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Steady free-surface flow at the stern of a ship

Benjamin James Binder
School of Mathematical Sciences, University of Adelaide, Adelaide, South Australia 5005, Australia

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New solutions for steady two-dimensional free-surface flow past a curved plate are considered here. They can be interpreted as approximations to the flow locally at the stern of a ship. Weakly nonlinear solutions are derived analytically and nonlinear solutions are computed by boundary integral equation methods. Analysis in the phase plane provides a way to determine the geometries of hulls that give rise to wave-free stern flows. These waveless flows are desirable as they reduce ship-drag. © *2010 American Institute of Physics*. [doi:10.1063/1.3275847]

I. INTRODUCTION

The hydrodynamic performance characteristics of a ship are due in large measure to the details of the flow near the bow and stern, and for this reason, numerous efforts have been made to analyze bow and stern flows. Broadly speaking there are two cases to consider: flows in infinite depth fluid,[1–8] for example, in the ocean, and flows in finite depth fluid,[9–14] such as in an open channel or canal. These flows can be further classified as having either smooth-detachment or stagnation-detachment from the ship. The focus of the study here is for smoothly detaching steady flows from the stern of a ship in finite depth fluid.

In general stern flows are three-dimensional and are difficult to treat mathematically. However, when the stern of the ship is wide, the resulting problem can be considered two-dimensional in the vicinity of the ship center-plane. Powerful complex variable methods can be used to formulate a boundary integral equation, which we solve numerically for the nonlinear problem. In addition, the Korteweg–de Vries (KDV) equation yields analytical weakly nonlinear solutions, which can be compared with our nonlinear computations. More importantly, however, analysis in the phase plane of the weakly nonlinear problem provides a systematic approach to identify all the possible types of solutions and the number of independent parameters. Shown in Fig. 1 is a sketch of two-dimensional flow at the stern of a ship in finite depth fluid.

The energy contained in the surface waves generated by a moving ship is an important component of the overall drag on a ship. Stern flows with very small waves or that are completely wave-free have been generated by varying flow parameters or the precise geometry of the stern in the case of infinite depth fluid.[1,3–6,8] In finite depth fluid waveless supercritical flows[9,11–14] have been investigated as well, and recently, Maleewong and Grimshaw[10] obtained a waveless nonlinear solution for supercritical flow past a semi-infinite flat plate. However, less attention has been paid to smoothly detaching waveless subcritical flows in finite depth fluid. We show that by considering a curved plate, we not only recover the supercritical solution[10] but also discover new waveless subcritical stern flows. These new wave-free solutions for subcritical flow are desirable as they reduce ship-drag. As free-surface potential flows are reversible, the waveless solutions are also approximations to splashless bow flows.[6]

II. FORMULATION

Consider the steady two-dimensional irrotational flow of an inviscid and incompressible fluid past the stern of a ship in an open channel (see Fig. 1). A frame of reference is taken with the ship that is moving at a constant speed from left to right in the channel. The direction of the fluid flow relative to the ship is then from right to left. The fluid is bounded from below by the bottom of the channel $A'D'$ and from above by the free-surface AB and the stern of the ship BCD. A system of Cartesian coordinates is defined (x^*,y^*) with the x^*-axis along the bottom of the channel and the y^*-axis passing through the separation point B $(x^*=0)$ at the stern of the ship. The flow is assumed to separate tangentially (smooth-detachment) from the stern of the ship at B. The acceleration due to gravity g is acting in the negative y^* direction.

We assume that under the horizontal bottom of the ship CD, as $x^*\to\infty$, there is uniform flow characterized by a constant speed U and constant depth H. The dimensionless depth-based Froude number can then be defined as

$$F=\frac{U}{(gH)^{1/2}}. \tag{1}$$

If the flow is also uniform as $x^*\to-\infty$ with constant speed V and depth D, we introduce an additional depth-based Froude number

$$F^*=\frac{V}{(gD)^{1/2}}. \tag{2}$$

The two Froude numbers can be related in the following way. First the conservation of mass implies

$$VD=UH. \tag{3}$$

Then Eqs. (1)–(3) yield

$$F^*=F\left(\frac{D}{H}\right)^{-3/2}. \tag{4}$$

The equation of the free-surface AB is $y^*=H+\eta^*$, where η^* is the elevation of the free-surface on top of the level H. Assuming that the pressure P_s^* along the horizontal bottom of

