Constructing quantum games from nonfactorizable joint probabilities

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A probabilistic framework is developed that gives a unifying perspective on both the classical and quantum versions of two-player games. We suggest exploiting peculiar joint probabilities involved in Einstein-Podolsky-Rosen (EPR) experiments to construct a quantum game when the corresponding classical game is obtained from factorizable joint probabilities. We analyze how nonfactorizability changes Nash equilibria in three well-known games of prisoner's dilemma, stag hunt, and chicken. In this framework we find that for the game of prisoner's dilemma even nonfactorizable EPR joint probabilities cannot be helpful to escape from the classical outcome of the game. For a particular version of the chicken game, however, we find that the two nonfactorizable sets of joint probabilities, which maximally violate the Clauser-Holt-Shimony-Horne sum of correlations, indeed result in new Nash equilibria.

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I. INTRODUCTION

The usual approach in the area of quantum games [1–9] consists of analyzing a quantum system maneuvered by participating agents, recognized as players, who possess the necessary means for their actions on parts of the system. The quantum system evolves to its final state, and players' payoffs, or utilities, mathematically expressed as expectation values of self-adjoint payoff operators, are generated from quantum measurement [10]. Thus the usual constructions of quantum games involve the concepts of quantum state vectors, entangled states, quantum measurement, expectation values, trace operation, and density operators, etc. This may seem normal because as being part of the research field of quantum computation [11] quantum games are expected to exploit relevant tools from quantum mechanics. However, in our experience, this noticeable reliance of the models of quantum games on the tools of quantum mechanics also succeeds in keeping many readers away from this interdisciplinary area of research. Ideally, they would like to see genuine quantum games constructed from elementary probabilistic concepts, as is the case with many examples in game theory [12]. We find this situation as an opportunity to present a probabilistic approach in which quantum games are constructed without referring to the tools of quantum mechanics.

While looking for the possibility of such an approach, it is encouraging to find that the most unusual character of quantum mechanics can be expressed in terms of probabilities [13] only. For example, Bell's inequalities [10,14–16] can be written in terms of constraints on joint probabilities relevant to pairs of certain random variables. As probabilities are central to the usual analyses in game theory, it seems natural to use the peculiar probabilities, responsible for the violation of Bell's inequalities, to construct quantum games. We therefore suggest the construction of quantum games from the probabilities arising in Einstein-Podolsky-Rosen (EPR) experiments [10,14,16–18] performed to test the violation of Bell's inequalities. The most unusual character of the EPR probabilities—that they may not be factorizable—motivates us, in this paper, to find how nonfactorizability can be used to construct quantum games. In other words, we search for the role of nonfactorizable probabilities in game-theoretic solution concepts when EPR experiments provide the sets of nonfactorizable probabilities.

This explicitly probabilistic approach toward quantum games is expected to be of interest to readers from such areas as economics [19] and mathematical biology [20], where game theory finds extensive applications and the tools of quantum mechanics are found to be rather alien. Second, because of its exclusively probabilistic content, this approach promises to provide a unified perspective for both classical and quantum games.

The rest of the paper is organized as follows. Section II describes standard EPR setup and using it to play two-player games. Section III discusses playing two-player games using coins and presents two- and four-coin setups to play the well-known games of prisoner's dilemma (PD), stag hunt (SH), and chicken [12]. Section IV describes playing two-player games using EPR setup. Section V develops a framework in which factorizable probabilities lead to the classical game whereas nonfactorizable probabilities result in the quantum game. Section VI discusses the results and presents a view for further work.

II. USING THE STANDARD EPR SETUP TO PLAY TWO-PLAYER GAMES

We use the standard EPR setup [10,18] to play two-player games. This setup consists of two spatially-separated participants, known as Alice and Bob, who share two-particle systems emitted by the same source. We denote Alice's parameter by a, which can be set either at S_1 or at S_2, and denote Bob's parameter by b, which can be set either at S_1' or at S_2'.

In a run, Alice sets her apparatus either at S_1 or at S_2 and, in either case, on receiving her particle she makes a measurement, the outcome of which is π_A, which is either +1 or −1. In the same run, Bob sets his apparatus either at S_1' or at S_2' and, in either case, on receiving his particle he makes a measurement, the outcome of which is π_B, which can be either +1 or −1. Alice and Bob record the outcomes of their measurements for many runs as they receive two-particle systems emitted from the same source.

We denote the probability that Alice obtains the outcome $\pi_A=+1$ or -1 by $\Pr(\pi_A;a)$ and, similarly, we denote the probability that Bob obtains the outcome $\pi_B=+1$ or -1 by $\Pr(\pi_B;b)$. Also, we denote the probability that Alice and Bob obtain the outcomes π_A and π_B, respectively, by $\Pr(\pi_A,\pi_B;a,b)$. These outcomes result from their choices of the parameters a and b—i.e., which one of the four pairs (S_1,S_1'), (S_1,S_2'), (S_2,S_1'), and (S_2,S_2') is realized in a run.

According to quantum theory, the outcomes π_A and π_B both are completely random and Alice and Bob can only find the probabilities of obtaining +1 or −1 as the outcomes of their measurements. Alice's and Bob's parameters a and b decide these probabilities.

In many runs, Alice can choose between S_1 or S_2 with some probability. Similarly, in many runs, Bob can choose between S_1' or S_2' with some probability.

Assume that the source emits a total of N two-particle systems. We denote by $N(\pi_A;a)$ the number of times Alice gets the outcome π_A when she may set her parameter a either at S_1 or at S_2. Similarly, we denote by $N(\pi_B;b)$ the number of times Bob gets the outcome π_B when he may set his parameter b either at S_1' or at S_2'. And we denote by $N(\pi_A,\pi_B;a,b)$ the number of times when Alice gets the outcome π_A and Bob gets the outcome π_B, wherever they may set their parameters a and b, respectively. When N is large, the ensemble probabilities are defined as

$$\Pr(\pi_A;a)=N(\pi_A;a)/N,$$

$$\Pr(\pi_A;b)=N(\pi_B;b)/N,$$

$$\Pr(\pi_A,\pi_B;a,b)=N(\pi_A,\pi_B;a,b)/N. \tag{1}$$

Now, factorizability states that

$$\Pr(\pi_A,\pi_B;a,b)=\Pr(\pi_A;a)\Pr(\pi_B;b). \tag{2}$$

Namely, the joint probabilities are arithmetic product of their respective marginals.

We recognize key features of an EPR setup being that these relate to a probabilistic system divided into two parts such that (a) each observer has access to one part of the system, (b) each observer can select between two available choices, (c) observers cannot communicate between themselves, (d) observers can make independent selections between the available choices, (e) probabilities relevant to each part of the system are normalized,[1] and (f) probabilities are sensible quantities.

It is worth mentioning here that the experimental testing of Bell's inequality involves four correlation experiments that correspond to combining S_1 with S_1', S_1 with S_2', S_2 with S_1', and S_2 with S_2', respectively. These experiments are mutually exclusive in the sense that for any given experiment Alice has to select between S_1 and S_2 and Bob has to select between S_1' and S_2'. That is, Alice (Bob) cannot go for S_1 (S_1') and S_2 (S_2') simultaneously because the corresponding observables are incompatible and cannot be measured simultaneously, whereas in the above derivation of the Bell's inequality, it is assumed that S_1,S_1',S_2,S_2' all have definite values which can be measured simultaneously in pairs.

To bring nonfactorizability into the realm of two-player games, we consider symmetric two-player, two-strategy, noncooperative games [12] represented by the matrices

$$\mathcal{A}=\text{Alice}\begin{matrix} \\ X_1 \\ X_2\end{matrix}\overset{\text{Bob}}{\begin{matrix} X_1' & X_2' \\ \end{matrix}}\begin{pmatrix} K & L \\ M & N\end{pmatrix},$$

$$\mathcal{B}=\text{Alice}\begin{matrix} \\ X_1 \\ X_2\end{matrix}\overset{\text{Bob}}{\begin{matrix} X_1' & X_2' \\ \end{matrix}}\begin{pmatrix} K & M \\ L & N\end{pmatrix}, \tag{3}$$

where all K,L,M,N are real numbers. Players can go for one of the two available strategies: X_1, X_2 for Alice and X_1', X_2' for Bob.

In this paper we construct quantum games from nonfactorizable probabilities that exploit the EPR setup. This rests on Fine's view [21] that violation of Bell's inequality in EPR experiments shows that quantum theory violates factorizability. This view allows us to construct quantum games for which factorizability always corresponds to the classical game.

