1-1-2007

Robust predictive control of constrained systems with actuating delay.

Ali Kebarighotbi

University of Windsor

Robust Predictive Control of Constrained Systems with Actuating Delay

by

Ali Kebarighotbi

A Thesis
Submitted to the Faculty of Graduate Studies through Electrical and Computer Engineering in Partial Fulfillment of the Requirements for the Degree of
Master of Applied Science at the
University of Windsor

Windsor, Ontario, Canada
2007

Your file Votre référence
ISBN: 978-0-494-35015-7
Our file Notre référence
ISBN: 978-0-494-35015-7

Abstract

Constraints, actuating delay, uncertainties and imperfect state information have many realizations in the actual applications. These phenomena affect the system analysis and controller design in that care should be taken in designing associated stabilizing controllers.

This thesis is dedicated to a setting where the constrained control of an input-delayed linear discrete-time system subject to bounded measurement noise and disturbance input is in question. Using a propagator-based delay compensation strategy and a set theoretic model predictive control scheme, a robust control synthesis for such a setting is introduced. More complications arise from the imperfect state information.

In this manuscript, a scheme to satisfy the constraints as well as to compensate for this delay is presented. It is also guaranteed that the closed-loop system's trajectory will remain at the vicinity of the origin at the steady state. A number of illustrative examples verify the theoretic results.

To my parents and Yasaman for their support and patience.

Acknowledgements

I would like to first thank my supervisor, Dr. Xiang Chen whose guidance has been keeping me on the right research track since the start of my educational career in Windsor. I would specially like to thank him for his attention to my sometimes sporadic thoughts and results on various subjects.

I want also to extend my complements to Dr. Guofeng Zhang with his patience in helping me to understand the basic ideas of the predictive control and in introducing me to the fruitful technical resources which were quite effective in helping me quickly get into the picture of the subject I was researching on.

Needless to say, this work could not be completed without the helpful comments which I received from my committee members Dr. B. Shahrava and Dr. Z. Hu. Therefore, I would like to thank them for their contribution to this work.

After all, I would like to thank all the administrative staff in the department of electrical engineering whose welcoming-ness have made my stay in the University of Windsor sweet and smooth.

Contents

List of Figures

List of Acronyms

AP	Analytical Predictor
AEE	Advanced Estimation Error
AUB	Asymptotically Ultimately Bounded
CNOMPC	Correlation None-Observant Model Predictive Control
COMPC	Correlation Observant Model Predictive Control
DAP	Discrete Analytical Predictor
DROS	Deviation-Robust One-Step
DRS	Deviation-Robust Stabilizable
GAP	Generalized Analytical Predictor
IMC	Internal Model Control
LMI	Linear Matrix Inequalities
LPLC	Linear Programming with Linear Constraints
MPC	Model Predictive Control
MPCCIC	Model Predictive Control with Contractive Invariance Constraints
PDI	Positively Disturbance Invariant
PD	Predicted Deviation
QPLC	Quadratic Programming with Linear Constraints
QPQC	Quadratic Programming with Quadratic Constraints

List of Symbols

The notation used in this thesis is as follows:

Capital Greek and Latin alphabets are used for representing various matrices. x_i refers to the i-th entry of the vector x or i-th row of matrix X depending on the context. A zero vector in $\mathbb{R}^n$ is represented by 0_n. Finally, $\mathbb{Z}^+$ is the set of all positive integers plus 0.

Notation	Definition
$\oplus$	Minkowski addition.
$\bigoplus$	Minkowski summation operator.
$\sim$	Pontryagin difference operator.
A^T	Transpose of a matrix A.
$\mathcal{A}^o$	Interior of set $\mathcal{A}$.
$\mathfrak{B}^\epsilon$	A ball with radius ϵ.
$\mathbb{R}^n$	n-dimensional Euclidean space.
cont(.)	Contraction function.
$\mathrm{d}_h(.,.)$	Hausdorff metric.
hull(.)	Convex hull of set of points in a vector space.
Pre(.)	Preimage of the set as its operand.
vert(.)	Set of vertices of a polyhedron.

Chapter 1

Introduction

1.1 Motivation

The areas of time delay systems and constrained control have long been under intensive investigations. Many advancements have been made in either topic and there is a myriad of publications pertinent to each subject. However, in actual applications (e.g. process control and networked control of constrained systems) there are many situations where the presence of neither of them can be neglected. Unfortunately, this fact has not received the attention it deserves and the main reason behind this work is to address this lack.

The time delay and constraints on the system shrink the domain of attraction of the closed-loop system. Domain of attraction is simply defined as the largest possible region in the state space in which the closed-loop system is asymptotically stable [71]. In order to have an effective control, this region should be enlarged as much as possible. More specifically, effective control demands a method which can address the adverse effect of both time delay and constraints on the closed-loop system. This, hence, motivates one to do an in-depth analysis in both areas and to try to address their combinatorial problem by means of the available design control techniques. This issue becomes more difficult when besides the stability of the system some performance specifications are also imposed on the design. These performance requirements usually appear in the problem setting as new constraints on the behavior of system dynamics. For example, it may be crucial in a design that how fast the closed-loop response of the system is going to be regulated. This kind of performance specification can be introduced as a set of contractive constraints on the state trajectory of the closed-loop system, for instance.

To make the motif behind the subject more applied, it is also wise to consider the effect of imperfect state information in control and to account the need for measuring the output of the plant to reproduce the system states, known as output feedback. This issue becomes difficult due to various sources of unmeasured uncertainty like, no perfect model of the plant under control, persistent state disturbances and measurement noise. The effect of these uncertainties makes it impossible to design a controller able to guarantee the exponential or even asymptotic stability in their original sense [71]. This makes another motivation which is aimed to design an estimator which together with the controller scheme can guarantee the a bounded steady state response known as ultimate boundedness [11] of the system.

On the other hand, control of a system suffering from a problem like actuating delay is effective when the computation of the control does not compromise the hardware infrastructure (e.g. faster CPU and more memory) and is fast enough so that it does not make another delay problem in the loop. This limits the set of applicable tools which can be used to tackle this issue.

Given these motivations and requirements for designing a safe estimator/controller combination, this thesis concentrates on incorporating the effect of delay and uncertainties in design of a controller which beside guaranteeing stability guarantees that the constraints will not be violated and compensates for the actuating delay after a short interval as if no delay is present in the control input.

The rest of this chapter is devoted to give the reader a background on the works already done for the time delay systems and constrained control. The intention is to highlight the motivations more specific to each field as well as to their overlap. A note on the thesis structure is also included at the end.

1.2 Systems with Actuating Delay

The control of time delay systems has been the hot spot for the last years and has already received a lot of attention. The motivation behind this push first came from the process industry since there were many examples of the delay systems which required a better control than the conventional memoryless PI controllers (e.g. heat exchangers and feeding/exhausting systems in the process plants). As a result of such motivation many theoretical advancements have been made during the last decades. See for example, [35, 78, 88] for comprehensive survey on the recent results in this realm. Interesting discussions on the controllability of actuating delay systems can be found in [44, 45, 46] for both continuous and discrete-time systems. For some books on the subject reader may be referred to [34, 48, 57, 58, 70]. Unfortunately, except for [70] which has a dedicated part for actuating delay most of the material available is on the systems with state delays, though the control of systems with actuating is no less challenging.

The developments of the theoretical ground for control of the systems with actuating delay has tracked two somehow independent ways for continuous-time and discrete-time systems but started within the same time frame. The credit for discussion in the continuous-time domain goes to [2, 54, 59] wherein authors advocate the use of propagator based controller (e.g. a controller which works on the future states of the system rather than the present states) to enlarge the domain of attraction of the resulting closed-loop dynamic system. Their approach is then tailored for robustness against uncertainties in many publications (e.g. see [47]) which usually consider a robust analysis for the systems with state feedback. Although not as fruitful as propagator-based control, some efforts have also been made on tailoring the existing non-delay methods or memoryless feedback schemes [20, 85, 91] in order to achieve a degree of robustness. However, it has been shown via simple analysis [80] that the memoryless controllers are far inferior to the propagated-based schemes.

By the advent of the digital control, some developments in the discrete-time systems was needed. Many propagator-based theories has been developed for the nominal systems. Smith predictor which has attracted so much attention in the industry was first introduced by [86]. In the nominal sense it could compensate for the actuating delay very easily. Due to the poor stability of smith predictor design other schemes addressing actuating delay have emerged. See for example, [27, 28, 29] for internal model control (IMC), [23, 67] for analytical and discrete analytical predictor (AP/DAP) schemes and generalized analytical predictor (GAP) [89, 90]. IMC also has got some attention in the industry due to better steady state performance as opposed to Smith predictor. However, because it utilizes the inverse of the plant model, the model of the plant should exactly be known. Also demerits of the Smith predictor for unstable and uncertain systems is still true for IMC. Analytical predictor schemes

AP, DAP, and GAP ameliorated these demerits and have shown better steady state response and better robustness to known uncertainties, however, they also fall short in case of unmeasured perturbations. To avoid scattered discussion an in-depth analysis cannot be given on these schemes here. A basic comparison of these schemes is given in [90] where it can be concluded that all the schemes fall short when it comes to the systems with uncertainties where f unmeasured disturbance or modeling error is present. Later, some researchers have attempted to modify these schemes in order to use them for the unstable systems [3, 66, 64, 87], though not with big success.

All in all, the result of the researches done to date justifies the fact that more or less there is not much one can do to control of the input-delayed systems when the they suffers from unknown or unmeasured uncertainties either in the form of modeling error or in the form of exogenous disturbance and noise. By using a propagator which is in a way related to DAP and the work done in [2], it will be shown in later chapters that the effect of this issue strains the controller design and certain cares should be taken while designing a controller to stabilize the general systems with actuating delay.

1.3 Constrained Systems and Predictive Control

1.3.1 An Overview

In terms of finding applications in industry, theories developed for the constrained systems come in the second position after the regular linear system theories. This is due to the fact that simply all the controlled systems have either implicitly or explicitly constraints on their input or states. In many applications like ship rudder control or compressor systems not attending to the existence of such constraints is tantamount to deadly accidents [32].

The constrained control via schemes other than predictive control is not common. However, some researches have been done in this direction. In [38] a procedure is presented in order to maximize the attractive region of the input-constrained closed-loop system with linear feedback. Using linear matrix inequalities (LMI) [13] a way to determine the maximal domain of attraction of linear systems, albeit in the form of an ellipsoid, is introduced. Differently, [75] has used the invariance set theory in order to characterize the domain of attraction in the form of a polytope. Each taken approach assumed linear state feedback law. This is because nonlinear control synthesis for such systems in optimal scheme requires finding a robust control lyapunov function (RCLF) [26] which is not an easy task for general nonlinear systems in most of the cases. Moreover, the design of the proposed linear law is done offline removing the chance to change it according to the changing online conditions.

On the other side of the spectrum comes the MPC which is related to the optimal control concept and is tailored mainly to consider constraints on the system's input and/or states. It is a recursive methodology wherein at each time instant an optimization is performed over a future control input trajectory rather than a control input alone. Implementation is done by applying only the first entry of such a trajectory to the plant. MPC removes the shortcomings of the other constrained control approaches by adopting a time varying control law which can adapt to condition changes and introducing variations with nonlinear law which are easy to implement and can have an immense effect in enlarging the domain of attraction of the closed-loop system [81]. These capabilities has turned MPC into a popular control approach with over 2000 reported applications [61]. Also it is known as the only advanced method with significant impact on the industry [56]. This is no surprise by knowing the fact that MPC is first used in the industry and then has attracted the academia.

For the cogency of discussion further notes on the origin of MPC is omitted here. A comprehensive note on MPC history can be found in [56].

1.3.2 Major MPC schemes

In this section the aim is to discuss the major MPC schemes which have found good merits in terms of a combination of robustness, optimality, and computational intensity. For comprehensive notes on various types of MPC schemes to date reader may consult several surveys and books on the model predictive control [6, 56, 68, 63, 77]. The nonlinear MPC schemes are discussed in [61, 63, 72]. Also, industry-oriented discussions can be found in [73, 74].

The MPC scheme which was first used in the industry had a finite horizon [1]. However, It is proven that finite horizon scheme falls short in stabilization of the systems [56]. As a remedy to this problem a dual mode MPC (DMMPC) was then introduced. In this scheme a terminal cost and a terminal constraint have been added to the finite horizon MPC in order to emulate an infinite horizon problem. The idea of using a terminal cost and constraint at once to guarantee nominal feasibility as well as stability was first introduced in [41], where the terminal constraint was chosen to be the origin, i.e. $\mathcal{T} = \{0\}$. However, this constraint reduces the size of the feasible set and could result in numerical convergence problems in the optimization, especially when working with nonlinear models [61]. Also it could not be extended to the case of systems with uncertainties.

One of the most popular MPC methods for guaranteeing robust stability is to choose an invariant terminal set [65]. Such a set has a feature that every state trajectory starting inside this set will remain in its interior for unlimited time. By choosing

[1] Discussions on finite and infinite horizon problems can be found in Chapter 2.

the terminal constraint to be such a set, rather than the origin, the size of the feasible region of the MPC optimization for a given horizon N is increased and most of the numerical convergence problems are addressed. After introduction of this noble approach to the academia almost all the robust MPC schemes proposed were in a way a subsidiary to it. Major schemes which have found considerable attentions are [5, 18, 51]. Among these schemes model predictive control with contractive invariance constraint (MPCCIC) [18] is chosen and extended in this thesis resulting in a whole new method. This scheme makes the grounding of the main discussion of the thesis which can be found later in Section 3.3. It is based on set invariance theory [12] and involves computation of problem-relevant invariant sets or attractive regions prior to doing any online optimization. This has the effect of less online computation which is amicable for actuating delay problem.

It is also important to point out that using this approach one can easily take the state estimation error and actuating delay into the consideration. This then can be seen as a remedy to problem of output feedback in MPC which has not lent itself to full disclosure yet. In fact, there are only few useful papers published on this issue [4, 55, 62, 83] which as a result make this area remain fairly open to new investigations. The cause of this issue is the strict dependence of MPC predictions on the current system state. Therefore, any error in the state measurement yields predictions which are not close to the actual plant state trajectory in future. An adequately updated survey on the output feedback MPC can be found in [25].

There are other DMMPC schemes which have considered the delay in the system. The major work is done in [51] which spawned a series of schemes based on LMIs [39, 40, 80]. However, LMI approach has the shortcoming in that it is not clear how to use it for the output feedback structure [56]. Moreover, LMI dimensions can increase rapidly with number of the states and amount of delay in the system which renders it an online-computationally demanding scheme[56].

1.3.3 Different MPC Optimizations

All the MPC schemes which have found considerable attention in the industry assume either linear or quadratic constraints for the processes under control. The reason is that using nonlinear constraints can yield to non-convex and/or nonlinear optimization problems which then put the computation resources under pressure. Common optimizations involve quadratic programming with linear constraints (QPLC) and linear programming with linear constraints (LPLC). However, there are instances of successful schemes using quadratic programming with quadratic constraints (QPQC) [52] albeit at the expense of heavier but tolerable computations. Since quadratic constraints, usually in terms of ellipsoids, can fall short in tightly approximating the actual constraints on the system, theoretical developments in this thesis has been grounded on linear constraints. Furthermore, the scheme proposed in this thesis is intended to compensate for delay and base a procedure to address faster applications like networked control [92]. Hence, using the quadratic constraints is not advocated.

1.4 Estimation

There are various estimation/reachability analysis technics using for examples ellipsoids [9, 16, 53, 82], zonotopes [1, 31] and parallelotopes [17] to define a guaranteed state estimation. However, each of these approaches has shortcomings when compared to the polytopic approach taken here. For example, each step in the estimation by ellipsoids requires outer-approximation of the resulting estimation error bound which is detrimental to precision of the analysis. Zonotopes and parallelotopes are special polytopes and this speciality makes them not as flexible as the general polytopes in estimation of an arbitrary region in the space. A brief comparison is given in [76].

It is known for a while that coupling an stable state estimator and a nominally exponentially stable MPC scheme will result in asymptotic closed-loop stability of the whole system given the disturbances and noises are decaying overtime [84]. However, when the problem deals with the persistent uncertainties in the form of unmeasured disturbance input, measurement noise, finding a control scheme to guarantee asymptotic stability is not possible. The reason is that when the perturbations are persistent using an stable estimator can only guarantee a bound on the state estimation error and the problem of coupling of such an estimator with a nominally exponentially stable MPC scheme does not guarantee even the asymptotically ultimately bounded (AUB) stability[2]. However, it is shown in [60] that by applying invariance theorem one can achieve the AUB stability of the closed-loop response.

