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ECONOMIC DESIGN OF X-BAR AND CUSUM CHARTS AS APPLIED TO NON-NORMAL PROCESSES.

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ECONOMIC DESIGN OF $\overline{X}$- AND CUSUM

CHARTS AS APPLIED TO NON-NORMAL PROCESSES

by

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ABSTRACT

This study is directed towards the development of models and procedures for the economic design of $\overline{X}$-charts, $\overline{X}$-charts with warning limits and cusum charts to control non-normal process means. The objective of the design is to determine the optimal values of the chart parameters by minimizing the expected loss-cost. Two alternative operating policies are considered. Under policy I, the process is allowed to continue in operation during the search for the single assignable cause. Under policy II, the process is shut-down immediately after the search for the assignable cause is initiated.

In developing these models, the non-normal probability density function of the process variable is expressed in terms of the first four terms of an Edgeworth series. The solution procedure for determining the design parameters of an $\overline{X}$-chart consists of an explicit equation for the sampling interval, and an implicit equation in sample size and the control limit coefficient. An optimization algorithm based on Hooke and Jeeve's pattern search technique is developed and employed to minimize the loss-cost function under both operating policies. A simplified scheme, which determines the design parameters by minimizing the loss-cost function subject to a specified level of consumer's risk, is also developed for both operating policies. Through numerical examples it is concluded that the resulting simplified scheme is close to the minimum control plan. The sensitivity analysis of the model

operating under policy II indicates that the model is highly sensitive to the shift parameter and the rate of occurrence of the assignable cause, moderately sensitive to the fixed and variable sampling costs, and relatively insensitive to the repair and search cost. The single assignable cause model is then extended to treat multiple assignable causes. The solution to the multiple cause model is found to be close to that of the 'matched' single cause model.

The economic design of $\bar{x}$-chart with warning limits is considered under policy II. In order to develop the loss-cost function, expressions for the average run lengths when the process is in control and when it is out of control are derived. The optimal values of the design parameters are obtained by using a two-stage optimization algorithm similar to that used for the economic design of $\bar{x}$-chart. Numerical examples are provided and the effects of the non-normality parameters on loss-cost function and design parameters are examined. Furthermore, a simplified scheme is devised subject to the condition that the assignable cause is detected after a specified average run length.

For the economic design of cusum charts, the average run lengths are derived by solving a system of linear equations which approximate the integral equations for the required quantities. Using the decision interval scheme, an iterative algorithm is developed to determine the optimal design parameters. A simplified version of the algorithm is also presented. From numerical studies it is observed that the effect of skewness is more marked than that of kurtosis.

A comparison of the performances of the three charts indicates that, for the shift in the process mean between 0.5σ and 1.5σ the performance of the cusum chart is better than that of the $\overline{x}$-chart with warning limits. However, the performance of the latter is better than that of $\overline{x}$-chart with only action limits. With the shift in the process mean above 1.5σ, the performance of the $\overline{x}$-chart is slightly better than those of $\overline{x}$-chart with warning limits and cusum charts.

Finally, the effect of human errors on the model is studied, and a simulation of the model behaviour under extreme cases of sample distribution is carried out.

To My Parents and wife Bilkis

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NOMENCLATURE

A_k	kth Gaussian coefficient
α	Probability of searching for an assignable cause when it does not exist
b	Fixed sampling cost
β	Probability of not searching for an assignable cause when it exists
β_1	Measure of skewness
β_2	Measure of kurtosis
d	Lead distance of a V-mask measured on a vertical scale
D	Average time taken to find an assignable cause for an $\bar{x}$-chart under policy I
δ	Shift parameter
ε	Efficiency of a control plan
e	Delay factor for an $\bar{x}$-chart under policy I
E(C)	Expected production cycle length
γ_1	Equal to $\sqrt{\beta_1}$
γ_2	Equal to $\beta_2 - 3$
h	Decision interval for one-sided cusum chart
H	Standardized decision interval for one-sided cusum chart
I	Expected net income per-hour
k	Control limit coefficient for an $\bar{x}$-chart
k_a	Action limit coefficient for an $\bar{x}$-chart with warning limits

k_w	Warning limit coefficient
K	Reference value of one-sided cusum chart
k_r	Expected repair cost
k_s	Expected search cost
L	Per-hour loss-cost
λ	Rate of occurrence of an assignable cause
m	Number of Gaussian points
μ_0	Process mean when in-control, acceptable quality level
μ_1	Process mean when out-of-control, rejectable quality level
n	Sample size
$N(z)$	Unconditional average sample number of Wald sequential test
P	Probability of true alarm
P'	Probability that a point falls below the warning limit
$P(z)$	Probability that a Wald test which starts at a distance z from the lower boundary ends on or below it
ϕ	Half angle of the V-mask
$\phi(x)$	Probability density function of a standardized normal variate x
$\Phi(x)$	Cumulative distribution function of a standardized normal variate x
q	The probability that a point falls between the warning and action limits
R_0	Average run length of a chart when the process is at acceptable quality level μ_0

R_1	Average run length of a chart when the process is at rejectable level μ_1
R_c	Critical run length in between the warning and action limits
s	Sampling interval
σ	Process standard deviation
τ_r	Expected repair time
τ_s	Expected search time
θ	Magnitude of variation in mean in a cusum chart
U	Loss rate
V	Expected cost of searching for an assignable cause when the process is in control
V_0	Per-hour income when the process is in control
V_1	Per-hour income when the process is out of control
W	Expected cost of searching for an assignable cause when it exists
z_k	kth Gaussian point

CHAPTER 1

INTRODUCTION

1.1 General Introduction

In any production process, some variations in product quality are unavoidable. These variations can be divided into two categories, (i) random variations and (ii) variations due to assignable causes. If the random variations exhibit a stable pattern, the process is said to be operating under a stable system of chance-causes, or simply, to be in a state of in-control. Variations that are not within the stable pattern of chance-causes are attributed to assignable causes and the process is then said to be in the state of out-of-control. It is desirable that, when there is evidence that assignable causes of variation are present, these causes be detected and removed from the process and, hence, the process be brought back to the in-control state. This is facilitated by the use of quality control charts.

A statistical quality control chart is a dynamic device which, on the basis of the process performance, determines operational criteria to distinguish between random and non-random variations in product quality and thereby provides a basis for taking action to eliminate removable causes of variations. Thus the two major uses of control charts are to establish operational criteria to bring a process under control and to maintain the existing state of control.

When the product quality is measured on a continuous scale, commonly used statistical quality control charts for controlling the process average are the $\bar{x}$-chart and the cumulative sum (cusum) chart. To use a control chart, the user must specify a sample size, a sampling interval and the control limits or critical region for the chart. Selection of these parameters is called the design of the control chart.

The design of the control chart with respect to economic criteria has been a subject of interest during the last three decades. The objective of the design has been either to minimize the inspection requirements or to maximize the income, i.e., to minimize the loss-cost. The assumption underlying the design has been the normality of the process mean.

1.2 Statement of the Problem

In many industrial processes, the process variables, which are the outputs of man-machine systems, do not always conform with the normality assumption. The measurable quality characteristic, which describes the product quality, is a random variable whose density function depends upon one or more parameters of the product quality and often has a non-normal distribution. In such cases, conventional charts, which are based on the normality assumption, could affect the probabilities associated with the control limits or critical regions and may wrongly indicate lack of control or out-of-control.

Generally, there can be as many causes as one can imagine for a process to be non-normal in nature. For instance, a process may have been screened for out-of-tolerance parts, resulting in a truncated distribution. The truncated distribution can also occur when the values of a measured variable can be accurately recorded only in a certain interval. This may be due to limitations in measuring instruments, or to the purely practical considerations of ease and speed of observation [Johnson and Leone, 1976].

Another type of non-normal distribution, known as mixture-distribution, arises when products from two or more separate sources are mixed. If m machines are making the same product and a quality characteristic x is distributed normally for the product from any machine, but with mean and/or standard deviation varying from machine to machine, a mixture of products from all m machines will not, in general, have a normal distribution of x [Johnson and Leone, 1976].

The powers and products of normal variates have distributions, which are in general skewed to the right [Haldane, 1942]. There are circumstances in which skewness must be regarded as being typical of a product variate [Morrison, 1958].

In general, the distribution of a product characteristic is unknown. Given a sample of measurements of a product characteristic, the general objectives of analysis are to estimate the parameters of the distribution and to make inferences. Common analyses consist of computing estimates of mean, variance, skewness and kurtosis of the underlying distribution. Judgements of normality or non-normality can be based on measures of skewness and kurtosis.

In setting the control limits or the critical regions for control charts, the assumption of normality is justified by the central limit theorem. The theorem essentially states that, under certain conditions, the distribution of the sample mean will approach normality for large sample sizes. But increasing the sample size increases the sampling costs. Decreasing the sample-size will increase the losses resulting from deviations of the mean from normality. The problem is to find the optimum. Solutions are known for the case where the product variable considered is normally distributed. However, there are cases where neither the product variable is normally distributed, nor the sample size is large enough to apply the central limit theorem. In these situations, the question arises as to what effect non-normality of various degrees will have upon the operation of $\overline{x}$-charts and cusum charts. The present study is an attempt in answering this question.

1.3 Objectives of the Study

The major objectives of the present study are to develop mathematical models and procedures for the optimal design of control charts to control non-normal process means based on economic criteria and to investigate the effects of non-normality on the design parameters and on the long run average loss-cost function developed in the models. The economic design of a control chart involves the optimal determination of design parameters so that the average loss-cost is minimum.

The investigation is confined to the design of $\overline{x}$-charts, $\overline{x}$-charts with warning limits and cusum charts for the control of the mean of a process when there is a single assignable cause and when the observational variables are independent and non-normally distributed. Furthermore, the study is concerned with quality control tests involving a single statistic, i.e., the sample mean.

The loss-cost incurred in a production cycle is assumed to consist of the search cost following a false alarm, the search and adjustment costs following a true alarm and the cost of maintaining the control chart.

In the case of $\overline{x}$-charts, the loss cost function depends on the probabilities of Type I and Type II errors; expressions for these probabilities are thus developed.

Similarly, expressions for average run lengths when the process is in control and when it is out of control are developed for the economic design of $\overline{x}$-charts with warning limits and of cusum charts.

It is also the objective of this study to investigate the effects of variations in the cost and shift parameters and in the rate of occurrence of the assignable cause on the values of the design parameters. Accordingly, the sensitivity of the model to errors in the estimation of these factors will be analyzed.

The present research is further concerned with the development of a simplified scheme, suitable for practical application at a factory level, for each of the underlying control charts.

Comparisons among the relative performances of these charts are made through numerical examples.

Finally, the effects of human error on the proposed models, as well as a simulation of model behaviour under extreme sample distribution, are discussed.

1.4 Outline of Proposed Study

After the introductory material of Chapter 1, the presentation is patterned as follows:

In Chapter 2 the historical background of control charts is presented, the literature on control charts is surveyed and the motivation for the present study is described.

Chapter 3 reviews the statistical properties and the design criteria of control charts.

Chapter 4 is devoted to the development of mathematical models for the economic design of $\bar{x}$-charts under two operating policies. Under policy I, the process is allowed to continue in operation during the search for the assignable cause. Under policy II, the process is shut down during the search for the assignable cause. An optimization algorithm,based on Hooke and Jeeve's pattern search technique,is employed to minimize the loss-cost function under both operating policies and to obtain the respective optimal design parameters (i.e., sample size, sampling interval and control limits coefficient). A simplified scheme, which determines the design parameters by minimizing the loss-cost function subject to a specified level of consumer's risk,is also developed. The chapter also includes a sensitivity analysis of the model under operating policy II and investigates the model behaviour when there is a multiplicity of assignable causes.

Chapter 5 describes the model of the economic design of $\bar{x}$-charts with warning limits under policy II and details the formulation of the loss-cost function, the determination of the average run length when the process is in control and when it is out of control and a two-stage optimization algorithm to determine the optimum design parameters. It includes a simplified version of the algorithm as well.

The economic design of cusum charts to control non-normal process means under policy II is described in Chapter 6. It contains the formulation of the loss-cost function, the determination of average run lengths by a system of linear algebraic equations and an iterative algorithm to obtain the optimal values of the design parameters. Also presented is a semi-economic scheme which allows the user to specify the value of the average run length at the rejectable quality level. Finally, the Chapter evaluates the relative performances of the three control charts developed in Chapters 4-6.

In Chapter 7, the effects of human error on the model and a simulation of the model behaviour under extreme cases of sample distributions are discussed.

A summary of the findings, conclusions, and recommendations for future research are presented.

CHAPTER 2

LITERATURE SURVEY AND MOTIVATION FOR PROPOSED STUDY

This Chapter reviews and classifies the existing literature on the subject of economic design of control charts. The present survey deals with the specific portion of the available literature which seems most relevant to the scope of this research. Furthermore, it provides the motivation for the development of the models described in later chapters.

2.1 Economic Design of $\bar{x}$-Charts to Control Normal Means

Based on the minimum cost criterion, Duncan [1956] proposed the single assignable cause model for the economic design of the $\bar{x}$-chart to control normal process means. He assumed that the occurrence time of the assignable cause is an exponential random variable. He developed an expression for an approximate per hour loss-cost function of the process. In developing this function, he considered relevant incomes when the process is in control and when it is out-of-control, cost of looking for an assignable cause when it exists and when it does not exist and the cost of maintaining a control chart. Based on several numerical approximations, Duncan developed an iterative procedure to find the near-optimum solutions of sample size n and control limit coefficient k. A closed form solution for s is given,using the optimal values of n and k. Duncan's model is simple and practical in some situations but not sufficiently general, as it does not allow the process to be shut down when a search for the assignable cause is being carried out and it does not include the time and cost of repairing the process if it is found out-of-control.

Cowden [1957] developed an economic design of an $\bar{x}$-chart and defined the total cost function as the sum of the operating cost, engineering cost and merchandising cost. His model assumes that every morning production starts in an unknown state. If a point on the $\bar{x}$-chart goes outside the control limits, a search is made to look for any trouble. If the trouble is detected, it is corrected immediately. Once the process has been corrected, no more troubles occur during the rest of the day. Cowden's model is not suitable for the study of the control chart, as the manufacturer may simply examine his process every morning, correct the trouble if found and then start the production of the day without using any control charts.

Gibra [1967] investigated the optimal economic design of an $\bar{x}$-chart used to monitor a process involving tool wear, in which the mean of the quality characteristic exhibits a linear trend. The optimal control procedure determines decision rules for adjustment due to drift, as well as for the occurrence of an assignable cause. The control rules minimize adjustment costs and costs due to the production of defective items.

Goel, et al. [1968] developed an iterative optimization algorithm to determine the exact optimal solution for Duncan's model.

Taylor [1968] developed a model which allows for process shut down during the search for the assignable cause and which includes the time and cost of repairing the process when it is out of control. But he omitted the cost of sampling.

Gibra [1971] has proposed a single assignable cause model of an $\bar{x}$-chart. His process model is similar to Duncan's model. In the development of the cost model, he proposed the concept of worst cycle quality level (WCQL). The optimal values of the design parameters are obtained by minimizing the expected cost function subject to constraints on the WCQL.

Baker [1971] has proposed two discrete-time models in which a sample size n is taken at the end of each period and a test statistic is plotted on the control chart with $\pm k\sigma$ limits. His first model assumed that the number of periods the process remains in the control state follows a geometric distribution, while his second model assumed that the number of periods the process remains in the control state follows a Poisson distribution. Furthermore, he pointed out that the optimal economic control chart design is relatively sensitive to the choice of process failure mechanism. Substantial cost penalities may be incurred if an incorrect process failure mechanism is assumed.

Chiu and Wetherill [1974] modified Duncan's and Taylor's models and proposed a semi-economic scheme for the design of an $\bar{x}$-chart by utilizing the concept of operating characteristic (OC) curves.

An assumption common to all the works cited above is that, when the process is disturbed by an assignable cause, only the mean changes while the variance remains unchanged. Krishnamorthi [1979] proposed a model under the assumption that the variance is also changed due to the occurrence of the assignable cause.

Control charts with warning limits were first introduced by Page [1955]. The chart includes warning limits which lie inside the action limits. A search for an assignable cause is undertaken if the last sample mean is in the out-of-control state (falls outside of the action limits) or if the last sample mean completes a run of length R_c which is in between the warning and action limits. Page [1962] modified his first model and measured the sensitivity of an $\bar{x}$-chart by developing the Mean Action Time of the chart using run theory.

Weindling, et al. [1970] proposed a Mean Action Time of an $\bar{x}$-chart with warning limits and discussed the effects on mean action time of an $\bar{x}$-chart of changes in the location of the action and warning limits and the critical run length.

Gordon and Weindling [1975] developed a cost model for the economic design of a warning limit control chart. They considered a single assignable cause model. The costs considered are those of inspection, defective production and searching for and correcting the assignable cause. The criterion is average cost per good produced. The process has only one out-of-control state and shifts to this state are governed by a Markov process. The model of Gordon and Weindling allows economically-optimal determination of the design parameters, k_a (action limit coefficient), k_w (warning limit coefficient), n (sample size) and s (sampling interval).

Chiu and Cheung [1977] extended the work of Gordon and Weindling by considering the cost of process shut-down. They made various comparisons among the minimum costs designs of $\overline{x}$-charts with and without warning limits. They also provided a simplified scheme for the determination of control parameters.

Knappenberger and Grandage [1969] proposed a model for the economic design of an $\overline{x}$-chart when there are multiple assignable causes. They minimized the expected cost per unit product. They assumed that the costs of investigating both real and false alarms are the same. This assumption is not practical.

Duncan [1971] extended his single assignable cause model to a multiple assignable cause model. The occurrence times of assignable causes are assumed to be independent exponential random variables. He assumed that once the process shifts to an out-of-control state due to the occurrence of an assignable cause, it remains in that out-of-control state and no further assignable causes occur until the process is brought back to the in-control state. This assumption is quite unrealistic. But Duncan also formulated the "double occurrence" model in the same work, under the assumption that after an initial shift, a second occurrence of the assignable cause is possible. He showed that this modification in the model has little effect on the optimum solution of the design parameters, but produced some changes in the behaviour of the cost surface. Both Duncan [1971] and Knappenberger and Grandage [1969] defined a "matched" single cause model and found that the optimum control plan of the matched single cause model approximated well the true optimum control plan for the original multiple cause model.

2.2 Economic Design of Cusum Charts to Control Normal Means

The economic design of the cusum chart has a shorter history than the economic design of the $\bar{x}$-chart. It was first investigated by Taylor [1968] for normal processes. Taylor's single assignable cause model expressed the expected loss-cost per unit time as a function of the sample size n, the sampling interval s and the V-mask design parameters d and half angle ϕ. The model allows for process shut-down during the search for the assignable cause and includes the time and the cost of repairing the process when it is out-of-control. However, the model assumes that n and s are specified and that the effect of the assignable cause is a function of the sample size. The cost of sampling is also omitted.

Goel and Wu [1973] developed a single assignable cause model for the optimum economic design of a cusum control chart for controlling normal process means. They utilized a cost model similar to Duncan's single cause $\bar{x}$-chart model and presented both V-mask and decision interval schemes to obtain the optimum values of the design parameters.

Under the assumption of normality of the process means and following the general modelling structure of both Duncan's $\bar{x}$-chart model and Taylor's cusum chart model, Chiu [1974] developed a single cause economic model for a cusum chart. He considered a one-sided decision interval scheme in formulating the per hour loss-cost function and presented both a numerical optimization method and a simplified approximate solution procedure to determine the optimal values of the design parameters.

2.3 Design of Control Charts for Non-Normal Processes

The design of control charts which are discussed in the above sections and the comprehensive surveys of recent developments in control chart techniques by Gibra [1975] and Montgomery [1980], reveal that a considerable amount of work has been undertaken for the economic design of control charts under the assumption that the process variables are normally distributed. It is common knowledge that industrial random variables do not always conform to the assumption of normality. In such cases conventional control charts, which are based on the normality assumption, may wrongly indicate that the process is out-of-control when it is actually in the in-control state. Similarly, they may wrongly indicate that the process is in the in-control state whereas it remains in the out-of-control state.

Delaporte [1951] demonstrated the effect of non-normality on control charts for sample means. Through numerical studies he has shown that the values of upper and lower control limits, obtained on the assumption of a normal population, differed substantially from the respective values obtained by studying the actual distribution of means.

Gayen [1953] discussed the need for correcting the normal theory control charts for measuring departures from normality and described some methods for calculating control limits for means and standard deviations in situations where their distributions depart significantly from the normal. One of his suggested methods was to express the non-normal probability density function in terms of an Edgeworth series, which provided a convenient alternative to the normal density function.

Moore [1957] has shown how certain departures from the normality assumption could affect the probabilities associated with control limits calculated by normal theory. Through numerical studies he cautioned against the risk of rigidity concerning the normality assumption.

Ferrel [1958] considered the case when the distribution of the quality characteristic is badly skewed and devised control charts for a log-normal population.

Singh [1966] investigated the effect of non-normality of the manufactured units on producer's and consumer's risks. He considered only the effect of the peakedness parameter, $\gamma_2 = \beta_2 - 3$ and he found that in the case of platykurtic populations ($\gamma_2 < 0$), if the specification limits are set near the mean, then both the producer's and consumer's risks will be greater than their respective normal theory values and if the specification limits are set far from the mean, both risks will be smaller than their corresponding normal theory values.

Hahn [1971], Schilling and Nelson [1976], Heiks [1977] and Gruska [1978] examined the conditions under which one might or might not expect process variables to be normally distributed and indicated the procedures which may be used to check the validity of the normality assumption. They suggested some very important transformations of the process variable to achieve a better method of approximation to normality and made several comments on the consequences of incorrectly assuming normality.

Following Duncan's work for normal processes, Nagendra and Rai [1971] developed an economic model of an $\bar{x}$-chart to control non-normal process means. They used several numerical approximations to derive the per hour loss-cost function of the model. Taking the first partial derivatives of the loss-cost function with respect to sample size n, sampling interval s and using some approximations, they obtained expressions for the design parameters n and s for a specified value of control limit coefficient k. Since k is not treated as a variable, the resulting plan may be far from optimal. Moreover, the study did not consider the cost of process shut-down and no attempt was made to study the effect of variations in the cost factors on the solution vectors.

Raouf, et al. [1979] used a direct search technique to obtain an optimal solution of the design parameters of an $\bar{x}$-chart to control non-normal process means and studied the effects of cost factors and non-normality parameters on the solution vectors.

Lashkari and Rahim [1979] developed an economic model of an $\bar{x}$-control chart to control non-normal process means, considering the cost of process shut-down. They also provided a simplified scheme to determine the values of design parameters.

Lashkari and Rahim [1980] also developed an economic design of cusum charts under the non-normality assumption.

Very recently, Rahim and Lashkari [1981] proposed an economic model for the design of an $\bar{x}$-chart with warning limits to control non-normal process means.

2.4 Motivation for Proposed Study

The literature survey leads to the conclusion that considerable attention has been devoted to the economic design of control charts under the assumption of normality of the process means. In many cases the normality assumption is applied without knowing the distribution of the process variable, or even when the process variable deviates from the normal distribution.

The theoretical justification for the normality assumption is based on the central limit theorem, which states that under very general conditions the distribution of the sum, and therefore of the average, of n independent observations will approach normality as the number of observations increases.

The question of how large a sample should be to apply the central limit theorem will have a bearing on the operating cost of a control chart. The operation of a control chart involves both fixed and variable sampling costs. Sampling cost increases with increase of sample size. Decreasing sample size will increase losses resulting from deviations from the mean. Therefore, it is desirable to find the optimum sample size that would balance the costs against the losses.

The solution for optimal determination of the sample size under the normality assumption is known either for given values of probabilities of Type I and Type II error [Knappenberger, 1966], or for given values of control limit coefficient and shift parameter [Weiller, 1952]. The probability that the sample point falls outside the control limits when the process is actually in the in-control state is known as the probability of Type I error, whereas the probability that a point falls inside the control limits when the process is in an out-of-control state due to the occurrence of an assignable cause is known as the probability of Type II error. In the design of sampling plans, the probabilities of Type I and Type II errors are known as the producer's and the consumer's risks, respectively. Here, then, the concern of the quality control engineer is to achieve a compromise between the values of the producer's and consumer's risks.

In many applications, data will seldom follow a normal distribution. We may also be confronted with an industrial situation where the assumption of normality is neither achievable nor desirable. For instance, often the data may be so badly skewed that the skewness itself produces outages and indicates the presence of an assignable cause of variation if the normality assumption is made [Morrison, 1958]. In such cases the skewness must be regarded as being typical of a variate. Otherwise, probabilities associated with control limits (probabilities of Type I and Type II errors) calculated by normal theory will provide erroneous results.

There are other situations where the process is non-normal in nature. For example, a process may have been screened for out-of-tolerance parts, resulting in a truncated distribution [Hald, 1952].

In general, due to limitations in the measuring instruments or due to purely practical considerations of ease and speed of measurement, the values of the measured variable are recorded accurately only in a certain interval which, consequently, causes a truncated distribution [Johnson and Leone, 1976].

A mixture-distribution [Johnson and Leone, 1976] is another type of non-normal distribution which arises when products from two or more different sources are mixed. For example, the quality characteristic of a product produced on any one of the m machines may be distributed normally, but if the mean or standard deviation of the process varies from machine to machine, a mixture of products from all m machines will not, in general, yield a normal distribution.

Also a process may be subject to tool wear [Duncan, 1974], resulting in non-normal process characteristics. Distributions of powers and products of normal variates are, in general, non-normal [Haldane, 1942]. Such situations are encountered frequently in the application of statistical control chart analysis to thermionic value test data [Morrison, 1958].

Although many industrial processes are non-normal in nature and despite the fact that the assumption of normality is very crucial in such circumstances, little attention has been paid to the economic design of control charts under the non-normality assumption. There is need for a procedure which will enable us to deal with non-normality of data, to design control charts accordingly and to continue our search for the assignable causes of variation.

CHAPTER 3

OPERATING PROPERTIES AND DESIGN CRITERIA OF CONTROL CHARTS

Statistical properties and design criteria of control charts which are relevant to the scope of this study are described in the following section.

3.1 Statistical Properties of Control Charts

It is current practice to use statistical techniques to monitor the variability of the quality of output of an industrial process. The rationale for this procedure is the classification of such variability into one of two types - variability due to inherent random fluctuations of the process, and variability due to changes in the process parameters. Process variability is of concern to the quality control engineer because the product must meet certain performance standards specified by the designer. Usually these standards are given in the form of specification limits within which a product's measurable characteristics must lie in order for the product to be considered acceptable. The production engineer must therefore attain and thereafter, maintain, the state of control of the process in which variability is due only to inherent random fluctuations. In other words, he must make the process behave as if each measurable property of the product comes from a single statistical population having stationary parameters (i.e., constant with respect to time). If these parameters do vary with time, such variation must be investigated and its cause must be discovered by the production engineer.

Any feature of the production cycle which causes a change in one or more of the process parameters is called an assignable cause. The presence of such an assignable cause is to be detected by the control chart and removed from the system by the quality control engineer. The $\bar{x}$-control chart and the cumulative sum (cusum) chart are two well-known statistical techniques which have been used for the last few decades in detecting an assignable cause under the assumption that the quality characteristics of the product are normally distributed.

In an $\bar{x}$-control chart, the control limits are set at $\pm$ k standard deviations of the sample mean from the target value. A sample of size n is taken from the process every s hours and the sample mean is plotted on the $\bar{x}$-chart. The process is subject to the occurrence of an assignable cause of variation which takes the form of a shift in the process mean from μ to $\mu + \delta\sigma$, where μ, σ and δ are, respectively, the process mean (target value), the process standard deviation and the shift parameter. The occurrence of sample means outside the control limits is regarded as an indication that the process is in an out-of-control state.

On the other hand, in a cumulative sum control chart, a sample of size n is taken at regular intervals of s hours. Successive values of the sample mean are compared with a predetermined reference value K and the cumulative sum of deviations from this value is plotted or tabulated on the cusum chart. If this sum exceeds a predetermined decision interval h, the indication is that a change has occurred in the mean level of the variable. Properties of a cusum test are described by a pair of average run lengths, R_0 and R_1, associated with the state of the process in-control and out-of-control, respectively.

The efficiency of both $\overline{x}$-charts and cusum charts in detecting the lack of control depends upon the values of the design parameters n, s and k for an $\overline{x}$-chart and n, s, h and K for a cusum chart.

3.2 Quality Control Chart as a Test of Hypothesis

As mentioned above, the function of a quality control procedure is to maintain a process in a state of control. This function is accomplished by periodically testing the null hypothesis that the process parameters are equal to the control values. The test is conducted by measuring the quality of a sample of the product produced by the process. The value of the test statistic is computed from the sample data. If this value falls in the critical region (i.e., outside the control limits), the null hypothesis is rejected and the process is investigated to determine and correct the condition which caused the process to go out of control. If the value of the test statistic is not in the critical region, the process is assumed to be in control and it is allowed to continue.

As in any hypothesis testing procedure, two types of error may occur. One type, generally called "Type I error", involves rejecting the null hypothesis when the process is in control. The second type, generally called "Type II error", involves failure to reject the null hypothesis when the process is out of control. Type II error leads to costs associated with an increase in the number of defective products produced by an out-of-control process. Costs of unnecessary investigation and loss of production arise from Type I error. Both of these costs can be decreased by increasing the sample size and decreasing the sampling interval; however, this reduction in error cost is accompanied by an increase in sampling and testing costs. Type I error costs can also be decreased by decreasing the critical region, thus increasing type II error costs.

3.3 Criteria for the Design of Control Charts

The design of control charts involves the optimum selection of design parameters. Selection criteria of these design parameters can be classified in the following categories and are discussed below:

1. Power Function Criterion,
2. Average Run Length Criterion,
3. Minimum Cost Criterion.

3.3.1 Power Function Criterion. The Power Function criterion, which is a method commonly employed for determining the parameters n and k of an $\overline{x}$-control chart, was first used by Knappenberger [1966]. The use of this criterion is equivalent to defining a test of hypothesis between two simple alternatives:

$$H_0: \mu = \mu_0 ,$$
$$H_1: \mu = \mu_0 \pm \delta\sigma, \quad \delta > 0 ,$$

where $\delta\sigma$ is the shift in the process mean. The statistic $\overline{x}$ has a normal distribution with mean μ and variance σ^2/n and the control limits of the $\bar{x}$-chart are at $\pm k\sigma/\sqrt{n}$. For a given probability α of Type I error, the values of n and k are chosen so that the power of the test, that is, the probability of rejecting H_0 when H_1 is true, is some specified value $(1 - \beta)$.

The power function approach is a straightforward and simple criterion to use. But an arbitrary choice of α and β does not reflect the cost and risk factors associated with the process and does not appear to be more logical than an arbitrary selection of n and k. Further, the sampling interval, s, is not taken into consideration.

3.3.2 Average Run Length Criterion. Page [1954] defined the term "Average Run Length" (ARL) as the average number of articles inspected between two successive occasions when some rectifying action is taken and employed it as a criterion for the design of $\bar{x}$-charts. Page showed that for a one-sided $\bar{x}$-chart, the ARLs, R_0, when the process is in control, and R_1, when the process is out-of-control are

$$R_0 = n/[1 - \Phi(k)] \tag{3.1}$$

and

$$R_1 = n/[1 - \Phi(k - \delta\sqrt{n})], \tag{3.2}$$

where $\Phi(x)$ is the cumulative distribution function of the standardized normal variate x.

When both negative and positive deviations in the process mean are equally important, the ARLs, R_0' and R_1' are

$$R_0' = n/2\ [1 - \Phi\ (k)] \tag{3.3}$$

and $$R_1' = n/\ [(1 - \Phi\ (k - \delta\sqrt{n}) + \Phi\ (-k - \delta\sqrt{n})]\ . \tag{3.4}$$

The methods used to design an $\bar{x}$-chart are either to fix R_0 and choose n and k for a given value of δ that will minimize R_1, or to specify R_1 and choose those values of n and k that will maximize R_0.

Weiler [1952] has shown that the average number of articles inspected before a change is given by

$$A(n) = n/[1 - \Phi\ (k - \delta\sqrt{n})]\ . \tag{3.5}$$

For given values of k and δ, he determined a sample size n which minimizes the function A(n). Weiler showed that for a given control limit coefficient k, the value of n that minimizes A(n) depends on the amount $\delta\sigma$ by which the population mean has changed.

Page [1954a] introduced the cumulative sum chart as an alternative to the $\bar{x}$-chart for controlling the mean of a normal process. The selection criterion of the design parameters, based on the average run length criterion, is as follows.

The design parameters n, h, K and s are generally selected to yield approximate ARLs R_0 and R_1 at acceptable and rejectable quality levels μ_0 and μ_1, respectively. Ideally, the ARL should be large when the process is operating at an acceptable quality level (AQL) and small when the process is operating at a rejectable quality level (RQL). Page [1954b] showed that the cusum chart scheme is equivalent to a sequence of Wald-sequential tests with horizontal boundaries (0,h) and initial score zero. He derived the following expression for the ARL:

$$ARL = N(0)/\{1-P(0)\}, \tag{3.6}$$

where $P(0)$ is the probability that the test starts on the lower boundary and ends on the lower boundary. $N(0)$ is the unconditional average number of samplings of the test. For known values of $N(0)$ and $P(0)$, the value of ARL can be found from the above expression. However, it is quite complicated to obtain the values of $P(0)$ and $N(0)$ from the integral equations $P(z)$ and $N(z)$, respectively. The expressions for $P(z)$ and $N(z)$ are as follows:

$$P(z) = \int_{-\infty}^{-z} \phi(x)\, dx + \int_0^h P(x)\, \phi(x-z)\, dx \tag{3.7}$$

$$N(z) = 1 + \int_0^h N(x)\, \phi(x-z)\, dx, \tag{3.8}$$

where $\phi(x)$ is the density function of the process variable x, distributed normally.

Kemp [1958] developed approximate solutions for $P(z)$ and $N(z)$ for the case when x has a normal distribution. Ewan and Kemp [1960], Goel and Wu [1971], and Goel [1971] provided nomograms which can be used for the selection of design parameters approximately satisfying the requirements of R_0 and R_1 for controlling the mean of normal processes. Ewan and Kemp [1960] suggested that the reference value, K, should be midway between the ARL and RQL. Kemp [1962] showed that the V-mask scheme with lead distance d and half angle ϕ is equivalent to a two-sided interval scheme.

3.3.3 Minimum Cost Criterion. The power function approach and the average run length approach discussed above are both statistical criteria. The design of control charts based on these criteria does not take into consideration the sampling interval, the cost and risk factors and various other parameters related to the process being controlled. From an industrial quality-control engineering point of view, a more realistic approach would be to use a criterion that would include the income and the cost figures associated with the process and the maintenance and operation of the control chart. However, to apply such decision criteria, the quality control engineer must know the characteristic of the product (measurable or attributive). He should have a knowledge of the state and nature, the failure mechanism, operating policy and the income and cost parameters of the production process.

3.4 Assumptions About Process Behaviour

In this section, the assumptions about the behaviour of the production process, which are required to formulate a model for the economic design of control charts are described.

3.4.1 The Production Process. A specified production process is considered. It is assumed that the quality characteristic of the process is a variable measurable on a continuous scale. The process variable is assumed to be non-normally distributed with probability density function $f(\mu_0, \sigma^2, \beta_1, \beta_2)$ with mean μ_0, variance σ^2, measure of skewness β_1 and measure of kurtosis β_2. The process starts in an in-control state and may be disturbed by the occurrence of an assignable cause which shifts the process mean from μ_0 to $\mu_0 + \delta\sigma$, where δ is the known shift parameter and σ, β_1 and β_2 are assumed to remain stable. The occurrence of the assignable cause is considered as a random shock acting on the system, that is, the probability of the process shift within a small interval of time is directly proportional to the length of the interval.

To determine the nature of the transitions between the in-control and out-of-control states, it is assumed that the assignable cause occurs according to a Poisson process with mean rate of occurrence λ. That is, the length of time the process remains in the in-control state is an exponential random variable distributed with mean $1/\lambda$ hours.

Two different operating policies of the process are considered.

1) Policy I, which assumes that the process is kept running until the assignable cause is discovered.

2) Policy II, which assumes that the process is shut down during the search for the assignable cause. With the aid of these two policies, the manufacturer can decide upon the appropriate models to be chosen for minimization of the process loss-cost.

The various incomes and costs that are associated with the operation of the process are: income when the process is in the in-control state, income when the process is in the out-of-control state, the cost of searching for an assignable cause when one exists, cost of searching for an assignable cause when none exists and the cost of maintaining the control chart.

3.4.2 The Loss-Cost Function. A production cycle is defined as the time period from beginning of production (or adjustment) to the detection or elimination of an assignable cause. The production cycle for the process models under operating policy I consists of four periods: 1) the in-control period, 2) the out-of-control period, 3) the time to take a sample and interpret the results, and 4) the time to find the assignable cause. Similarly, the production cycle under operating policy II also consists of four periods: 1) the in-control period, 2) the out-of-control period, 3) the search period due to false alarms, and 4) the search and repair period due to true alarms.

Considering the relevant income and cost parameters associated with each period of the production cycle, the expected cost per production cycle can easily be derived. Hence, the expected cost per unit time is defined as the ratio of the expected value of total cost incurred during the cycle to the expected length of the cycle.

CHAPTER 4

ECONOMIC DESIGN OF $\overline{X}$-CHARTS TO CONTROL NON-NORMAL PROCESS MEANS

In this Chapter Duncan's model is generalized using the Edgeworth approximation to the normal distribution.

To investigate the economic design of $\bar{x}$-charts, initially a basic single assignable cause process model under policy I is proposed and its expected loss-cost function is developed. An analytical solution to obtain the optimal value of the design parameters (i.e., sample size n, sampling interval s, control limits coefficient k) is not possible. An optimization technique based on Hooke and Jeeve's pattern search is developed to obtain the optimal design parameter values.

The fundamental assumptions in developing the process model under policy I are: (1) that the process is allowed to continue in operation, during the search for the assignable cause and (2) that the cost of eliminating the assignable cause is not charged against the net income for the production cycle. In many processes, these restrictions are unrealistic and it would be of interest to formulate a cost model based on different assumptions. Hence, a single assignable cause model for an $\bar{x}$-chart under policy II is also proposed and an expected loss-cost function is derived. The optimal design parameters are obtained by applying the same optimization technique developed for the cost model under policy I. A simplified scheme is also proposed for each of the cost models for selecting the design parameters so that the expected loss-cost is minimal for a specified level of consumer's risk.

Through numerical studies, a sensitivity analysis of the model, under policy II, is performed. Investigations are also made to examine the effects of errors in the estimation of data parameters on minimization of the loss-cost function for the proposed control plan.

Many production processes are affected by several assignable causes and, in such situations, a single assignable cause model is not applicable. The single assignable cause model under policy II is extended to a multiple assignable cause model. With numerical illustrations, it is demonstrated that a "matched" single assignable cause model can be proposed so that its optimal control plan approximates the exact optimum control plan for the original multiple assignable cause model obtained by a direct search technique. Hence, a simplified scheme such as that applied to the single assignable cause model is suggested for the "matched" single assignable cause model.

4.1 Characteristics of the Process Variable

To take into consideration the effects of non-normality on control chart design, it is assumed that the first four terms of the Edgeworth series expansion provide an adequate representation of the distribution of the quality characteristic [Gayen, 1953]. Denoting the quality characteristic by the random variable X, the probability density function (pdf) of the standardized variable $x = \dfrac{X - \mu_0}{\sigma}$ has the following form [Barton and Dennis, 1952; Kendall and Stuart, 1969]:

$$f(x|\mu_0) = \phi(x) - \frac{\sqrt{\beta_1}}{6}\phi^{(3)}(x) + \frac{(\beta_2-3)}{24}\phi^{(4)}(x) + \frac{\beta_1}{72}\phi^{(6)}(x) \qquad (4.1)$$

Define $\gamma_1 = E\{x^3\}$ and $\gamma_2 = E\{x^4\} - 3$. Recalling that $x = \dfrac{X - \mu_0}{\sigma}$, we have $\gamma_1 = \sqrt{\beta_1}$ and $\gamma_2 = \beta_2 - 3$. γ_1 is called the coefficient of skewness. Positive values of γ_1 usually correspond to pdf's with dominant tails on the right side and negative values to tails on the left side. γ_2 is called the coefficient of excess (or kurtosis). For normal distribution both γ_1 and γ_2 are equal to zero. In this study, γ_1 and γ_2 are to be used as the measures of non-normality parameters. The equation (4.1) can be represented in terms of γ_1 and γ_2 as follows

$$f(x|\mu_0) = \phi(x) - \frac{\gamma_1}{6}\phi^{(3)}(x) + \frac{\gamma_2}{24}\phi^{(4)}(x) + \frac{\gamma_1^2}{72}\phi^{(6)}(x) \qquad (4.2)$$

Utilizing the well known relations [Gayen, 1953]

$\gamma_1(\bar{x}) = \dfrac{\gamma_1(x)}{\sqrt{n}}$ and $\gamma_2(\bar{x}) = \dfrac{\gamma_2(x)}{n}$, the pdf of the standardized sample average $y = \dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$ is given by

$$f(\bar{x}|\mu_0) = f_n(y) = \phi(y) - \frac{\gamma_1}{6\sqrt{n}}\phi^{(3)}(y) + \frac{\gamma_2}{24n}\phi^{(4)}(y) + \frac{\gamma_1^2}{72n}\phi^{(6)}(y), \quad (4.3)$$

where $\phi(x)$ is the pdf of the standardized normal variable x and

$$\phi^{(r)}(x) = (\frac{d}{dx})^r \ \phi(x).$$

Barton and Dennis [1952] studied the values of γ_1 and γ_2, through $\beta_1 = \gamma_1^2$ and $\beta_2 = \gamma_2 + 3$, which make the Edgeworth series non-negative and unimodal. A graph showing the regions in which non-negative and unimodal properties are true in the (β_1, β_2) plane was also given by them. Berndt [1957] made a similar investigation when only γ_1 was used.

