Theta}_1(s) \\ \hdashline \vdots \\ \hdashline \boldsymbol{\Theta}_{n_c}(s) \end{bmatrix} &= - \begin{bmatrix} \boldsymbol{Y_{c}}_1(s) & & \\ & \ddots & \\ & & \boldsymbol{Y_{c}}_{n_c}(s) \end{bmatrix} \begin{bmatrix} \boldsymbol{\Psi}_1(s) \\ \hdashline \vdots \\ \hdashline \boldsymbol{\Psi}_{n_c}(s) \end{bmatrix} \\ &\quad + \begin{bmatrix} \boldsymbol{Y_{u}}_1(s) & & \\ & \ddots & \\ & & \boldsymbol{Y_{u}}_{n_c}(s) \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_1(s) \\ \hdashline \vdots \\ \hdashline \boldsymbol{U}_{n_c}(s) \end{bmatrix} \end{aligned} \tag{32}$$

where the first term on the righthand side of (32) is the pressure dependent term and the second term corresponds to the connection flows associated with the controlled nodal states. Substituting (32) into (31) provides the full expression relating the nodal pressures to the controlled nodal states and simple node flows

$$\left(\boldsymbol{Y_o}(s) + \begin{bmatrix} \boldsymbol{Y_{c1}}(s) & & & \\ & \ddots & & \\ & & \boldsymbol{Y_{cn_c}}(s) & \\ & & & \mathbf{0} \end{bmatrix}\right) \begin{bmatrix} \boldsymbol{\Psi}_1(s) \\ \hdashline \vdots \\ \hdashline \boldsymbol{\Psi}_{n_c}(s) \\ \hdashline \boldsymbol{\Psi}_s(s) \end{bmatrix} = \begin{bmatrix} \boldsymbol{Y_{u1}}(s) & & & \\ & \ddots & & \\ & & \boldsymbol{Y_{un_c}}(s) & \\ & & & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_1(s) \\ \hdashline \vdots \\ \hdashline \boldsymbol{U}_{n_c}(s) \\ \hdashline \boldsymbol{\Theta_s}(s) \end{bmatrix} \tag{33}$$

where the elementwise expression for the admittance matrix acting on the simple connection pressures can be derived as

$$\{\boldsymbol{Y_o}(s) + \text{diag}\,\{\boldsymbol{Y_{c1}}, \ldots, \boldsymbol{Y_{cn_c}}, \mathbf{0}\}\}_{i,k} = \begin{cases} \displaystyle\sum_{j\in\Lambda_i} \frac{\coth\Gamma_j(s)}{Z_j(s)} & \text{if } k = i \in \mathcal{N}_s \\ \displaystyle\sum_{j\in\Lambda_i} \frac{\coth\Gamma_j(s)}{Z_j(s)} + \{\boldsymbol{Y_{ci}}(s)\}_{\langle i,i\rangle_l} & \text{if } k = i \in \mathcal{N}_c \text{ and } i \in \mathcal{N}_l, l \in \mathcal{N}_c \\ -\dfrac{\text{csch}\Gamma_j(s)}{Z_j(s)} & \text{if } \lambda_j \in \Lambda_i \cap \Lambda_k, i, k \in \mathcal{N}_s \\ \{\boldsymbol{Y_{ci}}(s)\}_{\langle i,k\rangle_l} & \text{if } i, k \in \mathcal{N}_l, l \in \mathcal{N}_c \\ 0 & \text{otherwise} \end{cases}, \tag{34}$$

where $\mathcal{N}_l$ is the l-th compound node connection set, and $\langle\cdot\rangle_l$ maps from the ordering in the state vectors to the local ordering for the connections at compound node $l \in \mathcal{N}_c$. Here, unlike the pressure dependent outflow, off diagonal terms in the admittance matrix are changed in addition to the diagonal terms. The structure of (34) is consistent with that of (23) where the diagonal terms are comprised of sums of transfer functions, each associated with the connection between a node and its neighboring nodes, and the off-diagonal terms are comprised of single transfer functions, each associated with the connection between two nodes.

The matrix equation (33) represents the network admittance matrix for a compound node network of arbitrary configuration and is the main result of the paper. The following example demonstrates the construction process for the network in Figure 2(a).

Example 5. *Consider the network* $(\mathcal{G}(\mathcal{N},\Lambda),\mathcal{P},\mathcal{C})$ *in Figure 2(a), with*

$$\begin{aligned}
&\mathcal{N}=\{1,2,3,4,5,6\}\,,\ \textit{as the set of nodes}\\
&\Lambda=\{(1,2),(2,3),(2,4),(3,4),(3,5),(4,5),(5,6)\}\,,\ \textit{as the set of links}\\
&\mathcal{P}=\{\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3,\mathcal{P}_4,\mathcal{P}_5,\mathcal{P}_6,\mathcal{P}_7\}\,,\ \textit{as the set of link functions, and}\\
&\mathcal{C}=\{\phi_2,\phi_3,\phi_5\}\,,\ \textit{as the set of compound node functions,}
\end{aligned}$$

where the compound node set is $\mathcal{N}_c=\{2,3,5\}$ *and the simple node set is* $\mathcal{N}_s=\{1,4,6\}$. *Given the compound node realisation in Figure 2(b), the connection sets for the compound nodes can be expressed as* $\mathcal{N}_2=\{21,22\}$, $\mathcal{N}_3=\{31,32,33\}$, *and* $\mathcal{N}_5=\{51\}$, *which leads to the expanded simple node network* $(\mathcal{G}(\mathcal{N}_o,\Lambda_o),\mathcal{P}_o)$ *in Figure 2(c) defined by the following network sets*

$$\begin{aligned}
&\mathcal{N}_o=\{1,21,22,31,32,33,4,51,6\}\,,\\
&\Lambda_o=\{(1,21),(22,31),(22,4),(32,4),(33,51),(4,51),(51,6)\}\,,\\
&\mathcal{P}_o=\{\mathcal{P}_{o1},\mathcal{P}_{o2},\mathcal{P}_{o3},\mathcal{P}_{o4},\mathcal{P}_{o5},\mathcal{P}_{o6},\mathcal{P}_{o7}\}
\end{aligned}$$

where Λ_o *is constructed from* Λ *according to the relabeling function* (29)*, and* $\mathcal{P}_o$ *is the relabeled elements of* $\mathcal{P}$*, associated with* Λ_o. *For the expanded simple node network* $(\mathcal{G}(\mathcal{N}_o,\Lambda_o),\mathcal{P}_o)$*, the nodal states, ordered as in* (30)*, are*

$$\boldsymbol{\Psi}(s)=\begin{bmatrix}\Psi_{21}(s)\\ \Psi_{22}(s)\\ \hdashline \Psi_{31}(s)\\ \Psi_{32}(s)\\ \Psi_{33}(s)\\ \hdashline \Psi_{51}(s)\\ \hdashline \Psi_{1}(s)\\ \Psi_{4}(s)\\ \Psi_{6}(s)\end{bmatrix},\quad \boldsymbol{\Theta}(s)=\begin{bmatrix}\Theta_{21}(s)\\ \Theta_{22}(s)\\ \hdashline \Theta_{31}(s)\\ \Theta_{32}(s)\\ \Theta_{33}(s)\\ \hdashline \Theta_{51}(s)\\ \hdashline \Theta_{1}(s)\\ \Theta_{4}(s)\\ \Theta_{6}(s)\end{bmatrix}. \tag{35}$$

Given (23)*, the network admittance matrix* $\boldsymbol{Y}_o$ *for the expanded simple node network* $(\mathcal{G}(\mathcal{N}_o,\Lambda_o),\mathcal{P}_o)$ *can be* *expressed as*

$$\boldsymbol{Y_o}(s) =$$

$$\left[\begin{array}{cc|ccc|c|ccc}
t_1 & 0 & 0 & 0 & 0 & 0 & -s_1 & 0 & 0 \\
0 & \sum_{j=2,3} t_j & -s_2 & 0 & 0 & 0 & 0 & -s_3 & 0 \\
\hline
0 & -s_2 & t_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & t_4 & 0 & 0 & 0 & -s_4 & 0 \\
0 & 0 & 0 & 0 & t_5 & -s_5 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & -s_5 & \sum_{j=5,6,7} t_j & 0 & -s_6 & -s_7 \\
\hline
-s_1 & 0 & 0 & 0 & 0 & 0 & t_1 & 0 & 0 \\
0 & -s_3 & 0 & -s_4 & 0 & -s_6 & 0 & \sum_{j=3,4,6} t_j & 0 \\
0 & 0 & 0 & 0 & 0 & -s_7 & 0 & 0 & t_7
\end{array}\right] \tag{36}$$

where $t_j = t_j(s) = Z_c^{-1}(s)\coth\Gamma_j(s)$ *and* $s_j = s_j(s) = Z_c^{-1}(s)\text{csch}\Gamma_j(s)$*. Note that as* (36) *is a network admittance matrix, it is a square symmetric matrix, and the row and colum partitions of* (36) *correspond to the partitions of the state vectors in* (35)*. Assuming that node 2 has one controlled state, node 3 has two and node 5 has none, the admittance forms* (20) *of the compound node functions* ϕ_2*,* ϕ_3*, and* ϕ_5 *are*

$$\begin{aligned}
\begin{bmatrix} Q_{c21}(s) \\ Q_{c22}(s) \end{bmatrix} &= \boldsymbol{Y_c}_2(s) \begin{bmatrix} P_{c21}(s) \\ P_{c22}(s) \end{bmatrix} - \boldsymbol{Y_u}_2(s)U_2(s), \\
\begin{bmatrix} Q_{c31}(s) \\ Q_{c32}(s) \\ Q_{c33}(s) \end{bmatrix} &= \boldsymbol{Y_c}_3(s) \begin{bmatrix} P_{c31}(s) \\ P_{c32}(s) \\ P_{c33}(s) \end{bmatrix} - \boldsymbol{Y_u}_3(s) \begin{bmatrix} U_{31}(s) \\ U_{32}(s) \end{bmatrix}, \\
Q_{c51}(s) &= \boldsymbol{Y_c}_5(s)P_{c51}(s).
\end{aligned} \tag{37}$$