The idea of set invariance for designing an estimator first published in [24] where a method to define a polytopic bound on the estimation error is proposed for the systems with disturbance input. In this thesis, the idea in [24] is extended to the case where the measurements are contaminated by persistent but polytopically bounded noise.

1.5 Structure of Thesis

This thesis is organized as follows:

Chapter 2: Set Invariance and Robust Predictive Control

In Section 2.1, a preliminary definition on the MPC is given in both nominal case and

[2]Relevant discussion on AUB stability has been included in Section 2.2.4

general dual mode scheme. Section 2.2 is dedicated to give preliminaries regarding the polytopic objects and their representations. A group of set valued operations and tools is also defined and characterized in this section. In particular, Section 2.2.3 is devoted to the discussion on the basics of the set invariance in control. The way to design and implement the invariance sets under time-invariant linear feedback law as well as time-varying controller schemes is also discussed in this section. The final section of Chapter 2 regards to the MPCCIC scheme since it is needed for understanding the materials given in Chapter 3.

Chapter 3: Robust Predictive Control with Actuating Delay

The main contribution of this work is squeezed in this chapter. Chapter 3 begins with the analysis and design of a error-bounding state estimator which is a extension to the work done [24]. It also deals with introducing a linear set propagator which plays an important role in compensating for the actuating delay as well as in enlarging the domain of attraction of the resulting closed-loop system. The discussion on the proposed MPC approach is given in section 3.3. This includes the notes on designing various invariant sets like terminal constraint and a new feasibility and stability guaranteeing constraint set for the proposed MPC model. Section 3.4 is then intended to verify the theoretics developed in the previous sections of this chapter via a set of illustrative examples.

Chapter 4: Conclusions and future work

This chapter summarizes the contributions made by this thesis and outlines directions for future research.

Chapter 2

Set Invariance and Robust Predictive Control

In this chapter, first an overview of the model predicitve control methodology is given. The basic conceptual definitions are explained and different tools needed to operate on sets are characterized. General idea behind the set invariance theory and its relation to MPC is also discussed. In particular, The MPCCIC scheme is also introduced to provide prerequisites to help assimilate the discussion in the main part of this thesis.

2.1 Model Predictive Control

Assume a case in which no disturbance is present and exact state information is available for control-related computations. Let a system dynamics be summarized as the following:

$$x(k+1) = f(x(k), u(k), w(k)), \tag{2.1}$$

where k is the time step, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $w \in \mathbb{R}^n$ and $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is general time-invariant continuous function of state x, input u and the disturbance input w. It is also assumed that $f(\cdot,\cdot,\cdot)$ possesses a fixed point at origin, i.e. $0_n = f(0_n, 0_m, 0_n)$. The following set memberships are also held

$$x \in \mathcal{X} \ni 0_n, \tag{2.2}$$

$$u \in \mathcal{U} \ni 0_m, \tag{2.3}$$

$$w \in \mathcal{W} \ni 0_n. \tag{2.4}$$

where $\mathcal{X}$ is a generic set and $\mathcal{U}$ is compact. Model predictive control is a scheme in which a nominal copy of plant model (i.e. when $w = 0_n$) known as *internal model* is used to predict the future states and inputs of the plant. In this section, a brief discussion is dedicated to the MPC schemes defined for the system (2.1) under different conditions.

2.1.1 Nominal Regulation Problem for MPC

Consider the dynamics (2.1) where it is assumed that $w(k) = 0_n, \forall k \in \mathbb{Z}^+$. The nominal MPC problem can be described by the following procedure:

- At each instant k find the solution to the following constrained optimization problem:

$$\mathbf{u}_{opt} = \min_{\mathbf{u}} \sum_{i=0}^{N-1} L(\check{x}(k+i|k), \check{u}(k+i|k)), \tag{2.5a}$$

$$\mathbf{u} = [\check{u}(k|k)^T, \check{u}(k+1|k)^T, \ldots, \check{u}(k+N-1|k)^T]^T, \tag{2.5b}$$

subject to

$$\check{x}(k|k) = x(k), \tag{2.5c}$$

$$\check{x}(k+i+1|k) = f(\check{x}(k+i|k), \check{u}(k+i|k), 0_n), \tag{2.5d}$$

$$\check{x}(k+i|k) \in \mathcal{X}, \qquad i = 0, \ldots, N-1, \tag{2.5e}$$

$$\check{u}(k+i|k) \in \mathcal{U}, \qquad i = 0, \ldots, N-1, \tag{2.5f}$$

- Set the actual input $u(k) = \check{u}(k|k)$ and repeat the optimization with updated data at next sampling instance.

In the above optimization $\check{x}(k+i|k), \check{u}(k+i|k), i = 0, \ldots, N-1$ are the MPC's *predicted state* and *predicted control input* respectively. They are defined as predicted state and predicted input of the system for time step $k+i$ which are evaluated based on the state information at time k, i.e. $x(k)$. (2.5d) is the MPC's internal model used to do the predictions. $L(.,.)$ is called *stage cost function* which is a continuous, non-negative and time invariant function defined on $\mathcal{X} \times \mathcal{U}$.

It is known [56] that due to finite horizon nature of the problem (i.e. when $N < \infty$), optimization (2.5) cannot guarantee feasibility nor stability of closedloop dynamics in any sense. The following general scheme is then introduced to get over this problem.

2.1.2 Dual-mode MPC (DMMPC) - Need for Terminal Cost and Constraint

Considering (2.1), a generic DMMPC optimization can be represented by

$$\mathbf{u}_{opt} = \min_{\mathbf{u}} F(\check{x}(k+N|k)) + \sum_{i=0}^{N-1} L(\check{x}(k+i|k), \check{u}(k+i|k)), \tag{2.6a}$$

$$\mathbf{u} = [\check{u}(k|k)^T, \check{u}(k+1|k)^T, \ldots, \check{u}(k+N-1|k)^T]^T, \tag{2.6b}$$

subject to

$$\check{x}(k|k) = x(k), \tag{2.6c}$$

$$\check{x}(k+i+1|k) = f(\check{x}(k+i|k), \check{u}(k+i|k), 0_n), \tag{2.6d}$$

$$\check{x}(k+i|k) \in \mathcal{X}, \qquad i = 0, \ldots, N-1, \tag{2.6e}$$

$$\check{u}(k+i|k) \in \mathcal{U}, \qquad i = 0, \ldots, N-1, \tag{2.6f}$$

$$\check{u}(k+i|k) = g(\check{x}(k+i|k)) \in \mathcal{U}, \qquad i \geq N, \tag{2.6g}$$

$$\check{x}(k+i|k) \in \mathcal{T} \subseteq \mathcal{X}, \qquad i \geq N, \tag{2.6h}$$

where $F(.)$ is the *terminal cost* which is a non-negative, time invariant and continuous function on $\mathcal{X}$. $g(.)$ is a time invariant function on $\mathcal{X}$ which defines a time-invariant state feedback law inside the set $\mathcal{T}$. $\mathcal{T}$ itself is called the *terminal constraint.*

Remark 2.1: The reason for naming this scheme as *dual-mode* is caused by the fact that for the predicted control inputs with indices higher that that of horizon N, control law switches from MPC law to a fixed control law defined by $g(.)$. In the simplest case $g(.)$ can be a linear state feedback law which is will be discussed later in this chapter.

Remark 2.2: It had been known (e.g. see [10]) for quite some time that using Bellman's principle of optimality one can use the problem (2.5) with infinite horizon, i.e. setting $N = \infty$. However, it was not known how to handle constraints with infinite horizon since it makes (2.5) an infinte-dimensional problem for which a solution can not be conceived. DMMPC has solved this problem by doing the optimization over an N-tuple trajectory with matrix representation (2.6b) and relegating the rest of the problem to time-invariant control law (2.6g) for which a terminal cost $F(\cdot)$ can be defined.

2.2 Set Invariance and MPC

Invariance concept in control has been considered relatively early in modern control literature (e.g. see [22]) and is proven as a tool both in analysis and synthesis of control sysytems [12]. This section is dedicated to relation of set invariance and MPC methodology.

2.2.1 Polyhedrons, Polytopes and Their Representations

Definition 2.1 (Closed Half-space): Consider n-dimensional Euclidean space $\mathbb{R}^n$. Associated with any constant vector $\pi \in \mathbb{R}^n, \pi \neq 0_n$ and a constant $\theta \in \mathbb{R}$, there is a *Closed Half-space* defined by

$$\mathcal{H}(\pi, \theta) = \{x \in \mathbb{R}^n | \pi^T x \leq \theta\}. \tag{2.7}$$

Definition 2.2 (Polyhedron): A convex set, $\mathcal{A} \subset \mathbb{R}^n$, is called a *polyhedron* if it can be represented by a finite intersection of closed half-spaces.

$$\mathcal{A} = \bigcap_{i=1}^{n_{ha}} \mathcal{H}(\pi_i, \theta_i), \quad \pi_i \in \mathbb{R}^n, \theta_i \in \mathbb{R}$$

where n_{ha} is the number of half-spaces involved.

Using (2.7), a polyhedron can be represented by its *half-space representation* of the form

$$\mathcal{A} = \{a \in \mathbb{R}^n | \pi_i^T a \leq \theta_i, \quad i = 1, \ldots, n_{ha}\} = \{a \in \mathbb{R}^n | \Pi a \leq \Theta\}, \tag{2.8}$$

where $\Pi \in \mathbb{R}^{n_{ha} \times n}$ and $\Theta \in \mathbb{R}^{n_{ha}}$. In particular, for a polyhedron $\mathcal{A}$ containing the origin in its interior $\mathcal{A}^o$, a representation (2.8) exists where $\theta_i \geq 0, i = 1, \ldots, n_{ha}$[50].

Definition 2.3 (Extreme Point): Consider a convex set $\mathcal{A} \in \mathbb{R}^n$. A point $\bar{a}$ is an *extreme point* of $\mathcal{A}$ if and only if

$$\bar{a} \in \mathcal{A} : \nexists a_1,\ a_2 \in \mathcal{A},\ \nexists \lambda \in (0,\ 1) \quad \text{such that} \quad \bar{a} = (1-\lambda)a_1 + \lambda a_2.$$

Intuitively extreme points of a convex set are the corners or vertices of that set. The set of all extreme points of $\mathcal{A}$ is called *extreme set* of $\mathcal{A}$.

Definition 2.4 (Convex Hull): Consider a set of points $A = \{a_i \in \mathbb{R}^n,\ i = 1, \ldots, n\}$. The *convex hull* of A is a set $\mathcal{A}$ represented by

$$\mathcal{A} = \mathrm{hull}(A) = \{a \in \mathbb{R}^n | \exists \alpha = \{\alpha_i, i = 1, \ldots, n : \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1\} \text{ and } a = \sum_{i=1}^{n} \alpha_i a_i\} \tag{2.9}$$

and is intuitively the minimal convex envelope containing all points in A.

Theorem 2.1: [79], Consider a convex set $\mathcal{A}$ with a countable extreme set $\bar{A} = \{\bar{a}_i \in \mathcal{A}, i = 1, \ldots, n_{va}\}$. Let also $A = \{a_i \in \mathcal{A}, i = 1, \ldots, n\}$ be arbitrary. Then A has the minimum number of points such that $\mathrm{hull}(A) = \mathcal{A}$ if and only if $A = \bar{A}$.

Proposition 2.1: Let $\mathcal{A}$ be a polyhedron with extreme set $\bar{A} = \{\bar{a}_i, i = 1, \ldots, n_{va}\}$. Then $\mathcal{A}$ has also a *convex hull representation* of the form

$$\mathcal{A} = \mathrm{hull}(\bar{A}) = \mathrm{hull}\big(\{\bar{a}_i, i = 1, \ldots, n_{va}\}\big). \tag{2.10}$$

Remark 2.3: Representations (2.8) and (2.10) are interchangeable. However, as the dimension and the number of half-spaces grows finding all vertices of the polyhedron which involves a mix of search and linear programming (LP) can become quite demanding. The same is also true for finding a numerically robust algorithm which can compute the convex hull of a large number of points [21].

Definition 2.5: A set $\mathcal{A} \in \mathbb{R}^n$ is said to be bounded if and only if there exists a constant $r > 0$ such that $\mathcal{A} \subset \mathfrak{B}^r$, where $\mathfrak{B}^r = \{x \in \mathbb{R}^n : \|x\| \leq r\}$[1].

Definition 2.6 (Polytope): A bounded polyhedron is called a *polytope.*

Definition 2.7: A polytope $\mathcal{A}$ is said to be symmetric if and only if $\forall a \in \mathcal{A}, -a \in \mathcal{A}$. Briefly shown, $\mathcal{A}$ is symmetric if $\mathcal{A} = -\mathcal{A}$.

Remark 2.4: In order to have a compact and unified representation, a notation $\mathbb{K}^n$ is adopted in this thesis to define all compact and convex sets in $\mathbb{R}^n$. Notice that in this sense polytopes in $\mathbb{R}^n$ are members of $\mathbb{K}^n$.

2.2.2 Basic Set-induced Operations

Definition 2.8: Consider a polyhedron $\mathcal{A} \subset \mathbb{R}^n$ with half-space representation (2.8). Then *affine translation* of $\mathcal{A}$ with respect to the translation vector $v \in \mathbb{R}^n$ is a set $\mathcal{B} \subset \mathbb{R}^n$ and is defined by

$$\mathcal{B} = v + \mathcal{A} = \{b \in \mathbb{R}^n | \Pi b \leq \Theta + \Pi v\}. \tag{2.11}$$

Definition 2.9: Given two polyhedrons

$$\mathcal{A} = \{a \in \mathbb{R}^n | \Pi_1 a \leq \Theta_1\}, \qquad \mathcal{B} = \{b \in \mathbb{R}^n | \Pi_2 b \leq \Theta_2\},$$

their intersection is a set $\mathcal{C} \subset \mathbb{R}^n$ with the following representation

$$\mathcal{C} = \mathcal{A} \cap \mathcal{B} = \left\{ c \in \mathbb{R}^n | \begin{bmatrix} \Pi_1 \\ \Pi_2 \end{bmatrix} c \leq \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix} \right\}.$$

Remark 2.5: The matrix concatenation above may yield redundant inequalities. These inequalities may strain the computations involving these sets since they cause redundant computations. To tackle this problem some methods already are introduced to remove such redundancies [15, 42].

[1] $\|\cdot\|$ can be any vector norm defined in $\mathbb{R}^n$.

Definition 2.10: Let $\mathcal{A} \in \mathbb{K}^n$ be a polytope and $\mathrm{T} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. The *linear transformation* of $\mathcal{A}$ is a set $\mathcal{B} \subset \mathbb{R}^m$ with the following definition

$$\mathcal{B} = T(\mathcal{A}) = \{b \in \mathbb{R}^m | b = T(a),\ \forall a \in \mathcal{A}\}. \tag{2.12}$$

When the transformation is written in the form of a matrix in $\mathbb{R}^{m\times n}, m \geq n$, two situations might arise which are discussed by the following theorems.

Theorem 2.2 ($T \in \mathbb{R}^{m\times m}$ is invertible [8, 42]): Assuming $\mathcal{A}$ possesses the representation (2.8), the linear transformation of $\mathcal{A}$ can be written as

$$\mathcal{B} = T\mathcal{A} = \{b \in \mathbb{R}^n | \Pi T^{-1} b \leq \Theta\}.$$

Theorem 2.3 ($T \in \mathbb{R}^{m\times n}, m \geq n$ [42, 69]): Let $r = \mathrm{rank}(T)$. Then

$$\mathcal{B} = T\mathcal{A} = \{b \in \mathbb{R}^m | T_\perp b = 0, \Pi T_I b \leq \Theta\}$$

where rows of $T_\perp \in \mathbb{R}^{(m-r)\times m}$ form a basis for subspace of $\mathbb{R}^m$ which is orthogonal to the subspace spanned by columns of T and T_I is any matrix with property $T_I T = I_m$.

A more elaborated account on these theorems as well as their proofs can be found in [42]. As a general alternative to Theorems 2.2 and 2.3, the following theorem can be used to address the problem of linear transformation. However, according to Remark 2.3, this may cause heavy computations when the number of vertices is large.