The conditions given by Barton and Dennis on γ_1 and γ_2 regarding the positive definiteness and unimodality of f(x) are assumed in the present study. Further it is assumed that $\beta_2 \geq 1 + \beta_1$.

4.2 Single Assignable Cause Model - Policy I

Similar to Duncan's model for normal processes, a single assignable cause model for non-normal processes is presented in this section. Considering a numerical example, optimal values of the design parameters that are obtained using a direct search technique are compared with the corresponding approximate values provided by Nagendra and Rai [1971]. The approximate solution procedure, used by Nagendra and Rai, has also been improved. Furthermore, a simplified scheme is developed based on a prescribed 90 or 95 percent probability that the defective items found in a sample fall outside the control limits when the process is out of control.

4.2.1 Formulation of Loss-Cost Function. In order to formulate the loss-cost function for the economic design of an $\bar{x}$-chart, the characteristics that are to be derived are as follows.

i) The density function of the occurrence of the assignable cause is given by

$$f_{T_a}(t) = \lambda e^{-\lambda t}, \quad \lambda > 0, \quad t \geq 0$$
$$= 0 \qquad \text{otherwise}.$$

The average time required for the assignable cause to occur is

$$E(T_a) = \int_0^\infty t\, \lambda e^{-\lambda t} = 1/\lambda. \tag{4.4}$$

Hence the process remains in an in-control state with an average length of time of $1/\lambda$ hours.

ii) If the samples are taken at intervals of s hours, then, given the occurrence of the assignable cause in the interval between the jth and j+1st samples or between js and (j+1)s hours, the average time of occurrence of the assignable cause within an interval between samples is given by

$$\tau = E[T_b|A] = \frac{\int_{js}^{(j+1)s} \lambda e^{-\lambda t}\,(t-js)\,dt}{P(A)},$$

where T_b denotes the mean time of occurrence within an intersample interval and A denotes the event that the assignable cause occurs in the interval. Thus,

$$\tau = \frac{\int_{js}^{(j+1)s} \lambda e^{-\lambda t}\,(t-js)\,dt}{\int_{js}^{(j+1)s} \lambda e^{-\lambda t}\,dt}$$

$$= \frac{e^{-\lambda js} \int_0^s \lambda T_1 \, e^{-\lambda T_1} \, dT_1}{e^{-\lambda js} \int_0^s \lambda e^{-\lambda T_1} \, dT_1},$$

where $T_1 = t - js$. Thus

$$\tau = \frac{1 - (1 + \lambda s) e^{-\lambda s}}{\lambda \, (1 - e^{-\lambda s})} \tag{4.5}$$

$$= \frac{s}{2} - \frac{\lambda s^2}{12} + 0 \, (\lambda^3 s^4) \tag{4.6}$$

iii) Under the assumption that the control limits are set at $\pm k$ standard deviations of the sample mean from the target value, the probability that the assignable cause is detected when the process is in the out-of-control state is

$$P = 1 - \beta = \int_{-\infty}^{\mu_0 - k\sigma/\sqrt{n}} f(\bar{x}|\mu_1) \, d\bar{x} + \int_{\mu_0 + k\sigma/\sqrt{n}}^{\infty} f(\bar{x}|\mu_1) \, d\bar{x}, \tag{4.7}$$

where μ_1 is the process average when it is out of control and is equal to $\mu_0 + \delta\sigma$. Integrating (4.7), we obtain

$$\begin{aligned} P = 1 - \beta = {} & \Phi(-k-\delta\sqrt{n}) - \frac{\gamma_1}{6\sqrt{n}} \phi^{(2)}(-k-\delta\sqrt{n}) + \frac{\gamma_2}{24n} \phi^{(3)}(-k-\delta\sqrt{n}) \\ & + \frac{\gamma_1^2}{72n} \phi^{(5)}(-k-\delta\sqrt{n}) + 1 - \Phi(k-\delta\sqrt{n}) + \frac{\gamma_1}{6\sqrt{n}} \phi^{(2)}(k-\delta\sqrt{n}) \\ & - \frac{\gamma_2}{24n} \phi^{(3)}(k-\delta\sqrt{n}) - \frac{\gamma_1^2}{72n} \phi^{(5)}(k-\delta\sqrt{n}). \end{aligned} \tag{4.8}$$

P is known as the probability of true alarm. β is known as the probability of a

Type II error. Under the normality assumption, equation (4.8) is reduced to

$$P = 1 - \beta = 1 - \Phi(k-\delta\sqrt{n}) + \Phi(-k-\delta\sqrt{n}) . \qquad (4.9)$$

iv) When the assignable cause has not occurred, that is, when the process is in control, the probability of a sampling point falling outside the control limit or the probability of the false alarm is

$$\alpha = 1 - \int_{\mu_0 - k\sigma/\sqrt{n}}^{\mu + k\sigma/\sqrt{n}} f(\overline{x}|\mu_0)\, d\overline{x}$$

$$= 1 - [\ \Phi(k) - \frac{\gamma_1}{6\sqrt{n}}\phi^{(2)}(k) + \frac{\gamma_2}{24n}\phi^{(3)}(k) + \frac{\gamma_1^2}{72n}\phi^{(5)}(k)$$

$$- \Phi(-k) + \frac{\gamma_1}{6\sqrt{n}}\phi^{(2)}(-k) - \frac{\gamma_2}{24n}\phi^{(3)}(-k) - \frac{\gamma_1^2}{72n}$$

$$\phi^{(5)}(-k)]$$

$$= 1 - [\ \Phi(k) - \frac{\gamma_1}{6\sqrt{n}}\phi^{(2)}(k) + \frac{\gamma_2}{24n}\phi^{(3)}(k) + \frac{\gamma_1^2}{72n}\phi^{(5)}(k) - (1 - \Phi(k))$$

$$+ \frac{\gamma_1}{6\sqrt{n}}\phi^{(2)}(k) + \frac{\gamma_2}{24n}\phi^{(3)}(k) + \frac{\gamma_1^2}{72n}\phi^{(5)}(k)]$$

$$= 1 - \Phi(k) - \frac{\gamma_2}{12n}\phi^{(3)}(k) - \frac{\gamma_1^2}{36n}\phi^{(5)}(k) + 1 - \Phi(k)$$

$$= 2 - 2\Phi(k) - \frac{1}{36n}[3\gamma_2\ \phi^{(3)}(k) + \gamma_1^2\ \phi^{(5)}(k)]$$

$$= 2\Phi(-k) - \frac{1}{36n}\ [3\gamma_2 \cdot \phi^{(3)}(k) + \gamma_1^{\ 2}\ \phi^{(5)}(k)] \ldots \tag{4.10}$$

$$= 2\Phi(-k) - \alpha_c\ , \tag{4.11}$$

where $\alpha_c = \frac{1}{36n}\ [3\gamma_2\ \phi^{(3)}(k) + \gamma_1^{\ 2}\ \phi^{(5)}(k)]$.

α is the probability of a Type I error. For the normal case α is simply equal to $2\Phi(-k)$. This can also be noted from equation (4.11) that as $n \to \infty$, $\alpha \to 2\Phi(-k)$.

v) After the occurrence of the assignable cause, the probability that it will be detected on the jth sample is $(1-P)^{j-1}P$, which is the probability density function of the geometric distribution; then the expected number of samples taken before the assignable cause is detected is

$$= \sum_{j=1}^{\infty} j\ (1-P)^{j-1}P$$

$$= \frac{1}{P}\ . \tag{4.12}$$

Therefore, the expected time for the process to be out of control before a sample point falls outside the control limits is

$$\frac{s}{P} - \tau. \tag{4.13}$$

vi) The time required to take samples and to interpret their results has an average length of en.

vii) Let the time required to find the assignable cause have an average length of D.

Then the following statements are true.

viii) The expected length of time during which the process is out of control before the search for the assignable cause is concluded is given by

$$\frac{s}{p} - \tau + en + D. \qquad (4.14)$$

ix) The expected production cycle length of in-control/out-of-control is

$$E(C) = \frac{1}{\lambda} + \frac{s}{p} - \tau + en + D \,. \qquad (4.15)$$

Let V_0 be the per-hour income when the process is in control, V_1, the per-hour income when the process is out of control, b+cn the cost of taking a sample, V the expected cost of searching for the assignable cause when none exists and W the expected cost of searching for an assignable cause when it exists.

x) The expected number of false alarms per cycle before the process goes out of control will be α times the expected number of samples taken in the 'in-control' period. The expected number of false alarms per cycle will thus be

$$= \alpha \sum_{j=0}^{\infty} \int_{js}^{(j+1)s} j\lambda e^{-\lambda t}\, dt$$

$$= \alpha \sum_{j=0}^{\infty} j \, [e^{-j\lambda s} - e^{-(j+1)\lambda s}]$$

$$= \alpha \, [(e^{-\lambda s} - e^{-2\lambda s}) + 2(e^{-2\lambda s} - e^{-3\lambda s}) + 3(e^{-3\lambda s} - e^{-4\lambda s}) + ..]$$

$$= \alpha \, [e^{-\lambda s} + e^{-2\lambda s} + e^{-3\lambda s} + e^{-4\lambda s} + \dots]$$

$$= \frac{\alpha \, e^{-\lambda s}}{1 - e^{-\lambda s}} \; . \qquad (4.16)$$

Then the expected net income derived from the production cycle is

$$\frac{V_0}{\lambda} + V_1 \left(\frac{s}{p} - \tau + en + D\right) - W - V \alpha \, e^{-\lambda s}/(1-e^{-\lambda s}) - (b+cn)\frac{E(C)}{s} \; . \qquad (4.17)$$

Hence, the average net income per hour is

$$I = \frac{\text{expression (4.17)}}{\text{expression (4.15)}} \; .$$

Defining L as $L = V_0 - I$, we get after suitable simplification,

$$L = \frac{\lambda U B_1 + V B_0 + \lambda W}{1 + \lambda B_1} + \frac{(b+cn)}{s} \; , \qquad (4.18)$$

where $U = V_0 - V_1$,

$$B_1 = \frac{s}{p} - \tau + en + D \; ; \qquad (4.19)$$

and $B_0 = \alpha \, (1 - \lambda\tau)/s$ $\qquad (4.20)$

The function L represents the loss-cost per hour for the present model. The problem is to minimize the per-hour loss-cost function L with respect to the design parameters n, s and k.

4.2.2 Determination of the Optimal Design Parameters. An explicit solution of n, s and k is not possible. However, for a specified value of k, an approximate value of n can be obtained. A value of s can be approximated using the values of n and k. This can be accomplished as follows.

For simplicity we assume that $\delta > 0$. Thus the terms containing $(-k-\delta\sqrt{n})$ may be neglected in equation (4.8), reducing it to

$$P = 1 - \Phi(k-\delta\sqrt{n}) + \frac{\gamma_1}{6\sqrt{n}}\,\phi^{(2)}(k-\delta\sqrt{n}) - \frac{\gamma_2}{24n}\,\phi^{(3)}(k-\delta\sqrt{n})$$

$$- \frac{\gamma_1^2}{72n}\,\phi^{(5)}(k-\delta\sqrt{n}), \qquad \delta > 0 . \tag{4.21}$$

Letting $k-\delta\sqrt{n} = \zeta$,

$$P = 1 - \Phi(\zeta) + \frac{\gamma_1}{6\sqrt{n}}\,\phi^{(2)}(\zeta) - \frac{\gamma_2}{24n}\,\phi^{(3)}(\zeta)$$

$$- \frac{\gamma_1^2}{72n}\,\phi^{(5)}(\zeta) . \tag{4.22}$$

Moreover, λ is a small quantity and hence λB_1 is small compared to unity. Therefore, the term λB_1 can be omitted from the first denominator of equation (4.18). Thus we have

$$L \simeq L' = \lambda U B_1 + V B_0 + \lambda W + \frac{(b+cn)}{s} . \tag{4.23}$$

Then L' is partially differentiated with respect to n and s and equating the derivatives to zero, gives the following equations.

$$\frac{\partial L'}{\partial n} = \lambda U \frac{\partial B_1}{\partial n} + V \frac{\partial B_0}{\partial n} + \frac{c}{s} = 0 \qquad (4.24)$$

$$\frac{\partial L'}{\partial s} = \lambda U \frac{\partial B_1}{\partial s} + V \frac{\partial B_0}{\partial s} - \frac{b+cn}{s^2} = 0, \qquad (4.25)$$

where

$$\frac{\partial B_1}{\partial n} = -\frac{s}{p^2}\frac{\partial p}{\partial n} + e$$

$$\frac{\partial B_0}{\partial n} \approx \frac{\partial \alpha}{\partial n}\cdot\frac{1}{s} = \frac{\alpha_c}{ns}$$

$$\frac{\partial p}{\partial n} = \frac{\delta}{2\sqrt{n}}\phi(\zeta) + \frac{1}{144n^2}[-12\gamma_1 \{\delta n\, \phi^{(3)}(\zeta) + \sqrt{n}\, \phi^{(2)}(\zeta)\}$$

$$+ 3\gamma_2 \{\delta\sqrt{n}\, \phi^{(4)}(\zeta) + 2\, \phi^{(3)}(\zeta)\} + \gamma_1^2 \{\delta\sqrt{n}\, \phi^{(6)}(\zeta)$$

$$+ 2\, \phi^{(5)}(\zeta)\}]$$

$$\frac{\partial B_1}{\partial s} \approx \frac{1}{p} - \frac{1}{2} + \frac{\lambda s}{6}$$

$$\frac{\partial B_0}{\partial s} \approx -\frac{\alpha}{s^2}$$

From (4.24) and (4.25):

$$\lambda U\left(-\frac{s}{p^2}\frac{\partial P}{\partial n}+e\right)+V\frac{\alpha_c}{ns}+\frac{c}{s}=0 \tag{4.26}$$

$$\lambda U\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{\alpha V}{s^2}-\frac{b+cn}{s^2}=0 \tag{4.27}$$

From (4.27):

$$s=[(\alpha V+b+cn)/\{\lambda U\left(\frac{1}{p}-\frac{1}{2}\right)\}]^{1/2} \tag{4.28}$$

Substituting this value of s in equation (4.26):

$$\alpha_c V+n\left(c-\frac{\alpha V+b+cn}{p^2\left(\frac{1}{p}-\frac{1}{2}\right)}\frac{\partial p}{\partial n}+\lambda Ue\right)=0 \tag{4.29}$$

For a specified value of k, the value of n which satisfies equation (4.29) can be taken as an approximate sample size. Substituting this value of n in equation (4.28), an approximate value of s can be evaluated.

Similar types of expressions for n and s have been derived by Nagendra and Rai [1971]. However, in these derivations, the term λUe in equation (4.29) was not accounted for. They took the partial derivatives of L, instead of L', and performed much more complicated computations than the procedure described above. Moreover, they considered the derived values for n and s as the optimum solutions for the design parameters, which did not seem to be realistic. Preclusion of the term λUe in equation (4.29) may have some serious effects on the approximate solutions of n and s. Inclusion of λUe in their solution for n, could have improved the accuracy of the approximate control plan. Nevertheless, keeping one of the design variables fixed, the optimum design of the control plan cannot be achieved.

In the case of controlling the mean of a normal process, it may be feasible for practical purposes to specify the value of k, so as to attain a certain level of probability of Type I error. The reason behind this is that the expression for the probability of Type I error under normality assumption is independent of sample size n. But, under the non-normality assumption, the probability of Type I error is dependent on both sample size n and control limit coefficient k. The probability of Type II error for controlling both normal and non-normal means is a function of n and k. Therefore, for a specified value of k, the approximate value of the design parameters will not provide an optimum control plan. Therefore, a direct search technique is desirable to obtain the exact optimum control plan. Through numerical illustration it will be shown later in this Chapter how the approximate solutions are deviating from their corresponding optimum solutions obtained by using the direct search technique. However, for a specified value of k, approximate values of n and s could be used as a good initial point for a direct search.

Direct Search Solution

The pattern search technique of Hooke and Jeeves [1961] is employed to minimize the expected per hour loss-cost associated with the operation of an $\overline{x}$-chart. Pattern search is a direct search technique for minimizing function f, of a vector-valued variable $\underline{x}$. For the present case, $f = L$ and $\underline{x} = (n,s,k)$ is a three-dimensional vector with the components of $\underline{x}$ equal to the design parameters.

The search starts with a local exploration in small steps around the starting point. If the exploration is a success, i.e., if the loss-cost reduces during local exploration, the step size grows; if the exploration is a failure, the step size is reduced. If a change of direction is necessary, the method starts over again with a new pattern. The search is terminated when the step size is reduced to a specified value, or when the number of iterations equals to a predetermined value, whichever occurs first. However, due to the characteristics of function L, some modifications to the method have to be made in order to account for the inherent constraints on the sample size, and on the probabilities of Type I and Type II errors. These modifications are as follows.

i) n must be an integer value,

ii) the expressions for P and α, i.e., equation (4.8) and (4.11), are non-negative for given values of γ_1, γ_2 and δ.

The computer program 'PROGRAM XBAR' which incorporates these modifications is given in Appendix I.

In the past, under the normality assumption, the value of k was chosen to be either 2.5 or 3 in the conventional design of $\bar{x}$-charts [Shewhart, 1931; Dudding and Jennett, 1942]. One of these two values as an initial value of k is chosen for the search. The root of the equation (4.29) is then obtained utilizing external FUNCTION F1 and SUBROUTINE ZREALI.

Then this root is considered as an initial value of n. The initial value of s is then evaluated using equation (4.28). The optimum values of the design parameters are then obtained by the developed modified pattern search 'SUBROUTINE SUB'. During the search, the functional value is evaluated using SUBROUTINE COST.

Numerical Examples

To obtain the optimal design parameters, the search method assumes that the objective function is convex. Since it is not possible to analytically investigate the convexity of L, some analysis of its behaviour was conducted through numerical studies.

One such study is presented in Table 4.1 of Example 4.1. This indicates that the surface of L is approximately convex in the region around the optimal value.

Example 4.1 Consider a process having non-normality parameters γ_1 = -0.5 and γ_2 = -0.5, the shift parameter δ = 1, and the rate of occurrence of the assignable cause λ = 0.01. The cost parameters are assumed as follows: V_0 = 150, V_1 = 50, V = 50, W = 25, b = 0.5, c = 0.1, D = 2.0 and e = 0.05. The values of the loss-cost function L and the design parameters in the neighbourhood of the optimal point are shown in Table 4.1. The loss-cost function assumes a minimum value of L^* = 5.1953 at the following design parameter values:

sample size n = 12

sampling interval s = 1.7147

Table 4.1 Values of the Loss-Cost Function and Design Parameters in the Neighbourhood of Minimum Position

(λ=0.01, V_0=150, V_1=50, V=50, W=25, b=0.5, c=0.1, e=0.05, δ=1.0, γ_1=-0.5 γ_2=-0.5 , D = 2.0)

n	s	k	L
10	1.6147	2.5648	5.2107
11	1.7345	2.5675	5.1974
12	1.7147	2.6336	5.1953
13	1.7058	2.6745	5.2138
14	1.9342	2.6298	5.2377
15	1.9081	2.7491	5.2691

control limit coefficient $k = 2.6336$.

With the assumption of the convexity of the objective function, the optimal solution for the design parameters was determined for a wide range of non-normality parameters γ_1 and γ_2, and of the shift parameter δ. The cost parameters were fixed throughout the optimization procedure. For numerical illustrations, the optimal solutions for three sets of data are shown in Table 4.2. The values of δ, γ_1 and γ_2 that are assigned to the three sets are as follows:

Set 1: $\delta = 0.5$ $\gamma_1 = -0.5$ and $\gamma_2 = -0.5$;

Set 2: $\delta = 1.0$ $\gamma_1 = 1.0$ and $\gamma_2 = 0.5$;

Set 3: $\delta = 2.0$ $\gamma_1 = 0.5$ and $\gamma_2 = 1.0$.

The relevant cost parameters and the value of λ associated with Table 4.2 are the same as those in Table 4.1.

The optimal solutions obtained using direct search techniques are compared with the 'improved approximate' solutions computed using equations (4.29) and (4.28). In addition, comparison is also shown with the results obtained using the approximate procedure proposed by Nagendra and Rai.

Results in Table 4.2 show that in all cases the proposed search optimization method yields lower loss-costs than both the 'approximate' and 'improved approximate' methods where k is considered as a fixed quantity. Moreover, in the optimal search method, no terms are neglected for finding the solution. Therefore, it gives accurate and reliable optimum values for the design parameters.

Table 4.2 Comparison of Results by Approximate Solution and Optimal Solution

SET	k	δ	γ_1	γ_2	Approximate Solution						Optimal Solution			
					Nagendra and Rai's Algorithm			Proposed Algorithm						
					n	s	L	n	s	L	n*	s*	k*	L*
1	3.0	0.5	-0.5	-0.5	66	3.30	9.401	49	2.49	8.798	24	2.08	2.15	7.542
	2.5				50	3.01	8.389	36	2.24	7.909				
	2.0				42	3.33	8.119	32	2.82	7.720				
2	3.0	1.0	0.5	0.5	19	2.06	5.488	16	1.79	5.387	12	1.80	2.59	5.225
	2.5				16	2.18	5.384	13	1.95	5.2359				
	2.0				14	2.78	5.742	12	2.65	5.645				
3	3.0	2.0	0.5	1.0	6	1.56	4.111	5	1.47	4.041	5	1.48	3.10	4.039
	2.5				5	1.82	4.270	5	1.80	4.298				
	2.0				4	2.51	4.853	4	2.49	4.867				

4.2.3 Development of a Simplified Scheme. The essential characteristic of this plan is to specify P, the probability of a true alarm and its detection to be at least at a given level (typical values are .90 or .95). This probability corresponds to a point on the OC curves, specifying the maximum level for the consumer's risk (typical values are .10 or .05). Thus, the optimal values of design parameters could be obtained by minimizing the loss-cost function developed in equation (4.18), provided that the consumer's risk does not exceed a maximum level in order to attain a specified level of protection against deteriorated quality. From this point of view, the scheme that will be developed is a semi-economic scheme. The condition $P = 0.90$ or $P = 0.95$ is intuitively reasonable because it enables the manufacturer to detect an assignable cause rather quickly, on the average about 1.1 or 1.05 samples after its occurrence, so as to reduce the loss due to prolonged production of a large proportion of defectives.

For the sake of mathematical simplicity and practical convenience, some approximations in the minimization procedure are made. In practice, λ is a very small quantity, say $\lambda = 0.01$, and λB_1 is small compared with unity. Therefore, λB_1 can be omitted from the first denominator of equation (4.18). Thus, the approximate loss-cost function is

$$L \approx L' = \lambda B_1 U + \lambda W + \frac{V\alpha}{s} + \frac{b+cn}{s}, \tag{4.30}$$

where

$$B_1 = \left(\frac{1}{p} - \frac{1}{2} + \frac{\lambda s}{12}\right) s + en + D$$

Denoting

$$\delta\sqrt{n} - k = a \tag{4.31}$$

and eliminating the terms containg $-\delta\sqrt{n} - k$ in the expression for P, the equation (4.8) becomes

$$P = \Phi(a) + \frac{\gamma_1}{6\sqrt{n}}\,\phi^{(2)}(a) + \frac{\gamma_2}{24n}\,\phi^{(3)}(a) + \frac{\gamma_1^2}{72n}\,\phi^{(5)}(a) \tag{4.32}$$

From equation (4.31),

$$n = \frac{(a + k)^2}{\delta^2} \tag{4.33}$$

substituting n from equation (4.33) in equation (4.30) and noting that P is a constant appearing in B_1, the optimum values of k and s are obtained by equating to zero the partial derivatives of L' with respect to k and s:

$$\frac{\partial L'}{\partial k} = \lambda U \frac{\partial B_1}{\partial k} + \frac{V}{s}\,\frac{\partial \alpha}{\partial k} + \frac{2c(a+k)}{s\delta^2} = 0 \tag{4.34}$$

$$\frac{\partial L'}{\partial s} = \lambda U \left(\frac{1}{p} - \frac{1}{2}\right) - \frac{\alpha V}{s^2} - \frac{(b+cn)}{s^2} = 0 \tag{4.35}$$

Equations (4.34) and (4.35) yield the following

$$2(c + \lambda eUs)(a + k) + \delta^2 V \frac{\partial \alpha}{\partial k} = 0 \tag{4.36}$$

$$\lambda U \left(\frac{1}{p} - \frac{1}{2}\right) s^2 - \alpha V - (b+cn) = 0 \tag{4.37}$$

From equation (4.36):

$$\frac{-2(a+k)}{\frac{\partial\alpha}{\partial k}} = \frac{\delta^2\ V}{c+\lambda Ues} , \tag{4.38}$$

where

$$\frac{\partial\alpha}{\partial k} = -2\ \phi(k) - \frac{\partial\alpha_c}{\partial k}$$

$$= -2\ [\phi(k) + \frac{\partial\alpha_c}{2\partial k}]\ .$$

Hence,

$$\frac{(a+k)}{\phi(k) + \frac{\partial\alpha_c}{2\partial k}} = \frac{\delta^2 V}{c+\lambda Ues} \tag{4.39}$$

Let

$$\frac{(a+k)}{\phi(k) + \frac{\partial\alpha_c}{2\partial k}} = A^{**} \tag{4.40}$$

Thus,

$$A^{**} = \frac{\delta^2\ V}{c+\lambda Ues} \tag{4.41}$$

From equation (4.37):

$$s = \{(\alpha V + b + cn)/\{\lambda\ U(P^{-1} - \tfrac{1}{2})\}\}^{1/2} \tag{4.42}$$

In equation (4.41), the term λUes is a small quantity because e is often small and thus could be omitted, as Duncan [1956] has suggested.

But Goel, et al. [1968] have shown that the effects of omitting this term may be serious if e happens to be moderately large. The presence of s in this term makes equations (4.41) and (4.42) intractable. As suggested by Chiu and Wetherill [1974], λUes is replaced by λUe, which is a poor approximation, but which turns out to be better than the complete omission of the term. Thus, equation (4.41) is rewritten as

$$A^{**} = \frac{\delta^2 V}{c + \lambda Ue} \quad (4.43)$$

The values of n, s and k for the semi-economic plan are thus the solutions of equations (4.32), (4.33), (4.42) and (4.43).

By varying the values of k and n that satisfy equations (4.32) and (4.33), one attains an acceptable level for P. Those values of k and n which also satisfy equation (4.43) are to be selected and used in equation (4.42) to determine the corresponding value of s.

For practical application, a series of tables are constructed in which for given values of δ, γ_1 and γ_2, the optimal values of k and n are listed corresponding to the value of A^{**}. Also listed are the values of α and $(\frac{1}{p} - \frac{1}{2})$ which are used for evaluating s from equation (4.42). The application of such tables is demonstrated through the following numerical example.

Example 4.2 Consider the situation where sample means are non-normally distributed with non-normality parameters γ_1 and γ_2. Suppose that the process is kept running until an assignable cause is discovered;

then the loss-cost function is given by equation (4.18). The values of δ, λ and cost parameters are given as follows:

$\delta = 2$, $\lambda = 0.01$, $V_0 = 150$, $V_1 = 50$, $V = 50$, $W = 25$, $b = 0.5$, $D=2.0$, $c = 0.1$ and $e = 0.05$.

To determine the economic plan with $P \geq 0.95$, Table 4.3 will be applicable.

Procedure

1. Calculate A**. Here $A^{**} = \dfrac{\delta^2 V}{c + \lambda Ue} = 1333$.

2. Determine k and n. From Table 4.3, we find that the closest value to $A^{**} = 1333$, is $A^{**} = 1295$, and the corresponding value of k is 3.1 and n is 6.

3. Evaluate s. We observe that $\alpha = 0.003$ and $(\frac{1}{P} - \frac{1}{2}) = 0.531$.

   Thus, $s = \{(\alpha V + b + cn)/[\lambda U(\frac{1}{P} - \frac{1}{2})]\}^{1/2}$

   $= 1.5342$

4. Evaluate B_1. $B_1 = (\frac{1}{P} - \frac{1}{2} + \frac{\lambda s}{12})s + en + D$

   $= (0.531 + 0.00127) \times 1.5342 + 0.30 + 2$

   $= 3.1166$

5. Evaluate L. Equation (4.18) can be well approximated by

$$L = \frac{\lambda UB_1 + \alpha V/s + \lambda W}{1 + \lambda B_1} + \frac{b + cn}{s}$$

$= 4.076$

Table 4.3 Simplified Scheme for Determination of Control Parameters for the Economic Design of $\bar{X}$-Chart to Control Non-Normal Means for Which P > 0.95 (δ=2, γ_1=0.5, and γ_2=1.0)

δ =2		γ_1 =0.5		γ_2 =1.0
k	n	α	(1/P-1/2)	A**
1.0	2	0.30219	0.523	11
1.1	2	0.25709	0.531	13
1.2	2	0.21716	0.540	14
1.3	2	0.18218	0.552	16
1.4	3	0.15509	0.513	23
1.5	3	0.12847	0.517	27
1.6	3	0.10577	0.523	32
1.7	3	0.08659	0.530	38
1.8	3	0.07054	0.540	46
1.9	3	0.05722	0.551	56
2.0	4	0.04606	0.517	78
2.1	4	0.03689	0.522	96
2.2	4	0.02945	0.529	119
2.3	4	0.02344	0.538	149
2.4	4	0.01861	0.548	188
2.5	5	0.01428	0.519	262
2.6	5	0.01120	0.525	334
2.7	5	0.00876	0.532	426
2.8	5	0.00684	0.541	545
2.9	6	0.00506	0.518	773
3.0	6	0.00390	0.523	999
3.1	6	0.00300	0.531	1295
3.2	6	0.00230	0.539	1682
3.3	6	0.00175	0.550	2189
3.4	7	0.00124	0.524	3195
3.5	7	0.00093	0.532	4206

Note: A** in this Table is as defined by equation (4.43).

The exact solution to this problem is obtained through the direct search technique, which yields $n = 5$, $s = 1.48$, $k = 3.10$ and $L = 4.0390$. The variation is only 0.92 percent in the loss-cost function. It is interesting to note that a simplified scheme provides less cost than the cost obtained by the approximate solution method according to Nagendra and Rai.

To compute the value of A** and corresponding value of n, α and $(\frac{1}{P} - \frac{1}{2})$, a computer program 'SEMIXBAR' is developed and is presented in Appendix II.

4.3 Single Assignable Cause Model - Policy II

4.3.1 Formulation of Loss-Cost Function for Policy II. In practice, in some production processes, the machine has to be shut down during the search for ~~the~~ assignable cause; the repair cost is charged against the net income from the process, and the time to repair the process is taken into consideration. In order to develop the loss-cost functions, the following additional terms, in conjunction with the terms defined in sections 4.2.1 and 4.2.2, are used. Let the expected length of search time be τ_s hours, and the expected search cost be k_s. If the assignable cause does not exist, production is resumed after the search. If the assignable cause actually exists, it can always be detected and eliminated, but it takes a further expected repair time of τ_r hours and a further expected repair cost of k_r to restore the process in-control state. The process starts afresh in-control after the reparation. It is assumed that the time for taking samples is negligible.

Then it is straightforward to see that the average length of a production cycle consists of four parts:

1) The in-control period, with an average length of $1/\lambda$ hours,

2) The out-of-control period, with an average length of $\frac{s}{p} - \tau$,

3) The search time due to a false alarm, $\alpha\tau_s(\frac{1}{\lambda} - \tau)/s$,

4) The search and repair times due to the true alarm, $\tau_s + \tau_r$.

Thus the expected length of a production cycle under operating policy II is

$$\frac{1}{\lambda} + (\frac{s}{p} - \tau) + \alpha\tau_s\,(\frac{1}{\lambda} - \tau)/s + \tau_s + \tau_r\ . \tag{4.44}$$

Similarly, the expected income from a production cycle is:

$$\frac{V_0}{\lambda} + V_1\,(\frac{s}{p} - \tau) - \alpha k_s(\frac{1}{\lambda} - \tau)/s - (k_s + k_r)\ . \tag{4.45}$$

Hence the expected net income per hour is

$$I = \frac{\text{Expression (4.45)}}{\text{Expression (4.44)}}\ .$$

Defining $L = V_0 - I$ and after suitable simplification, the loss-cost function for Policy II is given as

$$L = \frac{\lambda B_1\ U + \lambda W + VB_0 + (b+cn)(1 + \lambda B_1)/s}{1 + \lambda B_1 + \tau_s B_0 + \lambda(\tau_r + \tau_s)} \tag{4.46}$$

where $U = V_0 - V_1$, $V = k_s + V_0\tau_s$, $W = k_r + k_s + V_0(\tau_r + \tau_s)$,

$$B_1 = \frac{s}{P} - \tau \quad \text{and}$$

$$B_0 = \alpha(1 - \lambda\tau)/s .$$

τ, α and P are defined in equations (4.5), (4.11) and (4.21) respectively. The function L represents the loss-cost per hour for the present model and is a function of the three design variables n, s and k. As in the economic design for the control plan under Policy I, the problem of an economic design for control plan under Policy II is the determination of the values of n, s and k for which L is minimum.

4.3.2 An Exact Algorithm. In order to determine the optimum values of the design parameters by minimizing the loss-cost function L, the algorithm that has been proposed in section 4.2.2 for Policy I is also recommended for Policy II.

4.3.3 Development of a Simplified Scheme for Policy II. A comparison between equations (4.18) and (4.46) shows that they have a similar mathematical form and that (4.18) appears to be a particular case of (4.46) when $\tau_r = \tau_s = 0$. Thus, following the same arguments of section 4.2.3 for the development of a simplified scheme for the present model, the following two equations can be derived as follows.

(a) Ignore λB_1 from the numerator and $\lambda(B_1 + \tau_r + \tau_s) + \tau_s B_0$ from the denominator in equation (4.46).

(b) Differentiate the resulting expression with respect to s and k; and equate to zero. Thus,

$$s = \{(\alpha\ V + b + cn)/\lambda U(p^{-1} - \tfrac{1}{2})\}^{1/2} \tag{4.47}$$

$$A^{**} = \frac{\delta^2 V}{c} \tag{4.48}$$

where the term A** is defined by equation (4,41). Hence, the tables which are constructed for a simplified scheme under Policy I, are also applicable for the present model. Application of one of such tables to the semi-economic design of an $\bar{x}$-chart to control non-normal process means under Policy II is shown through the following numerical example.

Example 4.3: Consider the case where the sample means are non-normally distributed with parameters $\gamma_1 = 1.0$ and $\gamma_2 = 2.0$. Suppose that the process is shut-down during the search for the assignable cause; the loss-cost function is then given by equation (4.46). The values of the shift parameter, the rate of occurrence of the assignable cause and the cost parameters are given as follows

$\delta = 2$, $\lambda = 0.01$, $V_0 = 100$, $V_1 = -100$, $k_r = 20$, $k_s = 10$,
$\tau_s = 0.2$, $\tau_s = 0.1$, $b = 0.5$ and $c = 0.1$.

To determine the economic plan with $P \geq 0.90$, Table 4.4 is applicable.

Procedure:

To make use of Table 4.4, first obtain the quantities that are needed for the loss-cost function, equation (4.46). These are as follows:
$\delta = 2.0$, $U = 200$, $\lambda = 0.01$, $V = 20$, $W = 60$.

Table 4.4 Simplified Scheme for Determination of Control Parameters for the Economic Design of $\bar{X}$-Chart to Control Non-Normal Means for Which $0.90 \leq P \leq 0.95$ (δ=2, γ_1=1.0, and γ_2=2.0)

δ =2		γ_1 =1.0		γ_2 =2.0
k	n	α	(1/P-1/2)	A**
1.2	2	0.23014	0.555	15
1.5	2	0.11490	0.571	21
1.6	2	0.09312	0.596	26
2.0	3	0.03650	0.558	77
2.1	3	0.03471	0.571	92
2.2	3	0.02858	0.594	117
2.6	4	0.01316	0.568	321
2.7	4	0.01103	0.589	394
3.0	5	0.00580	0.558	818
3.1	5	0.00480	0.576	946
3.2	5	0.00396	0.598	1125
3.4	6	0.00231	0.556	1917
3.5	6	0.00185	0.573	2314

Note: A** in this Table is as defined by equation (4.48).

1. Calculate A^{**}. $A^{**} = \frac{\delta^2 V}{c} = \frac{80}{0.1} = 800$

2. Determine k and n. From Table 4.4, we find that the closest value to $A^{**} = 800$, is $A^{**} = 818$, which corresponds to $k = 3$. Further, we find that $n = 5$ and $\alpha = 0.00580$.

3. Evaluate s. Observe that $P^{-1} - \frac{1}{2} = 0.558$. Thus,

   $s = \{(\alpha V + b + cn)/[\lambda M(P^{-1} - \frac{1}{2})]\}^{1/2} = 1.0$.

4. Estimate the average loss-cost. From equation (4.46), we compute the loss-cost for this plan to be $L = 2.8162$. The exact solution to this problem, obtained by a direct search method, yields $k = 2.89$, $n = 5$, $s = 1.032$, $\alpha = 0.0075$ and $L = 2.8078$.

<u>Example 4.4:</u> Suppose, in Example 4.3, the non-normality of the process is ignored. Accordingly, the values of the parameters γ_1 and γ_2 are equal to zero. Using Table 4.5 and following the standard procedure of Example 4.3, one arrives at the following plan with no difficulty; $n = 5$, $k = 3.0$, $s = 0.933$, and $L = 2.7329$.

However, if the process is, in fact, regarded as non-normal, the plan results in an actual hourly loss-cost of $L = 2.8162$, as noted in example 4.3. Thus, the use of conventional control plans with normality assumption, even though the production process is markedly non-normal, will result in misleading values of loss-cost. This would eventually amount to substantial losses over a long period of operation.

Table 4.5 Simplified Scheme for Determination of Control Parameters for the Economic Design of $\bar{x}$-Chart to Control Non-Normal Means for Which $0.90 \leq P \leq 0.95$ (δ=2, γ_1=0.0, and γ_2=0.0)

δ =2		γ_1 =0.0		γ_2 =0.0
k	n	α	(1/P-1/2)	A**
1.2	2	0.23014	0.555	15
1.3	2	0.19360	0.567	17
1.4	2	0.16151	0.583	19
1.5	2	0.13362	0.601	22
1.9	3	0.05743	0.563	53
2.0	3	0.04550	0.577	64
2.1	3	0.03573	0.594	79
2.4	4	0.01640	0.558	179
2.5	4	0.01242	0.572	228
2.6	4	0.00932	0.588	294
2.7	4	0.00693	0.607	384
2.9	5	0.00373	0.562	751
3.0	5	0.00270	0.576	1009
3.1	5	0.00194	0.593	1369
3.3	6	0.00097	0.558	2844
3.4	6	0.00067	0.572	3976
3.5	6	0.00047	0.588	5613

Note: A** in this Table is as defined by equation (4.48).

4.4 Efficiency of the Control Plan

In order to design an economically optimum $\bar{x}$-chart control plan, all the relevant data parameters must be estimated before the loss-cost function can be minimized. Unfortunately, little attention has been paid to the effect, on the optimality, of errors in estimating cost and data parameters. The manufacturer can use an economic approach with sufficient confidence only if he has prior knowledge of optimum data parameters. Depending on the individual circumstances, and the nature of the product, errors in estimation may occur in varying degrees. It is therefore desirable to investigate to what extent these errors affect the optimality or the economic design of $\bar{x}$-charts.

Recall the loss-cost function developed in section 4.3.1, which after simplification, may be written as

$$L = \frac{\lambda(V_0-V_1)B_1+(k_s+V_0\tau_s)B_0+\lambda\{k_s+k_r+V_0(\tau_s+\tau_r)\}+(b+cn)(1+\lambda B_1)/s}{1+\lambda B_1+\tau_s B_0+\lambda(\tau_s+\tau_r)} . \tag{4.49}$$

The formulation of the loss-cost function involves the following data parameters:

δ, λ, V_0, V_1, τ_s, τ_r, k_s, k_r, b, c, γ_1, γ_2.

To measure the efficiency of a non-optimum plan, the method of Hald [1964] is adopted in the present study. Hald's measure of efficiency has the advantage of being invariant to the choice of origin and the scale of losses, and of lying between 0 and 1. Consider the expression of L in equation (4.49) which represents the loss-cost borne by the manufacturer when he uses a particular control plan $\pi(n, h, k)$. This L has an unavoidable, minimum part, L_m, which corresponds to an imaginary, perfect, control procedure that detects the assignable cause as soon as it occurs without any sampling inspection and unnecessary halting of the production. It is clear that

$$L_m = \{\lambda[k_s + k_r + V_0(\tau_s + \tau_r)]\}/\{1 + \lambda(\tau_s + \tau_r)\}. \qquad (4.50)$$

We may then define the efficiency of a general control plan π relative to the optimum control plan π_0 to be

$$\varepsilon(\pi,\pi_0) = \{L(\pi_0) - L_m\}/\{L(\pi) - L_m\}. \qquad L(\pi) > L(\pi_0) \qquad (4.51)$$

$$\varepsilon(\pi,\pi_0) = \{L(\pi) - L_m\}/\{L(\pi_0) - L_m\}. \qquad L(\pi) < L(\pi_0) \qquad (4.52)$$

This efficiency, ε, is clearly invariant to the origin and the scale of losses, and it lies between 0 and 1. The better the control plan, the closer to unity the value of ε, and vice versa. The quantity $100(1-\varepsilon)$ expresses the saving in percent of the sampling costs and other losses for the control plan π by using π_0 instead of π. Using Hald's criterion for measuring the efficiency of a control plan, a sensitivity analysis of the model under Policy II is performed. Also, the attention of the user is drawn to the effects of errors in the estimation of the critical data parameter, on which a reliable optimum control plan is largely dependent.

Sensitivity Analysis

Effect of λ. The average number of assignable causes per hour is denoted by λ and an increase in λ is equivalent to a decrease in the average time for the assignable cause to occur. To study the effect of this increase in λ on the optimum design, the following data set is considered.

$V_0 = 150$, $V_1 = 50$, $k_r = 20$, $k_s = 10$,

$\tau_r = 0.2$, $\tau_s = 0.1$, $\gamma_1 = 0.5$, $\gamma_2 = 1.0$

and $\delta = 2.0$.

