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# Pfaffian Differential Expressions and Equations

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# PFAFFIAN DIFFERENTIAL EXPRESSIONS
# AND EQUATIONS

BY

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# INTRODUCTION

It is needless to point out the necessity and the importance of the study of Pfaffian differential expressions and equations. While it is interesting to consider from the pure mathematical point of view, their applications in many branches of applied mathematics are well known. To mention a few, one may observe that they arise in connection with line integrals (example, determination of work). They provide a more rational formulation of the foundations of thermodynamics as developed by the Greek mathematician Caratheodory. They also arise in the problem of determining the orthogonal trajectories. In many branches of engineering and other physical sciences they appear with problems concerning partial differential equations.

An expression of the form

$$\sum_{i=1}^{n} F_i(x_1, x_2, \ldots\ldots x_n)\ dx_i \tag{1}$$

where the $F_i$ $(i = 1,2.\ldots\ldots n)$ are functions of some or all of the n independent variables $x_1, x_2, \ldots\ldots x_n$ is called a Pfaffian differential form or expression in n variables. Similarly the relation

$$\sum_{i=1}^{n} F_i\ dx_i = 0 \tag{2}$$

is called a Pfaffian differential equation.

Thus a Pfaffian differential equation in three variables is of the form

$$P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = 0 \qquad (3)$$

where P, Q, and R are functions of x, y, and z. Given a Pfaffian differential equation, we are interested to find the solution of the form

$$f(x,y,z) = C \qquad (4)$$

where C is an arbitrary constant, which is one-parameter family of surfaces in three dimensional space.

Equivalently, if we write equation (4) in the form

$$z = g(x,y) \qquad (5)$$

and writing equation (3) in the form

$$dz = -\frac{P(x,y,z)}{R(x,y,z)}dx - \frac{Q(x,y,z)}{R(x,y,z)}dy \qquad (6)$$

we notice that (5) satisfies equation (6) identically so that we have

$$\frac{\partial z}{\partial x} = -\frac{P(x,y,z)}{R(x,y,z)} \qquad \frac{\partial z}{\partial y} = -\frac{Q(x,y,z)}{R(x,y,z)} \qquad (7)$$

which is a system of partial differential equations of the first order.

Thus, we notice that a Pfaffian differential equation may be treated as a system of partial differential equations. In some treatises on the subject they are dealt with before the partial differential equations which makes the understanding of arguments difficult for the beginner. This may provoke erroneous interpretations. Since the partial differential equations and Pfaffian equations are intermixed it is preferable to consider a Pfaffian equation as a system of partial differential equations when observed from the pure mathematical point of view.

If there exists a function $u(x,y,z)$ such that $du(x,y,z) = P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$ then the equation (3) is said to be exact, and the solution is given by

$$u(x,y,z) = C$$

where C is an arbitrary constant.

But even if the equation (3) is not exact, it may be possible to find a function $\mu(x,y,z)$, called the integrating factor, and a function $u(x,y,z)$ such that

$$\mu P\,dx + \mu Q\,dy + \mu R\,dz = du$$

In such a case we say that the equation is integrable. If the equation is integrable, it is always possible to find a solution.

A necessary and sufficient condition that the equation (3) is integrable is that the relation

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

holds.

If the equation is integrable, it is not necessary to determine the integrating factor in order to obtain the solution. In fact the determination of the integrating factor is more complicated than finding the solution of the problem itself.

When equation (3) is found to be integrable, various methods are available to find the solution. We shall recall the well known methods for solving a Pfaffian differential equation in "Survey of Well Known Methods."

Our main result deals with the solution of nonhomogeneous Pfaffian differential equations. Ulysses V. Ward showed that a nonhomogeneous differential equation in two variables x and y, when made homogeneous by assigning dimension 1 to x and its differential dx and dimension n to y and its differential dy, the substitution $y = v x^n$ renders the new equation in x and v separable and can be easily solved.*

In our main result we shall extend his method to a Pfaffian differential equation of three variables. For this we shall define quasi-homogeneous functions and incidentally are in a position to prove the extension of Euler's theorem to quasi-homogeneous functions. Then we shall define a quasi-homogeneous Pfaffian differential equation and give the solution of such an equation, which is our main result and it is believed to be original.

We shall also give a necessary condition in order that the given equation (3) may be quasi-homogeneous.

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*Ward, Ulysses V. "Linear First Order Differential Equations, <u>American Mathematics Monthly</u>, Vol. 67, No. 8, 1960.