Theorem 2.4: [79] Let $\mathcal{A} \in \mathbb{K}^n$ be a polytope. Assume $\bar{A} = \mathrm{vert}(\mathcal{A}) = \{\bar{a}_i, i = 1, \ldots, n_{va}\}$ is the extreme set of $\mathcal{A}$ and $T \in \mathbb{R}^{m\times n}$ is a linear matrix transformation. Then the set $\mathcal{B} = \mathrm{T}(\mathcal{A})$ is a subset of $\mathbb{K}^m$ with the set of vertices $\bar{B} = \{\bar{b}_j, j = 1, \ldots, n_{vb}\}$ such that

$$n_{vb} \leq n_{va} \text{ and } \forall \bar{b}_j \in \bar{B}, \exists \bar{a}_i \in \bar{A} : \bar{b}_j = T\bar{a}_i.$$

Theorem 2.5: [79] Consider a countable set of points A in $\mathbb{R}^n$. Let also B be a set of points in $\mathbb{R}^n$ such that $A \subset B$. Then $\text{hull}(A) \subseteq \text{hull}(B)$.

Definition 2.11: Consider a polytope $\mathcal{A} \in \mathbb{K}^n$ and a linear matrix transformation $T : \mathbb{R}^n \to \mathbb{R}^m$. Then, *pre-image* of $\mathcal{A}$ with respect to T is defined as

$$\text{Pre}(\mathcal{A}) = \{x \in \mathbb{R}^n | Tx \in \mathcal{A}\}.$$

Definition 2.12 (Support Function [36, 50, 79]): The support function of a set $\mathcal{A} \subset \mathbb{R}^n$ calculated at a given direction $\eta \in \mathbb{R}^n$ is defined by

$$h_{\mathcal{A}}(\eta) = \sup_{a \in \mathcal{A}} (\eta^T a).$$

Furthermore, if $\mathcal{A}$ is a polytope with representations (2.8) and (2.10), then

$$h_{\mathcal{A}}(\eta) = \max_{\bar{a}} (\eta^T \bar{a}), \qquad \bar{a} \in \text{vert}(\mathcal{A}). \tag{2.13}$$

Definition 2.13 (Minkowski Addition [36]): Consider $\mathcal{A}$ and $\mathcal{B}$ as two generic sets in an Euclidean space. The *Minkowski addition* of the two sets is described by

$$\begin{aligned} \mathcal{C} = \mathcal{A} \oplus \mathcal{B} &= \{c = a + b : a \in \mathcal{A}, b \in \mathcal{B}\}, \\ &= \bigcup_{b \in \mathcal{B}} (\mathcal{A} + b). \end{aligned} \tag{2.14}$$

The notation "$\bigoplus$" will also be used in order to show the sum operator for this type of addition.

Lemma 2.6: [33] When the operands $\mathcal{A}$ and $\mathcal{B}$ in (2.14) are polyhedrons, their Minkowski addition can be computed as per following,

$$\mathcal{C} = \{c \in \mathbb{R}^n | c \in \text{hull}(\bar{a} + \bar{b}), \forall \bar{a} \in \text{vert}(\mathcal{A}), \forall \bar{b} \in \text{vert}(\mathcal{B})\}. \tag{2.15}$$

Remark 2.6: Lemma 2.6 suggests that Minkowski addition can be done through finding the vertices of each operand and then, finding all combinations of addition of their vertices and taking the hull of the resulting combinations.

The following theorem characterize the relation between support function of the sets and their Minkowski addition.

Theorem 2.7: [79] Let $\mathcal{A}$ and $\mathcal{B}$ be two sets with their Minkowski addition defined in (2.14). For any direction $\eta \in \mathbb{R}^n$, we have

$$h_{\mathcal{C}}(\eta) \geq h_{\mathcal{A}}(\eta) + h_{\mathcal{B}}(\eta).$$

Definition 2.14 (Pontryagin Difference [36]): Given two generic sets $\mathcal{A} \subset \mathbb{R}^n$ and $\mathcal{B} \in \mathbb{K}^n$, the *Pontryagin difference* or shortly *p-difference* between $\mathcal{A}$ and $\mathcal{B}$ is defined by

$$\begin{aligned} \mathcal{C} = \mathcal{A} \sim \mathcal{B} &= \{\mathbf{c} \in \mathbb{R}^n | \mathbf{c} + \mathcal{B} \subset \mathcal{A}\} \\ &= \bigcap_{b \in \mathcal{B}} (\mathcal{A} - b). \end{aligned} \tag{2.16}$$

Theorem 2.8: [50] Let $\mathcal{A}$ and $\mathcal{B}$ be such that their p-difference (2.16) is defined. For any direction $\eta \in \mathbb{R}^n$, we have

$$h_{\mathcal{C}}(\eta) \leq h_{\mathcal{A}}(\eta) - h_{\mathcal{B}}(\eta).$$

Lemma 2.9: [50] Let $\mathcal{A}$ and $\mathcal{B}$ be sets in Euclidean space such that $\mathcal{A} \sim \mathcal{B} \neq \emptyset$. Let $\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2$, then

$$\mathcal{A} \sim \mathcal{B} = (\mathcal{A} \sim \mathcal{B}_1) \sim \mathcal{B}_2. \tag{2.17}$$

The following lemma shed light on the way with which the p-difference can be computed for polytopes.

Lemma 2.10: [50] Suppose $\mathcal{A} \subset \mathbb{R}^n$ is a polyhedron with representation (2.8), and assume $\mathcal{B} \in \mathbb{K}^n$ is a polytope for which $h_{\mathcal{B}}(\pi_i), i = 1, \ldots, n_{ha}$, then

$$\mathcal{A} \sim \mathcal{B} = \{c \in \mathbb{R}^n | \pi_i^T c \leq \theta_i - h_{\mathcal{B}}(\pi_i), i = 1, \ldots, n_{ha}\} \tag{2.18}$$

Definition 2.15 (Hausdorff Metric [14]): The distance between two compact sets $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^n$ can be discribed by *Hausdorff metric* which is defined as

$$\mathrm{d}_h(\mathcal{A}, \mathcal{B}) = \inf\{\epsilon > 0 | \mathcal{A} \subset \mathcal{B}_\epsilon \text{ and } \mathcal{B} \subset \mathcal{A}_\epsilon\}$$

where $\mathcal{A}_\epsilon$ and $\mathcal{B}_\epsilon$ denote the union of all closed balls of radius ϵ centered at points of $\mathcal{A}$ and $\mathcal{B}$ respectively.

Proposition 2.2: Let $\mathfrak{B}^\epsilon$ be a closed ball with radius ϵ centred at origin. Then, according to the definition 2.13, in defintion 2.15 we have

$$A_\epsilon = \mathcal{A} \oplus \mathfrak{B}^\epsilon \text{ and } B_\epsilon = \mathcal{B} \oplus \mathfrak{B}^\epsilon.$$

2.2.3 Set Invariance Basics

The following discussion grounds the idea behind the set invariance concept for linear discrete-time systems. The concepts given here will be extended in Chapter 3 to address the main contribution of this thesis.

Consider the following constrained linear discrete-time system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + w(k), \\ u(k) &= h(x(k)), \\ y(k) &= Cx(k). \end{aligned} \tag{2.19}$$

Here, $k \in \mathbb{Z}^+$ is the time step. $x(k)$ and $x(k+1) \in \mathbb{R}^n$ are the current and next states of the system. $u(k) \in \mathbb{R}^m$ is the current control input and $w(k) \in \mathbb{R}^n$ is the current realizaton of the disturbance input. Function $h : \mathbb{R}^n \to \mathbb{R}^m$ is the feedback control law on the perfect state information x and is assumed to be continuous. $y(k) \in \mathbb{R}^p$ is the system output. Moreover, it is assumed that the matrices A, B, and C have compatible dimensions. The system is subject to the following constraints,

$$x \in \mathcal{X}, \quad 0_n \in \mathcal{X} \tag{2.20}$$

$$u \in \mathcal{U}, \quad 0_m \in \mathcal{U}, \tag{2.21}$$

where $\mathcal{X}$ is assumed to be a polyhedron in $\mathbb{R}^n$ and $\mathcal{U} \in \mathbb{K}^n$ is assumed to be a polytope. Furthermore, no information is assumed on the disturbance input $w(k)$ except for a polytopic set membership

$$w \in \mathcal{W}, \quad 0_n \in \mathcal{W}. \tag{2.22}$$

Remark 2.7: Compared to (2.1), the state evolution in (2.19) can be found by setting $f(x(k), u(k), w(k)) = Ax(k) + Bu(k) + w(k)$. Also here, $\mathcal{X}$ is a polyhedron and $\mathcal{U}$ is a polytope which results in a more restricted condition.

For the system (2.19), basic invariant sets under different control laws are discussed in the sequel. A more general discussion for the case of nonlinear feedback law, $h(\cdot)$, can be found in [42, 43].

Invariant Sets under Linear Time-invariant State Feedback, $h(x) = Kx$

Let in (2.19) the control law be defined by,

$$u(k) = h(x) = Kx(k), \tag{2.23}$$

then

$$x(k+1) = (A + BK)x(k) + w(k), \tag{2.24}$$

where it is assumed that K is such that the matrix $\Phi = A + BK$ is stable, i.e. all its eigenvalues lie inside the unit circle.

Definition 2.16 (Input Admissible Set [43]): For the system (2.19), a set $\mathcal{X}_{ad} \subseteq \mathcal{X}$ is called *input admissible* under the linear state feedback law (2.23) if and only if

$$\mathcal{X}_{ad} = \{x : x \in \mathcal{X},\ Kx \in \mathcal{U}\}. \tag{2.25}$$

Definition 2.17 (Positively Disturbance Invariant(PDI) Set [49, 50]): For the system (2.19), a set $\mathcal{T}$ is called PDI under linear time-invariant state feedback if and only if for a time step k_0, we have

$$\forall x(k_0) \in \mathcal{T} \text{ and } \forall w(k) \in \mathcal{W},\ x(k) \in \mathcal{T} \text{ and } u(k) = Kx(k) \in \mathcal{U} \qquad \forall k > k_0. \tag{2.26}$$

Proposition 2.3: If $\mathcal{T}$ is a PDI set then by the condition (2.26) we have $\mathcal{T} \subseteq \mathcal{X}_{ad}$.

Proposition 2.4: $\mathcal{T}$ is a PDI set if the following condition is satisfied:

$$(A + BK)\mathcal{T} \subseteq \mathcal{T} \sim \mathcal{W}, \quad \mathcal{T} \subseteq \mathcal{X}_{ad}. \tag{2.27}$$

Remark 2.8: PDI set can be defined for any general nonlinear but time-invariant feedback law [42]. However, in this thesis PDI set alludes to linear state feedback law (2.23).

Invariant Sets under Affine Feedback, $h(x) = Kx + q$

Consider (2.19) with the control law defined by

$$u(k) = Kx(k) + q(k), \tag{2.28}$$

where $q(k)$ is a residual computed via a control law other than state feedback so as to give more flexibility in controlling (2.19). Then, (2.19), can be written as

$$x(k+1) = (A + BK)x(k) + Bq(k) + w(k), \quad k \geq 0. \tag{2.29}$$

Definition 2.18 (Robustly Stabilizable Set [7, 43]): Assume (2.29) admits a PDI set $\mathcal{T}$ under linear state feedback law, i.e. when $q(k) = 0_m$. Then, a set $\mathcal{S}_M(\mathcal{X}, \mathcal{T}) \subseteq$

$\mathcal{X}$, is called *M-step robustly stabilizable set* for (2.29) if and only if it contains all states in $\mathcal{X}$ for which there exists a time-varying feedback control law (2.28) which produces an input trajectory $\{u(k) = Kx(k) + q(k)\}_{k=0}^{M-1}$ which satisfies the input constraint (2.21) and drives the system state to $\mathcal{T}$ in M steps or less, while keeping the evolution (2.29) inside the state constraint $\mathcal{X}$. Mathematically speaking, this is equivalent to say

$$\begin{aligned}\mathcal{S}_M(\mathcal{X},\mathcal{T}) = \; & \{x(0) \in \mathbb{R}^n | \exists \{u(k) = Kx(k) + q(k) \in \mathcal{U}\}_{k=0}^{M-1}, \exists N \leq M : \\ & \{x(k) \in \mathcal{X}\}_{k=0}^{N-1}, \{x(k) \in \mathcal{T}\}_{k=N}^{M}, \forall \{w(k) \in \mathcal{W}\}_{k=0}^{M-1}\} . \end{aligned} \quad (2.30)$$

Proposition 2.5: The following condition is true for stabilizable sets. For any positive integer N,

$$\mathcal{S}_N(\mathcal{X},\mathcal{T}) \supseteq \mathcal{S}_{N-1}(\mathcal{X},\mathcal{T}) \supseteq \ldots \supseteq \mathcal{S}_1(\mathcal{X},\mathcal{T}) \supseteq \mathcal{S}_0(\mathcal{X},\mathcal{T}) = \mathcal{T}.$$

In this manuscript, the arguments $(\mathcal{X},\mathcal{T})$ may be excluded for brevity whenever it does not cause confusion.

2.2.4 Predicitve Control with Contractive Invariance Constraint

MPCCIC scheme is a variant of DMMPC. Because the controller scheme which is going to be proposed in this thesis is infact inspired by *model predicitve control with contractive invariance constraint*(MPCCIC)[18], here a brief section is dedicated to its background. Beforehand, the following definition is in order.

Definition 2.19 (Asymptotically Ultimately Bounded Stability [11, 60]): Consider a system resulting from (2.19) by omitting constraints (2.20) and (2.21) and $u = 0$. This system is called *asymptotically ultimately bounded*(AUB) if the system evolves asymptotically to a bounded set, i.e. there are finite constants β, $\gamma > 0$ such that the following condition is satisfied.

$$\forall \alpha \in (0, \gamma), \exists k^* \geq 0 : \|x(0)\| \leq \alpha \Rightarrow \|x(k)\| \leq \beta, \forall k > k^*$$

where $\|.\|$ can be any vector norm.

To give a definition for the case in which $u \neq 0$ and constraints (2.20) and (2.21) are present the following proposition is given.

Definition 2.20: The system (2.19) is said to be AUB stabilizable if there exists an initial state $x(0) \in \mathcal{X}$ and an infinite input trajectory $\{u(k) \in \mathcal{U}\}_{k=0}^{\infty}$ which can satisfy the following conditions disregarding all possible disturbance trajectories $\{w(k) \in \mathcal{W}\}_{k=0}^{\infty}$:

(i) $x(k) \in \mathcal{X}, k = 0, \ldots, \infty$.

(ii) $\exists k^* \geq 0$ such that for all $k \geq k^*$, $\|x(k)\| \leq \beta$, $\beta \geq 0$ is such that $\|x\| \leq \beta \Rightarrow x \in \mathcal{X}$.

The set of all initial states which admit an admissible input trajectory to guarantee these two conditions is called AUB stabilizable set.

MPCCIC definition

Here, a variant of MPC which is then extended in this thesis to form the main contribution is discussed. If the quadratic stage cost and terminal cost is considered and an affine control law (2.28) is assumed, the MPCCIC scheme at time step k can be implemented by a quadratic programming with linear constraints (QPLC) over a *predicted auxiliary input* trajectory $\{\check{q}(k+i|k)\}_{i=0}^{N-1}$ where N is the control horizon:

- Having the perfect state information $x(k)$, solve

$$\mathbf{q}_{op} = \arg \min_{\mathbf{q}} J(N),$$

$$J(N) = \Big\{ \check{x}^T(k+N|k)P\check{x}(k+N|k) + \sum_{i=0}^{N-1} [\check{x}^T(k+i|k)Q\check{x}(k+i|k) + \check{q}^T(k+i|k)R\check{q}(k+i|k)] \Big\}, \quad (2.31a)$$

$$\mathbf{q} = [\check{q}^T(k|k), \ldots, \check{q}^T(k+N-1|k)]^T, \quad (2.31b)$$

subject to

$$\check{x}(k|k) = x(k), \quad (2.31c)$$

$$\check{x}(k+i+1|k) = A\check{x}(k+i|k) + B\check{u}(k+i|k), \quad (2.31d)$$

$$\check{u}(k+i|k) = K\check{x}(k+i|k) + \check{q}(k+i|k) \in \mathcal{U}, \quad (2.31e)$$

$$\check{x}(k+i|k) \in \mathcal{X}, \quad (2.31f)$$

$$\check{x}(k+N|k) \in \mathcal{T}, \text{ or } \check{q}(k+i|k) = 0_m, \quad i \geq N, \quad (2.31g)$$

$$\check{x}(k+1|k) \in \mathcal{S}_{con} \subseteq \mathcal{X}, \quad (2.31h)$$

$$P \geq 0, \quad Q \geq 0, \quad R > 0, \quad (2.31i)$$

- Put $u(k) = Kx(k) + \check{q}(k|k)$ and repeat the optimization at th next time step.