The numerical values assigned to λ are 0.005, 0.008, 0.01, 0.05, 0.08, and 0.1. Suppose the true value of λ = 0.005 . For this, the exact optimum control plan $\pi_0 = (n,s,k) = (6, 2.11, 3.07)$ and the values of $L_m = 0.3744$ and $L = 1.4765$.

Let the other values of λ be incorrectly estimated. Thus, the error factor for these cases will be = estimated λ/ true λ. The effects of λ on the design parameters and on the loss-cost function, and the relative efficiencies (measured as 100e) are obtained using equation (4.51) and (4.52) corresponding to λ. These are given in Table 4.6 and depicted in Fig.4.1. In Fig.4.1, graphs are drawn on different scales to accommodate the relevant values of the variables.

It is observed that the only significant effect of an increase in λ is on design parameter s. For example, if λ increases from 0.005 to 0.01, i.e., if the average time for the assignable cause to occur reduces from

Table 4.6 Effect of λ on the Control Plan and Effect of Errors in the Estimated Values of λ.

λ	Optimal design n	s	k	L	L_m	Error factor	100ε
0.005	6	2.11	3.07	1.4765	0.3744		100
0.008	6	1.67	3.11	1.9924		1.6	68.11
0.01	6	1.50	3.08	2.3052		2.0	57.07
0.05	6	0.68	3.11	7.0784		10.0	16.43
0.08	6	0.54	3.12	10.0464		16.0	11.39
0.1	6	0.50	3.10	11.8952		20.0	9.57

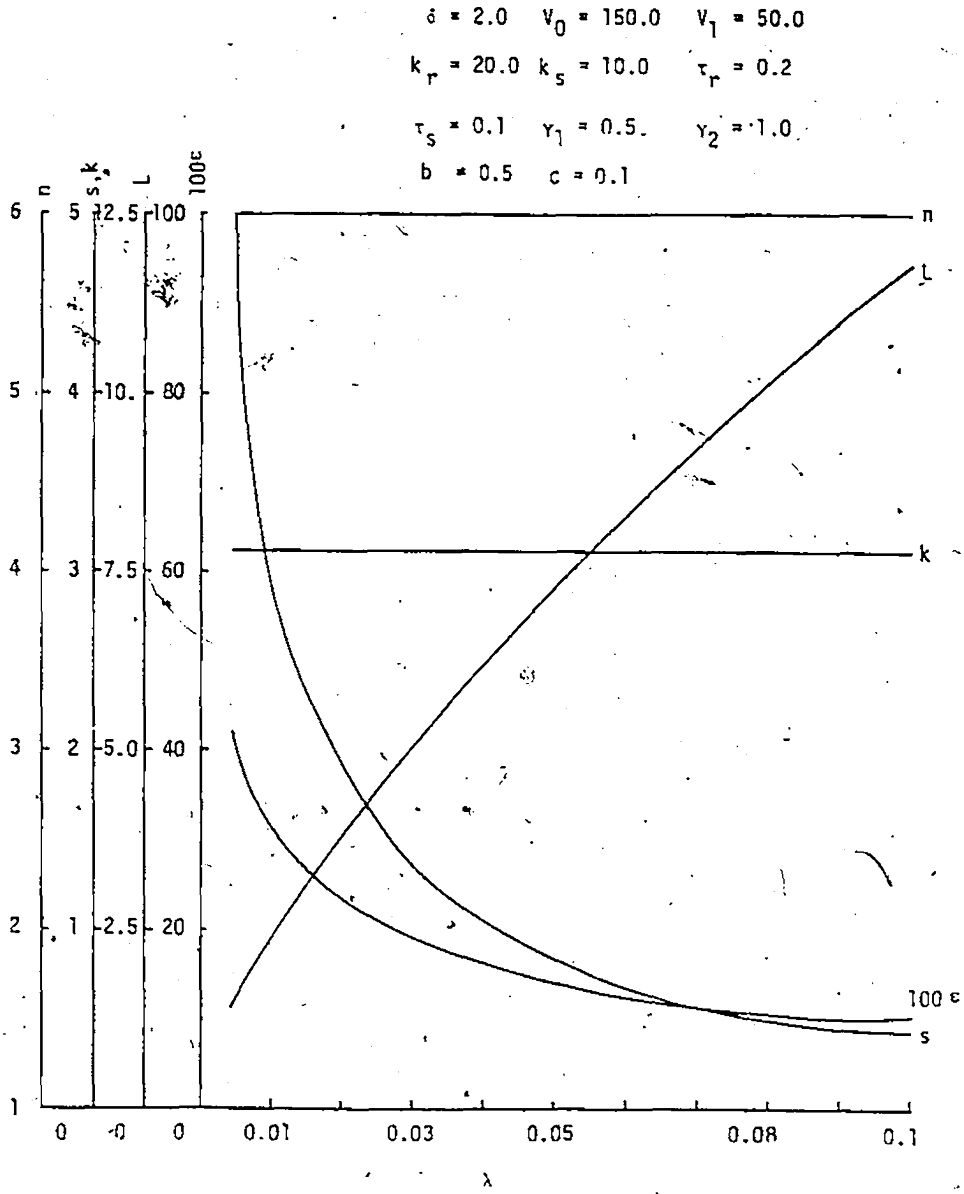

Fig. 4.1 Effect of λ on the Design Parameters and the Loss-Cost Function, and Effect of Errors in the Estimated Values of λ.

200 hours to 100 hours, the sampling interval changes from 2.11 to 1.50. The sample size n remains the same and there is a slightly effect on the control limit coefficient, k. However, with the increase of λ, the loss-cost function increases significantly and over-estimates of λ result in low efficiencies of the control plan. Thus λ is a critical data parameter.

Effect of δ. The shift parameter δ is related to the change in the process mean by an amount $\delta\sigma$. An error in estimating δ results in an incorrect estimate of the effect of the assignable cause ($\delta\sigma$) assuming that the estimate of σ is accurate. Consequently, the profit derived from the out-of-control state, V_1 is also incorrectly estimated. The effect of δ on the design variables and loss-cost function, and the consequences of incorrect estimation of δ, are shown in Fig. 4.2. As δ increases the value of sample size and sampling interval decreases, but the value of control limit coefficient k increases. The value of the loss-cost function decreases gradually with the increases of δ. The correct value of δ is assumed to be 0.5 in each case. The measure of efficiency 100e is low or very low despite the relatively small sizes of assumed error in the estimation (over estimation) of δ. This leads to the conclusion that δ is also a critical parameter.

Effect of Cost Factors b and c. The cost factors b and c determine the cost of maintaining the control chart, which is equal to (b+cn) per sample, where b is the cost of sampling and cn is the cost of plotting and computation. The effect of b on the design parameter

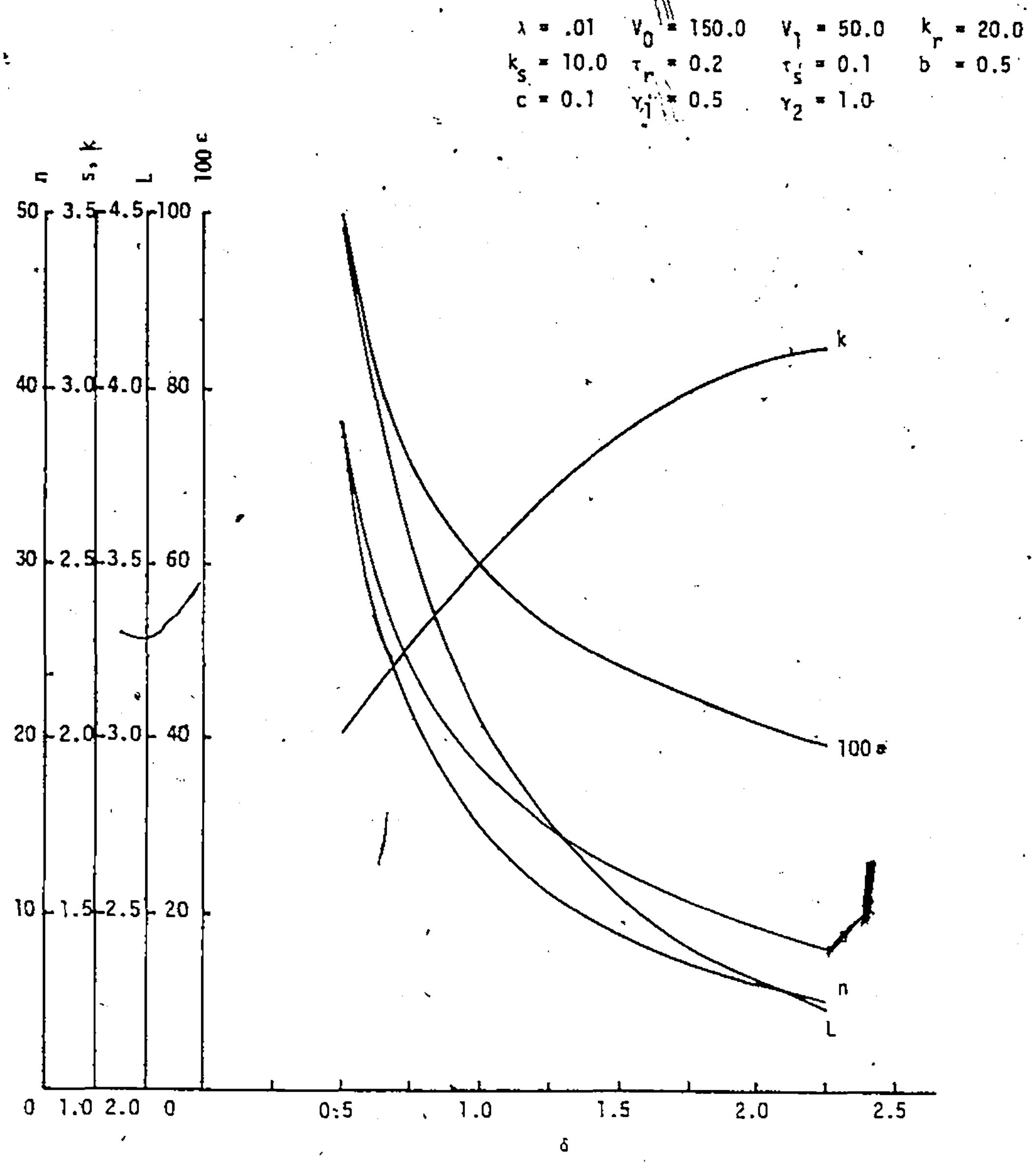

Fig. 4.2 Effect of δ on the Design Parameters and the Loss-Cost Function, and Effect of Errors in the Estimated Values of δ.

and on the loss-cost function, and the effects of errors in the estimated values of b are shown in Fig. 4.3. This indicates that as b increases, sample size and sampling interval increase, but the effect of b on control limit coefficient is insignificant.

The effect of c is depicted in Fig. 4.4 which shows that sample size decreases and sampling interval increases with the increase of c.

Effects of k_r and k_s. Figures 4.5 and 4.6 indicate that the effects of k_r and k_s on design parameters are insignificant. However, the loss-cost increases with the increase of both k_r and k_s. It may be noted that cost factor k_r was not considered by Duncan [1956]. However, no explanation was given for the omission of this factor in his model.

Effects of τ_r and τ_s. Effects of $\cancel{\tau_r}$ and τ_s are significant on the loss-cost but insignificant on the design parameter k as seen in Figs. 4.7 and 4.8. Their effects on the sampling interval and sample size n, are moderate.

Based on the above results, the following conclusions about the sensitivity of the model with respect to λ, δ and to the cost factors may be drawn:

The optimum design is:

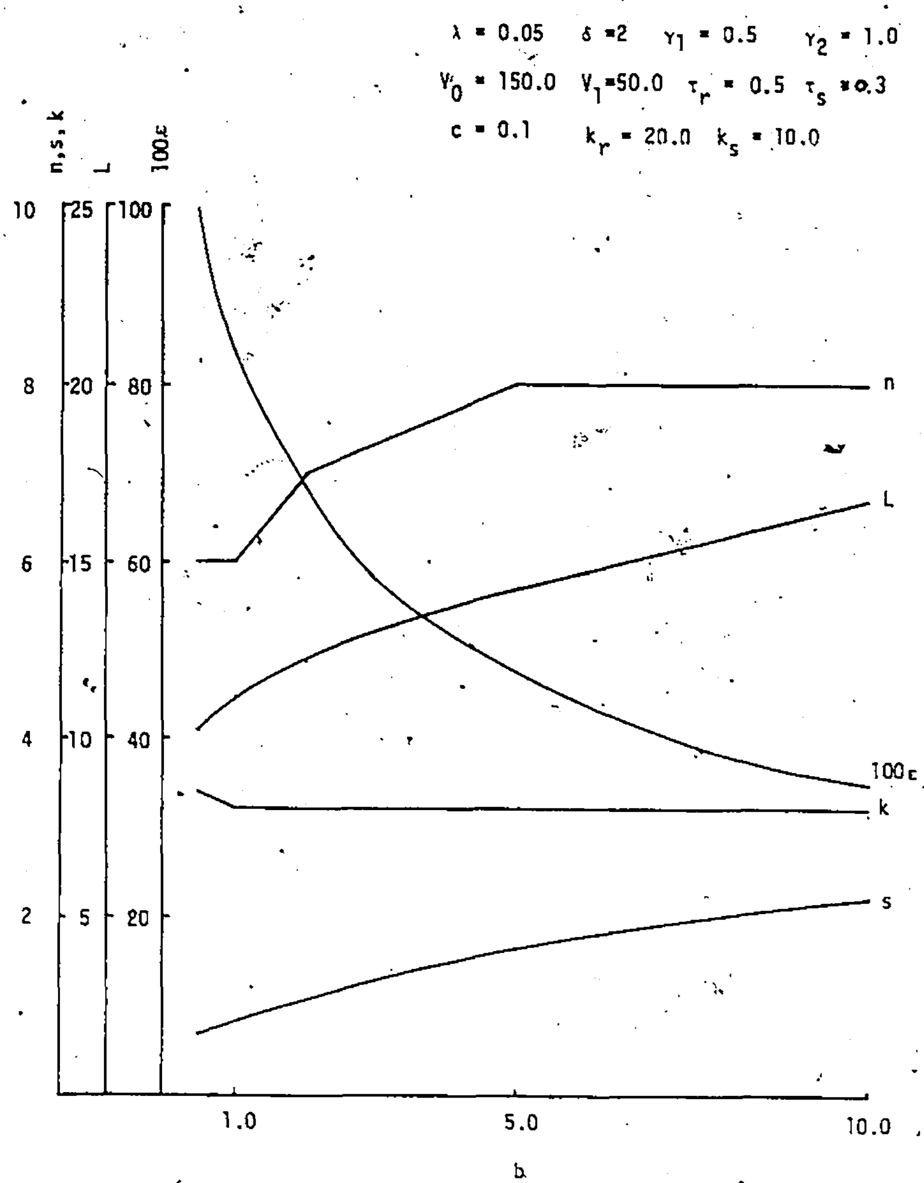

Fig. 4.3 Effect of b on the Design Parameters and the Loss-Cost Function, and Effect of Errors in the Estimated Values of b.

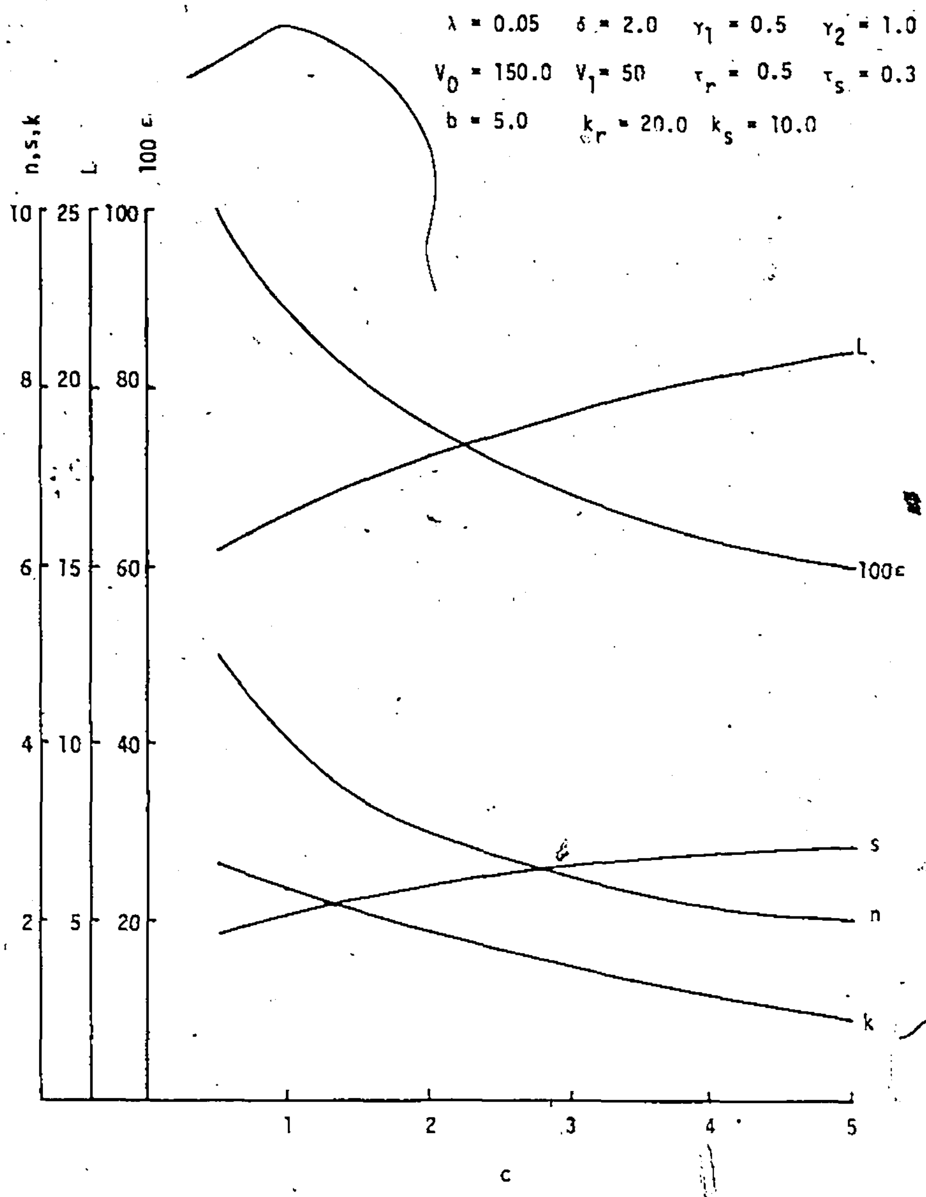

Fig. 4.4 Effect of c on the Design Parameters and the Loss-Cost Function, and Effect of Errors in the Estimated Value of c.

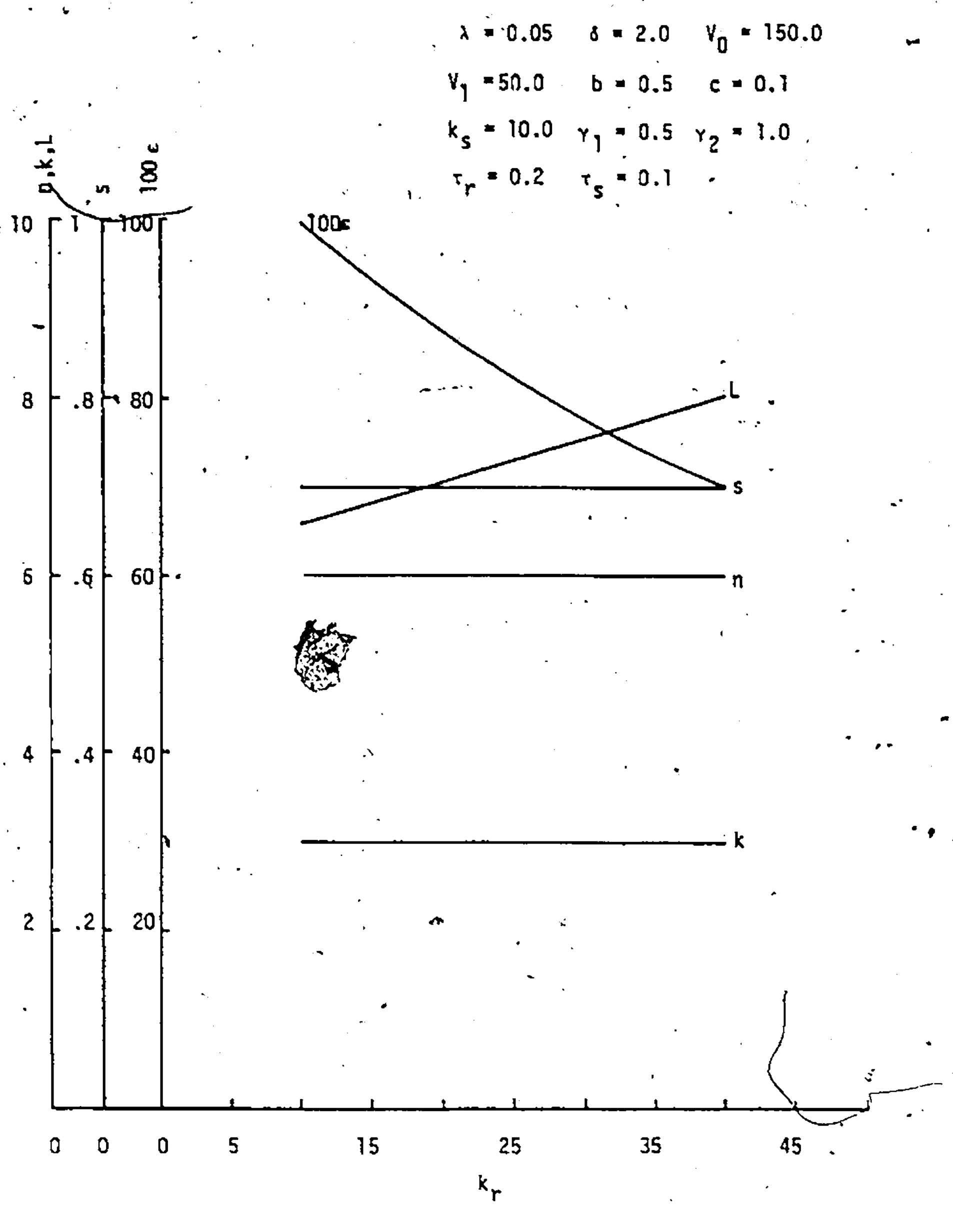

Fig. 4.5 Effect of k_r on the Design Parameters and the Loss-Cost Function, and Effect of Errors in the Estimated Values of k_r.

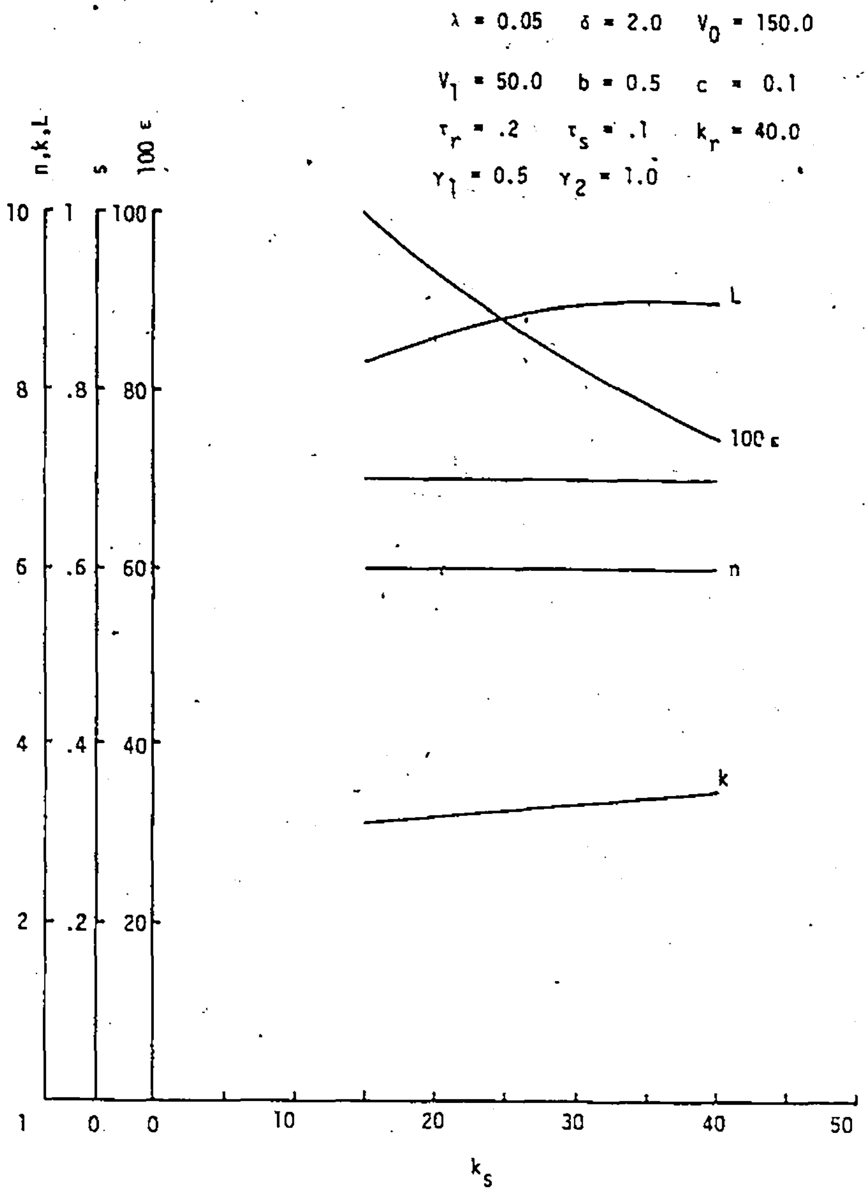

Fig. 4.6 Effect of k_s on the Design Parameters and the Loss-Cost Function, and Effect of Errors in the Estimated Values of k_s.

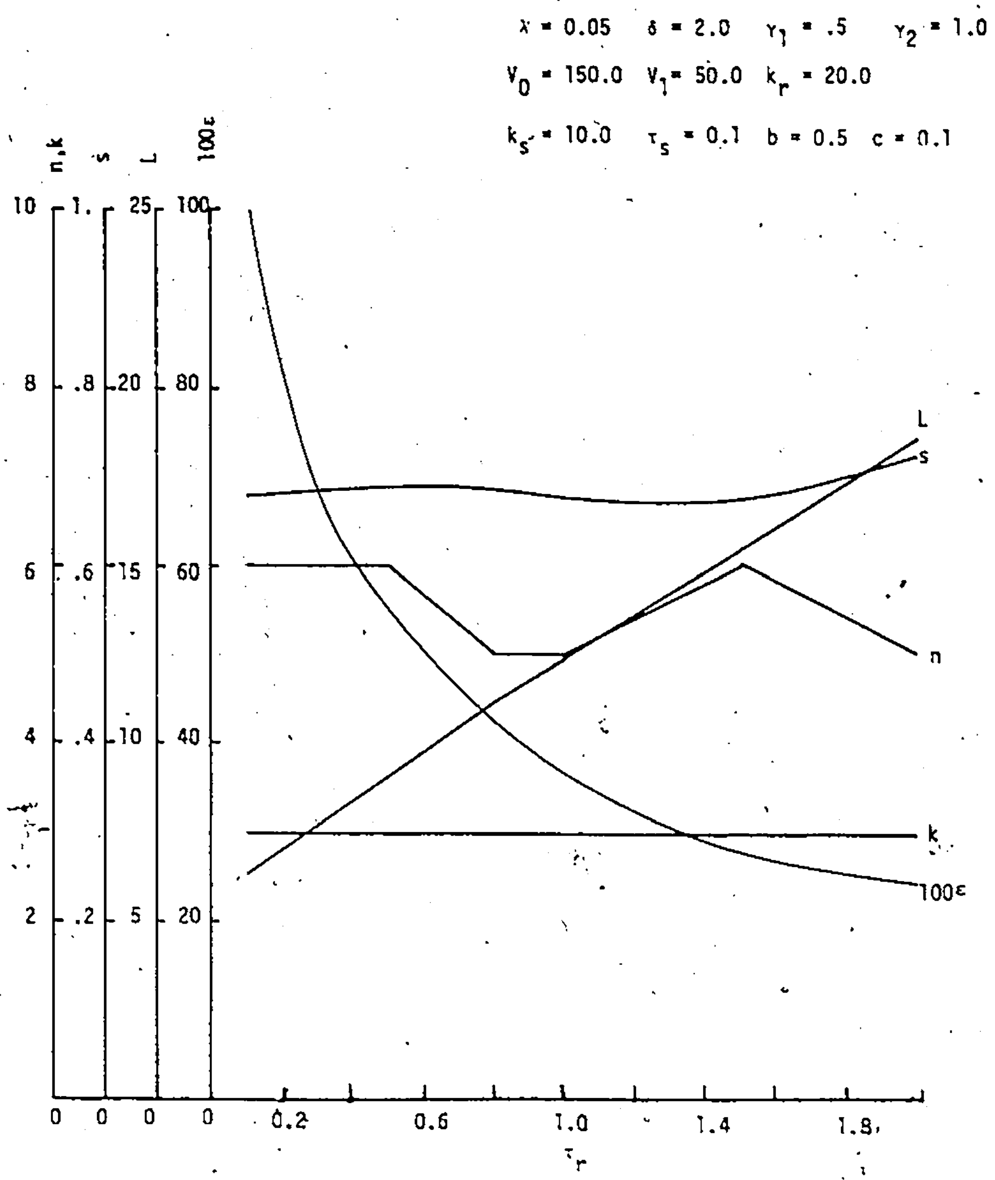

Fig. 4.7 Effect of τ_r on the Design Parameters and the Loss-Cost Function, and Effect of Errors in the Estimated Values of τ_r.

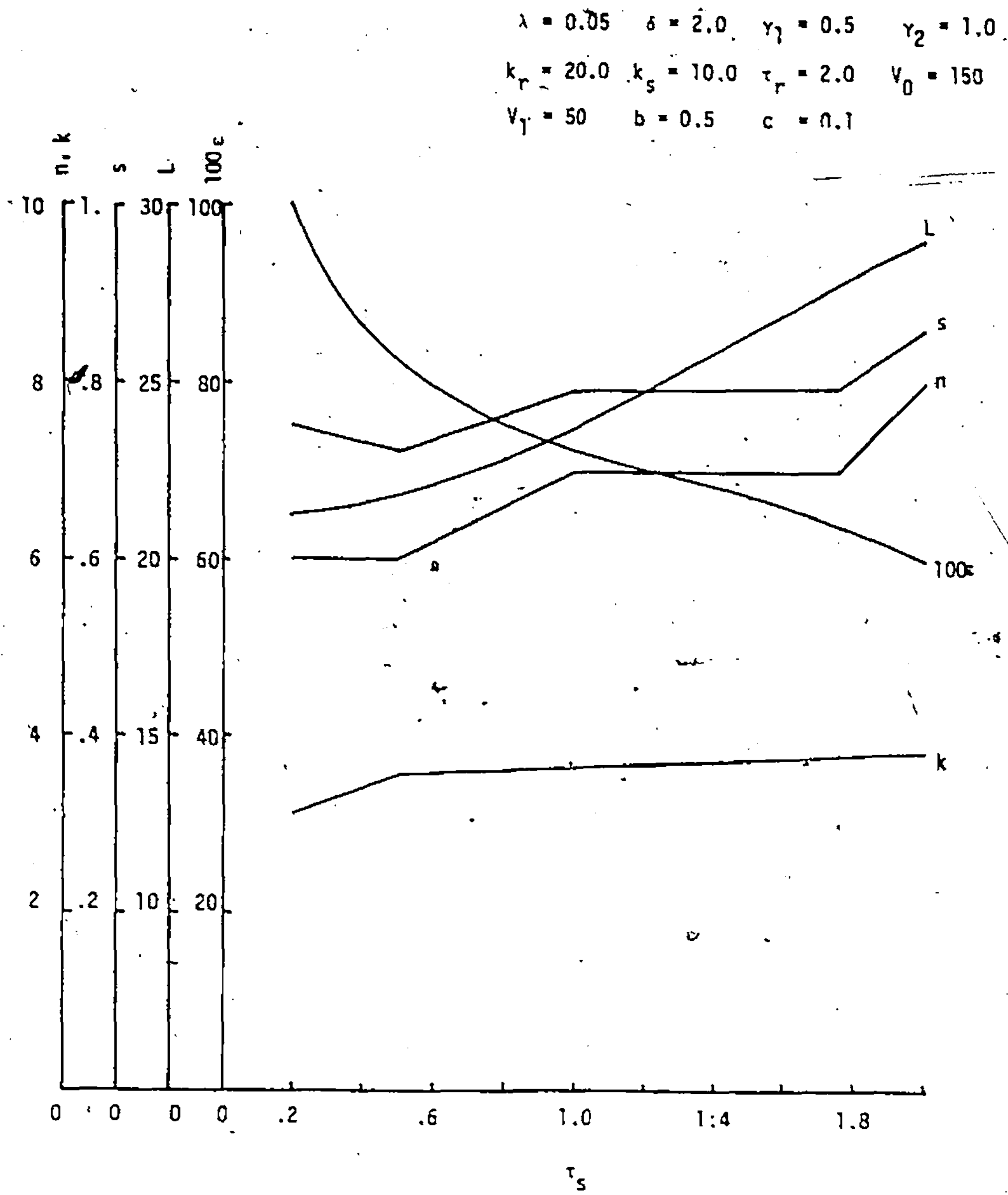

Fig. 4.8 Effect of τ_s on the Design Parameters and the Loss-Cost Function and Effect of Errors in the Estimated Values of τ_s.

- highly sensitive to errors in estimating the shift parameter δ, and the rate of occurrence of the assignable cause λ.
- moderately sensitive to the fixed cost and variable sampling costs.
- relatively insensitive to the repair and search costs.

The discussion about the effects of the non-normality parameters on the design variables and on the loss-cost function will be presented later in Chapter 6.

4.5 Multiple Assignable Cause Model

The fundamental assumption of the process model studied in previous sections is that there exists a single assignable cause which shifts the process mean by an amount $\delta\sigma$. In practice this assumption may not be satisfied, as it often occurs that a multiplicity of assignable causes may operate on the process.

The production processes considered in this section have an in-control state, and may jump to one of the several out-of-control states, each with an associated assignable cause. The process is assumed to start in the state of control with mean μ. It could be disturbed by the occurrence of an assignable cause A_j $(j = 1,2,..,n)$ which produces a shift in the process mean of $\delta_j\sigma$, where σ is the process standard deviation. It is assumed that when the process has been disturbed by a given assignable cause it is free from the occurence of other assignable causes. In other words, this is equivalent to Duncan's Model I for normal cases [Duncan, 1971].

4.5.1 Formulation of Loss-Cost Function. Following the assumptions regarding the operating conditions of the process stated above, let:

V_0 = profit when the process is in control.

V_j = profit when the process is in out-of-control state due to assignable cause A_j.

τ_s = expected time to search for an assignable cause.

k_s = expected per hour search cost.

τ_j = expected repair time, if the process is disturbed by the assignable cause A_j.

k_j = expected per hour repair cost, if the process is disturbed by the assignable cause A_j.

In a production cycle the time at which the process goes out of control is distributed as the minimum of n independent exponentially distributed random variables with means $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_j}$ and thus has an exponential distribution with mean $\frac{1}{\lambda}$ where $\lambda = \Sigma\lambda_j$. This means, that duration of the in-control state is, on the average, $\frac{1}{\lambda}$ hours.

Let,

α - probability of a false alarm,

P_j - probability of a true alarm when assignable cause A_j is operating.

Each false alarm has a search time of τ_s; thus, the total time spent on the search = $\tau_s B_0$, where

$$B_0 = \alpha(\exp(-\lambda s))/\{1 - \exp(-\lambda s)\}.$$

When the assignable cause occurs, it may occur at any time between two samples. The average occurrence time in the interval between two samples is

$$t_j = [1 - (1 + \lambda_j s)\exp(-\lambda_j s)]/[\lambda_j - \lambda_j \exp(-\lambda_j s)] \qquad (4.53)$$

The time before a true alarm is signalled is

$$B_j = s/P_j - t_j. \qquad (4.54)$$

Following this alarm, a further expected time τ_j is required for detecting and eliminating the assignable cause A_j. Now, since there may only be one assignable cause present in each production cycle and since the frequency of the assignable cause A_j is λ_j/λ, the expected time the process is out of control, counting from the occurrence of the assignable cause to the completion of the cycle, is

$$\Sigma \lambda_j (B_j + \tau_j)/\lambda \qquad (4.55)$$

Thus the expected length of a production cycle is

$$\frac{1}{\lambda}(1 + \Sigma \lambda_j B_j + \tau_s B_0 + \Sigma \lambda_j \tau_j) \qquad (4.56)$$

The expected net income per cycle is:

$$\frac{V_0}{\lambda} + [\Sigma\, \lambda_j V_j B_j - VB_0 - \Sigma\, \lambda_j W_j - (b+cn)(1 + \Sigma\, \lambda_j B_j)/s]/\lambda \qquad (4.57)$$

Analogous to the single assignable cause model, the loss-cost function for the multiple assignable cause model is derived as follows.

$$L = \frac{\Sigma\, \lambda_j U_j B_j + VB_0 + \Sigma\, \lambda_j W_j + (1 + \Sigma\, \lambda_j B_j)(b+cn)/s}{1 + \Sigma\, \lambda_j B_j + \tau_s B_0 + \Sigma\, \lambda_j \tau_j}, \qquad (4.58)$$

where $V = k_s + V_0\tau_s$,

$W_j = k_j + V_0\tau_j$,

and $U_j = V_0 - V_j$.

For an exact optimum design, the search method of section 4.2.2 can be used with suitable modifications. The initial position for the search can be given by the method explained in an example later.

4.5.2 Application of the Simplified Scheme. In this section a matched [Duncan, 1971] single assignable cause model is proposed so that its semi-economic plan will approximate the true optimum control plan for the original multiple cause model.

The proposed matched single cause model is defined as follows.

1. The shift δ_s produced by the single assignable cause is equal to the weighted mean for the multiple cause shifts; so that

$$\delta_s = \Sigma\, \lambda_j \delta_j/\lambda \qquad (4.59)$$

2. The rate of occurrence of the single assignable cause is the sum of the rates of occurrences of the individual assignable causes in the multiple cause model:

$$\lambda_s = \lambda = \Sigma\, \lambda_j \; . \tag{4.60}$$

3. The hourly profit induced by the occurrence of the single assignable cause (V_s):

$$V_s = \Sigma\, \lambda_j V_j / \lambda; \quad \text{so that } U_s = V_0 - V_s \; . \tag{4.61}$$

4. Average time taken to eliminate the single assignable cause is defined as

$$\tau_{rs} = \frac{\Sigma\, \lambda_j \tau_j}{\lambda} \; . \tag{4.62}$$

5. Average cost for the detection and elmination of the single assignable cause for true alarm is then,

$$k_{rs} = \frac{\Sigma\, \lambda_j k_j}{\lambda} \; . \tag{4.63}$$

6. The average cost of searching for a single assignable cause when it exists is thus,

$$W_s = k_{rs} + V_0\, \tau_{rs} \; . \tag{4.64}$$

To determine an approximately optimum plan by the simplified scheme of section 4.3.3, an example is considered below.

Example 4.5 Consider a non-normal multiple cause model defined by the quantities given in Table 4.7.

Table 4.7 A Triple Cause Model

λ_j	δ_j	V_j	k_j	τ_j
0.005	1.2	262	42	0.20
0.004	2.0	75	30	0.15
0.001	3.5	-110	20	0.10

Other parameters values are assumed as follows.

$V_0 = 350$, $k_s = 20$, $\tau_s = 0.15$, $b = 2.0$, $c = 0.3$, $\gamma_1 = 1.0$ and $\gamma_2 = 2.0$.

Determine an approximate plan for the triple cause model with $P \geq 0.95$.

For the matched model,

$\lambda_s = 0.01$, $\delta_s = 1.75$, $V_{1s} = 150$, $k_{rs} = 35$, $\tau_{rs} = 0.17$

so that

$U_s = V_0 - V_{1s} = 200$, $V_s = k_s + V_0\tau_s = 20 + 52.50 = 72.50$

$W_s = 94.5$.

Table 4.8 is applicable. Following the simplied scheme of section 4.3.3

1st step: $A^{**} = 370$

2nd step: Table 4.8 gives $n = 6$ and $k = 2.7$.

3rd step: Evaluate s. From Table 4.8 $\alpha = 0.00967$ and $P^{-1} - \frac{1}{2} = 0.544$.

Thus, $s = \{(\alpha V_s + b + cn)/[\lambda_s U_s(P^{-1} - \frac{1}{2})]\}^{1/2}$

$= 2.03$

4th step: Using equation (4.46), $L = 5.3254$.

For comparison purposes, the worked-out exact plan for the original triple cause model is given as:

$n = 7$, $s = 1.86$, $k = 2.89$ and $L = 5.3659$.

Table 4.8 Simplified Scheme for Determination of Control Parameters for the Economic Design of $\bar{X}$-Chart to Control Non-Normal Means for Which $P > 0.95$ (δ=1.75, γ_1=1.0, and γ_2=2.0)

δ =1.75		γ_1 =1.0		γ_2 =2.0
k	n	α	$(1/P-1/2)$	A**
1.2	2	0.23014	0.555	16
1.0	2	0.29715	0.544	9
1.1	3	0.25719	0.508	13
1.2	3	0.21571	0.514	14
1.3	3	0.17936	0.521	16
1.4	3	0.14793	0.530	19
1.5	3	0.12114	0.543	22
1.6	4	0.10136	0.513	31
1.7	4	0.08224	0.519	37
1.8	4	0.06647	0.527	46
1.9	4	0.05362	0.538	56
2.0	4	0.04325	0.551	70
2.1	5	0.03512	0.520	97
2.2	5	0.02827	0.528	123
2.3	5	0.02283	0.539	157
2.4	5	0.01853	0.552	201
2.5	6	0.01466	0.524	274
2.6	6	0.01188	0.532	350
2.7	6	0.00967	0.544	446
2.8	7	0.00749	0.522	604
2.9	7	0.00607	0.530	764
3.0	7	0.00492	0.540	960
3.1	7	0.00398	0.553	1197
3.2	8	0.00299	0.529	1622
3.3	8	0.00239	0.539	2018
3.4	8	0.00190	0.551	2503
3.5	9	0.00139	0.530	3440

Note: A** in this Table is as defined by equation (4.48).

