Investigation of the Properties of the Iterations of a Homeomorphism on a Metric Space

Murray B. Peterson, Jr.
Utah State University

INVESTIGATION OF THE PROPERTIES OF THE ITERATIONS OF A HOMEOMORPHISM ON A METRIC SPACE

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Murray B. Peterson, Jr.

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INTRODUCTION

Considerable study has been made concerning the properties of the iterations of a homeomorphism on a metric space. Much of this material is scattered throughout the literature and understood solely by a specialist. The main object of this paper is to put into readable form proofs of theorems found in G. T. Whyburn's Analytic Topology [8] pertaining to this topic in topology. Properties of the decomposition space of point-orbits induced by the iterations of a homeomorphism will compose a major part of the study. Some theorems will be established through series of lemmas required to fill in much of the detail lacking in proofs found in [8].

Although an elementary knowledge of topology is assumed throughout the paper, references are given for basic definitions and theorems used in developing some of the proofs.

The following symbols and notation will be used throughout the paper. X will denote a metric space with metric ρ, S a topological space, I the set of positive integers, A,B,C,... sets of points or elements. Small letters, such as a,b,c, x,y,z,... will designate elements or points of sets. U and V will denote open sets, $S_r(x)$ a spherical neighborhood of x with radius r. A' denotes the set of limit points of A, $\bar{A}$ the set of closure points of A. $\cup$, $\cap$, $\subset$ will denote union, intersection, and set inclusion respectively. The symbol $\in$ will mean "is an element of". $\emptyset$ denotes the empty set. S − A is the set of points in S which are not in A.

FINITE PROPERTY FOR COMPACT ORBITS

The main result to be established in this section is that in order for a homeomorphism defined on a compact metric space to be pointwise periodic, it is necessary and sufficient that the point-orbits be compact. The sufficiency part of the proof requires several lemmas. Before proceeding to obtain these results some essential definitions and theory will be developed.

A function f is a homeomorphism of a set onto itself provided that f is a 1-1 continuous mapping and the inverse function f^{-1} is continuous. See [4],p. 24. Let f be a homeomorphism of a metric space X onto itself. The iterations of the mapping f will be defined to be the integral powers of f, denoted by

$$f^{o}(x) = x$$

$$f(x) = f^{1}(x)$$

$$f\left[f^{n}(x)\right] = f^{n+1}(x) \qquad \text{for } n>o$$

$$f^{-1}\left[f^{-n}(x)\right] = f^{-n-1}(x) \qquad \text{for } n>o$$

where x is an element of X. It is clear that the laws of exponents are satisfied, ie.

$$f^{m}\left[f^{n}(x)\right] = f^{n+m}(x)$$

$$(f^{m})^{n}(x) = f^{mn}(x)$$

where m and n are integers.

The period of f at a point x in X is the least positive integer n such that $f^{n}(x) = x$. A homeomorphism f is said to be periodic on X if there exists an integer n such that symbol $\in$ will mean "is an element of". $\emptyset$ denotes the empty set. $S - A$ is the set of points in S which are not in A.

$f^n(x) = x$ for each $x \in X$. The least such positive integer n will denote the period of f on X. Now, f is said to be pointwise periodic on X provided f is periodic at each x in X.

From the above definitions it is easy to see that if f is periodic on X, f is pointwise periodic on X. However, the converse is not true. Consider the space X consisting of all ordered pairs of positive integers (i,j) such that $j \leq i$. Let f be the continuous mapping defined as follows:

$$f(i,j) = (i,j+1) \text{ if } j < i$$
$$f(i,j) = (i,1) \quad \text{ if } j = i$$

It is easy to see that f is pointwise periodic on X. However, f is not periodic on X since for each positive integer n, there exists a point with period $n + 1$, namely the point with coordinates $(n+1, 1)$.

Let f be a homeomorphism defined on a metric space X. Let x be a point of X. The orbit of the point x will be defined to be the set of all y in X such that $f^n(x) = y$ for some integer n. A_x will denote the orbit of x with respect to the homeomorphism f. If f is pointwise periodic at f, it is easy to see that A_x is finite since $A_x = \{x, f(x), \ldots, f^n(x)\}$ where $f^{n+1}(x) = x$ for some positive integer n.

Theorem 1.1. Let $x, y \in X$ such that $x \neq y$. Then a necessary and sufficient condition that $A_x \cap A_y \neq \emptyset$ is that $A_x = A_y$.

Proof. The sufficiency part of the proof is obvious. To prove necessity, let $z \in A_x \cap A_y$ and $x_1 \in A_x$. By the definition of a point-orbit there exist integers r,s such that $f^r(x) = f^s(y) = z$. Also there exists an integer t such that $f^t(x) = x_1$. Consider the case where $t > r$. Let $t - r = m$. Then $f^m[f^s(y)] = f^m[f^r(x)] = f^{r+m}(x) = f^t(x) = x_1$ and x_1 is an element of A_y. If $r = t$, $x_1 = f^s(y)$ and $x_1 \in A_y$. The case where $t < r$ is similar to the first case. Therefore $A_x \subset A_y$. Similarly, $A_y \subset A_x$. Hence $A_x = A_y$ which was to be shown.

Consider the metric space X and f a homeomorphism of X onto X. The orbit decomposition of X with respect to f is the set of all point-orbits A_x such that x is an element of X. Note that $\bigcup_{x \in X} A_x = X$ and by the previous theorem A_x and A_y are either disjoint or equal for x and y in X. Some properties of the decomposition of X into point-orbits will be proved in the next section.

Attention is now turned toward proving that a compact orbit is finite. A metric space X is compact provided that any infinite subset A of X has a limit point in X. See [4], p.38. Note that countable compactness and compactness are equivalent in a metric space, as is shown in [4], p. 109. The closure of a set A, denoted by $\bar{A}$, is defined to be the set $A \cup A'$.