III. PLAYING GAMES WITH COINS

The above-mentioned features are remindful of coins which, if distributed between players, are found to have all the above-mentioned properties. For coins factorizability has a straightforward meaning in that the associated probabilities remain factorizable. Hence, we develop an analysis of two-player games with nonfactorizable probabilities by first translating playing of three well-known games in terms of the games played when players share coins. It turns out that a version of this translation provides the right comparison with the probabilities involved in the EPR experiments and opens the way to the next step—i.e., to introduce nonfactorizable probabilities into the playing of two-player games.

A. Two-coin setup

We now consider pairs of coins and use it to play a two-player game (3). For example, this game can be played when each player receives a coin, heads up, and "to flip" or "not to flip" is a player's strategy. Both coins are then passed to a referee who rewards the players after observing the state of both coins.

Assume S_1 (to flip) and S_2 (not to flip) are Alice's strategies and S_1' (to flip) and S_2' (not to flip) are Bob's strategies. That is, with reference to the matrices (3), we make the association $S_1\sim X_1$, $S_2\sim X_2$ and $S_1'\sim X_1'$, $S_1'\sim X_1'$. In a two-coin setup, we assume that the strategies S_1 and S_1' represent Alice's and Bob's actions "to flip" the coin, respectively and,

[1] Its exact meaning will be described shortly.

similarly, S_2 and S_2' represent Alice's and Bob's actions "not to flip" the coin, respectively.

In repeated runs of the game players can play mixed strategies. Alice's mixed strategy $x \in [0,1]$ is the probability to choose S_1 over S_2, and similarly Bob's mixed strategy $y \in [0,1]$ is the probability to choose S_1' over S_2'. Players' payoffs are written as

$$\Pi_{A,B}(x,y) = \begin{pmatrix} x \\ 1-x \end{pmatrix}^T \begin{pmatrix} (K,K) & (L,M) \\ (M,L) & (N,N) \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix}, \qquad (4)$$

where T is for transpose and the subscripts A and B refer to Alice and Bob, respectively. The first and second entries in parentheses are Alice's and Bob's payoffs, respectively. Assuming the strategy pair $(x^\star, y^\star)$ to be a Nash equilibrium (NE) [12] then requires that

$$\Pi_A(x^\star,y^\star) - \Pi_A(x,y^\star) \geq 0,$$

$$\Pi_B(x^\star,y^\star) - \Pi_B(x^\star,y) \geq 0. \qquad (5)$$

In the rest of this paper we will use "NE" when we refer to either a Nash equilibrium or to Nash equilibria, assuming that the right meaning can be judged from the context. We identify this arrangement to play a two-player game using two coins as the *two-coin setup*.

B. Four-coin setup

The game (3) can also be played using four coins instead of two. It is arranged by assigning two coins to each player before the game is played. In a run each player has to choose one coin. Two coins out of four are, therefore, chosen by the players in each turn. These coins are then passed to a referee who tosses them together and observes the outcome. It is assumed that the players do not need to share fair coins.

We recall that in two-coin setup S_1 and S_1' are Alice's and Bob's strategies, respectively, which represent players' actions "to flip" the coin that a player receives in a turn. Instead of flipping or not flipping, in four-coin setup a player's strategy is to choose one out of the two coins that are made available to each player in a turn. The four-coin setup is relevant as, in a run, choosing a coin out of the two corresponds to choosing one of the two directions in which measurement is performed in standard EPR experiment, the outcome of which is $+1$ or -1.

In repeated games, a player's strategy is defined by the selection she makes over several runs of the game. For example, a player plays a pure strategy when she goes for the same coin over all the runs and plays a mixed strategy when she finds a probability to choose one coin out of the two over many runs. The referee rewards the players according to their strategies, the underlying statistics of four coins obtained from the outcomes of many tosses, each one of which follows every time the two players choose two out of the total four coins, and the matrices (3) representing the game being played.

We identify the arrangement using four coins to play a two-player game as the *four-coin setup*. Note that in four-coin setup the players' rewards depend on the outcomes of repeated tosses even for pure strategies. A large number of runs are, therefore, necessary whether a player plays a pure strategy or a mixed strategy. The four-coin setup provides an inherently probabilistic character to playing a two-player game and facilitates a probabilistic analysis when we seek to play the game (3) using EPR experiments.

As the four-coin setup uses a different definition of a strategy relative to the two-coin setup, we call S_1 and S_2 Alice's coins and S_1' and S_2' Bob's coins. When selecting a coin is a player's strategy and we want to play the game given by the matrices (3), it is reasonable to make the association $S_1 \sim X_1$, $S_2 \sim X_2$ and $S_1' \sim X_1'$, $S_1' \sim X_1'$.

We represent the head of a coin by $+1$ and its tail by -1 and adapt this convention in the rest of this paper. For coins, Alice's outcome of $\pi_A = +1$ or -1 (whether she goes for S_1 or S_2) is independent from Bob's outcome of $\pi_B = +1$ or -1 (whether he goes for S_1' or S_2') and relevant joint probabilities are factorizable.

Referring to the definition (2) of factorizability and noticing that probabilities associated with coins are factorizable, we use the same notation that is introduced in Sec. II to consider, for example, the probability $\Pr(\pi_A, \pi_B; S_1, S_1')$, which can be factorized as $\Pr(\pi_A; S_1)\Pr(\pi_B; S_1')$.

We define the probabilities $r, r' \in [0,1]$ by $r = \Pr(+1; S_1)$ and $r' = \Pr(+1; S_1')$, saying that r is the probability of getting a heads for Alice's first coin S_1 and r' is the probability of getting a heads for Bob's first coin S_1'. Factorizability then allows us to write $\Pr(+1,-1; S_1, S_1') = r(1-r')$ and $\Pr(-1,-1; S_2, S_2') = (1-s)(1-s')$ where $s = \Pr(+1; S_2)$ and $s' = \Pr(+1; S_2')$; i.e., s and s' are the probabilities of getting a heads for Alice's and Bob's second coin, respectively.

In four-coin setup we find it useful to have the following table:

			Bob			
			S_1'		S_2'	
			$+1$	-1	$+1$	-1
Alice	S_1	$+1$	rr'	$r(1-r')$	rs'	$r(1-s')$
		-1	$r'(1-r)$	$(1-r)(1-r')$	$s'(1-r)$	$(1-r)(1-s')$
	S_2	$+1$	sr'	$s(1-r')$	ss'	$s(1-s')$
		-1	$r'(1-s)$	$(1-s)(1-r')$	$s'(1-s)$	$(1-s)(1-s')$

(6)

from which we define payoff relations for the players:

$$\Pi_{A,B}(S_1,S_1') = \mathfrak{r}^T(\mathcal{A},\mathcal{B})\mathfrak{r}', \quad \Pi_{A,B}(S_1,S_2') = \mathfrak{r}^T(\mathcal{A},\mathcal{B})\mathfrak{s}',$$

$$\Pi_{A,B}(S_2,S_1') = \mathfrak{s}^T(\mathcal{A},\mathcal{B})\mathfrak{r}', \quad \Pi_{A,B}(S_2,S_2^1) = \mathfrak{s}^T(\mathcal{A},\mathcal{B})\mathfrak{s}', \tag{7}$$

where $\mathfrak{r}=\binom{r}{1-r}$, $\mathfrak{s}=\binom{s}{1-s}$, $\mathfrak{r}'=\binom{r'}{1-r'}$, and $\mathfrak{s}'=\binom{s'}{1-s'}$. For example, $\Pi_A(S_1,S_2')$ is Alice's payoff when, in repeated runs of coin tossing, she always goes for her first coin, i.e., S_1, while Bob goes for his second coin, i.e., S_2'.

As is the case with the two-coin setup, Alice's mixed strategy in the four-coin setup is the probability with which she chooses her pure strategy[2] S_1 over her other pure strategy S_2 during repeated runs of the experiment. Similarly, Bob's mixed strategy is the probability with which he chooses his pure strategy S_1' over his other pure strategy S_2' during repeated runs of the experiment. Assume that Alice plays S_1 with probability x and Bob plays S_1' with probability y; their mixed-strategy payoff relations are

$$\Pi_{A,B}(x,y) = \begin{pmatrix} x \\ 1-x \end{pmatrix}^T \begin{pmatrix} \Pi_{A,B}(S_1,S_1') & \Pi_{A,B}(S_1,S_2') \\ \Pi_{A,B}(S_2,S_1') & \Pi_{A,B}(S_2,S_2') \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix}. \tag{8}$$

The NE can then be found from (5), which is written as

$$(\mathfrak{r}-\mathfrak{s})^T\mathcal{A}\{y^\star(\mathfrak{r}'-\mathfrak{s}')+\mathfrak{s}'\}(x^\star - x) \geq 0,$$

$$\{x^\star(\mathfrak{r}-\mathfrak{s})^T+\mathfrak{s}^T\}\mathcal{B}(\mathfrak{r}'-\mathfrak{s}')(y^\star - y) \geq 0. \tag{9}$$

In the following, before we make a transition to playing our game using EPR experiments, we consider playing three well-known games using both the two- and four-coin setups.

