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A Survey in Mean Value Theorems

David A. Neuser
Utah State University

A SURVEY IN MEAN VALUE THEOREMS

by

David A. Neuser

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I sincerely thank Professor Stanley G. Wayment for his assistance, encouragement, and confidence.

David A. Neuser

To "Twig"

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ABSTRACT

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by

David A. Neuser, Master of Science

Utah State University, 1970

Major Professor: Dr. Stanley G. Wayment
Department: Mathematics

A variety of new mean value theorems are presented along with interesting proofs and generalizations of the standard theorems.

Three proofs are given for the ordinary Mean Value Theorem for derivatives, the third of which is interesting in that it is independant of Rolle's Theorem. The Second Mean Value Theorem for derivatives is generalized, with the use of determinants, to three functions and also generalized in terms of n-th order derivatives.

Observing that under certain conditions the tangent line to the curve of a differentiable function passes through the initial point, we find a new type of mean value theorem for derivatives. This theorem is extended to two functions and later in the paper an integral analog is given together with integral mean value theorems.

Many new mean value theorems are presented in their respective settings including theorems for the total variation of a function, the arc length of the graph of a function, and for vector-valued functions. A mean value theorem in the complex plane is given in which the difference quotient is equal to a linear combination of the values of the derivative. Using a regular derivative, the ordinary Mean Value Theorem for derivatives is extended into $R^n, n>1$.

(50 pages)

CHAPTER I

INTRODUCTION

Early in the study of calculus we learn of the ordinary mean value theorems for derivatives and integrals, and of their generalizations. However, there are many other less well known theorems which can be classified under the general heading of mean value theorems. In this paper we will present many of these theorems together with interesting proofs and generalizations of the standard theorems.

In Chapter II we restrict ourselves to mean value theorems for derivatives of real-valued functions. We present three proofs of the ordinary Mean Value Theorem: the first being the standard proof using Rolle's Theorem, the second using an area function, determinants, and Rolle's Theorem, and the third using a sliding interval technique by R. J. Easton and S. G. Wayment [2] which is, interestingly enough, independent of Rolle's Theorem. This same technique is then used to prove a theorem by Darboux [11] showing that the derivative possesses the intermediate value property. We will at times refer to this property of the derivative as the Darboux property.

The Second Mean Value Theorem for derivatives, often referred to as the Generalized Mean Value Theorem, is presented in a linear form, thus avoiding any assumptions about non-vanishing terms [1]. Devinatz [1] then generalizes the theorem to three functions with the use of determinants. Another generalization is given by D. H. Trahan [14] in terms of n-th order derivatives.

An new mean value theorem for derivatives is given by T. M. Flett [4] in which he observes that for a differentiable function f defined on [a,b], there is a point c in (a,b) at which $(c - a)f'(c) = f(c) - f(a)$, provided $f'(a) = f'(b)$. D. H. Trahan [13] generalizes Flett's results by relaxing the condition that $f'(a) = f'(b)$, and then follows this with an extension of the theorem to two functions.

In Chapter III we consider integral mean value theorems for real-valued functions. The ordinary Integral Mean Value Theorem is proved under a stronger hypothesis than is necessary. As a result, the theorem has an immediate generalization in which the requirement of continuity is relaxed. K. S. Miller [7] has extended the Integral Mean Value Theorem in what he refers to as the "First Mean Value Theorem" and the "Second Mean Value Theorem." We follow these theorems with two more general theorems, the proofs of which are relatively less involved.

Recently S. G. Wayment [16] submitted an integral analog to Flett's Theorem in which he observed that for a continuous function f defined on [a,b], there is a point w in (a,b) at which $(w - a)f(w) = \int_a^w f(x)dx$, provided $f(a) = f(b)$. We include his proof along with another proof for comparison. The additional proof is interesting in that it uses Flett's Theorem. Wayment's proof could be generalized if the sum of a continuous function and a function with the intermediate value property was in turn a function possessing the intermediate value property. However, we provide a counterexample, the construction of which was suggested by J. W. Cannon.

In Chapter IV we look at a variety of different mean value theorems in their respective settings. If f is an absolutely continuous function defined on [a,b] which is differentiable on (a,b), then S. G. Wayment has shown [17], with the use of the Lebesgue integral, that there exists mean value theorems for the total variation of f over [a,] and the arc length of the graph of f on [a,]. The proofs would also follow using the Riemann integral if, under the given hypotheses, the absolute value of the derivative was necessarily Riemann integrable. However, E. W. Hobson [5] provides us with a counterexample. The theorems are then generalized by relaxing the condition that f be absolutely continuous.

For a real-valued function f, the total variation of f and the arc length of the graph of f are quite different. However, for vector-valued functions it is customary to define the arc length of the graph of the function to be the total variation [12]. If we think of the real-valued function f rather as the vector-valued function given by $f:x \to (x,f(x))$, we find that its total variation and the arc length of its graph are equivalent. With the above definition in mind, similar proofs would follow if the norm of the derivative had the intermediate value property. However, a counterexample is given. We also provide a counterexample showing that the original theorems cannot be extended to $R^n, n \geq 2$.

If we consider the differentiable vector-valued function $v(t) = (t,f(t))$ defined on the unit interval [a,b], we find an interesting vector analog to the ordinary Mean Value Theorem for derivatives in that there is a point p in (a,b) at which $v'(p)$ has the same direction and magnitude as the vector $(b - a, f(b) - f(a))$. Under certain conditions we find similar results with the more general function $v(t) = (x(t),y(t))$. However, we find it easy to construct a differentiable vector-valued function v from [a,b] into R^3 whose derivative is different in magnitude and direction from $[v(b) - v(a)]/(b - a)$ at each point in (a,b).

For a holomorphic complex-valued function f defined on a connected open subset G of the complex plane where $z_1 \varepsilon G$, $z_2 \varepsilon G$, and A is an arc in G connecting z_1 and z_2, there may not exist a point $z \varepsilon A$ at which $(z_2 - z_1)f'(z) = f(z_2) - f(z_1)$. However, McLeod [6] gives us a mean value theorem in which the difference quotient is equal to a linear combination of the values of the derivatives.

The main problem in establishing a mean value theorem for functions defined on $R^n, n>1$, is that there are many ways of defining the derivative. Using a regular derivative [9], R. J. Easton and S. G. Wayment [3] give a mean value theorem with the additional hypothesis that the function be absolutely continuous with respect to Lebesgue measure. For notational purposes, the proof is given only for the case n = 2 and uses a sliding "interval" technique similar to that used in the third proof of the ordinary Mean Value Theorem for derivatives. We remark that L. Misik [8] gives the same theorem without the added condition of absolute continuity.

CHAPTER II

MEAN VALUE THEOREMS FOR DERIVATIVES

For completeness we begin with some elementary results.

Let D be the set of all real-valued functions f which are defined and continuous on [a,b] and such that f' exists at each point in (a,b).

<u>Theorem 2.1 (Rolle's Theorem)</u>

If $f \varepsilon D$ and $f(a) = f(b)$, then there exists a point $c \varepsilon (a,b)$ such that $f'(c) = 0$.

<u>Proof.</u> Since $f \varepsilon D$, there exists points m and M in [a,] such that $f(m) \leq f(x) \leq f(M)$ for each $x \varepsilon [a,b]$. If $f(m) = f(M)$, then f is constant on [a,b] implying that $f'(x) = 0$ for each $x \varepsilon (a,b)$. If $f(m) \neq f(M)$, assume $f(a) < f(M)$ and hence $M \varepsilon (a,b)$. Since $f \varepsilon D$, $f'(M)$ exists and

$$f'_+(M) = \lim_{h \to 0^+} (1/h)\,[f(m + h) - f(M)] = \lim_{h \to 0^-} (1/h)\,[f(M + h) - f(M)] = f'_-(M).$$

But $f'_+(M) \leq 0$ and $f'_-(M) \geq 0$ so it follows that $f'(M) = 0$.
The case $f(a) > f(m)$ is handled similarly.

The following theorem is a statement of the ordinary Mean Value Theorem for derivatives. We follow it with three proofs, the first two of which depend on Rolle's Theorem.

<u>Theorem 2.2 (Mean Value Theorem)</u>

If $f \varepsilon D$, then there exists a point $c \varepsilon (a,b)$ such that $(b - a)f'(c) = f(b) - f(a)$.

Proof 1. Let $g(x)$ be the equation of the straight line joining the points $(a,f(a))$ and $(b,f(b))$, that is

$$g(x) = [f(b) - f(a)]\,[(x - a)/(b - a)] + f(a).$$

Define $d(x) = f(x) - g(x)$. Since $f \varepsilon D$ and $g \varepsilon D$ it follows that $d \varepsilon D$. Now $d(a) = 0 = d(b)$ so applying Rolle's Theorem, there exists a point $c \varepsilon (a,b)$ such that $d'(c) = 0$. Thus

$$d'(c) = f'(c) - g'(c) = f'(c) - [f(b) - f(a)]/(b - a) = 0$$

and the result follows.