## SURVEY OF WELL KNOWN METHODS

When the Pfaffian differential equation (3) is exact, the following methods are available to determine the solution.

### Methods for an exact Pfaffian differential equation

*Method 1.* If the equation (3) is found to be exact, then there exists a function $f(x,y,z)$ such that

$$\frac{\partial f}{\partial x} = P, \quad \frac{\partial f}{\partial y} = Q \quad \frac{\partial f}{\partial z} = R$$

and the solution is found through a system of partial differential equations.

*Method 2.* If the equation (3) is exact, then the solution can be obtained through the application of the formula

$$\int_{x_0}^{x} P(x,y,z)\,dx + \int_{y_0}^{y} Q(x_0,y,z)\,dy + \int_{z_0}^{z} R(x_0,y_0,z)\,dz = C$$

which is based on the properties of line integrals.

*Method 3. By inspection.* It may sometimes by possible to obtain the solution of a Pfaffian differential equation by inspection.

*Method 4. Variables separable.* If it is possible to write the differential equation (3) in the form

$$P(x)\,dx + Q(y)\,dy + R(z)\,dz = 0$$

the solution is given by the equation

$$\int P(x)\,dx + \int Q(y)dy + \int R(z)dz = C$$

where C is an arbitrary constant.

## Methods for a Pfaffian differential equation which is integrable, but not exact

If the equation (3) is integrable, but not exact, the following methods are available in order to find the solution.

Method 5. One variable separable. If one of the variables, say z, is separable, the equation can be brought to the form

$$P(x,y)dx + Q(x,y)dy + R(z) = 0$$

In this case the condition of integrability reduces to

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

which means that

$$P\,dx + Qdy$$

is an exact differential, say du,

so that the given equation reduces to

$$du + R(z)dz = 0$$

and the solution is

$$u + \int R(z)dz = C$$

where C is an arbitrary constant.

Method 6. Homogeneous equations. If the equation (3) is homogeneous, that is, P,Q,R are homogeneous functions in x,y,z of the same degree, then the substitutions

$$y = ux \quad , \quad z = vx$$

render the new equation in x,u,v separable in x and can be solved by the method 5.

Method 7. Natani's method. In this method we treat one of the variables, say z, as though it were a constant and solve the resulting differential equation

$$P\,dx + Q\,dy = 0$$

Suppose the solution of this equation is

$$\phi(x,y,z) = C_1$$

where $C_1$ is a constant. The solution of equation (3) is then of the form

$$\Phi(\phi, z) = C_2$$

where $C_2$ is a constant and we can express this solution in the form

$$\phi(x,y,z) = \psi(z)$$

where $\psi(z)$ is function of z alone. To determine the function $\psi(z)$, we notice that if we give to the variable x, a fixed value (say $\alpha$), then

$$\phi(\alpha, y, z) = \psi(z) \tag{8}$$

is a solution of the differential equation

$$Q(\alpha,y,z)\,dy + R(\alpha,y,z)\,dz = 0 \tag{9}$$

Now we find a solution of this equation in the form

$$K(y,z) = C \tag{10}$$

by the method of first order differential equations.

Since equations (8) and (10) represent general solutions of the same differential equation (9), they must be equivalent. If we, therefore, eliminate the variable y between equations (8) and (10), we get an expression for the function $\psi(z)$. Hence we obtain the solution of the equation (3) on substituting the expression for $\psi(x)$

Note: We may conveniently choose a value for $\alpha$ such as 0 or 1 so that the equation (9) can be easily solved.

Method 8. Reduction to ordinary differential equations (Mayer's Method).* If the condition of integrability is satisfied, by this method we reduce the problem of finding the solution of a Pfaffian differential equation of the type (3) to that of integrating one ordinary differential equation of the first order in two variables.

If the equation (3) is integrable, it has a solution of the form

$$f(x,y,z) = C \tag{11}$$

which is a one-parameter family of surfaces in three dimensional space. These integral surfaces will be intersected in single infinity of curves by the plane

$$z = x + k\,y \tag{12}$$

where k is a constant. The curves thus formed will be solutions of the differential equation

$$p(x,y,k)\,dx + q(x,y,k)\,dy = 0 \tag{13}$$

obtained by eliminating z from the equations (11) and (12).