CHAPTER 5

AN ECONOMIC DESIGN OF $\overline{X}$-CHARTS WITH WARNING LIMITS TO CONTROL NON-NORMAL PROCESS MEANS

In this chapter, an expected cost model for a production process under the surveillance of an $\overline{x}$-chart with warning limits for controlling the non-normal process mean is developed. It is assumed that the process is subject to the occurrence of a single assignable cause and is operating under the policy II. The design parameters of a general control chart with warning limits are the sample size, the sampling interval, the action limit coefficient, the warning limit coefficient, and the critical run length. To develop the expected loss-cost function, expressions for the average run lengths, when the process is in control, and when the process is out of control, are derived. A direct search technique is employed to obtain the optimal values of the design parameters. Numerical examples are provided, and the effects of the non-normality parameters on the loss-cost function and on the design parameters are discussed. Conclusions are drawn about the relative efficiencies of the economic design of $\overline{x}$-charts with and without warning limits. A simplified form of the algorithm is also devised which could be useful for practical application at the workshop level.

5.1 Formulation of Loss-Cost Function

A production process which has two states, in-control and out-of-control, is considered. The process is assumed to start in a state of in-control. The quality characteristic of the process variable is measurable on a continuous scale and is non-normally distributed with the same density function as described in section 4.1.

The process is assumed to be shut-down during the search for the assignable cause. A sample of fixed size n is taken at regular intervals of time and the sample mean is plotted on a one-sided $\bar{x}$-chart with warning limits. The upper action limit is set at $\mu_0 + k_a\, \sigma/\sqrt{n}$, where k_a is the upper control limit coefficient. The upper warning limit is set at $\mu_0 + k_w\, \sigma/\sqrt{n}$ where $0 < k_w < k_a$. A search for the assignable cause is undertaken if the last sample mean falls outside the action limit, or if the last sample mean completes a critical run length R_c which is in between the warning and action limits.

Following the general outlines of the works of Duncan [1956] and, Chiu and Cheung [1977], the loss-cost function of the process under the surveillance of an $\bar{x}$-chart with warning limits for controlling the non-normal process means can be formulated as follows.

Let T_a be the random time during which the process operates under the state of control. By assumption, T_a has an exponential distribution with $E(T_a) = 1/\lambda$. Let M be the number of samples taken before the process goes out of control, and G the number of samples taken after the Mth sample and up to the moment the chart signals lack of control. Let N be the number of false alarms occurring among the first M samples. Then it is straightforward to see that the expected length of the production cycle consists of four parts: (a) the in-control period, (b) the search times due to false alarms, (c) the out of control period, and (d) the search and repair times due to true alarms.

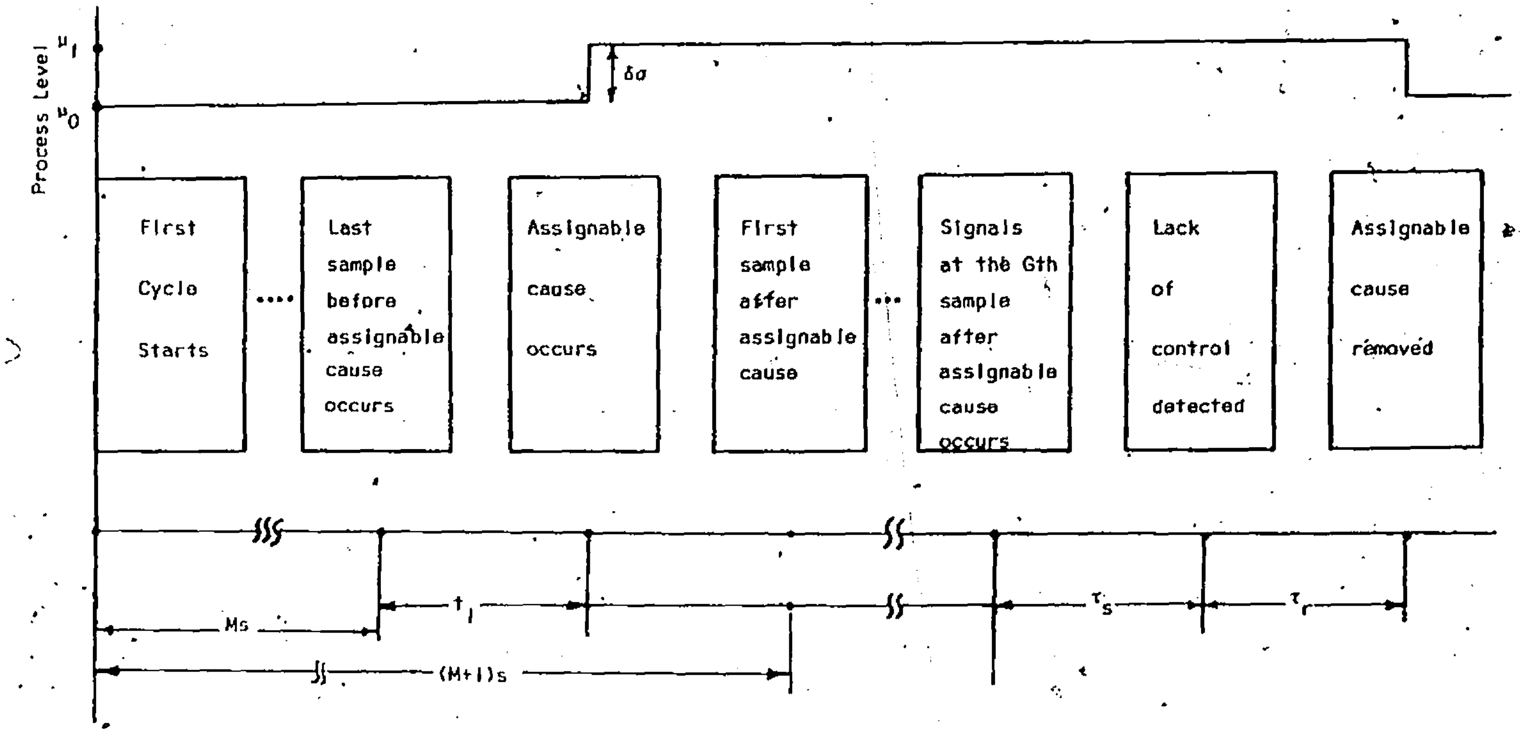

Fig. 5.1 Diagrammatic Representation of In-Control and Out-of-Control State of the Process.

Using the same terminology as defined in section 4.3, the expected length of a production cycle is

$$sE(M) + \tau_s E(N) + sE(G) + \tau_s + \tau_r , \tag{5.1}$$

and the expected income from a production cycle is

$$\frac{V_0}{\lambda} + V_1\ E(Ms+Gs-T_a) - E(N)k_s - (b+cn)E(M+G) - k_s - k_r . \tag{5.2}$$

Hence, the average net income per hour is

$$I = \frac{\text{Expression (5.2)}}{\text{Expression (5.1)}} . \tag{5.3}$$

The assignable cause occurs somewhere between the Mth and the (M+1)st samples, in the cycle. Then the average length of the time of the occurrence within this interval, measured from the beginning of the interval, is:

$$E(t_1) = E(T_a - Ms) = \{1 - (1+\lambda s)\exp(-\lambda s)\}/\{\lambda - \lambda\exp(-\lambda s)\} \approx \frac{s}{2} - \frac{1}{12}\lambda s^2 \tag{5.4}$$

Thus from equation (5.4):

$$E(M) = 1/\lambda s - \frac{1}{2} + \lambda s/12 \tag{5.5}$$

To determine the expected number of false alarms during the first M samples, we have for fixed M [Chiu, 1974]:

$$E(N|M) \approx M/R_0$$

where R_0 is the average run length (ARL) of $\overline{x}$-chart with warning limit at the acceptable quality level μ_0. Thus, from equation (5.5):

$$E(N) \approx E(M)/R_0 = \{1/\lambda s - \frac{1}{2} + \lambda s/12\}/R_0 \tag{5.6}$$

Taylor [1968] has shown, by computer simulation, that the dependence of E(G) on M is negligible, and that it could be written as

$$E(G) \approx R_1 \tag{5.7}$$

where R_1 is the ARL of the chart at the rejectable quality level,

$\mu_1 = \mu_0 + \delta\sigma$. Thus,

$$E(Ms+Gs-T_a) = R_1\, s - \frac{1}{2} s + \frac{1}{12}\lambda s^2 \tag{5.8}$$

and

$$E(M+G) = \frac{1}{\lambda s} - \frac{1}{2} + \frac{1}{12}\lambda s + R_1 . \tag{5.9}$$

Substituting equations (5.6) - (5.9) into equation (5.3) and defining

$$\begin{aligned} U &= V_0 - V_1; \\ V &= k_s + V_0\tau_s; \\ W &= k_r + k_s + V_0(\tau_r + \tau_s); \\ B_0 &= \left(\frac{1}{s} - \frac{\lambda}{2} + \frac{\lambda^2 s}{12}\right)/R_0; \\ B_1 &= \left(R_1 - \frac{1}{2} + \frac{\lambda s}{12}\right)s; \\ L &= V_0 - I; \end{aligned} \tag{5.10}$$

thus, L becomes after some simplification,

$$L = \frac{\lambda U B_1 + V B_0 + \lambda W + (b+cn)(1+\lambda B_1)/s}{1 + \lambda B_1 + \tau_s B_0 + \lambda(\tau_r + \tau_s)} \tag{5.11}$$

where L represents the average long-run per-hour loss-cost of the process.

5.2 Effect of Non-Normality on Loss-Cost Function

Before minimizing equation (5.11) to obtain the optimum design parameters of the $\bar{x}$-chart with warning limits, it is noted that the values of the average run lengths R_0 and R_1 are dependent on the probability density function of the process variable, which, by our assumption, is non-normal.

The average run length, for a one-sided $\bar{x}$-chart with warning limits for controlling a normal process mean, as given by Page [1962] is

$$ARL = (1 - q^{R_c})/[1 - q - P'(1-q^{R_c})], \tag{5.12}$$

where R_c is the critical run length, P' is the probability that a point falls below the warning limit, and q is the probability that a point falls between the warning and action limits. For controlling non-normal process means, when the process is out of control, the following expressions for p' and q are derived using equation (4.21) given in section 4.2.

$$p' = \Phi(k_w-\delta\sqrt{n}) - \frac{\gamma_1}{6\sqrt{n}}\phi^{(2)}(k_w-\delta\sqrt{n}) + \frac{\gamma_2}{24n}\phi^{(3)}(k_w-\delta\sqrt{n}) + \frac{\gamma_1^2}{72n}\phi^{(5)}(k_w-\delta\sqrt{n}) \tag{5.13a}$$

and

$$\begin{aligned} q = {} & \Phi(k_a-\delta\sqrt{n}) - \Phi(k_w-\delta\sqrt{n}) - \frac{\gamma_1}{6\sqrt{n}}[\phi^{(2)}(k_a-\delta\sqrt{n}) - \phi^{(2)}(k_w-\delta\sqrt{n})] \\ & + \frac{\gamma_2}{24n}[\phi^{(3)}(k_a-\delta\sqrt{n}) - \phi^{(3)}(k_w-\delta\sqrt{n})] + \frac{\gamma_1^2}{72n}[\phi^{(5)}(k_a-\delta\sqrt{n}) \\ & - \phi^{(5)}(k_w-\delta\sqrt{n})] \end{aligned} \tag{5.13b}$$

where Φ denotes the distribution function of the unit normal variate. Thus, R_1 is obtained by substituting equations (5.13a,b) into equation (5.12). Similarly, letting $\delta = 0$ in equations (5.13a,b) and substituting the resulting p' and q into equation (5.12) R_0 is obtained. These expressions will be used in equation (5.11) for locating the minimum position of L.

5.3 Determination of the Optimal Design Parameters

In order to obtain the optimum control plan, the objective function L given by equation(5.11)is minimized with respect to the design variables, i.e., the sample interval s, the coefficients of action and warning limits k_a and k_w, and the critical run length, R_c. The dependence of L on three parameters, k_a, k_w, and R_c, through equations (5.12) and (5.13a,b) precludes the use of any analytical optimization method. Rather, the direct search method of Hooke and Jeeves [1961] is employed to minimize L with respect to the vector of variables (n, s, k_a, k_w, R_c). However, due to the characteristics of function L, some modifications to the method have to be made in order to account for the inherent constraints on some of the design variables. These modifications are as follows.

(i) n and R_c assume integer values;

(ii) k_a and k_w maintain the relationship such that $0 < k_w < k_a$;

(iii) the expressions for R_1 and R_0 are non-negative for given values of γ_1, γ_2, and δ.

On the basis of these modifications, the computer program 'WARNING' is developed in order to minimize the loss-cost function and is given in Appendix III.

From the past studies on the design of $\bar{x}$-charts with warning limits [Chiu and Cheung, 1977] under the normality essumption, it has been found that the value of the critical run length R_c is either 1 or 2. Therefore, in the process of optimization, the range of values for R_c is from 1 to 4. In the conventional design of $\bar{x}$-charts with warning limit $k_a = 3$ and $k_w = 2$ have been considered [Chiu and Cheung, 1977]. Thus, the relation $k_w \doteq \frac{2}{3} k_a$ is used to specify the initial values of these two parameters.

Finally, an initial value for s is determined as follows. In practice, the values of λ and $1/R_0$ are very small. Hence, the quantity $\lambda B_1 + \tau_s B_0 + \lambda(\tau_r + \tau_s)$ in the denominator of equation (5.11) is very small compared with unity, and therefore it can be omitted; similarly, in the numerator, the term λB_1 is very small compared to unity and thus it can also be omitted. Consequently, equation (5.11) becomes

$$L \simeq \lambda U B_1 + V B_0 + \lambda W + (b+cn)/s \tag{5.14}$$

By differentiating equation (5.14) with respect to s and setting the results equal to zero, and omitting the terms λ^2 and λ^2/R_0, the following equation is obtained

$$s \simeq \{(\frac{V}{R_0} + b+cn)/[\lambda U(R_1 - \frac{1}{2})]\}^{1/2} \tag{5.15}$$

which will be used to determine an initial value for s after choosing the initial values of n, k_a and k_w.

NUMERICAL EXAMPLES

To obtain the optimal design parameters, the search method assumes that the objective function is convex. Since it is not possible to analytically investigate the convexity of L, some analysis of its behaviour was conducted through numerical studies, which indicated that the surface of L is approximately convex in the region around the optimal value.

With the assumption of convexity of the objective function, optimal plans were determined for a wide range of the non-normality parameters, γ_1 and γ_2, and of the shift parameter, δ; the cost parameters were fixed throughout the optimzation process.

Example 5.1. Consider a process having non-normality parameters $\gamma_1 = 0.5$ and $\gamma_2 = 1.0$, the shift parameter $\delta = 2$, and the rate of the occurrence of the assignable cause $\lambda = 0.01$. The cost parameters are assumed as follows: $V_0 = 150$, $V_1 = 50$, $k_r = 20$, $k_s = 10$, $\tau_r = 0.2$, $\tau_s = 0.1$, $b = 0.5$, and $c = 0.1$. The results of the optimization, presented in Table 5.1, indicate that the optimal plan is obtained at $R_c = 2$, $n = 5$, $s = 1.428$ hours, $k_a = 2.9062$, $k_w = 2.5034$, and the objective function value is $L = 2.2955$.

Table 5.1 Optimal Design for Example 5.1

R_c	n	s	k_a	k_w	L
1	5	1.431	3.2953	2.8917	2.2963
2	5	1.428	2.9062	2.5034	2.2955
3	5	1.430	2.8914	2.3081	2.2962
4	5	1.430	1.4291	2.1425	2.2963

To study the effects of non-normality parameters on the loss-cost function and the design parameters, Table 5.2 is prepared. The relevant cost parameters, the shift parameters, and the value of λ associated with Table 5.2 are the same as those in Table 5.1. However, parameter γ_1 is varied from -0.5 to 1.0 with increments of 0.5, and parameter γ_2 is varied from -0.5 to 2.0 with increments of 0.5. For given values of γ_1 and γ_2, the Table presents the optimal design parameters (n, s, k_a, k_w and R_c), and the optimal value of L.

It is evident from Table 5.2, that the effect of skewness is more marked than that of kurtosis. For given γ_2, the values of s, k_a, k_w and n increase as γ_1 increases. The same is true for L. However, the critical run length R_c remains unchanged, and the sample size n does not show marked changes.

From a non-economic point of view, Roberts [1966] and Weindling, et al. [1970] used the Average Run Length (ARL) criterion to compare the efficiencies of the economic designs of $\bar{x}$-chart with and without warning limits. They assumed the same ARL (R_0), when the process is in control for both charts. The ARL (R_1) values when the process is out of control are then compared for various shifts, $\delta\sigma$, in the process mean. They have reached the following conclusion; for a small shift, the $\bar{x}$-chart with warning limits has a shorter R_1 than the $\bar{x}$-chart with only action limits. From the manufacturer's point of view, the loss-cost value is more useful for assessing the effectiveness of a control chart as opposed to average run lengths. Thus, a minimum cost criterion is used to measure the relative performances of these charts in this study; the results are presented in section 6.7.

TABLE 5.2 Optimal Values of the Design Parameters and Lost-Cost Function of an Economic Design of $\bar{X}$-chart with Warning Limit.
($\lambda=0.01, V_0=150, V_1=50, k_r=20, k_s=10, \tau_r=0.2, \tau_s=0.1, b=0.5, c=0.1$)

γ_2	$\delta = 2.0$, $\gamma_1 = -0.5$	$\gamma_1 = 0.0$	$\gamma_1 = 0.5$	$\gamma_1 = 1.0$	
-0.5	4	5	5	5	n
	1.2782	1.4039	1.4240	1.4557	s
	2.5125	2.7734	2.8500	2.8797	k_a
	2.1698	2.3643	2.4831	2.5112	k_w
	2	2	2	2	R_c
	2.2032	2.2531	2.2761	2.2941	L
0.0	4	5	5	5	n
	1.2875	1.4077	1.4270	1.4540	s
	2.5250	2.7906	2.8687	2.8976	k_a
	2.1690	2.3706	2.4800	2.5159	k_w
	2	2	2	2	R_c
	2.2160	2.2609	2.2828	2.3003	L
0.5	4	5	5	6	n
	1.2992	1.4076	1.4284	1.5170	s
	2.5359	2.8132	2.8844	3.1187	k_a
	2.1675	2.3721	2.4847	2.7331	k_w
	2	2	2	2	R_c
	2.2286	2.2685	2.2893	2.3044	L
1.0	4	5	5	6	n
	1.3062	1.4099	1.4283	1.5153	s
	2.5516	2.8328	2.9062	3.1320	k_a
	2.1691	2.3768	2.5034	2.7276	k_w
	2	2	2	2	R_c
	2.2408	2.2758	2.2955	2.3081	L
1.5	4	5	5	6	n
	1.3162	1.4122	1.4282	1.5136	s
	2.5656	2.8547	2.9250	3.1554	k_a
	2.1675	2.4362	2.5112	2.7448	k_w
	2	2	2	2	R_c
	2.2528	2.2827	2.3014	2.3117	L
2.0	5	5	5	6	n
	1.4075	1.4121	1.4265	1.5126	s
	2.7320	2.8812	2.9484	3.1663	k_a
	2.3073	2.4612	2.5175	2.7620	k_w
	2	2	2	2	R_c
	2.2635	2.2893	2.3070	2.3151	L

However, it would be interesting to note that, when the process is in control, a large Average Run Length (R_0) corresponds to a small Type I error, and when the process is out of control, a small Average Run Length (R_1), corresponds to small Type II error. Thus, following similar arguments that have been used in section 4.3.3 for developing simplified scheme for an economic design of an $\bar{x}$-chart, a simplified scheme for an $\bar{x}$-chart with warning limits is proposed in the following section.

5.4 A Simplified Scheme

In this section a semi-economic scheme is presented which allows the user to specify the value of ARL at the rejectable quality level R_1, so that a desired level of protection against the deteriorated quality could be obtained. It is interesting to note that in Table 5.2, the ratio of the warning limit coefficient k_w to the action limit coefficient k_a lies between 0.80 and 0.90. Under the normality assumption, similar results were obtained by Chiu and Cheung [1977]. For the development of a simplified scheme, an average value of this ratio, i.e., $k_w/k_a = 0.85$ is considered. Thus, for a given value of R_1, R_0 may be treated as a function of k_a only.

Letting $\delta\sqrt{n} - k_a = a$, so that,

$$n = \frac{(a + k_a)^2}{\delta^2} \tag{5.16}$$

substituting this value of n in equation (5.14), L becomes

$$L' = \lambda UB_1 + VB_0 + \lambda W + \left(b + \frac{c(a + k_a)^2}{\delta^2}\right)/s \tag{5.17}$$

The near optimum value of k_a is obtained from:

$$\frac{\partial L'}{\partial k_a} = V \frac{\partial B_0}{\partial k_a} + \frac{2c(a + k_a)}{\delta^2 s} = 0 \tag{5.18}$$

substituting B_0:

$$V \frac{-(\frac{1}{s} - \frac{1}{2}\lambda + \frac{1}{12}\lambda^2 s)}{R_0^2} \frac{\partial R_0}{\partial k_a} + \frac{2c(a + k_a)}{\delta^2 s} = 0 \qquad (5.19)$$

That is, approximately

$$\frac{(a + k_a) R_0^2}{\frac{\partial R_0}{\partial k_a}} = \frac{\delta^2 V}{2c} \qquad (5.20)$$

Defining

$$A^* = \frac{(a + k_a) R_0^2}{\frac{\partial R_0}{\partial k_a}} \qquad (5.21)$$

For various values of k_a, one can find the corresponding values of A*, knowing the values of δ, γ_1 and γ_2 by the following procedure:

Step 1: Choose a set of values for k_a, say (k_{a1}, k_{a2},...,k_{am}), such that $k_{ai+1} > k_{ai} \geq 1$ for all i.

Step 2: For each value of k_{ai} find n_i for which $R_{1i} = R_1^*$ by using equation (5.12), where R_1^* is the set value of R_1 at the desired level of protection. Now compute R_{0i} using equation (5.12) for given value of R_{1i}.

Step 3: Having set values of R_0's (i.e., R_{01}, R_{02},...,R_{0m}) corresponding to k_a's (i.e., k_{a1}, k_{a2},...,k_{am}), compute the vector of derivatives $(\frac{\partial R_{01}}{\partial k_{a1}}, \frac{\partial R_{02}}{\partial k_{a2}}, \ldots, \frac{\partial R_{0m}}{\partial k_{am}})$ using numerical differentiation.

Hence, calculate the corresponding values of A^*'s (i.e., $A_1^*, A_2^*, \ldots, A_m^*$).

The values of k_a, n, R_0 and A^* are thus tabulated, using the computer program 'SEMIWARN' given in Appendix IV. It is noted that such a table corresponds to specific values of R_1, δ, γ_1 and γ_2. A series of such tables are thus prepared for a wide range of non-normality parameters γ_1 and γ_2, the shift parameter δ, and for a specified value of R_1. The application of one of these tables is now demonstrated through a numerical example.

An Example.

Consider the same example as in section 5.3, for which $\gamma_1 = 0$, $\gamma_2 = 0.5$ and $\delta = 2.0$. Table 5.3 is prepared for this example. The computations are performed in the following steps:

Step 1: Calculate A^*: $A^* = \dfrac{V\delta^2}{2c} = 500$

Step 2: Determine k_a, n and R_0: From Table 5.3, it is found $k_a = 2.80$, $n = 5$ and $R_0 = 328.8$.

Step 3: Calculate s: Using equation (5.15) $s = [(\frac{V}{R_0} + b+cn)/\{\lambda U(R_1 - \frac{1}{2})\}]^{1/2}$

$= 140$

Step 4: Calculate B_0: $B_0 = (\frac{1}{s} - \frac{\lambda}{2} + \frac{\lambda^2 s}{12})/R_0$

$= 0.002159$

Table 5.3 Semi-economic Scheme for Design of $\overline{X}$-chart with Warning-limits to Control Non-Normal Process Means.

R_1 =1.05	δ =2	γ_1 =0	γ_2 =0.5
k	n	R_0	A*
1.1	2	7.5	13.6
1.2	3	8.8	18.4
1.3	3	10.4	21.0
1.4	3	12.4	24.1
1.5	3	15.0	27.9
1.6	3	18.1	32.6
1.7	3	22.1	38.5
1.8	3	27.2	45.7
1.9	4	33.7	62.6
2.0	4	42.1	75.7
2.1	4	52.9	92.2
2.2	4	67.0	113.1
2.3	4	85.5	139.8
2.4	5	111.5	196.0
2.5	5	144.7	246.7
2.6	5	188.9	312.9
2.7	5	248.4	399.9
2.8	5	328.8	514.6
2.9	6	451.1	746.1
3.0	6	608.0	978.5
3.1	6	825.1	1292.9
3.2	6	1127.7	1721.0
3.3	7	1608.2	2566.9
3.4	7	2238.4	3482.9
3.5	7	3138.9	4762.4

Note: A* in this Table is as defined by equation (5.20).

Step 5: Calculate B_1: $B_1 = (R_1 - \frac{1}{2} + \frac{\lambda s}{12})s$

$= 0.7709$

Step 6: Calculate L: $L = \dfrac{\lambda U B_1 + V B_0 + \lambda W + (b+cn)(1+\lambda B_1)/s}{1 + \lambda B_1 + \tau_s B_0 + \lambda(\tau_r+\tau_s)}$

$= 2.2705$

Therfore, the semi-economic control plan specifies the parameter values as $n=5$, $s=1.40$, $k_a=2.80$, $k_w=0.85*k_a=2.38$ with the loss-cost function value of 2.2705, which is only 0.09 percent above the exact loss-cost value of 2.2685, given in Table 5.2.

CHAPTER 6

AN ECONOMIC DESIGN OF CUMULATIVE SUM CHARTS TO CONTROL NON-NORMAL PROCESS MEANS

Even though $\bar{x}$-charts have been popularly used for over fifty years, the increasing complexity of industrial processes have necessitated a search for more efficient and economical means of improving quality control. An important development in this direction was the introduction of Cusum charts by Page [1954a] which have gained wide application ever since. The major application of cumulative sum charts is in industrial quality control, where the results from testing and inspecting the product are received in sequence and a prompt decision is required when the process starts malfunctioning. In this Chapter, a single assignable cause model under operating policy II for an economic design of cusum charts is considered. The economic design of the cusum charts involves the determination of the design parameters that minimize a relevant cost function. The design parameters are the sample size n, sampling interval s, the reference value K, and the decision interval h. Approximating the non-normal probability density function of the process by an Edgeworth series, and deriving the average run lengths in cusum control schemes by the use of a system of linear algebraic equations, an expression for the expected loss-cost function for the process is defined. Using the decision interval scheme, an iterative algorithm is developed and used for near-optimal determination of design parameters. A simplified version of the algorithm is also devised. Finally, comparisons are made among the relative performances of economic design of $\bar{x}$-charts with and without warning limits and cusum charts.

6.1 The Assumption of the Process Model

The assumptions regarding the state, nature, and operating conditions of the process are the same as described in section 5.1. The operation of a cusum chart for controlling the mean of a process involves taking samples of size n at regular intervals of s hours and plotting the cumulative sums $S_r = \sum_{j=1}^{r} (\bar{x}_j - K)$ versus sample number r, where $\bar{x}_j$ is the sample mean of the jth sample, and K is the prespecified reference value. If the cumulative sum exceeds the decision interval h, it is concluded that an upward shift in the process mean has occurred. Thus the sample size n, sampling interval s, reference value K, and the decision interval h are the parameters required for designing one-sided cusum charts. To control both positive and negative deviations from the process mean a V-mask with lead distance d and half angle ϕ, or two one-sided cusum charts with reference values K_1, K_2 ($K_1 > K_2$) and with respective decision intervals h and -h may be utilized [Goel and Wu, 1973].

6.2 Formulation of Loss-Cost Function

Following the same procedure as described in section 5.2, for the development of loss-cost function for $\bar{x}$-chart with warning limits the loss-cost function of the process under a cusum control chart can be defined as follows

$$L = \frac{\lambda UB_1 + VB_0 + \lambda W + (b+cn)(1+\lambda B_1)/s}{1 + \lambda B_1 + \tau_s B_0 + \lambda(\tau_r + \tau_s)} , \quad (6.1)$$

where

$$B_0 = (\frac{1}{s} - \frac{\lambda}{2} + \frac{\lambda^2 s}{12})/R_0 \quad (6.2)$$

and

$$B_1 = (R_1 - \frac{1}{2} + \frac{\lambda s}{12})s . \quad (6.3)$$

B_0 and B_1 are calculated as in section 5.1 using the corresponding values of R_0 and R_1 from the cusum chart.

The objective is to minimize the per-hour loss-cost function L with respect to the design parameters s, h, n and K. However, it is noted that the function L also depends on R_0, and R_1 which are in turn, functions of h, n and K. Also the basic integral equations for evaluating R_0 and R_1 involve the non-normal distribution of the quality characteristic of the product. Thus an analytical solution for the design parameters seems difficult. In the following section, an iterative optimization algorithm is proposed which minimizes the loss-cost function L, and converges to near-optimal values of the design parameters. A simplified scheme to determine the design parameters is also presented, which is less complicated and therefore is more applicable at the shop level.

6.3 Determination of the Control Parameters

As noted before, the loss-cost function L depends on the ARL whose determination is one of the major difficulties in the design of cusum charts. In the past, a number of methods for obtaining the ARL have been reported which utilize either approximate expressions or numerical techniques [Barraclough and Page, 1959; Van Dobben De Bruyn, 1968; Kemp, 1958; Page, 1954b; Goel, 1971; Goel and Wu, 1971]. In the present study the basic integral equations are approximated by a system of linear algebraic equations [Goel and Wu, 1973], and solved numerically to obtain the ARL of the cusum charts for non-normal process means.

6.3.1 Determination of Reference Value K. There is strong numerical and theoretical evidence [Ewan and Kemp, 1960] that for given R_1, the value of R_0 approaches its maximum when K, the reference value, is chosen midway between the AQL and RQL. Thus,

$$K = \mu_0 + \frac{1}{2}\delta\sigma \tag{6.4}$$

6.3.2 Determination of the ARL by a System of Linear Algebraic Equations. Following the work of Page [1954a], the ARL of a one-sided cusum chart for controlling non-normal process means, with horizontal boundaries at (0,H) is defined as

$$\text{ARL} = \frac{N(0)}{1-P(0)} \tag{6.5}$$

where P(0) and N(0) are special cases at z=0 of P(z) and N(z), which are defined as follows.

$$P(z) = \int_{-\infty}^{-z} g_1(x)dx + \int_0^H P(x) \cdot g_1(x-z)dx, \quad 0 \le z \le H, \qquad (6.6)$$

and

$$N(z) = 1 + \int_0^H N(x) \cdot g_1(x-z)dx, \quad 0 \le z \le H, \qquad (6.7)$$

where $g_1(x)$ is the pdf of the standardized increments $\frac{\bar{x} - K}{\sigma/\sqrt{n}}$ in the cumulative sum, for the non-normal process with mean $\theta = \frac{\mu_0 - K}{\sigma/\sqrt{n}}$ and with non-normality parameters γ_1 and γ_2, and H is the standardized decision interval defined as

$$H = \frac{h}{\sigma/\sqrt{n}}$$

Substituting equation (4.3) into equation (6.6) we obtain

$$P(z) = \int_{-\infty}^{-z} \left[\frac{1}{\sqrt{2\pi}} e^{-1/2(x-\theta)^2} - \frac{\gamma_1}{6\sqrt{n}} \phi^{(3)}(x-\theta) + \frac{\gamma_2}{24n} \phi^{(4)}(x-\theta) + \frac{\gamma_1^2}{72n} \phi^{(6)}(x-\theta)\right]dx$$

$$+ \int_0^H \left[\frac{1}{\sqrt{2\pi}} e^{-1/2(x-\theta-z)^2} - \frac{\gamma_1}{6\sqrt{n}} \phi^{(3)}(x-\theta-z) + \frac{\gamma_2}{24n} \phi^{(4)}(x-\theta-z)\right.$$

$$\left. + \frac{\gamma_1^2}{72n} \phi^{(6)}(x-\theta-z)\right] P(x)dx$$

or,

$$P(z) = \Phi(-z-\theta) - \frac{\gamma_1}{6\sqrt{n}} \phi^{(2)}(-z-\theta) + \frac{\gamma_2}{24n} \phi^{(3)}(-z-\theta) + \frac{\gamma_1^2}{72n} \phi^{(5)}(-z-\theta)$$

$$+ \int_0^H \left[\frac{1}{\sqrt{2\pi}} e^{-1/2(x-\theta-z)^2} - \frac{\gamma_1}{6\sqrt{n}} \phi^{(3)}(x-\theta-z) + \frac{\gamma_2}{24n} \phi^{(4)}(x-\theta-z)\right.$$

$$\left. + \frac{\gamma_1^2}{72n} \phi^{(6)}(x-\theta-z)\right] P(x)dx \qquad (6.8)$$

where $\Phi(x)$ is the cumulative distribution function of the standardized normal variate x. Equation (6.8) is a Fredholm integral equation of the second kind, which may be reduced as follows, using the method given by Kantorovich and Krylov [1958].

$$\tilde{P}(z) = \Phi(-z-\theta) - \frac{\gamma_1}{6\sqrt{n}}\phi^{(2)}(-z-\theta) + \frac{\gamma_2}{24n}\phi^{(3)}(-z-\theta) + \frac{\gamma_1^2}{72n}\phi^{(5)}(-z-\theta)$$
$$+ \sum_{j=1}^{m} A_j \left[\frac{1}{\sqrt{2\pi}} e^{-1/2(z_j-\theta-z)^2} - \frac{\gamma_1}{6\sqrt{n}}\phi^{(3)}(z_j-\theta-z) + \frac{\gamma_2}{24n}\phi^{(4)}(z_j-\theta-z)\right.$$
$$\left. + \frac{\gamma_1^2}{72n}\phi^{(6)}(z_j-\theta-z)\right] \tilde{P}(z_j). \qquad (6.9)$$

In the above expression, z_j are the Gaussian points (the roots of the Legendre polynomials), A_j are the Gaussian coefficients (weights) for the interval (0,H), and m is the number of Gaussian points. To determine the values of $\tilde{P}(z_j)$, m linear algebraic equations are developed as follows. Since

$$\sum_{j=1}^{m} A_j = \text{upper limit} - \text{lower limit} = H - 0 = H,$$

and defining,

$$K(z_j,z_i) = \phi(z_j-\theta-z_i) - \frac{\gamma_1}{6\sqrt{n}}\phi^{(3)}(z_j-\theta-z_i)$$
$$+ \frac{\gamma_2}{24n}\phi^{(4)}(z_j-\theta-z_i) + \frac{\gamma_1^2}{72n}\phi^{(6)}(z_j-\theta-z_i) \qquad (6.10)$$

$$K(z_i,\theta) = \int \phi(-z_i-\theta) - \frac{\gamma_1}{6\sqrt{n}}\phi^{(2)}(-z_i-\theta) + \frac{\gamma_2}{24n}\phi^{(3)}(-z_i-\theta)$$
$$+ \frac{\gamma_1^2}{72n}\phi^{(5)}(-z_i-\theta) \tag{6.11}$$

we have the following system of linear equations.

$$\tilde{P}(z_1)-A_1K(z_1,z_1)\tilde{P}(z_1)-A_2K(z_2,z_1)\tilde{P}(z_2)\ldots-A_mK(z_m,z_1)\tilde{P}(z_m) = K(z_1,\theta)$$

$$\tilde{P}(z_2)-A_1K(z_1,z_2)\tilde{P}(z_1)-A_2K(z_2,z_2)\tilde{P}(z_2)\ldots-A_mK(z_m,z_2)\tilde{P}(z_m) = K(z_2,\theta)$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\tilde{P}(z_m)-A_1K(z_1,z_m)\tilde{P}(z_1)-A_2K(z_m,z_2)\tilde{P}(z_2)\ldots-A_mK(z_m,z_m)\tilde{P}(z_m) = K(z_m,\theta)$$

The above system can be written in a compact form by using the matrix notation. Let

$$\underline{A} = \begin{bmatrix} \{1 - A_1K(z_1,z_1)\} & \{-A_2K(z_2,z_1)\} & \ldots & \{-A_mK(z_m,z_1)\} \\ \{-A_1K(z_1,z_2)\} & \{1-A_2K(z_2,z_2)\} & \ldots & \{-A_mK(z_m,z_2)\} \\ \vdots & \vdots & & \vdots \\ \{-A_1K(z_1,z_m)\} & \{-A_2K(z_2,z_m)\} & \ldots & \{1-A_mK(z_m,z_m)\} \end{bmatrix}$$

$$\underline{P} = [\tilde{P}(z_1) \;\; \tilde{P}(z_2) \;\; \ldots \;\; \tilde{P}(z_m)]'$$

$$\underline{Y} = [K(z_1,\theta) \;\; K(z_2,\theta) \;\; \ldots \;\; K(z_m,\theta)]'$$

Hence,

$$\underline{A}\,\underline{P} = \underline{Y}$$

or $$\underline{P} = \underline{A}^{-1}\,\underline{Y}. \tag{6.12}$$

provided $\underline{A}$ is not singular.

Similarly for equation (6.7) we obtain

$$\tilde{N}(z) = 1 + [\sum_{j=1}^{m} A_j \frac{1}{\sqrt{2\pi}} e^{-1/2(z_j-\theta-z)^2} - \frac{\gamma_1}{6\sqrt{n}} \phi^{(3)}(z_j-\theta-z) + \frac{\gamma_2}{24n} \phi^{(4)}(z_j-\theta-z)$$

$$+ \frac{\gamma_1^2}{72n} \phi^{(6)}(z_j-\theta-z)] \tilde{N}(z_j) \tag{6.13}$$

which results in

$$\underline{N} = \underline{A}^{-1} \underline{I} \tag{6.14}$$

where

$$\underline{N} = [\tilde{N}(z_1) \quad \tilde{N}(z_2) \quad \ldots \quad \tilde{N}(z_m)]'$$

$$\underline{I} = [\ 1 \quad 1 \quad \ldots \quad 1\ ]'$$

The calculations of $\tilde{P}(z)$ and $\tilde{N}(z)$ are easily performed on a digital computer. The number of Gaussian points, m, is chosen to achieve the desired accuracy for a given problem. To obtain the ARL, z is set equal to zero in equations (6.9) and (6.13), and the values of $\tilde{P}(0)$ and $\tilde{N}(0)$ are then substituted for P(0) and N(0), respectively, in equation (6.5), with H and $\theta = \delta\sqrt{n}/2$ for R_1, and H and $\theta = -\delta\sqrt{n}/2$ for R_0.

6.3.3 Determination of s. The optimal value of s is obtained by setting $\frac{\partial L}{\partial s} = 0$ for given values of n, h and K. This yields

$$\lambda s^2 \{U+\tau_s B_0 U+\lambda U(\tau_r+\tau_s)-B_0 V-\lambda W\} (\frac{\partial B_1}{\partial s})$$

$$+s^2 \{V+\lambda B_1 V+\lambda V(\tau_r+\tau_s)-\lambda\tau_s B_1 U-\lambda\tau_s W\} (\frac{\partial B_0}{\partial s})$$

$$-(b+cn)[\{1+\lambda B_1+\tau_s B_0+\lambda(\tau_r+\tau_s)\}(1+\lambda B_1)$$

$$+\tau_s s(1+\lambda B_1)(\frac{\partial B_0}{\partial s})-\lambda s\{\tau_s B_0+\lambda(\tau_r+\tau_s)\}(\frac{\partial B_1}{\partial s})] = 0 \tag{6.15}$$

where

$$\frac{\partial B_0}{\partial s} = (-\frac{1}{s^2} + \frac{1}{12}\lambda^2)/R_0$$

$$\frac{\partial B_1}{\partial s} = R_1 - \frac{1}{2} + \frac{1}{6}\lambda_s$$

Equation (6.15) is a quadratic equation in s, which can be solved on a computer with an initial root derived below. In practice the values of λ and $1/R_0$ are very small. Hence the quantity $\lambda B_1+\tau_s B_0+\lambda(\tau_r+\tau_s)$ in the denominator of equation (6.1) is very small compared with unity, and therefore it can be omitted. Similarly, in the numerator, the term λB_1 is very small compared with unity and thus it can be omitted. Consequently,

$$L \simeq L' = \lambda UB_1 + VB_0 + \lambda W + (b+cn)/s \tag{6.16}$$

The equation $\frac{\partial L'}{\partial s} = 0$ is then an approximation to (6.15). Solving for s, and omitting the terms λ^2 and λ^2/R_0, one obtains:

$$s \simeq [(\frac{V}{R_0} + b+cn)/\{\lambda U(R_1 - \frac{1}{2})\}]^{1/2} \tag{6.17}$$

which serves as an initial root for the numerical evaluation of equation (6.15).

We are now in a position to outline the iterative optimization algorithm to find the design parameters of the cusum chart for non-normal processes.

6.3.4 The Algorithm.

(1) Set the initial value of sample size n_1, i.e., $n_1=1$

(2) Set the initial value of the standardized decision interval H_j, i.e., $H_1 > 0$

(3) Evaluate R_0 and R_1 from equation (6.5)

(4) Evaluate s from equation (6.15)

(5) Evaluate the loss-cost function L_j^i from equation (6.1)

(6) Increment the standardized decision interval by ΔH:

$H_{j+1} = H_j + \Delta H$

(7) Repeat steps (3) through (6) until, for some value of the index, j such as J, the following holds

$L_{J+1}^i > L_J^i < L_{J-1}^i$

Let $L^*(n_i) = L_J^i$. Thus, $L^*(n_i)$ is the minimum loss-cost function corresponding to the sample size n_i.