We note in the above proof that $d(x)$ attains either a relative maximum or a minimum value at the point $x = c$. Let A, X, and B be the points $(a,f(a))$, $(x,f(x))$, and $(b,f(b))$, respectively. Let $\overline{AB}$ denote the line segment joining A and B and L_x denote the line segment joining X and $(x,g(x))$. Since the acute angle formed by L_x and $\overline{AB}$ is constant for each $x \varepsilon (a,b)$, it follows that the perpendicular distance $h(x)$ from X to $\overline{AB}$ is in direct proportion to $d(x)$. Thus by maximizing (or minimizing) $d(x)$, we maximize (or minimize) $h(x)$ which results in maximizing (or minimizing) the area of the triangle having verticies at A, X, and B. This gives us a slightly similar but interesting way of proving the Mean Value Theorem.

Proof 2. Using the notation in the above paragraph, we find that the area $T(x)$ of the triangle having verticies at A, X, and B is given by $T(x) = (1/2)|uXv|$ where u and v are the vectors $(x - a, f(x) - f(a), 0)$ and $(b - a, f(b) - f(a), 0)$, respectively. Now

$$uXv = \begin{vmatrix} i & j & k \\ x - a & f(x) - f(a) & 0 \\ b - a & f(b) - f(a) & 0 \end{vmatrix} = k[(a - b)f(x) - (x - b)f(a) + (x - a)f(b)]$$

where k is the unit vector (0,0,1). Since $T(x) \geq 0$, we now define a somewhat similar but less restrictive area function A(x) as

$$A(x) = (1/2)|k|[(a - b)f(x) - (x - b)f(a) + (x - a)f(b)].$$

If we let

$$F(x) = \begin{vmatrix} f(x) & x & 1 \\ f(a) & a & 1 \\ f(b) & b & 1 \end{vmatrix}$$

then $A(x) = (1/2)F(x)$. Since $A \varepsilon D$ and $A(a) = 0 = A(b)$, it follows by Rolle's Theorem that there exists a point $c \varepsilon (a,b)$ such that $A'(c) = 0$. Thus

$$A'(c) = (1/2) \begin{vmatrix} f'(c) & 1 & 0 \\ f(a) & a & 1 \\ f(b) & b & 1 \end{vmatrix} = (1/2) \{(a - b)f'(c) - [f(a) - f(b)]\} = 0$$

and the result follows.

In most standard calculus text, the proof of the ordinary Mean Value Theorem is dependent on Rolle's Theorem. The third proof of Theorem 2.2 is dependent on the nested interval theorem rather than on Rolle's Theorem [2]. We first consider the following lemmas.

Lemma 2.1

Let a,b,c, and d be real numbers with $d>0$ and $b>0$. If $a/b \leq c/d$, then $a/b \leq (a + c)/(b + d) \leq c/d$.

Proof. Multiplying both sides of our given equality by $bd>0$, we obtain $ad \leq bc$. We now add cd to both sides and factor to obtain

$(a + c)d \leq (b + d)c$ and hence $(a + c)/(b + d) \leq c/d$. Similarly, if we add ab to both sides of the inequality $ad \leq bc$ and factor we obtain $a(b + d) \leq b(a + c)$ which in turn yields $a/b \leq (a + c)/(b + d)$. Therefore, $a/b \leq (a + c)/(b + d) \leq c/d$.

Lemma 2.2

If f' exists at each point of the open set G and $x \varepsilon G$, then $f'(x)$ exists if and only if

$$L = \lim_{\substack{a \to x^- \\ b \to x^+}} \frac{f(b) - f(a)}{b - a} \text{ exists}$$

Proof. If L exists and $b \equiv x$, then $L = f'_-(x)$. Similarly, if $a \equiv x$, then $L = f'_+(x)$ and hence $f'(x)$ exists.

If $f'(x)$ exists, then $f'(x) = f'_+(x) = f'_-(x)$. Now for a fixed a and b, we can assume without loss of generality that

$$\frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(x)}{b - a}.$$

By lemma 2.1 we have that

$$\frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(a)}{b - a} \leq \frac{f(b) - f(x)}{b - x}$$

and hence

$$f'_-(x) = \lim_{a \to x^-} \frac{f(x) - f(a)}{x - a} \leq L \leq \lim_{b \to x^+} \frac{f(b) - f(x)}{b - x} = f'_+(x).$$

Therefore $L = f'(x)$ and L exists.

We now establish some notation for use in the following lemma and in the third proof of Theorem 2.2. If $I_1 = [a,b]$, then let $F(I_1) = f(b) - f(a)$ and $\left|I_1\right| = b - a$. Observe that for $a \leq x \leq b$, if $I' = [a,x]$ and $I'' = [x,b]$, then $F(I_1) = F(I') + F(I'')$ and $|I_1| = |I'| + |I''|$. Also, let $h = |I_1|$.

Lemma 2.3

If f' exists at each point of the open set G and $I_1 = [a,b] \subset G$, then there exists a point $c \in I_1$, such that $f'(c) = F(I_1)/|I_1|$.

Proof. Divide I_1 into two equal intervals, namely $I_{11} = [a, a + h/2]$ and $I_{12} = [a + h/2, b]$. Since $F(I_1) = F(I_{11}) + F(I_{12})$, we can assume without loss of generality that $F(I_{11}) \le (1/2)F(I_1) \le F(I_{12})$. Define the auxiliary function

$$g(t) = f(a + h/2 + t) - f(a + t)$$

which is continuous on $[0, h/2]$. Since $g(0) = F(I_{11}) \le (1/2)F(I_1) \le F(I_{12}) = g(h/2)$, it follows from the intermediate value property for a continuous function that there exists a point $t_0 \in [0,h/2]$ at which $g(t_0) = (1/2)F(I_1)$. This value of t_0 determines the interval $I_2 = [a + t_0, a + h/2 + t_0] \subset I_1$ where $F(I_2) = (1/2)F(I_1)$ and $|I_2| = (1/2)|I_1|$. Thus

$$\frac{F(I_1)}{|I_1|} = \frac{F(I_2)}{|I_2|}.$$

We proceed inductively to obtain a nested sequence of closed intervals $\{I_i\}$ such that

$$F(I_{i+1}) = \frac{F(I_i)}{2} \quad \text{and} \quad |I_{i+1}| = \frac{|I_i|}{2}.$$

If we let $\bigcap_{i=1}^{\infty} I_i = \{c\}$, then it follows from lemma 2.2 that

$$f'(c) = \lim_{i\to\infty} \frac{F(I_i)}{|I_i|} = \frac{F(I_1)}{|I_1|}$$

We now turn to the third proof of Theorem 2.2.

Proof 3. In view of lemma 2.3, it will suffice to show that there exists a closed interval $I_0 \subset (a,b)$ with the property that $F(I_0)/|I_0| = F(I_1)/|I_1|$.

Divide I_1 into two equal intervals I_{11} and I_{12}. If $F(I_{11})<(1/2)F(I_1)<F(I_{12})$, then the procedure in the proof of lemma 2.3 yields an interval $I_2 = I_0 \subset (a,b)$ with the desired property. If $F(I_{12})<(1/2)F(I_1)<F(I_{11})$, define the auxiliary function g as

$$g(t) = f(b - t) - f(b - h/2 - t) \text{ where } t\varepsilon[0,h/2].$$

Then following the procedure described in the proof of lemma 2.3, we again obtain an interval $I_0 \subset (a,b)$ with the desired property. If $F(I_{11}) = (1/2)F(I_1) = F(I_{12})$, let $I_{11} = I_2$ and divide I_2 into two equal intervals I_{21} and I_{22}. If $F(I_{21})<(1/2)F(I_2)$ or $F(I_{21})>(1/2)F(I_2)$, then the procedure in lemma 2.3 gives an interval $I_3 = I_0 \subset (a,b)$ with the desired property. If $F(I_{21}) = (1/2)F(I_2) = F(I_{22})$, choose $I_{22} = I_0 \subset (a,b)$. Therefore, from the above arguments and lemma 2.3, there exists a point $c\varepsilon I_0 \subset (a,b)$ such that $(b - a)f'(c) = f(b) - f(a)$.

A function f defined on [a,b] is said to have the intermediate value property provided the closed interval from f(x) to f(y) is contained in the image of the closed interval from x to y for each x and y in [a,b]. In the following theorem, Darboux [11] has shown that if f' exists on [a,b], then f' has the intermediate value property. The proof will use a sliding interval technique [2] similar to that used in the third proof of Theorem 2.2.

Theorem 2.3 (Theorem of Darboux)

If f' exists on $[a,b]$, then f' has the intermediate value property.

Proof. Let $[x,y]$ be an arbitrary closed subinterval of $[a,b]$. Since the result is obvious when $f'(x) = f'(y)$, we will assume without loss of generality that $f'(x)<f'(y)$. For any positive $\varepsilon <[f'(y) - f'(x)]/2$, there exists an h such that $0<h<y - x$ and

$$\left|[f(x + h) - f(x)]/h - f'(x)\right|< \varepsilon \text{ and } \left|[f(y) - f(y - h)]/h - f'(y)\right|< \varepsilon.$$

If we define $F(t) = [f(x + h + t) - f(x + t)]/h$, we see that F is continuous on $[0,y - x - h]$ and thus has the intermediate value property. By the way ε was chosen, $F(0)<f'(x) + \varepsilon<f'(y) - \varepsilon<F(y - x - h)$ and hence $[f'(x) + \varepsilon,f'(y) - \varepsilon]\subset F([0,y - x - h])$. Now for each $t\varepsilon[0,y - x - h]$ we can apply the Mean Value Theorem to f on $[x + t,x + h + t]$ and obtain a $t'\varepsilon(x + t,x + h + t)$ such that $f'(t') = F(t)$ and hence $F([0,y - x - h]) \subset f'([x,y])$. Since ε can be made arbitrarily small, it follows that $(f'(x),f'(y))\subset f'([x,y])$ and thus $[f'(x),f'(y)]\subset f'([x,y])$.