Lemma 1.2. Let X be a compact metric space. Then X is not the union of a countable collection of closed subsets of X, no one of which contains an open non-empty subset of X.

Proof. The proof is by contradiction. Suppose $X = \bigcup_{i \in I} C_i$ and each C_i is closed and contains no non-empty open set of X. Thus each point of C_i is a limit point of $X - C_i$ for each i. This means $C_i \subset \overline{X - C_i}$. Since each neighborhood of each point of C_i contains a point of $X - C_i$, $X - C_i \neq \emptyset$. Let p_1 be a point of the open set $X - C_1$. Then for some r, there exists a spherical neighborhood $S_{2r_1}(p_1)$ of p_1 such that $S_{2r_1}(p_1) \cap C_1 = \emptyset$. Then $\overline{S_{r_1}(p_1)} \cap C_1 = \emptyset$, since $\overline{S_{r_1}(p_1)} \subset S_{2r_1}(p_1)$. Let n_1 be the first integer such that $C_{n_1} \cap S_{r_1}(p_1) \neq \emptyset$. There exists such an integer n_1 since $X = \bigcup_{i \in I} C_i$.

Now there exists a point p_2 in $S_{r_1}(p_1) - C_{n_1}$, for otherwise C_{n_1} would contain the open set $S_{r_1}(p_1)$ which would violate the assumption that each C_i contains no non-empty open subset. Let $S_{r_2}(p_2)$ be such that $\overline{S_{r_2}(p_2)} \cap \{[X - S_{r_1}(p_1)] \cup C_{n_1}\} = \emptyset$. Again, there exists such a neighborhood of p_2 since $X - S_{r_1}(p_1)$ and C_{n_1} are each closed and p_2 is a point of neither set. Let n_2 be the first integer such that $C_{n_2} \cap S_{r_2}(p_2) \neq \emptyset$.

This process may be continued and thus, in general, there exist a p_j and $S_{r_j}(p_j)$ such that

$$\overline{S_{r_j}(p_j)} \cap \{[X - S_{r_{j-1}}(p_{j-1})] \cup C_{n_{j-1}}\} = \emptyset$$

provided p_{j-1}, $S_{r_{j-1}}(p_{j-1})$, $C_{n_{j-1}}$ have each been defined. Let n_{j+1} be the first integer such that $C_{n_{j+1}} \cap S_{r_j}(p_j) \neq \emptyset$.

Now, for each i, $\overline{S_{r_i}(p_i)} \supset S_{r_i}(p_i) \supset \overline{S_{r_{i+1}}(p_{i+1})}$. So the sets $\overline{S_{r_1}(p_i)}$, $\overline{S_{r_2}(p_2)}$, $\overline{S_{r_3}(p_3)}$, ... form a decreasing sequence of compact sets and the set $\bigcap_{i=1}^{\infty} \overline{S_{r_i}(p_i)} \neq \emptyset$. See [8], p. 4. Let $p \in \bigcap_{i=1}^{\infty} \overline{S_i(p_i)}$. Since $X = \bigcup_i C_i$, there exists an integer k such that $p \in C_k$. Now, $n_i \geq i$ for each integer i. By the selection of each C_{n_i}, $\overline{S_{r_{k+1}}(p_{k+1})} \cap C_k \subset \overline{S_{r_k}(p_r)} \cap C_k = \emptyset$ which contradicts the fact that $p \in \overline{S_{r_i}(p_i)}$ for each i. Hence, the theorem is established.

A subset M of a space S is said to be perfect if M is closed and each point of M is a limit point of M, ie. $M = M'$.

Lemma 1.3. No compact metric space is both countable and perfect.

Proof. This lemma is an immediate consequence of Lemma 1.2, for suppose X is a countable perfect metric space. Then $X = \bigcup_{i \in I} \{x_i\}$ where x_i is a limit point of X. Each $\{x_i\}$ is a closed subset of X. Each $\{x_i\}$ is not open since each neighborhood of x_i contains points from $X - \{x_i\}$. Thus X is the union of a countable collection of closed subsets of X, no one of which contains an open subset of X. This contradicts the previous lemma. Hence no compact metric space is both countable and perfect.

Lemma 1.4. Let f be a homeomorphism on a metric space X. If A_x is compact, A_x is finite.

Proof. The proof will be by contradiction. Let A_x be a compact orbit of X under the iterations of f. Suppose A_x is not finite. Since A_x is compact, it contains a limit point z. Since f is a homeomorphism on X, f maps limit points into limit points. See [4], p. 101. Thus $f^n(z)$ is a limit point of A_x for each integer n. Since z is a point of A_x, $A_z = A_x$ by theorem 1.1. Then A_x is closed and thus, perfect. Since A_x is a compact subspace of X, by lemma 1.3 A_x cannot be both countable and perfect. Hence, a contradiction and A_x is finite.

The following result is thus obtained from the preceding lemmas.

Theorem 1.5. A necessary and sufficient condition for a homeomorphism f defined on a metric space X to be pointwise periodic is that each point-orbit A_x be compact.

ORBIT DECOMPOSITION AND CONTINUITY PROPERTIES

In this section it will be proved that given a metric space X and the point-orbit decomposition of X induced by the iterations of a homeomorphism f on X, a necessary and sufficient condition for the decomposition to be continuous is that f have equicontinuous powers. Continuity for a collection of sets will be defined in terms of upper and lower semi-continuity. A few consequences of the definition of lower semi-continuity will be developed as preparation for proving the theorem.