C. Examples

We analyze the games of PD, SH, and chicken in two- and four-coin setups and afterwards make a transition to the EPR setup. The PD is known to be representative of the problems of social cooperation [12] and has been one of the earliest [2] and favorite topics for quantum games. Hence it is worthwhile to analyze this game in the setup using nonfactorizable probabilities. Our second game is SH, which, like PD, describes conflict between safety and social cooperation. Our third game is chicken, also known as the hawk-dove game [12], which is considered an influential model of conflict for two players in game theory.

1. Prisoner's dilemma

The PD is a noncooperative game [12] that is widely known to economists and social and political scientists and in recent years to quantum physicists. It was one of the earliest games to be investigated in the quantum regime [2]. Its name comes from the following situation: two criminals are arrested after having committed a crime together. Each suspect is placed in a separate cell and may choose between two strategies: *to confess* (D) and *not to confess* (C), where C and D stand for cooperation and defection.

If neither suspect confesses, i.e., (C,C), they go free, which is represented by K units of payoff for each suspect. When one prisoner confesses (D) and the other does not (C), the prisoner who confesses gets M units of payoff, which represents freedom as well as financial reward, i.e., $M>K$, while the prisoner who did not confess gets L, represented by his ending up in prison. When both prisoners confess, i.e., (D,D), both are given a reduced term represented by N units of payoff, where $N>L$, but it is not so good as going free, i.e., $K>N$.

Referring to the matrices (3) we make the association X_1, $X_1'\sim C$ and X_2, $X_2'\sim D$ and require that $M>K>N>L$. We define $\Delta_1=(M-K)$, $\Delta_2=(N-L)$, and $\Delta_3=(\Delta_2-\Delta_1)$ which makes Δ_1, $\Delta_2>0$ for this game. In two-coin setup, the inequalities (5) give

$$\Pi_A(x^\star,y^\star)-\Pi_A(x,y^\star) = (y^\star\Delta_3-\Delta_2)(x^\star-x) \geq 0,$$

$$\Pi_B(x^\star,y^\star)-\Pi_B(x^\star,y) = (x^\star\Delta_3-\Delta_2)(y^\star-y) \geq 0, \tag{10}$$

with the result that the strategy pair $(x^\star,y^\star)=(0,0)$ comes out as a unique NE. At this equilibrium the players' payoffs are $\Pi_A(S_1,S_1')=N=\Pi_B(S_1,S_1')$.

In the four-coin setup, the PD game as defined above is played as follows. Using the mixed-strategy payoff relation (8), the pair of pure strategies (S_2,S_2') is represented by $(x^\star,y^\star)=(0,0)$. If we require this strategy pair to be a NE, then we also need to know about the constraints this requirement imposes on r, s, r', and s'. When $(x^\star,y^\star)=(0,0)$ the NE inequalities (9) for PD are reduced to $-x(s-r)\Delta_2\{(\Delta_1/\Delta_2-1)s+1\}\geq 0$ and $-y(s'-r')\Delta_2\{(\Delta_1/\Delta_2-1)s'+1\}\geq 0$ which require that $(s-r)\leq 0$ and $(s'-r')\leq 0$ which should be true both when $\Delta_1/\Delta_2\geq 1$ and when $\Delta_1/\Delta_2<1$. This, of course, is possible if

$$\Pr\{S_2(+1)\} = s = 0 = s' = \Pr\{S_2'(+1)\}, \tag{11}$$

which must be fulfilled if the strategy pair $(x^\star,y^\star)=(0,0)$ is to be a NE in PD. Along with this the probabilities $\Pr(\pi_A,\pi_B;a,b)$ are to be factorizable.

This result provides the basis on which the forthcoming argument for the quantum version of this game rests. Notice that, from (7), we obtain $\Pi_A(S_2,S_2')=\Pi_B(S_2,S_2')=N$ when Eq. (11) holds.

The constraint (11) appears when the strategy pair (S_2,S_2') is assumed to be the NE. One can assume other strategy pair—for example, (S_1,S_1')—to be a NE for which, instead of the requirement (11), we obtain $\Pr\{S_1(+1)\}=r=0=r'=\Pr\{S_1'(+1)\}$. However, it is found that this freedom does not affect the argument for a quantum game developed in this paper.

2. Stag hunt

Along with PD, the game of SH provides another interesting context to study problems of social cooperation. It

[2]Notice that our definition of a pure strategy corresponds to the usual mixed-strategy. This agrees with the result in quantum games that a product a pure state corresponds to a mixed-strategy classical game.

describes the situation when two hunters can either jointly hunt a stag (an adult deer that makes a large meal) or individually hunt a rabbit (which is tasty but makes a substantially small meal). Hunting a stag is quite challenging and hunters need to cooperate with each other, especially, it is quite unlikely that a hunter hunts a stag alone.

It is found that, in contrast to PD, which has a single, pure NE, the game of SH has three NE, two of which are pure and one is mixed. The two pure NE correspond to the situations when both hunters hunt the stag as a team and when each hunts rabbit by himself. The SH differs from PD in that mutual cooperation gives maximum reward to the hunters. When compared to PD, SH is considered a better model for the problems of (social) cooperation.

Referring to the matrices (3) the game of SH is defined by $K > M \geq N > L$ and $M+N > K+L$. In two-coin setup the NE inequalities for this game are the same as the inequalities (10) except that now we have $\Delta_3 > \Delta_2 > 0$ and $0 > \Delta_1$ instead of Δ_1, $\Delta_2 > 0$, which holds for PD. Here Δ_1, Δ_2, and Δ_3 are defined in Sec. III C 1. This leads to three NE:

$$(x^\star, y^\star)_1 = (0,0),$$

$$(x^\star, y^\star)_2 = (\Delta_2/\Delta_3, \Delta_2/\Delta_3),$$

$$(x^\star, y^\star)_3 = (1,1), \tag{12}$$

and the corresponding payoffs at these equilibria, obtained from Eqs. (4), are

$$\Pi_A(x^\star, y^\star)_1 = N = \Pi_B(x^\star, y^\star)_1,$$

$$\Pi_A(x^\star, y^\star)_2 = (\Delta_2/\Delta_3)^2\Delta_3 + (\Delta_2/\Delta_3)\Delta_4 + N = \Pi_B(x^\star, y^\star)_2,$$

$$\Pi_A(x^\star, y^\star)_3 = K = \Pi_B(x^\star, y^\star)_3,$$

where we define $\Delta_4 = L + M - 2N$.

Now consider playing this game within the four-coin setup in which the NE inequalities (9) reduce to

$$(r-s)[y^\star(r'-s')\Delta_3 + (s'\Delta_3 - \Delta_2)](x^\star - x) \geq 0,$$

$$(r'-s')[x^\star(r-s)\Delta_3 + (s\Delta_3 - \Delta_2)](y^\star - y) \geq 0. \tag{13}$$

From these inequalities the NE $(x^\star, y^\star)_1 = (0,0)$ results when $s = 0 = s'$ and, similarly, the NE $(x^\star, y^\star)_3 = (1,1)$ results when $r = 0 = r'$. Also, the inequalities (13) hold when $x^\star = (s\Delta_3 - \Delta_2)/(s-r)\Delta_3$ and $y^\star = (s'\Delta_3 - \Delta_2)/(s'-r')\Delta_3$, and for $(x^\star, y^\star)_2 = (\Delta_2/\Delta_3, \Delta_2/\Delta_3)$ to be a NE we require

$$s = 0, \quad r = 1 \quad \text{and} \quad s' = 0, \quad r' = 1. \tag{14}$$

These constraints on r, s, r', and s' hold along with the probabilities $\Pr(\pi_A, \pi_B; a, b)$ being factorizable.

3. Chicken game

The game of chicken is about two drivers who drive toward each other from opposite directions. One driver must turn aside, or both may die in a crash. If one driver turns aside but the other does not, he will be called a "chicken." While each driver prefers not to yield to the opponent, the outcome where neither driver yields is the worst possible one for both. In this anticoordination game it is mutually beneficial for parties to play different strategies.

Sometimes, chicken is also known as the "hawk-dove" game, which originates from the parallel development of the basic principles of this game in two different research areas: economics and mathematical biology. Economists, and political scientists too, refer [19] to this game as chicken, while mathematical biologists refer [20] to it as the hawk-dove game.

The game of chicken differs from PD in that in chicken the mutual defection (the crash when both players drive straight) is the most feared outcome, while in PD cooperation while the other player defects is the worst outcome.