Theorem 2.2 is a special case of a more general theorem. In contrast to the standard form of the theorem, the Second Mean Value Theorem (or Generalized Mean Value Theorem) is given here without any assumptions about non-vanishing terms [1].

Theorem 2.4 (Second Mean Value Theorem)

If $f\varepsilon D$ and $g\varepsilon D$, then there exists a point $c\varepsilon(a,b)$ such that

$$g'(c) [f(b) - f(a)] = f'(c) [g(b) - g(a)].$$

Proof. Define

$$F(x) = [g(b) - g(a)]\,[f(a) - f(x)] + [g(x) - g(a)]\,[f(b) - f(a)].$$

Since $f \varepsilon D$ and $g \varepsilon D$, it follows that $F \varepsilon D$ also. Now $F(a) = 0 = F(b)$ so applying Rolle's Theorem, there exists a point $c \varepsilon (a,b)$ such that $F'(c) = 0$. Thus

$$F'(c) = g'(c)\,[f(b) - f(a)] - f'(c)\,[g(b) - g(a)] = 0$$

and the result follows.

We remark that Theorem 2.2 follows as a corollary if g is the identity function defined on [a,b].

Theorem 2.4 has the following generalization with the use of determinants [1].

Theorem 2.5

If $f \varepsilon D$, $g \varepsilon D$, and $h \varepsilon D$, then there exists a point $c \varepsilon (a,b)$ such that $g'(c)[f(b)h(a) - f(a)h(b)] = f'(c)[g(b)h(a) - g(a)h(b)] + h'(c)$ $[f(b)g(a) - f(a)g(b)]$.

Proof. Define

$$F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} \quad \text{where } x \varepsilon [a,b].$$

Since $f \varepsilon D$, $g \varepsilon D$, and $h \varepsilon D$, it follows that $F \varepsilon D$. Now $F(a) = 0 = F(b)$ so applying Rolle's Theorem, there exists a point $c \varepsilon (a,b)$ such that $F'(c) = 0$. Thus

$$F'(c) = \begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0$$

and the result follows

We note that Theorems 2.4 and 2.5 are extensions of the Mean Value Theorem in terms of the number of functions involved. We now turn our attention to extending the Second Mean Value Theorem in terms of n - th order derivatives [14]. It will be convenient in the following theorem to let the symbol n!! represent 1!2!3! · · · n!.

Theorem 2.6

If f and g are continuous on [a,a + nh], h>0, the n-th derivatives of f and g exist on (a,a + nh), and $c_k = (-1)^k \binom{n}{k}$, then there exists a point $c \varepsilon (a, a + nh)$ such that $f^{(n)}(c)[c_0 g(a) + \cdots + c_n g(a + nh)] = g^{(n)}(c)[c_0 f(a) + \cdots + c_n f(a + nh)]$.

Proof. Define

$$\phi(x) = \begin{vmatrix} 1 & x & x^2 & \cdots & x^{n-1} & g(x) & f(x) \\ 1 & a & a^2 & \cdots & a^{n-1} & g(a) & f(a) \\ 1 & a + h & (a + h)^2 & \cdots & (a + h)^{n-1} & g(a + h) & f(a + h) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a + nh & (a + nh)^2 & \cdots & (a + nh)^{n-1} & g(a + nh) & f(a + nh) \end{vmatrix}.$$

and note that $\phi(a + ih) = 0$ for each i = 0,1,2, · · ·,n. Since ϕ is continuous on each [a + (j - 1)h,a + jh], j = 1,2,3, · · · ,n, and differentiable on each (a + (j - 1)h,a + jh), it follows by Rolle's Theorem that in each (a + (j - 1)h,a + jh) there exists a point b_j such that $\phi'(b_j) = 0$. Now $\phi^{(2)}$ exists on each $[b_k, b_{k+1}]$, k = 1,2, · · ·,n - 1, and hence ϕ' is continuous on each (b_k, b_{k+1}). Again applying Rolle's Theorem we find that in each (b_k, b_{k+1}) there exists a point p_k such that $\phi^{(2)}(p_k) = 0$. Continuing in this way we arrive at a unique point $c\varepsilon(a, a + nh)$ such that $\phi^{(n)}(c) = 0$. That is,

$$\phi^{(n)}(c) = \begin{vmatrix} 0 & 0 & 0 & \cdots 0 & g^{(n)}(c) & f^{(n)}(c) \\ 1 & a & a^2 & \cdots a^{n-1} & g(a) & f(a) \\ 1 & a + h & (a + h)^2 & \cdots (a + h)^{n-1} & g(a + h) & f(a + h) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a + nh & (a + nh)^2 & \cdots (a + nh)^{n-1} & g(a + nh) & f(a + nh) \end{vmatrix} = 0.$$

The minor of $f^{(n)}(c)$ can be calculated by expanding down its last column, for the subsequent minors of $g(a + kh), 0\leq k\leq n$, are Vandermonde determinants. The minor of $f^{(n)}(c)$ is

$$[h^{n(n-1)/2}(n - 1)!!][c_o g(a) + \cdots c_n g(a + hn)].$$

Since the minor of $g^{(n)}(c)$ is similar, it follows that

$$f^{(n)}(c)\,[c_o g(a) + \cdots + c_n g(a + nh)] - g^{(n)}(c)[c_o f(a) + \cdots + c_n f(a + nh)]$$

equals zero.

We note that Theorem 2.4 follows as an immediate corollary if $n = 1$ and $h = b - a$.

Corollary 2.1

If f is continuous on $[a, a + nh]$, $h>0$, the n-th derivatives of f exists on $(a, a + nh)$, and $c_k = (-1)^k\binom{n}{k}$, then there exists a point $c\varepsilon(a, a + nh)$ such that $h^n f^{(n)}(c) = c_o f(a + nh) + \cdots + c_n f(a)$.

Proof. In Theorem 2.6, if we let $g(x) = x^n/n!$, then $c_o g(a) + \cdots + c_n g(a + nh) = [c_o a^n + c_1(a + h)^n + \cdots + c_n(a + nh)^n]/n!$.

The corollary follows from the last equation and the equations below:

$$
\begin{array}{lllll}
1 & c_0 + c_1 + & c_2 + & \cdots + c_n = 0 \\
2 & c_1 + & 2c_2 + & \cdots + nc_n = 0 \\
3 & c_1 + & 2^2c_2 + & \cdots + n^2c_n = 0 \\
\cdots & \cdots & \cdots & \cdots \\
n & c_1 + 2^{n-1} & c_2 + & \cdots + n^{n-1}c_n = 0 \\
(n+1) & c_1 + & 2^nc_2 + & \cdots + n^nc_n = (-1)^n n!
\end{array}
$$

The above equations can be generated in the following manner. The binomial expansion of $(1 - x)^n$ gives

i) $c_o + c_1x + c_2x^2 + \cdots + c_nx^n = (1 - x)^n$

which when evaluated at $x = 1$ yields equation 1. By taking the derivative of i) and multiplying through by x we obtain

ii) $c_1x + 2c_2x^2 + \cdots + nc_nx^n = - nx(1 - x)^{n - 1}$

which when evaluated at $x = 1$ yields equation 2. By taking the derivative of ii) and multiplying through by x we obtain

iii) $c_1x + 2^2c_2x^2 + \cdots + n^2c_nx^n = n(n - 1)x(1 - x)^{n - 2} - n(1 - x)^{n - 1}$

which when evaluated at $x = 1$ yields equation 3. Continuing in this way we obtain the above $n + 1$ equations.

T. M. Fleet [4] first observed that for a differentiable function f on [a,b] where $f'(a) = f'(b)$, that at some point in the interval, the tangent to the curve at that point passes through the initial point $(a,f(a))$. Thus we have the following new type of mean value theorem for derivatives.

Theorem 2.7 (Flett's (Theorem)

If f' exists on [a,b] and f'(a) = f'(b), then there exists a point $c\varepsilon(a,b)$ such that (c - a)f'(c) = f(c) - f(a).

Proof. Consider the function g defined by

$$g(x) = \begin{cases} \dfrac{f(x) - f(a)}{x - a} - f'(a), & a<x\leq b \\ 0, x = a \end{cases}$$

We note that g is continuous on [a,b] and differentiable on (a,b], and

$$g'(x) = -\frac{f(x) - f(a)}{(x - a)^2} + \frac{f'(x)}{x - a} \quad \text{for } a<x\leq b.$$

It will therefore be sufficient to prove that there exists some point $c\varepsilon(a,b)$ such that g'(c) = 0.

If g(b) = 0, the result follows immediately by Rolle's Theorem. Suppose then that g(b)>0, so that g'(b) = - g(b)/(b - a)<0. Thus there exists a point $x_1\varepsilon(a,b)$ such that $g(x_1)>g(b)$. Since g is continuous on $[a,x_1]$ and $g(a)<g(b)<g(x_1)$, there exists a point $x_2\varepsilon(a,x_1)$ such that $g(x_2) = g(b)$. Now applying Rolle's Theorem to the function g on $[x_2,b]$, there exists a point $c\varepsilon(x_2,b)$ such that g'(c) = 0. A similar argument applies if g(b)<0 and the proof is completed.