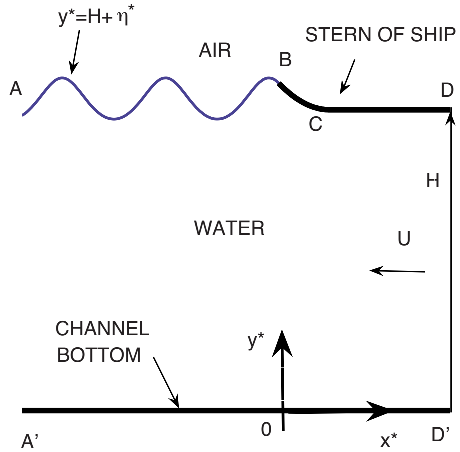

FIG. 1. (Color online) Sketch of flow problem.

the ship CD approaches a constant as $x^* \to \infty$, which is greater than the (constant) atmospheric pressure P_a on the free-surface AB, Bernoulli's equation can be written as

$$\frac{1}{2}(u^{*2}+v^{*2})+gy^*=\frac{1}{2}U^2+gH+\frac{P^*}{\rho}. \tag{5}$$

Here, u^* and v^* are the horizontal and vertical components of the fluid velocity, $P^*=P_s^*-P_a^*>0$, and ρ is the constant fluid density.

Two approaches are used to solve the flow problem. The first is a nonlinear numerical theory based on boundary-equation methods. The second is an analytical weakly nonlinear theory.

A. Nonlinear boundary integral equation

The nonlinear flow problem is solved numerically via a boundary integral equation technique, similar to that used by Binder *et al.*[15–17] Essential details only are presented here.

The dimensionless quantities $(x,y)=(x^*,y^*,\eta^*)/H$, $(u,v)=(u^*,v^*)/U$, and $P=P^*/\rho g H$ are defined by taking H as the reference length and U as the reference velocity. Equation (5) then takes the form

$$\frac{1}{2}(u^2+v^2)+\frac{1}{F^2}y=\frac{1}{2}+\frac{1}{F^2}(1+P). \tag{6}$$

Note that P is the dimensionless difference between the pressure applied on the bottom the ship and the atmospheric pressure.

Since the fluid is incompressible and the flow is irrotational, we introduce the complex potential function f and the complex velocity w,

$$f=\phi+\mathrm{i}\psi, \quad w=\frac{df}{dz}=u-\mathrm{i}v. \tag{7}$$

Without loss of generality we choose $\psi=0$ on the streamline $ABCD$, and it follows that $\psi=-1$ on the streamline $A'D'$. We choose $\phi=0$ $(x=0)$ at B and let $\phi=\phi_c$ $(x=x_c)$ at C. We define the analytic function $\tau-\mathrm{i}\theta$ as

$$w=u-\mathrm{i}v=e^{\tau-\mathrm{i}\theta} \tag{8}$$

and obtain the integral equation

$$\tau(\phi)=\int_{-\infty}^{\infty}\frac{\theta(\phi_0)e^{\pi\phi_0}}{e^{\pi\phi_0}-e^{\pi\phi}}d\phi_0, \tag{9}$$

which relates the values of τ and θ along $\psi=0$.

On the free-surface Eq. (6) becomes the dynamic boundary condition

$$e^{2\tau}+\frac{2}{F^2}y=1+\frac{2}{F^2}(1+P) \quad \text{on} \quad AB, \tag{10}$$

and we prescribe the geometry of the hull by

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}\eta}{\mathrm{d}x}=\tan\,\theta=G(\eta) \quad \text{on} \quad BC. \tag{11}$$

Along the flat horizontal bottom of the ship CD, we have $\theta=0$ and $y=1$.

We relate the values of x and y on $\psi=0$ by integrating numerically the identity

$$x_\phi+\mathrm{i}y_\phi=\frac{1}{u-\mathrm{i}v}=e^{-\tau+\mathrm{i}\theta} \tag{12}$$

and equating real and imaginary parts. This gives a parametric representation $x=x(\phi)$, $y=y(\phi)$ for the streamline $\psi=0$.