The pressure dependent term in the compound node network admittance matrix (33) *can be constructed as*

$$\text{diag}\left\{\boldsymbol{Y_{c}}_2(s),\boldsymbol{Y_{c}}_3(s),\boldsymbol{Y_{c}}_5(s),\mathbf{0}\right\} =$$

$$\left[\begin{array}{cc:ccc:c:ccc}
\{\boldsymbol{Y_c}_2\}_{1,1} & \{\boldsymbol{Y_c}_2\}_{1,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\{\boldsymbol{Y_c}_2\}_{2,1} & \{\boldsymbol{Y_c}_2\}_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\hdashline
0 & 0 & \{\boldsymbol{Y_c}_3\}_{1,1} & \{\boldsymbol{Y_c}_3\}_{1,2} & \{\boldsymbol{Y_c}_3\}_{1,3} & 0 & 0 & 0 & 0\\
0 & 0 & \{\boldsymbol{Y_c}_3\}_{2,1} & \{\boldsymbol{Y_c}_3\}_{2,2} & \{\boldsymbol{Y_c}_3\}_{2,3} & 0 & 0 & 0 & 0\\
0 & 0 & \{\boldsymbol{Y_c}_3\}_{3,1} & \{\boldsymbol{Y_c}_3\}_{3,2} & \{\boldsymbol{Y_c}_3\}_{3,3} & 0 & 0 & 0 & 0\\
\hdashline
0 & 0 & 0 & 0 & 0 & \boldsymbol{Y_c}_5 & 0 & 0 & 0\\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

and organising the state vector on the right side of (33) *as*

$$\left[\begin{array}{c:cc:ccc} U_{21}(s) & U_{31}(s) & U_{32}(s) & \Theta_1(s) & \Theta_4(s) & \Theta_6(s)\end{array}\right]^T,$$

the matrix operator on this state vector is given as

$$\text{diag}\left\{\boldsymbol{Y_u}_2(s),\boldsymbol{Y_u}_3(s),\boldsymbol{Y_u}_5(s),\boldsymbol{I}\right\} = \left[\begin{array}{c:cc:ccc}
\{\boldsymbol{Y_u}_2\}_{1,1} & 0 & 0 & 0 & 0 & 0\\
\{\boldsymbol{Y_u}_2\}_{2,1} & 0 & 0 & 0 & 0 & 0\\
\hdashline
0 & \{\boldsymbol{Y_u}_3\}_{1,1} & \{\boldsymbol{Y_u}_3\}_{1,2} & 0 & 0 & 0\\
0 & \{\boldsymbol{Y_u}_3\}_{2,1} & \{\boldsymbol{Y_u}_3\}_{2,2} & 0 & 0 & 0\\
0 & \{\boldsymbol{Y_u}_3\}_{3,1} & \{\boldsymbol{Y_u}_3\}_{3,2} & 0 & 0 & 0\\
\hdashline
0 & 0 & 0 & 0 & 0 & 0\\
\hdashline
0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right],$$

(i.e. *as there are no controlled states for compound node 5,* $\boldsymbol{Y_u}_5$ *is a matrix of one row and zero columns). Given these matrix identities, the network admittance equation* (33) *for the compound node network* $(\mathcal{G}(\mathcal{N},\Lambda),\mathcal{P},\mathcal{C})$ *can be constructed.*

Formulation of a Computable Model

Despite the qualitative understanding enabled by the representation (33), it is not suitable for numerical implementation. The computational utility of the compound node network model requires a mapping from the inputs (known nodal states) to the outputs (unknown nodal states). Consider the network $(\mathcal{G}(\mathcal{N},\Lambda),\mathcal{P},\mathcal{C})$ with compound nodes $\mathcal{N}_c$, and simple nodes $\mathcal{N}_s$ that can be partitioned as $\mathcal{N}_s = \mathcal{N}_J \cup \mathcal{N}_d \cup \mathcal{N}_r$ where $\mathcal{N}_J$ are junctions, $\mathcal{N}_d$ are the demand nodes (flow control nodes) and $\mathcal{N}_r$ are the reservoirs (pressure control nodes). The inputs for such a setup are the controlled node states $\boldsymbol{U}_i$ for each $i \in \mathcal{N}_c$, the controlled nodal demands Θ_{di} for each $i \in \mathcal{N}_d$ and the known reservoir pressures Ψ_{ri} for each $i \in \mathcal{N}_r$. Defining the nodal set

$$\mathcal{N}_D = \bigcup_{i\in\mathcal{N}_c} \mathcal{N}_i \cup \mathcal{N}_J \cup \mathcal{N}_d \tag{38}$$

(33) can be expressed as

$$\left[\begin{array}{c:c} \boldsymbol{Y}_{DD}(s) & \boldsymbol{Y}_{\boldsymbol{o}Dr}(s) \\ \hdashline \boldsymbol{Y}_{\boldsymbol{o}rD}(s) & \boldsymbol{Y}_{\boldsymbol{o}rr}(s) \end{array}\right] \left[\begin{array}{c} \boldsymbol{\Psi}_{\boldsymbol{D}}(s) \\ \hdashline \boldsymbol{\Psi}_{\boldsymbol{r}}(s) \end{array}\right] = \left[\begin{array}{cccc} \boldsymbol{Y}_{\boldsymbol{u}}(s) & & & \\ & \boldsymbol{I} & & \\ & & \boldsymbol{I} & \\ & & & \boldsymbol{I} \end{array}\right] \left[\begin{array}{c} \boldsymbol{U}(s) \\ \hdashline \boldsymbol{\Theta}_{\boldsymbol{J}}(s) \\ \hdashline \boldsymbol{\Theta}_{\boldsymbol{d}}(s) \\ \hdashline \boldsymbol{\Theta}_{\boldsymbol{r}}(s) \end{array}\right] \tag{39}$$

where $\boldsymbol{Y}_{\boldsymbol{u}} = \text{diag}\ \{\boldsymbol{Y}_{\boldsymbol{u}1}, \ldots, \boldsymbol{Y}_{\boldsymbol{u}n_c}\}$, $\boldsymbol{U}$ is the vector concatenation of the $\boldsymbol{U}_i$ vectors, $\boldsymbol{Y}_{\boldsymbol{o}rD}$, $\boldsymbol{Y}_{\boldsymbol{o}Dr}$, and $\boldsymbol{Y}_{\boldsymbol{o}rr}$ are the partitions of $\boldsymbol{Y}_{\boldsymbol{o}}$ corresponding to the node sets $\mathcal{N}_D$ and $\mathcal{N}_r$, that is

$$\left[\begin{array}{c:c} \boldsymbol{Y}_{\boldsymbol{o}DD}(s) & \boldsymbol{Y}_{\boldsymbol{o}Dr}(s) \\ \hdashline \boldsymbol{Y}_{\boldsymbol{o}rD}(s) & \boldsymbol{Y}_{\boldsymbol{o}rr}(s) \end{array}\right] = \boldsymbol{Y}_{\boldsymbol{o}}(s),$$

and $\boldsymbol{Y}_{DD}$ incorporates the compound node dynamics and is given by

$$\boldsymbol{Y}_{DD}(s) = \boldsymbol{Y}_{\boldsymbol{o}DD}(s) + \text{diag}\ \{\boldsymbol{Y}_{\boldsymbol{c}1}(s), \ldots, \boldsymbol{Y}_{\boldsymbol{c}n_c}(s), \boldsymbol{0}\}\,.$$

Noting that the nodal flow for junctions $\boldsymbol{\Theta}_{\boldsymbol{J}} = \boldsymbol{0}$, (39) can be reorganised to map from the input to the output as

$$\left[\begin{array}{c} \boldsymbol{\Psi}_D(s) \\ \hdashline \boldsymbol{\Theta}_r(s) \end{array}\right] = \boldsymbol{H}(s) \left[\begin{array}{c} \boldsymbol{U}(s) \\ \hdashline \boldsymbol{\Theta}_d(s) \\ \hdashline \boldsymbol{\Psi}_r(s) \end{array}\right], \tag{40}$$

where $\boldsymbol{H}$ is given by

$$\boldsymbol{H}(s) = \left[\begin{array}{c|c|c} \boldsymbol{Z}_{Du}(s)\boldsymbol{Y}_{\boldsymbol{u}}(s) & \boldsymbol{Z}_{Dd}(s) & -\boldsymbol{Z}_{DD}(s)\boldsymbol{Y}_{\boldsymbol{o}Dr}(s) \\ \hline \boldsymbol{Y}_{\boldsymbol{or}D}(s)\boldsymbol{Z}_{Du}(s)\boldsymbol{Y}_{\boldsymbol{u}}(s) & \boldsymbol{Y}_{\boldsymbol{or}D}(s)\boldsymbol{Z}_{Dd}(s) & \boldsymbol{Y}_{\boldsymbol{or}r}(s) - \boldsymbol{Y}_{\boldsymbol{or}D}(s)\boldsymbol{Z}_{DD}(s)\boldsymbol{Y}_{Dr}(s) \end{array}\right]$$

where $\boldsymbol{Z}_{DD}$ is an impedance matrix given by $\boldsymbol{Z}_{DD} = \boldsymbol{Y}_{DD}^{-1}$ and is partitioned as

$$\boldsymbol{Z}_{DD}(s) = \left[\begin{array}{c|c|c} \boldsymbol{Z}_{Du}(s) & \boldsymbol{Z}_{DJ}(s) & \boldsymbol{Z}_{Dd}(s) \end{array}\right]$$

where each partition corresponds to the nodes of the sets $\bigcup_{i\in\mathcal{N}_c}\mathcal{N}_i$, $\mathcal{N}_J$ and $\mathcal{N}_d$, respectively.