We note that in the cases g(b)>0 and g(b)<0, the existance of the point c is also guaranteed by Darboux's Theorem. For if we assume g(b)>0, then there exists a point $x_0\varepsilon(a,b)$ such that $g'(x_0)>0$, for if not then g'(x)<0 for each $x\varepsilon(a,b)$. But since g(a) = 0, this would imply that g(b)<0, a contradiction. Now $g'(x_0)>0>g'(b)$, so by Darboux's Theorem there exists a point $c\varepsilon(x_0,b)$ such that g'(c) = 0. A similar argument would apply if g(b)<0.

D. H. Trahan [13] has generalized Flett's result in the following two theorems. We first consider the following lemma.

Lemma 2.4

If f is continuous on [a,b] and f' exists on (a,b], and $f'(b)\cdot[f(b) - f(a)] \leq 0$, then there exists a point $c \in (a,b]$ such that $f'(c) = 0$.

Proof. If $f(a) = f(b)$, the result follows from Rolle's Theorem. If $f'(b) = 0$, then choose $c = b$. If $f'(b)[f(b) - f(a)] < 0$, then f assumes a maximum or minimum value at some point $c \in (a,b)$ and $f'(c) = 0$.

Theorem 2.8

If f' exists on [a,b] and

$$\left[f'(b) - \frac{f(b) - f(a)}{b - a}\right]\left[f'(a) - \frac{f(b) - f(a)}{b - a}\right] \geq 0,$$

then there exists a point $c \in (a,b]$ such that $(c - a)f'(c) = f(c) - f(a)$.

Proof. Define the auxiliary function

$$h(x) = \begin{cases} \dfrac{f(x) - f(a)}{x - a}, & a < x \leq b \\ f'(a), & x = a \end{cases}$$

We note that h is continuous on [a,b] and differentiable on (a,b], and

$$h'(x) = \frac{(x - a)f'(x) - [f(x) - f(a)]}{(x - a)^2}.$$

Since $h'(b)[h(b) - h(a)] = \left[\dfrac{(b - a)f'(b) - [f(b) - f(a)]}{(b - a)^2}\right]\left[\dfrac{f(b) - f(a)}{b - a} - f'(a)\right] = \dfrac{-1}{(b - a)}\left[f'(b) - \dfrac{f(b) - f(a)}{b - a}\right]\left[f'(a) - \dfrac{f(b) - f(a)}{b - a}\right] \leq 0$,

it follows from lemma 2.4 that there exists a point $c \in (a,b]$ such that $h'(c) = 0$ and hence $(c - a)f'(c) = f(c) - f(a)$.

We observe that Flett's Theorem follows as a corollary to Theorem 2.8. For if $(b - a)f'(b) = f(b) - f(a)$ we consider the function h as defined in the above proof and note that since $f'(a) = f'(b)$, then $h(a) = h(b)$. Applying Rolle's Theorem to h on $[a,b]$ we find a point $c \in (a,b)$ such that $h'(c) = 0$ and the result follows. On the other hand, if $(b - a)f'(b) \neq f(b) - f(a)$, then either i) $(b - a)f'(b) < f(b) - f(a)$ or ii) $(b - a)f'(b) > f(b) - f(a)$, and we note that $c \neq b$. If i) holds , then

$$\left[f'(b) - \frac{f(b) - f(a)}{b - a}\right]\left[f'(a) - \frac{f(b) - f(a)}{b - a}\right] \geq 0$$

and the result follows from Theorem 2.8. A similar argument applies if ii) holds.

The following corollary is a stronger statement of Flett's Theorem in that the condition $f'(a) = f'(b)$ is relaxed. The proof follows directly from Theorem 2.8 and will be deleted.

Corollary 2.2

If f' exists on $[a,b]$ and $f'(a)$ and $f'(b)$ are both less than or both greater than $[f(b) - f(a)]/(b - a)$, then there exists a point $c \in (a,b)$ such that $(c - a)f'(c) = f(c) - f(a)$.

Theorem 2.9

If f' and g' exist on $[a,b]$, $g'(a) \neq 0$, $g(x) \neq g(a)$ for all $x \in (a,b]$, and

$$\left[\frac{f'(a)}{g'(a)} - \frac{f(b) - f(a)}{g(b) - g(a)}\right]\Big[[g(b) - g(a)]\, f'(b) - [f(b) - f(a)]\, g'(b)\Big] \geq 0,$$

then there exists a point $c \in (a,b]$ such that $[g(c) - g(a)]\, f'(c) = [f(c) - f(a)]\, g'(c)$.

Proof. Define the auxiliary function

$$h(x) = \begin{cases} \dfrac{f(x) - f(a)}{g(x) - g(a)}, & a < x \leq b \\[2ex] \dfrac{f'(a)}{g'(a)}, & x = a \end{cases}$$

We note that h is continuous on [a,b] and h' exists on (a,b], and

$$h'(x) = \frac{[g(x) - g(a)]\, f'(x) - [f(x) - f(a)]\, g'(x)}{[g(x) - g(a)]^2}$$

Since $h'(b)[h(b) - h(a)] \leq 0$, it follows from lemma 2.4 that there exists a point $c \varepsilon (a,b]$ such that $h'(c) = 0$ and hence

$[g(c) - g(a)]\, f'(c) = [f(c) - f(a)]\, g'(c)$.

We note that Theorem 2.8 follows as an immediate corollary if $g(x) = x$ for each $x \varepsilon [a,b]$.

Corollary 2.3

If f' and g' exist on [a,b], $g'(a) \neq 0, g(x) \neq g(a)$ for each $x \varepsilon (a,b]$, $g'(b)[g(b) - g(a)]$ 0, and $f'(a)/g'(a) = f'(b)/g'(b)$, then there exists a point $c \varepsilon (a,b)$ such that $[g(c) - g(a)]\, f'(c) = [f(c) - f(a)]\, g'(c)$.

Proof. If

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(b)}{g'(b)},$$

define h as in the proof of Theorem 2.9. Since $f'(a)/g'(a) = f'(b)/g'(b)$, it follows that $h(a) = h(b)$. Thus by Rolle's Theorem, there exists a point $c \varepsilon (a,b)$ such that $h'(c) = 0$ and the conclusion follows. If the above equality does not hold, the conclusion follows from Theorem 2.9 and the fact that $g'(b)[g(b) - g(a)] > 0$.

CHAPTER III

MEAN VALUE THEOREMS FOR INTEGRALS

The integral mean value theorems of introductory calculus are usually proved under stronger hypotheses than are required. Continuity is often demanded when only a consequence of continuity is necessary.

Theorem 3.1 (Integral Mean Value Theorem)

If f is continuous on [a,b], then there exists a point $w \varepsilon (a,b)$ such that $(b - a)f(w) = \int_a^b f(x)dx$.

Proof. Consider the inequality

$$\inf f(x) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \sup f(x).$$

If equality holds on either side of this inequality, then f(x) = inf f(x) a.e. or f(x) = sup f(x) a.e. and thus there exists at least one point $w \varepsilon (a,b)$ with the desired property. If strict inequality holds on both sides of this inequality, then by the intermediate value property of a continuous function there exists a point $w \varepsilon (a,b)$ such that $(b - a)f(w) = \int_a^b f(x)dx$.

We note that continuity is required only to insure that f be integrable and have the intermediate value property. The function f defined by $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$ does not have a continuous derivative on [- 1,1] but the derivative does have the intermediate value property and is Riemann integrable. Hence the conditions of the ordinary Integral Mean Value Theorem are not met. We also note that there is a function F whose derivative exists at each point in [a,b] but F' is not Riemann integrable (see example 4.1). In view of these results, the Integral Mean Value Theorem has the following immediate generalization. The proof is similar to the proof of Theorem 3.1 and will be omitted.

Theorem 3.2

If f has the intermediate value property and is integrable (Riemann or Lebesgue), then there exists a point $w \varepsilon (a,b)$ such that $(b - a)f(w) = \int_a^b f(x)dx$.

It is interesting to note that there exists a Lebesgue integrable function which possesses the intermediate value property and is discontinuous at each point. Such a function is given in example 3.1.

We next present a generalization of a theorem due to K. S. Miller [7] and note that Theorem 3.1 follows as an immediate corollary if $g(x) \equiv 1$ for each $x \varepsilon [a,b]$.

Theorem 3.3

If f is continuous on [a,b] and g is Riemann integrable on [a,b] and either $g(x) \geq 0$ or $g(x) \leq 0$ for all $x \varepsilon [a,b]$, then there exists a point $w \varepsilon (a,b)$ such that $f(w)\int_a^b g(x)dx = \int_a^b f(x)g(x)dx$.

Proof. We will prove only the case $g(x) \geq 0$. Let $M = \sup f(x)$ and $m = \inf f(x)$, then

$$m\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq M \int_a^b g(x)dx.$$

If equality holds on either side of the above inequaltiy, then $f(x) = m$ a.e. or $f(x) = M$ a.e. which implies that there exists at least one point $w \varepsilon (a,b)$ such that $f(w)\int_a^b g(x)dx = \int_a^b f(x)g(x)dx$. If strict inequality holds on both sides of the above inequality, then it follows by the intermediate value property of a continuous function that there exists a point $w\varepsilon(a,b)$ such that $f(w)\int_a^b g(x)dx = \int_a^b f(x)g(x)dx$.

