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An Investigation of the Range of a Boolean Function

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AN INVESTIGATION OF THE RANGE OF A BOOLEAN FUNCTION

by

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INTRODUCTION TO BOOLEAN ALGEBRA

The purpose of this section is to define a boolean algebra and to determine some of the important properties of it.

A boolean algebra is a set B with two binary operations, join and meet, denoted by + and juxtaposition respectively, and a unary operation, complementation, denoted by ', which satisify the following axioms:

(1) for all $a,b \in B$ (that is, for all a,b elements of B) $a + b = b + a$ and $a\ b = b\ a$, (the commutative laws),

(2) for all $a,b,c \in B$, $a + b\ c = (a + b)\ (a + b)$ and $a\ (b + c) = a\ b + a\ c$, (the distributive laws),

(3) there exists $0 \in B$ such that for each $a \in B$, $a + 0 = a$, and there exists $1 \in B$ such that for each $a \in B$, $a\ 1 = a$,

(4) for each $a \in B$, $a + a' = 1$ and $a\ a' = 0$.

If $a + e = a$ for all a in B then $0 = 0 + e = e + 0 = e$, so that there is exactly one element in B which satisfies the first half of axiom 3, namely 0. Similarly there is exactly one element in B which satisfies the second half of axiom 3, namely 1.

The 0 and 1 as defined above will be called the distinguished elements.

If in the statement of any of the four axioms join, meet, 0, and 1 are replaced by meet, join, 1, 0 respectively, the axiom remains unchanged. Thus it follows that if a statement can be proved from the axioms, then the statement with join, meet, 0, and 1 replaced by meet, join, 1, and 0 respectively can also be proved. The two statements are called dual statements.

From the axioms it follows that: $a\,a = a\,a + 0 = a\,a + a\,a' = a\,(a + a') = a\,1 = a$; thus the idempotent laws follow:

(i) $a\,a = a$ and its dual

(i') $a + a = a$

Let $c + x = 1$ and $c\,x = 0$, then $c' = c'\,1 = c'\,(x + c) = c'\,x + c'\,c = x\,c' + x\,c = x\,(c' + c) = x\,1 = x$. Thus axiom 4 defines the operation complementation in the respect that $x = c'$ if and only if $c + x = 1$ and $c\,x = 0$.

The above property is also useful to show two elements are equal.

Since $1 = a + a' = a' + a$ and $0 = a\,a' = a'\,a$ it follows from above that:

(ii) $(a')' = a$

Notice that (ii) is a self-dual statement.

It follows that $1 = b + 1$, since $b\,(1 + b) = b\,1 + b\,b = b + b = b$ and $b'\,(1 + b) = b'\,1 + b'\,b = b' + 0 = b'$ implies that $1 = b + b' = b\,(1 + b) + b'\,(b + 1) = (b + b')\,(1 + b) = 1\,(1 + b) = 1 + b$, therefore:

(iii) $1 = b + 1$ and its dual

(iii') $0 = b\,0$

The absorption laws thus follow: $a + a\,b = a\,1 + a\,b = a\,(1 + b) = a\,1 = a$:

(iv) $a + a\,b = a$ and its dual

(iv') $a\,(a + b) = a$

Also: $a + b\,a' = (a + b)\,(a + a') = (a + b)\,1 = a + b$. Hence:

(v) $a + b = a + b\,a'$ and its dual

(v') $a\,b = a\,(b + a')$

Next the associative law for join will be established. Using the fact $a'\,(b\,a) = a'\,(b\,a) + a\,a' = a'\,(b\,a + a) = a'\,a = 0$ in the form $x'\,[(y + z)\,x] = 0$ it follows that:

$(x + y) + z = (x + x'\,y) + z = [x + (0 + x'\,y)] + z =$

$\left(x + \{x'\,[(y + z)\,x] + x'\,y\}\right) + z =$

$\{x + x'[(y + z)\,x + y]\} + z = \{x + [(y + z)\,x + y]\} + z =$

$\{x + [(y + z)\,x + (y + z)\,y]\} + (x\,z + z) =$

$[x\,(x + y) + (y + z)\,(x + y)] + [x\,z + (y + z)\,z] =$

$[x + (y + z)]\,(x + y) + [x + (y + z)]\,z =$

$[x + (y + z)]\,[(x + y) + z]$. Thus $a + (b + c) = (c + b) + a = [c + (b + a)]\,[(c + b) + a] = [a + (b + c)]\,[(a + b) + c] = (a + b)\,c$, and the associative laws are proved:

(vi) $a + (b + c) = (a + b) + c$ and its dual

(vi') $a\,(b\,c) = (a\,b)\,c$

Define $a + b + c$ to be the common value of $a + (b + c)$ and $(a + b) + c$. Also define $a\,b\,c = a\,(b\,c) = (a\,b)\,c$.