A version of the chicken game is obtained from the matrices (3) when

$$K = 0, \quad L = \alpha, \quad M = \beta,$$

$$N = 0, \quad 0 < \alpha < (\alpha + \beta). \tag{15}$$

Playing this game in two-coin setup the inequalities (5) are reduced to $\{\alpha - y^\star(\alpha+\beta)\}(x^\star - x) \geq 0$ and $\{\alpha - x^\star(\alpha+\beta)\}(y^\star - y) \geq 0$ and three NE emerge:

$$(x^\star, y^\star)_1 = (1,0),$$

$$(x^\star, y^\star)_2 = (\alpha/(\alpha+\beta), \alpha/(\alpha+\beta)),$$

$$(x^\star, y^\star)_3 = (0,1). \tag{16}$$

The corresponding payoffs at these equilibria, obtained from Eqs. (4), are $\Pi_A(x^\star, y^\star)_1 = \alpha$, $\Pi_B(x^\star, y^\star)_1 = \beta$, $\Pi_A(x^\star, y^\star)_2 = \alpha\beta/(\alpha+\beta) = \Pi_B(x^\star, y^\star)_2$, $\Pi_A(x^\star, y^\star)_3 = \beta$, and $\Pi_B(x^\star, y^\star)_3 = \alpha$. Now we play this game using the four-coin setup. The NE inequalities come out to be the same as the ones given in (13) except that now we have $\Delta_3 = -(\alpha+\beta)$ and $\Delta_2 = -\alpha$. Then for $(x^\star, y^\star)_1 = (1,0)$ we require $r = 0$ and $s' = 0$. Similarly, for $(x^\star, y^\star)_3 = (0,1)$ we require $r' = 0$ and $s = 0$.

At $(x^\star, y^\star)_2 = (\alpha/(\alpha+\beta), \alpha/(\alpha+\beta))$ the inequalities (13) reduce to $(r-s)(\alpha - \alpha r' - \beta s')[\alpha/(\alpha+\beta) - x] \geq 0$ and $(r'-s') \times (\alpha - \alpha r - \beta s)[\alpha/(\alpha+\beta) - y] \geq 0$ which puts constraint on r, s, r', and s' given as

$$\alpha(1-r') = \beta s', \quad \alpha(1-r) = \beta s. \tag{17}$$

A special case is the one when $\alpha = \beta$ and the strategy pair $(x^\star, y^\star) = (1/2, 1/2)$ becomes a NE which imposes certain constraints on r, s, r', and s'. For this NE the inequalities (9), for the game defined by (3) and (15), are reduced to $(r-s)\{-(\alpha+\beta)(r'+s')/2 + L\}(1/2 - x) \geq 0$ and $(r'-s')\{-(\alpha+\beta)(r+s)/2 + L\}(1/2 - y) \geq 0$. This requires $r+s = 1 = r'+s'$ if the strategy pair $(x^\star, y^\star) = (1/2, 1/2)$ is to be a NE in this game. Along with this, the probabilities $\Pr(\pi_A, \pi_B; a, b)$ are to be factorizable.

IV. PLAYING GAMES WITH EPR EXPERIMNTS

Section III B describes playing a two-player game with four coins such that choosing a coin is a strategy while players' payoffs are given by their strategies, the matrix of the game, and the underlying statistics of the coins. This facilitates transition to playing the *same* game using EPR experiments.

In the EPR setup, Alice and Bob are spatially separated and are unable to communicate with each other. In an individual run, both receive one-half of a pair of particles originating from a common source. In the same run of the experiment both choose one from two given (pure) strategies. These strategies are the two directions in space along which spin or polarization measurements can be made.

Keeping the notation for the coins, we denote these directions to be S_1, S_2 for Alice and S_1', S_2' for Bob. Each measurement generates +1 or −1 as the outcome, as is the case with coins after their toss in the four-coin setup. Experimental results are recorded for a large number of individual runs of the experiment, and payoffs are awarded depending on the directions the players go for over many runs (defining their strategies), the matrix of the game they play, and the statistics of the measurement outcomes.

For EPR experiments, we retain Cereceda's notation [22] for the associated probabilities:

$$p_k = \Pr(\pi_A, \pi_B; a, b), \tag{18}$$

with

$$k = 1 + \frac{(1-\pi_B)}{2} + 2\frac{(1-\pi_A)}{2} + 4(b-1) + 8(a-1).$$

In this notation, for example, we write p_1 for the probability $\Pr(+1,+1;S_1,S_1')$ and p_8 for the probability $\Pr(-1,-1;S_1,S_2')$. One can then construct the following table of probabilities:

			Bob			
			S_1'		S_2'	
			$+1$	-1	$+1$	-1
Alice	S_1	$+1$	p_1	p_2	p_5	p_6
		-1	p_3	p_4	p_7	p_8
	S_2	$+1$	p_9	p_{10}	p_{13}	p_{14}
		-1	p_{11}	p_{12}	p_{15}	p_{16}

(19)

This table allows us to transparently see how the probabilities $p_i (1 \le i \le 16)$ are linked to the probabilities $\Pr(\pi_A, \pi_B; a, b)$, where we recall that a can be set at S_1 or at S_2 and, similarly, b can be set at S_2 or at S_2'. In Cereceda's notation the EPR probabilities p_i are normalized as they satisfy the following relations:

$$p_1 + p_2 + p_3 + p_4 = 1, \quad p_5 + p_6 + p_7 + p_8 = 1,$$
$$p_9 + p_{10} + p_{11} + p_{12} = 1, \quad p_{13} + p_{14} + p_{15} + p_{16} = 1. \tag{20}$$

Notice that the factorizable probabilities (6) are also normalized and (20) holds for them.

Payoff relations (7) are originally constructed when the game given by the matrices (3) is played with four coins and their mathematical form convinces one to use the following recipe [23,24] to reward players when the same game is played using EPR probabilities (19):

$$\Pi_A(S_1,S_1') = Kp_1 + Lp_2 + Mp_3 + Np_4,$$
$$\Pi_A(S_1,S_2') = Kp_5 + Lp_6 + Mp_7 + Np_8,$$
$$\Pi_A(S_2,S_1') = Kp_9 + Lp_{10} + Mp_{11} + Np_{12},$$
$$\Pi_A(S_2,S_2^1) = Kp_{13} + Lp_{14} + Mp_{15} + Np_{16}. \tag{21}$$

Here $\Pi_A(S_1,S_2')$, for example, is Alice's payoff when she plays S_1 and Bob plays S_2'. As is the case with four coins, the payoff relations for Bob are obtained from (21) by the transformation $L \leftrightarrows M$ in Eqs. (21).

When p_i are factorizable in terms of r, r', s, s', a comparison of (21) with (7) requires

$$p_1 = rr', \quad p_2 = r(1-r'), \quad p_3 = r'(1-r),$$
$$p_4 = (1-r)(1-r'),$$
$$p_5 = rs', \quad p_6 = r(1-s'), \quad p_7 = s'(1-r),$$
$$p_8 = (1-r)(1-s'),$$
$$p_9 = sr', \quad p_{10} = s(1-r'), \quad p_{11} = r'(1-s),$$
$$p_{12} = (1-s)(1-r'),$$
$$p_{13} = ss', \quad p_{14} = s(1-s'), \quad p_{15} = s'(1-s),$$
$$p_{16} = (1-s)(1-s'). \tag{22}$$

That is, the factorizability of p_i in terms r, r', s, and s' makes the game played by EPR probabilities equivalent to the one played by using coins.

However, the EPR probabilities p_i, appearing in (7), may not be factorizable in terms of r, s, r', and s', whereas for both the payoff relations (7) and (21) the normalization (20) continues to hold.

V. TWO-PLAYER GAMES USING NONFACTORIZABLE PROBABILITIES

As is the case with the coin game, Alice's mixed strategy is defined to be the probability to choose between S_1 and S_2 and we can use, once again, the payoff relations (8), which, however, now correspond to the possible situation when p_i may not be factorizable So that relations (7) can be replaced with relations (21) in Alice's mixed-strategy payoff relation in (8). The same applies to Bob's payoff relations.

Note that when p_i are factorizable, using (22) allows the probabilities r, r', s, s' to be expressed in terms of p_i:

$$r = p_1 + p_2, \quad s = p_9 + p_{10},$$
$$r' = p_1 + p_3, \quad s' = p_5 + p_7, \tag{23}$$

which are useful relations for the forthcoming argument for a quantum game.