K. S. Miller [7] states the Second Mean Value Theorem with a stronger hypothesis than is necessary. We therefore follow his result with a more general theorem.

Theorem 3.4 (Second Mean Value Theorem)

If f is a continuous, monotone increasing function defined on [a,b], and $g(x)\geq 0$ is integrable on [a,b], then there exists a point $w\varepsilon[a,b]$ such that

$$\int_a^b f(x)g(x)dx = f(a)\int_a^w g(x)dx + f(b)\int_w^b g(x)dx.$$

Proof. By Theorem 3.4, there exists a point $p\varepsilon(a,b)$ such that $\int_a^b f(x)g(x)dx = f(p)\int_a^b g(x)dx$. Define $\phi(x) = f(x) - f(a)$ and note that ϕ is a continuous, nonnegative, monotone increasing function on [a,b]. Now

$$\phi(p)\int_a^b g(x)dx = f(p)\int_a^b g(x)dx - f(a)\int_a^b g(x)dx,$$

so

$$f(p)\int_a^b g(x)dx = \phi(p)\int_a^b g(x)dx + f(a)\int_a^b g(x)dx.$$

If we define $G(x) = \int_x^b g(x)dx$, then G is a continuous, nonnegative, monotone decreasing function on [a,b]. Since $G(b) = 0$,

$$\phi(p)\int_a^b g(x)dx = \phi(p)G(a) = \phi(p)G(a) + [\phi(b) - \phi(p)]G(b).$$

But $\phi(p)\geq 0$, $\phi(b) - \phi(p)\geq 0$, so there exists a number α with $G(b)\leq \alpha \leq G(a)$ such that

$$\phi(p)\int_a^b g(x)dx = \alpha\ \{\phi(p) + [\phi(b) - \phi(p)]\} = \alpha\phi(b).$$

By the intermediate value property of a continuous function, it follows that there exists some point $w\varepsilon[a,b]$ such that $G(w) = \alpha$ and thus

$$\phi(p)\int_a^b g(x)dx = G(w)\phi(b) = [f(b) - f(a)]\int_w^b g(x)dx.$$

Therefore,

$$\int_a^b f(x)g(x)dx = f(p)\int_a^b g(x)$$

$$= \phi(p)\int_a^b g(x)dx + f(a)\int_a^b g(x)dx$$

$$= [f(b)\int_w^b g(x)dx - f(a)\int_w^b g(w)dx] + [f(a)\int_a^w g(x)dx + f(a)\int_w^b g(x)dx]$$

$$= f(a)\int_a^w g(x)dx + f(b)\int_w^b g(x)dx$$

which completes the proof.

By relaxing the condition that f be a monotone increasing function on [a,b] we obtain the following more general theorem, the proof of which is similar to the proof of Theorem 3.4 and thus will be deleted.

Theorem 3.5

If f is continuous on [a,b] and $f(a)\leq f(x)\leq f(b)$ for each $x\varepsilon[a,b]$, and if $g(x)\geq 0$ is integrable on [a,b], then there exists a point $w\varepsilon[a,b]$ such that

$$\int_a^b f(x)g(x)dx = f(a)\int_a^w g(x)dx + f(b)\int_w^b g(x)dx.$$

We now turn our attention to an integral analog to Flett's result (Theorem 2.5). S. G. Wayment [16] first posed the following theorem and we include his proof (Proof 2) here along with an additional proof for comparison.

Theorem 3.6

If $f(t)$ is a continuous function defined on $[a,b]$ and $f(a) = f(b)$, then there exists a point $w\varepsilon(a,b)$ such that $(w - a)f(w) = \int_a^w f(t)dt$.

Proof 1. Let $F(x) = \int_a^x f(t)dt$. Since f is continuous, we have that F is differentiable and $F'(x) = f(x)$ for all $x\varepsilon[a,b]$. Now $F'(a) = f(a) = f(b) = F'(b)$, so applying Flett's result, there exists a point $w\varepsilon(a,b)$ such that $(w - a)F'(w) = F(w) - F(a)$. This reduces to $(w - a)f(w) = \int_a^w f(t)dt$, the desired result.

We remark that Proof 1 uses the full power of continuity to insure that $F'(x) = f(x)$ for all $x\varepsilon[a,b]$.

Proof 2. Since f is continuous, there exist points t_1 and t_2 in $[a,b]$ such that $f(t_1) \geq f(x)$ and $f(t_2) \leq f(x)$ for all $x\varepsilon[a,b]$. Now $(t_1 - a)f(t_1) \geq \int_a^{t_1} f(t)dt$ and $(t_2 - a)f(t_2) \leq \int_a^{t_2} f(t)dt$. Define $F(x) = \int_a^x f(t)dt - (x - a)f(x)$. Since f is continuous, we have that F is continuous also. Now $F(t_1) \leq 0$ and $F(t_2) \geq 0$ so by the intermediate value property of a continuous function, there exists a point $w\varepsilon(a,b)$ such that $F(w) = 0$ and hence $(w - a)f(w) = \int_a^w f(t)dt$.

Proof 2 can be generalized if F has the intermediate value property whenever f does. This, in turn, would be implied if the sum of a continuous function and a function with the intermediate value property were necessarily a function with the intermediate value property. The following example shows that such is not the case. The techniques used in the construction of this example are somewhat standard in topology and were suggested by J. W. Cannon.

Example 3.1. We first generate a sequence $\{C_i\}$ of Cantor sets in the following way. Construct C_1 on $[0,1]$ and let L_1 be the largest number for which there exists at least one open interval (a_1,b_1) in $\sim C_1$ with length L_1. Construct C_2 on the closed middle third of (a_1,b_1). Let L_2 be the largest number for which there exists at least one open interval (a_2,b_2) in $\sim(C_1 \cup C_2)$ with length $L_2 \leq L_1$. Construct C_3 on the closed middle third of (a_2,b_2). In general, construct C_n on the closed middle third of the open interval in $\sim(\bigcup_{i=1}^{n} C_i)$ with length L_n, where L_n is the largest number for which there exists at least one open interval in $\sim(\bigcup_{i=1}^{n} C_i)$ with length $L_n \leq L_{n-1}$.

For each i, let g_i be a continuous, monotone increasing function which maps C_i onto [0,1]. The construction of such functions can be found in [10]. Define.

$$g(x) = \begin{cases} g_i(x) & \text{if } x \varepsilon C_i \\ 0 & \text{if } x \varepsilon [0,1] - \bigcup_{i=1}^{\infty} C_i. \end{cases}$$

We note that g has the intermediate value property, for given any arbitrary open interval (a,b) contained in [0,1], there is an i such that $C_i \subset (a,b)$ and $g(C_i) = [0,1]$.

For a real valued function p(x) defined on [0,1], let $\|p\| = \sup |p(x)|$. We next define a sequence $\{f_i\}$ of continuous functions on [0,1] such that $\sum_{i=1}^{\infty} \|f_i\| < \infty$ and hence $f = \sum_{i=1}^{\infty} f_i$ is continuous on [0,1]. The $\{f_i\}$ will be constructed in such a way that $(f + g)(x) \neq 1/2$ for any value of x in [0,1]. Since g is continuous on C_1, there is a finite collection U_1 of open intervals whose closures are pairwise disjoint which covers C_1 and such that if x and y are in $O_{1i} \varepsilon U_1$ for some i, then $|g(x) - g(y)| < \varepsilon_1 < 1/3$. Let V_1 be the collection of those open intervals in U_1 which contain values of x satisfying $g(x) = 1/2$.

Define $f_1(x) = \varepsilon_1$ for the values of x covered by V_1, and let $f_1(x) = 0$ for the values of x which are covered by U_1 but not covered by V_1. Extend f_1 to be continuous on [0,1] with functional values between 0 and ε_1. If $x\varepsilon C_1$ and is covered by V_1, then $g(x)>1/2 - \varepsilon_1$ and hence $(f_1 + g)(x)>1/2$. Thus $f_1 + g$ is continuous on C_1 and $(f_1 + g)(x)\neq 1/2$ for any $x\varepsilon C_1$. Let δ_1 be the distance from 1/2 to the image of C_1 under $f_1 + g$, that is, $\delta_1 = \rho(1/2,[f_1 + g](C_1))$. We note that $\delta_1 \leq \varepsilon_1$ and choose ε_2 such that $0<\varepsilon_2<\delta_1/3$. Let $h_2 = f_1 + g$ on C_2. Since h_2 is continuous on C_2, there is a finite collection U_2 of open intervals whose closures are pairwise disjoint which covers C_2 and such that if x and y are in $O_{2i}\varepsilon U_2$ for some i, then $|h_2(x) - h_2(y)|<\varepsilon_2$. Let V_2 be the collection of those open intervals in U_2 which contain values of x satisfying $h_2(x) = 1/2$. Define $f_2(x) = \varepsilon_2$ for the values of x covered by V_2, and let $f_2(x) = 0$ for the values of x which are covered by U_2 but not covered by V_2. Extend f_2 to be continuous on [0,1] with functional values between 0 and ε_2. If $x\varepsilon C_2$ and is covered by V_2, then $h_2(x)>1/2 - \varepsilon_2$ and hence $(f_2 + h_2)(x)>1/2$. Thus $f_2 + h_2$ is continuous on C_2 and $(f_2 + h_2)(x)\neq 1/2$ for any $x\varepsilon C_2$. Let $\delta_2 = \rho(1/2, [f_2 + h_2](C_2))$. We note that $\delta_2\leq\varepsilon_2$ and choose ε_3 such that $0<\varepsilon_3<\delta_2/3$. Let $h_3 = f_2 + h_2$ on C_3. We proceed inductively to define $\{\varepsilon_i\}$, $\{\delta_i\}$, $\{f_i\}$, and $\{h_i\}$ such that $\delta_i = \rho(1/2,[f_i + h_i](C_i))$, noting that $\delta_i\leq\varepsilon_i$, and $0<\varepsilon_{i+1}<\delta_i/3$. If $f = \sum_{i=1}^{\infty} f_i$, then since $\rho(1/2,[f_j + h_j](C_j)) = \rho(1/2,[g + \sum_{i=1}^{\infty} f_i](C_j)) = \delta_i$ and