(8) Increment the sample size by 1:

$n_{i+1} = n_i + 1$

(9) Repeat steps (2) through (8) until, for some value of the index i, such as I, the following holds

$L^*(n_{I+1}) > L^*(n_I) < L^*(n_{I-1})$

Let $L^{**} = L^*(n_I)$. Thus L^{**} is the overall minimum value of the loss-cost function, and the values of R_0, R_1, s and H corresponding to L^{**} are the near-optimal values of the design parameters.

(10) The decision interval h is obtained from $h = H\sigma/\sqrt{n}$, and the reference value K from:

$K = \mu_0 + \frac{1}{2}\delta\sigma.$

The computer program 'CUSUM' for the above algorithm is developed and listed in Appendix V .

6.4 Numerical Illustration

In this section, the application of the algorithm is demonstrated via a numerical example. Also the properties of the optimal solution of the control chart parameters, obtained over a wide range of the values of the non-normality parameters γ_1 and γ_2, and the shift parameter δ, are discussed.

A Numerical Example

A non-normal process with mean $\mu_0 = 25$, variance $\sigma^2 = 1.2$ and non-normality parameters $\gamma_1 = 0.5$ and $\gamma_2 = 1.0$ is considered. Other parameters are assumed as follows: $\lambda = 0.05$, $\delta = 2$, $V_0 = 150$, $V_1 = 50$, $k_r = 20$, $k_s = 10$, $\tau_r = 0.2$, $\tau_s = 0.1$, $b = 0.5$ and $c = 0.1$.

The values of the loss-cost function L and the design parameters in the neighbourhood of the optimal point are shown in Table 6.1 and depicted in Fig. 6.1. The loss-cost function assumes a minimum value of

$$L^* = 7.0268$$

at the following design parameter values:

sample size $n = 5$

sampling interval $s = 0.648$ hours

standardized decision interval $H \doteq 0.70$

ARL at acceptable quality level $R_0 = 288.48$

ARL at rejectable quality level $R_1 = 1.056$

Therefore, the decision interval h is

$$h = H \frac{\sigma}{\sqrt{n}} = (0.70)(\sqrt{1.2})/\sqrt{5} = 0.34$$

Table 6.1 Values of the Loss-Cost Function and Design Parameters in the Neighbourhood of Minimum Position.

n	s	H	R_0	R_1	L
3	0.596	0.75	80.30	1.117	7.3666
	0.581	0.80	89.21	1.186	7.3592
	0.566	0.85	99.35	1.203	7.3559
	0.552	0.90	110.96	1.220	7.3565
	0.539	0.95	124.28	1.238	7.3609
4	0.633	0.65	130.86	1.083	7.1030
	0.621	0.70	147.77	1.092	7.0966
	0.611	0.75	167.47	1.102	7.0941
	0.600	0.80	190.54	1.113	7.0954
	0.590	0.85	217.76	1.124	7.1005
5	0.662	0.60	219.50	1.044	7.0309
	0.655	0.65	251.10	1.050	7.0273
	0.648	0.70	288.48	1.056	7.0268
	0.641	0.75	332.99	1.063	7.0294
	0.633	0.80	386.47	1.070	7.0350
6	0.693	0.55	347.41	1.023	7.0549
	0.688	0.60	400.41	1.026	7.0523
	0.684	0.65	463.44	1.030	7.0521
	0.679	0.70	538.88	1.034	7.0543
	0.674	0.75	629.88	1.038	7.0586

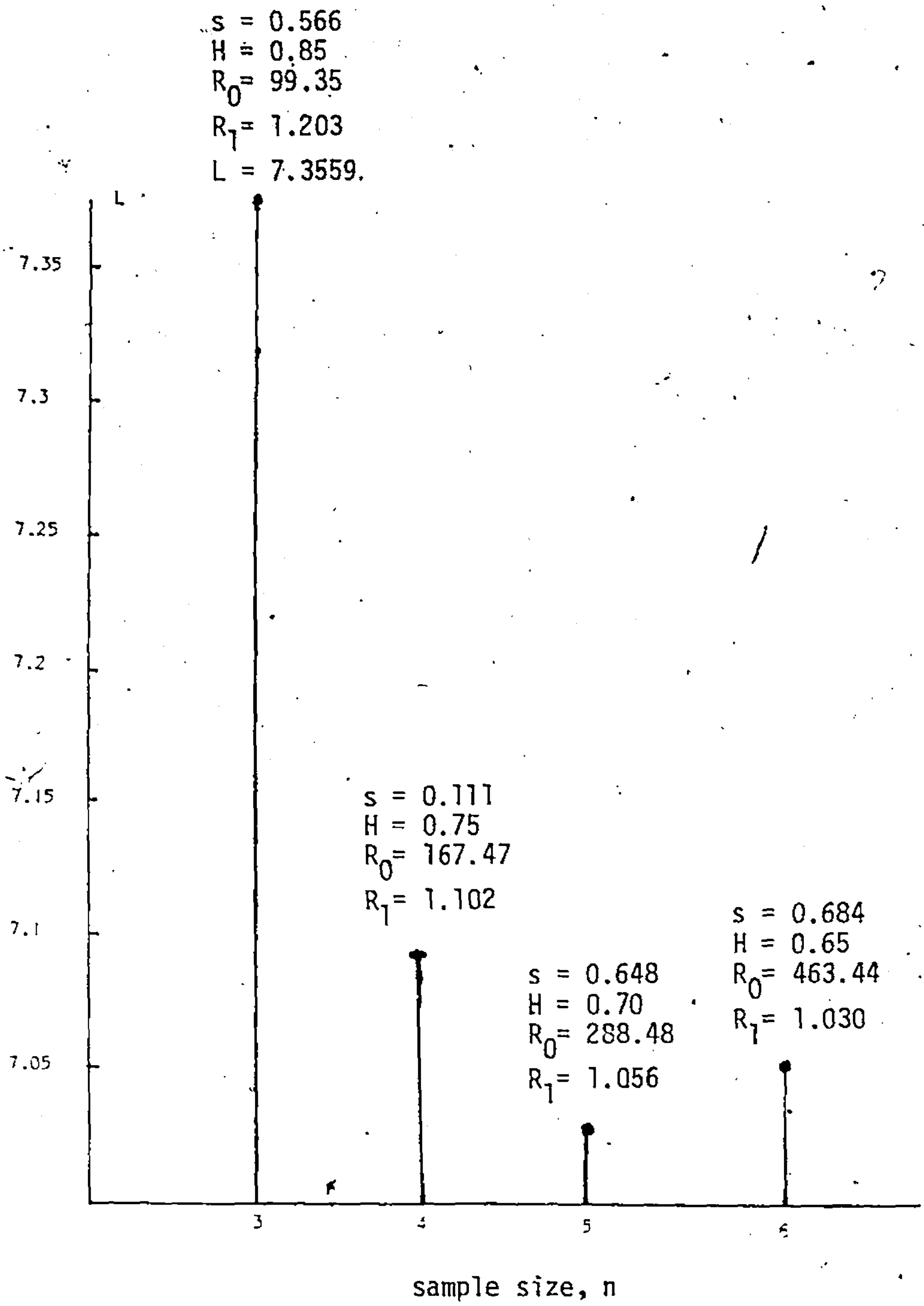

Fig. 6.1 Loss-Cost Function and Design Parameter in the Neighbourhood of Minimum Position

and the reference value is

$$K = \mu_0 + \frac{1}{2}\delta\sigma = 25 + \frac{1}{2}(2)(\sqrt{1.2}) \doteq 26.1$$

Properties of the Optimum Solution

The near-optimal values of the control chart parameters are obtained over a wide range of non-normality parameters γ_1 and γ_2, and the shift parameter δ, as shown in Table 6.2. The rate of occurrence of the assignable cause λ, and the relevant cost parameters associated with Table 6.2 are the same as those in Table 6.1. The numerical values assigned to γ_1 are -0.5, 0.0, 0.5 and those assigned to γ_2 are -0.5, 0.0, 0.5, 1.0, 1.5 and 2.0. The shift parameter δ is assumed to vary from 0.5 to 2.25 with increments of 0.25. For specific values of γ_1, γ_2, and δ, the optimal sample size n, the standardized decision interval H, the sampling interval s, and the ARLs, R_0 and R_1, are obtained by minimizing the per hour loss-cost function L using the algorithm described in section 6.3.4.

Based on the optimization results, given in Table 6.2, the following observations regarding the properties of the optimal solutions may be made.

For the pair of values of γ_1 and γ_2, the loss-cost function L decreases with the increase of δ in all cases. As δ increases, the sample size n decreases which results in the decrease of variable cost associated with sample inspection. The sampling interval also decreases as δ increases, which will increase the sampling cost. But with an increase in δ, the average

Table 6.2 Values of the Design Parameters and Loss-Cost Function of an Economic Design of Cusum Chart to Control Non-Normal Process Means.

(λ=.05, V_0=150, V_1=50, k_r=20, k_s=10, τ_r=0.2, τ_s=0.1, b=0.5, c=0.1)

γ_2	δ = 0.50, γ_1 = -0.5	δ = 0.50, γ_1 = 0.0	δ = 0.50, γ_1 = 0.5	δ = 0.75, γ_1 = -0.5	δ = 0.75, γ_1 = 0.0	δ = 0.75, γ_1 = 0.5	
-0.5	28	29	29	17	18	19	n
	1.211	1.186	1.186	1.169	1.142	1.129	R_1
	24.62	21.86	20.67	57.32	47.72	46.37	R_0
	0.400	0.350	0.350	0.500	0.450	0.450	H
	1.154	1.205	1.213	0.924	0.976	1.006	s
	11.0521	11.1057	11.1533	9.3290	9.4007	9.4563	L
0.0	28	29	29	17	18	19	n
	1.210	1.186	1.186	1.169	1.142	1.128	R_1
	24.63	21.87	20.68	56.87	47.43	46.08	R_0
	0.400	0.350	0.350	0.500	0.450	0.450	H
	1.155	1.205	1.213	0.925	0.977	1.007	s
	11.0491	11.1028	11.1505	9.3294	9.4012	9.4571	L
0.5	28	29	29	17	18	19	n
	1.210	1.185	1.185	1.168	1.141	1.128	R_1
	24.64	21.88	20.69	56.42	47.13	45.79	R_0
	0.400	0.350	0.350	0.500	0.450	0.450	H
	1.155	1.205	1.214	0.926	0.978	1.008	s
	11.0462	11.1000	11.1476	9.3298	9.4017	9.4579	L
1.0	27	29	29	17	18	19	n
	1.218	1.185	1.185	1.167	1.140	1.127	R_1
	23.37	21.89	20.70	55.99	46.84	45.51	R_0
	0.400	0.350	0.350	0.500	0.450	0.450	H
	1.142	1.206	1.214	0.927	0.979	1.008	s
	11.0429	11.0971	11.1447	9.3302	9.4022	9.4587	L
1.5	27	29	29	17	18	19	n
	1.217	1.185	1.185	1.166	1.140	1.127	R_1
	23.38	21.90	20.71	55.56	46.56	45.23	R_0
	0.400	0.350	0.350	0.500	0.450	0.450	H
	1.143	1.206	1.214	0.928	0.980	1.009	s
	11.0396	11.0943	11.1419	9.3306	9.4028	9.4596	L
2.0	27	29	29	17	18	19	n
	1.217	1.184	1.184	1.166	1.139	1.127	R_1
	23.39	21.91	20.72	55.13	46.28	44.96	R_0
	0.400	0.350	0.350	0.500	0.450	0.450	H
	1.143	1.206	1.215	0.929	0.981	1.010	s
	11.0363	11.0914	11.1390	9.3310	9.4033	9.4604	L

Table 6.2 Continued...

γ_2	δ = 1.00, γ_1 = -0.5	1.00, 0.0	1.00, 0.5	1.25, -0.5	1.25, 0.0	1.25, 0.5	
-0.5	12	13	13	8	9	10	n
	1.123	1.104	1.101	1.127	1.100	1.077	R_1
	106.08	94.15	75.30	159.19	133.73	128.02	R_0
	0.500	0.500	0.500	0.550	0.550	0.550	H
	0.817	0.857	0.873	0.706	0.752	0.793	s
	8.3754	8.4560	8.5122	7.7767	7.8611	7.9182	L
0.0	12	12	13	8	9	10	n
	1.122	1.130	1.100	1.126	1.099	1.076	R_1
	103.35	86.61	73.99	150.00	127.92	123.41	R_0
	0.500	0.550	0.500	0.550	0.550	0.550	H
	0.819	0.824	0.875	0.708	0.755	0.795	s
	8.3798	8.4601	8.5163	7.7858	7.8690	7.9251	L
0.5	12	12	13	8	9	10	n
	1.121	1.129	1.100	1.124	1.098	1.076	R_1
	100.75	84.74	72.71	141.81	122.60	119.11	R_0
	0.500	0.550	0.500	0.550	0.550	0.550	H
	0.821	0.826	0.876	0.711	0.757	0.797	s
	8.3841	8.4643	8.5204	7.7948	7.8769	7.9319	L
1.0	12	12	13	8	9	10	n
	1.121	1.128	1.099	1.123	1.098	1.076	R_1
	98.27	82.96	71.48	134.46	117.70	115.11	R_0
	0.500	0.550	0.500	0.550	0.550	0.550	H
	0.823	0.828	0.878	0.714	0.760	0.799	s
	8.3885	8.4685	8.5244	7.8038	7.8848	7.9387	L
1.5	12	12	13	8	9	10	n
	1.120	1.128	1.099	1.122	1.097	1.075	R_1
	95.92	81.24	70.29	127.84	113.18	111.37	R_0
	0.500	0.550	0.500	0.550	0.550	0.550	H
	0.824	0.829	0.879	0.717	0.762	0.801	s
	8.3928	8.4726	8.5285	7.8127	7.8926	7.9455	L
2.0	12	12	13	9	9	10	n
	1.119	1.127	1.098	1.092	1.096	1.075	R_1
	93.67	79.60	69.14	142.98	108.99	107.86	R_0
	0.500	0.550	0.500	0.500	0.550	0.550	H
	0.826	0.831	0.881	0.754	0.764	0.803	s
	8.3972	8.4767	8.5326	7.8212	7.9003	7.9523	L

Table 6.2 Continued...

Y_2	1.50			1.75			
Y_1	-0.5	0.0	0.5	-0.5	0.0	0.5	
	6	7	8	5	6	6	n
	1.113	1.081	1.055	1.092	1.058	1.057	R_1
	242.81	188.09	182.17	519.86	316.54	212.10	R_0
-0.5	0.550	0.550	0.550	0.550	0.550	0.600	H
	0.647	0.700	0.743	0.614	0.671	0.682	s
	7.3702	7.4582	7.5155	7.0727	7.1714	7.2229	L
	6	7	8	5	6	6	n
	1.112	1.080	1.054	1.091	1.058	1.057	R_1
	215.88	174.46	171.98	397.65	279.11	195.45	R_0
0.0	0.550	0.550	0.550	0.550	0.550	0.600	H
	0.651	0.703	0.746	0.619	0.674	0.685	s
	7.3849	7.4699	7.5247	7.0922	7.1851	7.2353	L
	7	7	8	5	6	6	n
	1.078	1.079	1.054	1.090	1.057	1.057	R_1
	262.08	162.67	162.88	321.96	249.60	181.23	R_0
0.5	0.500	0.550	0.550	0.550	0.550	0.600	H
	0.692	0.706	0.748	0.624	0.677	0.688	s
	7.3992	7.4815	7.5339	7.1115	7.1987	7.2475	L
	7	7	8	5	6	6	n
	1.077	1.087	1.060	1.089	1.063	1.063	R_1
	235.69	172.97	173.86	270.48	257.09	189.47	R_0
1.0	0.500	0.600	0.600	0.550	0.600	0.650	H
	0.695	0.699	0.741	0.628	0.673	0.682	s
	7.4118	7.4925	7.5427	7.1305	7.2122	7.2591	L
	7	7	8	5	6	6	n
	1.077	1.086	1.060	1.088	1.063	1.062	R_1
	214.12	161.75	165.01	233.20	233.05	176.79	R_0
1.5	0.500	0.600	0.600	0.550	0.600	0.650	H
	0.698	0.702	0.743	0.633	0.676	0.685	s
	7.4243	7.5034	7.5511	7.1495	7.2247	7.2702	L
	7	7	8	5	6	6	n
	1.076	1.085	1.060	1.087	1.062	1.062	R_1
	196.17	151.90	157.01	204.95	213.12	165.70	R_0
2.0	0.500	0.600	0.600	0.550	0.600	0.650	H
	0.701	0.705	0.746	0.638	0.679	0.688	s
	7.4368	7.5142	7.5595	7.1682	7.2372	7.2813	L

Table 6.2 Continued...

Y_2	Y_1 = 2.00: -0.5	2.00: 0.0	2.00: 0.5	2.25: -0.5	2.25: 0.0	2.25: 0.5	
-0.5	4	5	5	3	4	4	n
	1.078	1.047	1.045	1.096	1.046	1.048	R_1
	736.48	446.65	265.20	316.41	494.98	304.18	R_0
	0.500	0.550	0.600	0.550	0.550	0.650	H
	0.586	0.640	0.653	0.538	0.608	0.617	s
	6.8497	6.9570	7.0048	6.6805	6.7869	6.8417	L
0.0	4	5	5	3	4	4	n
	1.077	1.047	1.051	1.094	1.046	1.048	R_1
	476.28	370.96	269.73	703.55	387.40	264.69	R_0
	0.500	0.550	0.650	0.550	0.550	0.650	H
	0.592	0.644	0.649	0.547	0.612	0.620	s
	6.8768	6.9732	7.0194	6.7189	6.8081	6.8590	L
0.5	4	5	5	4	4	4	n
	1.084	1.047	1.050	1.040	1.051	1.054	R_1
	426.82	317.19	243.43	647.99	365.26	263.57	R_0
	0.550	0.550	0.650	0.400	0.600	0.700	H
	0.591	0.647	0.652	0.607	0.611	0.617	s
	6.9030	6.9894	7.0326	6.7525	6.8278	6.8758	L
1.0	4	5	5	4	4	5	n
	1.083	1.052	1.050	1.044	1.056	1.025	R_1
	326.56	315.56	221.80	530.06	348.27	440.93	R_0
	0.550	0.600	0.650	0.450	0.650	0.650	H
	0.597	0.645	0.656	0.608	0.609	0.654	s
	6.9284	7.0045	7.0458	6.7773	6.8470	6.8908	L
1.5	4	5	5	4	4	5	n
	1.082	1.052	1.056	1.049	1.055	1.028	R_1
	264.45	277.97	228.02	452.02	297.51	449.66	R_0
	0.550	0.600	0.700	0.500	0.650	0.700	H
	0.603	0.648	0.651	0.608	0.613	0.651	s
	6.9534	7.0193	7.0583	6.8016	6.8645	6.9001	L
2.0	4	5	5	4	4	5	n
	1.087	1.057	1.055	1.049	1.060	1.029	R_1
	249.79	279.47	210.17	356.14	291.37	408.25	R_0
	0.600	0.650	0.700	0.500	0.700	0.700	H
	0.602	0.645	0.654	0.613	0.611	0.653	s
	6.9779	7.0333	7.0700	6.8245	6.8812	6.9087	L

run length R_0 increases. This results in a decrease in the expected number of false alarms. Hence, the search cost is reduced and an increase in sampling cost is compensated. Gradual increases in the decision interval with increasing δ, is obvious. These results are depicted in Figs. 6.2 and 6.3.

For given values of δ and γ_2, the optimum values of sample size and sampling interval increase with increasing γ_1. A large sampling interval should decrease the total fixed cost of sampling. Moreover, the average run length at the rejectable quality level R_1, decreases with increasing γ_1, which results in reducing the loss-cost due to extended duration of off-target product. But, it appears from Table 6.2 that the loss-cost function is increasing with the increase of γ_1. The reason is that as γ_1 increases, the average run length, R_0 at the acceptable quality level decreases, resulting in a large number of false alarms and thus a higher loss-cost function value. One such case is depicted in Fig. 6.4. It is observed that variations in loss-cost function and in the design parameter due to variation of γ_2, are consistent in the range of -0.5 to 0.5 and 1.0 to 2.0, but its effects on R_0 is quite remarkable as shown in Fig. 6.5.

In the economic design of control charts, smaller probabilities of Type II error are desirable, since they result in smaller ARL values at the rejectable quality level, R_1. For example, for a probability of Type II error equal to 0.05, the probability of true alarm is 0.95 which corresponds to an R_1 value of $1/0.95 \approx 1.05$. This indicates that the

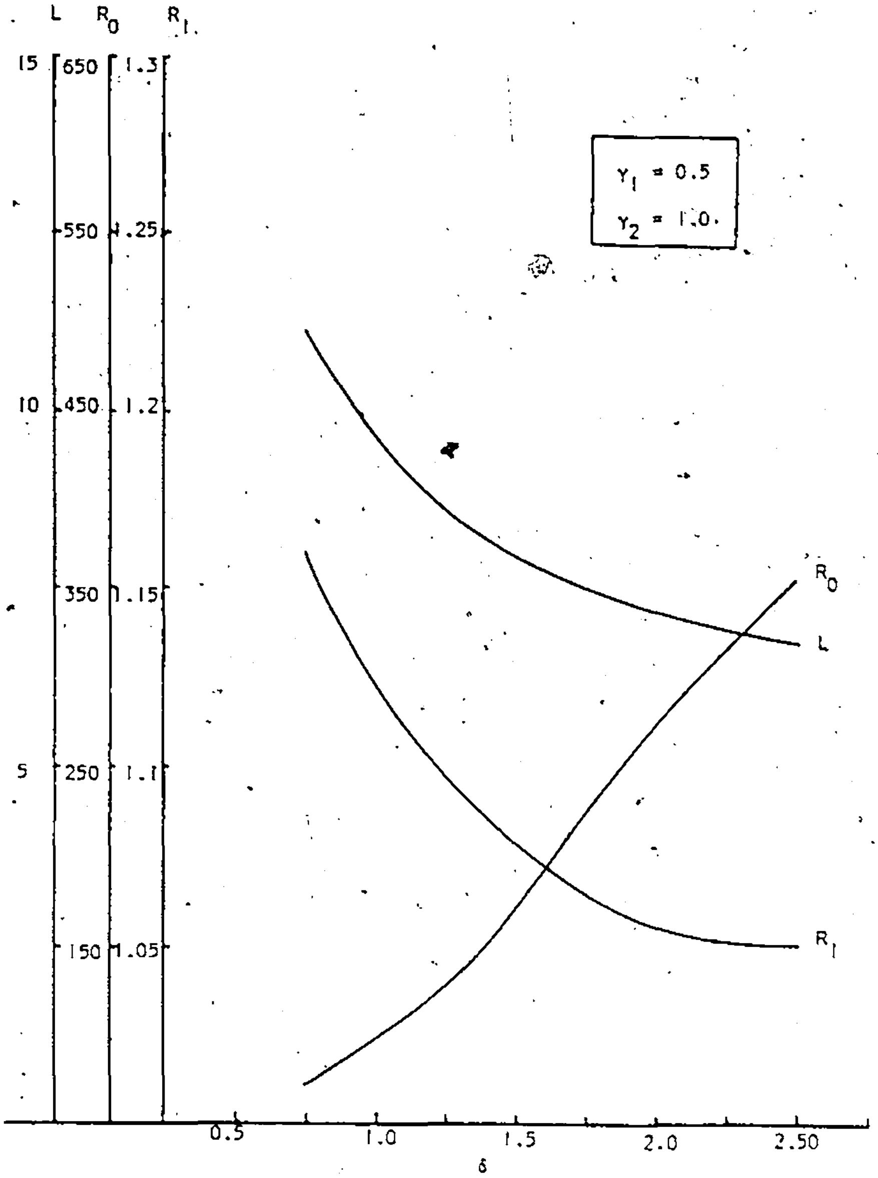

Fig. 6.2 Effect of δ for Given Values of Y_1 and Y_2 on L, R_0 and R_1.

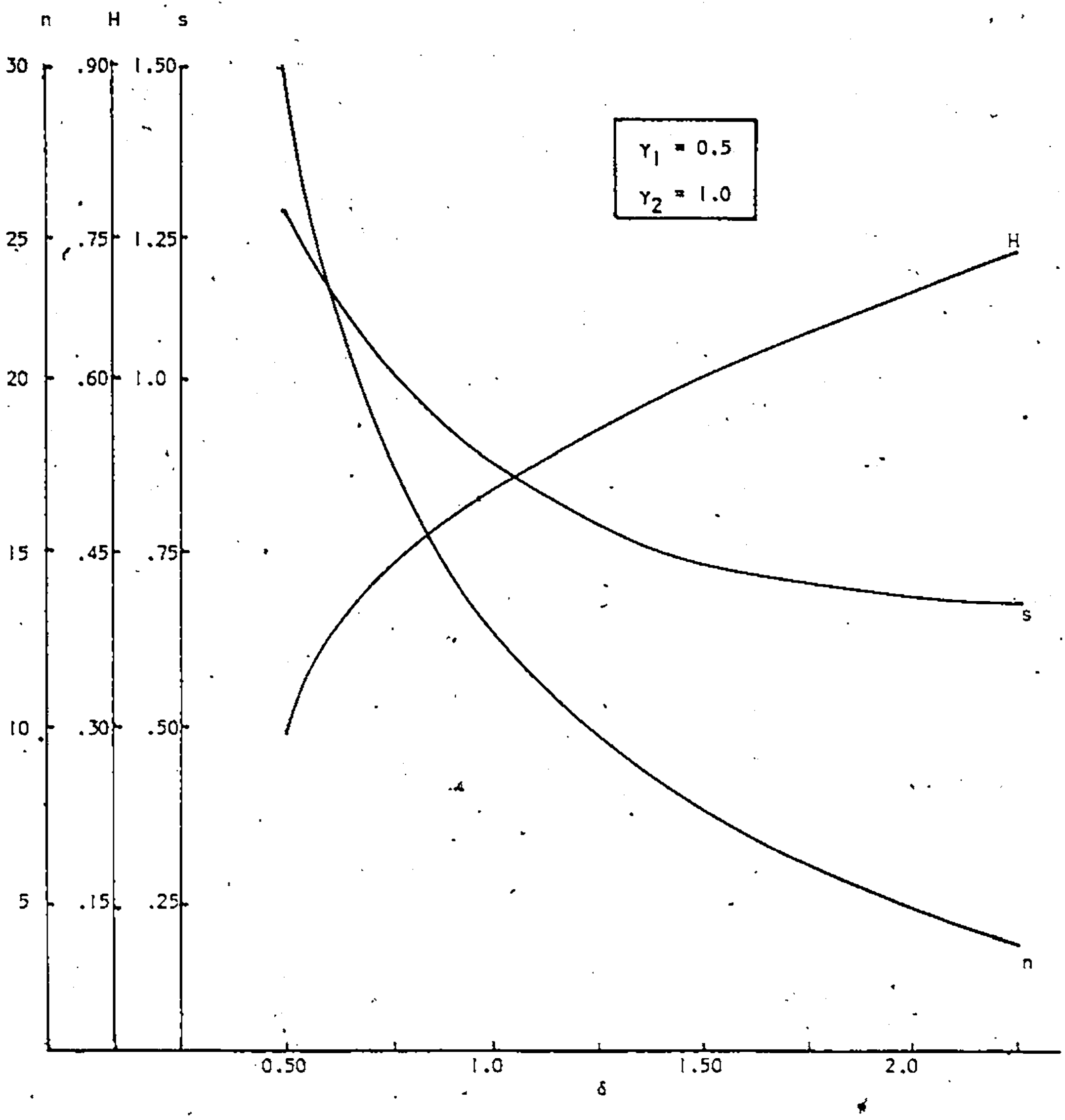

Fig. 6.3 Effect of δ for Given Values of γ_1 and γ_2 on Design Parameters.

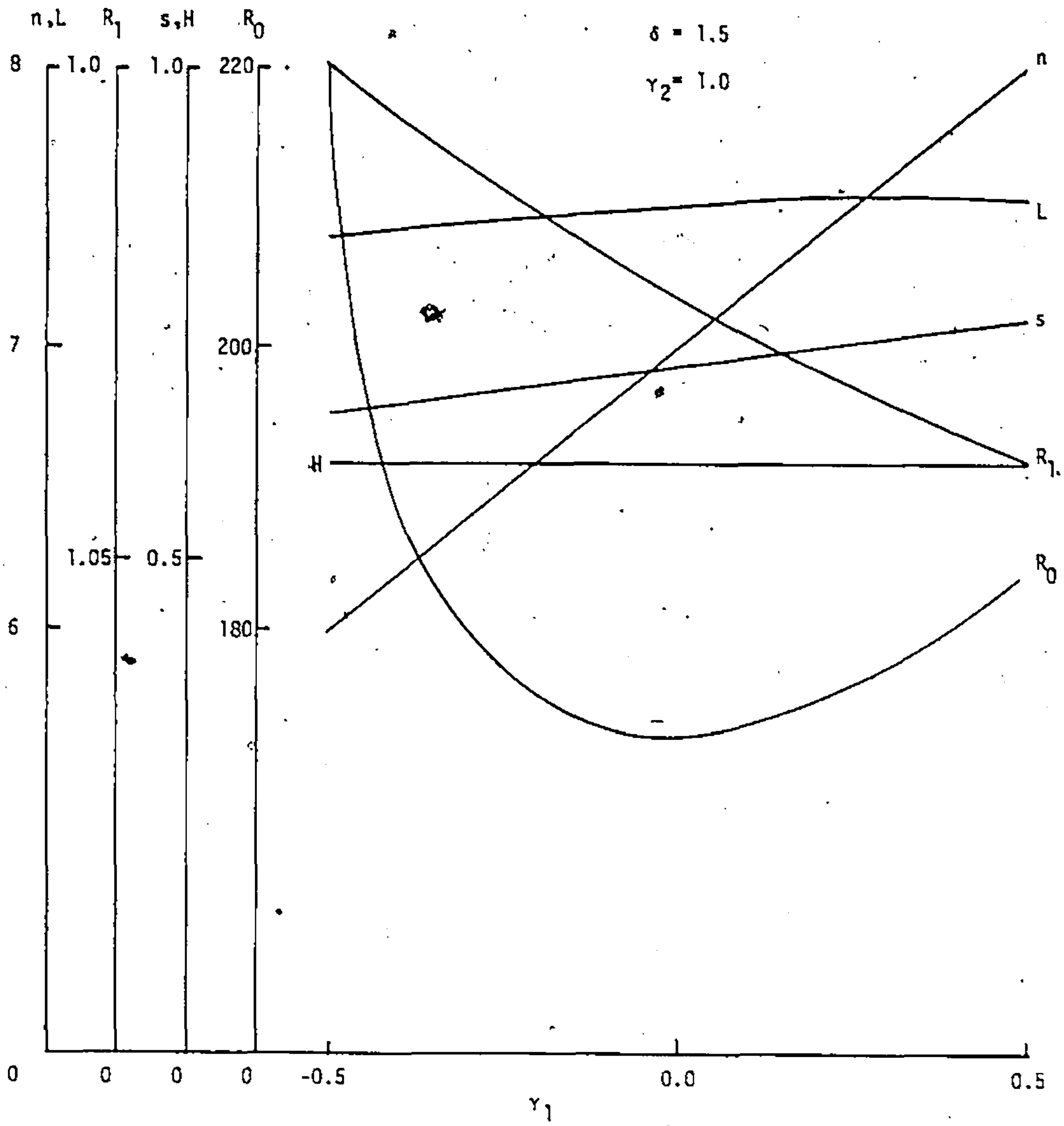

Fig. 6.4 Effect of γ_1 for Given (γ_2,δ) on Design Parameters, ARLs and Loss-Cost Function

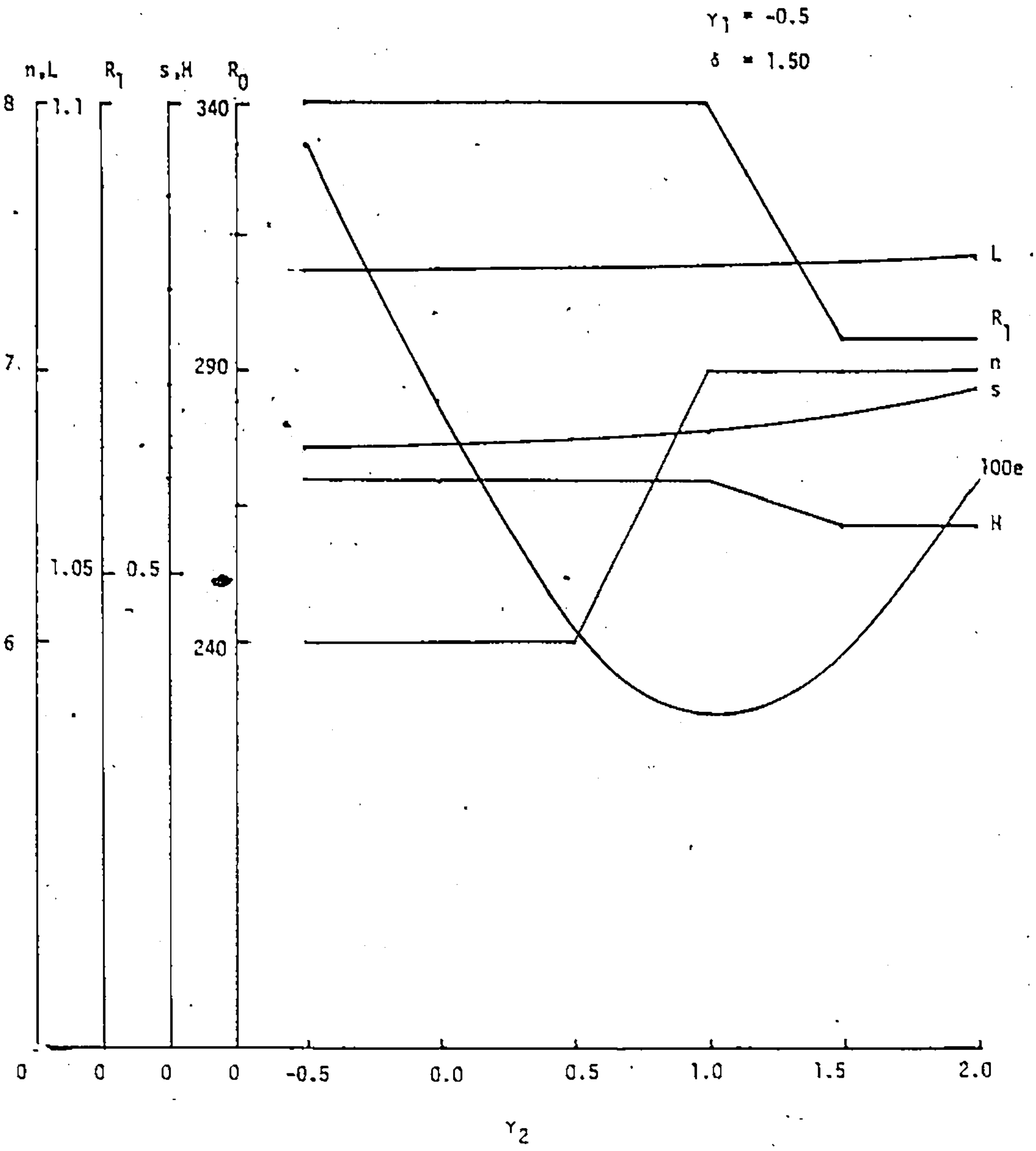

Fig. 6.5 Effect of γ_2 for Given (γ_1, δ) on Design Parameters, ARLs and Loss-Cost Function

assignable cause can be detected during a length of time equal, on the average, to 1.05 times the sampling intervals. The quick detection of the assignable cause tends to reduce the loss-cost due to prolonged production of off-target product.

6.5 A Simplified Scheme

In this section a semi-economic scheme is presented which allows the user to specify the value of the ARL at the rejectable quality level R_1, so that a desired level of protection against the deteriorated quality could be obtained. Therefore, for a given R_1, R_0 may be treated as a function of θ only.

Considering the definition of θ:

$$\theta = \frac{(\mu_0 - K)}{\sigma/\sqrt{n}} \tag{6.18}$$

and substituting the value of K from equation (6.4), we obtain

$$\theta = -\frac{1}{2}\,\delta\sqrt{n} \tag{6.19}$$

Thus;

$$n = \frac{4\theta^2}{\delta^2}\,. \tag{6.20}$$

Substituting this value of n in equation (6.16), we obtain

$$L' = \lambda UB_1 + VB_0 + \lambda W + (b+c\,.\,\frac{4\theta^2}{\delta^2})/s\,. \tag{6.21}$$

Applying the same approximation used in deriving equation (6.17), the near-optimal value of θ is obtained from:

$$\frac{\partial L'}{\partial \theta} = V \frac{\partial B_0}{\partial \theta} + \frac{8c\theta}{\delta^2 s} = 0 \tag{6.22}$$

Substituting B_0 from equation (6.2):

$$V \frac{-(\frac{1}{s} - \frac{1}{2}\lambda + \frac{1}{12}\lambda^2 s)}{R_0^2} \cdot \frac{\partial R_0}{\partial \theta} + \frac{8c\theta}{\delta^2 s} = 0$$

That is,

$$(R_0^2\theta)/(\frac{\partial R_0}{\partial \theta}) = \frac{V\delta^2}{8c} \tag{6.23}$$

Define:

$$\hat{D} = \frac{(R_0^2\theta)}{(\frac{\partial R_0}{\partial \theta})} \tag{6.24}$$

For various values of θ, one can easily compute the corresponding values of $\hat{D}$, knowing the values of δ, γ_1 and γ_2 by the following procedure:

Step 1: Choose a set of values for θ, say $(\theta_1, \theta_2, .., \theta_m)$, such that $\theta_{i+1} > \theta_i$, and $\theta_i \geq 0$, for all i.

Step 2: For each value of θ, find the corresponding value of n, using equation (6.20).

Step 3: For each pair of (n_i, θ_i) values, find H_i, using the algorithm described in section 6.3.4 for which $R_{1i} = R_1^*$ where R_1^* is the set value of R_1 at the desired level of protection. R_{0i} is now computed for the given value of R_{1i}.

Step 4: Having set the values of R_0's (i.e., R_{01}, $R_{02}, \ldots, R_{0m}$) corresponding to θ's (i.e., θ_1, $\theta_2, \ldots, \theta_m$), compute the vector of derivatives $\left(\frac{\partial R_{01}}{\partial \theta_1}, \frac{\partial R_{02}}{\partial \theta_2}, \ldots, \frac{\partial R_{0m}}{\partial \theta_m}\right)$ numerically. Hence calculate the corresponding values of $\hat{D}$'s (i.e., $\hat{D}_1$, $\hat{D}_2, \ldots, \hat{D}_m$).

Based on this algorithm a computer program 'CUSUM SEMI' is developed and is listed in Appendix VI . The values of $\hat{D}$, θ, H, and R_0 thus obtained are tabulated for later use. It is noted that such a table corresponds to specific values of R_1, δ, γ_1, and γ_2. A series of such tables are thus prepared for a wide range of non-normality parameters γ_1 and γ_2, the shift parameter δ, and for specified values of R_1. The application of these tables is now demonstrated through a numerical example.

An Example

Consider the same example as in section 6:5, for which $\mu_0 = 25$, $\sigma^2 = 1.2$, $\gamma_1 = 0.5$, $\gamma_2 = 1.0$ and $\delta = 2.0$. Table 6.3 is prepared for this example, in which $R_1 = 1.05$ by assumption. The computations are performed in the following steps:

Step 1: Calculate D: $\hat{D} = \frac{V\delta^2}{8c} = 125$.

Step 2: Determine n: From Table 6.3, find an initial value of $\theta = 2.35$ corresponding to $\hat{D} \doteq 125$. Hence $n = \frac{4(2.35)^2}{2} = 5.52$; since n must be integer, let n = 6, for which $\theta = 2.45$ from equation (6.20).

Table 6.3 Simplified Scheme for Determination of Control Parameters

$R_1=1.05$	$\delta=2.00$	$\gamma_1=.50$	$\gamma_2=1.00$
$\hat{D}$	θ	H	R_0
16.34	1.70	0.297	24.17
17.50	1.75	0.350	27.60
19.23	1.80	0.402	31.79
21.32	1.85	0.454	37.05
23.94	1.90	0.505	43.70
27.19	1.95	0.555	52.21
31.08	2.00	0.604	63.25
35.90	2.05	0.653	77.96
42.46	2.10	0.702	97.97
50.85	2.15	0.750	125.43
61.77	2.20	0.798	164.48
76.52	2.25	0.846	221.79
97.92	2.30	0.894	309.12
125.78	2.35	0.941	446.23
169.52	2.40	0.989	681.15
229.66	2.45	1.036	1103.08
335.45	2.50	1.084	1979.23
490.91	2.55	1.131	4022.55
929.65	2.60	1.179	10384.39
4761.11	2.65	1.227	34180.96

Note: $\hat{D}$ in this Table is as defined by equation (6.23).

Step 3: For $\theta = 2.45$, find $H = 1.036$, $R_0 = 1103.08$. Hence $h = 1.036\sigma/\sqrt{n} = 0.46$

Step 4: Calculate s from equation (6.17): $s = [(\frac{25}{1103.08} + 0.5 + 0.6)/(5\text{x}0.55)]^{1/2} = 0.64$ hours

Step 5: Calculate B_0 from equation (6.2): $B_0 = (1.5625 - 0.0250 + 0.0001)/1103.08 = 0.00139$

Step 6: Calculate B_1 from equation (6.3): $B_1 = (1.05 - 0.5 + 0.00267) \times 0.64 = 0.35371$

Step 7: Calculate the loss-cost function from equation (6.1)

$$L = \frac{1.7686 + 0.03475 + 3.75 + 1.7491}{1 + 0.017686 + 0.000139 + 0.015} = 7.0704$$

Step 8: Calculate $K' = 25 + \frac{1}{2}(2)(\sqrt{1.2}) \approx 26.1$

It is seen that the loss-cost for the semi-economic control plan is only 0.62 percent above the loss-cost value of 7.0268 given in Table 6.2

This simplified scheme can be easily handled by the workshop supervisor. It may also provide a good initial position for direct search for an exact optimum plan.