Let A be a subset of a metric space X. S(A,d) will be defined to be the set of all $x \in X$ such that $\rho(x,a) < d$ for some point a in A. A collection $\mathcal{G}$ of point-orbits A_x is said to be lower semi-continuous if given a point-orbit A_x and $\epsilon > o$ there exists $\delta > o$ such that $A_x \subset S(A_y, \epsilon)$ whenever $A_y \cap S(A_x, \delta) \neq \emptyset$. A collection $\mathcal{G}$ of point-orbits A_x is said to be upper semi-continuous if given a point-orbit A_x and $\epsilon > o$ there exists $\delta > o$ such that $A_y \subset S(A_x, \epsilon)$ whenever $A_y \cap S(A_x, \delta) \neq \emptyset$.

Theorem 2.1. If f is a pointwise periodic homeomorphism on a metric space X, the orbit decomposition is lower semi-continuous.

Proof. Let x be any arbitrary point of X and let A_x denote the orbit of x under the iterations of f. Since f is pointwise periodic, let n denote the period of f at x.

Now, f^i is continuous for each integer i since f^i is the composition function determined by i iterations of f and f is continuous. See [4], p. 72.

Let $\epsilon > o$ be given. By the continuity of f^i for each integer i, there exists

δ_i^o and if $\rho(x,y) < \delta_i^o$, $\rho\left[f^i(x), f^i(y)\right] < \epsilon$ for $o < i \leq n$,

δ_i^1 and if $\rho\left[f(x), y\right] < \delta_i^1$, $\rho\left[f^i(x), f^{i-1}(y)\right] < \epsilon$ for $1 < i \leq n+1$,

δ_i^2 and if $\rho\left[f^2(x), y\right] < \delta_i^2$, $\rho\left[f^i(x), f^{i-2}(y)\right] < \epsilon$ for $2 < i \leq n+2$,

. . . .

δ_i^{n-1} and if $\rho\left[f^{n-1}(x), y\right] < \delta_i^{n-1}$, $\rho\left[f^i(x), f^{i-n+1}(y)\right] < \epsilon$ for $n-1 < i \leq 2n-1$.

Let $\delta = \min \left\{\delta_i^j, \epsilon\right\}$ where $o \leq j \leq n-1$, $o < i \leq 2n-1$. Suppose $A_y \cap S(A_x, \delta) \neq \emptyset$. Let $z \in A_y \cap S(A_x, \delta)$. Then $\rho(w,z) < \delta$ for some $w \in A_x$. Let u be a point of A_x. Now, u has period n since each point of an orbit has the same period. There exists a positive integer $m \leq n$ such that $f^m(u) = w$. Then $\rho\left[f^m(u), z\right] < \delta$. By the way δ was chosen

$$\rho\left[f^{n-m}\left[f^m(u)\right], f^{n-m}(z)\right] = \rho\left[f^n(u), f^{n-m}(z)\right] = \rho\left[u, f^{n-m}(z)\right] < \epsilon$$

Hence $u \in S(A_y, \epsilon)$. Thus $A_x \subset S(A_y, \epsilon)$ which was to be proved.

Consider the homeomorphism f defined on a metric space X. If f is pointwise periodic, let p(x) denote the period of x where $x \in X$. Note that p(x) is also a function defined on X. Thus, p(x) will be called the period function associated with f.

Corollary 2.2. Let f be a pointwise periodic homeomorhpism on a metric space X. Then for each $x \in X$ there exists a $\delta > o$ such that if $\rho(x,y) < \delta$, then $p(y) \geqslant p(x)$.

Proof. The proof will be by contradiction. Let $x \in X$ such that for each $\delta > o$ there exists a $y \in X$ such that $\rho(x,y) < \delta$ and $p(y) < p(x)$. Let x have period n. Let $3\epsilon = \min \{\rho(f^i(x), f^j(x))\}$ for $o < i,j \leqslant n$, $i \neq j$. Since f^i is continuous for each i, there exists $\delta_i < \epsilon$ such that if $\rho(x,y) < \delta_i$ then for $\rho[f^i(x), f^i(y)] < \epsilon$.
Let $\delta' = \min \{\delta_i\}$ $o < i \leqslant n$.

Now there exists y_1 such that $\rho(x,y_1) < \delta'$ and $p(y_1) = k$ where $k < n$. Using the triangle inequality for a metric space it is seen that

$$\begin{aligned} 3\epsilon \leqslant \rho[x, f^k(x)] &\leqslant \rho[x, f^k(y_1)] + \rho[f^k(y_1), f^k(x)] \\ &= \rho(x, y_1) + \rho[f^k(y_1), f^k(x)] \\ &< \epsilon + \epsilon = 2\epsilon, \end{aligned}$$

a contradiction. Thus, corollary is proved.

A real-valued function f defined on a metric space X is lower semi-continuous at a point $a \in X$ provided that for each $\epsilon > o$ there exists a $\delta > o$ such that $f(y) \geq f(a) - \epsilon$ whenever $\rho(y,a) < \delta$.

Corollary 2.3. Let f be a pointwise periodic homeomorphism defined on a metric space X. The period function p(x) associated with f is lower semi-continuous.

Proof. This result is an immediate consequence of corollary 2.2. Let $x \in X$ and $\epsilon > o$ be given. Then there exists a $\delta < \epsilon$ such that $p(y) \geqslant p(x)$ whenever $\rho(x,y) < \delta$. Hence the period function is lower semi-continuous.

A homeomorphism f on a metric space X is said to have equicontinuous powers provided that for each x in X and each $\epsilon > o$ there exists a $\delta > o$ such that if $\rho(x,y) < \delta$, then for each integer n, $\rho\left[f^n(x), f^n(y)\right] < \epsilon$.

The point-orbit decomposition of a metric space X, generated by a homeomorphism f is said to be continuous if it is both lower semi-continuous and upper semi-continuous.