$\sum_{i=j+1}^{\infty} \|f_i\| < \sum_{i=1}^{\infty} \delta_j(1/3^i) = \delta_j/2$, it follows by the triangle inequality that $(f + g)(x)\neq 1/2$ for any x in C_j. If $x\varepsilon \sim\bigcup_{i=1}^{\infty} C_i$, then $f(x)\leq \sum_{i=1}^{\infty} \|f_i\| < \sum_{i=1}^{\infty} 1/3^i = 1/2$.

We remark that it is possible to construct a function f which has the intermediate value property, is Riemann integrable, and such that $\int_a^x f(x) - (x - a)f(x)$ does not have the intermediate value property.

CHAPTER IV

OTHER MEAN VALUE THEOREMS

For a real-valued function f defined on [a,b], let $L_a^b f$ represent the arc length of the graph of f on [a,b] and let $V_a^b f$ represent the total variation of f over [a,b]. Intuitively, $L_a^b f$ is the total distance a particle would travel along the graph of f from point (a,f(a)) to point (b,f(b)), whereas $V_a^b f$ is the total distance a particle's projected image onto the y-axis would travel as the particle itself moves along the graph of f from point (a,f(a)) to point (b,f(b)).

We note that the above relationship between $V_a^b f$ and $L_a^b f$ is the exception rather than the rule. For if $f:[a,b]\to R^n$ is a rectifiable curve, i.e., continuous and of bounded variation, it is customary to define the arc length of the graph of f to be $V_a^b f$ [12]. Thus for the real-valued function f defined on [a,b], we find that $V_a^b f = L_a^b f$ if we consider f as the mapping from [a,b] into R^2 given by $f:x\to(x,f(x))$.

We now turn our attention to establishing mean value theorems for $V_a^b f$ and $L_a^b f$ [15,17].

Let D be the set of all real-valued functions f which are continuous on [a,b] and such that f' exists on (a,b), and let A be the set of all real-valued functions f which are absolutely continuous on [a,b].

<u>Theorem 4.1</u>

If $f\varepsilon A\cap D$, then there exists a point $p\varepsilon(a,b)$ such that $V_a^b f = (b - a)|f'(p)|$.

Proof. Since f is absolutely continuous, it follows that $\int_a^b |f'| = V_a^b f$ where the integral is the Lebesgue integral. Now

$$(b - a)\inf_{t\varepsilon(a,b)}|f'(t)| \leq \int_a^b |f'| \leq (b - a)\sup_{t\varepsilon(a,b)}|f'(t)|.$$

If equality holds on either side of this inequality, then

$$|f'(t)| = \sup_{t\varepsilon(a,b)}|f'(t)| \text{ a.e. or } |f'(t)| = \inf_{t\varepsilon(a,b)}|f'(t)| \text{ a.e.}$$

Thus $V_a^b f = (b - a)|f'(t)|$ a.e. If strict inequality holds on both sides of this inequality, it follows from Darboux's Theorem (Theorem 2.3) that there exists a point $p\varepsilon(a,b)$ such that $V_a^b f = (b - a)|f'(p)|$.

Theorem 4.2

If $f\varepsilon A\cap D$, then there exists a point $p\varepsilon(a,b)$ such that $L_a^b f = (b - a)\sqrt{1 + |f'(p)|^2}$.

Proof. We can say that $L_a^b f = \int_a^b \sqrt{1 + |f'|^2}$ provided this Lebesgue integral exists [12]. Since

$$|\Delta y| \leq \sqrt{(\Delta x)^2 + (\Delta y)^2} \leq |\Delta x| + |\Delta y|,$$

it follows that

$$V_a^b f \leq \int_a^b \sqrt{1 + |f'|^2} \leq (b - a) + V_a^b f.$$

Thus for a continuous function we can conclude that $V_a^b f$ exists (that is, f is of bounded variation) if and only if $L_a^b f$ exists. Since $f\varepsilon A\cap D$, it follows that $L_a^b f$ exists. By the Darboux property, f' has the intermediate value property and hence $|f'|$, $|f'|^2$, $1 + |f'|^2$, and $\sqrt{1 + |f'|^2}$ do also. We note that in general the sum of a continuous function and a function with the Darboux property need not be a function with the Darboux property (see example 3.1). Now

$$(b - a)\inf_{t\varepsilon(a,b)} \sqrt{1 + |f'(t)|^2} \leq \int_a^b \sqrt{1 + |f'|^2} \leq (b - a) \sup_{t\varepsilon(a,b)} \sqrt{1 + |f'(t)|^2}.$$

If equality holds on either side of this inequality, then either

$$\sqrt{1 + |f'(t)|^2} = \inf_{t\varepsilon(a,b)} \sqrt{1 + |f'(t)|^2} \quad \text{a.e.}$$

or

$$\sqrt{1 + |f'(t)|^2} = \sup_{t\varepsilon(a,b)} \sqrt{1 + |f'(t)|^2} \quad \text{a.e.}$$

Thus we have that $L_a^b f = (b - a)\sqrt{1 + |f'(t)|^2}$ a.e. If strict inequality holds on both sides, then by the Darboux property for $\sqrt{1 + |f'|^2}$, there exists a point $p\varepsilon(a,b)$ such that

$$\int_a^b \sqrt{1 + |f'|^2} = (b - a)\sqrt{1 + |f'(p)|^2}.$$

Thus this integral exists and we have that $L_a^b f = (b - a)\sqrt{1 + |f'(p)|^2}$.

We remark that the proofs of Theorems 4.1 and 4.2 would follow using the Riemann integral provided $|f'|$ is Riemann integrable. Since $|f'|$ is bounded and if $|f'|$ is Riemann integrable, then $|f'|$ is Lebesgue integrable. However, example 4.1 provides a function $f\varepsilon A \cap D$ such that $|f'|$ is not Riemann integrable [5].

Example 4.1. Let G be a perfect non-dense set of points in the interval (a,b) and such that its measure is greater than zero. Let (α, β) be an arbitrary open interval in the complement of G and define the following function on (α, β):

$$\phi(x,\alpha) = (x - \alpha)^2 \sin \frac{1}{x - \alpha} .$$

Thus $\phi'(x,\alpha) = 2(x - \alpha)\sin \frac{1}{x - \alpha} - \cos \frac{1}{x - \alpha}$.

We note that $\phi'(x,\alpha) = 0$ at an infinite number of points in (α, β). Let $\alpha + \lambda = \max \{x | x<(1/2)(\alpha + \beta)$ and $\phi'(x,\alpha) = 0\}$. Define the following function on each component (α, β) of the complement of G:

$$F(x) = \begin{cases} \phi(x,\alpha) \text{ if } x\varepsilon(\alpha, \beta) \text{ and } \alpha \leq x \leq \alpha + \lambda \\ \phi(\alpha + \lambda,\alpha) \text{ if } \alpha + \lambda \leq x \leq \beta - \lambda \\ -\phi(x,\beta) \text{ if } \beta - \lambda \leq x \leq \beta \end{cases}$$

and $F(x) = 0$ for each $x\varepsilon G$. The function F is continuous and has a bounded derivative on the interval (a,b). If $x\varepsilon G$, we note that $F'(x) = 0$ and that in any σ-neighborhood of x there is an interval in the complement of G in which there are an infinite number of points at which F' is greater than 1. Therefore, F' is discontinuous at each point of G. Since G has positive measure and since $F'(x) = |F'(x)| = 0$ at each $x\varepsilon G$, it follows that $|F'|$ is not Riemann integrable.

Theorems 4.1 and 4.2 can be made somewhat stronger if we relax the requirement that f be absolutely continuous.

Theorem 4.3

If $f\varepsilon D$ and f is of bounded variation on $[a,b]$, then there exists a point $p\varepsilon(a,b)$ such that $V_a^b f = (b - a)|f'(p)|$.

Proof. Assume that there does not exist a point $p\varepsilon(a,b)$ such that $V_a^b f = (b - a)|f'(p)|$. Since f' has the Darboux property on (a,b) and there is not point $p\varepsilon(a,b)$ at which $|f'(p)| = V_a^b f/(b - a)$, it follows that either i) $|f'(x)|>V_a^b f/(b - a)$ for each $x\varepsilon(a,b)$, or ii) $|f'(x)|<V_a^b f/(b - a)$ for each $x\varepsilon(a,b)$.