From (iii) and (iii') it follows that $1 = 0 + 1$ and $0 = 1\ 0 = 0\ 1$; hence:

(vii) $1 = 0'$ and its dual

(vii') $0 = 1'$

Since $(a + b)\ a'\ b' = a\ a'\ b' + b\ a'\ b' = 0 + 0 = 0$ and $(a + b) + a'\ b' = (a + b + a')\ (a + b + b') = 1\ 1 = 1$, the De Morgan laws follow:

(viii) $(a + b)' = a'\ b'$ and its dual

(viii') $(a\ b)' = a' + b'$

Define $x < y$ to mean $x + y = y$.

If $x < y$ and $y < z$ then $x + y = y$ and $y + z = z$, thus $x + z = x + y + z = y + z = z$ and $x < z$. Therefore the relation $<$ is transitive.

If $x < y$ and $y < x$ then $x + y = y$ and $y + x = x$ thus $x = y$. Hence the relation $<$ is a partial order in a boolean algebra. A partial order is a reflexive, anti-symmetric, transitive relation.

Define $x > y$ to mean $x\ y = y$. In a sense $>$ is the dual relation of $<$.

If $x < y$ then $x + y = y$ and $y\ x = (x + y)\ x = x$ thus $y > x$. Similarly if $x > y$ then $y < x$.

Since $x\ y = x$ is a necessary and sufficient condition that $x + y = y$, either condition will be used for $x < y$. Furthermore $x < y$ and $y > x$ will be used interchangeably.

By (iii) and (iii'), it follows that $0 < a < 1$ for all elements a in B.

Let $x < y$. Then $x + y = y$, or by De Morgan's law $x' y' = y'$ and $y' < x'$. Also $a (x + y) = a y$ or $a x + a y = a y$ thus $a x < a y$. Furthermore $a + x + y = (a + x) + (a + y) = a + y$ thus $a + x < a + y$.

If $x < y$ then $x y' = (x y) y' = 0$, also if $x y' = 0$ then $x y = x y + x y' = x (y + y') = x$ or $x < y$. Similarly if $x < y$ then by De Morgan's law on the above result $x' + y = 1$, and if $x' + y = 1$ then $x < y$.

Let $x_1, \cdots, x_n$ be variables whose common domain is a subset D of a boolean algebra B. A function f is called a boolean function if the rule for the function, $f(x_1, \cdots, x_n)$, can be be built up from the variables $x_1, \cdots, x_n$ and elements of B by a finite number of operations meet, join, and complemention. The range of the boolean function f is the set $R = \{r : r \in B$, and there exist $d_1, \cdots, d_n \in D$ for which $f(d_1, \cdots, d_n) = r\}$. $\{x : P(x)\}$ means the set of all x such that x has property P; hence the set R is the set of all elements r of B for which there exist $d_1, \cdots, d_n$ elements of D and $f(d_1, \cdots, d_n) = r$.

Theorem I.1

Every boolean function of one variable in B has a rule of the form $f(x) = f(1) x + f(0) x'$.

Proof: Since $a = a (x + x') = a x + a x'$, and $x = 1 x + 0 x'$, the statement is true if $f(x) = a$ or $f(x) = x$.

If $g(x)$ and $h(x)$ are of the required form, that is if $g(x) = g(1)\, x + g(0)\, x'$ and $h(x) = h(1)\, x + h(0)\, x'$, then $[g(x)]'$, $g(x) + h(x)$, and $g(x)\, h(x)$ are also of the required form, since:

$[g(x)]' = [g(1)\, x + g(0)\, x']' =$

$\{[g(1)]' + x'\}\ \{[g(0)]' + x\} =$

$[g(1)]'\, [g(0)]' + [g(1)]'\, x + [g(0)]'\, x' =$

$[g(1)]'\, [g(0)]'\, x + [g(1)]'\, [g(0)]'\, x' + [g(1)]'\, x + [g(0)]'\, x' =$

$[g(1)]'\, x + [g(0)]'\, x'$. Also, $g(x) + h(x) =$

$g(1)\, x + g(0)\, x' + h(1)\, x + h(0)\, x' =$

$[g(1) + h(1)]\, x + [g(0) + h(0)]\, x'$. And, $g(x)\, h(x) =$

$[g(1)\, x + g(0)\, x']\, [h(1)\, x + h(0)\, x'] =$

$g(1)\, h(1)\, x + g(0)\, h(0)\, x'$.