Theorem. If f is a pointwise periodic homeomorphism on a metric space X, in order that the orbit decomposition be continuous it is necessary and sufficient that f have equicontinuous powers.

Proof. To prove the sufficiency it will suffice to show that the orbit decomposition is upper semi-continuous since Theorem 2.1 established lower semi-continuity for the orbit decomposition. Let $\epsilon > o$ be given. Since f has equicontinuous powers at $p \in X$, there exists a $\delta > o$ such that if $\rho(p,x) < \delta$, then $\rho\left[f^n(x), f^n(p)\right] < \epsilon$ for each integer n. Thus, $A_x \subset S(A_p, \epsilon)$ and the orbit decomposition is upper semi-continuous. Hence, the orbit decomposition is continuous.

In order to prove necessity, let $x \in X$, $\epsilon > o$ be given and let n denote the period of f at x.
Let $\delta' = \min_{i \neq j} \left\{ \rho\left[f^i(x), f^j(x)\right] \right\}$ and $\delta'' = \frac{1}{2} \min \left\{ \epsilon, \delta' \right\}$.
For each i, $-n \leqslant i \leqslant n$, f^i is continuous at x and there is a δ_i such that $\rho\left[f^i(x), f^i(y)\right] < \delta''$ whenever $\rho(x,y) < \delta_i$. Let $\delta''' = \min \left\{ \delta'', \delta_i \right\}$ for $-n \leqslant i \leqslant n$. By the upper semi-

continuity there is a $\delta < \delta'''$ such that if $A_y \cap S(A_x, \delta) \neq \emptyset$, then $A_y \subset (A_x, \delta''')$.

Now let $\rho(x,y) < \delta$ and examine $\rho\left[f^i(x), f^i(y)\right]$. First since $\delta < \delta_n$, $\rho\left[f^n(x), f^n(y)\right] < \delta'' < \epsilon$, and since $\delta < \delta_{-n}$, $\rho\left[f^{-n}(x), f^{-n}(y)\right] < \delta'' < \epsilon$. Then since $f^n(y) \in S(A_x, \delta''')$ and $f^{-n}(y) \in S(A_x, \delta''')$ and from the way δ''' was chosen, $\rho\left[f^i(x), f^n(y)\right] > \delta'''$ for $o < i < n$ and $\rho\left[f^i(x), f^{-n}(y)\right] > \delta'''$ for $-n < i < o$. Thus $\rho\left[f^n(x), f^n(y)\right] < \delta'''$ and $\rho\left[f^{-n}(x), f^{-n}(y)\right] < \delta'''$. Similarly then, since $\delta''' < \delta''$, it follows that $\rho\left[f^{kn}(x), f^{kn}(y)\right] < \delta'''$ for any integer k. Furthermore, since $\delta''' < \delta_j$ for $o \leqslant j < n$, then $\rho\left[f^{kn+j}(x), f^{kn+j}(y)\right] < \delta'' < \epsilon$ so that f has equicontinuous powers at x.

COMPONENT-ORBITS AND CONVERGENCE PROPERTIES

In developing the properties of point and component-orbits in a metric space, an interesting result is that a convergent sequence of point or component-orbits converges to a component-orbit. This section concentrates on establishing this result. A few consequences of this property will complete the section. The following definitions are essential for the theory which follows.

$A_1, A_2, \ldots.$ is a decreasing sequence of sets provided that $A_{n+1} \subset A_n$ for each positive integer n. A chain of sets is a finite sequence of sets $A_1, A_2, \ldots, A_k$ such that $A_i \cap A_{i+1} \neq \emptyset$ for $i = 1,2, \ldots, k-1$. An ϵ-chain in a metric space is a finite sequence of points $a_1, a_2, \ldots., a_k$, such that $\rho(a_i, a_{i+1}) < \epsilon$ for $i = 1,2, \ldots, k-1$. A metric space X is ϵ-connected if for each pair of points $a,b \in X$, there exists an ϵ-chain $a_1, a_2, \ldots., a_k$ in X such that $a = a_1$ and $b = a_k$. Two points a and b are connected in X provided that there exists a connected set $A \subset X$ such that $a \in A$, $b \in A$. Let $L = A \cup B$ be a subset of a space X. $A \cup B$ is said to be a separation of L if $\bar{A} \cap B = \bar{B} \cap A = \emptyset$.

Lemma 3.1. If a compact metric space X is ϵ-connected for each $\epsilon > 0$, then X is connected.

Proof. The proof is by contradiction. Suppose X is not connected. Then $X = A \cup B$ is a separation of X. Since A and B are each closed and hence, compact in X, $\rho(A,B) = \epsilon$, $\epsilon > 0$ for some ϵ. This is shown in [4], p. 90. Clearly no $\epsilon/2$-chain can join a point of A to a point of B. Thus X is not $\epsilon/2$-connected which contradicts the hypothesis.

Remark. If a subset of a metric space is connected, it is ϵ-connected for each ϵ.

Proof of remark. Suppose there exists $\epsilon > o$ such that A is not ϵ-connected. Then there exist points $a, b, \in A$ such that there is no ϵ-chain between a and b. Let B be the set of all $x \in A$ such that there exists an ϵ-chain between a and x. Let C be the set A - B. Each point y in C is greater then ϵ distance from a point in B. It is easily seen that $A = B \cup C$ is a separation of A. This contradicts the fact A is connected. Hence, the remark is true.

Lemma 3.2. Let A_1, A_2, ... be a decreasing sequence of compact, ϵ-connected sets. Then $A = \bigcap_i A_i$ is 2ϵ-connected.