Equations (9)–(12) define a nonlinear integrodifferential equation for the unknown function $\theta(\phi)$ on AB. This equation is discretized, and the resulting equations are solved by Newton's method.

When the flow as $x^* \to -\infty$ is uniform, there is an additional relationship between the constant depth far downstream $y(-\infty)=1+\eta(-\infty)$ and the Froude number F. Using Eq. (3) we substitute for the velocity in Eq. (6) to obtain the equation

$$y(-\infty)^3-\left(\frac{F^2}{2}+P+1\right)y(-\infty)^2+\frac{F^2}{2}=0. \tag{13}$$

One root corresponds to waveless subcritical flow and another to supercritical flow. The remaining third root corresponds to a physically unrealistic depth downstream and is of no interest to us here. When attempting to compute a waveless solution, we force the free-surface flat by imposing the condition that $\theta(-\infty)=0$. The extra condition given by Eq. (13) is then satisfied implicitly.

B. Weakly nonlinear theory

Analogous to similar previous studies,[10,15–18] we derive a new steady forced KDV equation to model the free-surface flow at the stern of a ship in finite depth fluid. The derivation is based on long wavelength asymptotics. If L denotes a typical horizontal length scale and H is the constant depth as $x^* \to \infty$, we introduce the small parameter $\epsilon=(H/L)^2 \ll 1$, the dimensionless spatial variables $(\hat{x},\hat{y})=(\epsilon^{1/2}x^*,y^*)/H$, and the free-surface elevation $\epsilon\hat{\eta}=\eta^*/H$. The dimensionless difference in pressure is then $\hat{P}=\epsilon^{-2}P^*/\rho g H=\epsilon^{-2}P$. The Froude number F is written as $F=1+\epsilon\mu$.