Along with the normalization (20), the EPR probabilities p_i $(1 \leq i \leq 16)$ also satisfy certain other constraints imposed by the requirements of causality. Cereceda [22] writes these constraints as

$$p_1 + p_2 - p_5 - p_6 = 0, \quad p_1 + p_3 - p_9 - p_{11} = 0,$$

$$p_9 + p_{10} - p_{13} - p_{14} = 0, \quad p_5 + p_7 - p_{13} - p_{15} = 0,$$

$$p_3 + p_4 - p_7 - p_8 = 0, \quad p_{11} + p_{12} - p_{15} - p_{16} = 0,$$

$$p_2 + p_4 - p_{10} - p_{12} = 0, \quad p_6 + p_8 - p_{14} - p_{16} = 0, \quad (24)$$

which is referred to as the *causal communication constraint* [22]. Notice that the constraints (24), of course, also hold when p_i are factorizable and are written in terms of r, s, r', and s' as in (22). Essentially, these constraints state that, on measurement, Alice's probability of obtaining particular outcome (+1 or −1), when she goes for S_1 or S_2, is independent of how Bob sets up his apparatus (i.e., along S_1' or along S_2'). The same applies to Bo—i.e., on measurement his probability of obtaining a particular outcome (+1 or −1) when he goes for S_1' or S_2'—is independent of how Alice sets up her apparatus (i.e., along S_1 or along S_2). Other authors may like to call the constraints (24) by some different name; for example, Winsberg and Fine [25] have described them as the *locality constraint*.

Notice that because of normalization (20) half of Eqs. (24) are redundant, which makes 8 among 16 probabilities p_i independent. A convenient solution [22] of the system (20) and (24), for which the set of variables

$$\upsilon = \{p_2, p_3, p_6, p_7, p_{10}, p_{11}, p_{13}, p_{16}\} \quad (25)$$

is expressed in terms of the remaining set of variables

$$\mu = \{p_1, p_4, p_5, p_8, p_9, p_{12}, p_{14}, p_{15}\}, \quad (26)$$

is given as follows:

$$p_2 = (1 - p_1 - p_4 + p_5 - p_8 - p_9 + p_{12} + p_{14} - p_{15})/2,$$

$$p_3 = (1 - p_1 - p_4 - p_5 + p_8 + p_9 - p_{12} - p_{14} + p_{15})/2,$$

$$p_6 = (1 + p_1 - p_4 - p_5 - p_8 - p_9 + p_{12} + p_{14} - p_{15})/2,$$

$$p_7 = (1 - p_1 + p_4 - p_5 - p_8 + p_9 - p_{12} - p_{14} + p_{15})/2,$$

$$p_{10} = (1 - p_1 + p_4 + p_5 - p_8 - p_9 - p_{12} + p_{14} - p_{15})/2,$$

$$p_{11} = (1 + p_1 - p_4 - p_5 + p_8 - p_9 - p_{12} - p_{14} + p_{15})/2,$$

$$p_{13} = (1 - p_1 + p_4 + p_5 - p_8 + p_9 - p_{12} - p_{14} - p_{15})/2,$$

$$p_{16} = (1 + p_1 - p_4 - p_5 + p_8 - p_9 + p_{12} - p_{14} - p_{15})/2. \quad (27)$$

The relationships (27) between joint probabilities arise because both the normalization condition (20) and the causal communication constraint (24) are fulfilled.

From Eqs. (27) one can obtain other constraints considering that the sum of any combination of probabilities from the set υ must be non-negative. In the following are some results to be used later in this paper. In (27) the sum $p_2 + p_7$ is non-negative and it requires that $p_1 + p_8 \leq 1$. In (27) the sum $p_3 + p_{10}$ is non-negative and it requires that $p_1 + p_{12} \leq 1$. These inequalities are found useful when we develop a quantum version of PD. Similarly, the sum $p_6 + p_{13}$ is non-negative and it requires that $p_8 + p_{13} \leq 1$. This inequality is found useful in developing a quantum version of SH.

Using (21) in (8), with the assumption that $(x^\star, y^\star)$ is a NE, one obtains

$$\Pi_A(x^\star, y^\star) - \Pi_A(x, y^\star) = (x^\star - x)[y^\star\{K\Omega_1 + L\Omega_2 + M\Omega_3 + N\Omega_4\} + \{K(p_5 - p_{13}) + L(p_6 - p_{14}) + M(p_7 - p_{15}) + N(p_8 - p_{16})\}] \geq 0, \quad (28)$$

$$\Pi_B(x^\star, y^\star) - \Pi_B(x^\star, y) = (y^\star - y)[x^\star\{K\Omega_1 + M\Omega_2 + L\Omega_3 + N\Omega_4\} + \{K(p_9 - p_{13}) + M(p_{10} - p_{14}) + L(p_{11} - p_{15}) + N(p_{12} - p_{16})\}] \geq 0, \quad (29)$$

where $\Omega_1 = p_1 - p_5 - p_9 + p_{13}$, $\Omega_2 = p_2 - p_6 - p_{10} + p_{14}$, $\Omega_3 = p_3 - p_7 - p_{11} + p_{15}$, and $\Omega_4 = p_4 - p_8 - p_{12} + p_{16}$.

Now use (27) to write (28) and (29) in terms of the probabilities appearing in the set μ given in (26) to obtain

$$\Pi_A(x^\star, y^\star) - \Pi_A(x, y^\star) = (1/2)(x^\star - x)[y^\star\Delta_3(1 + p_1 + p_4 - p_5 - p_8 - p_9 - p_{12} - p_{14} - p_{15}) - \{\Delta_3(1 - p_5 - p_8 - p_{14} - p_{15}) + (\Delta_1 + \Delta_2)(p_1 - p_4 - p_9 + p_{12})\}] \geq 0, \quad (30)$$

$$\Pi_B(x^\star, y^\star) - \Pi_B(x^\star, y) = (1/2)(y^\star - y)[x^\star\Delta_3 \times (1 + p_1 + p_4 - p_5 - p_8 - p_9 - p_{12} - p_{14} - p_{15}) - \{\Delta_3(1 - p_9 - p_{12} - p_{14} - p_{15}) + (\Delta_1 + \Delta_2)(p_1 - p_4 - p_5 + p_8)\}] \geq 0. \quad (31)$$

Notice that the probabilities associated with the EPR experiments can be factorized only for certain directions of measurements even for singlet states. For these directions the game played using EPR experiments can thus be interpreted within the four-coin setup.

Essentially, we obtain a quantum game from the classical as follows. Referring to the four-coin setup developed in the Sec. III B, the factorizability of associated probabilities in terms of r, s, r', and s' allows us to translate the requirement that the resulting game have a classical interpretation into certain constraints on r, s, r', and s'. We find that from factorizability the relations (23) follow and from these relations the constraints on r, s, r', and s' can be reexpressed in terms of p_i $(1 \leq i \leq 16)$. We now obtain a quantum version of the game by retaining these constraints and afterwards allowing p_i to become nonfactorizable. In this procedure retaining the constraints ensures that a classical outcome results when the probabilities become factorizable.

In the following we consider the impact of nonfactorizable probabilities on the NE in PD, SH, and chicken game.

A. Prisoner's dilemma

Recall that Sec. III A states the result that when PD is played with four coins we require the condition (11) to hold if the strategy pair (S_2, S_2') is to exist as a NE. Along with this the probabilities p_i are to be factorizable.

This motivates us to construct a quantum version of PD when probabilities p_i are not factorizable while the constraint (11) remains valid. The condition (11) ensures that with factorizable probabilities the game can be interpreted classically.

Notice that when the probabilities p_i are factorizable—i.e., they can be written as in (22)—the constraint (11) can hold when numerical values are assigned to certain probabilities among p_i:

$$p_5 = 0, \quad p_7 = 0, \quad p_9 = 0, \quad p_{10} = 0, \quad p_{16} = 1, \tag{32}$$

where, because of the normalization (20), $p_{16}=1$ requires that $p_{13}=0$, $p_{14}=0$, and $p_{15}=0$. This can also be noticed more directly from (23). This assignment of values to certain probabilities reduces Eqs. (20) and (24) to

$$p_1 + p_2 + p_3 + p_4 = 1, \quad p_1 + p_2 = p_6, \quad p_1 + p_3 = p_{11},$$

$$p_3 + p_4 = p_8, \quad p_{11} + p_{12} = 1, \quad p_2 + p_4 = p_{12}, \quad p_6 + p_8 = 1. \tag{33}$$

Substituting (32) into (30) and (31) gives

$$\Pi_A(x^\star,y^\star) - \Pi_A(x,y^\star) = (1/2)(x^\star - x)[y^\star\Delta_3(1 + p_1 + p_4 - p_8 - p_{12}) - \{\Delta_3(1 - p_8) + (\Delta_1 + \Delta_2)(p_1 - p_4 + p_{12})\}] \geq 0, \tag{34}$$

$$\Pi_B(x^\star,y^\star) - \Pi_B(x^\star,y) = (1/2)(y^\star - y)[x^\star\Delta_3(1 + p_1 + p_4 - p_8 - p_{12}) - \{\Delta_3(1 - p_{12}) + (\Delta_1 + \Delta_2)(p_1 - p_4 + p_8)\}] \geq 0. \tag{35}$$

Note that from (33) we obtain $(1-p_8)=p_6=p_1+p_2$, which, for factorizable probabilities, becomes equal to r when we refer to Eqs. (23). Similarly, from (33) we obtain $(p_1-p_4+p_{12})=p_1+p_2=r$. Substituting these into (34) along with the condition $x^\star=0=y^\star$ gives $\Pi_A(0,0)-\Pi_A(x,0)=xr\Delta_2\geq 0$. In a similar way we find from (33) that $(1-p_{12})=p_{11}=p_1+p_3$ which for factorizable probabilities becomes equal to r' when we use Eqs. (23). Likewise, from (33) we obtain $(p_1-p_4+p_8)=p_1+p_3=r'$. Substituting these into (35), along with the condition $x^\star=0=y^\star$, gives $\Pi_B(0,0)-\Pi_B(0,y)=yr'\Delta_2\geq 0$. This can be described as follows: When probabilities p_1, p_4, p_8, and p_{12} are factorizable and the values assigned to them in (32) hold, the inequalities (34) and (35) ensure that the strategy pair (S_2, S_2') becomes a NE.