NUMERICAL EXAMPLES

Two example networks are considered in the following. Network-1 (Figure 2) is a 7-pipe/6-node network adapted from Zecchin *et al.* [2009], and Network-2 (Figure 3) is a 51-pipe/34-node network adapted from Vítkovský [2001]. The numerical experiments compare the frequency responses as calculated by the proposed admittance matrix method, and that calculated from the discrete time-domain method of characteristics (MOC) model via the discrete Fourier transform (DFT). As expected, and verified by many experiments, the admittance matrix methodology yields the exact solution for linear networks. Hence, comparisons involving linear networks are not presented. A question of greater practical interest is *how well does the method approximate systems comprised of nonlinear components?* It is for this reason that the results presented are for numerical experiments performed on nonlinear systems. The MOC results for network-1 were obtained from a frequency sweep, where the system was excited into a steady oscillatory state, one frequency at a time. In contrast, the MOC results for network-2 were obtained from a transient excitation, where the frequency response was computed using the entire transient response.

Network-1 in Steady-Oscillatory State

For the numerical study of network-1 (Figure 2), the network parameters are as follows; pipe diameters = $\{60, 50, 35, 50, 35, 50, 60\}$ mm, pipe lengths = $\{31, 52, 34, 41, 26, 57, 28\}$ m, the wavespeeds and the Darcy-Weisbach friction factors were set to 1000 m/s and 0.02, respectively, for all pipes, and the compound node details are given in Tables 1 and 2. The demand at node 1 was taken as a sinusoidal form of amplitude 0.2 L/s about a base demand level of 10 L/s. For the MOC model, a frequency sweep was performed for 200 frequencies up to

20 Hz. Figure 4 presents the amplitude of the sinusoidal pressure fluctuations observed at node 6 as computed by the Laplace-domain admittance matrix and the DFT of the MOC in steady oscillatory state. Figure 5 presents the same results for the sinusoidal flow fluctuations into the reservoir at node 1. Within both figures, the error between the Laplace-domain and MOC approaches is presented in the bottom subfigure.

Despite the nonlinearities of pipe friction and the valve pressure loss, extremely good matches between the two methods are observed as the errors for both the pressure and flow are more than three orders of magnitude less than the amplitude of the response oscillations (as seen in Figures 4 and 5). The larger errors occur at the networks harmonic frequencies, where the linear admittance matrix model slightly over estimates the amplitude of the nonlinear MOC model.

Network-2 in Transient State

The original formulation for network-2 in Vítkovský [2001] was modified as follows: pipe lengths were rounded to the nearest meter and the wavespeeds were all made to be 1000 m/s to ensure a Courant number of 1, which was required to preserve the accuracy of the MOC; the nodal demands were doubled to increase the flow through the network; nodes 7, 9, 11, 19, 22, 23, 25, and 34 were converted to compound nodes, the details of which are given in Tables 1 and 2. For brevity, the network details are not given here, but the range of network parameters are $[450, 895]$ m for pipe lengths, $[304, 1524]$ mm for pipe diameters, and $[80, 280]$ L/s for nodal demands (for case study details, the reader is referred to [Vítkovský, 2001]).

In order to avoid burdensome computational requirements, network-2 was analyzed in the transient state as opposed to the steady-oscillatory state used for network-1. This meant that the frequency response was computed from a single MOC simulation of the system for the entire response time of the system. The network was excited into a transient state by a perturbing the flow at nodes $\{14, 17, 28\}$. Results for two types of excitations are presented. The first type of excitation involved a pulse perturbation, which was achieved by reducing the nodal flows by a magnitude of $\{70, 50, 100\}$ L/s for a duration $\{0.055, 0.025, 0.075\}$ s only. The second involved a step perturbation, which was achieved by reducing the nodal flows by a magnitude of $\{70, 50, 100\}$ L/s.

A plot of the frequency response at nodes 14 and 18 for network-2 with the pulse perturbation is given in Figures 6 and 7 where the top subfigure gives the frequency response of the Laplace-domain method, and the bottom subfigure shows the magnitude of the error between the Laplace-domain method and the MOC. Due to the densely distributed harmonics, only the range 0 - 4 Hz is shown.

Figures 6 and 7 show that the error between the DFT of the MOC and the proposed Laplace-domain admittance matrix method is small in comparison to the spectral amplitude of the frequency response. This illustrates that even for a network of a large size containing nonlinear elements such as emitters, valves, and accumulators, the linear admittance matrix model provides an extremely good approximation of the nonlinear MOC model. Similarly with network-1, the larger errors occur at the networks harmonics. There is also a slight trend of increasing in magnitude with increasing frequency. Despite this, the matches are excellent.

A plot of the frequency response at nodes 14 and 18 for the step excitation is given in Figures 8 and 9, where the top subfigure gives the frequency response and the bottom subfigure shows the magnitude of the error between the two methods. The plots are presented with a log scale on the vertical axis as the excitation energy for a step input reduces rapidly for increasing frequency.

It is observed from Figures 8 and 9 that the error between the methods is over an order of magnitude less than the spectral amplitude of the frequency response. This error is surprisingly low, given that for the step input the operating point of the linearization for the Laplace-domain model (*i.e.* the initial steady-state) is different to the final operating position of the network due to the permanent change in the nodal flows. The change of the steady-state operating point is the cause for the error peak near the zero frequency point. The linear Laplace-domain model is seen to yield a good approximation of the nonlinear system even when the operating point for the system shifts.

CONCLUSIONS

Existing methods for modeling the frequency-domain behavior of a transient fluid line system have either been limited by the configuration of network types that they can model, or are limited by the hydraulic element types that they can encompass. Within this paper, a completely new formulation is derived that is able to deal with networks of an arbitrary configuration containing an extremely broad class of hydraulic elements, termed *compound nodes*, namely those that yield an admittance type representation.

An analytic representation of the network admittance matrix has been presented in this paper, which not only yields significant qualitative information about the network, but also serves as a basis for numerical implementation of the proposed method. An interesting finding presented in this paper is that the admittance matrix for a compound node network can be expressed as the addition of two matrix terms, one pertaining to its simple node network structure, and the other containing the compound node dynamics.

The proposed new method has been verified by numerical examples with a 7-pipe network, and a 51-pipe network. For these case studies, the proposed method provided excellent agreement with the frequency response as calculated by the method of characteristics. This result was particularly interesting as the networks contained nonlinear elements (emitters, valves, accumulators, and turbulent pipes).

This proposed new approach allows complete flexibility with regard to the topological structure of a network and the types of hydraulic elements. As such, it overcomes previous limitations in frequency-domain modeling of pipe networks, and provides general basis for future research utilizing the Laplace-domain representation of fluid line systems.

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CAPTIONS FOR FIGURES AND TABLES

Figures

Example of a compound node consisting of a capacitive dead end branch and an offtake bounded by valves A and B. (a) The physical layout demonstrating the link end states of pressure and flow for links [a], [b], and [c], and the internal node states of the internal pressure ψ_o, the capacitive flow θ_o and the offtake flow θ_d. (b)The simple connection configuration, where the compound node is observed to have two simple connections with [a] incident to one and [b], and [c] incident to another. (c) The transformed simple connection states, where P_{ciA} and P_{ciB} are the pressures at connections A and B, and Q_{ciA} and Q_{ciB} are the aggregated flows into connections A and B. (d) The expanded simple node network representation of the compound node with simple node pressures Ψ_A and Ψ_B, and flows Θ_A and Θ_B. For this example, the variables are related as follows: $P_{ciA}(s) = \Psi_A(s) = P_a(l_a, s)$; $Q_{ciA}(s) = -\Theta_A(s) = Q_a(l_a, s)$;, $P_{ciB}(s) = \Psi_B(s) = P_b(l_b, s) = P_c(l_c, s)$ and $Q_{ciB}(s) - \Theta_B(s) = Q_b(l_b, s) - Q_c(0, s)$.