Suppose the general solution of equation (13) is

$$\phi(x,y,k) = \text{constant.} \tag{14}$$

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*Goursat, E. Differential Equations; translated by E. R. Hedrick. New York: Ginn and Co., 1916. New York: Dover Publications, 1959. p. 229.

Since a point on the axis of the planes (12) is determined by $y = 0$, $x = c$(a constant), if the curve (14) passes through this point, we must have

$$\phi(x,y,k) = \phi(c,0,k) \tag{15}$$

When k varies, equation (14) represents a one-parameter family of curves through this point $y = 0$, $x = c$. By varying c we obtain the family of curves through each point on the axis of equation (12). Eliminating k between equations (15) and (12) we get the general surfaces in the form

$$\phi(x,y,\frac{z-x}{y}) = \phi(c,0\frac{z-x}{y})$$

which are solutions of the Pfaffian differential equation (3).

Note: This method is better than Natani's method in the sense that it involves the solution of only one ordinary differential equation in two variables. But this ordinary differential equation in two variables is often more difficult to solve than the equations in Natani's method.

Method 9. Bertrand's method.* In this method we first solve the linear partial differential equation

$$X(f)=(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y})\frac{\partial f}{\partial x}+(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z})\frac{\partial f}{\partial y}+(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x})\frac{\partial f}{\partial z} = 0 \tag{16}$$

If u and v are two independent integrals of equation (16)

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*Goursat, E. Differential Equations; translated by E. R. Hedrick. New York: Ginn and Co., 1916. New York: Dover Publications, 1959. p. 232.

then

$$\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right)\frac{\partial u}{\partial x} + \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\frac{\partial u}{\partial y} + \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\frac{\partial u}{\partial z} = 0 \qquad (17)$$

$$\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right)\frac{\partial v}{\partial x} + \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\frac{\partial v}{\partial y} + \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\frac{\partial v}{\partial z} = 0 \qquad (18)$$

Since the condition of integrability is satisfied,

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \qquad (19)$$

Eliminating

$$\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}, \quad \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}$$

from the equations (17), (18), and (19), we get

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ P & Q & R \end{vmatrix} = 0 \qquad (20)$$

From equation (20) we obtain two function $\lambda$ and $\mu$ such that

$$P = \lambda \frac{\partial u}{\partial x} + \mu \frac{\partial v}{\partial x}$$

$$Q = \lambda \frac{\partial u}{\partial y} + \mu \frac{\partial v}{\partial y} \qquad (21)$$

$$R = \lambda \frac{\partial u}{\partial z} + \mu \frac{\partial v}{\partial z}$$

By virtue of equation (21), equation (3) will reduce to

$$\lambda \, du + \mu \, dv = 0 \qquad (22)$$

Since $\lambda/\mu$ depends only on the variables u and v, equation (22) is a differential equation in u and v, and it can be solved.

This method is preferable when the equation is symmetric in x,y and z.

## MAIN RESULTS

### Quasi-homogeneous functions

Definition. We shall call a function $f(x,y,z)$ to be quasi-homogeneous in $x,y$ and $z$ of order $m$, if there exist $p,q,r$ such that

$$f(xt^p, yt^q . zt^r) = t^m f(x,y,z) \tag{23}$$

and call $p,q,r$ the dimensions of $x,y,z$ respectively.

Theorem 1. If $f(x,y,z)$ is a quasi-homogeneous function in $x,y,z$ of order $m$, then

$$f(x,y,z) = x^{m/p} \; f(1, \frac{y}{x^{q/p}}, \frac{z}{x^{r/p}})$$

where $p,q,r$ are dimensions of $x,y,z$ respectively.

Proof. By definition,

$$f(xt^p, yt^q, zt^r) = t^m f(x,y,z)$$

Let $t^p = 1/x$ so that $t^q = x^{-q/p}$, $t^r = x^{-r/p}$

Substituting these in the above identity, we get

$$f(1, y\, x^{-q/p}, z\, x^{-r/p}) = x^{-m/p} \; f(x,y,z)$$

Multiplying by $x^{m/p}$, we obtain

$$x^{m/p} \; f(1, y\, x^{-q/p}, z\, x^{-r/p}) = f(x,y,z)$$

which is the desired result.