Proof. Let $a, b \in A$ where $A = \bigcap_i A_i$. Choose n large enough so that $A_n \subset S(A, \epsilon/2)$. See [6], p. 47. Let $a = a_o, a_1, \ldots, a_k = b$ be an ϵ-chain in A_n. Let $b_i \in S_{\epsilon/2}(a_i)$ for $1 \leqslant i \leqslant k-1$. Then, using the triangle inequality for a metric space, it is easily seen that $a, b_1, b_2, \ldots, b_{k-1}, b$ is a 2ϵ-chain in A.

It should be remarked here that if a set is ϵ-connected, it is ϵ'-connected for each $\epsilon' > \epsilon$.

Lemma 3.3. If for each $\epsilon > o$, the points a and b of a compact set B can be joined by an ϵ-chain in B, then they are connected in B.

Proof. Let a and b be points of a compact set B such that they can be joined by an ϵ-chain for each $\epsilon > o$. Then let A_n be the set of points of B which can be joined to a by a $1/n$-chain in B for each positive integer n. By hypothesis, $b \in A_n$ for each n. Hence, $b \in \bigcap_{n=1}^{\infty} A_n$. The set $B - A_n$ is open for each n since if $x \in B - A_n$, $S_{1/n}(x) \subset B - A_n$. Otherwise, x would be an element of A_n by the way A_n is defined. Thus, each A_n is closed and compact. The sequence A_n is a decreasing sequence, ie. $A_{n+1} \subset A_n$ for each n. This follows from the remark which precedes this lemma.

Now, A_n is $1/n$-connected for each n by the way A_n is defined. A_n is $1/n_o$-connected if $n > n_o$ where n_o is fixed. By lemma 3.2, $\bigcap_{n=1}^{\infty} A_n = \bigcap_{n_o}^{\infty} A_n = A$ is $2/n_o$-connected for each n_o and therefore is ϵ-connected for each ϵ since for each $\epsilon > o$ there exists an n_o' such that $2/n_o' < \epsilon$. Thus, A is connected by lemma 3.1. Now $a, b, \in A$. Hence, a and b are connected in B.

Lemma 3.4. Let a and b be points of a compact set L which are not connected in L. Then there exists a separation $L = A \cup B$ of L such that $a \in A$, $b \in B$.

Proof. By the contrapositive of the preceding lemma there exists an $\epsilon > o$ such that a and b cannot be joined in L by an ϵ-chain. Let A be the set of points of L which can be joined to a by an ϵ-chain and let $B = L - A$. Now $\rho(A,B) \geq \epsilon > o$ by the way A is defined. Thus, it easily follows that $\bar{A} \cap B = \bar{B} \cap A = \emptyset$. Hence, $A \cup B$ is a separation of L such that $a \in A$, $b \in B$.

A subset K of a space S is said to be a component of S provided that K is a maximal connected subset of S, ie. K is contained in no connected subset of S other than itself. A subset G of S is a component-orbit if $G = \bigcup_{-\infty}^{\infty} f^n(K)$ where K is a component of G and f is a homeomorphism on S. Now, if f(S) = S, then a subset Y of S is said to be invariant provided f(Y) = Y. If f is pointwise periodic on S, a closed invariant subset G in S is said to be a component-orbit provided given any component K in G, there exists a positive integer n such that $f^n(K) = K$ and $\bigcup_{i=1}^{n} f^i(K) = G$.

Lemma 3.5. Let L be a compact metric space and K a component of L such that $p \in L - K$. Then there exists a separation $L = A \cup B$ such that $K \subset A$, $p \in B$.

Proof. Let a be a point of K. Since K is a component and $p \in L - K$, then a and p are not connected. By lemma 3.4 there exists a separation $L = A \cup B$ of L such that $a \in A$ and $p \in B$. Since K is connected and $a \in A$, $K \subset A$.

Lemma 3.6. Let L be a compact metric space and K_i a component of L, $p \in L - K_i$ for i = 1,2,..., n. Then there exists a separation $L = A \cup B$ such that $\bigcup_{i=1}^{n} K_i \subset A$, $p \in B$. Furthermore, there exist disjoint open sets V and U such that $A \subset V$, $B \subset U$.

Proof. By lemma 3.5, there exists a separation $L = A_i \cup B_i$ of L such that $K_i \subset A_i$ and $p \in B_i$ for i = 1,2,...,n. Then $\bigcup_{i=1}^{n} K_i \subset \bigcup_{i=1}^{n} A_i$ and $p \in \bigcap_{i=1}^{n} B_i$. Let $A = \bigcup_{i=1}^{n} A_i$ and $B = \bigcap_{i=1}^{n} B_i$. (Note that from [4], page. 70, given sets C and D, $\overline{C \cup D} = \overline{C} \cup \overline{D}$ and $\overline{C \cap D} \subset \overline{C} \cap \overline{D}$.)

Hence, the following is true:

$$\left(\bigcup_{i=1}^{n} A_i\right) \cap \left(\bigcap_{j=1}^{n} B_j\right) = \left(\bigcup_{i=1}^{n} \overline{A}_i\right) \cap \left(\bigcap_{j=1}^{n} B_j\right)$$

$$= \bigcup_{i=1}^{n} \left[\overline{A}_i \cap \left(\bigcap_{j=1}^{n} B_j\right)\right]$$

$$= \bigcup_{i=1}^{n} \bigcap_{j=1}^{n} (\overline{A}_i \cap B_j)$$

$$= \emptyset \text{ since } \overline{A}_i \cap B_j = \emptyset \text{ for } i = j.$$

$$\left(\bigcup_{i=1}^{n} A_i\right) \cap \left(\bigcap_{i=1}^{n} B_j\right) \subset \left(\bigcup_{i=1}^{n} A_i\right) \cap \left(\bigcap_{i=1}^{n} \overline{B}_j\right)$$

$$= \bigcup_{i=1}^{n} \left[A_i \cap \left(\bigcap_{j=1}^{n} \overline{B}_j\right)\right]$$

$$= \bigcup_{i=1}^{n} \bigcap_{j=1}^{n} (A_i \cap \overline{B}_j)$$

$$= \emptyset \text{ since } A_i \cap \overline{B}_j = \emptyset \text{ for } i = j.$$

Therefore A and B are separated sets.