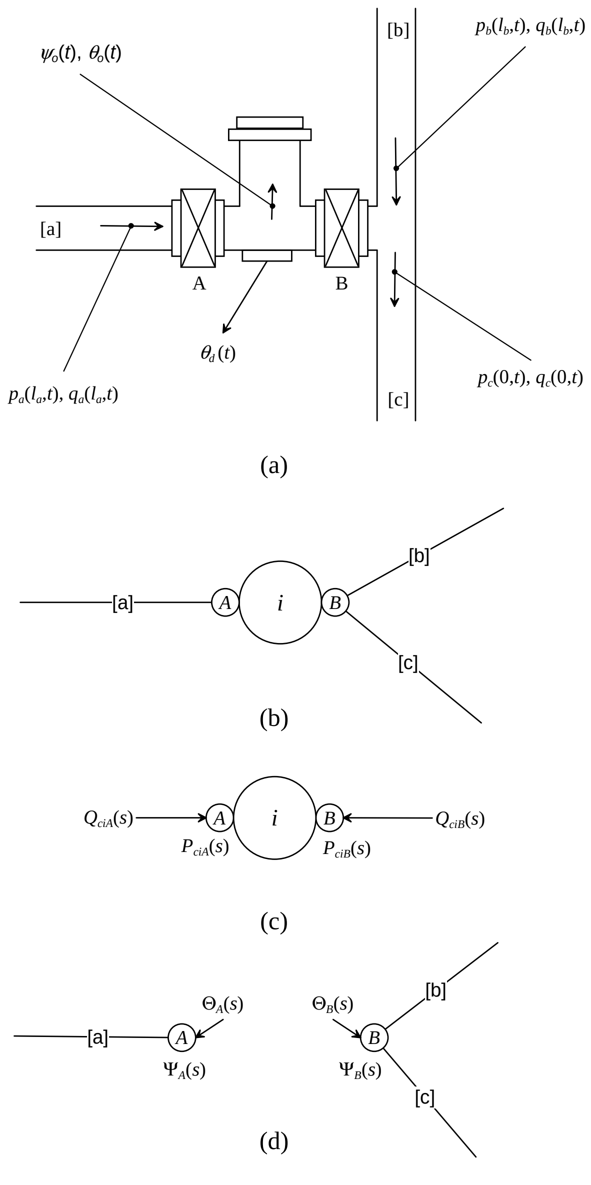


Figure 1: Caption

Example network-1 adapted from Zecchin *et al.* [2009], with controlled demand as node 1, a single valve at node 2, two valves at node 3 and capacitance branch at node 5. (a) The physical configuration of the system. (b) The compound nodes' connection configurations. (c) The simple connection expanded network, where the denoted Θ_i's are the flows into the simple connection expanded network from the compound nodes.

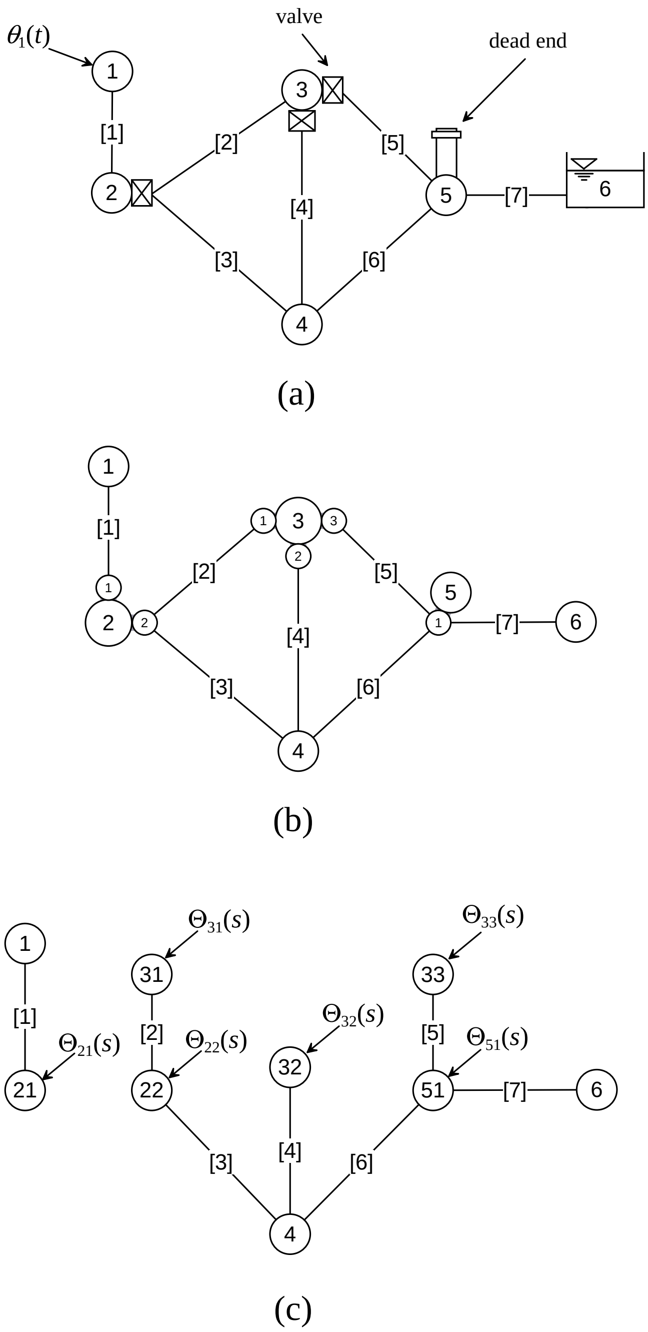
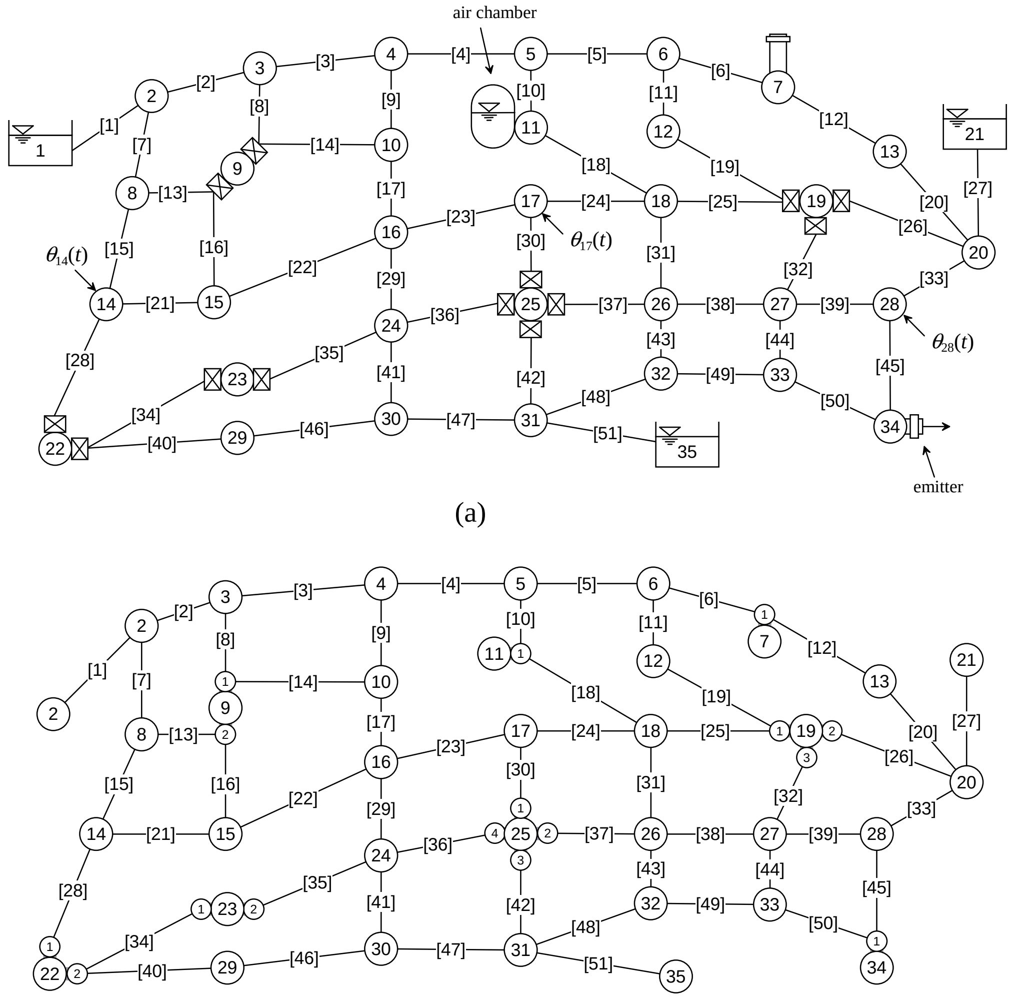


Figure 2: Caption



(a)

(b)

Figure 3: Caption

Example network-2, adapted from Vítkovský [2001], with compound nodes as described in Table 1. (a) The physical layout of the network, and (b) shows the compound nodes’ connection configurations.