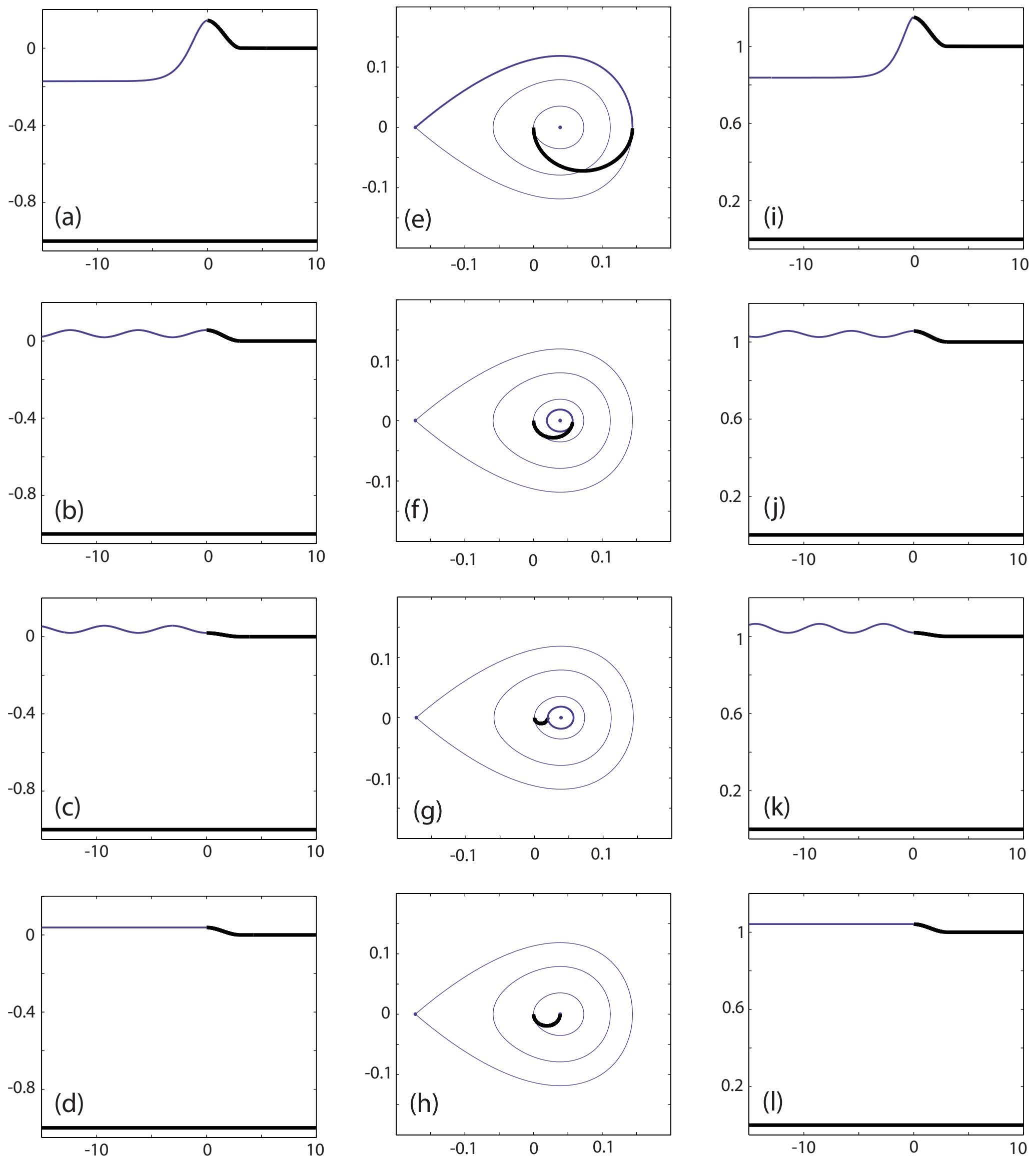

FIG. 2. (Color online) Free-surface profiles with phase plane analysis for $F=0.90$ and $P=0.01$. [(a)–(d)] Weakly nonlinear profiles, η versus x. [(e)–(h)] Weakly nonlinear phase portraits, η_x vs η. [(i)–(l)] Nonlinear profiles, $y=\eta+1$ vs x. (a) $\eta(0)=0.14$, $\eta(-\infty)=\eta_2=-0.17$, and $F^*=1.19$. (b) $\eta(0)=0.057$. (c) $\eta(0)=0.019$. (d) $\eta(0)=\eta_1=0.038$ and $F^*=0.85$. (i) $\eta(0)=0.15$, $\eta(-\infty)=-0.16$, and $F^*=1.17$. (j) $\eta(0)=0.057$. (k) $\eta(0)=0.019$. (l) $\eta(0)=0.041$ and $F^*=0.84$.

Substituting expansions in powers of ϵ into the exact potential equations (rewritten in terms of the new scaled variables), the forced KDV equation is derived by equating coefficients of the powers of ϵ. The forced KDV equation (rewritten in terms of the variables $x=\epsilon^{-1/2}\hat{x}$ and $\eta=y-1=\epsilon\hat{\eta}$ used in the nonlinear computations) is

$$\eta_{xx}+\tfrac{9}{2}\eta^2-6(F-1)\eta=3P \quad \text{on} \quad AB. \tag{14}$$

This equation is asymptotically valid for $F\sim 1$, $|\eta|\ll 1$, and $P\ll 1$.

Equation (14) can be rewritten in the form of a two-dimensional nonlinear dynamical system, and the fixed points in the phase plane (η,η_x) are characterized by $\eta_x=0$ and

$$\eta_1=\tfrac{2}{3}(F-1)+\sqrt{\tfrac{4}{9}(F-1)^2+\tfrac{2}{3}P} \tag{15}$$

or

$$\eta_2=\tfrac{2}{3}(F-1)-\sqrt{\tfrac{4}{9}(F-1)^2+\tfrac{2}{3}P}. \tag{16}$$

The fixed point $(\eta_1,0)$ is a center, and the fixed point $(\eta_2,0)$ is a saddle point.

Integrating Eq. (14) yields

$$\eta_x^2=6(F-1)\eta^2-3\eta^3+6P\eta+C_s \quad \text{on} \quad AB, \tag{17}$$

where C_s is a constant. This equation gives the trajectories or orbits of the phase portrait in the phase plane. Equations (14)–(17) are similar to those of Binder *et al.*,[16] who considered flow past a step in the bottom of a channel with height $h=-P$.