It is shown in [4], p. 110, that if A and B are any two disjoint separated sets of a metric space X, then there exist disjoint open sets U and V such that $A \subset U$, $B \subset V$. Thus, the second part of the lemma is established.

Lemma 3.7. If U and V are two disjoint open subsets of a metric space X, then U and V are separated.

Proof. Since $U \cap V = \emptyset$, $U \subset (X - V)$. Then by [4], p.70, $\overline{U} \subset \overline{(X - V)} = (X - V)$. Hence, $\overline{U} \cap V \subset (X - V) \cap V = \emptyset$. Therefore $\overline{U} \cap V = \emptyset$. Similarly, $U \cap \overline{V} = \emptyset$. Thus, U and V are separated.

Let S be a topological space and consider G to be any finite collection of non-empty subsets of S. The set of all points x of S such that each neighborhood of x contains a point from each of infinitely many sets of G is called the limit superior of G and is denoted by lim sup G. The set of all points x in S such that each neighborhood of x contains a point from all but a finite number of the sets of G is called the limit inferior of G and is denoted by lim inf G. If lim inf G = lim sup G, the collection G is said to converge to the limit set of G, denoted by lim G, where lim G = lim inf G = lim sup G.

Let X be a compact metric space and f a pointwise periodic homeomorphism defined on X. Let G_1, G_2, ... be a sequence of component-orbits converging to a limit set L in X. Suppose K is a component-orbit in L such that $K \neq L$. Then there exists a point p in L - K. By lemma 3.7 there exists a separation $L = A \cup B$ of L such that $K \subset A$, $p \in B$. Also, there exist separated open sets V and U such that $A \subset V$, $B \subset U$. Based on these preliminary remarks, the following lemma will be established.

Lemma 3.8. If x_1, x_2, ... is a sequence of points converging to a point x in K where $K \subset V$, and $x_i \in G_i$ for each positive integer i, then there exists i_0 such that if $i > i_0$, then $f^n(x_i) \in X - V$ for some n.

Proof. The proof is by contradiction. Suppose there exists a sequence i_1, i_2, ..., $(i_j < i_{j+1})$ such that $f^n(x_{i_j}) \in V$ for each n.

Since U and V each contain points of L, U and V each intersect G_i for all but a finite number of the G_i's.

In order to facilitate the proof, suppose U and V intersect all the G_i's. Now $x_{i_j} \in G_{i_j}$. Let $G_{i_j}^{(k)}$ be a component of G_{i_j} such that $G_{i_j}^{(k)} \cap U \neq \emptyset$. But $f^n(x_{i_j}) \in V$ for each n and $f^{n_j}(x_{i_j}) \in G_{i_j}^{(k)}$ for some n_j. Hence $G_{i_j}^{(k)} \cap V \neq \emptyset$. Since, by lemma 3.7, V and U are separated sets and $G_{i_j}^{(k)}$ is connected for each j, there exist points z_j such that $z_j \in G_{i_j}^{(k)} - (U \cup V)$. The sequence z_1, z_2, ... forms an infinite set. Since X is compact, X - (U ∪ V) is compact and some subsequence z_{j_1}, z_{j_2}, ... of the z_j's converges to a point z in X - (U ∪ V). Since each neighborhood of z contains all but a finite number of the z_{j_ℓ}'s then each neighborhood of z contains a point from each of infinitely many of the G_{i_j}'s, and thus of the G_i's. Hence, $z \in L$, a contradiction since $L \subset (U \cup V)$. This completes the proof of the lemma.

Lemma 3.9. Let S be a space and f a homeomorphism defined on X. If G_1, G_2, ... is a convergent sequence of invariant sets in X, the limit set L is invariant under f, ie. f(L) = L.

Proof. Let $y \in f(L)$. Thus, there exists an x L such that f(x) = y. Hence, there exists a sequence x_1, x_2, ... converging to x such that $x_i \in G_i$ for each i. The sequence $f(x_1)$, $f(x_2)$, ... converges to y as shown in [4], p. 101. Now, for each i, $f(x_i) \in G_i$ since $f(G_i) = G_i$. Therefore, $y \in L$. This proves $f(L) \subset L$.

Since f^{-1} is also a homeomorphism, $f^{-1}(L) \subset L$. Thus,

$f\left[f^{-1}(L)\right] \subset f(L)$. But $f\left[f^{-1}(L)\right] = L$. Therefore, $L \subset f(L)$. Hence $f(L) = L$.

Theorem 3.10. Let X be a compact metric space and f a pointwise periodic homeomorphism defined on X. Then if $G_1, G_2, \ldots$ is a convergent sequence of point or component-orbits in X, the limit set L is a component-orbit.

Proof. Since $f(G_i) = G_i$ for each i, $f(L) = L$ by lemma 3.9. Let K_o be a component of L. Let $f(K_o) = K_1$, $f(K_1) = K_2$, ..., $f(K_{n+1}) = K_o$ so that $K = \bigcup_{i=1}^{n} K_i$ is a component-orbit in L.

Suppose $K \neq L$. L is closed and therefore compact since X is compact. By lemma 3.6 there exists a separation $L = A \cup B$ such that $K \subset A$. Also there exist disjoint open sets V and U such that $A \subset V$ and $B \subset U$.