In this procedure,$\mathbf{q}$ is the matrix representation of the predicted auxiliary input trajectory. $\mathcal{T}$ is the terminal constraint and assumed to be PDI under state feedback (2.23). (2.31i) simply means matrices P and Q are positively semidefinite and R is assumed positively definite. This assures that the stage cost and cost satisfy the reuirements discussed in Section 2.1.1. $\mathcal{S}_{con}$ is called *contractive invariance constraint* which involves a contraction of a proper stabilizable set defined in(2.30) and is calculated by the following algorithm:

Algorithm 2.1: At each time instant k, considering $x(k)$ as the true plant state

1. Check if $x(k) \in \mathcal{S}_N$, where N is considered the smallest integer which makes this condition true.

2. Impose an constraint on MPC which makes $x(k+1) \in \mathcal{S}_{N-1}$ true. By (2.29) this can be written in terms of the MPC predicted state (2.31d) which is the same as (2.29) but with no disturbances $w(k) \in \mathcal{W}$,i.e.

$$\check{x}(k+1|k) \in \mathcal{S}_{N-1} \sim \mathcal{W} \Rightarrow x(k+1) \in \mathcal{S}_{N-1} \Rightarrow \mathcal{S}_{con} = \mathcal{S}_{N-1} \sim \mathcal{W}. \quad (2.32)$$

Remark 2.9: AUB stability of the MPCCIC scheme comes from the fact that each stabilizable set $\mathcal{S}_i, i = 0, \ldots, N-1$ satisfied the AUB condition in Proposition 2.20. See [60] for the relevant but scattered discussion and proof.

Chapter 3

Robust Predictive Control with Actuating Delay

This chapter embraces the major contribution and results of this thesis. It starts with the problem formulation. Then using a combination of disturbance invariance concept [24] and the set-membership membership estimation [82] an estimator is designed which guarantees a polytopic bound on the estimation error. Later, this bound is used in designing an output feedback MPC scheme which together with the proposed estimator guarantees the AUB stability. More improvements are given by considering the observer dynamics and a correlation between the uncertainties. The proposed scheme also compensates for the delay in the control input.

3.1 Problem Formulation

First, let us introduce two new notations.

Definition 3.1: Consider a polytope $\mathcal{A} \in \mathbb{R}^n$ and a trajectory $\{a(k) \in \mathcal{A}\}_{k=0}^{\infty}$. Then a *time-stamped* polytope $\mathcal{A}(k) \in \mathbb{R}^n$ is a set such that $\forall k \in \mathbb{Z}^+$:

- It is shape-wise time-invariant, i.e. $\mathcal{A}(k) = \mathcal{A}$.
- Only $a(k) \in \mathcal{A}(k)$.

Definition 3.2: Consider a trajectory $\{a(k)\}_{k=0}^{\infty}$. Then, a polytope denoted by $\mathcal{A}_t(k)$ with k and t as integers and $t \leq k$, refers to a polytope computed at time t such that it defines a set at time k: $\mathcal{A}_t(k) \ni a(k)$.

3.1.1 System specifications

Consider a system described by

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k-\tau) + w(k), \\ y_m(k) &= Cx(k) + v(k). \end{aligned} \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the plant state and $u \in \mathbb{R}^m$ is a retarded control input vector with a finite delay described by $\tau \in \mathbb{Z}^+$. $y_m \in \mathbb{R}^p$ is the measurement output which is contaminated by the noise signal $v \in \mathbb{R}^p$. The following set of assumptions are made on the system (3.1).

Assumption 1: Polyhedral constraints on the input and state of the system are assumed by

$$u \in \mathcal{U} \in \mathbb{K}^m, \qquad x \in \mathcal{X} \subset \mathbb{R}^n \tag{3.2}$$

where $0_n \in \mathcal{X}$ and $0_m \in \mathcal{U}$.

Assumption 2: Disturbance input and measurement noise admit the following set memberships

$$w(k) \in \mathcal{W}(k) \in \mathbb{K}^n, \qquad v(k) \in \mathcal{V}(k) \in \mathbb{K}^p, \tag{3.3}$$

where $\mathcal{W}(k)$ and $\mathcal{V}(k)$ are symmetric polytopes time-stamped as per Definition 3.1.

Assumption 3: If $\tau = 0$, then the resulting system is controllable and observable following the general definitions of these terms [71].

Assumption 4: A Luenberger linear estimator is coupled with y_m to make an estimate $\hat{x}(k)$ of the actual state $x(k)$. The estimator dynamics can be defined by

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k-\tau) + L(y_m(k) - C\hat{x}(k)) \tag{3.4}$$

in which $L \in \mathbb{R}^{p\times n}$ is called the estimator gain. Moreover, it is primarily assumed that the estimator is stable, i.e. the matrix $\Psi = A - LC$ has spectral radius $\rho(\Psi)$ less than 1.

Assumption 5: At time step k the following trajectory of the system

$$\{u(k-\tau+i) \in \mathcal{U}\}_{i=0}^{\tau-1} \tag{3.5}$$

is assumed to be known.

Assumption 6: Let the initial input trajectory be defined by setting $k = 0$ in (3.5). It is assumed that this trajectory along with the initial state $x(0)$ make an *admissible set of initial conditions*, i.e they satisfy the following requirement,

$$x(\tau) \in \mathcal{X}, \qquad \forall\{w(i) \in \mathcal{W}\}_{i=0}^{\tau-1}, \tag{3.6}$$

where by iterating state equation in (3.1), we have

$$x(\tau) = A^{\tau}x(0) + \sum_{i=0}^{\tau-1} A^{\tau-1-i}Bu(-\tau+i) + \sum_{i=0}^{\tau-1} A^{\tau-1-i}w(i). \tag{3.7}$$

Remark 3.1: In the sequel, Assumptions 1-6 are implied whenever (3.1) is referenced. Any exception will be stated.

3.1.2 Requirements

General Synthesis Problem: Devise a new MPCCIC scheme which together with the estimator (3.4) ensures the followings.

- Guarantees the AUB stability of the closed-loop against the estimation error, disturbance input, and measurement noise.
- Compensates for the actuating delay, i.e. makes the closed-loop response behave as if $\tau = 0$.
- Respects the constraints in Assumption 1 for all-time, given Assumptions 2-6 are given.

The above problem can be described with three inter-connected smaller problems: Estimation, delay compensation and control.

Let the estimation error be defined by

$$e(k) = x(k) - \hat{x}(k), k \in \mathbb{Z}^{+}.$$

Then the estimation error admits the following dynamics,

$$e(k+1) = \Psi e(k) + w(k) - Lv(k). \tag{3.8}$$

Estimation Problem: Consider the system (3.3) with the corresponding state estimator (3.4). The aim is to design L such that for a given symmetrically polytopic estimation error bound denoted by $\mathcal{E} \in \mathbb{K}^n$ the following condition is satisfied,

$$e(k_0) \in \mathcal{E}, \text{ for some } k_0 \geq 0 \Rightarrow e(k) \in \mathcal{E}, \forall k \geq k_0. \tag{3.9}$$

Delay Compensation Problem: In order to compensate for the actuating delay τ, a method should be specified to provide an estimate of the system state $x(k)$ at the earlier time $k - \tau$.

Lemma 3.1: Consider the state evolution (3.1) at time $k-\tau$ which can be described by

$$x(k-\tau+1) = Ax(k-\tau) + Bu(k-2\tau) + w(k-\tau). \tag{3.10}$$

Let $\hat{x}(k-\tau)$ be the estimate of $x(k-\tau)$ computed by estimator (3.4). Then an *advanced estimate* of $x(k)$ calculated at time $k-\tau$ and its corresponding *advanced estimation error* can be characterized by the following set of propagations

$$\hat{x}_{k-\tau}(k) = A^{\tau}\hat{x}(k-\tau) + \sum_{i=0}^{\tau-1} A^{\tau-1-i}Bu(k-2\tau+i), \tag{3.11}$$

$$\hat{e}_{k-\tau}(k) = x(k) - \hat{x}_{k-\tau}(k) \in \hat{\mathcal{E}}_{k-\tau}(k), \quad k \in \mathbb{Z}^+,$$

$$\hat{\mathcal{E}}_{k-\tau}(k) = A^{\tau}\mathcal{E}(k-\tau) \oplus \bigoplus_{i=0}^{\tau-1} A^{\tau-1-i}\mathcal{W}(k-\tau+i). \tag{3.12}$$

Proof. Iterating (3.10) gives

$$x(k) = A^{\tau}x(k-\tau) + \sum_{i=0}^{\tau-1} A^{\tau-1-i}Bu(k-2\tau+i) + \sum_{i=0}^{\tau-1} A^{\tau-1-i}w(k-\tau+i). \tag{3.13}$$

Now comparing (3.11) and (3.13) yields the following advanced estimation error

$$\hat{e}_{k-\tau}(k) = A^{\tau}e(k-\tau) + \sum_{i=0}^{\tau-1} A^{\tau-1-i}w(k-\tau+i). \tag{3.14}$$

The actual values of $\{w(k-\tau+i)\}_{i=0}^{\tau-1}$ and $e(k-\tau)$ are not known to compute (3.14). A set containing all realizations of (3.14) can be defined by

$$\hat{\mathcal{E}}_{k-\tau}(k) = \{\hat{e}_{k-\tau}(k) \text{ is evaluated by (3.14) } \forall \{w(k-\tau+i) \in \mathcal{W}\}_{i=0}^{\tau-1} \text{ and } \forall e(k-\tau) \in \mathcal{E}\}.$$

Using Definition 2.13, (3.12) is derived. □

Proposition 3.1: Let a control input calculated according to $\hat{x}_{k-\tau}(k)$ in (3.20) be denoted by $u_{k-\tau}(k)$. Then setting

$$u(k-\tau) = u_{k-\tau}(k) = h(\hat{x}_{k-\tau}(k)), \tag{3.15}$$

defines a delay compensating control law[1].

[1]The function $h : \mathbb{R}^n \to \mathbb{R}^m$ can be a time varying control law which is specifically defined later in Section 3.3.1.

Proof. Inserting (3.15) into (3.1) yields to the following expression.

$$x(k+1) = Ax(k) + Bh(\hat{x}_{k-\tau}(k)) + w(k)$$

which clearly shows that $x(k+1)$ is depended upon information on time step k only. □

Proposition 3.2: Since (3.12) consists of Minkowski addition of shapewise time-invariant sets also is $\hat{\mathcal{E}}_{k-\tau}(k)$ and $\hat{\mathcal{E}}_{k-\tau}(k) = \hat{\mathcal{E}}$, $\forall k$. We then have

$$\hat{\mathcal{E}} = A^{\tau}\mathcal{E} \oplus \bigoplus_{i=0}^{\tau-1} A^{\tau-1-i}\mathcal{W}. \tag{3.16}$$

Definition 3.3: The set $\hat{\mathcal{E}}$ is called *advanced estimation error* (AEE) *set.*

Control Problem: Define a new MPCCIC scheme such that, given Assumption 6 is satisfied, guarantees that the closed-loop response regulates to origin and respects the constraints defined in Assumption 1. This feature should be invariant of the adverse effect of uncertainties in Assumption 2 and AEE set(3.16). The control problem can be solved by

- defining an invariant terminal constraint set for the new MPCCIC to assure ultimate boundedness of response (steady state requirement).
- finding a feasibility region in $\mathcal{X}$ for any point of which new MPCCIC scheme guarantees convergence to the invariant terminal constraint set in less than or equal N time steps (transient response requirement).

The system setup is shown in Fig.3.1. It is worth noting that by feeding the control with the advanced estimation (3.11), it is actually possible to split the plant into a delay-free part and a part consisting of delay units only. Also note that at each instant $k-\tau$ the MPC optimization is finding $u(k) = u_{k-\tau}(k)$. At time $k-\tau$, this input has not yet been applied to the plant.

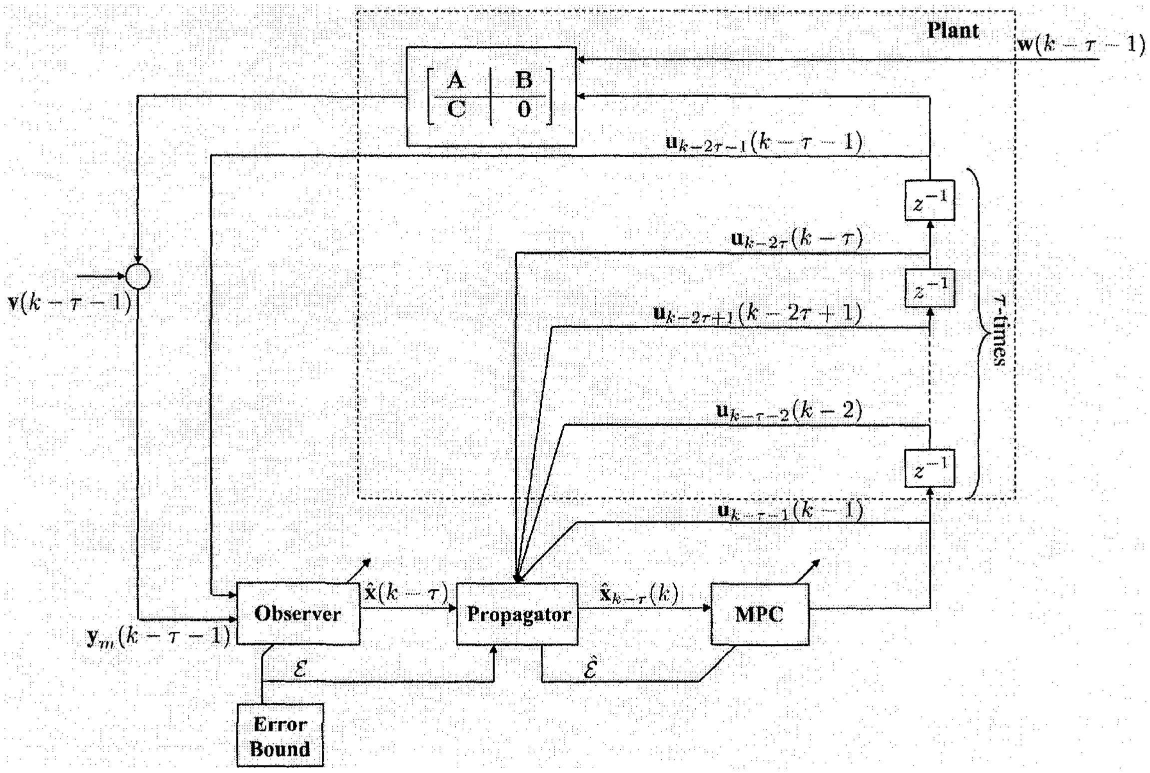

Figure 3.1: Problem formulation

3.2 Error Bounding Estimator

At each instant, the only information available for state estimation are the current input, measured output and the bounds (3.3). In order to have a well defined structure, it is imperative to guarantee a predefined set bound in (3.9) on the estimation error.

Proposition 3.3: Associated with a predefined error bound set in (3.9), and the uncertainties defined in Assumption 2, there exists a $\mathcal{L} \subset \mathbb{R}^{n \times p}$ which, if not empty, contains the admissible estimator gains which solve the estimation problem in Section 3.1.2. Such a set is defined as

$$\mathcal{L} = \left\{ L \in \mathbb{R}^{n \times p} | \Psi\mathcal{E} \subseteq \mathcal{E} \sim \hat{\mathcal{W}},\ \hat{\mathcal{W}} = \mathcal{W} \oplus L\mathcal{V} \right\}, \tag{3.17}$$

Proof. Consider the estimation error dynamics (3.8). Define $\hat{w}(k) = w(k) - Lv(k)$. Then estimation problem in Section 3.1.2 implies that given $e(k) \in \mathcal{E}$ for some integer k, the outcome of the following condition should be true

$$e(k+1) = \Psi e(k) + \hat{w}(k) \in \mathcal{E}, \quad \forall w(k) \in \mathcal{W}, \forall v(k) \in \mathcal{V}.$$

However, by Definition 2.13 , this implies that $e(k+1) \in \mathcal{E}, \quad \forall \hat{w}(k) \in \hat{\mathcal{W}}$, where $\hat{\mathcal{W}} = \mathcal{W} \oplus (-L\mathcal{V})$. By symmetry of $\mathcal{V}$, we arrive at (3.17). □

Proposition 3.4: Consider the system (3.1). Assume $\mathcal{E}$ is a symmetric polytope such that

$$\mathcal{E} = \left\{ e \in \mathbb{R}^n | \eta_i^T e \leq \xi_i, \ \xi_i > 0, \ i = 0, \dots, n_{he} \right\},$$

where $\eta_i \neq 0_n$. Let also the set of vertices of $\mathcal{E}$ be known and be defined as

$$\text{vert}(\mathcal{E}) = \{\bar{e}_j \in \mathbb{R}^n, j = 1, \dots, n_{ve}\}.$$

Then by Lemmas 2.9 and 2.10 it is possible to have the following alternative for (3.17).

$$\mathcal{L} = \Big\{ L \in \mathbb{R}^{n \times p} | \eta_i^T \Psi \bar{e}_j \leq \xi_i - \max_{\bar{w}_k}(\eta_i^T \bar{w}_k) - \max_{\bar{v}_\ell}(-\eta_i^T L \bar{v}_\ell), \bar{w}_k \in \text{vert}(\mathcal{W}), \bar{v}_\ell \in \text{vert}(\mathcal{V}), \\ \forall i = 1, \dots, n_{he}, j = 1, \dots, n_{ve} \Big\}. \quad (3.18)$$

Remark 3.2: Note that in (3.18) there is a maximization over a phrase which involves the unknown variable L. Implementing this maximization is not possible. To get over this problem let us introduce the following lemma.