Suppose i) holds. Let $a = x_0 < x_1 < \cdots < x_n = b$ be an arbitrary partition of [a,b], and let p_i be the point in (x_{i-1}, x_i) guaranteed by the Mean Value Theorem such that $f(x_i) - f(x_{i-1}) = f'(p_i)\Delta x_i$ = where Δx_i $x_i - x_{i-1}$. Then taking the suprema over all possible partitions of [a,b], we find that

$$V_a^b f = \sup \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

$$= \sup \sum_{i=1}^{n} |f'(p_i)|\Delta x_i$$

$$> \sup \sum_{i=1}^{n} |V_a^b f/(b - a)|\Delta x_i = V_a^b f$$

which is clearly a contradiction.

Suppose ii) holds. This implies that $|f'|$ is bounded and hence f is absolutely continuous. By Theorem 4.1, there exists a point $p \varepsilon (a,b)$ such that $V_a^b f = (b - a)|f'(p)|$ which contradicts our original assumption.

Theorem 4.4

If $f \varepsilon D$ and f is of bounded variation on [a,b], then there exists a point $p \varepsilon (a,b)$ such that $L_a^b f = (b - a)\sqrt{1 + |f'(p)|^2}$.

Proof. As in the proof of Theorem 4.2, we note that for a continuous function f, $V_a^b f$ exists if and only if $L_a^b f$ exists. Since $f \varepsilon D$ and is of bounded variation,we have that $L_a^b f$ exists. Since f' has the Darboux property, it follows that $\sqrt{1 + |f'|^2}$ does also. If we now assume that there does not exist a point $p \varepsilon (a,b)$ at which $L_a^b f =$ $(b - a)\sqrt{1 + |f'(p)|^2}$, the proof will follow by similar arguments as used in the proof of Theorem 4.3.

In view of the discussion prior to Theorem 4.1, it would seem clear that similar results to Theorems 4.1 through 4. 4could be obtained if f is a function on [a,b] into R^n such that f is absolutely continuous on [a,b], f' exists on (a,b), and $\|f'\|$ has the intermediate value property. However, the following example shows that $\|f'\|$ need not have the intermediate value property.

Example 4.2. For a <0 and b>0, define

$$F(x) = \begin{cases} (x^2\sin(1/x), x^2\cos(1/x), x) & \text{for } x \varepsilon [a,0) \cup (0,b] \\ (0,0,0) & \text{for } x = 0 \end{cases}$$

Then

$$F'(x) = \begin{cases} (2x\sin(1/x) - \cos(1/x), 2x\cos(1/x) + \sin(1/x), 1) & \text{for } \\ x\varepsilon[a,0) \cup (0,b] \\ (0,0,1) & \text{for } x = 0 \end{cases}$$

We observe that $\|F'(0)\| = 1$, whereas $\|F'(x)\| = \sqrt{4x^2 + 2} \geq \sqrt{2}$ for $x\varepsilon[a,0) \cup (0,b]$.

A vector-valued function f is said to be absolutely continuous if and only if each of its components is absolutely continuous [12]. In the function F defined in example 4.2, if we let f_i represent the i - th component of F, then $|f_i(x)| \leq 3$ and hence f_i is absolutely continuous. Thus $\int_a^b \|F'\| = \int_a^b \sqrt{4x^2 + 2} = V_a^b F$. Now $\sqrt{4x^2 + 2}$ is a continuous function on [a,b] and thus by Theorem 3.1, there exists a point $p\varepsilon(a,b)$ at which $\int_a^b \sqrt{4x^2 + 2} = (b - a)\sqrt{4p^2 + 2}$. So we have that $V_a^b f = (b - a)\sqrt{4p^2 + 2} = (b - a)\|F'(p)\|$ even though $\|F'\|$ does not have the intermediate value property on [a,b].

One might ask at this point if f being a differentiable and absolutely continuous function from [a,b] into R^n is sufficient for there to exist a point $p \varepsilon (a,b)$ such that $V_a^b f = (b - a)\|f'(p)\|$. Unfortunately, the answer to this question is no, as the following example shows.

Example 4.3. Define

$$g_\varepsilon(x) = \begin{cases} (\varepsilon x^2 \sin(1/x), \varepsilon x^2 \cos(1/x), x), & \text{if } x \varepsilon [-1,0) \\ (x^2 \sin(1/x), x^2 \cos(1/x,x), & \text{if } x \varepsilon (0,1] \\ (0,0,0), & \text{if } x = 0 \end{cases}$$

then

$$g_\varepsilon'(x) = \begin{cases} (2\varepsilon x \sin(1/x) - \cos(1/x), 2\varepsilon x \cos(1/x) + \sin(1/x), 1), & \text{if } x \varepsilon [-1,0) \\ (2x \sin(1/x) - \cos(1/x), 2x \cos(1/x) + \sin(1/x), 1), & \text{if } x \varepsilon (0,1] \\ (0,0,1), & \text{if } x = 0 \end{cases}$$

and

$$\|g_\varepsilon'(x)\| = \begin{cases} \sqrt{4x^2 \varepsilon^2 + \varepsilon^2 + 1} & \text{if } x\ [-1,0) \\ \sqrt{4x^2 + 2} & \text{if } x \varepsilon (0,1] \\ 1 & \text{if } x = 0 \end{cases}$$

Let $x \varepsilon [-1,0)$, then for any $\delta > 0$ there exists an $\varepsilon > 0$ such that $1 \leq \|g_\varepsilon'(x)\| \leq 1 + \delta$. Let $F(t) = \left[V_{-1}^t g_\varepsilon(x)\right] / [t - (-1)]$, then F is a continuous function of t on [0,1]. Now

$$F(o) = V_{-1}^0 g_\varepsilon(x) = \int_{-1}^0 \|g_\varepsilon'(x)\| < \int_{-1}^0 (1 + \delta) = 1 + \delta$$

and

$$F(1) = (1/2) V_{-1}^1 g_\varepsilon(x) = (1/2)\int_{-1}^0 \|g_\varepsilon'(x)\| + (1/2) \int_0^1 \|g_\varepsilon'(x)\|$$

$$\geq (1/2) \int_{-1}^0 1 + (1/2) \int_0^1 \sqrt{2}$$

$$= \frac{1 + \sqrt{2}}{2}.$$

Thus for sufficiently small $\delta > 0$, there exists a point $t \varepsilon (0,1)$ such that $1 + \delta < F(t) < \frac{1 + \sqrt{2}}{2} < \sqrt{2}$ and consequently there cannot exist a point $p \varepsilon (-1, t)$ at which

$$V_{-1}^{t} g_{\varepsilon}(x) = (t + 1) \|g_{\varepsilon}'(x)\| .$$

The ordinary Mean Value Theorem for derivatives of a real-valued function f from [a,b] into R guarantees the existence of a point in (a,b) at which the derivative has the same slope as the line joining the points (a,f(a)) and (b,f(b)). If we consider f as a vector-valued function from [a,b] into R^2, we get the following interesting results.

Theorem 4.5

If v is a differentiable vector-valued function defined on [a,b], given by v(t) = (t,f(t)) where f is a real-valued function from [a,b] into R, then there exists a point $p \varepsilon (a,b)$ such that v'(p) has the same direction as the vector (b - a,f(b) - f(a)).

Proof. Since v is differentiable on [a,b], it follows that f is differentiable and that v'(t) = (1,f'(t)). By Theorem 2.2, there exists a point $p \varepsilon (a,b)$ such that f'(p) = [f(b) - f(a)]/(b - a). Thus

$$v'(p) = (b - a, f(b) - f(a)) \cdot [1/(b - a)].$$

Since 1/(b - a) is positive, it follows that v'(p) has the same direction as the vector (b - a,f(b) - f(a)).

Corollary 4.1

If b - a = 1, then v'(p) will also have the same magnitude as the vector (b - a,f(b) - f(a)).

We might ask if similar results can be obtained if v is a more general vector-valued function from [a,b] into R^2. The answer is yes if we place a few more restrictions on v.

Theorem 4.6

If v is a differentiable vector-valued function from [a,b] into R^2 given by $v(t) = (x(t),y(t))$ such that $v'(t)$ is nowhere zero on [a,b] and $v(a) \neq v(b)$, then there exists a point $p \in (a,b)$ such that $v'(p)$ is parallel to the vector $u = (x(b) - x(a), y(b) - y(a))$.

Proof. Since v is differentiable on [a,b], we have that $v'(t) = (x'(t),y'(t))$. Also, since $v(a) \neq v(b)$, it follows that $u \neq (0,0)$ and hence either $x(b) - x(a) \neq 0$ or $y(b) - y(a) \neq 0$.

Suppose $x(b) - x(a) \neq 0$. Define the auxillary function

$$\phi(t) = y(t) - y(a) - \frac{y(b) - y(a)}{x(b) - x(a)}[x(t) - x(a)]$$

then $\phi(a) = 0 = \phi(b)$. Also, ϕ is continuous and differentiable on [a,b] since v is. Thus

$$\phi'(t) = y'(t) - \frac{y(b) - y(a)}{x(b) - x(a)} \cdot x'(t)$$

exists for all $t \in [a,b]$. By Rolle's Theorem, there exists a point $p \in (a,b)$ such that $\phi'(p) = 0$ and hence

$$y'(p) = \frac{y(b) - y(a)}{x(b) - x(a)} \cdot x'(p).$$

Therefore $v'(p) = u \cdot \frac{x'(p)}{x(b) - x(a)}$ which implies that $v'(p)$ and u are parallel vectors.