<u>Extension of Euler's theorem</u>

<u>Theorem 2</u>. If x,y,z are of dimensions p,q,r respectively and f(x,y,z) is quasi-homogeneous function of order m then

$$px\frac{\partial f}{\partial x} + qy\frac{\partial f}{\partial y} + rz\frac{\partial f}{\partial z} = mf(x,y,z)$$

<u>Proof</u>. By theorem 1,

$$f(x,y,z) = x^{m/p}\, f(1,y\,x^{-q/p},z\,x^{-r/p})$$

that is,

$$f(x,y,z) = x^{m/p}\, f(1,u,v)$$

where

$$u = y\,x^{-q/p} \quad \text{and} \quad v = z\,x^{-r/p}$$

Now

$$\frac{\partial f}{\partial x} = \frac{m}{p}\,x^{(m-p)/p} f(1,u,v) + x^{m/p}\,\frac{\partial f(1,u,v)}{\partial u}\left[y\left(\frac{-q}{p}\right)x^{-(q+p)/p}\right] + x^{m/p}\,\frac{\partial f(1,u,v)}{\partial v}z\left(\frac{-r}{p}\right)x^{-(r+p)/p}$$

$$= \frac{m}{p}x^{(m-p)/p}\, f(1,u,v) - \frac{q}{p}\,x^{(m-q-p)/p}\,y\frac{\partial f(1,u,v)}{\partial u} - \frac{r}{p}\,x^{(m-r-p)/p}\,z\frac{\partial f(1,u,v)}{\partial v}$$

$$\frac{\partial f}{\partial y} = x^{m/p}\frac{\partial f(1,u,v)}{\partial u}\,x^{-q/p} = \frac{\partial f(1,u,v)}{\partial u}\,x^{(m-q)/p}$$

and $$\frac{\partial f}{\partial z} = x^{m/p}\,\frac{\partial f(1,u,v)}{\partial v}x^{-r/p} = x^{(m-r)/p}\,\frac{\partial f(1,u,v)}{\partial v}$$

Therefore,

$$px \frac{\partial f}{\partial x} + qy \frac{\partial f}{\partial y} + rz \frac{\partial f}{\partial z}$$

$$= m\, x^{m/p}\, f(1,u,v) - q\, x^{(m-q)/p}\, y \frac{\partial f(1,u,v)}{\partial u}$$

$$- \frac{r}{p}\, x^{(m-r)/p} z \frac{\partial f(1,u,v)}{\partial u} + q\, x^{(m-q)/p} y \frac{\partial f(1,u,}{\partial u}$$

$$+ r\, x^{(m-r)/p}\, z \frac{\partial f(1,u,v)}{\partial v}$$

$$= m\, x^{m/p}\, f(1,u,v)$$

$$= mf(x,y,z).$$

Hence,

$$px \frac{\partial f}{\partial x} + qy \frac{\partial f}{\partial y} + rz \frac{\partial f}{\partial z} = m\, f(1,y,z)$$

<u>Definition of quasi-homogeneous Pfaffian differential equation</u>

$$P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = 0$$

is said to be quasi-homogeneous of order m, if P,Q,R are quasi-homogeneous functions in x,y,z of orders m-p, m-q, and m-r respectively.

<u>Solution of a quasi-homogeneous Pfaffian differential equation</u>

Consider a quasi-homogeneous Pfaffian differential equation

$$P(x,yz)dz + Q(x,y,z)dy + R(x,y,z)dz = 0 \qquad (24)$$

of order m where p,q,r are dimensions of x,y,z respectively.

Then, by definition, P,Q,R are quasi-homogeneous functions of orders m-p, m-q, and m-r respectively.

Then by theorem 1,

$$P(x,y,z) = x^{(m-p)/p}\, P(1, yx^{-q/p}, zx^{-r/p}) \tag{25}$$

$$Q(x,y,z) = x^{(m-q)/p}\, Q(1, yx^{-q/p}, zx^{-r/p}) \tag{26}$$

$$R(x,y,z) = x^{(m-r)/p}\, R(1, yz^{-q/p}, zx^{-r/p}) \tag{27}$$

If we set

$$y = ux^{q/p} \quad , \qquad z = vx^{r/p} \tag{28}$$

we have

$$dy = x^{q/p}\, du + \frac{q}{p}\, x^{(q-p)/p} udx \tag{29}$$

and

$$dz = x^{r/p}\, dv + \frac{r}{p}\, x^{(r-p)/p}\, vdx \tag{30}$$

By virtue of equations (25), (26), (27), (28), (29), and (30), equation (24) reduces to

$$x^{(m-p)/p}\, P(1,u,v)\, dx + x^{(m-q)/p}\, Q(1,u,v)\, (x^{q/p}\, du + \frac{q}{p}\, x^{(q-p)/p} udx) + x^{(m-r)/p}\, R(1,u,v)\, (x^{r/p} dv + \frac{r}{p}\, x^{(r-p)/p}\, vdx) = 0$$