Choose a sequence of points $x_1, x_2, \ldots$ such that $x_i \in G_i$ and x_i converges to $x \in K_o$. Since all but a finite number of the x_i's are in V, there is no loss of generality in assuming that all are in V. For each i, let y_i be the first point in the sequence $f(x_i), f^2(x_i), \ldots$ which belongs to X - V. If the G_i's are point-orbits, y_i exists for each i since $G_i \cap (X - V) \neq \emptyset$ for all but a finite number of the G_i's and it may be assumed $G_i \cap (X - V) \neq \emptyset$ for all G_i's. If $G_1, G_2, \ldots$ is a sequence of component-orbits, y_i exists for each i by lemma 3.8.

Since X is compact and thus, also X - V, some subsequence $y_{i_1}, y_{i_2}, \ldots$ of the y_i's converges to a point $y \in (X - V) \cap L$. The point $f^{-1}(y_{i_j})$ is in V for each j by the way the sequence $y_1, y_2, \dots$ was defined. Thus, since f^{-1} is a homeomorphism, the sequence $f^{-1}(y_{i_1}), f^{-1}(y_{i_2}), \dots$ converges to $f^{-1}(y) \in V \cap L$.

Now $f^{-1}(y) \in V \cap (L - K)$ for if $f^{-1}(y)$ K, then $f\left[f^{-1}(y)\right] = y$ would be an element of K which it is not. Since $f^{-1}(y) \neq x$ and the sequence $f^{-1}(y_{i_1}), f^{-1}(y_{i_2}), \dots$ can not converge to x, there exists an N_1 such that $f^{-1}(y_{i_j}) \neq x_{i_j}$ for $j > N_1$.

By the way the sequence $y_1, y_2, \dots$ was chosen, the points $f^{-2}(y_{i_j})$ for $j > N_1$ are in V. This follows since for each $j > N_1$, $f^{-2}(y_{i_j})$ is still some non-negative power of f at x_{i_j}. Now, only a finite number of the $f^{-2}(y_{i_j})$'s for $j > N_1$ can equal x_{i_j}. This follows since $f^{-2}(y) \in V \cap (L - K)$ and the sequence $f^{-2}(y_{i_1}), f^{-2}(y_{i_2}), \dots$ converges to $f^{-2}(y)$. Hence, there exists an $N_2 > N_1$ such that $f^{-2}(y_{i_j}) \neq x_i$, $j > N_2$. Continuing this process $f^{-n}(y) \in V$ for each positive integer n. This is contrary to the fact that f is periodic at y and $f^{-n_o}(y) = y \in X - V$ for some n_o. Therefore K = L and the theorem is established.

Corollary 3.11. If L contains a fixed point, then L is connected.

Proof. The proof will be by contradiction. Suppose L is not connected. Then let $L = L_1 \cup L_2$ be a separation of L. Let K_o be a component of L such that $p \in K_o$ and p is invariant under f, ie. $f(p) = p$.
Hence, $p \in f^i(K_o)$ $(i = o,1,2,\dots, n-1)$. Since K_o is connected,

without loss of generality suppose $K_0 \subset L_1$. Then $p \in L_1$. By the previous theorem K = L is the component-orbit of K_0. Since $L_2 \neq \emptyset$, then $f^i(K_0) \subset L_2$ for some i. Thus, p also is in L_2, a contradiction since $L_1 \cap L_2 = \emptyset$. Therefore, L is connected.

Corollary 3.12. If L contains an invariant connected non-empty subset, then L is connected.

Proof. Let A be an invariant connected non-empty subset of L. Then A contains an invariant point p. Hence, by corollary 3.11, L is connected.

The following lemma makes use of the fact that a compact metric space has a countable basis, which is shown in [4], pp. 106-109.

Lemma 3.13. Let X be a compact metric space. Then every infinite sequence of distinct subsets of X contains a convergent subsequence.

Proof. Let $A_1, A_2, \ldots$ be a sequence of distinct subsets of X. Let $R_1, R_2, \ldots$ be a countable basis for X. Form an array of sequences of sets as follows.

Let $\{A_i^1\}$ denote the sequence $A_1, A_2, \ldots$. For each positive integer n, $\{A_i^{n+1}\}$ is obtained from $\{A_i^n\}$ in the following manner. If there exists a subsequence $\{A_{i_j}^n\}$ of $\{A_i^n\}$ such that $\limsup \{A_{i_j}^n\} \cap R_n = \emptyset$, let $\{A_i^{n+1}\}$ be one such subsequence. If $\{A_{i_j}^n\} \cap R_n \neq \emptyset$ for each subsequence $\{A_{i_j}^n\}$, let $\{A_i^{n+1}\} = \{A_i^n\}$.

Thus, the following array:

$$\left\{A_i^1\right\} = A_1^1, A_2^1, A_3^1, \ldots\ldots\ldots$$

$$\left\{A_i^2\right\} = A_i^2, A_2^2, A_3^2, \ldots\ldots\ldots$$

$$\left\{A_i^3\right\} = A_i^3, A_2^3, A_3^3, \ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\left\{A_i^m\right\} = A_i^m, A_2^m, A_3^m, \ldots, A_m^m \ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\left\{A_i^n\right\} = A_i^n, A_2^n, A_3^n, \ldots, A_n^n \ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

The proof, by contradiction, will establish the convergence of the diagonal sequence $\left\{A_n^n\right\}$. $\left\{A_n^n\right\}$ is a subsequence of $\left\{A_i^1\right\}$. Suppose $\left\{A_n^n\right\}$ is not convergent. This means the lim sup $\left\{A_n^n\right\} \neq$ lim inf $\left\{A_n^n\right\}$. But lim inf $\left\{A_n^n\right\} \subset$ lim sup $\left\{A_n^n\right\}$ as shown in [8], p. 10. Thus, there must exist a point x in lim sup $\left\{A_n^n\right\}$ such that x is not in lim inf $\left\{A_n^n\right\}$. Since X is second countable, there exists a neighborhood R_m of x and an infinite subsequence $\left\{A_{n_i}^{n_i}\right\}$ of $\left\{A_n^n\right\}$ such that $R_m \cap \left\{A_{n_i}^{n_i}\right\} = \emptyset$.