Since a boolean function of one variable is a finite number of applications of meet, join, and complementation on x and elements of B, it follows by induction that all boolean functions of one variable are in the form $f(x) = f(1)\, x + f(0)\, x'$.

Define, in a boolean algebra B, $\sum_{(e)_B} f(e_1, \cdots, e_n)$ and $\prod_{(e)_B} f(e_1, \cdots, e_n)$ to be the join and meet respectively over all combinations such that either $e_i = 0$ or $e_i = 1$, where 0 and 1 are the distinguished elements of B.

Example: $\sum_{(e)_B} f(e_1,e_2) = f(1,1) + f(1,0) + f(0,1) + f(0,0)$

and $\prod_{(e)_B} f(e_1,e_2) = f(1,1)\, f(1,0)\, f(0,1)\, f(0,0)$.

Let e equal 0 or 1, define x^e to be x if $e = 1$ and x' if $e = 0$.

It then follows that a boolean function of one variable is in the form $f(x) = f(1)\,x + f(0)\,x' = \sum_{(e)_B} f(e_1)\,x^{e_1}$.

Theorem I.2

If f is a boolean function of n variables then

$$f(x_1,\cdots,x_n) = \sum_{(e)_B} f(e_1,\cdots,e_n)\, x_1^{e_1} \cdots x_n^{e_n}.$$

Proof: The theorem will be proved by induction. If $n=1$, then from above the statement is true. Assume that for $n=k$ the statement is true. Then since $f(x_1,\cdots,x_k,x_{k+1})$ can be thought of as a boolean function of one variable x_{k+1}, $f(x_1,\cdots,x_{k+1}) = f(x_1,\cdots,x_k,1)\,x_{k+1} + f(x_1,\cdots,x_k,0)\,x'_{k+1}$. By the induction hypothesis, $f(x_1,\cdots,x_{k+1}) =$

$$\sum_{(e)_B} f(e_1,\cdots,e_k,1)\, x_1^{e_1} \cdots x_k^{e_k}\, x_{k+1} + \sum_{(e)_B} f(e_1,\cdots,e_k,0)\, x_1^{e_1} \cdots x_k^{e_k}\, x'_{k+1} =$$

$\sum\limits_{(e)_B} f(e_1, \cdots, e_{k+1})\, x_1^{e_1} \cdots x_{k+1}^{e_{k+1}}$. Thus the statement is true for $k+1$ whenever it is true for k, and hence true for all positive integers.

Theorem I.3

$$\sum_{(e)_B} a_1^{e_1} \cdots a_n^{e_n} = 1, \; a_i \quad B.$$

Proof: This also will be proved by induction. If $n=1$ then $\sum\limits_{(e)_B} a_1^{e_1} = a_1 + a_1' = 1$. Assume the statement is true for $n=k$. Then $\sum\limits_{(e)_B} a_1^{e_1} \cdots a_k^{e_k} a_{k+1}^{e_{k+1}} =$

$$\Big[\sum_{(e)_B} a_1^{e_1} \cdots a_k^{e_k}\Big] a_{k+1} + \Big[\sum_{(e)_B} a_1^{e_1} \cdots a_k^{e_k}\Big] a_{k+1}' =$$

$a_{k+1} + a_{k+1}' = 1$.

DEFINITION OF A RELATIVE BOOLEAN ALGEBRA

This section will define the concept of a relative boolean algebra of a boolean algebra, and give some properties of them.

Let a boolean algebra B and a, b elements of B, be given. Define the set $S_{a,b}$ by $S_{a,b} = \{x : x \in B,\ a < x < b\}$. Hence $S_{a,b}$ is the set of all elements x of B where $a < x < b$.

Note that if $a < b$ then $a, b \in S_{a,b}$. Let $x \in S_{a,b}$, by transitivity $a < b$. Thus $S_{a,b} \neq \varphi$, the empty set, if and only if $a < b$. In section I it was shown that for all $x \in B$, $0 < x < 1$; thus $S_{0,1} = B$.

Let $S_{a,b} \neq \varphi$, that is $a < b$. Since $a\,(b\,x + a) = a$ and $b\,(b\,x + a) = b\,x + a\,b = b\,x + a$, $a < b\,x + a < b$. Thus for all $x \in B$, $b\,x + a \in S_{a,b}$. If $x \in S_{a,b}$, then $a < x < b$ or $b\,x + a = x + a = x$.

Throughout this section it will be assumed that a boolean algebra B has been given and that all elements are elements of B.