Suppose $y(b) - y(a) \neq 0$. The proof is similar to the proof in the case above if we define our auxillary function to be

$$\Theta(t) = x(t) - x(a) - \frac{x(b) - x(a)}{y(b) - y(a)} [y(t) - y(a)].$$

Corollary 4.2

If y'(p) and y(b) - y(a), or x'(p) and x(b) - x(a) have the same sign, then v'(p) has the same direction as u.

Corollary 4.3

If y'(p) = y(b) - y(a) or x'(p) = x(b) - x(a), then v'(p) has the same magnitude as u.

If we try to generalize the above results to a differentiable vector-valued function v from [a,b] into R^n, n>2, we find that there need not exist a point $p \varepsilon (a,b)$ at which v'(p) has the same direction or magnitude as the vector [v(b) - v(a)]/(b - a). The following is an example of such a function.

Example 4.4. Define v(t) = (cos t, sin t, t) on $[0,2\Pi]$. Now [v(b) - v(a)]/(b - a) = (0,0,1) whereas v'(t) = (- sin t, cos t,1). In order for these two vectors to be parallel, one must be a multiple of the other for some $t \varepsilon [0,2\Pi]$ which is clearly impossible.

Since the magnitude of v'(t) = $\sqrt{2}$ for all $t \varepsilon [0,2\Pi]$ and the magnitude of [v(b) - v(a)]/(b - a) = 1, it follows that there does not exist a point $p \varepsilon [0,2\Pi]$ at which v'(p) and [v(b) - v(a)]/(b - a) have the same magnitude.

If we consider f as a holomorphic complex-valued function defined on a subset of the complex plane C, we find that a mean value theorem in the form of the ordinary Mean Value Theorem (Theorem 2.2) may not always be possible. However, McLeod [6] has shown that for a holomorphic function f defined on a connected open set $G \subset C$ where z_1 and

z_2 are points in G such that the segment joining them is also in G, that

$$f(z_2) - f(z_1) = (z_2 - z_1)(\lambda_1 f'(p_1) + \lambda_2 f'(p_2))$$

for some p_1 and p_2 on the segment joining z_1 and z_2, and some non-negative real numbers λ_1 and λ_2 such that $\lambda_1 + \lambda_2 = 1$. This seems intuitively clear since as z_2 gets "close" to z_1,

$[f(z_2) - f(z_1)]/(z_2 - z_1)$, $f'(p_1)$, $f'(p_2)$, and hence $\lambda_1 f'(p_1) + \lambda_2 f'(p_2)$ get "close" to $f'(z_1)$.

We note that the linear expression $\lambda_1 f'(p_1) + \lambda_2 f'(p_2)$ cannot in general be replaced by a vlaue $f'(p)$. For example, if we define $f(z) = e^z$ and choose $z_2 = z_1 + 2\Pi i$, we find that $f(z_2) - f(z_1) = 0$ whereas $(z_2 - z_1)\, f'(p) = 2\Pi ie^p \neq 0$ for any p. However, if we let $\lambda_1 = 1/2 = \lambda_2$ and choose $p_1 = z_1$ and $p_2 = z_1 + \Pi i$, we see that McLeod's result holds.

The main difficulty in establishing a mean value theorem in $R^n, n>1$, is that there are many different ways of defining a derivative in this setting. Before we present a mean value theorem in R^n we first make the following definitions relative to R^n [9]. The set $I \subset R^n$ is a closed interval if $I = \{(x_1, x_2, \cdots, x_n) \mid a_i \leq x \leq b_i, i = 1, 2, \cdots, n\}$, and the set $J \subset R^n$ is a closed cube if J is a closed interval having equal, non-zero sides. The diameter of I is given by $d(I) = \sup\{\rho(x,y) \mid x \in I, y \in I\}$ where $\rho(x,y)$ represents the distance between points x and y, and its Lebesgue measure is given by $\mu(I) = \prod_{i=1}^{n} (b_i - a_i)$. A sequence $\{I_i\}$ of closed intervals is said to converge to $x \in R^n$, denoted $I_i \to x$, if $x \in I_i$ for each i, and $\lim_{i\to\infty} d(I_i) = 0$. A sequence $\{I_i\}$ is said to be a regular sequence if $I_i \to x$, and if there is a constant $\alpha>0$, called the parameter of regularity, such that for each i, there is a cube $J_i \supset I_i$ for which

$$\frac{\mu(I_i)}{\mu(J_i)} \geq \alpha$$

Intuitively, a regular sequence is one in which the intervals do not become too "thin." If we let T be a finitely additive set function defined on at least the closed intervals, then the upper and lower regular derivatives of T are given by

$$\overline{T}'^{*}(x) = \sup_{I_i\to x} \overline{\lim_{i\to\infty}} \frac{T(I_i)}{\mu(I_i)} \quad \text{and} \quad \underline{T}'^{*}(x) = \inf_{I_i\to x} \underline{\lim_{i\to\infty}} \frac{T(I_i)}{\mu(I_i)} ,$$

respectively, where the sup and inf are taken over regular sequences which converge to x. The regular derivative, denoted $T'^{*}(x)$, is said to exist at the point x if $\overline{T}'^{*}(x) = \underline{T}'^{*}(x)$.

L. Misik has shown [8] that if T'^{*} exists at each point of an interval in R^n, then a mean value theorem holds. However, we will give here a simpler and more direct proof of his result, as given by R. J. Easton and S. G. Wayment [3], using the additional hypothesis that T be absolutely continuous with respect to μ.

For notational purposes, we will restrict the proof of the following theorem to R^2. In this setting, a closed interval becomes a closed rectangle and its Lebesgue measure becomes its area. For this reason, we will use a more suggestive notation by letting $R = [a,b;c,d] = \{(x,y) | x\varepsilon[a,b] \text{ and } y\varepsilon[c,d]\}$ represent an arbitrary interval in R^2 and

$A(R)$ represent its area. Also, let $h = b - a$ and $k = d - c$. We define the auxiliary function $g_R(x,y) = T(R)$, where (x,y) is the midpoint of R. We note that since T is absolutely continuous with respect to μ, that g_R is a continuous function of x and y. In general, we would define g_I on the closed intervals in R^n to be a function of n-variables. The following proof will use a sliding interval technique similar to that used in the third proof of Theorem 2.2.

Theorem 4.7

If $T'^*(x)$ exists at each point w of a closed rectangle R and T is absolutely continuous with respect to μ, then there exists a point $p \varepsilon R$ such that $T'^*(p) = T(R)/A(R)$.

Proof. Let $R_1 = R = [a,b;c,d]$ and divide R_1 into four similar rectangles, namely $R_{11} = [a,a + h/2;c,c + k/2]$, $R_{12} = [a,a + h/2;d - k/2,d]$, $R_{13} = [b - h/2,b;d - k/2,d]$, and $R_{14} = [b - h/2,b;c,c + k/2]$. Since

$$T(R_1) = \sum_{i=1}^{4} T(R_{1i}),$$

we can assume without loss of generality that $T(R_{11}) \leq (1/4)T(R_1) \leq T(R_{13})$. The other cases follow in a similar manner. Let $\alpha = k/h$ and define the auxiliary function

$$f(t) = g_{R_{11}}(a + t + h/4, c + \alpha t + k/4)$$

which is continuous on $[0,h/2]$. Since $f(0) = T(R_{11}) \leq (1/4)T(R_1) \leq T(R_{13}) = f(h/2)$, it follows from the intermediate value property for a continuous function that there exists a point $t_0 \varepsilon [0,h/2]$ such that $f(t_0) = (1/4)T(R_1)$. This value of t_0 determines a rectangle

$$R_2 = [a + t_0, a + t_0 + h/2; c + \alpha t_0, c + \alpha t_0 + k/2] \subset R_1$$

where $T(R_2) = (1/4)T(R_1)$ and $A(R_2) = (1/4)A(R_1)$.

Thus

$$\frac{T(R_1)}{A(R_1)} = \frac{T(R_2)}{A(R_2)} .$$

We proceed inductively to obtain a nested sequence of closed rectangles $\{R_i\}$, each of which is geometrically similar to R_1 and having sides parallel to R_1, such that

$$T(R_{i+1}) = \frac{T(R_i)}{4} \text{ and } A(R_{i+1}) = \frac{A(R_i)}{4} .$$

By the nested interval theorem there exists exactly one point $p \in \bigcap_{i=1}^{\infty} R_i$, and

$$T'^{*}(p) = \lim_{i \to \infty} \frac{T(R_i)}{A(R_i)} = \frac{T(R_1)}{A(R_1)} .$$

We note that by changing the selection process slightly for R_3 and R_4, we can insure $p \in \mathrm{Int}(R_1)$.

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VITA

David A. Neuser

Candidate for the Degree of

Master of Science

Thesis: A Survey in Mean Value Theorems

Major Field: Mathematics

Biographical Information:

Personal Data: Born May 20, 1944, in Eau Claire, Wisconsin.

Education: Graduate of Memorial High School, Eau Claire, Wisconsin in 1962. Bachelor of Science degree in Mathematics from Wisconsin State University, Eau Claire, Wisconsin, June, 1966.