Next we extend the definition of η along the curved part of the ship,

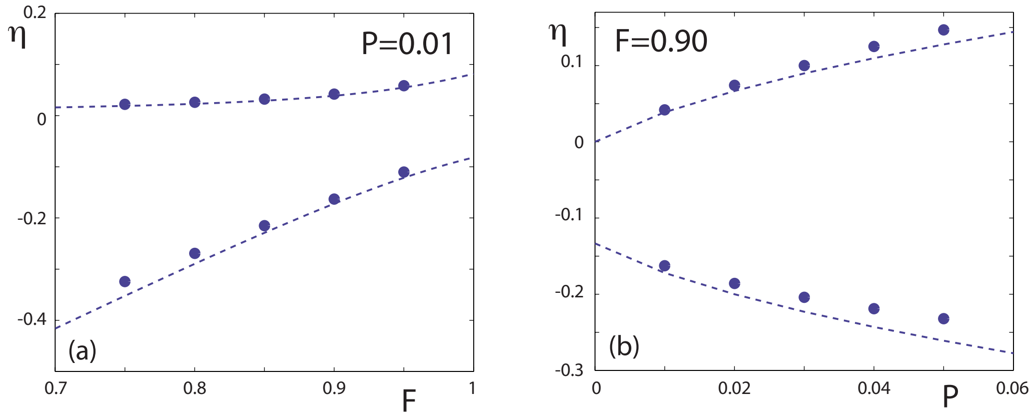

FIG. 3. (Color online) Nonlinear (dots) and weakly nonlinear (broken curves) values of the wave-free free-surface elevation as $x \to -\infty$. The top broken curves [given by Eq. (15)] correspond to wave-free subcritical flow. The bottom broken curves [given by Eq. (16)] correspond to supercritical flow. (a) Free-surface elevation $\eta(-\infty)$ vs F for a fixed value of $P=0.01$. (b) Free-surface elevation $\eta(-\infty)$ vs P for a fixed value of $F=0.90$.

$$\eta_x = G(\eta) = \left(\left(\frac{\alpha}{2} \right)^2 - \left(\eta - \frac{\alpha}{2} \right)^2 \right)^{1/2} \quad \text{on} \quad BC, \tag{18}$$

where α is a constant. For the flat horizontal part of the ship bottom, we have

$$\eta = 0 \quad \text{and} \quad \eta_x = 0 \quad \text{on} \quad CD. \tag{19}$$

We check that Eq. (18) is compliant with Eq. (19) at C or $x=x_c$ by solving Eq. (18) with $\eta_x=0$, which yields

$$\eta(\alpha - \eta) = 0. \tag{20}$$

Equation (18) is then satisfied at C with $\eta(x_c)=0$, and we choose $\alpha=\eta(0)$. This choice of α will become clear later when we examine the solution for wave-free subcritical flow in Fig. 2.

To complete the weakly nonlinear analysis, we solve the differential equation, Eq. (18), with $\eta(x_c)=0$ to obtain

$$\eta(x) = \frac{\alpha}{2}(1 - \cos(x - x_c)) \quad \text{on} \quad BC. \tag{21}$$

The condition that $\eta(0)=\alpha$ then determines the value of $x_c=\pi$ at C.

Weakly nonlinear solutions can now be constructed in the phase plane, which determines the number, type, and approximate values of the independent parameters when computing nonlinear solutions. The idea is to combine the trajectories associated with Eq. (17) on the free-surface with Eqs. (18) and (19) for the ship-hull. As we shall see the geometry of the hull described by Eq. (18) will enable us to seek both weakly nonlinear and nonlinear wave-free solutions.

III. DISCUSSION OF RESULTS

We begin our discussion of the results with the general layout in Fig. 2. All of the results are characterized by fixed values of $F=0.90$ and $P=0.01$, which will assist us in identifying the qualitatively different types of solutions. In the first column of Figs. 2(a)–2(d) are weakly nonlinear solutions with their corresponding phase plane analysis illustrated in the second column of Figs. 2(e)–2(h). Shown in the third column of Figs. 2(i)–2(l) are fully nonlinear profiles, which compare well with their associated weakly nonlinear profiles (in the first column).