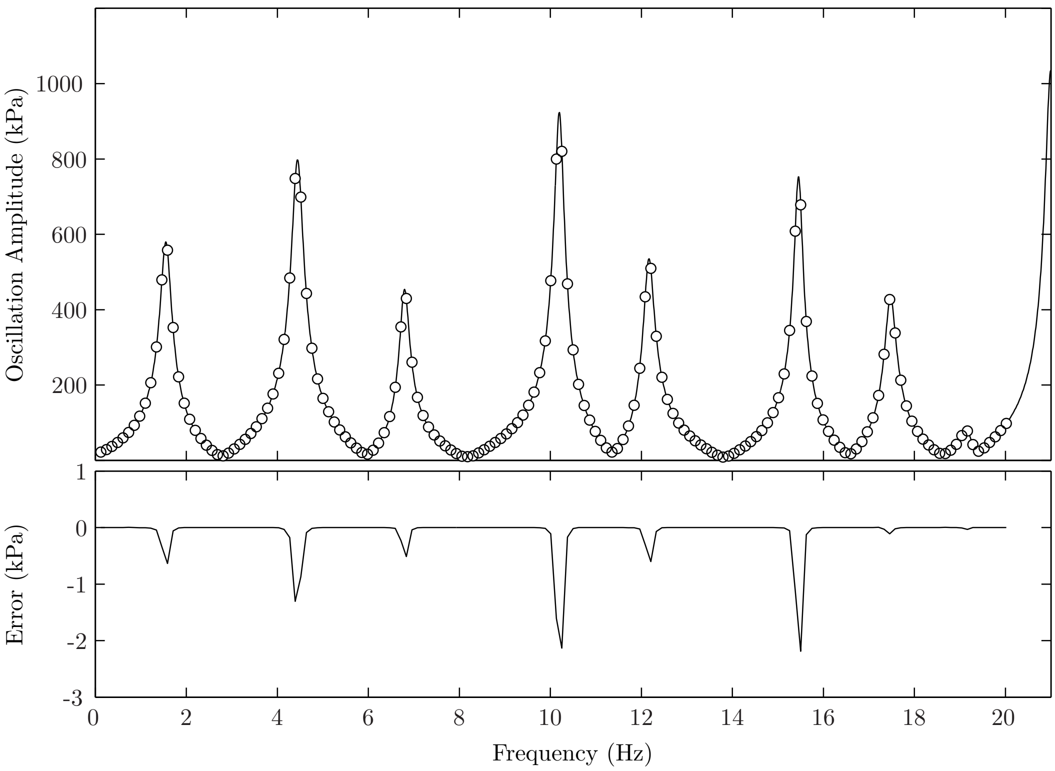


Figure 4: Caption

Sinusoidal pressure amplitude response for network-1 at node 6 for the admittance matrix model (continuous line) and the method of characteristics in steady oscillatory state (○ points). The error between the two methods is given in the bottom figure.

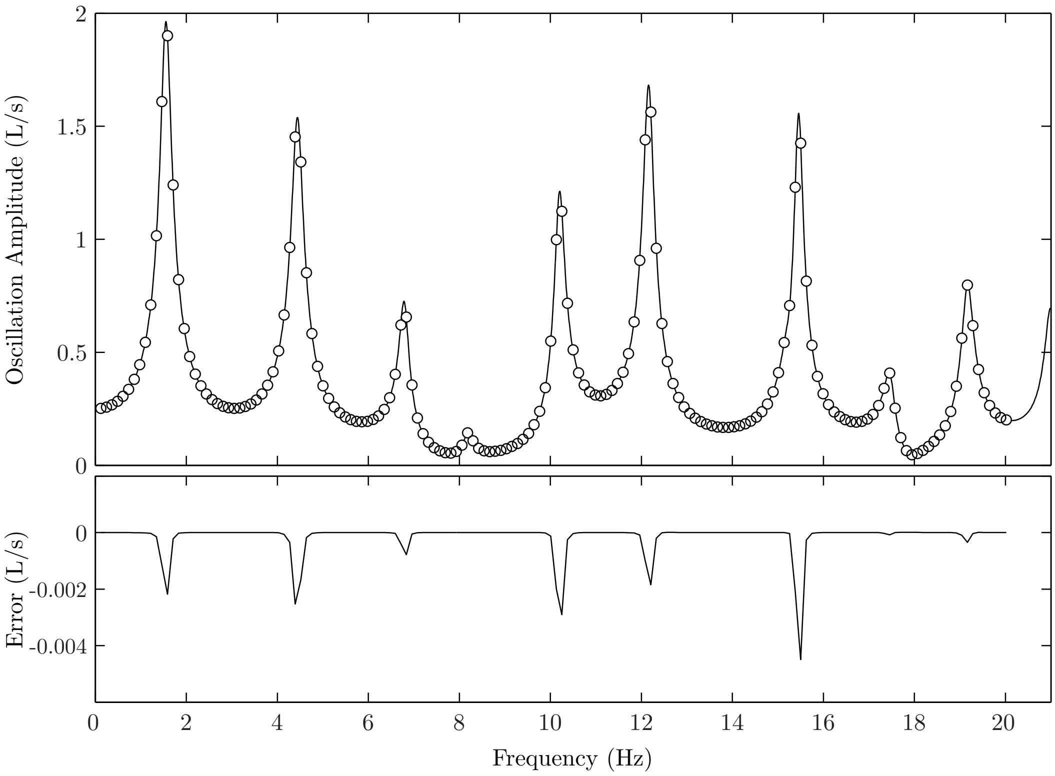


Figure 5: Caption

Sinusoidal flow amplitude response for network-1 at node 1 for the admittance matrix model (continuous line) and the method of characteristics in steady oscillatory state (○ points). The error between the two methods is given in the bottom figure.

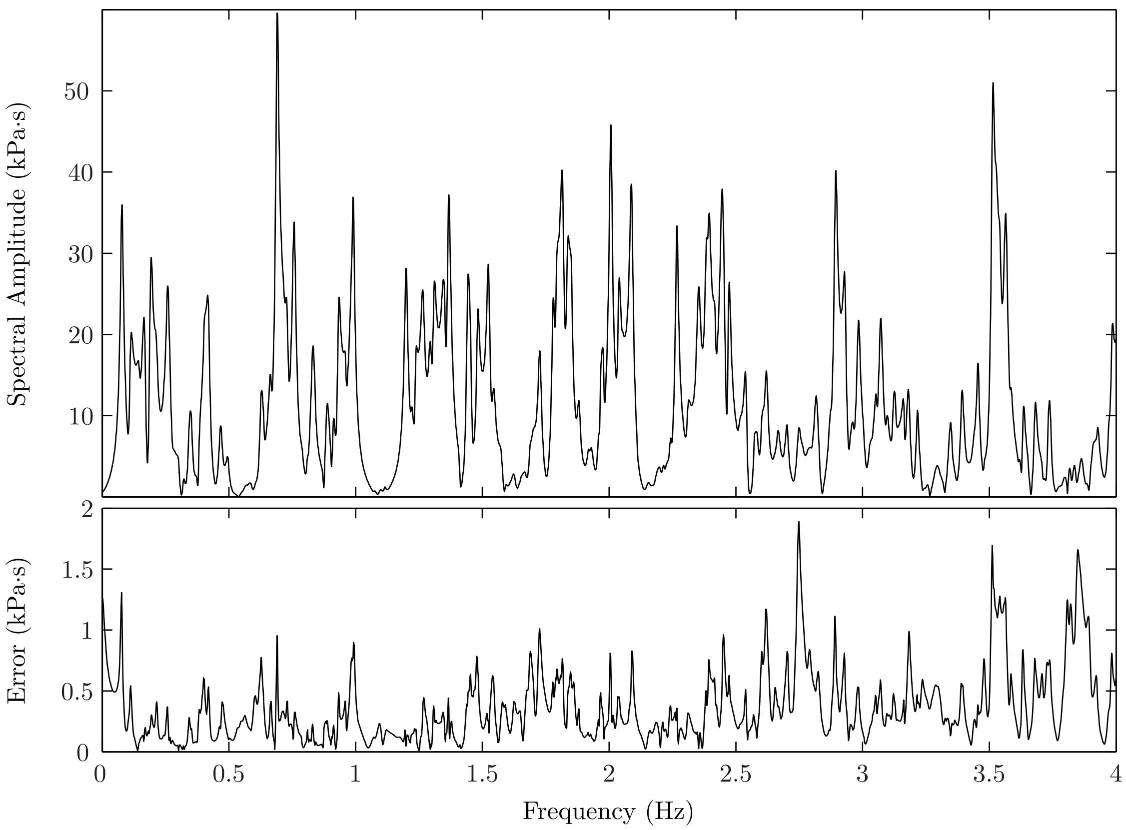


Figure 6: Caption

Pressure frequency response magnitudes for network-2 at node 14 for the admittance matrix model pulse perturbation. The lower figure gives the magnitude of the error between the admittance matrix and MOC methods (the admittance matrix minus the DFT of the MOC).

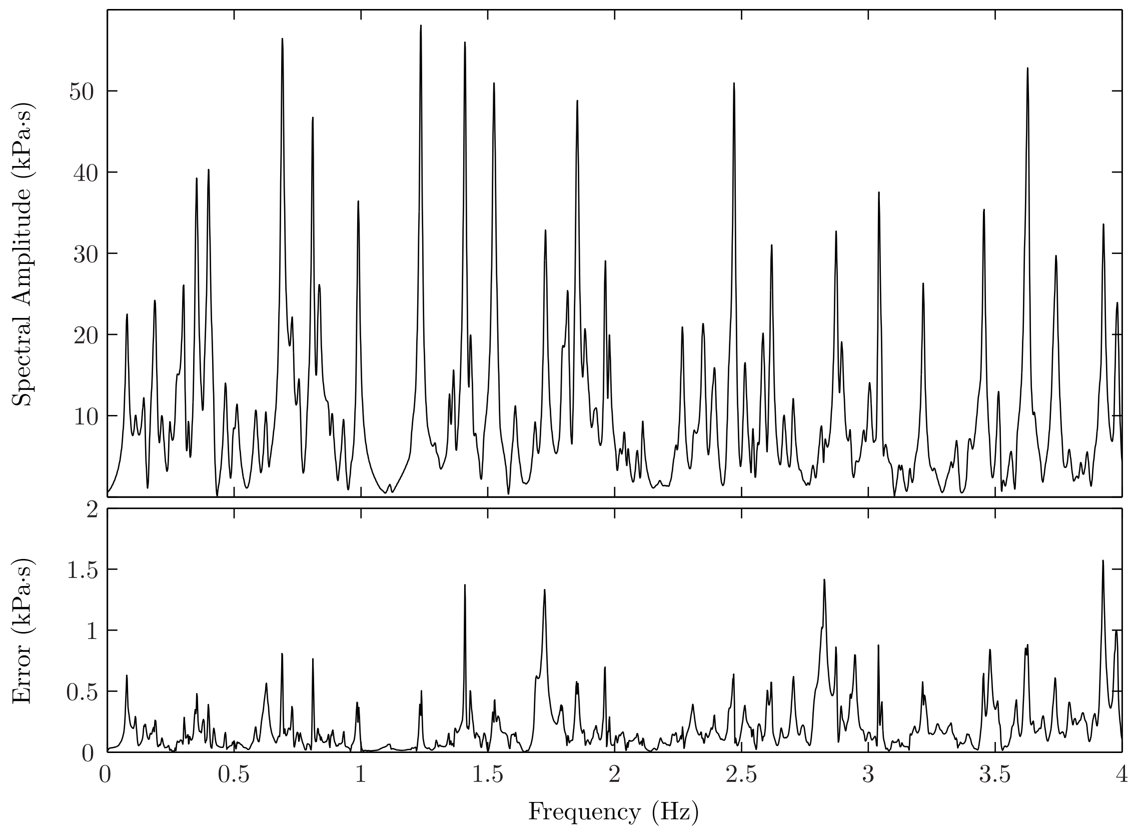


Figure 7: Caption

Pressure frequency response magnitudes for network-2 at node 18 for the admittance matrix model for the pulse perturbation. The lower figure gives the magnitude of the error between the admittance matrix and MOC methods (the admittance matrix minus the DFT of the MOC).