6.6 Application of Simplified Scheme to Two-Sided Charts

In this section we discuss how the above simplified scheme can be easily applied to a cusum chart with two-sided decision interval.

Consider a cusum chart which has an upper standardized decision interval H with central reference value K, and a lower standardized decision interval -H with a central reference value -K. Then it is well known that ARL's for the chart are

$$R_0' = \frac{1}{2} R_0 \quad \text{at AQL}$$

and

$$R_1' = R_1 \quad \text{at RQL}$$

R_0 and R_1 are the corresponding ARL's for the one-sided cusum chart specified by H and $\theta = (\mu_0 - K)/\sigma/\sqrt{n}$. It is clear that if R_0 is replaced by $\frac{1}{2} R_0$ in equation (6.5), (6.23), a simplifed scheme can be derived for the two-sided case, analogous to the one-sided case developed in section 6.5. The modified procedure for applying Table 6.3to the present case, therefore, is

Equate $\hat{D}$ to $\dfrac{V\delta^2}{4c}$.

Obtain θ, H, n, and k by the procedure of the example of section 6.5.

For the purpose of calculating s, use the formula:

$$s = [(\frac{2V}{R_0} + b + cn)/\{\lambda U(R_1 - \frac{1}{2}]\}]^{1/2}.$$

Translate H and θ into h and K by the same method used in section 6.5.

6.7 A Relative Comparison Among the Economic Design of $\bar{x}$-chart, $\bar{x}$-chart with Warning Limits and Cusum Chart

Because of its simplicity and ease of operation, the $\bar{x}$-chart with action limits has been in use for about fifty years. The $\bar{x}$-chart with warning limits has become popular during the recent years since it is generally believed that it is more efficient than the $\bar{x}$-chart for detecting the shifts in the process mean. But there are certain disadvantages in the operation of the $\bar{x}$-chart with warning limits. The use of warning limits, in addition to the control limits, implies that a certain number, R_c, of successive means must fall between the warning and control limits to take action. If, for example $R_c = 3$, then two points in this area, followed by a third mean between the centre line and the warning limit, would cause no action. Even another point between the same warning limit and control limit would not constitute convincing evidence of a shift in the process mean. In other words, there is no cumulative effect of the mean points that deviate from the expected value unless R_c number of points ($R_c = 3$ for this example) successively appear in between the warning and the control limits. Now it is certainly possible that a shift could occur in the process mean and remain undetected for a substantial period.

In contrast, the cusum chart is based on all sample points rather than the last few samples. The chart deals with retrospective examination of the past samples to detect the occurrence of significant changes in the process mean.

The purpose of this section is to make a comprehensive comparison of the performances of the $\bar{x}$-chart, the $\bar{x}$-chart with warning limits and the cusum chart at various degrees of shift in the process level. In the past, using the average run length criterion under the normality assumption, many authors [Goldsmith and Whitfield, 1961; Roberts, 1966; Goel, 1968] have studied the the performance characteristics of $\bar{x}$- and cusum charts. They compared their relative efficiencies in detecting lack of control. In this section, under non-normality assumption the minimum loss cost criterion is employed for each chart as a measure of its performance. The cost factors associated with each chart are assumed to be equal. All the three charts are considered to be one-sided.

Consider a process operating under Policy II. The shift parameter δ, mean μ_0, variance σ^2, and non-normality parameters (γ_1, γ_2) of the process along with the relevant cost factors are known. The optimum design parameters and loss-cost function of the $\bar{x}$-chart, $\bar{x}$-chart with warning limits and cusum chart are obtained by the methods given in sections 4.3, 5.3 and 6.3, respectively. The loss-costs for these charts have been compared in Table 6.4 for various shifts in the process mean.

In addition to the optimum design parameters, and loss-cost for each of the control charts at various levels of shift parameter δ, the corresponding average run lengths R_0 and R_1 are also shown.

The rate of occurrence of the assignable cause λ, mean μ_0, variance σ^2, the non-normality parameters γ_1, γ_2 and the cost factors, associated with Table 6.4 are

$\lambda = 0.05$, $\mu_0 = 0.0$, $\sigma^2 = 1.0$, $\gamma_1 = 0.5$, $\gamma_2 = 0.5$, $V_0 = 150.0$, $V_1 = 50$, $k_r = 20$, $k_s = 10$, $\tau_r = 0.2$, $\tau_s = 0.1$, $b = 0.5$ and $c = 0.1$.

It can be seen from Table 6.4 and Fig. 6.6 that for the shifts in the process mean between 0.5σ and 1.5σ the loss-cost for the cusum chart is slightly less than that of the $\bar{x}$-chart with warning limits. However, the loss-cost due to the $\bar{x}$-chart with warning limits is lower

Table 6.4 Comparison of Three Minimum-Cost Control Procedures
(λ=0.05, γ_1=0.5, γ_2=1.0, V_0=150, V_1=50, k_r=20, k_s=10, τ_r=0.2, τ_s=0.1, b=0.5, c=0.1)

δ	$\bar{x}$-Charts					$\bar{x}$-Charts With Warning Limits					Cusum Charts				
	n	s	R_0	R_1	L	n	s	R_0	R_1	L	n	s	R_0	R_1	L
0.50	30	1.26	20	1.17	11.14687	30	1.24	20	1.17	11.1443	29	1.21	21	1.19	11.1447
0.75	19	1.01	43	1.12	9.4623	19	1.01	44	1.12	9.4593	19	1.01	46	1.13	9.4587
1.00	13	0.88	70	1.10	8.5272	13	0.88	73	1.10	8.5250	13	0.88	72	1.10	8.5244
1.25	10	0.80	111	1.07	7.9397	10	0.80	114	1.07	7.9391	10	0.80	115	1.08	7.9387
1.50	8	0.76	161	1.06	7.5742	8	0.74	165	1.06	7.5426	8	0.74	174	1.06	7.5227
1.75	6	0.69	175	1.06	7.2590	6	0.68	182	1.06	7.2592	6	0.68	190	1.06	7.2591
2.00	5	0.66	222	1.05	7,0449	5	0.65	230	1.05	7.0458	5	0.66	222	1.05	7.0458
2.25	5	0.66	434	1.83	6.8885	4	0.62	238	1.05	6.8915	5	0.65	441	1.03	6.8908

than that of the $\bar{x}$-chart with action limits. But when the shift in the process mean is above 1.5σ, the loss-cost for the $\bar{x}$-chart is slightly less than that of the $\bar{x}$-chart with warning limit and cusum charts. It is also observed from Table 6.4 that, for each of these charts, the relatively large sample size and large sampling interval are ~~more~~ economical for a small shift in the process mean. However, for the shifts greater than 1.5σ, small sample size and frequent sampling are desirable.

A professional quality control engineer is always concerned with the optimal selection of producer's and consumer's risks which are, respectively, analogous to the probabilities of Type I and Type II errors for an $\bar{x}$-chart. The desirability of small or optimal values of the Type I and Type II errors for an $\bar{x}$-chart can be translated into the desirability of a large average run length R_0, when the process is in control and a short average run length R_1, when the process is out of control, respectively, for an $\bar{x}$-chart with warning limits and for a cusum chart. Extensive numerical studies based on Table 6.4 reveal that there are no appreciable differences in the average run length R_1 among these three charts.

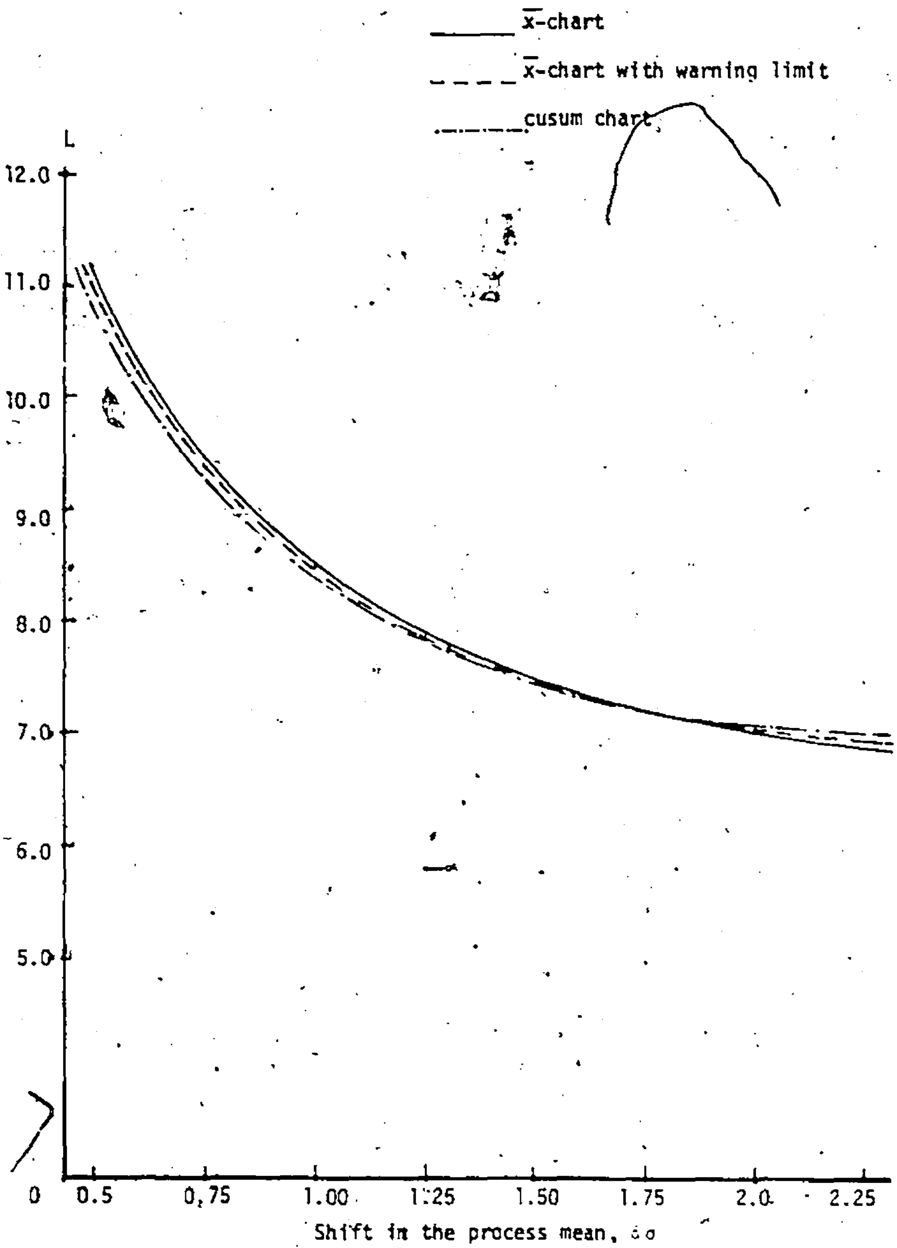

Fig. 6.6 Loss-Cost vs $\delta\sigma$ for $\overline{x}$-chart, $\overline{x}$-chart with Warning Limits and Cusum Chart

CHAPTER 7

MODEL BEHAVIOUR UNDER HUMAN ERROR AND EXTREME SAMPLE DISTRIBUTIONS

This chapter addresses the effects of human errors on the models developed for the economic design of control charts in the present study. It also includes the discussions on a simulation of the model behaviour under extreme cases of sample distributions.

7.1 The Effects of Human Errors

It has been mentioned earlier in this dissertation that the industrial products are the outputs of man-machine systems. In developing the mathematical models for the economic design of control charts in chapters 4-6, the presence of inspection or measurement errors was not considered. The assumption was that these errors did not occur or, if they did, their frequency was low enough that they had no practical importance in the models. In practice, there may be some situations where the inspection tasks or measurements are not error free. In such situations, the presence of inspection or measurement errors may seriously affect the level of protection afforded by a statistical quality control procedure [Dorris and Foote, 1979]. For these reasons, errors should not be ignored. Such errors not only severely distort quality objectives but also increase the loss-costs. However, once it is known that inspection error or measurement error is present, the quality control engineer may use revised training procedures or introduce new equipment to reduce the inaccuracies in the measurements. But these actions alone will not erase them completely [Jacobson, 1952]. Therefore, in order to make the quality control procedure more accurately representative, one should incorporate these errors in designing the underlying control plan.

Since the impact of human factors are ever-increasing in the present industrial environment, it would be fair to present the following discussion on the work done in this area which has not been included in the literature survey in chapter 2.

Effects of human errors on various aspects of attributes acceptance sampling plans have been considered in detail by several authors [Ayoub, et al., 1970 a,b; Biegel, 1974, Case, et al., 1975; Collins, et al., 1973,1978; Collins and Case, 1976; Dorris and Foote, 1978,1979; Drury, 1978; etc.]. Two types of inspection errors are possible in attribute sampling plans. These are: an item which is good may be classified as a defective (Type I error) or an item which is defective may be classified as good (Type II error). The performance measures such as probability of acceptance, average outgoing quality (AOQ), lot tolerance percent defective (LTPD), total cost per lots and others have been extensively treated in the above mentioned works.

The effects of measurement errors on variables acceptance sampling plans have also been widely studied. Among them were the work of Diviney and David [1963]; and Mei, et al., [1975] are noteworthy. There are two types of error involved in variable measurement. These are bias and imprecision.

Bias: Bias is considered as the difference between the true measurement of a product and the long run average of repeated measurements made on the product. Mathematically, bias may be expressed as: $\mu_e = E(\hat{\mu}_0) - \mu_0$, where $\hat{\mu}_0$ represents an observed measurement and μ_0 is the true measurement of a specified unit.

Imprecision: When the measuring procedure is sufficiently sensitive, repeated readings on the same unit of product will show a certain amount of scatter, whether or not there is a bias or calibration error. This second type of error can be assumed to be normally distributed and to be, at least approximately, independent of the true value of the product. The standard deviation of these scattered points is known as the imprecision error. The usual, well-known remedies of these errors are [Juran, 1951]:

1. use more precise measuring equipment;
2. institute an extensive operator-measuring-training program;
3. use average rather than single measurements.

Diviney and David [1963] presented the relationships that exist between measurement error and variable acceptance, and demonstrated a corrective procedure which effectively minimized unnecessary rejection in the variable acceptance plan. Bias, imprecision, and their combined effects on the operating characteristic curve are examined in detail by Mei, et al., [1975]. They presented a method which is explicitly designed for compensating measurement error and provides the desired operating curve.

Only two published works on the control charts under errors are available in the current quality control literature. Both of these papers assume that the measured variables are normally distributed. One of these, the P control chart under inspection error, was presented by Case [1980] and the other one by Abraham [1977] covering $\overline{x}$, R and cusum charts under the assumption of imperfect inspection. Under the assumption of the normality of the measured variables x_i and measurement error e_i, Abraham computed the ARL at the acceptable quality level for both $\overline{x}$-charts and cusum charts. These results were then compared with the corresponding values of ARL obtained when there was no measurement error. The effects of measurement error on the economic design of control charts under the assumption of non-normality of measured variables have not been studied yet.

The following assumptions are made to incorporate measurement errors in the economic model of the control charts for controlling non-normal process means.

1. The measured variables are non-normally distributed with mean μ_0, variance σ^2, measure of skewness β_1, and measure of kurtosis β_2.

2. Each measurement involves some deviation from the true value. This deviation, characterized by bias and imprecision, is the random variable normally distributed with mean 0 and variance

σ_e^2, i.e., $N(0,\sigma_e^2)$.

3. The lot distribution and the measurement error distribution are independent.

Let the observed measured variable be denoted by:

$$x_0 = x + x_e \tag{7.1}$$

where x is the true value and x_e is the measurement error. Then the probability distribution of the observed value is the convolution of the lot distribution and the measurement error distribution. That is,

$$f(x_0) = f_1(x) * f_2(x_e) \tag{7.2}$$

where * denotes convolution, $f_1(x)$ is the probability distribution of true value and $f_2(x_e)$ is the probability distribution of measurement error. The mean, standard deviation, the measure of skewness, and the measure of kurtosis of the observed x_0 are as follows:

$$x_0 = x + x_e$$

$$E(x_0) = E(x + x_e) = \mu \tag{7.3}$$

$$V(x_0) = V(x) + V(x_e)$$

$$= \sigma^2 + \sigma_e^2 \tag{7.4}$$

$$\mu_3(x_0) = \mu_3(x) \tag{7.5}$$

$$\mu_4(x_0) = \mu_4(x) + 6\sigma^2\ \sigma_e^2 + 3\sigma_e^4 \tag{7.6}$$

$$\beta_1(x_0) = \mu_3^2(x)/\ (\sigma^2 + \sigma_e^2)^3$$

$$\beta_2(x_0) = \frac{\mu_4(x_0)}{(\sigma^2 + \sigma_e^2)^2}$$

where μ_r denotes the γth corrected moments.

Therefore the non-normality parameters $\gamma_1(x_0)$ and $\gamma_2(x_0)$ of the observed values are:

$$\gamma_1(x_0) = \frac{\gamma_1(x)}{(1 + \sigma_e^2/\sigma^2)^{3/2}} \tag{7.7}$$

$$\gamma_2(x_0) = \frac{\gamma_2(x)}{(1 + \sigma_e^2/\sigma^2)^2} \tag{7.8}$$

Substituting $\gamma_1(x_0)$ and $\gamma_2(x_0)$ in equations (4.10 & 4.22) for $\gamma_1(x)$ and $\gamma_2(x)$ respectively and, thus compensating for measurement errors, one could easily proceed with the analysis of economic design of $\bar{x}$-charts for controlling non-normal process means.

7.2 Application of the Model to the Simulated Distributions

In this section, the application of one of the models developed in chapters 4-6 is illustrated through two simulated non-normal distributions. These distributions are members of the following non-normal family of distributions [Box and Tiao, 1962]:

$$f(x;\ \mu,\ \sigma,\ \eta) = \omega\, e^{-\frac{1}{2}\{|\frac{x-\mu}{\sigma}|^{2/(1+\eta)}\}}$$

where η is a "measure of non-normality", and

$$\omega^{-1} = \Gamma\{1 + \tfrac{1}{2}(1+\eta)\}\ 2^{\{1 + \frac{1}{2}(1+\eta)\}}\sigma$$

$$(-\infty < x < \infty,\quad 0 < \sigma < \infty,\quad -\infty < \mu < \infty,\quad -1 < \eta < 1).$$

In particular, when $\eta = 0$, the parent distribution becomes normal; when $\eta = 1$, the parent distribution becomes double exponential; and when $\eta \to -1$, the parent distribution tends to uniform.

This non-normal family of distributions, however, has the following limitations. In using η as above, the parent distribution considers only non-zero fourth moments and assumes a symmetric distribution. With the knowledge of the non-normality parameters of the simulated distribution, the average run length R_1, is obtained at different levels of sample size n. If the result of R_1 for given sample size n is in good agreement with the corresponding result obtained from the analytical solution, one can justify the validity of the underlying model proposed in this study. These are accomplished as follows:

I. Consider a two-parameter double exponential distribution with probability density function

$$f(x; \mu, \sigma) = \frac{1}{2\sigma} e^{-|x-\mu|/\sigma} \quad -\infty \leq x \leq \infty \quad \ldots$$

The cumulative distribution is given by

$$F(x) = \frac{1}{2} e^{(x-\mu)/\sigma} \qquad x \leq \mu$$

$$F(x) = 1 - \frac{1}{2} \cdot e^{-(x-\mu)/\sigma} \qquad x > \mu$$

The double exponential random variate x can easily be generated by means of the following steps:

(i) generate uniformly random numbers u in the interval [0,1]

(ii) if u is less than or equal to 0.5, set $u = F(x) = \frac{1}{2} e^{(x-\mu)/\sigma}$ so that $x = \mu + \sigma \log (2u)$.

(iii) if u is greater than 0.5, set $u = 1 - \frac{1}{2} e^{-(x-\mu)/\sigma}$ so that $x = \mu - \sigma \log \{2(1-u)\}$.

Having generated the random variates x_i, $(i = 1,\ldots,N)$ one can find the values of the non-normality parameters γ_1 and γ_2. Substituting these values of γ_1 and γ_2 in equation (4.22) for a given value of control limit coefficient k, and shift parameter δ, the values of n at different levels of P could be determined; hence when the process is out of control, the corresponding average run lengths of $R_1 = \frac{1}{P}$ can be found. For given values of n, k and δ, the values of P considering the actual distribution [i.e., double exponential in this case] of the process also can be obtained from:

$$P = \int_{-\infty}^{\mu_0 - k\sigma/\sqrt{n}} f(\bar{x};\, \mu_1,\, \sigma/\sqrt{n})\, d\bar{x} + \int_{\mu_0 + k\sigma/\sqrt{n}}^{\infty} f(\bar{x};\, \mu_1,\, \sigma/\sqrt{n})\, d\bar{x} \ldots$$

$$= \int_{-\infty}^{-k-\delta\sqrt{n}} f(z;\, 0,1)\, dz + \int_{k-\delta\sqrt{n}}^{\infty} f(z;\, 0,1)\, dz \qquad (7.9)$$

Numerical Illustration

Consider a double exponential population with mean of 0 and unit standard deviation. A sample distribution of this population is generated, and it is tabulated in Table 7.1, and depicted in Figure 7.1. Therefore, the values of the non-normality parameters $\gamma_1 = 0.05$ and $\gamma_2 = 2.19$ are obtained. Substituting the values of γ_1 and γ_2 in equation (4.22) the values of P and the corresponding R are obtained for different sample sizes, as shown in Table 7.2. It is assumed that the values of the control limit coefficient k and the shift parameter δ are fixed. When $k = 3$, $\delta = 2$ and $n = 5$, from Table 7.2, the average run length $R_1 = 1.07$. This value of R_1 is considered to be the model value.

The value of R_1 can also be obtained analytically using equation (7.9).

$$P = \frac{1}{2} e^{-7.47} + \int_{-1.47}^{\infty} \frac{1}{2} e^{-|z|}\, dz$$

$$= 0.003 + 1 - \frac{1}{2} e^{-1.47}$$

$$= 0.888$$

Hence, the analytical result of $R_1 = 1.12$.

It is interesting to note that the value of P can also be calculated from column 5 of Table 7.1; it is equal to $(1 - 0.109) = 0.881$.

Table 7.1 Sample Distribution of a Simulated Double Exponential Population with Mean 0 and Unit Standard Deviation.

X	FREQUENCY	CUM FREQ	PERCENT	CUM PERCENT
-5.9	1	1	0.100	0.100
-5	2	3	0.200	0.300
-4.2	1	4	0.100	0.400
-4	1	5	0.100	0.500
-3.9	1	6	0.100	0.600
-3.7	2	8	0.200	0.800
-3.5	2	10	0.200	1.000
-3.4	4	14	0.400	1.400
-3.3	6	20	0.600	2.000
-3.2	3	23	0.300	2.300
-2.9	3	26	0.300	2.600
-2.8	4	30	0.400	3.000
-2.7	2	32	0.200	3.200
-2.6	4	36	0.400	3.600
-2.5	5	41	0.500	4.100
-2.4	6	47	0.600	4.700
-2.3	4	51	0.400	5.100
-2.2	7	58	0.700	5.800
-2.1	5	63	0.500	6.300
-2	6	69	0.600	6.900
-1.9	5	74	0.500	7.400
-1.8	4	78	0.400	7.800
-1.7	10	88	1.000	8.800
-1.6	9	97	0.900	9.700
-1.5	12	109	1.200	10.900
-1.4	11	120	1.100	12.000
-1.3	11	131	1.100	13.100
-1.2	12	143	1.200	14.300
-1.1	11	154	1.100	15.400
-1	13	167	1.300	16.700
-0.9	25	192	2.500	19.200
-0.8	30	222	3.000	22.200
-0.7	28	250	2.800	25.000
-0.6	26	276	2.600	27.600
-0.5	30	306	3.000	30.600
-0.4	30	336	3.000	33.600
-0.3	35	371	3.500	37.100
-0.2	38	409	3.800	40.900
-0.1	38	447	3.800	44.700

Cont. Table 7.1

0	48	495	4.800	49.500
0.1	51	546	5.100	54.600
0.2	39	585	3.900	58.500
0.3	46	631	4.600	63.100
0.4	32	663	3.200	66.300
0.5	35	698	3.500	69.800
0.6	21	719	2.100	71.900
0.7	22	741	2.200	74.100
0.8	24	765	2.400	76.500
0.9	17	782	1.700	78.200
1	14	796	1.400	79.600
1.1	28	824	2.800	82.400
1.2	15	839	1.500	83.900
1.3	15	854	1.500	85.400
1.4	19	873	1.900	87.300
1.5	10	883	1.000	88.300
1.6	13	896	1.300	89.600
1.7	8	904	0.800	90.400
1.8	11	915	1.100	91.500
1.9	6	921	0.600	92.100
2	7	928	0.700	92.800
2.1	4	932	0.400	93.200
2.2	6	938	0.600	93.800
2.3	6	944	0.600	94.400
2.4	3	947	0.300	94.700
2.5	8	955	0.800	95.500
2.6	11	966	1.100	96.600
2.7	2	968	0.200	96.800
2.8	1	969	0.100	96.900
2.9	4	973	0.400	97.300
3	4	977	0.400	97.700
3.1	3	980	0.300	98.000
3.2	2	982	0.200	98.200
3.3	4	986	0.400	98.600
3.5	2	988	0.200	98.800
3.6	1	989	0.100	98.900
3.7	1	990	0.100	99.000
3.9	1	991	0.100	99.100
4.1	1	992	0.100	99.200
4.3	1	993	0.100	99.300
5	1	994	0.100	99.400
5.1	2	996	0.200	99.600
5.2	1	997	0.100	99.700
5.4	1	998	0.100	99.800
6.4	2	1000	0.200	100.000

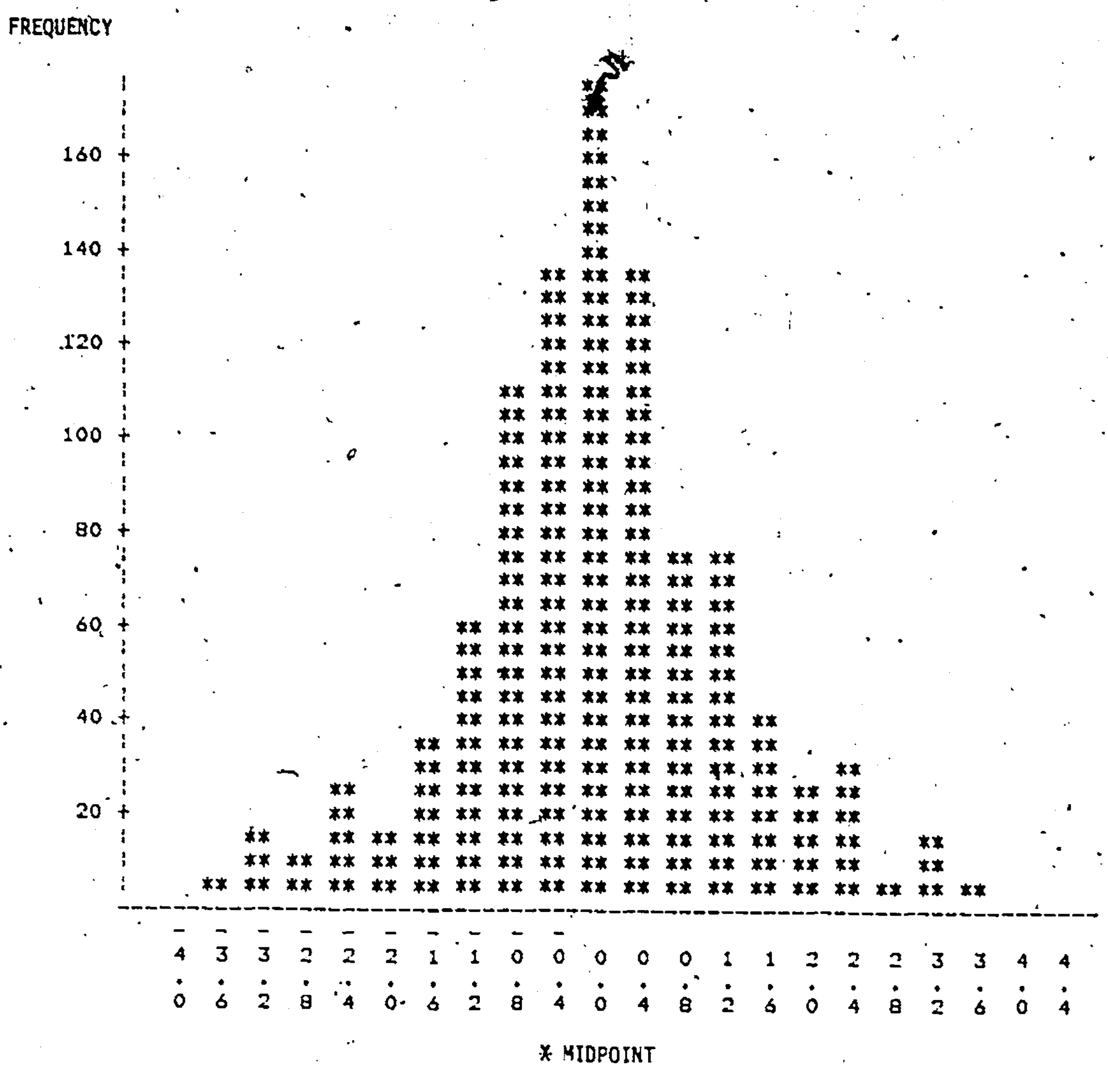

Fig. 7.1 Frequency Curve of the Simulated Double Exponential Sample Distribution.

Table 7.2 Average Run Length R_1 When a Process is Double Exponentially Distributed

$\gamma_1 = 0.05$ $\gamma_2 = 2.19$ $\delta = 2$ $k = 3$		
Sample Size n	Power of the Test P	Average Run Length R_1
1	0.11	9.01
2	0.42	2.39
3	0.69	1.45
4	0.85	1.18
5	0.93	1.07
6	0.97	1.03
7	0.98	1.02
8	0.99	1.01

which results in $R_1 = 1.14$. This indicates that the analytical result of the average run length R_1, is very close to both the simulated result and the result obtained from equation (4.22) used in the underlying model.

II. Consider a rectangular population with density function

$$f(x) = \frac{1}{B-A} \qquad A \le x \le B$$
$$= 0 \qquad \text{otherwise}$$

The cumulative distribution is given by:

$$F(x) = \int_A^x \frac{dx}{B-A} = \frac{x-A}{B-A}.$$

Assuming $A = -100$ and $B = 100$ and following a similar procedure as applied to the two-parameter double exponential distribution, the uniform random variates are generated and the sample distribution of these generated random variates are shown in Table 7.3, and depicted in Fig. 7.2. The values of the non-normality parameter are $\gamma_1 = -0.5$ and $\gamma_2 = -1.21$. Substituting these in equation (4.22), the average run length of the process under the rectangular distribution for different sample sizes is evaluated and shown in Table 7.4. It is assumed that $k = 3$, $\delta = 2$.

From Table 7.4, for $n = 5$, the average run length $R_1 = 1.08$ the corresponding value obtained from the analytical solution is 1.03. Therefore, since the values of R_1 for various sample sizes of both of these underlying simulated process distributions do not differ significantly from the corresponding analytical values, validation of the models developed in this study is quite justifiable.

Table 7.3 Sample Distribution of a Simulated Rectangular Population with Mean 0 and Unit Standard Deviation.

X	FREQUENCY	CUM FREQ	PERCENT	CUM PERCENT
-1.7	23	23	2.300	2.300
-1.6	28	51	2.800	5.100
-1.5	24	75	2.400	7.500
-1.4	24	99	2.400	9.900
-1.3	30	129	3.000	12.900
-1.2	21	150	2.100	15.000
-1.1	20	170	2.000	17.000
-1	37	207	3.700	20.700
-0.9	37	244	3.700	24.400
-0.8	28	272	2.800	27.200
-0.7	30	302	3.000	30.200
-0.6	26	328	2.600	32.800
-0.5	31	359	3.100	35.900
-0.4	22	381	2.200	38.100
-0.3	27	408	2.700	40.800
-0.2	19	427	1.900	42.700
-0.1	32	459	3.200	45.900
0	27	486	2.700	48.600
0.1	30	516	3.000	51.600
0.2	31	547	3.100	54.700
0.3	32	579	3.200	57.900
0.4	29	608	2.900	60.800
0.5	30	638	3.000	63.800
0.6	33	671	3.300	67.100
0.7	32	703	3.200	70.300
0.8	19	722	1.900	72.200
0.9	32	754	3.200	75.400
1	24	778	2.400	77.800
1.1	27	805	2.700	80.500
1.2	41	846	4.100	84.600
1.3	32	878	3.200	87.800
1.4	35	913	3.500	91.300
1.5	25	938	2.500	93.800
1.6	39	977	3.900	97.700
1.7	23	1000	2.300	100.000

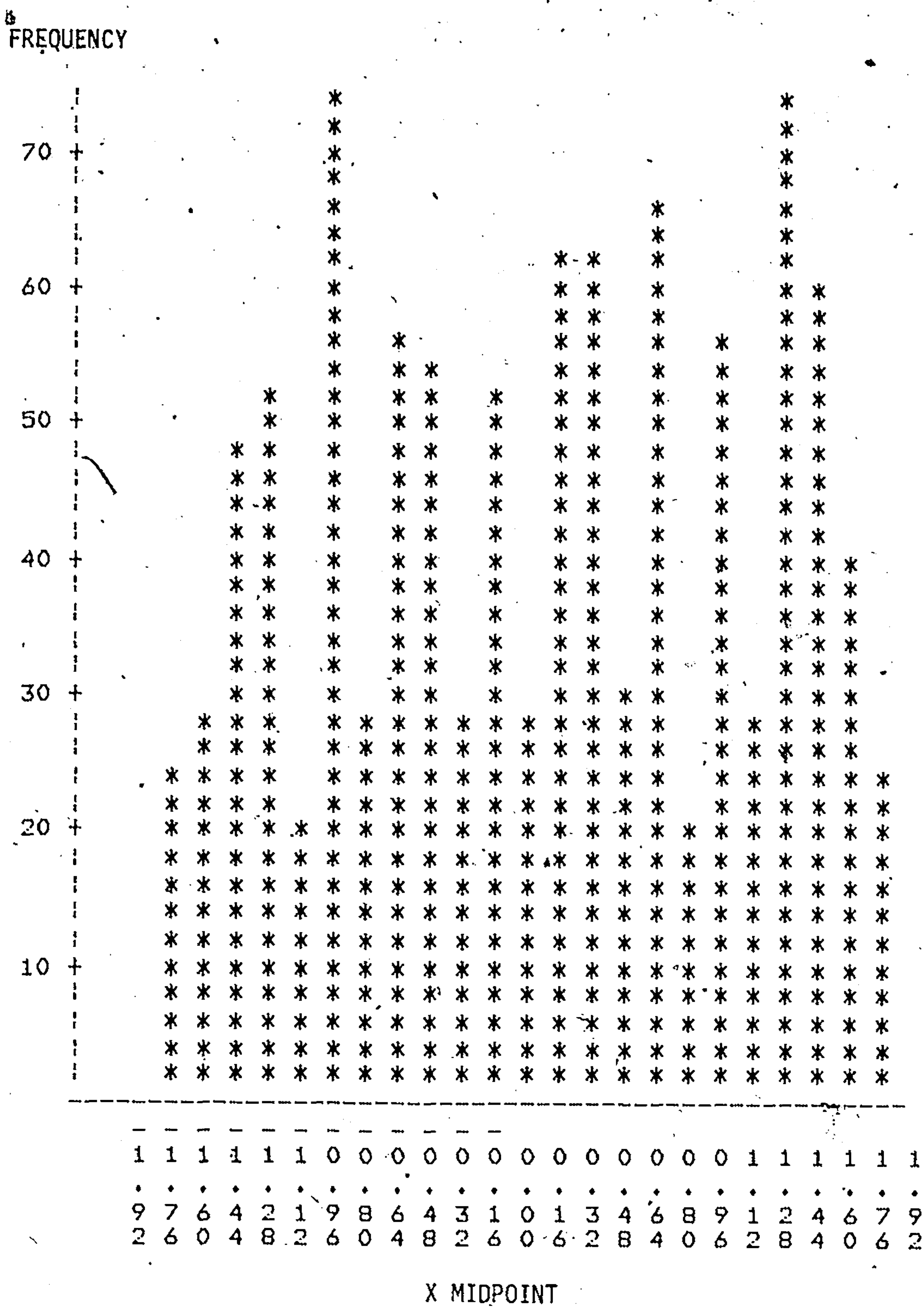

Fig. 7.2. Frequency Curve of the Simulated Rectangular Sample Distribution.

Table 7.4 Average Run Length R_1 When a Process is Rectangularly Distributed

$\gamma_1 = -0.05$ $\gamma_2 = -1.21$ $\delta = 2$ $k = 3$		
Sample Size n	Power of the Test P	Average Run Length R_1
1	0.18	5.56
2	0.43	2.33
3	0.67	1.49
4	0.83	1.20
5	0.92	1.08
6	0.97	1.03
7	0.98	1.02
8	0.99	1.01

Furthermore, the optimum designs under these distributions are obtained for a given set of cost factors, rate of occurrence of the assignable cause λ, and shift parameter δ. The corresponding design under the normality assumption is also obtained for the purpose of comparison. These are shown in Table 7.5. The results indicate that under these two extreme sample distributions, only optimal values of the average run length R_0 deviate significantly from the corresponding value obtained under the normality assumption. Although the difference in per-hour loss-cost function under normal and non-normal distributions is not significant, over a long period of operation the difference in total loss-cost may become quite considerable.

Table 7.5 Comparison of the Economic Design of $\bar{x}$-chart for Normal and Non-Normal Processes.
($\lambda=0.05$, $\delta=2$, $V_0=150$, $V_1=50$, $\tau_r=0.2$, $\tau_s=0.1$, $K_r=20$, $K_s=10$, $b=0.5$, $c=0.1$)

Process	γ_1	γ_2	n	s	k	P	R_1	α	R_0	L
Double-exponential	0.05	2.19	5	0.65	2.87	0.947	1.06	0.0039	251	7.0415
Normal	0.0	0.0	5	0.65	2.77	0.955	1.05	0.0028	357	6.9719
Rectangular	-0.05	-1.21	5	0.63	2.73	0.959	1.04	0.0018	588	6.9245

CHAPTER 8

CONCLUSIONS AND RECOMMENDATIONS

The contributions of the present research may be summarized as follows.

The research presents economic models for the design of $\bar{x}$-charts, of $\bar{x}$-charts with warning limits, and of cusum charts to control non-normal process means. Appropriate search optimization algorithms are devised and are employed on the loss-cost function, derived for the relevant control chart, to obtain the optimal values of the design parameters. In addition, a simplified scheme, applicable at the workshop level, is developed for each of the control charts. A sensitivity analysis is carried out to demonstrate the effect on the optimal solution of varying the model parameters and cost factors. Subsequently, the effect of non-normality on the design of control charts is studied. Relative performances of the three charts are compared. Furthermore, model behaviour under human error is investigated. Finally, validation of the model is justified using simulated non-normal distributions.

8.1 CONCLUSIONS

Findings of this research bring forth a number of conclusions, which are described below.

1. Economic Design of $\bar{x}$-Charts. A process with a single assignable cause of variation is considered. The loss-cost function for the process is developed under two operating policies. Policy I assumes that the process is not allowed to continue in operation during the search for an assignable cause. Policy II assumes that the process is allowed to continue in operation during the search. An optimization algorithm, based on Hooke and Jeeve's pattern search technique, is developed and employed to minimize the loss-cost function under both operating policies and thus the corresponding optimal values of the design parameters are obtained. The search technique assumes that the objective function is convex. Since it is very difficult, if not impossible, to verify analytically that the objective function is convex, some analysis of its behaviour is conducted through numerical studies and it is found that the surface of the objective function is approximately convex in the region around the optimal values. Although the above search scheme results in the most economic design, it requires a good knowledge of mathematics, statistics and computer programming. Therefore there is clearly a need for a simple and concise method that would be applicable at the workshop level. The simplified scheme developed here would serve this purpose. The tables provided for the simplified scheme can be used to determine the design parameters which minimize loss-cost for a specified level of consumer's risk (typical values are 0.1 or 0.05). Specifying the consumer's risk point to be 0.1 or 0.05, enables the manufacturer to detect the assignable cause about 1.1 or 1.05 samples, on the average, after its occurrence.

Numerical studies show that the resulting simplified scheme is close to the minimum control plan. In addition to the optimal and simplified schemes, an approximate solution procedure is also presented. However, this solution procedure considers the value of the control limit coefficient as a fixed factor. Moreover, it does not take into account the average time required to discover, and the cost of searching for the assignable cause, when it exists. Nevertheless, it could be used as a good initial point of the suggested optimum search algorithm and it will reduce computational time by a considerable amount.

A sensitivity analysis of the model reveals that the model is highly sensitive to the shift parameter and the rate of occurrence of the assignable cause, moderately sensitive to fixed and variable sampling costs, and relatively insensitive to repair and search costs. Analysis of the results showed that smaller samples should be taken more frequently to detect large shifts in the process means and large samples should be taken less frequently for smaller shifts. The solutions to the multiple assignable cause model are found very close to those of the 'matched' single assignable cause model. This widens the applicability of the proposed simplifed scheme.

2. Economic Design of $\bar{x}$-Charts with Warning Limits. An optimum $\bar{x}$-chart design with warning limits under Policy II is obtained for some sets of data. It is found that the most economic choice of critical run length R_c is equal to 2. It is also noticed that the effect of skewness is more marked than that of kurtosis. It is observed from the analysis that the ratio of warning limit coefficient K_w to action

limit coefficient k_a lies between 0.80 and 0.90. Accordingly, the average value of this ratio, i.e., 0.85, is considered and a simplified scheme is developed under the restriction that the assignable cause is detected, on the average, 1.1 or 1.05 samples after its occurrence.