That is,

$$p\, x^{(m-p)/p}\, P(1,u,v)\, dx + p\, x^{m-q)/p}\, Q(1,u,v) x^{q/p} du + q\, x^{(m-q)/p}\, Q(1,u,v)\, x^{(q-p)/p}\, udx$$

$$+ p\, x^{(m-r)/p} \;\; R(1,u,v)\; x^{r/p} dv$$

$$+ r\, x^{(m-r)/p} \;\; R(1,u,v)\; x^{(r-p)/p} \;\; v\, dx = 0$$

That is,

$$x^{(m-p)/p} \left[ pP(1,u,v) + q\, u\, Q(1,u,v) + r\, v\, R(1,u,v) \right] dx$$

$$+ p\, x^{m/p} Q(1,u,v) du + P\, x^{m/p} \; R(1,u,v)\; dv = 0$$

Dividing by

$$x^{m/p} \left[ pP(1,u,v) + q\, u\, Q(1,u,v) + rv\, R(1,u,v) \right]$$

we get

$$\frac{x^{(m-p)/p} \; dx}{x^{m/p}} + \frac{pQ(1,u,v) du}{pP(1,u,v) + quQ(1,u,v) + rvR(1,u,v)}$$

$$+ \frac{pR(1,u,v)\; dv}{p\, P(1,u,v) + quQ(1,u,v) + rv\, R(1,u,v)} = 0$$

That is,

$$\frac{dx}{x} + A(u,v) du + B(u,v) dv = 0 \qquad (31)$$

where

$$A(u,v) = \frac{p\;\; Q(1,u,v)}{pP(1,u,v) + q\; u\; Q(1,u,v) + r\; v\; R(1,u,v)}$$

and

$$B(u,v) = \frac{p\;\; R(1,u,v)}{pP(1,u,v) + q\; u\; Q(1,u,v) + r\; v\; R(1,u,v)}$$

Now equation (31) can be easily solved since one of the variables is separated from the other two.

<u>A necessary condition for the quasi-homogeneity of a Pfaffian differential equation</u>

A necessary condition may be obtained as follows:

Let

$$P = \sum_{i=1}^{n_1} a_i \, x^{\alpha_i} \, y^{\beta_i} \, z^{\gamma_i}$$

$$Q = \sum_{j=1}^{n_2} b_j \, x^{\lambda_j} \, y^{\mu_j} \, z^{\nu_j}$$

$$R = \sum_{k=1}^{n_3} c_k \, x^{\varepsilon_k} \, y^{\eta_k} \, z^{\zeta_k}$$

so that the equation (3) is of the form

$$\left(\sum_{i=1}^{n_1} a_i x^{\alpha_i} y^{\beta_i} z^{\gamma_i}\right) dx + \left(\sum_{j=1}^{n_2} b_j x^{\lambda_j} y^{\mu_j} z^{\nu_j}\right) dy$$

$$+ \left(\sum_{k=1}^{n_3} c_k x^{\varepsilon_k} y^{\eta_k} z^{\zeta_k}\right) dz = 0 \qquad (32)$$

where $a_i, b_j, c_k$ are coefficients and the indices

$$\alpha_i, \beta_i, \gamma_i \,;\ \lambda_j, \mu_j, \nu_j \ \text{ and } \ \varepsilon_k, \eta_k, \zeta_k$$

are integers or fractions, positive or negative or zero.

If we replace x by $xt^p$, y by $yt^q$ and z by $zt^r$ the powers of t in the different terms of equation (32) are given by

$$p\,\alpha_i + q\,\beta_i + r\,\gamma_i \qquad i = 1,2,\ldots\ldots n_1$$

$$p\,\lambda_j + q\,\mu_j + r\,\nu_j \qquad j = 1,2,\ldots\ldots n_2 \qquad (33)$$

$$p\,\varepsilon_k + q\,\eta_k + r\,\zeta_k \qquad k = 1,2,\ldots\ldots n_3$$

Now we have $n_1 + n_2 + n_3$ linear expressions in p,q and r. In order that equation (32) is quasi-homogeneous of order n, we must have P,Q,R as quasi-homogeneous functions of orders n-p, n-q, n-r respectively. In other words the power of t in each term of P in n-p, the power of t in each term of Q in n-q, and the power of t in each term of R is n-r.