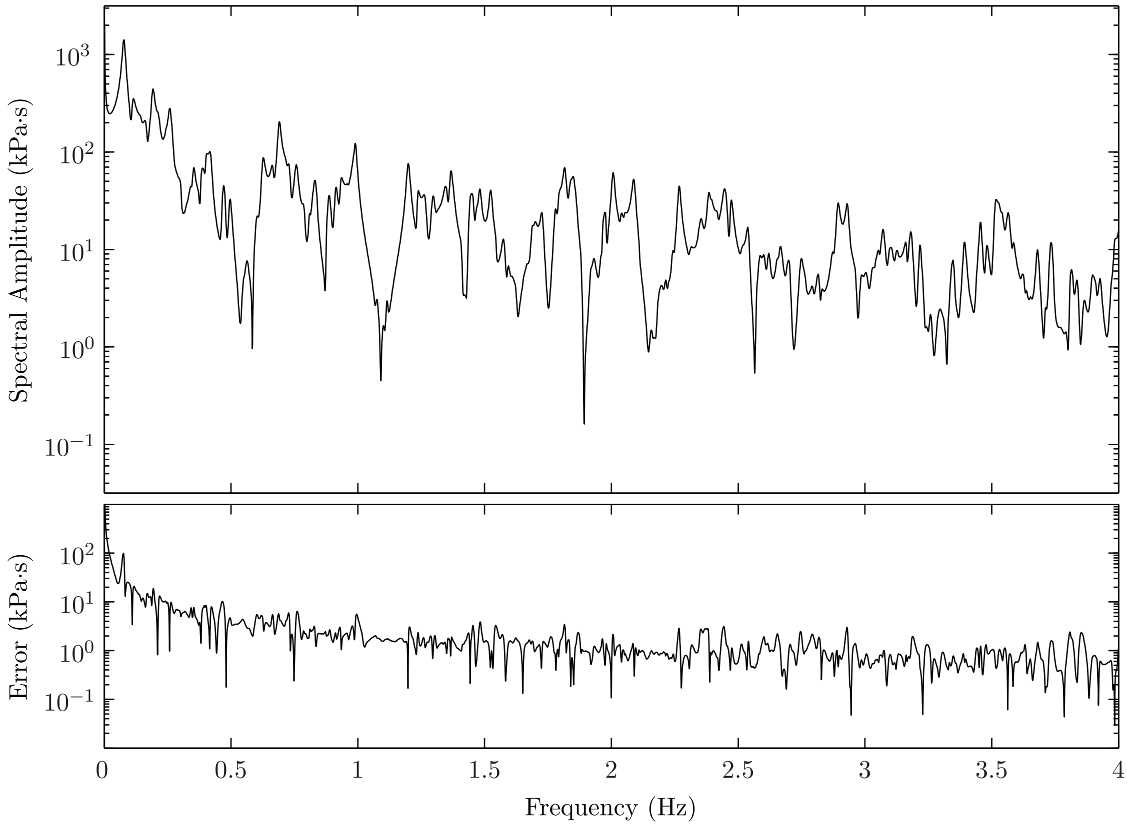


Figure 8: Caption

Pressure frequency response magnitudes for network-2 at node 14 for the admittance matrix model for the step perturbation. The lower figure gives the magnitude of the error between the admittance matrix and MOC methods (the admittance matrix minus the DFT of the MOC).

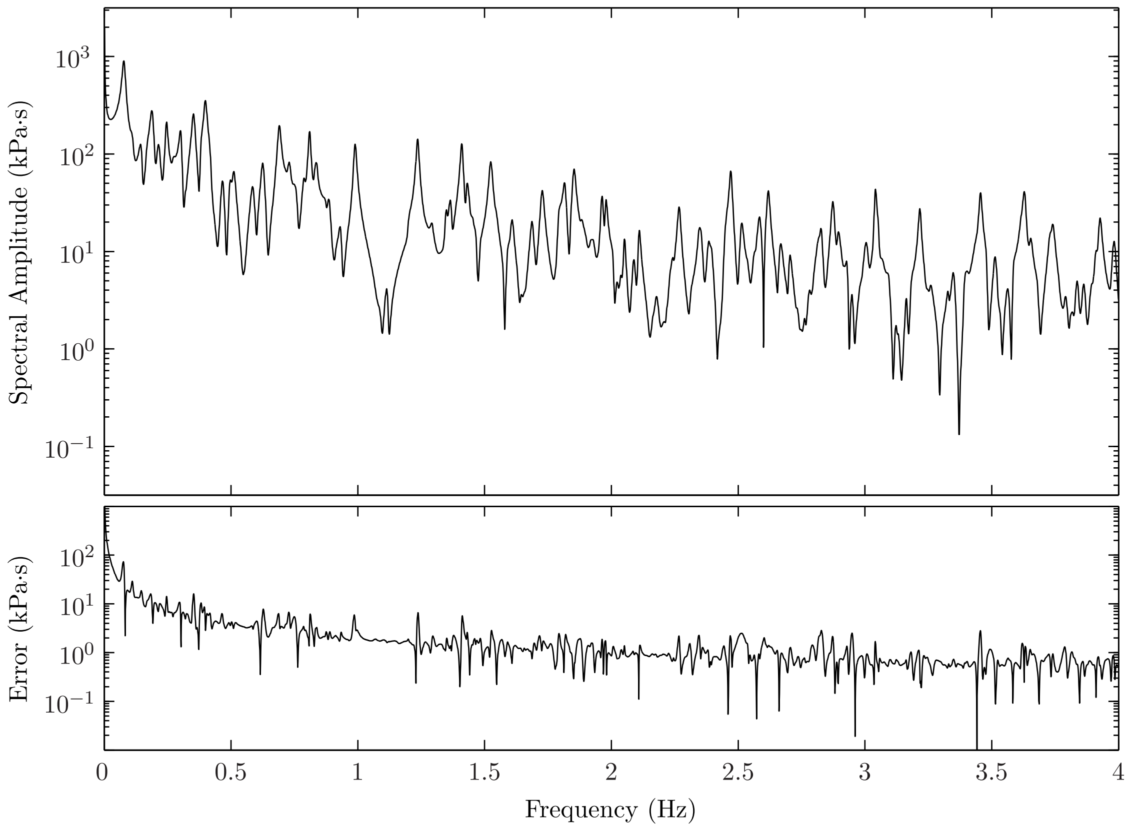


Figure 9: Caption

Pressure frequency response magnitudes for network-2 at node 18 for the admittance matrix model for the step perturbation. The lower figure gives the magnitude of the error between the admittance matrix and MOC methods (the admittance matrix minus the DFT of the MOC).

Table Captions

Table 1: Compound element parameters for numerical studies for networks 1 and 2. Capacitors are physical models for short dead end sections, and accumulators are the models for air chambers.

Table 2: Compound node element details. All elements are discussed at greater depth in Wylie and Streeter [1993].