3. Economic Design of Cusum Charts. The design of cusum charts involves much more mathematical complexities than the $\bar{x}$-chart with and without warning limits. Optimal values of the control chart parameters are obtained over a wide range of non-normality and shift parameters. The optimization algorithm enables one to locate the minimum where the cost surface is either strictly convex or relatively flat around the optimum. The following observations regarding properties of the optimal solution may be made. For pairs of values of γ_1 (measure of skewness) and γ_2 (measure of kurtosis), the loss-cost function L, sample size n and sampling interval s decrease with an increase of δ. By increasing δ, the average run length R_0 increases, which subsequently decreases the number of false alarms. Due to decreases in s, the expected sampling cost increases, but this increased cost trades off with the reduced search cost. For given values of δ and γ_2, the optimum values of sample size, sampling interval and loss-cost increases with increasing γ_1. But average run length, R_0, decreases with increasing γ_1. The variations in loss-cost function and in design parameters due to variation in γ_2 are not remarkable. The simplified scheme developed here can be easily handled by a quality control practitioner. It may provide a good initial point for the proposed search algorithm

and may reduce computational complexities by a considerable amount. A comparison of the performances of the three charts indicates that, for a shift in the process mean between $.5\sigma$ and 1.5σ, the performance of the cusum chart is better than that of the $\bar{x}$-chart with warning limits. However, the performance of the latter is better than that of the $\bar{x}$-chart with only action limits. With the shift in the process mean above 1.5σ, the performance of the $\bar{x}$-chart is slightly better than that of the $\bar{x}$-chart with warning limits and of the cusum chart.

4. Human Factors. Industrial products are the outputs of man-machine systems. In practice, there may be some situations where inspection tasks or measurements are not error-free. In such situations, these errors may seriously affect the level of protection afforded by the quality control procedure. In order to incorporate these errors in the models, developed in this research, the required expressions for mean, variance, measure of skewness and measure of kurtosis are derived. Under measurement errors, it is noticed that the non-normality parameters decrease with increasing σ_e^2/σ^2 (ratio of measurement error variance to process variance).

5. Application of the Models. Validation of the model is justified by the use of two non-normal simulated distributions (viz. double exponential and rectangular distributions) encountered in industry. The optimum designs under these two distributions are obtained for a given set of cost factors, rate of occurrence of the assignable cause λ, and shift parameter δ. The results indicate that

under these two extreme sample distributions only optimal values of the average run length R_0 deviate significantly from the corresponding value obtained under the normality assumption. Although, the differences in per-hour loss-cost function under normal and non-normal distributions is not significant, over a long period of operation the difference in total loss-cost may become quite considerable.

8.2 RECOMMENDATIONS

As a result of this investigation, several additional research topics may be proposed.

1. The assumption that the occurrence time of an assignable cause follows an exponential distribution could be relaxed. If the probability of a process shift within a small interval of time is directly proportional to the length of the interval, then this assumption is appropriate. However, if the assignable cause occurs as a result of the cumulative effects of heat, vibration, shock and other similar phenomena, or as a result of improper set-up or excessive stress during the process start up, then use of the exponential distribution in the model may not be appropriate [Montgomery, 1980], and serious economic consequences may result from this model assumption [Baker, 1971]. Investigation of this aspect is suggested.

2. The models developed in the present investigation require that a shift in process mean be specified. It could be of considerable interest to investigate the sensitivity of the model by assuming δ as a random

variable with a known probability density function.

3. It is assumed in this study that when the process is disturbed by an assignable cause, only the mean changes while the variance and non-normality parameters remain unchanged. It might be interesting to investigate how the $\bar{x}$-chart and σ-chart perform together as a composite unit and to determine how optimal they are under different conditions of changed mean and standard deviation.

4. An investigation can be carried out of the joint economic design of the $\bar{x}$-chart and R-chart for non-normal processes. This could be done using the present study and the works of Saniga [1977] and Singh [1970].

5. The simultaneous control of two or more related, measurable variables is of considerable importance in the field of statistical quality control when a function of the product depends on the joint effect of these variables, rather than on the separate effects of each. Under the normality assumption the problem has been considered by Jackson [1959], and Montgomery and Kalatt [1972]. Analogous to these, an attempt could be made to extend the present work into a multicharacteristic control chart.

APPENDICES

APPENDIX I

PROGRAM XBAR

PROGRAM DESCRIPTION

```
*************************************************************
!  This program is used for finding the optimal design       !
!parameters of an x-chart by minimizing the loss-cost        !
!function L. The program consists of two stages of search .  !
!In the first stage it provides an approximate solution of   !
!the design parameters. This approximate solution is used    !
!as an initial point for the second stage search. These are  !
!accomplished as follows:                                    !
!  First stage:                                              !
! An approximate solution of the sample size is obtained     !
!solving equation (4.29) for a specified value of a control  !
!limit coefficient. Function F1  presents the equation       !
!(4.29) and its root is evaluted through IMSL(International  !
!Mathematical & Statistical Libraries) routine  ZREAL1.      !
!An approximate value of sampling interval conditioned upon  !
!the sample size and control limit coefficient is evaluted   !
!using equation (4.28).                                      !
!  Second stage:                                             !
! In the second stage ,the program starts search for optimal!
!design parameters by Hook and Jeeve's pattern search.       !
!During the search, the functional value is evaluated using  !
!subroutine COST.                                            !
!                                                            !
! NOMENCLATURE                                               !
! ------------                                               !
! Model Parametrs          Descriptions                      !
! ALPHA                Type I error                          !
! B                    Fixed sampling cost, b                !
! C                    Variable sampling cost, c             !
! D                    Average time required to find an assign!
!                      able cause after a true alarm under   !
!                      policy I                              !
! DELTA                Shift parameter                       !
! E                    Time required to take and inspect a   !
!                      sample for the model operating under  !
!                      policy I                              !
! LAMDA                Rate of occurrence of assignable cause !
! KR                   Averge repair cost ,Kr under policy II !
! KS                   Average search cost ,Ks under policy II!
```

```
! R1                     Measure of skewness                        !
! R2                     Measure of Kurtosis                        !
! TR                     Average time to repair,Tr                  !
! TS                     Average time to search ,Ts                 !
! P                      Probability of true alarm                  !
! U                      Loss rate                                  !
! V                      Average cost of looking for an assign      !
!                        able cause                                 !
! V0                     Income per hour when process is in         !
!                        control                                    !
! V1                     Income per hour when the process is        !
!                        in out of control                          !
! W                      Average cost of looking for an assign      !
!                        able when none exists                      !
! Variable Name                                                     !
! First stage                                                       !
! X1                     Initial value of sample size ,n            !
! X3                     Initial value of control limit ,k          !
! X(1)                   Current value of sample size               !
! X(2)                   Current value of sampling interval,s       !
! X(3)                   Current value of control limit coeffi-     !
!                        cient                                      !
! Second stage                                                      !
! XO(I)                  Location of initial base points, I=1,3     !
! XM(I)                  Location of current base points, I=1,3     !
! XT(I)                  Location of temporary base points,I=1,3    !
! FXB                    Functional value at initial base point     !
! FXE                    Functional value at current base point     !
! FXT                    Functional value at temporary base point   !
! ICMAX                  Maximum number of iterations               !
! IC                     Number of iterations                       !
!                                                                   !
! OUTPUT DESCRIPTION                                                !
! ------------------                                                !
! XM(1)                 Sample size, n                              !
! XM(2)                 Sampling interval, s                        !
! XM(3)                 Control limit coefficient, k                !
! P                     Probability of true alarm, P                !
! ALPHA                 Probability of false alarm, Alpha           !
**************************************************************
C                                 PROGRAM LISTING
C     **************************************************************
C     * THE ECONOMIC DESIGN OF X-CHARTS TO CONTROL NON-NORMAL        *
C     *PROCESS MEANS . THE PRORAM IS MEANT FOR BOTH OPERATING        *
C     *POLICIES. THE NUMERICAL VALUES ASSIGNED TO R1 ARE -0.5,0.0,   *
C     *0.5, 1.0 AND TO R2 ARE -0.5,0.0,0.5,1.0,1.5 AND 2.0 . THE     *
C     *SHIFT PARAMETER DELTA ASSUMED TO 0.5 TO 2.25 WITH INCREMENTS*
C     *OF 0.25.                                                      *
C     **************************************************************
      EXTERNAL F1
```

```
      INTEGER NSIG,ITMAX,LL,IER,NN
      REAL LAMDA,KS,KR,F1,EPS,EPS2,BTA,X4(1)
      DIMENSION X(3),XO(3),XM(3),TABLE(8,4,6,6),DEL(3)
      COMMON DELTA,LAMDA,R1,R2,V,C,TS,W,TR,B,D,E,ALPH,FP,DK,U,KR,KS
C     *****************************************************************
C     *                          SET MODEL PARAMETERS                 *
C     *****************************************************************
      LAMDA=0.05
      VO=150.0
      V1=50.00
      KR=20.0
      KS=10.0
      TR=0.2
      TS=0.1
      V=KS+VO*TS
      W=KR+KS+VO*(TR+TS)
      U=VO-V1
      B=0.5
      C=0.1
      D=0.0
      E=0.0
      DELTA=0.5
      IDELTA=1
  400 CONTINUE
      R1=-0.5
      IR1=1
  300 CONTINUE
      R2=-0.5
      IR2=1
  200 CONTINUE
C     **********************************************************
C     *                     FIRST STAGE SEARCH                 *
C     **********************************************************
      IF(DELTA.EQ.0.5) X1=45.0
      IF(DELTA.EQ.0.75) X1=35.0
      IF(DELTA.EQ.1.0) X1=25.0
      IF(DELTA.EQ.1.25) X1=15.0
      IF(DELTA.GE.1.5) X1=5.0
      IF(DELTA.LE.1.5) X3=2.50
      IF(DELTA.GT.1.50) X3=3.0
      DK=X3
      X4(1)=X1
C     ************************************************************
C     * COMPUTE THE ROOT OF EQUATION (4.29) USING IMSL ROUTINE*
C     * ZREAL1. ARGUMENT REPRESENT: F1-A FUNCTION SUBPROGRAM  *
C     * WRITTEN BY USER, EPS -FIRST STOPPING CRITERION,EPS2-  *
C     * SPREAD CRITERIA FOR MULTIPLE ROOTS, NSIG-2ND STOPPING *
C     * CRITERION, ITMAX-MAXIMUM NUMBER OF ITERATIONS,LL-NO.  *
C     * OF ROOTS TO BE FOUND, IER-ERROR PARAMETER,X4-ROOT     *
C     ************************************************************
```

```
      EPS=1.0E-5
      EPS2=1.0E-5
      BTA=1.0E-3
      NSIG=5
      ITMAX=1000
      LL=1
      CALL ZREAL1(F1,EPS,EPS2,BTA,NSIG,LL,X4,ITMAX,IER)
      NN1=X4(1)+0.5
      X(1)=NN1
      X(2)=SQRT((ALPH*T+B+C*X(1))/(LAMDA*U*FP))
      X(3)=X3
C     ************************************************************
C     *                    SECOND STAGE SEARCH                   *
C     ************************************************************
      XO(1)=X(1)
      XO(2)=X(2)
      XO(3)=X(3)
      DEL(1)=1.0
      DEL(2)=0.5
      DEL(3)=0.5
      NN=3
      CALL SUB(DEL,NN,XT,XO,XM,FXB,P,ALPHA)
      TABLE(IDELTA,IR1,IR2,1)=XM(1)
      TABLE(IDELTA,IR1,IR2,2)=XM(2)
      TABLE(IDELTA,IR1,IR2,3)=XM(3)
      TABLE(IDELTA,IR1,IR2,4)=P
      TABLE(IDELTA,IR1,IR2,5)=ALPAH
      TABLE(IDELTA,IR1,IR2,6)=FXB
550   R2=R2+0.5
      IR2=IR2+1
      IF(R2.GT.2.0) GO TO 600
      GO TO 200
600   R1=R1+0.5
      IR1=IR1+1
      IF(R1.GT.1.0) GO TO 700
      GO TO 300
700   DELTA=DELTA+0.25
      IDELTA=IDELTA+1
      IF(DELTA.GT.2.25) GO TO 800
      GO TO 400
800   PRINT 1,LAMDA,VO,V1,KR,KS,TR,TS,B,C,D,E
      IM=1
      IN=2
      PRINT 2
      DO 1100 IJ=1,4
      IF(IJ.NE.1) PRINT 5
      XIN=IN
      DELTA1=0.25*XIN
      DELTA2=DELTA1+0.25
      PRINT 3,DELTA1,DELTA2
```

```
      PRINT 4
      R2=-0.5
      DO 1000 IR2=1,6
      PRINT 11,((TABLE(IDELTA,IR1,IR2,1),IR1=1,4),IDELTA=IM,IN)
      PRINT 12,((TABLE(IDELTA,IR1,IR2,2),IR1=1,4),IDELTA=IM,IN)
      PRINT 13,R2,((TABLE(IDELTA,IR1,IR2,3),IR1=1,4),IDELTA=IM,IN)
      PRINT 14,((TABLE(IDELTA,IR1,IR2,4),IR1=1,4),IDELTA=IM,IN)
      PRINT 15,((TABLE(IDELTA,IR1,IR2,5),IR1=1,4),IDELTA=IM,IN)
      PRINT 16,((TABLE(IDELTA,IR1,IR2,6),IR1=1,4),IDELTA=IM,IN)
      IF (IR2.EQ.6) GO TO 1000
      PRINT 4
      R2=R2+0.5
1000   CONTINUE
      IM=IM+2
      IN=IN+2
      R2=-0.5
      PRINT 2
1100   CONTINUE
      STOP
1     FORMAT('1'////////'LAMDA=',F4.2,2X,'PO=',F6.1,2X,'P1=',F6.1,2X,
     *'KR=',F5.1,2X,'KS=',F5.1,2X,'TR=',F4.2,2X,'TS=',F3.1,2X,'B=',
     *F4.2,2X,'C=',F4.2,2X,'D=',F3.1,2X,'E=',F3.1)
2      FORMAT(' +',80('-'),'+')
3      FORMAT(' |      |',74X,'|'/
     *' |      |',31X,'DELTA',38X,'|'/
     *' |      |',14X,F4.2,30X,F4.2,22X,'|'/
     *' |      |  ',2(30('-'),4(' ')),'     |'/
     *' |      |',74X,'|'/
     *' | R2   |',33X,'R1',39X,'|'/
     *' |      | ',2('  -0.5       0.0       0.5       1.0     '),
     *'     |'/
     *' |      | ',2(4(' ------ '),'  '),'      |')
4      FORMAT(' |-----|',2(4('--------'),'-|'),'------|')
 5     FORMAT('1'////////' +',80('-'),'+')
11     FORMAT(' |      |',2(4F8.0,' |'),'   N  |')
12     FORMAT(' |      |',2(4F8.2,' |'),'   S  |')
13     FORMAT(' |',F4.1,' |',2(4F8.2,' |'),'   K  |')
14     FORMAT(' |      |',2(4F8.4,' |'),'   P  |')
15     FORMAT(' |      |',2(4F8.4,' |'),' ALPHA|')
16     FORMAT(' |      |',2(4F8.4,' |'),'   F  |')
      END
C     **************************************************
C     *         HOOK AND JEEVES SEARCH ROUTINE          *
C     **************************************************
      SUBROUTINE  SUB(DEL,NN,XT,XO,XM,FXB,P,ALPHA)
      DIMENSION XX1(3),XT(3),XO(3),XM(3),DEL(3)
      ICMAX=1000
      XM(1)=XO(1)
      XM(2)=XO(2)
      XM(3)=XO(3)
```

```
      XM1=XM(1)
      XM2=XM(2)
      XM3=XM(3)
      CALL COST(XM1,XM2,XM3,P,ALPHA,FXB)
      IC=0
   21 KK=0
      DO 11 I=1,NN
      TEMP=XM(I)
      XM(I)=XM(I)+DEL(I)
      XM1=XM(1)
      XM2=XM(2)
      XM3=XM(3)
      CALL COST(XM1,XM2,XM3,P,ALPHA,FXE)
      IF((FXE.LT.FXB).AND.(P.GT.0.0).AND.(ALPHA.
     *GE.0.0)) GO TO 12
      XM(I)=TEMP
      XM(I)=XM(I)-DEL(I)
      XM1=XM(1)
      XM2=XM(2)
      XM3=XM(3)
      CALL COST(XM1,XM2,XM3,P,ALPHA,FXE)
      IF((FXE.LT.FXB).AND.(P.GT.0.0).AND.(ALPHA.
     *GE.0.0)) GO TO 12
      XM(I)=TEMP
      GO TO 11
   12 FXB=FXE
      KK=1
   11 CONTINUE
      IF(KK.EQ.0)GO TO 18
      DO 16 I=1,NN
   16 XX1(I)=XM(I)
      DO 13 I=1,NN
   13 XT(I)=2.*XX1(I)-XO(I)
      XT1=XT(1)
      XT2=XT(2)
      XT3=XT(3)
      CALL COST(XT1,XT2,XT3,P,ALPHA,FXT)
      DO 14 I=1,NN
   14 XO(I)=XX1(I)
      IF((FXT.LT.FXB).AND.(P.GT.0.0).AND.(ALPHA.
     *GE.0.0)) GO TO 15
      GO TO 21
   15 FXB=FXT
      DO 17 I=1,NN
   17 XM(I)=XT(I)
      IC=IC+1
      IF(IC.GE.ICMAX) GO TO 19
      GO TO 21
```

```
18     DO 28 I=2,NN
       IF(DEL(I).GT.0.001) GO TO 30
       GO TO 29
30     DEL(I)=DEL(I)/2.0
29     DEL(I)=DEL(I)
28     CONTINUE
       IF((DEL(2).LE.0.001).AND.(DEL(3).LE.0.001)) GO TO 19
       GO TO 21
   19  CONTINUE
       RETURN
       END
C      **************************************************************
C      *    A SUBROUTINE FOR COMPUTING LOSS-COST FUNCTION           *
C      **************************************************************
       SUBROUTINE COST(X1,X2,X3,P,ALPHA,F)
       REAL LAMDA,KS,KR
       COMMON DELTA,LAMDA,R1,R2,V,C,TS,W,TR,B,D,E,ALPH,FP,DK,U,KR,KS
       ALPHA=ALPH
       IF((X1.LE.0.0).OR.(X2.LE.0.0).OR.(X3.LE.0.0).OR.(ALPHA.LE.
      *0.0)) GO TO 84
       Y1=X3-DELTA*SQRT(X1)
       CALL EDGW(Y1,X1,H2X,H3X,H4X,H5X,H6X,FIX,P,R1,R2)
       TT=(1.0-(1.0+LAMDA*X2)*EXP(-X2*LAMDA))/
      1(LAMDA-LAMDA*EXP(-LAMDA*X2))
       B1=X2/P-TT+E*X1+D
       Y3=X3
       CALL EDGW(Y3,X1,H2K,H3K,H4K,H5K,H6K,DIX,P,R1,R2)
       ALPHA1=DIX
       P1=(H2K*R1)/(6.0*SQRT(X1))-(H3K*R2)/(24.0*X1)-H5K*(R1*R1
      1)/(72.0*X1)
       ALPHA2=P1
       ALPHA=ALPHA1+ALPHA2
       B0=ALPHA*(1.0-LAMDA*TT)/X2
       U1=LAMDA*B1*U+LAMDA*W+T*B0+(B+C*X1)*(1.0+LAMDA*B1)/X2
       U2=(1.0+LAMDA*B1+TS*B0+LAMDA*(TR+TS))
       F=U1/U2
84     RETURN
       END
       REAL FUNCTION F1(X4)
       REAL X4
       REAL LAMDA,KR,KS
       COMMON DELTA,LAMDA,R1,R2,V,C,TS,W,TR,B,D,E,ALPH,FP,DK,U,KR,KS
       IF(X4.LE.0.0) GO TO 86
       X2=DK
       CALL EDGW(Y1,X4,H2X,H3X,H4X,H5X,H6X,FIX,P,R1,R2)
       IF(P.GT.0.0) GO TO 85
       GO TO 86
       Y1=X2-DELTA*SQRT(X4)
85     FP=1/P-0.5
       Y3=X2
       CALL EDGW(Y3,X4,H2K,H3K,H4K,H5K,H6K,DIX,P,R1,R2)
```

```
      ALPHA1=DIX
      P4=(H2K*R1)/(6.0*SQRT(X4))-(H3K*R2)/(24.0*X4)-(H5K*R1*R1
     1)/(72.0*X4)
      ALPHA2=P4
      ALPH=ALPHA1+ALPHA2
      IF(ALPH.LE.0) GO TO 86
      DPN1=(DELTA)/(2.0*SQRT(X4))*ZX
      DPN2=-12.0*R1*(DELTA*X4*H3X+SQRT(X4)*H2X)
      DPN3=3.0*R2*(DELTA*SQRT(X4)*H4X+2.0*H3X)
      DPN4=R1*R1*(DELTA*SQRT(X4)*H6X+2.0*H5X)
      DPN5=(DPN2+DPN3+DPN4)/(144.0*X4*X4)
      DPN=DPN1+DPN5
      F1=ALPHA2*T+X4*(C-DPN*(ALPH*T+B+C*X4)/(P**2*FP)+LAMDA*U*E)
86    RETURN
      END
C     ********************************************************
C     * COMPUTE THE FIRST FOUR TERMS OF AN EDGEWORTH SERIES *
C     * AND PROBABLITY OF TYPE I ERROR                       *
C     ********************************************************
      SUBROURTINE EDGW(Y1,X1,H2X,H3X,H4X,H5X,H6X,FIX,P,R1,R2)
      IF(Y1.EQ.0.0) GO TO 89
      ZX=0.39894228*EXP(-(Y1*Y1)/2.0)
      GO TO 91
   89 ZX=0.39894228
91    H2X=(Y1*Y1-1.0)*ZX
      H3X=-(Y1**3-3.0*Y1)*ZX
      H4X=(Y1**4-6.0*Y1**2+3.0)*ZX
      H5X=-(Y1**5-10.0*Y1**3+15.0*Y1)*ZX
      H6X=(Y1**6-15.0*Y1**4+45.0*Y1**2-15.0)*ZX
C     **********************************************************
C     * COMPUTE THE AREA UNDER NORMAL CURVE USING IMSL ROUTINE *
C     * MSMRAT(Y1,RM,IER). ARGUMENT REPRESENT: Y1-VARIATE, RM- *
C     * THE RATIO OF THE ORDINATE TO THE UPPER TAIL AREA ,IER- *
C     * ERROR PARAMETER                                        *
C     **********************************************************
      CALL MSMRAT(Y1,RM,IER)
      FIX=ZX/RM
      P=FIX+H2X*R1/(6.0*SQRT(X1))-H3X*R2/(24.0*X1)-H5X*(R1*R1
     1)/(72.0*X1)
      RETURN
      END
```

APPENDIX II

PROGRAM SEMIXBAR

```
                     PROGRAM DESCRIPTION
*************************************************************
! This program is used for generating user's manual to      !
!determine the design parameters for an x-chart to control  !
!both normal and non-normal means. The fundamental objective !
!of this plan is to detect the assignable cause,when it     !
!occurs, with probability of 0.90 Or 0.95.                  !
!                                                           !
! NOMENCLATURE                                              !
!                                                           !
!  Variable name              Description                   !
!    NN                   Maximum sample size, n            !
!    DK                   Control limit coefficient,k       !
!    DELTA                Shift parameter                   !
!    R1                   Measure of skewness               !
!    R2                   Measure of Kurtosis               !
!    X1                   Specified value of true alarm     !
!                                                           !
! OUTPUT                                                    !
!    DK                   Control limit coeffiecient, k     !
!    I                    Sample size , n                   !
!    P(I)                 Probability of true alarm having  !
!                         sample size I                     !
!    ALPHA(I)             Probability of type I error       !
!                         conditioned upon the true alarm P(I) !
!    A(I)                 Presents equation (4.40)          !
*************************************************************
C                         PROGRAM LISTING
      EXTERNAL ALPH
      DIMENSION ALPHA(125),A(125),ALPHA4(125),P(125)
C     ***********************************************************
C     * SET MODEL PARAMETERS R1,R2,DELTA AND MAXIMUM ALLOWABLE  *
C     * SAMPLE SIZE NN                                          *
C     ***********************************************************
      NN=125
      DELTA=1.75
      R1=1.0
      R2=2.0
      DK=1.0
100   DO 300 I=1,NN
      BI=I
      X=DELTA*SQRT(BI)-DK
      CALL EDWOR(X,ZX,H2X,H3X,H4X,H5X,H6X,FIX,R1,R2)
      P(I)=1.0-FIX+H2X*R1/(6.0*SQRT(BI))+H3X*R2/(24.0*BI)+H5X*(R1*R1)
     1/(72.0*BI)
```

```
      X1=0.95
      IF(P(I).GE.X1) GO TO 200
      GO TO 300
200   PP=1.0/P(I)-0.5
      CALL ALPH(DELTA,DK,I,R1,R2,ALPHA,ALPHA4)
      A(I)=(DELTA*SQRT(BI))/ALPHA4(I)
      PRINT 101, DK, I, ALPHA(I),PP, A(I)
101   FORMAT(' ',5X,F3.1,5X,I3,5X,F8.5,5X,F8.3,5X,F8.0)
      GO TO 400
300   CONTINUE
400   DK=DK+0.1
      IF(DK.GT.3.5) GO TO 900
      GO TO 100
900   STOP
      END
C     ************************************************
C     * SUBROUTINE FOR CALCULATING EQUATION  (4.40) *
C     ************************************************
      SUBROUTINE ALPH(DELTA,DK,I,R1,R2,ALPHA,ALPHA4)
      DIMENSION ALPHA(125),ALPHA4(125)
      BI=I
      X=DK
      CALL EDWOR(X,ZK,H2K,H3K,H4K,H5K,H6K,FIX,R1,R2)
      ALPHA1=2.0*FIX
      ALPHA2=(3.0*R2*H3K+R1*R1*H5K)/(36.0*BI)
      AA=DELTA*SQRT(BI)
      ALPHA3=(3.0*R2*H4K+R1*R1*H6K)*AA
      ALPHA5=-2.*(3.0*R2*H3K+R1*R1*H5K)
      ALPHA6=(ALPHA3-ALPHA5)/72.
      ALPHA7=(DELTA*DELTA)*ALPHA6/(AA*AA*AA)
      ALPHA(I)=ALPHA1-ALPHA2
      ALPHA4(I)=ZK+ALPHA7
      RETURN
      END
      SUBROUTINE EDWOR(X,ZX,H2X,H3X,H4X,H5X,H6X,FIX,R1,R2)
      IF(X.EQ.0) GO TO 85
      ZX=0.39894228*EXP(-(X*X)/2.0)
      GO TO 86
85    ZX=0.39894228
86    H1X=X*ZX
      H2X=(X*X-1.0)*ZX
      H3X=-(X**3-3.0*X)*ZX
      H4X=(X**4-6.0*X**2+3.0)*ZX
      H5X=-(X**5-10.0*X**3+15.0*X)*ZX
      H6X=(X**6-15.0*X**4+45.0*X**2)*ZX
      CALL MSMRAT(X,RM,IER)
      FIX=ZX/RM
      RETURN
      END
```

APPENDIX III

PROGRAM WARNING

```
                     PROGRAM DESCRIPTION
****************************************************************
!  This program is used for finding the optimal design          !
!parameters of an x-chart with warning limits by minimizing     !
!the loss-cost function. The program consists of two stages.    !
!In the first stage it calculates an approximate                !
!solution of sampling interval for given value of sample        !
!size,control limit coefficient and warning limit coeffici-     !
!ent. In second stage the program starts search for optimal     !
!design parameters through four dimensional Hook and Jeeves'    !
!search technique.                                              !
!                                                               !
! NOMENCLATURE                                                  !
! ------------                                                  !
! The nomenclature for the model parameters is the same as      !
! that of APPENDIX I.                                           !
! Variable Name                                                 !
! First stage                                                   !
! X1                 Initial value of sample size ,n            !
! X3                 Initial value of control limit ,ka         !
! X4                 Initial value of warning limit,kw          !
! X2                 Sampling interval ,s                       !
! Second stage                                                  !
! XO(I)              Location of initial base points, I=1,4     !
! XM(I)              Location of current base points, I=1,4     !
! XT(I)              Location of temporary base points,I=1,4    !
! FXB                Functional value at initial base point     !
! FXE                Functional value at current base point     !
! FXT                Functional value at temporary base point   !
! ICMAX              Maximum number of iterations               !
! IC                 Number of iterations                       !
!                                                               !
! OUTPUT DESCRIPTION                                            !
! ------------------                                            !
! XM(1)              Sample size, n                             !
! XM(2)              Sampling interval, s                       !
! XM(3)              Control limit coefficient, ka              !
! XM(4)              Warning limit coefficient ,kw              !
! ARLO               Average run length when process is in      !
!                    control,RO                                 !
! ARL1               Average run lensh when process is in       !
```

```
      !                         out of control,R1                   !
      !--FXB                    Loss-cost function , L              !
      ************************************************************
C                                PROGRAM LISTING
C     ************************************************************
C     * THE ECONOMIC DESIGN OF X-CHARTS WITH WARNING LIMIT TO       *
C     * CONTROL NON-NORMAL PROCESS MEANS UNDER POLICY II .          *
C     * THE NUMERICAL VALUES ASSIGNED TO R1 ARE -0.5,0.0, 0.5, 1.0,*
C     *AND TO R2 ARE -0.5,0.0,0.5,1.0,1.5 AND 2.0 . THE             *
C     *SHIFT PARAMETER DELTA ASSUMED TO 0.5 TO 2.25 WITH INCREMENTS*
C     *OF 0.25.                                                     *
C     *************************************************************
      REAL LAMDA,KS,KR
      DIMENSION XO(4),XT(4),XM(4),DEL(4),TABLE(8,4,6,7)
      COMMON LAMDA,DELTA,TR,TS,V,U,W,B,C,R1,R2,KR,KS
C     *************************************************************
C     *                           SET MODEL PARAMETERS              *
C     *************************************************************
      LAMDA=0.01
      V0=150.0
      V1=50.0
      KR=20.00
      KS=10.0
      TR=0.2
      TS=0.1
      V=KS+V0*TS
      W=KR+KS+V0*(TR+TS)
      U=V0-V1
      B=0.5
      C=0.1
      DELTA=0.5
      IDELTA=1
  400 CONTINUE
      R1=-0.5
      IR1=1
  300 CONTINUE
      R2=-0.5
      IR2=1
  200 CONTINUE
C     ***********************************
C     *    FIRST STAGE                  *
C     ***********************************
      IF(DELTA.EQ.0.5) X1=35.
      IF(DELTA.EQ.0.75) X1=25.
      IF(DELTA.EQ.1.0)  X1=15.
      IF(DELTA.GE.1.25) X1=5.0
      IF(DELTA.EQ.0.5) X3=2.0
      IF(DELTA.EQ.0.75) X3=2.25
      IF(DELTA.EQ.1.0) X3=2.5
      IF(DELTA.GE.1.25) X3=3.0
```

```
      X4=0.66*X3
      Y1=X3-DELTA*SQRT(X1)
      Y2=X4-DELTA*SQRT(X1)
      Y3=X3
      Y4=X4
      CALL AUY(X1,Y1,R1,R2,P1)
      CALL AUY(X1,Y2,R1,R2,P2)
      CALL AUY(X1,Y3,R1,R2,P3)
      CALL AUY(X1,Y4,R1,R2,P4)
      P0=P1-P2
      PP=P3-P4
      ARL1=(1.0-P0**2)/(1.0-P0-P2*(1.-P0**2))
      ARL0=(1.0-PP**2)/(1.0-PP-P4*(1.-PP**2))
      H1=V/ARL0+B+C*X1
      H2=LAMDA*U*(ARL1-0.5)
      H3=H1/H2
      X2=SQRT(H3)
C     *********************************************
C     *            SECOND STAGE                   *
C     *********************************************
      X0(1)=X1
      X0(2)=X2
      X0(3)=X3
      X0(4)=X4
      NN=4
      DEL(1)=1.0
      DEL(2)=0.5
      DEL(3)=0.5
      DEL(4)=0.5
      CALL SUB(DEL,NN,XT,X0,XM,FXB,ARL0,ARL1)
      TABLE(IDELTA,IR1,IR2,1)=XM(1)
      TABLE(IDELTA,IR1,IR2,2)=XM(2)
      TABLE(IDELTA,IR1,IR2,3)=XM(3)
      TABLE(IDELTA,IR1,IR2,4)=XM(4)
      TABLE(IDELTA,IR1,IR2,5)=ARL0
      TABLE(IDELTA,IR1,IR2,6)=ARL1
      TABLE(IDELTA,IR1,IR2,7)=FXB
  550 R2=R2+0.5
      IR2=IR2+1
      IF(R2.GT.2.0) GO TO 600
      GO TO 200
  600 R1=R1+0.5
      IR1=IR1+1
      IF(R1.GT.1.0) GO TO 700
      GO TO 300
  700 DELTA=DELTA+0.25
      IDELTA=IDELTA+1
      IF(DELTA.GT.2.25) GO TO 800
      GO TO 400
  800 WRITE(6,1)LAMDA,V0,V1,KR,KS,TR,TS,B,C
```

```
      IM=1
      IN=2
      WRITE(6,2)
      DO 1100 IJ=1,4
      XIN=IN
      DELTA1=0.25*XIN
      DELTA2=DELTA1+0.25
      WRITE(6,3) DELTA1,DELTA2
      WRITE(6,4)
      R2=-0.5
      DO 1000 IR2=1,6
      WRITE(6,11) ((TABLE(IDELTA,IR1,IR2,1),IR1=1,4),IDELTA=IM,IN)
      WRITE(6,12)((TABLE(IDELTA,IR1,IR2,2),IR1=1,4),IDELTA=IM,IN)
      WRITE(6,13)((TABLE(IDELTA,IR1,IR2,3),IR1=1,4),IDELTA=IM,IN)
      WRITE(6,14)R2,((TABLE(IDELTA,IR1,IR2,4),IR1=1,4),IDELTA=IM,IN)
      WRITE(6,15)((TABLE(IDELTA,IR1,IR2,5),IR1=1,4),IDELTA=IM,IN)
      WRITE(6,16) ((TABLE(IDELTA,IR1,IR2,6),IR1=1,4),IDELTA=IM,IN)
      WRITE(6,17)((TABLE(IDELTA,IR1,IR2,7),IR1=1,4),IDELTA=IM,IN)
      IF (((IJ.NE.2).OR.(IJ.NE.4)).AND.(IR2.NE.6)) GO TO 900
      WRITE(6,2)
      IF(IJ.EQ.2) GO TO 1000
      WRITE(6,7)
      GO TO 1000
 900  WRITE(6,4)
      R2=R2+0.5
1000  CONTINUE
      IM=IM+2
      IN=IN+2
      R2=-0.5
      IF(IJ.NE.2) GO TO 1100
      WRITE(6,5)
      WRITE(6,2)
1100  CONTINUE
      STOP
1     FORMAT('1',5X,'LAMDA=',F4.2,2X,'P0=',F6.1,2X,'P1=',F6.1,2X,
     *'KR=',F5.1,2X,'KS=',F5.1,2X,'TR=',F4.2,2X,'TS=',F3.1,2X,'B=',
     *F4.2,2X,'C=',F4.2)
2     FORMAT(' +',80('-'),'+')
3     FORMAT(' ¦       ¦',74X,'¦'/
     *' ¦       ¦',31X,'DELTA',38X,'¦'/
     *' ¦       ¦',14X,F4.2,30X,F4.2,22X,'¦'/
     *' ¦       ¦  ',2(30('-'),4(' ')),'     ¦'/
     *' ¦       ¦',74X,'¦'/
     *' ¦ R2    ¦',33X,'R1',39X,'¦'/
     *' ¦       ¦ ',2('  -0.5      0.0      0.5      1.0    '),
     *'      ¦'/
     *' ¦       ¦ ',2(4(' ------ '),'  '),'      ¦')
4     FORMAT(' ¦-----¦',2(4('---------'),'-¦'),'------¦')
7     FORMAT('1+',80('-'),'+')
11    FORMAT(' ¦       ¦',2(4F8.0,' ¦'),'   N  ¦')
```

```
12    FORMAT(' |      |',2(4F8.2,' |'),'   S  |')
13    FORMAT(' |      |',2(4F8.2,' |'),'   KA |')
14    FORMAT(' |',F4.1,' |',2(4F8.2,' |'),'   KW |')
15    FORMAT(' |      |',2(4F8.0,' |'),' ARLO |')
16    FORMAT(' |      |',2(4F8.2,' |'),' ARL1 |')
17    FORMAT(' |      |',2(4F8.4,' |'),'   L  |')
      END
C     ***********************************
C     *  SUBROUTINE FOR PATTERN SEARCH  *
C     ***********************************
      SUBROUTINE SUB(DEL,NN,XT,XO,XM,FXB,ARLO,ARL1)
      DIMENSION XX1(4),XT(4),XO(4),XM(4),DEL(4)
      ICMAX=500
      XM(1)=XO(1)
      XM(2)=XO(2)
      XM(3)=XO(3)
      XM(4)=XO(4)
      XM1=XM(1)
      XM2=XM(2)
      XM3=XM(3)
      XM4=XM(4)
      CALL COST(XM1,XM2,XM3,XM4,PO,PP,ARLO,ARL1,FXB)
      IC=0
   21 KK=0
      DO 11 I=1,NN
      TEMP=XM(I)
      XM(I)=XM(I)+DEL(I)
      IF(XM(3).GT.XM(4)) GO TO 22
      XM(4)=0.85*XM(3)
22    XM1=XM(1)
      XM2=XM(2)
      XM3=XM(3)
      XM4=XM(4)
      CALL COST(XM1,XM2,XM3,XM4,PO,PP,ARLO,ARL1,FXE)
      IF((FXE.LT.FXB).AND.(PP.GT.0.0).AND.(PO.GT.0.0)) GO TO 12
      XM(I)=TEMP
      XM(I)=XM(I)-DEL(I)
      IF(XM(3).GT.XM(4)) GO TO 25
      XM(4)=0.85*XM(3)
25    XM1=XM(1)
      XM2=XM(2)
      XM3=XM(3)
      XM4=XM(4)
      CALL COST(XM1,XM2,XM3,XM4,PO,PP,ARLO,ARL1,FXE)
      IF((FXE.LT.FXB).AND.(PP.GT.0.0).AND.(PO.GT.0.0)) GO TO 12
      XM(I)=TEMP
      GO TO 11
   12 FXB=FXE
      KK=1
   11 CONTINUE
```

```
      IF(KK.EQ.0)GO TO 18
      DO 16 I=1,NN
   16 XX1(I)=XM(I)
      DO 13 I=1,NN
   13 XT(I)=2.*XX1(I)-XO(I)
      IF(XT(3).GT.XT(4)) GO TO 26
      XT(4)=0.85*XT(3)
26    XT1=XT(1)
      XT2=XT(2)
      XT3=XT(3)
      XT4=XT(4)
      CALL COST(XT1,XT2,XT3,XT4,PO,PP,ARLO,ARL1,FXT)
      DO 14 I=1,NN
   14 XO(I)=XX1(I)
      IF((FXT.LT.FXB).AND.(PP.GT.0.0).AND.(PO.GT.0.0)) GO TO 15
      GO TO 21
   15 FXB=FXT
      DO 17 I=1,NN
   17 XM(I)=XT(I)
      IC=IC+1
      IF(IC.GE.ICMAX) GO TO 19
      GO TO 21
18    DO 27 I=2,NN
      IF(DEL(I).GT.0.001) GO TO 28
      GO TO 29
   28 DEL(I)=DEL(I)/2.0
   29 DEL(I)=DEL(I)
   27 CONTINUE
      IF((DEL(2).LE..001).AND.(DEL(3).LE.001)
     *.AND.(DEL(4).LE.0.001)) GO TO 19
      GO TO 21
   19 CONTINUE
      RETURN
      END
C     ***********************************************************
C     * SUBROUTINE FOR CALCULATING LOSS-COST FUNCTION *
C     ***********************************************************
      SUBROUTINE COST(X1,X2,X3,X4,PO,PP,ARLO,ARL1,F)
      REAL LAMDA,KR,KS
      COMMON LAMDA,DELTA,TR,TS,V,U,W,B,C,R1,R2,KR,KS
      IF((X1.LE.0).OR.(X2.LE.0).OR.(X3.LE.0).OR.(X4.LE.0)) GO TO 86
      Y1=X3-DELTA*SQRT(X1)
      CALL AUY(X1,Y1,R1,R2,P1)
      Y2=X4-DELTA*SQRT(X1)
      CALL AUY(X1,Y2,R1,R2,P2)
      PO=P1-P2
      T2=PO+P2*(1.0-PO**2)
48    ARL1=(1.0-PO**2)/(1.0-T2)
      Y3=X3
      CALL AUY(X1,Y3,R1,R2,P3)
```

```
      Y4=X4
      CALL AUY(X1,Y4,R1,R2,P4)
      PP=P3-P4
      T1=PP+P4*(1.-PP**2)
49    ARLO=(1.0-PP**2)/(1.-T1)
      H1=V/ARLO+B+C*X1
      BO=(1.0/X2-0.5*LAMDA+1./12.*(LAMDA**2)*X2)/ARLO
      B1=(ARL1-0.5+1./12*(LAMDA*X2))*X2
      U1=LAMDA*U*B1+V*BO+LAMDA*W+(B+C*X1)*(1.0+LAMDA*B1)/X2
      U2=1.0+LAMDA*B1+TS*BO+LAMDA*(TS+TR)
      F=U1/U2
86    RETURN
      END
C     ************************************************
C     * SUBROUTINE FOR CALCULATING THE PROBABILITIES *
C     * ASSOCIATED WITH AVERAGE RUN LENGTHS          *
C     ************************************************
      SUBROUTINE AUY(X1,Y,R1,R2,P)
      ZY=0.39894228*EXP(-Y*Y/2.0)
      H2Y=(Y*Y-1.0)*ZY
      H3Y=-(Y**3-3.0*Y)*ZY
      H4Y=(Y**4-6.0*Y**2+3.0)*ZY
      H5Y=-(Y**5-10.0*Y**3+15.0*Y)*ZY
      H6Y=(Y**6-15.0*Y**4+45.0*Y**2-15.0)*ZY
      CALL MSMRAT(Y,RM,IER)
      FIX=ZY/RM
      P=1.0-FIX-(H2Y*R1)/(6.0*SQRT(X1))+(H3Y*R2)/(24.0*X1)+(H5Y*R1*R1)
     1/(72.0*X1)
      RETURN
      END
```