Now we ask about the fate of the NE strategy pair (S_2, S_2') when in (32) the values assigned to certain probabilities, resulting from the requirement (11), hold while p_i do not remain factorizable in terms of r, s, r', and s'. Allow the probabilities p_1, p_4, p_8, and p_{12} not to be factorizable and use (32) in (27) to get $1-p_1+p_4-p_8-p_{12}=0$ and the inequalities (34) and (35) take the form

$$\Pi_A(x^\star,y^\star) - \Pi_A(x,y^\star) = (x^\star - x)[\Delta_2\{y^\star - (1 - p_8)/p_1\} - \Delta_1 y^\star]p_1 \geq 0, \tag{36}$$

$$\Pi_B(x^\star,y^\star) - \Pi_B(x^\star,y) = (y^\star - y)[\Delta_2\{x^\star - (1 - p_{12})/p_1\} - \Delta_1 x^\star]p_1 \geq 0, \tag{37}$$

where Δ_1 and Δ_2 are defined in the Sec. III C 1. Note that as $(p_1+p_8)\leq 1$ and $(p_1+p_{12})\leq 1$ we have $1\leq(1-p_8)/p_1$ and $1\leq(1-p_{12})/p_1$ so that

$$\{y^\star - (1 - p_8)/p_1\} \leq 0, \quad \{x^\star - (1 - p_{12})/p_1\} \leq 0, \tag{38}$$

which results, once again, in the strategy pair $(x^\star, y^\star) = (0,0)$ being a NE, which is the classical outcome of the game.

Notice that this NE emerges for nonfactorizable EPR probabilities along with our requirement that factorizable probabilities must lead to the classical game. This result for PD appears to diverge away from the reported results in quantum games [2]. We believe that part of the reason resides with how payoff relations and players' strategies are defined in the present framework, which exploits the EPR setup for playing a quantum game.

B. Stag hunt

Section III C 2 describes playing SH in the four-coin setup for which three NE emerge. For each of these three NE there correspond constraints on r, s, r', and s' for factorizable probabilities. In the following we first translate these constraints in terms of the EPR probabilities p_i and afterwards allow p_i to assume nonfactorizable values when the constraints on r, s, r', and s', expressed in terms of p_i, continue to hold. In the following we follow this procedure for each individual NE that arises when SH is played in the four-coin setup.

$(x^\star, y^\star)_1=(0,0)$. Refer to (12) in Sec. III C 2 and consider the NE $(x^\star, y^\star)_1=(0,0)$ for which the constraint on probabilities are (11) as is the case with PD. The analysis for the quantum PD from Sec. V A, therefore, remains valid and we can directly use the inequalities (36)–(38) to obtain

$$\Pi_A(x^\star,y^\star) - \Pi_A(x,y^\star) = (x^\star - x)\{y^\star\Delta_3 - \Delta_2(1 - p_8)/p_1\}p_1 \geq 0,$$

$$\Pi_B(x^\star,y^\star) - \Pi_B(x^\star,y) = (y^\star - y)\{x^\star\Delta_3 - \Delta_2(1 - p_{12})/p_1\}p_1 \geq 0, \tag{39}$$

where $\Delta_3 > \Delta_2 > 0$. This gives rise to three equilibria

$$(x^\star, y^\star)_1^{Q_a} = (0,0),$$

$$(x^\star, y^\star)_2^{Q_a} = \{(\Delta_2/\Delta_3)(1 - p_{12})/p_1, \ (\Delta_2/\Delta_3)(1 - p_8)/p_1\},$$

$$(x^\star,y^\star)_3^{Q_a}=(1,1), \tag{40}$$

where the superscript Q refers to "quantum." From the relations (39) and (40) and the inequalities $(p_1+p_8)\leq 1$ and $(p_1+p_{12})\leq 1$ it turns out that $(x^\star,y^\star)_1^{Q_a}$ emerges without any further constraints apart from the ones given by the reexpressed form of (11), i.e., (32); $\{(x^\star)_2^{Q_a},(y^\star)_2^{Q_a}\}$ emerges when nonfactorizable probabilities are such that, apart from (32) to hold, both $(\Delta_2/\Delta_3)(1-p_{12})/p_1$ and $(\Delta_2/\Delta_3)(1-p_8)/p_1$ have values in the interval [0, 1]; and $(x^\star,y^\star)_3^{Q_a}$ emerges when, apart from (32) being true, both $\{\Delta_3-\Delta_2(1-p_8)/p_1\}$ and $\{\Delta_3-\Delta_2(1-p_{12})/p_1\}$ are non negative.

$(x^\star,y^\star)_2=(\Delta_2/\Delta_3,\Delta_2/\Delta_3)$. Refer to Sec. III C 2 and use (23) and (24), along with the normalization (20), to express the constraints (14) as

$$p_1=1=p_6 \quad \text{and} \quad p_{11}=1=p_{16}. \tag{41}$$

The normalization (20), then, assigns zero value to the remaining 12 probabilities. Now substitute the constraints (41) in Eqs. (30) and (31) to obtain the NE inequalities that will correspond to the nonfactorizable probabilities: $\Pi_A(x^\star,y^\star)-\Pi_A(x,y^\star)=(x^\star-x)[y^\star\Delta_3-\Delta_2]\geq 0$ and $\Pi_B(x^\star,y^\star)-\Pi_B(x^\star,y)=(y^\star-y)[x^\star\Delta_3-\Delta_2]\geq 0$. This result is identical to the classical situation and the three NE $(x^\star,y^\star)_1^{Q_b}=(0,0)$, $(x^\star,y^\star)_2^{Q_b}=(\Delta_2/\Delta_3,\Delta_2/\Delta_3)$, and $(x^\star,y^\star)_3^{Q_b}=(1,1)$ and emerges when (41) hold.

$(x^\star,y^\star)_3=(1,1)$. For this NE in (12) the constraint on the probabilities is $r=0=r'$. For factorizable probabilities, this constraint can be rewritten using normalization (20) along with (23) and (24) as

$$p_5=0,\quad p_6=0,\quad p_9=0,\quad p_{11}=0,$$
$$p_7+p_8=1=p_{10}+p_{12},\quad p_4=1, \tag{42}$$

from which using the normalization (20) it then follows that $p_1=0$, $p_2=0$, and $p_3=0$. The constraints (42) reduce the Nash inequalities (30) and (31) to

$$\Pi_A(x^\star,y^\star)-\Pi_A(x,y^\star)=(1/2)(x^\star-x)[y^\star\Delta_3(2-p_8-p_{12}-p_{14}-p_{15})-\{\Delta_3(1-p_8-p_{14}-p_{15})+(\Delta_1+\Delta_2)(-1+p_{12})\}]\geq 0, \tag{43}$$

$$\Pi_B(x^\star,y^\star)-\Pi_B(x^\star,y)=(1/2)(y^\star-y)[x^\star\Delta_3(2-p_8-p_{12}-p_{14}-p_{15})-\{\Delta_3(1-p_{12}-p_{14}-p_{15})+(\Delta_1+\Delta_2)(-1+p_8)\}]\geq 0. \tag{44}$$

Using the constraints (42) in the seventh equation in (27) results in $p_{13}=(2-p_8-p_{12}-p_{14}-p_{15})/2$, which then simplifies the Nash inequalities (43) and (44) to

$$\Pi_A(x^\star,y^\star)-\Pi_A(x,y^\star)=(x^\star-x)\{-(1-y^\star)p_{13}\Delta_3+(1-p_{12})\Delta_2\}\geq 0,$$

$$\Pi_B(x^\star,y^\star)-\Pi_B(x^\star,y)=(y^\star-y)\{-(1-x^\star)p_{13}\Delta_3+(1-p_8-p_{13})\Delta_2\}\geq 0, \tag{45}$$

and gives rise to three NE that are described below.