Lemma 3.2: labelestlemThe following definition of $\mathcal{L}$ is equivalent to one in (3.18),

$$\mathcal{L} = \Big\{ L \in \mathbb{R}^{n \times p} | \eta_i^T \Psi \bar{e}_j - \eta_i^T L \bar{v}_\ell \leq \xi_i - \max_{\bar{w}_k}(\eta_i^T \bar{w}_k), \bar{w}_k \in \text{vert}(\mathcal{W}), \\ \forall i = 1, \dots, n_{he}, j = 1, \dots, n_{ve}, \ell = 1, \dots, n_{vv} \Big\}. \quad (3.19)$$

where n_{vv} is the number of vertices of the measurement noise set $\mathcal{V}$.

Proof. By symmetry of $\mathcal{V}$, we have $-\max_{\bar{v}_\ell}(-\eta_i^T L\bar{v}_\ell) = \max_{\bar{v}_\ell}(\eta_i^T L\bar{v}_\ell)$. This implies that (3.18) can be rewritten as

$$\mathcal{L} = \Big\{L \in \mathbb{R}^{n\times p} | \eta_i^T \Psi \bar{e}_j - \max_{\bar{v}_\ell}(\eta_i^T L\bar{v}_\ell) \leq \xi_i - \max_{\bar{w}_k}(\eta_i^T \bar{w}_k), \bar{w}_k \in \mathrm{vert}(\mathcal{W}), \bar{v}_\ell \in \mathrm{vert}(\mathcal{V}), \\ \forall i = 1, \ldots, n_{he}, j = 1, \ldots, n_{ve}\Big\}.$$

Also we have

$$\forall \eta_i \in \mathbb{R}^n, \max_{\bar{v}_\ell}(\eta_i^T L\bar{v}_\ell) \in V, \text{ where } V = \{a_\ell : a_\ell = \eta_i^T L\bar{v}_\ell, \forall \ell = 1, \ldots, n_{vv}\}.$$

The proof is complete if it is noted that for all i, j defined in (3.19), the following condition is held,

$$\bigcap_{\ell=1}^{n_{vv}} \mathcal{H}(\eta_i^T \Psi \bar{e}_j, \xi_i - \max_{\bar{w}_k}(\eta_i^T \bar{w}_k)) = \mathcal{H}(\eta_i^T \Psi \bar{e}_j - \max_{\bar{v}_\ell}(\eta_i^T L\bar{v}_\ell), \xi_i - \max_{\bar{w}_k}(\eta_i^T \bar{w}_k)).$$

□

3.3 MPC Structure

3.3.1 Correlation None-observant MPC(CNOMPC)

In order to keep the true plant state $x(k)$ inside the state constraint (3.2) using this imperfect information, an effective control algorithm should be able to handle the effect of the AEE set (3.12). Using (3.11) this can be implemented in part and in terms of set operations by

$$\hat{x}_{k-\tau}(k) \in \mathcal{X}_{sh} = \mathcal{X} \sim \hat{\mathcal{E}} \Rightarrow x(k) \in \mathcal{X}, \tag{3.20}$$

which would be the first shot on properly defining the MPC optimization constraints.

The MPC Optimization Problem P(N)

Define $k_\tau = k - \tau$. The proposed MPC optimization to be solved can be described by the following

(i) Solve at each time instant k_τ,

$$\begin{aligned}\mathbf{q}_{op} &= \arg\min_{\check{\mathbf{q}}} \Big\{ \check{x}^T(k+N|k_\tau)P\check{x}(k+N|k_\tau) + \\ &\qquad \sum_{i=0}^{N-1} \big[\check{x}^T(k+i|k_\tau)Q\check{x}(k+i|k_\tau) + \\ &\qquad\qquad \check{q}^T(k+i|k_\tau)R\check{q}(k+i|k_\tau)\big] \Big\}, && (3.21a)\\ \mathbf{q} &= [\check{q}^T(k|k_\tau), \ldots, \check{q}^T(k+N-1|k_\tau)]^T, && (3.21b)\end{aligned}$$

subject to

$$\begin{aligned}&\check{x}(k|k_\tau) = \hat{x}_{k-\tau}(k), && (3.21c)\\ &\check{x}(k+i+1|k_\tau) = A\check{x}(k+i|k_\tau) + B\check{u}(k+i|k_\tau), && (3.21d)\\ &\check{u}(k+i|k_\tau) = K\check{x}(k+i|k_\tau) + \check{q}(k+i|k_\tau) \in \mathcal{U}, && (3.21e)\\ &\check{x}(k+i|k_\tau) \in \mathcal{X}_{sh}, \text{as per (3.20)} && (3.21f)\\ &\check{x}(k+N|k_\tau) \in \mathcal{T}, \text{ or } \check{q}(k+i|k_\tau) = 0_m, \quad i \geq N, && (3.21g)\\ &\check{x}(k+1|k_\tau) \in \mathcal{S}_{con} \subseteq \mathcal{X}_{sh}, && (3.21h)\\ &P \geq 0, \quad Q \geq 0, \quad R > 0. && (3.21i)\end{aligned}$$

where $\mathcal{T}$ is the terminal constraint and $\mathcal{S}_{con}$ is a *contractive constraint* which should be designed in order to ensure feasibility of the problem for all time steps later than the optimization time k_τ.

(ii) Apply

$$u(k) = h(\hat{x}_{k-\tau}(k)) = \check{u}(k|k_\tau) = K\check{x}(k|k_\tau) + \check{q}_{op}(k|k_\tau). \qquad (3.22)$$

Like Section (2.1.1), MPC predictions can be defined as follows. $\check{x}(k+i|k_\tau), \check{u}(k+i|k_\tau)$ and $\check{q}(k+i|k_\tau)$ are the MPC's predicted state, predicted control input and predicted auxiliary input for time $k+i$ which are calculated based on information at the optimization time k_τ. It is assumed that $\Phi = A + BK$ is a stable matrix and P

can for example be the solution to a discrete Lyapunov equation, i.e.

$$P : \Phi P \Phi^T - P + Q = 0_{n \times n}.$$

Since the MPC internal model (3.21g) is nominal, its decision making can become completely wrong for the plant. To tackle this problem it is wise to consider the deviation between MPC model trajectory and the that of the plant. Inserting (3.14) into (3.1) and considering $i \in \mathbb{Z}^+$ yields

$$\begin{aligned} x(k+i) = \Phi^i x(k) &- \sum_{j=0}^{i-1} \Phi^{i-1-j} BK \hat{e}_{k-\tau+j}(k+j) \\ &+ \sum_{j=0}^{i-1} \Phi^{i-1-j} B \check{q}(k+j|k_\tau) + \sum_{j=0}^{i-1} \Phi^{i-1-j} w(k+j). \end{aligned} \tag{3.23}$$

Definition 3.4: The point-wise differences between trajectories (3.23) and those obtained by recurring (3.21d), can be characterized by

$$d(k+i|k_\tau) = x(k+i) - \check{x}(k+i|k_\tau), i \geq 0.$$

They are called *predicted deviations* (PD) and can be presented by

$$\begin{aligned} d(k+i|k_\tau) &= \Phi^{i-1} A \hat{e}_{k-\tau}(k) - \sum_{j=1}^{i-1} \Phi^{i-1-j} BK \hat{e}_{k-\tau+j}(k+j) + \sum_{j=0}^{i-1} \Phi^{i-1-j} w(k+j), \\ d(k) &= \hat{e}_{k-\tau}(k). \end{aligned} \tag{3.24}$$

Since (3.24) involves unknown entities, the set terminology is utilized to define a set bound on each PD realization, i.e.

$$\begin{aligned} & d(k+i|k_\tau) \in \mathcal{D}(k+i|k_\tau), \\ & \mathcal{D}(k+i|k_\tau) = \Phi^{i-1} A \hat{\mathcal{E}}_{k-\tau}(k) \oplus \bigoplus_{j=1}^{i-1} \Phi^{i-1-j} BK \hat{\mathcal{E}}_{k-\tau+j}(k+j) \oplus \bigoplus_{j=0}^{i-1} \Phi^{i-1-j} \mathcal{W}(k+j), \\ & \mathcal{D}(k|k_\tau) = \hat{\mathcal{E}}_{k-\tau}(k). \end{aligned} \tag{3.25}$$

Remark 3.3: Using time invariance in (3.3), and slight abuse of notation an equivalent form of (3.25) can be introduced where the dependency on time k is removed. For $i \in \mathbb{Z}^+$,

$$\begin{aligned} \mathcal{D}(i) &= \Phi^{i-1} A\hat{\mathcal{E}} \oplus \bigoplus_{j=1}^{i-1} \Phi^{i-1-j} BK\hat{\mathcal{E}} \oplus \bigoplus_{j=0}^{i-1} \Phi^{i-1-j}\mathcal{W}, \\ \mathcal{D}(0) &= \hat{\mathcal{E}}. \end{aligned} \tag{3.26}$$

where the minus sign is removed from the summand involving $\hat{\mathcal{E}}$ due to the fact that $\hat{\mathcal{E}}$ is a result of Minkowski addition of individually symmetric sets.

Remark 3.4: The PD bound sets admit the following linear dynamics, i.e.

$$\mathcal{D}(i+1) = \Phi\mathcal{D}(i) \oplus BK\hat{\mathcal{E}} \oplus \mathcal{W}. \tag{3.27}$$

After discussing the MPC dynamics, we now are ready to give explanation on the procedures needed to design terminal and contractive constraints which make the whole closed-loop AUB stable.

Terminal Constraint $\mathcal{T}$

- *General setup*

In order to have an acceptable closed-loop behavior under the linear control we should guarantee

$$x(k+i) \in \mathcal{X}_{ad}, \quad i \in \mathbb{Z}^+,$$

where $\mathcal{X}_{ad}$ is defined in (2.25). Translation of this condition to its equivalent for the MPC's predicted states gives,

$$\forall k, \quad x(k+i) \in \mathcal{X}_{ad} \;\rightarrow\; \check{x}(k+i|k_\tau) \in \mathcal{X}_{ad} \sim \mathcal{D}(k+i|k_\tau), \quad i \in \mathbb{Z}^+.$$

where by (3.21g) $\breve{x}(k+i|k_\tau) = \Phi^i \breve{x}(k|k_\tau), i \in \mathbb{Z}^+$. The set $\mathcal{T}$ can be characterized as follows

$$\mathcal{T} = \bigcap_{i=0}^{\infty} \mathcal{Q}_i, \tag{3.28a}$$

$$\mathcal{Q}_i = \{\breve{x} \in \mathbb{R}^n | \Phi^i \breve{x} \in \mathcal{X}_{ad} \sim \mathcal{D}(k+i|k_\tau)\}. \tag{3.28b}$$

The analysis can be done by first introducing the following lemma which is a direct extension to the Theorem 4.1 in [50].

Lemma 3.3: Assume Φ is asymptotically stable. Consider (3.25). There exists a compact set, $\mathcal{D} \subset \mathbb{R}^n$ such that $\forall k, \lim_{i\to\infty} \mathcal{D}(k+i|k_\tau) = \mathcal{D}$, i.e.

$$\exists \epsilon > 0 \text{ and } i^* \in \mathbb{Z}^+ : \mathrm{d}_h(\mathcal{D}, \mathcal{D}(k+i^*|k_\tau)) < \epsilon.$$

A proof can be find in [50].

As discussed in the Chapter 1, for computational purposes [81], it is of interest to find a terminal constraint which is not empty and is bigger than just the origin $\{0\}$. To this end, let us introduce a new notation.

Definition 3.5: Consider a generic set $\mathcal{A} \in \mathbb{R}^n$, then

$$\mathcal{A} \not\doteq \emptyset \Leftrightarrow \mathcal{A} : \mathcal{A} \neq \emptyset \text{ and } \mathcal{A} \neq \{0_n\}.$$

Similarly,

$$\mathcal{A} \doteq \emptyset \Leftrightarrow \mathcal{A} : \mathcal{A} = \emptyset \text{ or } \mathcal{A} = \{0_n\}.$$

Theorem 3.4: $\mathcal{T} \not\doteq \emptyset$ if and only if $\forall i \in \mathbb{Z}^+, \mathcal{D}(k+i|k_\tau) \subset \mathcal{X}_{ad}$ and also $\mathcal{D} \subset \mathcal{X}_{ad}$, where $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}(k+i|k_\tau), \forall k$.

Proof. The necessity follows by noting that if

$$\exists i \in \mathbb{Z}^+ : \mathcal{D}(k+i|k_\tau) \not\subset \mathcal{X}_{ad} \Rightarrow \mathcal{X}_{ad} \sim \mathcal{D}(k+i|k_\tau) \doteq \emptyset,$$

This plus (3.28b) gives $\mathcal{T} \doteq \emptyset$. Also $\mathcal{D} \not\subset \mathcal{X}_{ad}$ contradicts the invariance of $\mathcal{T}$ if it is assumed that $\mathcal{T} \neq \{0_n\}$.
On the other hand, if $\mathcal{D}, \mathcal{D}(k+i|k_\tau) \subset \mathcal{X}_{ad}, i \in \mathbb{Z}^+$ then by (3.28b) $\nexists i \in \mathbb{Z}^+ : \mathcal{Q}_i \doteq \emptyset$. Extending this to the limit gives $\mathcal{T} \not\doteq \emptyset$. □

Although above theorem gives a necessary and sufficient condition for existence of $\mathcal{T}$ it does not give a practical tool to test it. The following discussion is intended to address this issue.

Theorem 3.5: Consider $\mathcal{X}_{ad}$ and $\hat{\mathcal{E}} \subset \mathcal{X}_{ad}$. Then $\mathcal{T} \not\doteq \emptyset$ exists if a small positive real number $\delta > 0$ exists such that $\mathcal{X}_{ad}$ is a PDI set with respect to the effective disturbance

$$\hat{\mathcal{W}} = BK\hat{\mathcal{E}} \oplus \mathcal{W} \oplus \mathfrak{B}^\delta.$$

Proof. If $\mathcal{X}_{ad}$ is PDI, by definition (2.27) it can be said that

$$\Phi \mathcal{X}_{ad} \subseteq \mathcal{X}_{ad} \sim \hat{\mathcal{W}}.$$

If the initial deviation $\mathcal{D}(k|k_\tau) = \hat{\mathcal{E}} \subset \mathcal{X}_{ad}$, then by (2.17) and (3.27)

$$\Phi \mathcal{D}(k|k_\tau) \subset \mathcal{X}_{ad} \sim BK\hat{\mathcal{E}} \sim \mathcal{W} \sim \mathfrak{B}^\delta \Rightarrow \mathcal{D}(k+1|k_\tau) \subset \mathcal{X}_{ad} \sim \mathfrak{B}^\delta \subset \mathcal{X}_{ad}.$$

Continuing this way leads to

$$\mathcal{D}(k+i|k_\tau) \subset \mathcal{X}_{ad} \sim \mathfrak{B}^\delta, i \in \mathbb{Z}^+$$

and $\mathcal{D} \subseteq \mathcal{X}_{ad} \sim \mathfrak{B}^\delta \subset \mathcal{X}_{ad}$. By Theorem 3.4 $\mathcal{T} \not\doteq \emptyset$ exists and proof is complete. □

Remark 3.5: The condition which has been set by Theorem 3.5 might be very tight in many circumstances. In fact, there are many situations in which one can find a prestabilizing feedback gain K which does not meet this condition whereas it can define a PDI terminal constraint by using the procedure (3.28). This issue will become clear later by illustrative examples.