Now, by the way in which the above array of sequences is defined, the sequence $\left\{A_n^n\right\}$ for $n > m$, and hence, also the sequence $\left\{A_{n_i}^{n_i}\right\}$ for $i > m$, are subsequences of $\left\{A_i^m\right\}$. Since $\left\{A_{n_i}^{n_i}\right\}$ for $i > m$ does not intersect R_m, lim sup $\left\{A_i^{m+1}\right\} \cap R_m = \emptyset$. But for $n > m$, $\left\{A_n^n\right\} \subset \left\{A_i^{m+1}\right\}$ and hence

$$\limsup \{A_n^n\} \subset \limsup \{A_i^{m+1}\}$$

$$\limsup \{A_n^n\} \cap R_m \subset \limsup \{A_i^{m+1}\} \cap R_m = \emptyset$$

But $x \in \limsup \{A_n^n\} \cap R_m \subset \emptyset$, a contradiction.

Therefore A_n^n converges which was to be proved.

Theorem 3.14. Let X be a compact metric space and f a pointwise periodic homeomorphism of X onto X. Let L be a disconnected invariant closed subset of X. Then for each separation $L = L_1 \cup L_2$, there exists a positive integer N_1 such that $f^{N_1}(L_1) \subset L_1$.

Proof. The proof will be by contradiction. Assume that for each positive integer n there exists a point $y_n \in L_1$ such that $f^n(y_n) \notin L_1$, that is $f^n(y_n) \in L_2$ since L is invariant. Let $x_n = y_{n!}$ so that $f^{n!}(x_n) \in L_2$. Since L is closed and thus compact, the sequence $x_1, x_2, \ldots$ has a subsequence $x_{n_1}, x_{n_2}, \ldots$ converging to a point x of L. Since L_1 and L_2 are separated and each x_{n_i} is in L_1, then x is in L_1. Let k denote the period of f at x and let $g(p) = f^k(p)$ for each $p \in X$. If m_p denotes the period of f at p, then $g^{m_p}(p) = f^{km_p}(p) = f^{m_p} f^{m_p} \ldots f^{m_p}(p) = p$, where the iteration is taken k times so that g is also pointwise periodic on X.

For each i, let G_i denote the point-orbit of x_{n_i} under g. By lemma 3.13 the sequence $G_1, G_2, \ldots$ contains a subsequence $G_{i_1}, G_{i_2}, \ldots$ converging to a set G. Now $x \in G$ since $x_{n_{i_j}} \in G_{i_j}$ and each neighborhood of x contains all but a finite number of the x_{n_i}'s and consequently of the $x_{n_{i_j}}$'s.

since $g(x) = f^k(x) = x$, it follows from corollary 3.11 that G is connected, and since L_1 and L_2 are separated with $x \in L_1$, then $G \subset L_1$.

If $n_{i_j} \geqslant k$ then $n_{i_j}!/k$ is an integer, which may be denoted by m_{i_j}. Then

$$f^{n_{i_j}!}(x_{n_{i_j}}) = f^{km_j}(x_{n_{i_j}}) = g^{m_j}(x_{n_{i_j}}) \in G_{i_j}.$$

Also $f^{n_{i_j}}(x_{n_{i_j}}) \in L_2$ by the way the x_n's were selected so that $G_{i_j} \cap L_2 \neq \emptyset$ for $n_{i_j} \geqslant k$. Let $t_j \in G_{i_j}$ such that $t_j \in L_2$. Since L_2 is compact there exists a subsequence t_{j_h} converging to t in L_2. But also $t \in G$ so that $G \cap L_2 \neq \emptyset$, which contradicts $G \subset L_1$. This establishes the theorem.

Theorem 3.15. Using the hypotheses of theorem 3.14, then for each separation $L = L_1 \cup L_2$, there exists a positive integer N such that $f^N(L_1) = L_1$ and $f^N(L_2) = L_2$.

Proof. By theorem 3.14 there exist positive integers N_1 and N_2 such that $f^{N_1}(L_1) \subset L_1$, and $f^{N_2}(L_2) \subset L_2$ respectively. Now, let $N = N_1N_2$. Then $f^{N_1N_2}(L_2) \subset L_2$ and $f^{N_1N_2}(L_1) \subset L_1$. $L_1 \cap L_2 = \emptyset$ and so

$$f^{N_1N_2}(L_1) \cap f^{N_1N_2}(L_2) \subset L_1 \cap L_2 = \emptyset.$$

Also, since L is invariant

$$f^{N_1N_2}(L_1) \cup f^{N_1N_2}(L_2) = L_1 \cup L_2.$$

Then

$$L_1 \cap \left[f^{N_1N_2}(L_1) \cup f^{N_1N_2}(L_2)\right] = L_1 \cap (L_1 \cup L_2) = L_1$$

$$\left[L_1 \cap f^{N_1N_2}(L_1)\right] \cup \left[L_1 \cap f^{N_1N_2}(L_2)\right] = L_1$$

$$f^{N_1N_2}(L_1) \quad \cup \quad \emptyset \qquad = L_1$$

Thus, $f^{N_1N_2}(L_1) = L_1$. Similarly, $f^{N_1N_2}(L_2) = L_2$. Hence, $f^N(L_1) = L_1$ and $f^N(L_2) = L_2$.

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