NOMENCLATURE

$\mathcal{C}$	Set of compound node functions
$\boldsymbol{H}(s)$	Transfer function matrix for the input/output network model
n_c	Number of compound nodes
n_d	Number of demand (flow controlled) nodes
n_r	Number of reservoir (pressure controlled) nodes
n_{si}	Number of simple connections for compound node i
n_Λ	Number of links
$\boldsymbol{N_d}$	Incidence matrix for downstream nodes
$\boldsymbol{N_u}$	Incidence matrix for upstream nodes
$\boldsymbol{N_{d}}_i$	Compound node incidence matrix for downstream nodes
$\boldsymbol{N_{u}}_i$	Incidence matrix for upstream nodes
$\mathcal{N}$	Set of nodes
$\mathcal{N}_c$	Set of compound nodes
$\mathcal{N}_d$	Set of demand (flow controlled) nodes
$\mathcal{N}_D$	Set of flow controlled nodes within the simple node expanded network, see (38)
$\mathcal{N}_i$	Set of connections for compound node i
$\mathcal{N}_J$	Set of junctions
$\mathcal{N}_o$	Set of nodes for the simple node expanded network
$\mathcal{N}_r$	Set of reservoir (pressure controlled) nodes
$\mathcal{N}_s$	Set of simple nodes
$p_i(x,t)$	Pressure for pipe i
$\boldsymbol{p}(\boldsymbol{x},t)$	Vector of pipeline pressures
$P_i(x,s)$	Laplace transform of pressure for pipe i
$\boldsymbol{P}(\boldsymbol{x},s)$	Vector of Laplace transform of pipeline pressures
$\boldsymbol{P_{c}}_i(s)$	Vector of Laplace transform of connection pressures for compound node i
$\boldsymbol{P_d}(s)$	Vector of pipeline pressures at upstream point
$\boldsymbol{P_u}(s)$	Vector of pipeline pressures at downstream point
$\mathcal{P}$	Set of pipeline functions
$q_i(x,t)$	Axial flow rate for pipe i

$\boldsymbol{q}(\boldsymbol{x},t)$	Vector of pipeline axial flow rates
$Q_i(x,s)$	Laplace transform of axial flow rate for pipe i
$\boldsymbol{Q}(\boldsymbol{x},s)$	Vector of Laplace transform of pipeline axial flow rates
$\boldsymbol{Q_c}_i(s)$	Vector of Laplace transform of connection flows for compound node i
$\boldsymbol{Q_d}(s)$	Vector of pipeline axial flow rates at upstream point
$\boldsymbol{Q_u}(s)$	Vector of pipeline axial flow rates at downstream point
$\boldsymbol{u}_i(t)$	Controlled internal nodal states for compound node i
$\tilde{\boldsymbol{u}}_i(t)$	Dependent internal nodal states for compound node i
$\boldsymbol{U}_i(s)$	Laplace transform of $\boldsymbol{u}_i(t)$
$\widetilde{\boldsymbol{U}}_i(s)$	Laplace transform of $\tilde{\boldsymbol{u}}_i(t)$
$\boldsymbol{Y}(s)$	Network admittance transfer matrix
$\boldsymbol{Y}_j(s)$	Admittance transfer matrix for hydraulic element j
$\boldsymbol{Y_c}_i(s)$	Admittance matrix operating on the compound node connection pressures for canonical form of compound node dynamics
$\boldsymbol{Y_u}_i(s)$	Admittance matrix operating on the compound node controlled states for canonical form of compound node dynamics
$Z_c(s)$	Series impedance
$\boldsymbol{\phi}$, $[\boldsymbol{\Phi}(s)]$	Compound node equation [and its Laplace transform]
Λ	Set of links within a graph or network
Λ_i	Set of links incident to node i
Λ_{di}	Set of links for which the downstream node is node i
Λ_{ui}	Set of links for which the upstream node is node i
$\psi(t)$	Nodal pressure
$\psi_r(t)$	Nodal pressure at reservoir node
$\boldsymbol{\Psi}(s)$	Vector of Laplace transformed network nodal pressures
$\boldsymbol{\Psi}_d(s)$	Vector of Laplace transformed network demand (flow control) node pressures
$\boldsymbol{\Psi}_D(s)$	Vector of Laplace transformed node pressures for flow controlled nodes within the simple node expanded network
$\boldsymbol{\Psi}_i(s)$	Vector of Laplace transformed of nodal pressures for compound node i for the simple node expanded network

$\mathbf{\Psi}_r(s)$	Vector of Laplace transformed network reservoir pressures
$\theta(t)$	Nodal flows (flow injections)
$\theta_d(t)$	Nodal flows at demand nodes
$\mathbf{\Theta}(s)$	Vector of Laplace transformed network nodal flows
$\mathbf{\Theta}_d(s)$	Vector of Laplace transformed network demand node flows
$\mathbf{\Psi}_i(s)$	Vector of Laplace transformed of nodal flows for compound node i for the simple expanded node network
$\mathbf{\Theta}_r(s)$	Vector of Laplace transformed network reservoir flows
$\Gamma(s)$	Fluid line propagation operator
$\mathbf{\Gamma}(s)$	Fluid line propagation operator matrix for hydraulic network

APPENDIX: ADMITTANCE REPRESENTATION OF A COMPOUND NODE

The three phased derivation of the compound node admittance form (20) from the Laplace-transform of the linearised original compound node equation (17) is outlined below, followed by an example.

Connection representation of a Compound Node

The representation of a compound node as a hydraulic component with connections is a fundamental representation of the compound node, and is independent of the connectivity of the compound node with the wider network. It is convenient to denote the set of connections for compound node $i \in \mathcal{N}_c$ by $\mathcal{N}_i$. As with a simple node, each connection $k \in \mathcal{N}_i$ has two states, the nodal pressure P_{cik}, and the nodal flow into the compound nodes component Q_{cik}, where, as stated above, the pressure P_{cik} is the common pressure shared by all link ends incident to the connection, and the flow Q_{cik} is the aggregated flow into the component from all links incident to the connection. Organising these states as (19), they can be related to the incident link states $\boldsymbol{P}_i$ and $\boldsymbol{Q}_i$, from (17) by

$$\boldsymbol{P}_i(s) = \left[\boldsymbol{N}_{\boldsymbol{u}i} + \boldsymbol{N}_{\boldsymbol{d}i}\right]^T \boldsymbol{P}_{\boldsymbol{c}i}(s), \quad \boldsymbol{Q}_{\boldsymbol{c}i}(s) = \left[\boldsymbol{N}_{\boldsymbol{u}i} - \boldsymbol{N}_{\boldsymbol{d}i}\right] \boldsymbol{Q}_i(s) \tag{41}$$

where $\boldsymbol{N}_{\boldsymbol{u}i}$ and $\boldsymbol{N}_{\boldsymbol{d}i}$ are compound node incidence matrices, describing the topology of the node, and are defined by

$$\{\boldsymbol{N}_{\boldsymbol{u}i}\}_{kj} = \begin{cases} 1 & \text{if } j\text{-th link in } \Lambda_{ui} \text{ is in } \Lambda_{uik} \\ 0 & \text{otherwise} \end{cases},$$

$$\{\boldsymbol{N}_{\boldsymbol{d}i}\}_{kj} = \begin{cases} 1 & \text{if } j\text{-th link in } \Lambda_{di} \text{ is in } \Lambda_{dik} \\ 0 & \text{otherwise} \end{cases},$$

where the link sets Λ_{uik} and Λ_{dik} are associated with the sets Λ_{ui} and Λ_{di}, respectively, and contain the upstream and downstream links that are incident to the connection k. The equations (41) are analogous to the simple node constraints for networks (7) and (9), and are explained as follows. Concerning the connection pressure P_{cik}, as there is no pressure variation within a connection, the pressure of any links incident to the same connection k will be equal. Concerning the connection flow Q_{cik}, as there is no accumulation of mass within a connection, the total flow through the connection k is equal to the sum of the inflows from the links in Λ_{dik} minus the outflows from the links in Λ_{uik}.

The existence of the relationships (41) implies that there exists a lower dimensional form of $\boldsymbol{\Phi}_i$ incorporating the component dynamics that is just dependent on the connection states $\boldsymbol{P_c}_i$ and $\boldsymbol{Q_c}_i$. This lower dimensional form can be expressed as

$$\boldsymbol{\Phi_s}_i(s) \left[\begin{array}{c} \boldsymbol{P_c}_i(s) \\ \hdashline \boldsymbol{Q_c}_i(s) \\ \hdashline \boldsymbol{U}_i(s) \\ \hdashline \widetilde{\boldsymbol{U}}_i(s) \end{array}\right] = \mathbf{0} \tag{42}$$

where $\boldsymbol{\Phi_s}_i$ is a $(n_{si} + n_{\tilde{u}i}) \times (2n_{si} + n_{ui} + n_{\tilde{u}i})$ matrix of stable transfer functions. The matrix system $\boldsymbol{\Phi_s}_i$ has $(n_{si} + n_{\tilde{u}i})$ rows as it must contain enough equations to determine one state at each connection, and all the internal response states.

Decoupled nodal equations

By definition, as $\widetilde{\boldsymbol{u}}_i$ is a nodal response variable, it can be uniquely determined from the other nodal states. Hence there exists a stable Laplace-domain transfer function mapping from the transformed connection pressures and flows ($\boldsymbol{P_c}_i$, $\boldsymbol{Q_c}_i$) and the transformed controlled nodal states ($\boldsymbol{U}_i$) to the transformed nodal response states $\tilde{\boldsymbol{U}}_i$. That is, $\boldsymbol{\Phi_s}_i$ can be partitioned as

$$\boldsymbol{\Phi_s}_i(s) = \left[\begin{array}{c:c:c:c} \boldsymbol{\Phi_{op}}_i(s) & \boldsymbol{\Phi_{oq}}_i(s) & \boldsymbol{\Phi_{ou}}_i(s) & \boldsymbol{\Phi_{o\tilde{u}}}_i(s) \\ \hdashline \boldsymbol{\Phi_{1p}}_i(s) & \boldsymbol{\Phi_{1q}}_i(s) & \boldsymbol{\Phi_{1u}}_i(s) & \boldsymbol{\Phi_{1\tilde{u}}}_i(s) \end{array}\right] \tag{43}$$

where the blocks correspond to their subscripted variables, and $\boldsymbol{\Phi_{1\tilde{u}}}_i$ is a $n_{\tilde{u}i} \times n_{\tilde{u}i}$ matrix that possesses a stable inverse. Formally, the matrix function $\boldsymbol{A}(s) : \mathbb{C} \mapsto \mathbb{C}^{n\times n}$ possesses a stable inverse if $\det \boldsymbol{A}(s) > 0$ for $\mathbb{R}e\left\{s\right\} \geq 0$. Therefore, a n_{si} order system exists that relates the states $\boldsymbol{P_c}_i$, $\boldsymbol{Q_c}_i$ and $\boldsymbol{U}_i$ can be decoupled from $\tilde{\boldsymbol{U}}_i$, which can be expressed as