• *Finite Determinedness*

One of the most important conditions which should be satisfied in order to make the construction (3.28) implementable is the *finite determinedness* of the set $\mathcal{T}$, i.e. existence of an index $i^* \in \mathbb{Z}^+$:

$$\mathcal{T} = \bigcap_{i=0}^{i^*} \mathcal{Q}_i.$$

i^* is called *determinedness index* [30] and ensures the procedure for computing terminal constraint $\mathcal{T}$ takes finite number of iterations.

Theorem 3.6: Assuming $\mathcal{X}_{ad}$ is a compact set and $\mathcal{T} \not\supseteq \emptyset$ exists, then

(i) $\mathcal{Q}_i, \quad i \in \mathbb{Z}^+$, are also compact.

(ii) $\exists i^* \in \mathbb{Z}^+ : \mathcal{T}_{i^*} = \mathcal{T}$. where $\mathcal{T}_{i^*} = \bigcap_{i=0}^{i^*} \mathcal{Q}_i$.

Proof. The proof of (i) comes naturally from (3.28) and compactness of $\mathcal{X}_{ad}$ since (3.28) involves linear transformation and set intersection which preserves the compactness. To prove (ii), we know

$$\forall k,\ \forall \epsilon > 0, \exists \underline{i} \in \mathbb{Z}^+ :$$
$$\mathrm{d}_h(\mathcal{X}_{ad} \sim \mathcal{D}, \mathcal{X}_{ad} \sim \mathcal{D}(k+i|k_\tau)) < \epsilon,\ \forall i \geq \underline{i}.$$

Let us define an outer/inner-approximation of the sets $\mathcal{Q}_i$, $\forall i \geq \underline{i}$ respectively by

$$\overline{\mathcal{Q}}_i = \{\breve{x} \in \mathbb{R}^n | \Phi^i \breve{x} \in \mathcal{X}_{\overline{ad}}\}, \mathcal{X}_{\overline{ad}} = \mathcal{X}_{ad} \sim \mathcal{D} \oplus \mathfrak{B}^\epsilon,$$

$$\underline{\mathcal{Q}}_i = \{\breve{x} \in \mathbb{R}^n | \Phi^i \breve{x} \in \mathcal{X}_{\underline{ad}}\}, \mathcal{X}_{\underline{ad}} = \mathcal{X}_{ad} \sim \mathcal{D} \sim \mathfrak{B}^\epsilon.$$

Two issues should be cleared here:

- By Theorem 3.4, $\mathcal{X}_{ad} \sim \mathcal{D} \not\supseteq \emptyset$ and by definition $(\mathcal{X}_{ad} \sim \mathcal{D})^o \ni 0_n$, hence if $\mathcal{X}_{ad} \sim \mathcal{D} \not\supseteq \emptyset \Rightarrow \exists \epsilon > 0 : \mathcal{X}_{\underline{ad}} \not\supseteq \emptyset$.

  In this discussion assume ϵ satisfies such a property.

- By p-difference and Minkowski addition properties [36] $\mathcal{X}^o_{\overline{ad}} \ni 0_n$, $\mathcal{X}^o_{\underline{ad}} \ni 0_n$ which implies

$$\exists \lambda \in (0,1): \; \lambda \mathcal{X}_{\overline{ad}} \subset \mathcal{X}_{\underline{ad}}.$$

Now assume the spectral radius of Φ be denoted by $\rho < 1$, then for an arbitrary convex and compact shape $\mathcal{C} \in \mathbb{K}^n$ [37]

$$\exists \mu \in [1 \; \infty): \; \forall \ell \in \mathbb{Z}, \; \Phi^\ell \mathcal{C} \subset \mu \rho^\ell \mathcal{C}.$$

Now $\exists \ell^* : \forall \ell \geq \ell^*, \; \mu\rho^\ell < \lambda$, then it is trivial to show that $\forall \ell > \ell^*$:

$$\mathcal{Q}_{\underline{i}+\ell} \supset \underline{\mathcal{Q}}_{\underline{i}+\ell} = \{\check{x} \in \mathbb{R}^n | \Phi^{\underline{i}+\ell}\check{x} \in \mathcal{X}_{\underline{ad}}\} \supset \overline{\mathcal{Q}}_{\underline{i}} \supset \mathcal{Q}_{\underline{i}}.$$

By (3.28a) this is tantamount to $\mathcal{T}_{\underline{i}+\ell} = \mathcal{T}_{\underline{i}+\ell+1}$ and by construction $\mathcal{T}_{\underline{i}+\ell} = \mathcal{T}$. This proves that $i^* = \underline{i} + \ell^*$ is a determinedness index (though might not minimal) for $\mathcal{T}$, hence, $\mathcal{T}$ is finitely determined. □

Contractive Invariance Constraint

To implement (3.21h) at time instant $k_\tau = k - \tau$ it is necessary to find a contractive constraint for $\check{x}(k+1|k_\tau)$ in such a way that the initial condition of MPC optimization at the next step $\check{x}(k+1|k_\tau+1)$ still falls inside the feasible region of the optimization. This guarantees the feasibility of optimization in future or all-time feasibility. The following theorem helps in addressing this problem.

Theorem 3.7: Consider the set $\mathcal{A} \in \mathbb{K}^n$. Suppose the aim is to find a contraction $\mathcal{A}_{con} = \text{cont}(\mathcal{A})$ such that $\check{x}(k+1|k_\tau) \in \mathcal{A}_{con}$ implies $\check{x}(k+1|k_\tau+1) \in \mathcal{A}$, then

$$\text{cont}(\mathcal{A}) = \mathcal{A} \oplus \mathcal{D}(k|k_\tau) \sim \mathcal{D}(k+1|k_\tau) \tag{3.29}$$

is such a contraction.

Proof. Suppose $\check{x}(k+1|k_\tau+1) \in \mathcal{A}$ then by (3.25) it is true that

$$x(k+1) - \check{x}(k+1|k_\tau+1) \in \mathcal{D}(k|k_\tau) \Rightarrow x(k+1) \in \mathcal{A} \oplus \mathcal{D}(k|k_\tau) \tag{3.30}$$

On the other hand, (3.25) suggests $x(k+1)-\check{x}(k+1|k_\tau) \in \mathcal{D}(k+1|k_\tau)$ which implies $x(k+1) \in \check{x}(k+1|k_\tau)+\mathcal{D}(k+1|k_\tau)$. Now (3.30) is true if $\check{x}(k+1|k_\tau)+\mathcal{D}(k+1|k_\tau) \subseteq \mathcal{A} \oplus \mathcal{D}(k|k_\tau)$. This is equal to say

$$\check{x}(k+1|k_\tau) \in \mathcal{A} \oplus \mathcal{D}(k|k_\tau) \sim \mathcal{D}(k+1|k_\tau)$$

and (3.29) is immediate. $\square$

In order to quantify the stabilizable sets for the proposed scheme, let us start from the terminal constraint (3.28) and find the sets of initial states $\check{x}(k|k_\tau) \in \mathbb{R}^n$ which can be driven to it in $1,\ldots,M$ steps despite the bounded PDs (3.25). Let $\tilde{\mathcal{S}}_i(\mathcal{X}_{sh},\mathcal{T}) \subseteq \mathcal{X}_{sh}, i=0,\ldots,M$ denote such sets and call them *deviation-robust stabilizable (DRS) sets*. It is assumed that $\tilde{\mathcal{S}}_0(\mathcal{X}_{sh},\mathcal{T}) = \mathcal{T}$. In order to characterize these sets the following definition is delivered.

Definition 3.6 (Deviation-Robust One-Step (DROS) Set): Consider system (3.1) with constraints and uncertainties (3.2) and (3.3), respectively. Given a compact set $\mathcal{A} \subset \mathcal{X}_{sh}$, $\mathcal{Q}(\mathcal{A})$ is called a DROS set, if

$$\mathcal{Q}(\mathcal{A}) = \{\check{x}(k|k_\tau) \in \mathcal{X}_{sh} | \exists u \in \mathcal{U} : \check{x}(k+1|k_\tau+1) \in \mathcal{A}\}.$$

By (3.29),

$$\mathcal{Q}(\mathcal{A}) = \{\check{x}(k|k_\tau) \in \mathcal{X}_{sh} | \exists u \in \mathcal{U} : \check{x}(k+1|k_\tau) \in \mathrm{cont}(\mathcal{A})\}. \tag{3.31}$$

Equation (3.31) implies

$$\tilde{\mathcal{S}}_{i+1}(\mathcal{X}_{sh},\mathcal{T}) = \mathcal{Q}(\tilde{\mathcal{S}}_i(\mathcal{X}_{sh},\mathcal{T})), \quad i=0,\ldots,M.$$

By (3.21d),

$$\tilde{\mathcal{S}}_{i+1}(\mathcal{X}_{sh},\mathcal{T}) = \{\check{x}(k|k_\tau) \in \mathcal{X}_{sh} | \exists \check{u}(k|k_\tau) \in \mathcal{U} : A\check{x}(k|k_\tau) + B\check{u}(k|k_\tau) \in \mathrm{cont}(\tilde{\mathcal{S}}_i(\mathcal{X}_{sh},\mathcal{T}))\}. \tag{3.32}$$

Proposition 3.5: Equation (3.32), can be rewritten as

$$\tilde{\mathcal{S}}_{i+1}(\mathcal{X}_{sh}, \mathcal{T}) = \mathcal{X}_{sh} \cap \mathrm{Pre}(\mathrm{cont}(\tilde{\mathcal{S}}_i(\mathcal{X}_{sh}, \mathcal{T})) \oplus -B\mathcal{U}).$$

where the pre-image is taken with respect to A (see Definition 2.11).

Proof. The necessity of intersection with $\mathcal{X}_s h$ is obvious from the first condition in (3.32). The second condition implies that $A\check{x}(k|k_\tau) \in \mathrm{cont}(\tilde{\mathcal{S}}_i(\mathcal{X}_{sh}, \mathcal{T} - B\check{u}(k|k_\tau)$. The proof is immediate. □

Now the following algorithm gives the procedure to compute the DRS sets:

Algorithm 3.1 (Construction of DRS sets): The stabilizable sets, then can be characterized by the following recursions.

$$\begin{aligned} \tilde{\mathcal{S}}_0 &= \mathcal{T}, \\ \tilde{\mathcal{S}}_{i+1} &= \mathcal{X}_{sh} \cap \mathrm{Pre}(\mathrm{cont}(\tilde{\mathcal{S}}_i) \oplus -B\mathcal{U}). \end{aligned}$$

Here, arguments $(\mathcal{X}_{sh}, \mathcal{T})$ are omitted for brevity.

Algorithm 3.2 (Proposed MPC Control): Assume a maximum control horizon N and the corresponding problem defined by (3.21a). At each instant $k - \tau$,

1. Set $M = 0$. If $\check{x}(k|k_\tau) \in \tilde{\mathcal{S}}_0 = \mathcal{T}$, set $\check{u}(k|k_\tau) = K\check{x}(k|k_\tau)$, else continue.

2. Set $M = M + 1$,
   if $M \leq N$,

   - if $\check{x}(k|k_\tau) \in \tilde{\mathcal{S}}_M$ set

     $$\mathcal{S}_{con} = \mathrm{cont}(\tilde{\mathcal{S}}_{M-1})$$

     and run the optimization (3.21a), else go to step 2.

   else run a feasibility recovery algorithm [83] which is not in the scope of this thesis.

3.3.2 Correlation Observant MPC (COMPC)

Consider (3.25). Although for every k the affine control law (3.22) guarantees the boundedness of $d(k+i|k_\tau)$, $i \in \mathbb{Z}^+$, it is still possible for a system with reasonable deadtime τ, that the PD sets become unnecessarily large, rendering (3.21a) infeasible. The cause of this problem is oversight of the fact that (3.25) consists of Minkowski addition over expressions sharing the same realizations. For example, assuming $\tau \geq 1$, it is readily seen that using (3.12) the sets $\hat{\mathcal{E}}_{k-\tau}(k)$ and $\hat{\mathcal{E}}_{k-\tau+1}(k+1)$ share on the phrase $\mathcal{W}(k-1)$. The main problem of ignoring this fact and attempting to make Minkowski summations in (3.25) is that the result may become unnecessarily large. The following discussion is devoted to give a reason for this phenomena and address a design in order to ameliorate its effect.

Theorem 3.8: Let $\mathrm{T}_1 : \mathbb{R}^n \to \mathbb{R}^\ell$ and $\mathrm{T}_2 : \mathbb{R}^n \to \mathbb{R}^\ell$ be two linear transformations. Then for arbitrary compact set $\mathcal{C} \subset \mathbb{R}^n$,

$$(\mathrm{T}_1 + \mathrm{T}_2)\mathcal{C} \subseteq \mathrm{T}_1\mathcal{C} \oplus \mathrm{T}_2\mathcal{C}. \tag{3.33}$$

Proof. To be specific to this text, we only prove the case when $\mathcal{C}$ is a polyhedron and the transformations can be presented by matrices. Let

$$\bar{\mathcal{C}} = \mathrm{vert}(\mathcal{C}) = \{\bar{c}_i \in \mathbb{R}^n, i = 1, \ldots, n_{vc}\}.$$

It can be shown [79] that

$$\mathrm{vert}((\mathrm{T}_1 + \mathrm{T}_2)\mathcal{C}) \subseteq (\mathrm{T}_1 + \mathrm{T}_2)\mathrm{vert}(\mathcal{C}) = \{(\mathrm{T}_1 + \mathrm{T}_2)\bar{c}_i : \bar{c}_i \in \bar{\mathcal{C}}\},$$

which implies that during the linear transformation some vertices may be removed. On the other hand by Theorem 2.5,

$$\begin{aligned} &\mathrm{T}_1\mathcal{C} \oplus \mathrm{T}_2\mathcal{C} = \mathrm{hull}(\mathcal{A}), \\ &\mathcal{A} = \{a_{i,j} \in \mathbb{R}^n | a_{i,j} = \mathrm{T}_1\bar{c}_i + \mathrm{T}_2\bar{c}_j, \forall i, j = 1, \ldots, n_{vc}\}. \end{aligned}$$

Now

$$\mathrm{vert}((\mathrm{T}_1+\mathrm{T}_2)\mathcal{C}) \subset \mathcal{A} \Rightarrow \mathrm{hull}(\mathrm{vert}((\mathrm{T}_1+\mathrm{T}_2)\mathcal{C}) \subseteq \mathrm{hull}(\mathcal{A})$$

which is another explanation of (3.33). □

As a result of this theorem, the aim would be to regroup those phrases in (3.25) which are in the form of righthand side of (3.33). Inserting (3.14) and (3.8) into (3.25), and regrouping those terms which has correlations gives

$$\begin{aligned} d(k+i|k_\tau) = {} & (\Phi^{i-1}A^{\tau+1} - \sum_{j=1}^{i-1}\Phi^{i-1-j}\Gamma\Psi^j)e(k-\tau) \\ & + \sum_{j=0}^{i-2}(\Phi^{i-2-j}A^{\tau+1} - \sum_{\ell=j+1}^{i-2}\Phi^{i-2-\ell}\Gamma\Psi^{\ell-j})w(k-\tau+j) \\ & + \sum_{j=0}^{\tau}A^{\tau-j}w(k+i-1-\tau+j) \\ & + \sum_{j=0}^{i-2}(\sum_{\ell=j}^{i-2}\Phi^{i-2-\ell}\Gamma\Psi^{\ell-j}L)v(k-\tau+j). \end{aligned}$$

where $\Gamma = BKA^\tau$. Using set terminology, this can be rewritten as

$$\begin{aligned} \mathcal{D}(k+i|k_\tau) = {} & (\Phi^{i-1}A^{\tau+1} - \sum_{j=1}^{i-1}\Phi^{i-1-j}\Gamma\Psi^j)\mathcal{E} \\ & \oplus \bigoplus_{j=0}^{i-2}(\Phi^{i-2-j}A^{\tau+1} - \sum_{\ell=j+1}^{i-2}\Phi^{i-2-\ell}\Gamma\Psi^{\ell-j})\mathcal{W} \\ & \oplus \bigoplus_{j=0}^{\tau}A^{\tau-j}\mathcal{W} \\ & \oplus \bigoplus_{j=0}^{i-2}(\sum_{\ell=j}^{i-2}\Phi^{i-2-\ell}\Gamma\Psi^{\ell-j}L)\mathcal{V}. \end{aligned} \tag{3.34}$$

Remark 3.6: All the theorems discussed previously in Section 3.3.1 are also applicable to PD sets defined by (3.34). In fact, it will be shown via illustrative examples that smaller PD sets gives us the ability to apply the results to bigger uncertainties/delays. In terms of offline computation of the terminal constraint $\mathcal{T}$ this yield usually to smaller determinedness index as opposed to the scheme described previously.