$(\boldsymbol{x}^\star,\boldsymbol{y}^\star)_1^{Q_c}$. For this NE we have $(x^\star,y^\star)_1^{Q_c}=(0,0)$ and the inequalities (45) become

$$\Pi_A(0,0)-\Pi_A(x,0)=xp_{13}\Delta_3\{1-(\Delta_2/\Delta_3)(1-p_{12})/p_{13}\}\geq 0,$$

$$\Pi_B(0,0)-\Pi_B(0,y)=yp_{13}\Delta_3\{1-(\Delta_2/\Delta_3)(1-p_8-p_{13})/p_{13}\}\geq 0. \tag{46}$$

As $(p_1+p_{12})\leq 0$ and $(p_8+p_{13})\leq 0$ we have $(1-p_{12})\geq 0$ and $(1-p_8-p_{13})\geq 0$. That is, $(x^\star,y^\star)_1^{Q_c}=(0,0)$ will be a NE when p_8, p_{12}, and p_{13} are such that $1\geq(\Delta_2/\Delta_3)(1-p_{12})/p_{13}$ and $1\geq(\Delta_2/\Delta_3)(1-p_8-p_{13})/p_{13}$ hold true along with the constraints (42).

$(\boldsymbol{x}^\star,\boldsymbol{y}^\star)_2^{Q_c}$. From the inequalities (45) the strategy pair $(x^\star,y^\star)_2^{Q_c}=(x^\star,y^\star)$ where the strategy pair $x^\star=1-(\Delta_2/\Delta_3)(1-p_8-p_{13})/p_{13}$ and $y^\star=1-(\Delta_2/\Delta_3)(1-p_{12})/p_{13}$ can exist as a NE when p_8, p_{12}, and p_{13} are such that $x^\star,y^\star\in[0,1]$. Together with this the constraints (42) hold.

$(\boldsymbol{x}^\star,\boldsymbol{y}^\star)_3^{Q_c}$. For the possible NE $(x^\star,y^\star)_3^{Q_c}=(1,1)$ the inequalities (45) become

$$\Pi_A(x^\star,y^\star)-\Pi_A(x,y^\star)=(1-x)(1-p_{12})\Delta_2\geq 0,$$

$$\Pi_B(x^\star,y^\star)-\Pi_B(x^\star,y)=(1-y)(1-p_8-p_{13})\Delta_2\geq 0, \tag{47}$$

which are to hold along with that the constraints (42) being true. Using $\Delta_1,\Delta_1>0$ with $(p_1+p_{12})\leq 0$ and $(p_8+p_{13})\leq 0$ we find that the inequalities (47) will always hold and the only requirement for the strategy pair $(x^\star,y^\star)_3^{Q_c}=(1,1)$ to be a NE is that the constraints (42) hold.

So the list of possible NE that can arise in the quantum SH consists of five members: i.e., (0, 0), (1, 1), $\{\Delta_2(1-p_{12})/\Delta_3p_1,\ \Delta_2(1-p_8)/\Delta_3p_1\}$, $(\Delta_2/\Delta_3,\ \Delta_2/\Delta_3)$, and $[\{1-\Delta_2(1-p_8-p_{13})/\Delta_3p_{13}\},\ \{1-\Delta_2(1-p_{12})/\Delta_3p_{13}\}$. Which one, or more, from this list is going to arise depends on the set of nonfactorizable probabilities. For example, we notice that there exist [22] two sets of nonfactorizable probabilities that maximally violate the quantum prediction of the Clauser-Holt-Shimony-Horne (CHSH) [26] sum of correlations. The first set is

$$p_j=(2+\sqrt{2})/8 \quad \text{for all } p_j\in\mu,$$
$$p_k=(2-\sqrt{2})/8 \quad \text{for all } p_k\in\upsilon, \tag{48}$$

and the second set is

$$p_j=(2-\sqrt{2})/8 \quad \text{for all } p_j\in\mu,$$
$$p_k=(2+\sqrt{2})/8 \quad \text{for all } p_k\in\upsilon, \tag{49}$$

where υ and μ are defined in (25) and (26), respectively. The probabilities in these sets are nonfactorizable because for both sets a solution of Eqs. (22) will involve one or more of the probabilities r, s, r', and s' being negative or greater than 1. Now for SH the requirement that factorizable probabilities are to lead to a classical game gives rise to three sets of constraints on EPR probabilities given by (32), (41), and (42). These sets of constraints correspond to the NE

$(x^\star,y^\star)_1=(0,0)$, $(x^\star,y^\star)_2=(\Delta_2/\Delta_3,\Delta_2/\Delta_3)$, and $(x^\star,y^\star)_3=(1,1)$ respectively. Unfortunately, the probabilities from either of the two sets (48) and (49), which maximally violate the quantum prediction of the CHSH sum of correlations, do not satisfy these constraints. Stated otherwise, the probabilities from the sets (48) and (49) are in conflict with the requirement that factorizable probabilities must lead to the classical game of SH. However, other sets of nonfactorizable probabilities can be found that are consistent with this requirement and, depending on the elements of a set, one or more out of five possible NE may emerge. This situation can be described by saying that in SH nonfactorizability can lead to NE, but unfortunately, either set (48) or (49) cannot be used for this purpose.

C. Chicken game

Refer to Sec. III C 3 and use Eqs. (17) to express the constraints on r, s, r', and s' in this setup. These constraints are imposed for the strategy pair $x^\star=\alpha/(\alpha+\beta)=y^\star$, which is to be a NE. Use Eqs. (23) and the normalization (20) to translate these constraints in terms of the EPR probabilities p_i: $\alpha(p_2+p_4)=\beta(p_5+p_7)$ and $\alpha(p_3+p_4)=\beta(p_9+p_{10})$. Adding and subtracting the second equation from the first one gives $\alpha(p_2+p_3+2p_4)=\beta(p_5+p_7+p_9+p_{10})$ and $\alpha(p_2-p_3)=\beta(p_5+p_7-p_9-p_{10})$ which can be reexpressed, using Eqs. (27), in terms of the probabilities in set μ, defined in (26), to obtain $(\alpha/\beta-1)(1-p_1+p_4)=p_5-p_8+p_9-p_{12}$ and $(1+\beta/\alpha)(-p_{14}+p_{15})=p_5-p_8-p_9+p_{12}$. Using these equations two probabilities can be eliminated from the inequalities (30) and (31). We select (arbitrarily) these to be p_{12} and p_{15} and express them in terms of other probabilities in the set μ, i.e.,

$$p_{12}=p_5-p_8+p_9-(\alpha/\beta-1)(1-p_1+p_4),$$

$$p_{15}=p_{14}+\frac{2(p_5-p_8)-(\alpha/\beta-1)(1-p_1+p_4)}{1+\beta/\alpha}. \quad (50)$$

Notice that for the chicken game, defined in (15), the definition of $\Delta_{1,2}$ in Sec. III C 1 gives $\Delta_1=\beta$ and $\Delta_2=-\alpha$. Now eliminate p_{12} and p_{15} from the inequalities (30) and (31) using Eqs. (50) and substitute for $\Delta_{1,2}$. The inequalities (30) and (31) then read

$$\Pi_A(x^\star,y^\star)-\Pi_A(x,y^\star)=\alpha(x^\star-x)[-y^\star\{(p_1-2p_5-p_9-p_{14}+p_8)+(1-p_1+p_4)\alpha/\beta+(p_1-p_5-p_9-p_{14})\beta/\alpha\}+\{(1-p_5-p_{14})-(p_5+p_{14})\beta/\alpha\}], \quad (51)$$

$$\Pi_B(x^\star,y^\star)-\Pi_B(x^\star,y)=\alpha(y^\star-y)[-x^\star\{(p_1-2p_5-p_9-p_{14}+p_8)+(1-p_1+p_4)\alpha/\beta+(p_1-p_5-p_9-p_{14})\beta/\alpha\}+\{(p_1-p_4-2p_5+2p_8-p_9-p_{14})+(1-p_1+p_4)\alpha/\beta-(p_9+p_{14})\beta/\alpha\}]. \quad (52)$$