Thus we have $n_1 + n_2 + n_3$ linear equations.

$$
\begin{aligned}
(\alpha_i + 1)P + \beta_i q + \gamma_i r - n &= 0 \qquad i = 1,2,\ldots n_1 \\
\lambda_j P + (\mu_j + 1) q + \nu_j r - n &= 0 \qquad j = 1,2,\ldots n_2 \\
\varepsilon_k P + \eta_k q + (\zeta_k + 1) r - n &= 0 \qquad k = 1,2,\ldots n_3
\end{aligned}
\tag{34}
$$

Since we have $n_1 + n_2 + n_3$ equations in four unknowns p,q,r and n, these equations in general may not be consistent, but if they are consistent then there exist unique values of p,q,r and, therefore, of n.

Thus a necessary condition that the given equation is quasi-homogeneous is that the above $n_1 + n_2 + n_3$ linear equations are consistent.

If the $n_1 + n_2 + n_3$ equations are consistent, we may as well divide each one of them by p and unique values of q/p, r/p, n/p may be obtained.

Example

Consider the Pfaffian differential equation

$$(5x^3 + 2y^4 + 2y^2z + 2z^3)dx + (4xy^3 + 2xyz)dy$$
$$+ (xy^2 + 2xz)dz = 0$$

Here

$$P = 5x^3 + 2y^4 + 2y^2z + 2x^3$$

$$Q = 4xy^3 + 2xyz$$

$$R = xy^2 + 2xz$$

$$P(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}) + Q(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}) + R(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})$$

$$= (5x^3 + 2y^4 + 2y^2z + 2z^3)\ (2xy-2xy)$$

$$+ (4xy^3 + 2xyz)\ (y^2 + 2z-2y^2-4z)$$

$$+ (xy^2 + 2xz)\ (8y^3 + 4yz-4y^3-2yz)$$

$$= 0$$

Condition of integrability is satisfied.

The equations (34) give

$$4p \qquad -n = 0$$

$$p + 4q \qquad -n = 0$$

$$p + 2q + r \qquad -n = 0$$

$$p + \qquad 3r \qquad -n = 0$$

$$p + 4q \qquad -n = 0$$

$$p + 2q + r \quad -n = 0$$

$$p + \qquad 2r \quad -n = 0$$

In order that these equations are consistent, we must have

$$4p = p + 4q = p + 2q + r = p + 3r$$

Therefore,

$$2q = r \quad \text{and} \quad 3p = 4q$$

Take

$$p = 1$$

then

$$q = 3/4$$

$$r = 3/2$$

The substitution is

$$y = ux^{3/4} \qquad z = vx^{3/2}$$

The given equation reduces to

$$(5x^3 + 2u^4x^3 + 2u^2x^{3/2} + vx^{3/2} + 2v^2x^3)dx$$
$$+ (4x\, u^3x^{9/4} + 2xux^{3/4}vx^{3/2})\ (x^{3/4}du + \frac{3}{4}\, x^{-1/4}udx)$$
$$+ (xu^2x^{3/2} + 2xvx^{3/2})\ (v\, \frac{3}{2}\, x^{1/2}dx + x^{3/2}\, dv) = 0$$

That is

$$x^3(5 + 5u^4 + 5u^2v + 5v^2)dx + x^4(4u^3 + 2uv)du$$

$$+ x^4(u^2 + 2v)dv = 0$$

That is

$$\frac{dx}{x} + \frac{(4u^3 + 2uv)du}{5(1 + u^4 + u^2v + v^2)} \qquad \frac{(u^2 + 2v)dv}{5(1 + u^4 + u^2v + v^2)} = 0$$

That is

$$\frac{dx}{x} + \frac{d(1 + u^4 + u^2v + v^2)}{5(1 + u^4 + u^2v + v^2)} = 0$$

Integrating,

$$\log x + \frac{1}{5}\log(1 + u^4 + u^2v + v^2) = \text{constant.}$$

That is,

$$x^5(1 + u^4 + u^2v + v^2) = C$$

$$x^5\left(1 + \frac{y^4}{x^3} + \frac{y^2\ z}{x^{3/2}x^{3/2}} + \frac{z^2}{x^3}\right) = C$$

That is,

$$x^5 + x^2y^4 + x^2y^2z + x^2z^2 = C$$

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