3.4 Illustrative Examples

Consider (3.1) with unstable openloop dynamics

$$A = \begin{bmatrix} 1.2 & 0.5 \\ 0 & 0.7 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \quad C = [1 \quad 0] \tag{3.35}$$

Assume uncertainties

$$\|w\|_\infty \leq 0.1, \quad \|v\|_\infty \leq 0.1$$

and the constraints

$$\|u\|_\infty \leq 2, \quad x \in \text{hull}\left(\begin{bmatrix} 10 \\ 0 \end{bmatrix}, \begin{bmatrix} -10 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \end{bmatrix}\right).$$

It is desired that the estimation error satisfies the condition $\mathcal{E} : \|e\|_\infty \leq 0.5$. Let K be the solution to the discrete LQR problem where in (3.21a) $Q = 2I$, $R = 10$. This gives $K \approx [-0.50 - 0.38]$. Moreover, using the discrete Lyapunov equation $P : (A + BK)P(A + BK)^T - P + Q = 0_{n \times n}$. A proper terminal cost weight matrix for (3.21a) can be found $P \approx \begin{bmatrix} 3.82 & -0.70 \\ -0.70 & 3.26 \end{bmatrix}$. Also using (3.19)

$$\mathcal{L} : \begin{bmatrix} 0 & -1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} L \lessapprox \begin{bmatrix} 0.083 \\ -1.125 \\ 1.250 \\ 0.083 \end{bmatrix}. \tag{3.36}$$

It is allowable to choose any $L \in \mathcal{L}$. In this example it is chosen to be the Tchebychev center of $\mathcal{L}$ i.e. $L \approx [1.188 \quad - 0.021]^T$. Figure 2, shows the estimation error realizations assuming $e(k) = [0.5 \quad 0.5]^T \in \text{vert}(\mathcal{E})$.

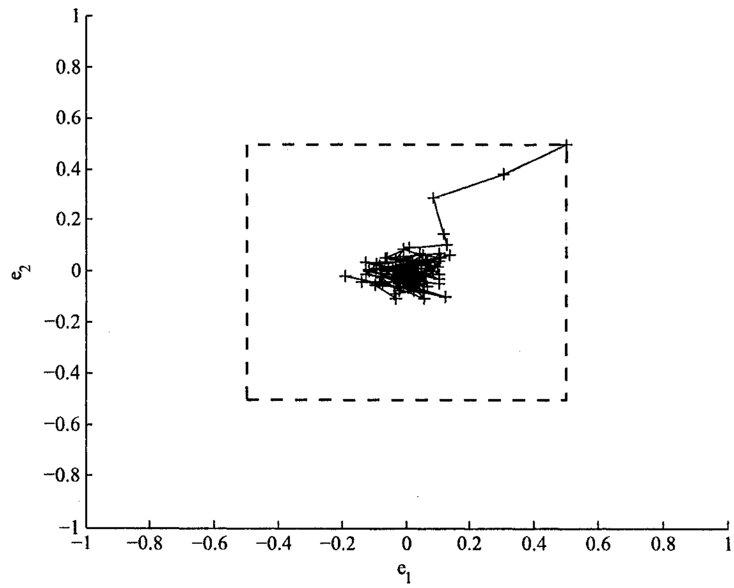

Figure 3.2: Estimation error trajectory

3.4.1 Simulation Based on DP Sets (3.25)

Assume the system suffers from delay $\tau = 3$ and the MPC optimization (3.14) has the maximum control horizon $N = 5$. Also consider the following initial conditions

$$x(k) = [1.1 \quad 2.2]^T, \qquad u(k) = u_{k-3}(k) = 1,$$
$$u(k+1) = u_{k-2}(k+1) = -1, \quad u(k+2) = u_{k-1}(k+2) = 1.$$

The result of simulations has been depicted in Fig 3.3 and Fig 3.4. Figure 3.3 shows both the state response of the plant and the control inputs before and after the MPC is utilized. It has also shown via horizontal dashed lines that input constraints have been satisfied.

In Fig 3.4, the plant state trajectories $x(k+\tau)$, $k \geq 0$ and propagated state trajectories $\hat{x}_k(k+\tau) = \check{x}(k+\tau|k_\tau+\tau)$, $k \geq 0$, denoted by plus signs and asterisks, have been compared to each other. The two outer sets shown via dashed boundaries present the original state constraints $\mathcal{X}$ and $\mathcal{X}_{sh}$ and the invariant sets $\tilde{\mathcal{S}}_i, 0 \leq i \leq 5$ have been shown via dashed/dotted style. Apart from conversion and ultimate bounded-

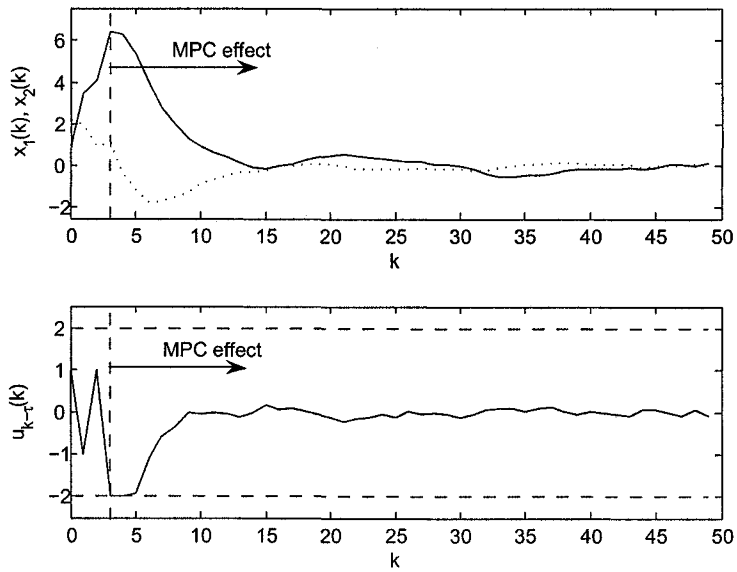

Figure 3.3: Plant states response for Example 3.4.1, $\tau = 3,\ N = 5$.

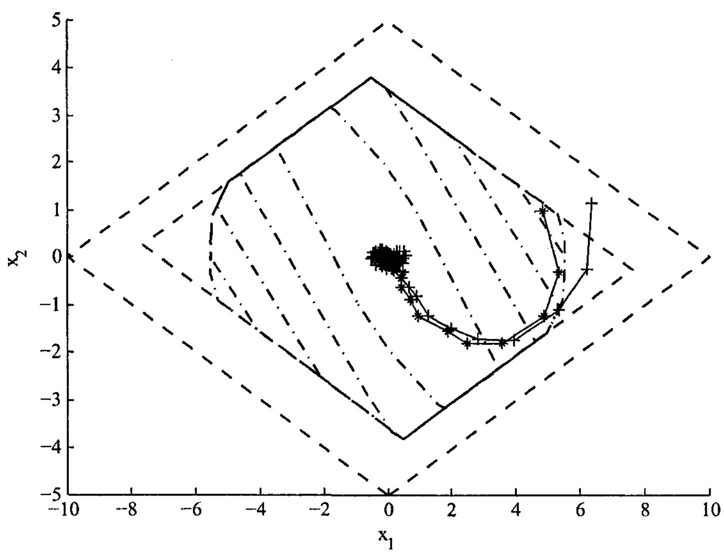

Figure 3.4: Invariant sets, propagated state and true future plant state trajectories pertaining to Example 3.4.1.

ness, Fig 3.4 verifies that keeping the propagated states inside the DRS sets is equal to keeping the plant states inside the original state constraints defined by (3.2) and example data. Note that in Fig 3.4 the DRS sets $\tilde{\mathcal{S}}_4, \tilde{\mathcal{S}}_5$ are so close that they cannot be discerned. It is also verified that having an admissible initial state the proposed algorithm is capable of driving the trajectory inside $\tilde{\mathcal{S}}_0 = \mathcal{T}$ in less than $N = 5$ steps.

3.4.2 Simulation Based on DP Sets (3.34)

Figure 3.5 shows the evolutions of DP sets defined by (3.25) and (3.34) for the system. It can be seen how the method proposed in Section 3.3.2 has shrunk the PD sets. The effect of such an improvement will be bigger upper bounds on delay and/or uncertainties. In fact, running the Example 3.4.1 with $\tau = 4$ results in empty terminal set which is tantamount to failure in design introduced in Section 3.3.1. However, COMPC can tolerate such a delay. Fig 3.6 and Fig 3.7 show the results of simulation for COMPC, using the same control horizon as Example 3.4.1 and $\tau = 4$ and the following set of initial conditions:

$$
\begin{aligned}
&x(k) = [0 \quad 3.4]^T, \\
&u(k) = u_{k-4}(k) = 1, \qquad\qquad u(k+1) = u_{k-3}(k+1) = -1, \\
&u(k+2) = u_{k-2}(k+2) = -1, \quad u(k+2) = u_{k-1}(k+3) = 1.
\end{aligned}
$$

It is worth noting that comparing Fig 3.7 with Fig 3.4 shows the terminal constraint set pertaining to COMPC scheme possess a simpler polytopic shape compared to that in Fig 3.4. This is due to the fact that smaller DP sets yield to lower finite determinedness index in computing the terminal set which is equal to savings in offline computations, though of not much interest.

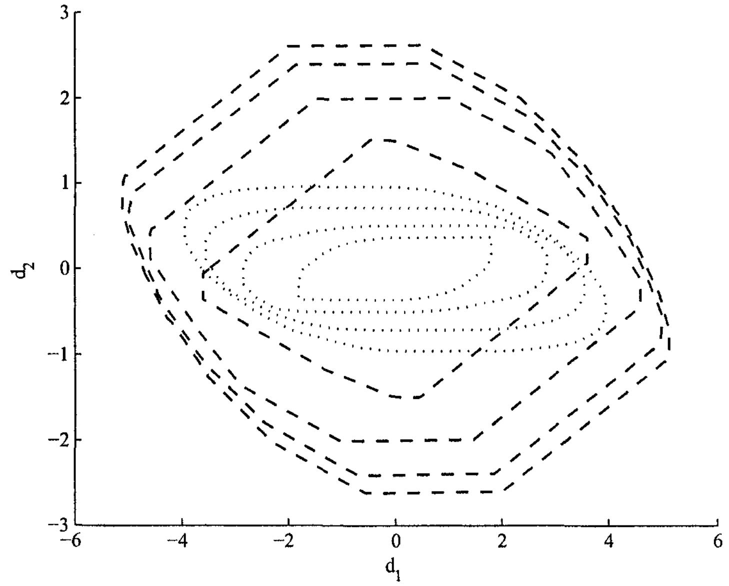

Figure 3.5: Comparing the DP sets computed by (3.25)(dashed style) and (3.34)(dotted style).

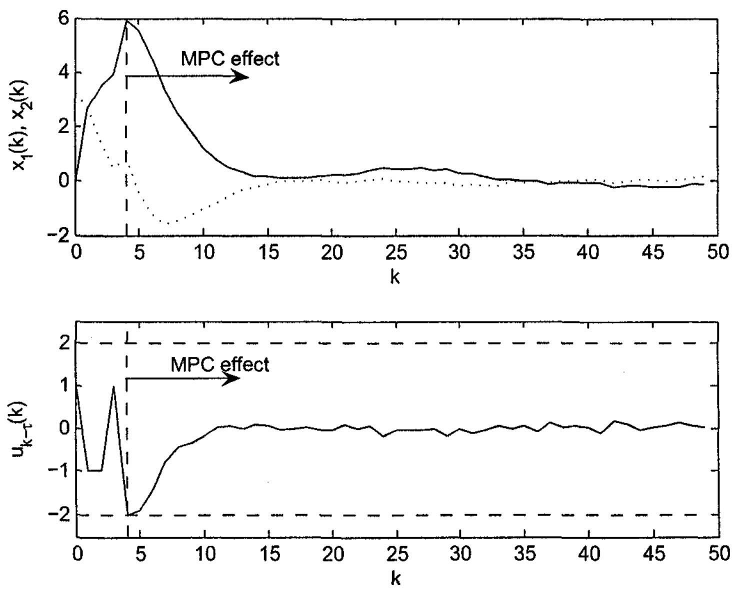

Figure 3.6: Plant states response for Example 3.4.2, $\tau = 4$, $N = 5$.

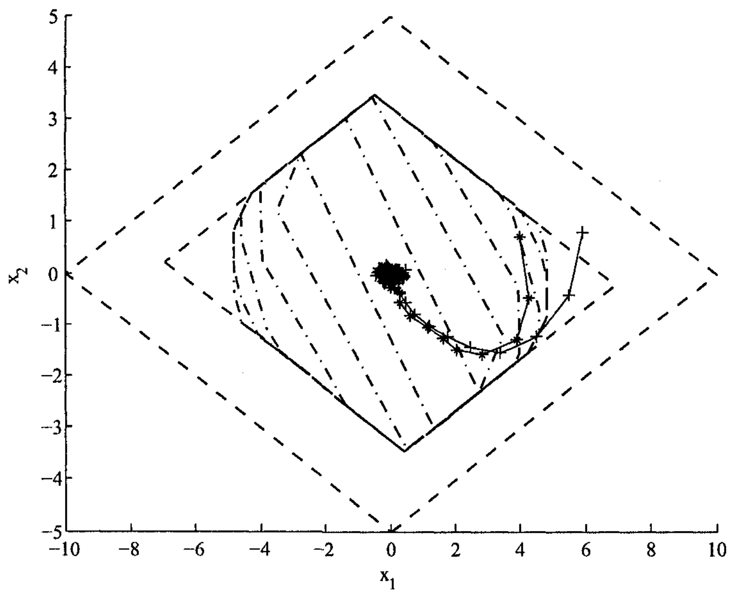

Figure 3.7: Invariant sets, propagated state and true future plant state trajectories pertaining to Example 3.4.2.

Chapter 4

Conclusions and Future Work

The main contributions of this thesis are summarized in this chapter and suggestions for possible future directions are outlined.

4.1 Contributions

The central idea of this thesis was to develop an output feedback methodology addressing the control of an input-delay linear system subject to polytopic constraints on its input and state. Development of the ideas included some contributions which are enumerated as follows.

4.1.1 Estimator Design

- An error-bounding estimator has been designed which can guarantee a given bound by finding a set of all estimator gains in such a way that any gain selected inside such a set can guarantee the specified error bound by making it a positively disturbance invariant set. It extends the result first given in [24] by accommodating measurement noise in the synthesis (Theorem ??).

4.1.2 Controller Design

A novel MPC structure capable of handling the effect of actuating delay, persistent uncertainties and imperfect state information is presented. The pertinent contributions are as follows:

- Development of an optimistic procedure to characterize the future possible deviations between true system trajectory and that of the MPC internal model (Equation (3.25)). This characterization then could be an avail for the rest of synthesis development since it guarantees the boundedness of the deviations by incorporating the notion of pre-stabilized predictions.
- Giving a procedure to characterize the terminal constraint and cost necessary to ensure feasibility and stability of the problem. The sufficient conditions for existence of the terminal constraint set are discussed (Theorems 3.4 and 3.5).
- Giving sufficient conditions for the finite determination of the terminal constraint sets under mild conditions (Theorem 3.6).
- Defining a new contractive invariance constraint with which the all time constraint satisfaction (feasibility) as well as stability of the scheme given admissible initial conditions is achievable (Theorem 3.7, Algorithms 3.1 and 3.2).
- Making improvement in the basic scheme primarily proposed by incorporating the correlation among the different prediction deviation (PD) sets given by (3.25). This leads to ability to deal with combinations of bigger uncertainties, error bounds, and actuating delay (Theorem 3.8 and Equation (3.34)).

4.2 Future Work

History has shown that controlling the systems with delay in their control input is a challenging subject. This is because not many synthesis schemes have been developed for the control of such systems and most of the work has been done for the analysis of such systems (see e.g. [34]). Here the possible future research directions , though not all of them are given to facilitate the future investigation in continuation to the work done in this thesis.

- In [19] a procedure is presented which can address the problem of trajectory tracking of piecewise constant references. Although this result is now only valid for MPCCIC, it may be extended to CNOMPC or COMPC scheme proposed here to give more applications to these scheme .

- It is interesting to consider the statistical information if available. Knowing statistical information on the uncertainties give the ability to tailor the current work in a way to get better performances, e.q. better steady state performance.