The inequalities (51) and (52) ensure that for factorizable probabilities the classical NE $x^\star=\alpha/(\alpha+\beta)=y^\star$ comes out as the outcome of the game. What is the fate of this equilibrium when probabilities are not factorizable? To address this question we consider a special case when $\alpha=\beta$, for which $x^\star=1/2=y^\star$ is the classical mixed-strategy outcome of the game. To obtain this outcome within the four-coin setup the constraints on r, s, r', and s' are $r+s=1=r'+s'$. The inequalities (51) and (52) reduce to

$$\Pi_A(x^\star,y^\star)-\Pi_A(x,y^\star)=\alpha(x^\star-x)\{-y^\star(1+p_1+p_4-3p_5+p_8-2p_9-2p_{14})+(1-2p_5-2p_{14})\}, \quad (53)$$

$$\Pi_B(x^\star,y^\star)-\Pi_B(x^\star,y)=\alpha(y^\star-y)\{-x^\star(1+p_1+p_4-3p_5+p_8-2p_9-2p_{14})+(1-2p_5+2p_8-2p_9-2p_{14})\}. \quad (54)$$

Notice that the inequalities (45) do not allow either of the strategy pairs $(x^\star,y^\star)=(1,0)$ or $(x^\star,y^\star)=(0,1)$ to be NE. Like it has been the case with the quantum SH, we now ask which of these nine possible NE will emerge when probabilities become nonfactorizable. To answer this we refer to the set (48) of probabilities and assign the value $(2+\sqrt{2})/8$ to each of the probabilities p_1, p_4, p_5, and p_8, p_9, p_{14}. Using Eqs. (50) the assumption that $\alpha=\beta$ then also assigns the same value, i.e., $(2+\sqrt{2})/8$, to both p_{12} and p_{15}. The inequalities (53) and (54) are

$$\Pi_A(x^\star,y^\star)-\Pi_A(x,y^\star)=(\alpha/\sqrt{2})(x^\star-x)(y^\star-1)\geq 0,$$

$$\Pi_B(x^\star,y^\star)-\Pi_B(x^\star,y)=(\alpha/\sqrt{2})(y^\star-y)(x^\star-1)\geq 0, \quad (55)$$

with the result that the strategy pairs $(x^\star,y^\star)=(1,1),(0,0)$ emerge as the new equilibria,[3] so that, in this case, the set (48) of nonfactorizable probabilities indeed leads to the new equilibria of the game. Using (8) one finds that at the equilibrium (1, 1) both players get $\alpha(2-\sqrt{2})/4$ while at (0,0) both players get $\alpha(2+\sqrt{2})/4$ as their payoffs.

Note that, when reexpressed in terms of the EPR probabilities p_i using Eqs. (23), the constraints $r+s=1=r'+s'$ can be written as

$$p_1+p_2+p_9+p_{10}=1=p_1+p_3+p_5+p_7, \quad (56)$$

which, of course, continue to hold for the set (48) when the probabilities p_i are allowed to be nonfactorizable.

Similarly, referring to the second set (49), we assign the value $(2-\sqrt{2})/8$ to the probabilities p_1, p_4, p_5, p_8, p_9, and p_{14} that appear in (53) and (54). Equations (50), with the assumption that $\alpha=\beta$, then assign the value of $(2-\sqrt{2})/8$ to

[3]Referring to (16) we recall that when $\alpha=\beta$ there are three equilibria, i.e., (1,0), (0,1), and (1/2, 1/2), in the classical chicken game, at which they get rewarded by $(\alpha,\beta),(\beta,\alpha)$ and $(\alpha\beta/(\alpha+\beta),\alpha\beta/(\alpha+\beta))$, respectively. Here the first and second entries refer to Alice's and Bob's rewards, respectively.

both p_{12} and p_{15} and the inequalities (53) and (54) read

$$\Pi_A(x^\star,y^\star) - \Pi_A(x,y^\star) = -(\alpha/\sqrt{2})(x^\star - x)(y^\star - 1) \geq 0,$$
$$\Pi_B(x^\star,y^\star) - \Pi_B(x^\star,y) = -(\alpha/\sqrt{2})(y^\star - y)(x^\star - 1) \geq 0, \quad (57)$$

which are the same as the ones given in (55), apart from extra negative signs. This results in three strategy pairs $(x^\star,y^\star)=(1,1),(1,0),(0,1)$ to come out as the equilibria. Once again, using (8) one finds that at all of these three equilibria both players get equally rewarded by the amount $\alpha(2+\sqrt{2})/4$. Hence, while referring to (16), we find that in this special case when $\alpha=\beta$ the set (49) of probabilities leads to new equilibria of the game. Notice that, as is the case with the set (48) of probabilities, the constraints (56) continue to hold also for the set (49) when p_i are allowed to be nonfactorizable.

VI. DISCUSSION

In the typical quantization procedure [2,5] of a two-player game, two quantum bits (qubits) are in a quantum-correlated (entangled) state and are given to the players Alice and Bob. The players' strategies consist of performing unitary actions on their respective qubits. The classical game remains a subset of the quantum game in that both players can play quantum strategies that correspond to the strategies available classically.

As more choices are allowed to the players, which now also include superpositions of their classical moves, it gives ground to the argument[4] that van Enk and Pike [27] have put forward. The setup proposed in this paper uses EPR experiments to play a two-player game and a quantum game is associated with a classical game such that it becomes hard to construct an Enk-Pike-type argument as both the payoff relations and the players' sets of strategies remain identical [28] in the classical and associated quantum games.

In the present setup it is nonfactorizability—responsible for the violation of Bell's inequality in EPR experiments—that gives rise to the new solutions in quantum games. When players play a game using a physical system for which joint probabilities are factorizable, the classical game always results. In other words, the role of nonfactorizable probabilities is sought in the game-theoretic solution concept of a NE, when the physical realization for these probabilities is provided by the EPR experiments. This analysis introduces a viewpoint in the area of quantum games in which nonfactorizability gets translated into the language of game theory.

The argument put forward in this paper can be described as follows. First, players' payoffs are reexpressed in the form $\Pi_{A,B}(p_i,x,y,\mathcal{A},\mathcal{B})$ where p_i are the 16 joint probabilities; x,y are players' strategies, and $\mathcal{A},\mathcal{B}$ are players' payoff matrices defined in (3). Second, Nash inequalities are used to impose constraints on p_i that ensure that with factorizable p_i the game has a classical outcome and the resulting payoffs can be interpreted in terms of a classical mixed-strategy game. It is achieved by playing the game in the four-coin setup and using Nash inequalities to obtain constraints on the coin probabilities r, s, r', and s' which reproduce the outcome of the classical mixed-strategy game. Using (23), which results from factorizability, these constraints on r, s, r', and s' are then translated in terms of constraints on p_i. Third, while referring to the EPR setup, p_i are allowed to be nonfactorizable when the constraints on p_i continue to hold. Fourth, and last, it is observed how nonfactorizability leads to the emergence of new solutions of the game.

Note that for a game different sets of constraints are defined depending on which NE is to be the solution of the game. For example, for three NE in chicken we require three different sets of constraints on r, s, r', and s'. Considering one of these three sets at a time, we repeat the four steps stated above. The same procedure is then repeated for other sets of constraints corresponding to other NE.

That is, in this setup not all solutions of a game are reexpressed in terms of a single set of constraints on r, s, r', and s'. Instead, a separate set of constraints is found for each NE. It seems that the four-coin setup is the minimal arrangement that allows one to introduce, in a smooth way, the EPR probabilities into a gamelike setting. We suggest that with increasing the number of coins, shared by each of the two players, all the NE of a game can be translated to a single constraint on the underlying coin probabilities which are subsequently translated in terms of p_i. This will then allow one to see the role of nonfactorizability in solution of a game from a single set of constraints. However, this will be obtained at a price: First, more coins will be involved, resulting in more mathematical complexity; second, for more coins, the player's strategy will need to be redefined such that it permits the incorporation of EPR probabilities.

Note that the usual approach uses entangled states to construct quantum games and this paper uses nonfactorizability to the same end. Mathematically, nonfactorizability comes out to be a stronger condition than the condition that translates entanglement into constraints on joint probabilities. That is, a nonfactorizable set of probabilities always corresponds to some entangled state, but an entangled state can produce a factorizable set of probabilities. For example, in the case of a singlet state the outcomes of two measurements violate Bell's inequality only along certain directions, and not along other directions. In other words, in a quantum game exploiting entangled states, the joint probabilities may still be factorizable, but for a quantum game, resulting from nonfactorizable probabilities, Bell's inequality is bound to be violated. Nonfactorizability being a stronger condition may well be suggested as the reason why it cannot be helpful to escape from the classical outcome in PD.

The proposed setup demonstrates how nonfactorizability can change the outcome of a game. We suggest an extension [29] of this setup to analyze multiplayer quantum games [3] where players share physical systems for which joint probabilities cannot be factorized.

[4]The argument of Enk and Pike [27] can be described as follows. The extended set of players' moves allows us to construct an extended payoff matrix that includes extra available moves. Enk and Pike interpret this by saying that the "essence" of a quantized game can be captured by a different classical game and it is the new game that is constructed and solved and *not* the original classical game.

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