APPENDIX IV

PROGRAM SEMIWARN

```
                         PROGRAM DESCRIPTION
*********************************************************************
! This program is used for generating tables based on semi-        !
!economic scheme useful at the workshop level to determine         !
!the design parameters of an x-chart with warning limits .         !
!The essential characteristic of the semi-economic scheme          !
!is that the assignable cause is detected,on the average,          !
!1.1 or 1.05 sample after its occurrence.                          !
!                                                                  !
! NOMENCLATURE                                                     !
!                                                                  !
!  Variable name                  Description                      !
!    DN                     Sample size , n                        !
!    DK                     Action limit coefficient, ka           !
!    DELTA                  Shift parameter                        !
!    R1                     Measure of skewness                    !
!    R2                     Measure of Kurtosis                    !
!    ARLO                   Average Run length when process is     !
!                           in control                             !
!    ARL1                   Average Run length when process is     !
!                           in out of control                      !
!    SARL                   Specified value of Average Run         !
!                           Length, ARL1                           !
! OUTPUT                                                           !
!    I                      Number of action limit coefficients    !
!    DK(I)                  Ith action limit coefficient           !
!    DN(I)                  Sample size corresponding to ith       !
!                           action limit coefficient               !
!    BARLO(I)               ARLO corresponds to ith action limit   !
!    AK(I)                  Equal to DELTA*SQRT(DN(I))-DK(I)       !
!    DL(I)                  Derivatives of ARLO(I) with respect    !
!                           to DK(I)                               !
!    AA(I)                  Represents equation (5.21)             !
*********************************************************************
C                          PROGRAM LISTING
      DIMENSION DK(32),DN(32),BARLO(32),AA(32),DL(32),AK(32)
      COMMON DELTA,R1,R2
C     ************************************************************
C     * SET MODEL PARAMETERS R1,R2,DELTA AND MAXIMUM ALLOWABLE   *
C     * ACTION LIMIT COEFFICIENTS                                *
C     ************************************************************
```

```
      DELTA=2.0
      R1=0.0
      R2=0.0
      DO 119 I=1,32
      X1=0.0
  116 X1=X1+1.0
      X2=1.0+I*0.1
      X3=0.85*X2
      CALL SUBS(Y1,X1,X2,X3,P0,PP,ARL0,ARL1)
      SARL=1.05
      IF(ARL1.LT.SARL) GO TO 115
      GO TO 116
  115 DK(I)=X2
      DN(I)=X1
      BARL0(I)=ARL0
      AK(I)=DELTA*SQRT(X1)
 119  CONTINUE
      NDIM=32
      CALL DGT3(DK,BARL0,DL,NDIM,IER)
      DO 113 I=1,32
      AA(I)=(AK(I)/DL(I))*BARL0(I)*BARL0(I)
      PRINT120,DK(I),DN(I),BARL0(I),AA(I)
120   FORMAT(' ',5X,F5.2,10X,F5.0,10X,F8.1,10X,F8.1)
113   CONTINUE
      STOP
      END
C     *****************************************************************
C     * SUBROUTINE FOR COMPUTING  A  VECTOR OF DERIVATIVE            *
C     * VALUES FOR GIVEN VECTOR OF ARGUMENT VALUES AND                *
C     * CORRESPONDING FUNCTIONAL VALUES. FOR REFERENCE SEE            *
C     * F.B. HILDEBRAND, INTRODUCTION TO NUMERICAL ANALYSIS           *
C     * ,MCGRAW-HILL  , NEW YORK, 1956.                               *
C     * DESCRIPTION OF ARGUMENTS: X- GIVEN VECTOR OF ARGUMENT*
C     * VALUES(DIMENSION NDIM), Y-GIVEN VECTOR OF FUNCTIONAL *
C     * VALUES CORRESPONDING TO X, Z-RESULTING VECTOR OF             *
C     * DERIVATIVE VALUES                                             *
C     *****************************************************************
      SUBROUTINE  DGT3(X,Y,Z,NDIM,IER)
      DIMENSION X(NDIM),Y(NDIM),Z(NDIM)
      IER=-1
      IF(NDIM-3) 8,1,1
1     A=X(1)
      B=Y(1)
      I=2
      DY2=X(2)-A
      IF(DY2) 2,9,2
2     DY2=(Y(2)-B)/DY2
      DO 6 I=3,NDIM
      A=X(I)-A
      IF(A) 3,9,3
```

```
3     A=(Y(I)-B)/A
      B=X(I)-X(I-1)
      IF(B) 4,9,4
    4 DY1=DY2
      DY2=(Y(I)-Y(I-1))/B
      DY3=A
      A=X(I-1)
      B=Y(I-1)
      IF(I-3) 5,5,6
    5 Z(1)=DY1+DY3-DY2
    6 Z(I-1)=DY1+DY2-DY3
      IER=0
      I=NDIM
    7 Z(I)=DY2+DY3-DY1
    8 RETURN
9     IER=I
      I=I-1
      IF(I-2) 8,8,7
      END
C     **************************************************
C     *  SUBROUTINE FOR CALCULATING AVERAGE RUN LENGTHS *
C     **************************************************
      SUBROUTINE SUBS(Y1,X1,X2,X3,P0,PP,ARL0,ARL1)
      COMMON DELTA,R1,R2
      IF((X1.LE.0).OR.(X2.LE.0).OR.(X3.LE.0)) GO TO 86
      Y1=X2-DELTA*SQRT(X1)
      CALL AUY(X1,Y1,R1,R2,P1)
      Y2=X3-DELTA*SQRT(X1)
      CALL AUY(X1,Y2,R1,R2,P2)
      P0=P1-P2
      T2=P0+P2*(1.0-P0**2)
48    ARL1=(1.0-P0**2)/(1.0-T2)
      Y3=X2
      CALL AUY(X1,Y3,R1,R2,P3)
      Y4=X3
      CALL AUY(X1,Y4,R1,R2,P4)
      PP=P3-P4
      T1=PP+P4*(1.-PP**2)
49    ARL0=(1.0-PP**2)/(1.-T1)
86    RETURN
      END
```

APPENDIX V

PROGRAM CUSUM

```
C                      PROGRAM DESCRIPTION
C       ************************************************************!
C       ! PURPOSE OF THE PROGRAM:                                   !
C       ! ----------------------                                    !
C       ! THE MAIN PURPOSES OF THIS  PROGRAM ARE AS FOLLOWS:        !
C       !1. TO MINIMIZE THE OBJECTIVE FUNCTION REPRESENTED BY FO(THE !
C       !EXPECTED PER HOUR COST ASSOCIATED WITH THE OPERATION OF    !
C       !A CUSUM CHART TO CONTROL NON-NORMAL PROCESS MEANS) AND TO  !
C       !FIND THE OPTIMUM VALUES OF THE DESIGN VARIABLES. THE PROGRAM!
C       !IS  ABLE TO LOCATE THE MIMIMA WHERE THE COST SURFACE IS    !
C       !EITHER STRICTLY CONVEX OR RELATIVELY FLAT AROUND THE       !
C       !OPTIMUM.                                                   !
C       !2. TO DETERMINE AVERAGE RUN LENGTH(ARL) OF A CUSUM CHART TO !
C       !CONTROL NON-NORMAL PROCESS MEANS BY SOLVING A SYSTEM OF    !
C       !LINEAR ALGEBRAIC EQUATIONS.THE SETS OF EQUATIONS USED FOR  !
C       !CALCULATING P(Z),N(Z) AND L(Z) ARE GIVEN BY EQUATIONS(6.8), !
C       !(6.13),AND (6.5).CALCULATION OF ARL FOR THE CUSUM CHART IS !
C       !A SPECIAL  CASE WITH Z=0. AVERAGE RUN LENGTH WHEN THE      !
C       !PROCESS IS IN CONTROL, DENOTED BY ARLO, IS CALCULATED FROM !
C       !STANDARDIZED DECISION INTERVAL H AND THETA=-DELTA*SQRT(N)  !
C       !*0.5, AND AVERAGE RUN LENGTH WHEN THE PROCESS IS OUT OF    !
C       !CONTROL,  ARL1, IS CALCULATED FROM H AND THETA=DELTA*SQRT  !
C       !(N)*0.5                                                    !
C       !3. TO ASSESS THE EFFECTS OF NON-NORMALITY PARAMETERS  AND  !
C       !SHIFT PARAMETER ON THE LOSS-COST AND ON THE DESIGN VARIABL- !
C       !ES AT VARIOUS LEVELS, FOR A GIVEN  SET OF COST AND RISK    !
C       !FACTORS.                                                   !
C       !                                                           !
C       ! SUBROUTINES NEEDED                                        !
C       ! -----------------------                                   !
C       ! THE PROGRAM USES THE FOLLOWING SUBROUTINES AND FUNCTIONS: !
C       !                                                           !
C       !  1. FUNCTION F:  CALCULATES THE SAMPLING INTERVAL         !
C       !  2. SUBROUTINE AUX: COMPUTES  FIRST FOUR TERMS OF THE EDG- !
C       !  EWORTH SERIES.                                           !
C       !  3. SUBROUTINE LEQT2F: SOLVES A SET OF LINEAR EQUATIONS   !
C       !  4. SUBROUTINE ZREAL1: FINDS THE REAL ROOTS OF A REAL FUN- !
C       !  CTION.                                                   !
C       !  5. SUBROUTINE MSMRAT: COMPUTES THE AREA UNDER NORMAL CUR- !
C       !  VE.                                                      !
C       ! SUBROUTINES #3, #4, AND #5 ARE IMSL(INTERNATIONAL MATHE-  !
C       ! MATHEMATICAL AND STATISTICAL LIBRARIES ) ROUTINES. FOR    !
C       ! DETAILS SEE REFERENCE MANNUAL IMSL LIBRARY-2,REVISED EDI-  !
```

```
C       ! TION ,JANUARY 1978. IMSL,SIXTH FLOOR,GNB BUILDING,7500         !
C       ! BELLAIRE BOULEVARD, HOUSTON, TEXAS 77036.                      !
C       ! EQUIVALENT SUBROUTINES TO #3, #4,AND #5 IN THE IBM SYSTEM/     !
C       ! 360 SCIENTIFIC SUBROUTINE PACKAGE ARE, RESPECTIVELY,           !
C       !-SUBROUTINE SIMQ: SOLUTION OF SIMULTANEOUS LINEAR ALGEBRAIC     !
C       ! EQUATION.                                                      !
C       !-SUBROUTINE RTNI OR DRTNI : ESTIMATING THE ROOT OF A FUNCT-     !
C       ! TION BY NEWTON'S ITERATION.                                    !
C       !-SUBROUTINE NDTR : FOR NORMAL DISTRIBUTION FUNCTION.            !
C       ! FOR DETAILS SEE REFERENCE IBM SYSTEM/360 MANUAL, IBM COR-      !
C       ! PORATION, TECHNICAL PUBLICATIONS DEPARTMENT, 112 EAST POST     !
C       ! ROAD, WHITE PLAINS,NEW YORK 10601.                             !
C       !                                                                !
C                                                                        !
C       !                                                                !
C       ! NOMENCLATURE                                                   !
C       ! ---------------                                                !
C       !   VARIABLE NAME                         DESCRIPTION            !
C       !     NORD                      NUMBER OF GAUSSIAN FIONTS        !
C       !     ZK                        GAUSSIAN POINTS                  !
C       !     AK                        GAUSSIAN COEFFICENTS             !
C       !     LAMDA                     RATE OF OCCURRENCE OF ASSIGN-    !
C       !                               ABLE CAUSE                       !
C       !     V0                        INCOME WHEN IN-CONTROL           !
C       !     V1                        INCOME WHEN OUT-OF-CONTROL       !
C       !     KR                        REPAIR COST PER HOUR             !
C       !     KS                        SEARCH COST PER HOUR             !
C       !     TR                        AVERAGE-TIME TO REPAIR           !
C       !     TS                        AVERAGE-TIME TO SEARCH           !
C       !     BB                        FIXED SAMPLING COST              !
C       !     CC                        VARIABLE SAMPLING COST           !
C       !     DELTA                     SHIFT PARAMETER                  !
C       !     R1                        MEASURE OF SKEWNESS              !
C       !     R2                        MEASURE OF KURTOSIS              !
C       !     THETA                     MAGNITUDE OF VARIATION IN MEAN   !
C       !     H                         STANDARDIZED DECISION INTERVAL   !
C       !     NH                        MAX. NO. OF DECISION INTERVALS   !
C       !     NI                        MAXIMUM SAMPLE SIZE              !
C       !                                                                !
C       ! INPUT DATA CARDS                                               !
C       ! ----------------                                               !
C       !                                                                !
C       ! ORDER OF CARDS              FORMAT          VARIABLE  NAME     !
C       !   1                         (8F10.5)           ZK              !
C       !   2                         (8F10.5)           AK              !
C       !   3                         (9F8.2)         LAMDA,V0,V1,KR     !
C       !                                             KS,TR,TS,BB,CC     !
C       !                                                                !
C       ! OUTPUT DESCRIPTION                                             !
C       ! ------------------                                             !
```

```
C      !                                                          !
C      ! VARIABLE NAME                        DESCRIPTION         !
C      !                                                          !
C      !   PZ1                                  P(Z)              !
C      !   SNZ1                                 N(Z)              !
C      !   PZ0                                  P(0)              !
C      !   SNZ0                                 N(0)              !
C      !   NI                                   N                 !
C      !   ARLZRC                               ARL0              !
C      !   ARLONC                               ARL1              !
C      !   HC                                   H                 !
C      !   GGC                                  S                 !
C      !   FFC                                  L                 !
C      ***********************************************************************
C                        PROGRAM LISTING
C      *************************************************************
C      * THE ECONOMIC DESIGN OF CUSUM CHARTS TO CONTROL NON-       *
C      * NORMAL PROCESS MEANS.                                     *
C      *************************************************************
       EXTERNAL F
       INTEGER N,NN,IA,IDGT,IER,M,K,NSIG,ITMAX,LL,KK,JJ
       INTEGER KKK,MM,NI,NH,NORD,II,III,JJJ,IMIN,IFLAG,IIII
       REAL F,EPS,EPS2,BTA,G(2)
       REAL LAMDA,KR,KS
       REAL C(3,3),B(3,1),WKAREA(30),SN(3,1),D(3,3)
       COMMON LAMDA,ARL0,ARL1,U,V,TR,TS,W,BB,CC,BI,KR,KS
       DIMENSION ZK(3),AK(3),Z(3),A(3)
       DIMENSION X(3,3),Q(3,3),FT(3,3)
       DIMENSION XX(3,1),Q1(3,1),FT1(3,1)
       DIMENSION X1(3,1),Q2(3,1),FT2(3,1)
       DIMENSION X2(1,1),Q3(1,1),FT3(1,1)
       DIMENSION ZX1(3,3),ZX2(3,3),ZX4(3,3),ZX5(3,3),ZX6(3,3)
       DIMENSION ETA(2),H(30),FF(30),ARLZR(30),ARLON(30)
       DIMENSION GG(30),FFC(30),HC(30),GGC(30),TABLE(8,3,6,6)
       DIMENSION ARLONC(30),ARLZRC(30)
       IFLAG=0
       NORD=3
       NI=30
       NH=16
       K=NORD
C      ***********************************************************************
C      * READ GAUSSIAN POINTS(ROOTS OF LEGENDER'S POLYNOMIALS) AND  *
C      * GAUSSIAN COEFFICENTS IN THE INTERVAL -1 TO +1 ,COST AND    *
C      * RISK FACTORS FROM THE DATA CARDS.                          *
C      ***********************************************************************
       READ(5,20) (ZK(II),II=1,K)
       READ(5,20) (AK(II),II=1,K)
20     FORMAT(3F10.5)
       READ(5,101) LAMDA,V0,V1,KR,KS,TR,TS,BB,CC
101    FORMAT(9F8.2)
```

```
      DELTA=0.5
      IDELTA=1
401   CONTINUE
      R1=-0.5
      IR1=1
300   CONTINUE
      R2=-0.5
      IR2=1
200   CONTINUE
      ETA(1)=0.50*DELTA
      ETA(2)=-0.50*DELTA
      DO 1000 I=1,NI
      BI=I
      DO 990 KK=1,NH
      H(KK)=0.15+0.05*KK
      DO 900 J=1,2
      ATA=ETA(J)
      THETA=ATA*SQRT(BI)
C     ***********************************************************
C     * CALCULATE THE GAUSSIAN COEFFICENTS AND GAUSSIAN POINTS *
C     * FOR THE INTERVAL 0 TO STANDARDIZED DECISION INTERVAL H.*
C     ***********************************************************
      DO 25 III=1,K
      Z(III)=(ZK(III)+1.0)*H(KK)*0.5
      A(III)=AK(III)*H(KK)*0.5
C     *****************************************************************
C     *CALCULATE THE ELEMENTS OF THE MATRIX 'A' USED IN EQUATION(6.12)*
C     *****************************************************************
25    CONTINUE
      DO 75 II=1,K
      DO 75 JJ=1,K
      X(II,JJ)=Z(II)-Z(JJ)-THETA
      CALL AUX(X,R1,R2,Q,FT,BI,II,JJ)
      IF(II.EQ.JJ) GO TO 60
      C(II,JJ)=-A(II)*Q(II,JJ)
      GO TO 65
60    C(II,JJ)=(1.0-Q(II,JJ)*A(II))
65    D(JJ,II)=C(II,JJ)
75    CONTINUE
C     *************************************************************
C     *CALCULATE THE ELEMENTS OF THE MATRIX 'Y' IN EQUATION (6.12)*
C     *************************************************************
      DO 80  II=1,K
      XX(II,1)=-Z(II)-THETA
      JJ=1
      CALL AUX(XX,R1,R2,Q1,FT1,BI,II,JJ)
      B(II,1)=FT1(II,1)
80    CONTINUE
      NN=3
      M=1
```

```
      IA=3
      IDGT=4
C    ****************************************************************
C    *CALCULATE INVERSE OF MATRIX 'A' USING IMSL SUBROUTINE LEQT2F *
C    *AND TO OBTAIN MATRIX 'P' IN EQUATION (6.12)                   *
C    ****************************************************************
      CALL LEQT2F(D,M,NN,IA,B,IDGT,WKAREA,IER)
      SUM=0.0
      DO 120 II=1,K
      X1(II,1)=Z(II)-THETA
      JJ=1
      CALL AUX(X1,R1,R2,Q2,FT2,BI,II,JJ)
120   SUM=SUM+Q2(II,1)*B(II,1)*A(II)
      X2(1,1)=-THETA
      II=1
      JJ=1
      CALL AUX(X2,R1,R2,Q3,FT3,BI,II,JJ)
      PZ1=SUM+FT3(1,1)
      DO 130 II=1,K
      SN(II,1)=1.0
130   CONTINUE
      CALL LEQT2F(D,M,NN,IA,SN,IDGT,WKAREA,IER)
      SUM1=0.0
C     ********************************************
C     * CALCULATE MATRIX N(Z) USING EQUATION (6.14)*
C     ********************************************
      DO 400 II=1,K
400   SUM1=SUM1+SN(II,1)*Q2(II,1)*A(II)
C     ****************************************************
C     * CALCULATE P(0),N(0),ARLO AND ARL1                *
C     ****************************************************
      SNZ1=1.0+SUM1
      IF(PZ1.GE.1.0) PZ1=0.999
      ARL=SNZ1/(1.0-PZ1)
      IF(ATA.EQ.ETA(2)) GO TO 600
      ARL1=ARL
      SNZ1=SNZ1
      PZ1=PZ1
      GO TO  900
600   ARLO=ARL
      SNZ0=SNZ1
      PZ0=PZ1
900   CONTINUE
      U=V0-V1
      V=KS+V0*TS
      W=KR+KS+V0*(TR+TS)
      GA=SQRT((V/ARLO+BB+CC*BI)/(LAMDA*U*(ARL1-0.5)))
      G(1)=GA
      G(2)=2.0
      EPS=1.0E-3
```

```
      EPS2=1.0-3
      BTA=1.0E-2
      NSIG=5
      ITMAX=100
      LL=2
C     ***********************************************************
C     *CALCULATE SAMPLING INTERVAL DENOTED HERE BY G USING        *
C     *EQUATION(6.15) AND SOLVED BY USING IMSL SUBROUTINE ZREAL1*
C     *FOR GIVEN VALUE OF SAMPLE SIZE.CALCULATE B0,B1 AND L=F0  *
C     *REPRESENTING EQUATIONS (6.2),(6.3) AND(6.1) RESPECTIVELY.*
C     ***********************************************************
      CALL ZREAL1(F,EPS,EPS2,BTA,NSIG,LL,G,ITMAX,IER)
      R=G(1)
      B0=(1.0/R-0.50*LAMDA+1.0/12.0*LAMDA**2*R)/ARL0
      B1=(ARL1-0.5+1.0/12.0*LAMDA*R)*R
      F0=(LAMDA*U*B1+V*B0+LAMDA*W+(BB+CC*BI)*(1.0+LAMDA*B1)/R)/
     1(1.0+LAMDA*B1+TS*B0+LAMDA*(TR+TS))
      H(KK)=H(KK)
      FF(KK)=F0
      ARLZR(KK)=ARL0
      ARLON(KK)=ARL1
      GG(KK)=R
      IF(KK.GT.3) GO TO 989
      GO TO 990
989   KK1=KK-1
      KK2=KK-2
      IF((FF(KK1).GT.FF(KK2)).AND.(FF(KK1).GT.FF(KK))) GO TO 990
      IMIN=KK-1
      GO TO 1101
990   CONTINUE
      XMIN=1000.0
      DO 1100 JJJ=1,NH
      IF(FF(JJJ).GT.XMIN) GO TO 1100
      XMIN=FF(JJJ)
      IMIN=JJJ
1100  CONTINUE
C     **********************************************************
C     *CALCULATE OVERALL OPTIMUM SAMPLE SIZE,SAMPLING INTERVAL *
C     *ARL0,ARL1 AND STANDARDIZED DECISION INTERVAL H, AND PER *
C     * HOUR LOSS-COST.                                        *
C     **********************************************************
1101  FFC(I)=FF(IMIN)
      HC(I)=H(IMIN)
      GGC(I)=GG(IMIN)
      ARLONC(I)=ARLON(IMIN)
      ARLZRC(I)=ARLZR(IMIN)
      IF(I.GT.3)  GO TO 1102
      GO TO 1000
1102  IF((FFC(I-1).GT.FFC(I-2)).AND.(FFC(I-1).GT.FFC(I))) GO TO 1000
      IFLAG=1
```

```
      IIII=I-1
      GO TO 1300
1000  CONTINUE
      XMIN=1000.0
      DO 1200 KKK=1,NI
      IF(FFC(KKK).GT.XMIN) GO TO 1200
      XMIN=FFC(KKK)
      I=KKK
1200  CONTINUE
1300  IF(IFLAG.EQ.1)    I=IIII
      TABLE(IDELTA,IR1,IR2,1)=I
      TABLE(IDELTA,IR1,IR2,2)=ARLONC(I)
      TABLE(IDELTA,IR1,IR2,3)=ARLZRC(I)
      TABLE(IDELTA,IR1,IR2,4)=HC(I)
      TABLE(IDELTA,IR1,IR2,5)=GGC(I)
      TABLE(IDELTA,IR1,IR2,6)=FFC(I)
      R2=R2+0.5
      IR2=IR2+1
      IF(R2.GT.2.0) GO TO 601
      GO TO 200
601   R1=R1+0.5
      IR1=IR1+1
      IF(R1.GT.0.5) GO TO 750
      GO TO 300
750   DELTA=DELTA+0.25
      IDELTA=IDELTA+1
      IF(DELTA.GT.2.25) GO TO 800
      GO TO 401
800   PRINT 1
      PRINT 1025,LAMDA,V0,V1,KR,KS,TR,TS,BB,CC
1025  FORMAT(' ',10X,'LAMDA=',F3.2,2X,'V0=',F6.1,2X,'V1=',F5.1,2X,
     1'KR=',F5.2,2X,'KS=',F5.2,2X,'TR=',F5.3,2X,'TS=',F4.2,2X,'B='
     2,F4.2,2X,'C=',F4.2)
      PRINT 2
      PRINT 3
      PRINT 4
      R2=-0.5
      DO 100 IR2=1,6
      PRINT 11,((TABLE(IDELTA,IR1,IR2,1),IR1=1,3),IDELTA=1,4)
      PRINT 12,((TABLE(IDELTA,IR1,IR2,2),IR1=1,3),IDELTA=1,4)
      PRINT 13,((TABLE(IDELTA,IR1,IR2,3),IR1=1,3),IDELTA=1,4)
      PRINT 14,R2,((TABLE(IDELTA,IR1,IR2,4),IR1=1,3),IDELTA=1,4)
      PRINT 15,((TABLE(IDELTA,IR1,IR2,5),IR1=1,3),IDELTA=1,4)
      PRINT 16,((TABLE(IDELTA,IR1,IR2,6),IR1=1,3),IDELTA=1,4)
      IF (IR2.EQ.6) GO TO 100
      R2=R2+0.5
      PRINT 4
100   CONTINUE
      PRINT 2
      PRINT 10
```

```
      PRINT 2
      PRINT 5
      PRINT 4
      R2=-0.5
      DO 110 IR2=1,6
      PRINT 11,((TABLE(IDELTA,IR1,IR2,1),IR1=1,3),IDELTA=5,8)
      PRINT 12,((TABLE(IDELTA,IR1,IR2,2),IR1=1,3),IDELTA=5,8)
      PRINT 13,((TABLE(IDELTA,IR1,IR2,3),IR1=1,3),IDELTA=5,8)
      PRINT 14,R2,((TABLE(IDELTA,IR1,IR2,4),IR1=1,3),IDELTA=5,8)
      PRINT 15,((TABLE(IDELTA,IR1,IR2,5),IR1=1,3),IDELTA=5,8)
      PRINT 16,((TABLE(IDELTA,IR1,IR2,6),IR1=1,3),IDELTA=5,8)
      IF (IR2.EQ.6) GO TO 110
      R2=R2+0.5
      PRINT 4
110   CONTINUE
      PRINT2
1     FORMAT('1'//////////// PROBLEM   ',94X,'PAGE 1')
2     FORMAT(' +-------------------------------------------------------------
     *--------------------------------------------------------------+')
3     FORMAT(' |      |',110X,'|'/
     *' |      |',49X,'DELTA',56X,'|'/
     *' |      |',11X,4H0.50,22X,4H0.75,22X,4H1.00,22X,4H1.25,17X,'|'/
     *' |      |',4('  ---------------------   '),'      |'/
     *' |      |',110X,'|'/
     *' | R2   |',51X,'R1',57X,'|'/
     *' |      | ',4(26H  -0.5      0.0      0.5     ),'      |'/
     *' |      | ',4(3(' ------- '),'  '),'      |'/
     *' |      |',110X,'|')
4     FORMAT(' |------|',4(3('--------'),'-|'),'------|')
5     FORMAT(' |      |',110X,'|'/
     *' |      |',49X,'DELTA',56X,'|'/
     *' |      |',11X,4H1.50,22X,4H1.75,22X,4H2.00,22X,4H2.25,17X,'|'/
     *' |      |',4('  ---------------------   '),'      |'/
     *' |      |',110X,'|'/
     *' | R2   |',51X,'R1',57X,'|'/
     *' |      | ',4(26H  -0.5      0.0      0.5     ),'      |'/
     *' |      | ',4(3(' ------- '),'  '),'      |'/
     *' |      |',110X,'|')
10    FORMAT('1'//////////// PROBLEM   ',94X,'PAGE 2'/)
11    FORMAT(' |      |',4(3F8.3,' |'),'   N  |')
12    FORMAT(' |      |',4(3F8.3,' |'),'  L1  |')
13    FORMAT(' |      |',4(3F8.2,' |'),'  L0  |')
14    FORMAT(' |',F4.1,' |',4(3F8.3,' |'),'   H  |')
15    FORMAT(' |      |',4(3F8.3,' |'),'   S  |')
16    FORMAT(' |      |',4(3F8.4,' |'),'   L  |')
      STOP
      END
      SUBROUTINE AUX(X,R1,R2,Q,ET,BI,I,J)
C     ****************************************************************
C     * CALCULATION OF FIRST FOUR TERM OF AN EDGEWORTH SERIES        *
```

```
C      * AND THE ELEMENTS OF EQUATION (6.10) AND (6.11).IMSL SUBROU-*
C      *TINE  MSMRAT IS USED TO CALCULATE THE AREA OF NORMAL CURVE. *
C      *************************************************************
       DIMENSION X(3,3),Q(3,3),FT(3,3),Y(3,3)
       IF(X(I,J).EQ.0.0) GO TO 85
       ZX1=0.39894228*EXP(-X(I,J)*X(I,J)/2.0)
       GO TO 86
85     ZX1=0.39894228
86     Y(I,J)=ABS(X(I,J))
       ZX2=(Y(I,J)**2-1.0)*ZX1
       ZX3=-(X(I,J)**3-3.0*X(I,J))*ZX1
       ZX4=(Y(I,J)**4-6.0*Y(I,J)**2+3.0)*ZX1
       ZX5=-(X(I,J)**5-10.0*X(I,J)**3+15.0*X(I,J))*ZX1
       ZX6=(Y(I,J)**6-15.0*Y(I,J)**4+45.0*Y(I,J)**2-15)*ZX1
       Q(I,J)=ZX1-R1/(6.0*SQRT(BI))*ZX3+R2/(24.0*BI)*ZX4+(R1*R1)/(72.0*BI
      1)*ZX6
       Z=X(I,J)
       CALL MSMRAT(Z,RM,IER)
       FIX=ZX1/RM
       FT(I,J)=1.0-FIX-R1/(6.0*SQRT(BI))*ZX2+R2/(24.0*BI)*ZX3+(R1*R1)/(72
      1.0*BI)*ZX5
       RETURN
       END
       REAL  FUNCTION F(G)
C      ***********************************************************
C      * EXTERNAL ROUTINE FOR CALCULATION OF SAMPLING INTERVAL AS*
C      * A ROOT OF THE EQUATION (6.15).                          *
C      ***********************************************************
       REAL G,LAMDA,KR,KS
       COMMON LAMDA,ARLO,ARL1,U,V,TR,TS,W,BB,CC,BI,KR,KS
       BO=(1./G-0.5*LAMDA+1./12.*LAMDA**2*G)/ARLO
       B1=(ARL1-0.5+1./12.*LAMDA*G)*G
       DBO=-(1./(G*G)-1./12.*LAMDA**2)/ARLO
       DB1=ARL1-0.5+1./6.*LAMDA*G
       T1=LAMDA*G**2*(U+TS*BO*U+LAMDA*U*(TR+TS)-BO*V-LAMDA*W)*DB1
       T2=G**2*(V+LAMDA*B1*V+LAMDA*V*(TR+TS)-LAMDA*TS*B1*U-LAMDA*TS*W)*
      1DBO
       T3=(1.0+LAMDA*B1+TS*BO+LAMDA*(TR+TS))*(1.0+LAMDA*B1)*(BB+CC*BI)
       T4=TS*G*(1.0+LAMDA*B1)*DBO*(BB+CC*BI)
       T5=(BB+CC*BI)*LAMDA*G*(TS*BO+LAMDA*(TR+TS))*DB1
       F=T1+T2-T3-T4+T5
       RETURN
       END
```

DATA

```
-0.7746     0.0000     0.7746
0.5556      0.8889     0.5556
    0.05   150.0    50.0    20.0    10.0    0.2    0.1    0.5    0.1
```

APPENDIX VI

PROGRAM CUSUMSEMI

```
C                      PROGRAM DESCRIPTION
C       **************************************************************!
C       ! PURPOSE OF THE PROGRAM:                                     !
C       ! ---------------------                                       !
C       ! THE MAIN PURPOSE  OF THIS  PROGRAM IS TO PROVIDE            !
C       !TABLES FOR THE SIMPLIFIED SCHEME FOR AN ECONOMIC DESIGN      !
C       !OF CUSUM CHART TO CONTROL NON-NORMAL PROCESS MEANS, UNDER    !
C       !A SPECIFIED VALUE OF ARL1 AT THE REJECTABLE QUALITY LEVEL.   !
C       ! NOMENCLATURE                                                !
C       ! ------------                                                !
C       !   VARIABLE NAME                    DESCRIPTION              !
C       !     NORD                     NUMBER OF GAUSSIAN PIONTS      !
C       !     ZK                       GAUSSIAN POINTS                !
C       !     AK                       GAUSSIAN COEFFICENTS           !
C       !     DELTA                    SHIFT PARAMETER                !
C       !     R1                       MEASURE OF SKEWNESS            !
C       !     R2                       MEASURE OF KURTOSIS            !
C       !     THETA                    MAGNITUDE OF VARIATION IN MEAN !
C       !     H                        STANDARDIZED DECISION INTERVAL !
C       !     NH                       MAX. NO. OF DECISION INTERVALS !
C       !     BI                       SAMPLE SIZE                    !
C       !     X1                       SPECIFIED VALUE OF ARL1=1.05   !
C       !     X2                       SPECIFIED VALUE OF ARL1=1.1    !
C       ! INPUT DATA CARDS                                            !
C       ! ----------------                                            !
C       !                                                             !
C       ! ORDER OF CARDS              FORMAT        VARIABLE  NAME    !
C       !   1                         (8F10.5)          ZK            !
C       !   2                         (8F10.5)          AK            !
C       !   3                         (3F10.5)       DELTA,R1,R2      !
C       !                                                             !
C       ! OUTPUT DESCRIPTION                                          !
C       ! ------------------                                          !
C       !                                                             !
C       ! VARIABLE NAME                          DESCRIPTION          !
C       !                                                             !
C       !   BH(I)                       ITH DECISION INTERVAL.        !
C       !   BTHETA(I)                   VALUE OF THETA CORRESPONDS    !
C       !                               TO THE ITH DECISION INTERVAL.!
C       !   BARLO(I)                    VALUE OF ARLO CORRESPONDS     !
C       !                               TO ITH VALUE OF THETA.        !
C       !   DL(I)                       DERIVATIVES OF BARLO(I)       !
C       !                               WITH RESPECT TO BTHETA(I).    !
```

```
C       !   D                             D*=((BARLO(I)**2)*THETA)/   !
C       !                                 DEL(I).                     !
C                         PROGRAM LISTING
C       *************************************************************
C       * A SIMPLIFIED ECONOMIC SCHEME FOR THE DESIGN OF CUSUM      *
C       * CHARTS TO CONTROL NON-NORMAL PROCESS MEANS.               *
C       *************************************************************
        DIMENSION BTHETA(28),BARLO(28),BH(28),DL(28),TABLE(28,4,2)
        INTEGER K,NDIM,IER
        COMMON  R1,R2,ZK(3),AK(3),BI
        NORD=3
        K=NORD
        NH=28
        I=NH
C       ***************************************************************
C       * READ GAUSSIAN POINTS (ROOTS OF LEGENDER'S POLYNOMIAL) AND  *
C       * GAUSSIAN COEFFICENTS IN THE INTERVAL -1 TO +1, NO-NORMALITY*
C       * PARAMETERS  AND DELTA.                                      *
C       ***************************************************************
        READ(5,20) (ZK(II),II=1,K)
        READ(5,20) (AK(II), II=1,K)
        READ(5,21) R1,R2,DELTA
 20     FORMAT(3F10.5)
 21     FORMAT(3F10.5)
C       *************************************************************
C       * SPECIFICATION OF DESIRED LEVEL OF ARL1 (TYPICAL VALUES    *
C       * ARE 1.05 OR 1.1).                                         *
C       *************************************************************
        X1=1.05
        X2=1.1
        THETA=1.65
        H=0.032
        X=X1
        IFLAG=1
800     DO 904 I=1,NH
        THETA=THETA+0.05
C       ***************************************************************
C       * CALCULATE SAMPLE SIZE USING EQUATION (6.20) FOR GIVEN THETA.
C       ***************************************************************
        BI=(4*(THETA*THETA))/(DELTA*DELTA)
899     CONTINUE
        CALL SUB(THETA,H,ARL)
        IF(ARL.GE.X) GO TO 901
        H=H+0.001
        GO TO 899
901     THETA=-THETA
C       **************************************************************
C       * CALCULATE ARLO AND H UNDER THE CONDITION THAT ARL1=1.05   *
C       * OR 1.1 FOR A GIVEN THETA.                                  *
C       **************************************************************
```

```
      CALL SUB(THETA,H,ARL)
      THETA=-THETA
      BTHETA(I)=THETA
      BH(I)=H
      BARLO(I)=ARL
904   CONTINUE
C     **********************************************************
C     * FIND THE DERIVATIVES OF ARLO'S WITH RESPECT TO THE     *
C     * CORRESPONDING VALUE OF THETA.                          *
C     **********************************************************
      NDIM=NH
      CALL DGT3(BTHETA,BARLO,DL,NDIM,IER)
      DO 911 I=1,NH
      TABLE(I,1,IFLAG)=((BARLO(I)**2)*BTHETA(I))/DL(I)
      TABLE(I,2,IFLAG)=BTHETA(I)
      TABLE(I,3,IFLAG)=BH(I)
      TABLE(I,4,IFLAG)=BARLO(I)
911   CONTINUE
      IF (IFLAG.EQ.2) GO TO 915
      IFLAG=2
      THETA=1.35
      X=X2
      H=0.03
      GO TO 800
915   PRINT 1,DELTA,R1,R2,X1,X2
      DO 920 I=1,NH
      PRINT 2
      PRINT 3,((TABLE(I,II,III),II=1,4),III=1,2)
920   CONTINUE
      PRINT 2
      PRINT 4
      STOP
1     FORMAT('1',26X,82('-')/
     *26X,' |',80X,'|'/
     *26X,' |',14X,'DELTA=',F4.2,14X,'R1=',F4.2,14X,'R2=',F4.2,14X,'|'/
     *26X,' |',80X,'|'/
     *26X,' |',80('-'),'|'/
     *26X,' ',2('|',39(' '),'|')/
     *26X,' ',2('|',16X,'L1=',F4.2,16X,'|')/
     *26X,' ',2('|',39(' '),'|')/
     *26X,' ',2('|',39('-'),'|')/
     *26X,' ',2('|',11X,'|',7X,'|',7X,'|',11X,'|')/
     *26X,' ',2('|      D*   | THETA |   H   |   ARLO    |')/
     *26X,' ',2('|',11X,'|',7X,'|',7X,'|',11X,'|')/
     *26X,' ',2('|',11('-'),'|',7('-'),'|',7('-'),'|',11('-'),'|'))
2     FORMAT(' ',26X,2('|',11X,'|',7X,'|',7X,'|',11X,'|'))
3     FORMAT(' ',26X,2('|',F10.1,' |',F6.2,' |',F6.3,' |',F10.1,' |'))
4     FORMAT(' ',26X,82('-'))
      END
C     *********************************************************
```

```
C      * SUBROUTINE FOR CALCULATION OF AVERAGE RUN LENGTH ARL*
C      ****************************************************
       SUBROUTINE SUB(THETA,H,ARL)
       REAL F,EPS,EPS2,BTA
       REAL C(3,3),B(3,1),WKAREA(30),SN(3,1),D(3,3)
       INTEGER N,NN,IA,IDGT,IER,M,K,NSIG,ITMAX,L,KK,II,III,JJJ,IMIN
       DIMENSION Z(3),A(3)
       DIMENSION X(3,3),Q(3,3),FT(3,3)
       DIMENSION XX(3,1),Q1(3,1),FT1(3,1)
       DIMENSION X1(3,1),Q2(3,1),FT2(3,1)
       DIMENSION X2(1,1),Q3(1,1),FT3(1,1)
       DIMENSION ZX1(3,3),ZX2(3,3),ZX4(3,3),ZX5(3,3),ZX6(3,3)
       COMMON R1,R2,ZK(3),AK(3),BI
       K=3
       DO 25 III=1,K
       Z(III)=(ZK(III)+1.0)*H*0.5
       A(III)=AK(III)*H*0.5
25     CONTINUE
       DO 75 II=1,K
       DO 75 JJ=1,K
       X(II,JJ)=Z(II)-Z(JJ)-THETA
       CALL AUX(X,R1,R2,Q,FT,BI,II,JJ)
       IF(II.EQ.JJ) GO TO 60
       C(II,JJ)=-A(II)*Q(II,JJ)
       GO TO 65
60     C(II,JJ)=(1.0-Q(II,JJ)*A(II))
65     D(JJ,II)=C(II,JJ)
75     CONTINUE
       DO 80  II=1,K
       XX(II,1)=-Z(II)-THETA
       JJ=1
       CALL AUX(XX,R1,R2,Q1,FT1,BI,II,JJ)
       B(II,1)=FT1(II,1)
 80    CONTINUE
       NN=3
       M=1
       IA=3
       IDGT=4
       CALL LEQT2F(D,M,NN,IA,B,IDGT,WKAREA,IER)
       SUM=0.0
       DO 120 II=1,K
       X1(II,1)=Z(II)-THETA
       JJ=1
       CALL AUX(X1,R1,R2,Q2,FT2,BI,II,JJ)
120    SUM=SUM+Q2(II,1)*B(II,1)*A(II)
       X2(1,1)=-THETA
       II=1
       JJ=1
       CALL AUX(X2,R1,R2,Q3,FT3,BI,II,JJ)
       PZ1=SUM+FT3(1,1)
```

```
      DO 130 II=1,K
      SN(II,1)=1.0
130   CONTINUE
      CALL LEQT2F(D,M,NN,IA,SN,IDGT,WKAREA,IER)
      SUM1=0.0
      DO 400 II=1,K
400   SUM1=SUM1+SN(II,1)*Q2(II,1)*A(II)
      SNZ1=1.0+SUM1
      ARL=SNZ1/(1.0-PZ1)
      RETURN
      END
DATA
-0.7746       0.0000      0.7746
0.5556        0.8889      0.5556
0.5           1.0         2.0
```

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