- Although the class of uncertainties addressed here have made a good stride on achieving more applicable scheme for real world problems, other cases of uncertainties can also be investigated. These cases include but are not limited to the structured uncertainties, measured disturbances which includes constant disturbance rejection, and modeling errors.

- The last but not the least is the developments for the nonlinear systems. Although current research has shown improvements in the softwares to implement the nonlinear optimizations and invariance, extending the ideas available for linear systems to the nonlinear case demands more investigations. Availability of such tools can guarantee achieving bigger domain of attractions for the closed-loop setup.

References

[1] T. Alamo, J.M. Bravo, and E.F. Camacho. Guaranteed state estimation by zonotopes. In *42th IEEE. Conference on Decision and Control*, pages 5831–5836, December 2003.

[2] Z. Artstein. Linear systems with delayed controls: a reduction. *IEEE Transactions on Automatic Control*, AC-27:869–879, 1982.

[3] M. Basin and J. Rodrigues-Gonzalez. Optimal control for linear systems with time delay in control input based on duality principle. In *Proc. American Control Conference*, pages 2144–2148, June 2003.

[4] A. Bemporad and A. Garulli. Output-feednack predictive control of constrained linear systems using via set-membership state estimation. *International Journal of Control*, 73(8):655–665, 2000.

[5] A. Bemporad, M. Morari, V. Dua, and E.N. Pistikopoulos. The explicit linear quadratic regulator for constrained systems. *Automatica*, 38:3–20, 2002.

[6] A. Bemporad, M. Morari, A. Garulli, A. Tesi, and A. Vicino. Robust model predictive control: A survey. *Lecture notes in control and information sciences*, pages 207–226, 1998.

[7] A. Bemporad, E.D. Tomsi, and M. Morari. *Lecture Notes in Computer Science: Optimizationbased verification and stability characterization of piecewise affine and hybrid systems.* Springer- Verlag, 2000.

[8] D.P. Bertsekas and I.B. Rhodes. On the minmax reachability of target sets and target tubes. *Automatica*, 7:233–247, 1971.

[9] D.P. Bertsekas and I.B. Rhodes. Recursive state estimation for a set-membership description of uncertainty. *IEEE Transactions on Automatic Control*, 16:117–128, 1971.

[10] R.R. Bitmead, M. Gevers, and V. Wertz. *Adaptive optimal control: The thinking man's GPC.* Prentice Hall, Englewood Cliffs, NJ, 1990.

[11] F. Blanchini. Ultimate boundedness control for uncertain discrete-time systems via set-induced lyapunov functions. In *30th IEEE. Conference on Decision and Control*, pages 1755–1760, 1991.

[12] F. Blanchini. Set invariance in control. *Automatica*, 35:1747–1767, 1999.

[13] S. Boyd, L. El-Ghaoui, E. Feron, and V. Balakrishnan. Linear matrix inequalities in system and control theory. *SIAM Journal of Control and Optimization*, 1994.

[14] A.M. Bruckner, J.B. Bruckner, and B.S. Thomson. *Real Analysis.* Prentice Hall, 1997.

[15] R.J. Caron, J.F. McDonald, and C.M. Ponic. A degenrate extreme point strategy for classification of linear constraints as redundant or necessary. *Journal of Optimization Theroy and Appications*, 62(2):225–237, August 1989.

[16] F.L. Chernousko. *State estimation of dynamic systems.* CRC Press, Boca Raton, 1994.

[17] L. Chisci, A. Garulli, and G. Zappa. Recursive state bounding by parallelotopes. *Automatica*, 32(7):1049–1055, 1996.

[18] L. Chisci and G. Zappa. Robust predictive regulation with invariance constraint. *European Journal of Control*, 8:2–14, 2002.

[19] L. Chisci and G. Zappa. Dual mode predictive tracking of piecewise constant references for constrained linear systems. *International Journal of Control*, 76(1):61–72, 2003.

[20] H.H. Choi and M.J. Chung. Memoryless stabilization of uncertain dynamic systems with time varying delayed states and controls. *Automatica*, 31:1349–1351, 1994.

[21] V. Chvátal. *Linear Programming.* W.H. Freeman and Company, 1983.

[22] M.C. Delfour and S.K. Mitter. Reachability of perturbed systems and min sup problems. *SIAM Journal of Control*, 7(4):521–533, 1969.

[23] J.E. Doss and C.F. Moore. The discrete analytical predictor - A generalized deadtime compensation technique. *ISA Transactions*, 20:77, 1982.

[24] F.Blanchini. Feedback control for linear time-invariant systems with state and control bounds in the presence of disturbances. *IEEE Transactions on Automatic Control*, 35(11):1231–1234, 1990.

[25] R. Findeisen, L. Imsland, F. Allgöver, and B.A. Foss. State and output feedback nonlinear model predictive control. *European J. Control*, 2003.

[26] R.A. Freeman and P.V. Kokotovic. Inverse optimality in robust stabilization. *SIAM Journal of Control and Optimization*, 34(4):1365–1391, 1996.

[27] C.E. Garcia and M. Morari. Internal model control. 1. A unifying review and some new results. *Ind. Eng. Chem. Process Des. Dev.*, 21(2):308–323, 1982.

[28] C.E. Garcia and M. Morari. Internal model control. 1. Design procedure for multivariable systems. *Ind. Eng. Chem. Process Des. Dev.*, 24(2):472–484, 1985.

[29] C.E. Garcia and M. Morari. Internal model control. 1. Multivariable control law computation and tuning guidlines. *Ind. Eng. Chem. Process Des. Dev.*, 24(2):484–494, 1985.

[30] E.G. Gilbert and K.T. Tan. Linear systems with state and control constraints: The theory and application of maximal output admissible sets. *IEEE Transactions on Automatic Control*, 36(9):1008–1020, 1991.

[31] A. Girard. Reachability of uncertain linear systems using zonotopes. In *Hybrid systems: Computation and control*, volume 3414 of *Lecture notes in computer science*, pages 291–305, 2005.

[32] G.C. Goodwin. Introduction to constrained control, available online at http//www.eng.newcastle.edu.au/eecs/cdsc/books/cce/lecture_slides.html, 2004.

[33] P. Gritzmann and B. Stumfels. Minkowski addition of polytopes: Computational complexity and applications to Gröbner bases. *SIAM Journal of Discrete Mathematics*, 6(2):246–269, May 1993.

[34] K. Gu, V.L. Kharitonov, and Jie Chen. *Stability of time-delay systems.* Birkhäuser Boston, first edition, 2003.

[35] Keqin Gu and S.I. Niculescu. Survey on recent results in the stability and control of time-delay systems. *Transactions of the ASME*, 125:158–165, June 2003.

[36] H. Hadwiger. *Vorlesungen über inhalt, oberflaäche und isoperimetrie.* Springer - Berlin, 1957.

[37] N.J. Higham and P.A. Knight. Matrix powers in finite precision arithmetic. *SIAM Journal of Matrix Analysis and Applications*, 16(2):343–358, 1995.

[38] T. Hu, Z. Lin, and B.M. Chen. An analysis and design method for linear systems subject to actuator saturation and disturbance. *Automatica*, 38:351–359, 2002.

[39] X.B. Hu and W.H. Chen. Model predictive control for constrained systems with uncertain state-delays. *International Journal of Robust and Nonlinear Control*, 14:1421–1432, October 2004.

[40] S.C. Jeong and P.G. Park. Constrained mpc algorithm for uncertain time-varying systems with state-delay. *IEEE Transactions on Automatic Control*, 50(2):257–263, 2005.

[41] S.S. Keerthi and E.G. Gilbert. Optimal infinite-horizon feedback laws for a general class of constrained discrete-time systems: Stability and moving-horizon approximations. *Journal of Optimization Theory and Applications*, 57(2):265–293, 1988.

[42] E.C. Kerrigan. *Robust constraint satisfaction: Invariant sets and predictive control.* PhD thesis, University of Cambridge, November 2000.

[43] E.C. Kerrigan and J.M. Maciejowski. Invariant sets for constrained nonlinear discrete-time systems with application to feasibility in model predictive control. In *Proc. 39th Conf. Decision and Control*, volume 5, pages 4951–4956, 2000.

[44] J. Klamka. Controllability of nonlinear systems with delay in control. *IEEE Transactions on Automatic Control*, pages 702–704, October 1975.

[45] J. Klamka. Relative controllability and minimum energy control of linear systems with distributed delays in control. *IEEE Transaction on Automatic Control*, pages 594–595, August 1976.

[46] J. Klamka. Controllability of nonlinear discrete-time systems. In *Proc. American Control Conference*, pages 4670–4671, May 2002.

[47] A. Kojima, K. Uchida, E. Shimemura, and S. Ishijima. Robust stabilization of a system with delayed control. *IEEE Transactions on Automatic Control*, AC-39:1694–1698, 1994.

[48] V.B. Kolmanovskii and A.D. Myshkis. *Introduction to theory and application of functional differential equations.* Kluwer, New York, 1999.

[49] I. Kolmanovsky and E.G. Gilbert. Maximal output admissible sets for discrete-time systems with disturbance inputs. In *Proc. American Control Conference*, pages 1995–2000, 1995.

[50] I. Kolmanovsky and E.G. Gilbert. Theory and computation of disturbance invariant sets for discrete-time linear systems. *Mathematical Problems in Engineering*, 4:317–367, 1998.

[51] M.V. Kothare, V. Balakrishnan, and M. Morari. Robust constrained model predictive control using linear matrix inequalities. *Automatica*, 32(10):1361–1379, 1996.

[52] B. Kouvaritakis, J.A. Rossiter, and J. Schuurmans. Efficient robust predictive control. *IEEE. Transactions on Automatic Control*, 45(8):1545–1549, 2000.

[53] A. Kurzhanskii and P. Varaya. Ellipsoidal techniques for reachability analysis of discrete-time linear systems. *IEEE Transactions on Automatic Control*, 52(1):26–38, 2005.

[54] W.H. Kwon and A.E. Pearson. Feedback stabilization of linear system with delayed control. *IEEE Transactions on Automatic Control*, AC-25:266–269, 1980.

[55] Y.I. Lee and B. Kouvaritakis. Receding horizon output feedback control for linear systems with input saturation. In *Proc. IEE. Control. Applications*, volume vol.148, pages 109–115, March 2001.

[56] J.M. Maciejowski. *Predictive control with constraints.* Prentice Hall, 2002.

[57] M.S. Mahmoud. *Robust control and filtering for time-delay systems.* Marcel Dekker, 2000.

[58] M. Malek-Zavarei and M. Jamshidi. *Time-delay systems: Analysis, optimization and applications.* North-Holland, Amsterdam, 1987.

[59] A.Z. Manitius and A.W. Olbrot. Finite spectrom assignement problem for systems with delays. *IEEE Transactions on Automatic Control*, AC-24(4):541–553, 1979.

[60] D.L. Marruendo, T. Álamo, and E.F. Camacho. Stability analysis of systems with bounded additive uncertainties based on invariant sets : Stability and feasibility of mpc. In *Proc. American Control Conference*, pages 364–369, Anchorage, AK, May 2002.

[61] D.Q. Mayne. Control of constrained dynamic systems. *Europearn Journal of Control*, 7:87–99, 2001.

[62] D.Q. Mayne, S.V. Raković, R. Findeisen, and F. Allgöver. Robust output feedback model predictive control of constrained linear systems. *Automatica*, 42:1217–1222, 2006.

[63] D.Q. Mayne, J.B. Rawlings, C.V. Rao, and P.O.M. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36(6):789–814, 2000.

[64] C. Meyer, D.E. Seborg, and R.K. Wood. A comparison of the Smith predictor and conventional feedback control. *Chem. Eng. Sci.*, 31:775–778, 1976.

[65] H. Michalska and D.Q. Mayne. Robust receding horizon control of constrained nonlinear systems. *IEEE Transaction on Automatic Control*, 38(11):1623–1633, 1993.

[66] W. Michiels and D. Roose. Time-delay compensation in unstable plants using delayed state feedback. In *40th IEEE Conference of Decision and Control*, pages 1433–1437, December 2001.

[67] C.F. Moore, C.L. Smith, and P.W. Murrill. Improved algorithm for direct digital control. *Instrumentation and Control Systems*, 43:70, 1970.

[68] M. Morari and H.J. Lee. Model predictive control : past, present and future. *Computers and chemical engineering*, 23(4-5):667–682, 1999.

[69] R.J.T. Morris and R.F. Brown. Extension of validity of GRG method in optimal control calculation. *IEEE Transactions on Automatic Control*, pages 420–422, June 1976.

[70] S.I. Niculescu and K. Gu. *Advances in time-delay systems.* Springer, first edition, 2004.

[71] K. Ogata. *Discrete-time control systems.* Prentice Hall, second edition, 1995.

[72] S. Qin and T. Badgwell. An overview of nonlinear model predictive control applications, 2000.

[73] S.J. Qin and T.A. Badgewell. An overview of industrial model predictive control technology. *In Chemical Process Control - AIChe Symposium Series, American Institute of Chemical Engineers*, 93(316):232–256, 1997.

[74] S.J. Qin and T.A. Badgewell. A survey of industrial model predictive control technology. *Control Engineering Practice*, 11(7):733–764, July 2003.

[75] S.V. Raković and D.Q. Mayne. Set robust control invariance for linear discrete time systems. In *43rd IEEE. Conerence on Decision and Control*, pages 3557–3562, December 2004.

[76] S.V. Raković and D.Q. Mayne. Set estimation for piecewise affine, discrete time systems with bounded disturbances. In *44th IEEE. Conerence on Decision and Control and the European Control Conference*, pages 975–980, December 2005.

[77] J.B. Rawlings. Tutorial overview of model predictive control. *IEEE Transaction on Automatic Control*, 20(3):38–52, June 2000.

[78] J.P. Richard. Time-delay systems:an overview of some recent advances and open problems. *Automatica*, 39:1667–1694, 2003.

[79] R.T. Rockfellar. *Convex Analysis*. Princeton University Press, 1972.

[80] Y.H. Roh. Robust stability of predictor-based control systems with delayed control. *International Journal of Systems Science*, 33(2):81–86, 2002.

[81] J.A. Rossiter, M.J. Rice, and B. Kouvaritakis. A numerically robust state-space approach to stable predictive control strategies. *Automatica*, 34:65–73, 1998.

[82] F.D. Schweppe. Recursive state estimation: Unknown but bounded errors and systm inputs. *IEEE Transactions on Automatic Control*, AC-13(1):22–28, February 1968.

[83] P.O.M. Scokaert and J.B. Rawlings. Feasibility issues in model predictive control. *AIChE Journal*, 45(8):1649 1659, 1999.

[84] P.O.M. Scokaert, J.B. Rawlings, and E.S. Meadows. Discrete-time stability with perturbations: Applications to model predictive control. *Automatica*, 33(3):463–470, 1997.

[85] J.C. Shen, B.S. Chen, and F.C. Kung. Memoryless stabilization of uncertain dynamic delay systems: Riccati equation approach. *IEEE Transactions on Automatic Control*, AC-36:638–640, 1991.

[86] O.J.M. Smith. Closer control of loops with deadtime. *Ind. Eng. Prog.*, 53:217, 1957.

[87] K. Watanabe and Y. Ishiyama. Modified smith predictor control for multivariable systems with delays and unmeasurable step disturbances. *International Journal of Control*, 37(5):959–973, 1983.

[88] K. Watanabe, E. Nobuyama, and K. Kojima. Recent advances in control of time-delay systems: A tutorial review. In *35th IEEE Conference on Decision and Control*, pages 2083–2089, December 1996.

[89] M.C. Wellons and T.F. Edgar. A generalized analytical predictor for process control. page 637, 1985.

[90] M.C. Wellons and T.F. Edgar. The generalized analytical predictor. *Ind. Eng. Chem. Res.*, 26:1523–1536, 1987.

[91] L. Yu, J. Chu, and H. Su. Robust memoryless H_∞ controller design for linear time delay systems with norm-bounded time-varying uncertainty. *Automatica*, 32:1759–1762, 1996.

[92] W. Zhang, M.S. Branicky, and S.M. Phillips. Stability of networked control systems. IEEE Control Systems Magazine, 2001.

VITA AUCTORIS

Ali Kebarighotbi was born in Tehran, Tehran, Iran, on November 30, 1978. He received his B.A.Sc. degree in electrical engineering in 2002 from the Amirkabir University of Technology (Tehran Polytechnic). He is currently a candidate in the electrical and computer engineering M.A.Sc. program at the University of Windsor. His current research area include mathematical system theory, Constrained control and delay compensation in the constrained actuating delay systems.