The results in the first row of Fig. 2 are qualitatively similar to those of Maleewong and Grimshaw[10] and others who obtained waveless nonlinear solutions for supercritical flow. This solution type can be referred to as a surfing flow[17,19] because the level of the flat horizontal bottom of the ship is higher than the free-surface as $x \to -\infty$.

The weakly nonlinear analysis in the phase plane that determines the number of independent parameters for this type of supercritical solution can be described as follows. We start at the origin in the phase plane [Fig. 2(e)], which represents the flat horizontal bottom of the ship CD and then move along the black semicircle, the curved part of the hull BC, until we reach the solitary wave orbit. To guarantee the intersection of the black curve with the solitary wave orbit, the value of $\eta(0)$ (or α) must then come as part of the solution. The journey in the phase plane now continues in an anticlockwise direction along the solitary wave orbit to the saddle point, $(\eta_2, 0)$. Therefore the number of independent parameters is two, and they can be taken as F and P.

As the flow is uniform as $x \to -\infty$ in the profiles in Figs. 2(a) and 2(i), we can calculate the value of the additional depth-based Froude number using Eq. (4). For the weakly nonlinear profile,

$$F^* = \frac{F}{(1 + \eta_2)^{3/2}} = 1.19,$$

and for the nonlinear profile,

$$F^* = \frac{F}{y(-\infty)^{3/2}} = 1.17.$$

Thus the flow is indeed supercritical as $x \to -\infty$, and the two values of F^* give a quantitative measure of agreement, which is excellent, between the weakly nonlinear and nonlinear theories. This agreement improves as $F \to 1$ and $P \to 0$.

Now consider the two wavy subcritical flows shown in the second and third rows of Fig. 2. They depend on an extra parameter $\eta(0)$ (or α), which determines the inner periodic orbit in the phase plane that in turn describes the shape of the free-surface in the weakly nonlinear profiles. This type of solution has three independent parameters and they can be chosen as F, P and $\eta(0)$. In the wavy weakly nonlinear profiles we chose values of $\eta(0)$ so that the waves on the free-surface have the same amplitude but are out of phase. The fact that the waves are out of phase is a necessary condition for the existence of a subcritical solution with no waves.

This leads us to examine the waveless subcritical flow shown in the last row of Fig. 2. For such a flow the movement from the black curve must be into the center $(\eta_1, 0)$ in the weakly nonlinear phase plane. Therefore, $\eta(0)=\eta_1$ comes as part of the solution and is no longer an independent parameter, unlike in the wavy subcritical flows. The number of independent parameters for a waveless subcritical solution is then two, F and P. Also, the choice of $\alpha=\eta(0)$ should now be clear [see Eqs. (18) and (19)] as we require that $\eta_x(0)=0$, where the free-surface separates from the ship at B. The weakly nonlinear and nonlinear values of $F^*=0.85$ and $F^*=0.84$ are in excellent agreement as well.

Finally, we present in Fig. 3 a comparison between computed nonlinear and weakly nonlinear values of the waveless downstream free-surface elevation for waveless subcritical flow and supercritical flow. As expected the agreement between the nonlinear and weakly nonlinear theories is excellent when $F\sim 1$, $|\eta|\ll 1$, and $P\ll 1$. It is interesting to see that this agreement is good even when F is not close to 1 in the case of waveless subcritical flow, for $P=0.01$ [top broken curve and dots in Fig. 3(a)]. An explanation for this is given by considering the limit as $F\to 0$ in Eqs. (13) and (15). For the nonlinear and weakly theory, this yields $\eta=P$ and $\eta=P/2+\mathcal{O}(P^2)$, respectively.

IV. CONCLUDING REMARKS

We have shown the existence of both weakly and nonlinear solutions for waveless subcritical stern flows, provided that care is taken when prescribing the geometry of the ship-hull. These flows reduce a major component of the overall drag on a ship. Our broader study involves computing further nonlinear solutions for values of F not close to 1 and P not close to 0, which are outside the range of validity of the weakly nonlinear theory. These results will be presented elsewhere.

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