$$\boldsymbol{\Phi_c}_i(s) \left[\begin{array}{c} \boldsymbol{P_c}_i(s) \\ \hdashline \boldsymbol{Q_c}_i(s) \\ \hdashline \boldsymbol{U}_i(s) \end{array}\right] = \mathbf{0} \tag{44}$$

where $\boldsymbol{\Phi_{c}}_i$ is an $n_{si} \times (2n_{si} + n_{ui})$ matrix of stable complex functions, given by

$$\begin{aligned}\boldsymbol{\Phi_{c}}_i(s) &= \left[\begin{array}{c:c:c} \boldsymbol{\Phi_{cp}}_i(s) & \boldsymbol{\Phi_{cq}}_i(s) & \boldsymbol{\Phi_{cu}}_i(s) \end{array}\right] \\ &= \left[\begin{array}{c:c:c} \boldsymbol{\Phi_{op}}_i(s) & \boldsymbol{\Phi_{oq}}_i(s) & \boldsymbol{\Phi_{ou}}_i(s) \end{array}\right] \\ &\quad -\boldsymbol{\Phi_{o\tilde{u}}}_i(s)\boldsymbol{\Phi_{1\tilde{u}}}_i^{-1}(s)\left[\begin{array}{c:c:c} \boldsymbol{\Phi_{1p}}_i(s) & \boldsymbol{\Phi_{1q}}_i(s) & \boldsymbol{\Phi_{1u}}_i(s) \end{array}\right]\end{aligned} \quad . \tag{45}$$

The matrix $\boldsymbol{\Phi_{c}}_i$ represents a minimal state matrix for the compound node i as it yields equivalent dynamics to $\boldsymbol{\Phi_{s}}_i$ but with a reduced number of states. An example of the derivation of the form of (45) is given in Example 6.

Admittance Representation of $\boldsymbol{\Phi_{c}}_i(s)$

Given the decoupled connection representation (45), the criteria for the existence of the admittance representation (20) can be precisely stated as

$$\text{rank}\left\{\boldsymbol{\Phi_{cq}}_i(s)\right\} = n_{si} \quad \text{for } \mathbb{R}e\{s\} \geq 0. \tag{46}$$

The significance of (46) is that it defines the criteria under which the compound nodes simple connection flows $\boldsymbol{Q_{c}}_i$ can be resolved from the simple connection pressures $\boldsymbol{P_{c}}_i$ and the compound nodes controlled states $\boldsymbol{U_{c}}_i$. The transfer matrix $\boldsymbol{\Phi_{cq}}_i$ is $n_{si} \times n_{si}$, therefore, the constraint (46) can be interpreted as $\boldsymbol{\Phi_{cq}}_i$ begin full rank without diminishing rank on $\mathbb{R}e\{s\} \geq 0$. In this instance $\boldsymbol{\Phi_{cq}}_i$ is a $n_{si} \times n_{si}$ transfer matrix with a stable inverse, and the admittance matrices from (20) can be given as

$$\boldsymbol{Y_{c}}_i(s) = -\left[\boldsymbol{\Phi_{cq}}_i(s)\right]^{-1}\boldsymbol{\Phi_{cp}}_i(s), \quad \boldsymbol{Y_{u}}_i(s) = \left[\boldsymbol{\Phi_{cq}}_i(s)\right]^{-1}\boldsymbol{\Phi_{cu}}_i(s). \tag{47}$$

It turns out that (46) is not a very restrictive, but that all components considered within this research adhere to this requirement. Consider the following example.

Example 6. *Revisiting the compound node from Figure 1 in Examples 1 and 2, it is recognised that there are* $n_{\lambda i} = 3$ *links,* $n_{ui} = 1$ *controlled node state,* $n_{\tilde{u}i} = 2$ *response node states, where the order of the* ϕ *is clearly 5. This compound node is recognised as having two connections (*i.e. $n_{si} = 2$*), one just outside valve* A *and the other outside valve* B*. Denoting these connections as* A *and* B*, the connection states are as given in* (21)*, and the topological matrices are*

$$\boldsymbol{N_{u}}_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{N_{d}}_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

By identifying and removing the connection equations (41)*,* (18) *can be converted into the form* (42) *as*

$$\boldsymbol{\Phi}_i(s) = \left[\begin{array}{cc:cc:c:cc} 0 & 1 & 0 & -c_B & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & -1 \\ \hdashline 0 & 0 & 0 & 0 & 0 & cs & -1 \\ 1 & 0 & -c_A & 0 & 0 & -1 & 0 \end{array}\right] \tag{48}$$

where the partitions correspond to those in (43)*, where*

$$[\boldsymbol{\Phi}_{\tilde{\boldsymbol{u}}\mathbf{1}i}(s)]^{-1} = \begin{bmatrix} cs & -1 \\ -1 & 0 \end{bmatrix}^{-1} = -\begin{bmatrix} 0 & 1 \\ 1 & cs \end{bmatrix} \tag{49}$$

clearly exists for $\mathbb{R}e\left\{s\right\} \geq 0$*. Given the expressions in* (45)*, the decoupled representation* (44) *is given by*

$$\boldsymbol{\Phi}_{\boldsymbol{c}i}(s) = \left[\begin{array}{cc:cc:c} -1 & 1 & c_A & -c_B & 0 \\ -cs & 0 & 1 + c_A cs & 1 & 1 \end{array}\right]$$

where the partitions are according to (45)*. Recognising from* (50) *that*

$$\boldsymbol{\Phi}_{\boldsymbol{cq}_i}(s) = \begin{bmatrix} c_A & -c_B \\ 1 + c_A cs & 1 \end{bmatrix}$$

the criteria (46) *holds if*

$$\det\left\{\boldsymbol{\Phi}_{\boldsymbol{cq}_i}(s)\right\} = c_A + c_B + c_A c_B cs \neq 0, \quad \text{on } \mathbb{R}e\left\{s\right\} \geq 0.$$

This clearly holds as c*,*c_A*, and* c_B *are all positive real numbers. Therefore, it can be demonstrated from* (47) *that the admittance matrices for* (20) *are given by*

$$\boldsymbol{Y}_{\boldsymbol{c}i}(s) = \frac{1}{\det\left\{\boldsymbol{\Phi}_{\boldsymbol{cq}_i}(s)\right\}}\begin{bmatrix} 1 + c_B cs & -1 \\ -1 & 1 + c_A cs \end{bmatrix}, \quad \boldsymbol{Y}_{\boldsymbol{u}i}(s) = -\frac{1}{\det\left\{\boldsymbol{\Phi}_{\boldsymbol{cq}_i}(s)\right\}}\begin{bmatrix} c_B \\ c_A \end{bmatrix}.$$

TABLES FOR MANUSCRIPT

Table 1: Compound element parameters for numerical studies for networks 1 and 2. Capacitors are physical models for short dead end sections, and accumulators are the models for air chambers.

Network	Node	Element Type	Parameters
1	2	1-valve junction	$\{C_d, d_v\} = \{1.5, 40 \text{ mm}\}$
1	3	2-valve junction	$\{C_d, d_v\} = \{1.5, 30 \text{ mm}\}$
1	5	Capacitor	$\{V_0, K_e\} = \{1\text{L}, 1.5 \text{ GPa}\}$
2	7	Capacitor	$\{V_0, K_e\} = \{10 \text{ L}, 1.5 \text{ GPa}\}$
2	9	2-valve junction	$\{C_d, d_v\} = \{0.9, 300 \text{ mm}\}$
2	11	Accumulator	$\{V_0, n\} = \{5 \text{ L}, 1.2\}$
2	19	3-valve junction	$\{C_d, d_v\} = \{0.9, 300 \text{ mm}\}$
2	22	2-valve junction	$\{C_d, d_v\} = \{0.9, 300 \text{ mm}\}$
2	23	2-valve junction	$\{C_d, d_v\} = \{0.9, 300 \text{ mm}\}$
2	25	4-valve junction	$\{C_d, d_v\} = \{0.9, 300 \text{ mm}\}$
2	34	Emitter	$\{C_d, d_e, \psi_0\} = \{0.9, 10 \text{ mm}, 0 \text{ Pa}\}$

Table 2: Compound node element details. All elements are discussed at greater depth in Wylie and Streeter [1993].

Element	States, $\tilde{\boldsymbol{u}}$	Parameter Set	Equations comprising $\boldsymbol{\phi}$
Emitter	$[\psi\ \ \theta]^T$	$\{C_d, d_e, \psi_0\}$	$\theta(t) - C_d A_e \sqrt{\dfrac{\psi(t) - \psi_0}{\rho}}$
Capacitor	$[\psi\ \ \theta]^T$	$\{V_0, K_e\}$	$\theta(t) - \dfrac{V_0}{K_e}\dfrac{d\psi}{dt}$
Accumulator	$[\psi\ \ \theta\, V]^T$	$\{V_0, n\}$	$\begin{bmatrix} \psi(t)V^n(t) - C_a \\ \theta(t) - \frac{dV}{dt} \end{bmatrix}$
Valve[a]	$[\psi_u\ \ \psi_d\ \ \theta]^T$	$\{C_d, d_v\}$	$\theta(t) - \text{sign}\,\{\Delta\psi(t)\}\, C_d A_v \sqrt{\dfrac{\lvert\Delta\psi(t)\rvert}{\rho}}$

[a] Note that $\Delta\psi(t) = \psi_u(t) - \psi_d(t)$.