Theorem II.1

$S_{a,b}$ is closed under meet and join of B; that is if $x, y \in S_{a,b}$, then $x + y$, $x\,y \in S_{a,b}$.

Proof: Let $x, y \in S_{a,b}$. From above $a < x, y < b$ and it follows that $(x + y) + b = (x + b) + (y + b) = b$, and $(x + y) a = x a + y a = a$; therefore $a < x + y < b$. Similarly, $x y + b = (x + b) (y + b) = b$, $x y a = (x a) (y a) = a$ and $a < xy < b$. Hence $x + y$ and $x y$ are elements of $S_{a,b}$.

If $S_{a,b} = \varphi$ then $S_{a,b}$ is closed under meet and join.

From above note that $C(x) = b x' + a \in S_{a,b}$ if $S_{a,b} \neq \varphi$. The element $C(x)$ will be called the relative complement in B with respect to $S_{a,b}$, or simply the relative complement if no confusion will result. $C(x)$ is defined, of course, for each element x of B. For convenience, x° or x^{+} will be used to denote $C(x)$. $C(x)$ will be shown to have properties of complementation in the set $S_{a,b}$.

Theorem II.2

$S_{a,b}$ is a boolean algebra, with distinguished elements a, b, where the meet and join are the same as in B and complementation being the relative complement in B with respect to $S_{a,b}$, if and only if $S_{a,b} \neq \varphi$.

Proof: Assume $S_{a,b} \neq \varphi$. From above, the three operations, meet, join, and relative complement, are closed. Meet and join are commutative and distributive since they are the operations in B. If $x \in S_{a,b}$, then $x + a = x$ and $x b = x$, thus a and b are the distinguished elements of $S_{a,b}$ as defined by axiom 3. Let $x^{\circ} = b x' + a$, the relative complement of x. For each s of $S_{a,b}$, $x x^{\circ} = x (b x' + a) = b x x' + a x = a$ and $x + x^{\circ} = x + b x' + a = x + b = b$. Thus axiom 4 is satisfied. Therefore the set $S_{a,b}$ together with the operations meet, join, and the relative complement in B with respect to $S_{a,b}$ is a boolean algebra.

If $S_{a,b}$ together with the three operations form a boolean algebra, then $S_{a,b} \neq \varphi$.

Since the meet and join in the boolean algebra $S_{a,b}$ are the same as in B, the partial order, <, is also the same, that is if $x, y \in S_{a,b}$ and if $x < y$ in B then $x < y$ in $S_{a,b}$.

A relative boolean algebra of a boolean algebra B is a subset, $S_{a,b}$, of B, $a < b$, together with the operations meet and join of B and the relative complement in B with respect to $S_{a,b}$. From theorem II.2, a relative boolean algebra is a boolean algebra.

Theorem II.3

Let $S_{a,b}$ be a relative boolean algebra of B and $\overline{S}_{c,d}$ be a relative boolean algebra of $S_{a,b}$.

Then $\bar{S}_{c,d}$ is a relative boolean algebra of B.

Proof: Since $\bar{S}_{c,d}$ is a relative boolean algebra of $S_{a,b}$, $a < c < d < b$. If $x \in B$ and $c < x < d$ then $x \in S_{a,b}$, thus $\bar{S}_{c,d} = \{x : x \in S_{a,b},\ c < x < d\} = \{x : x \in B,\ c < x < d\}$. The meet and join are the same operations in both $S_{a,b}$ and $\bar{S}_{c,d}$, and in B. Thus it remains to be shown that the relative complement in $S_{a,b}$ with respect to $\bar{S}_{c,d}$ is the same as the relative complement in B with respect to $\bar{S}_{c,d}$.

Let $x^{\circ} = b\,x' + a$ be the complement in $S_{a,b}$. Then the relative complement in $S_{a,b}$ with respect to $\bar{S}_{c,d}$ is $d\,x^{\circ} + c = d\,(bx' + a) + c =$ $d\,b\,x' + d\,a + c = d\,x' + c$ since $a < c < d < b$. Thus the complement in $\bar{S}_{c,d}$ is the relative complement in B with respect to $\bar{S}_{c,d}$.

Theorem II.4

If $S_{a,b}$ and $S_{c,d}$ are relative boolean algebras of B and $S_{c,d} \subset S_{a,b}$, [$\subset$ means is a subset of], then $S_{c,d}$ is a relative boolean algebra of $S_{a,b}$.

Proof: Since $S_{a,b}$ and $S_{c,d}$ are relative boolean algebras of B, the meet and join are the same as in B. Thus, since $S_{a,b} \subset S_{c,d}$, the meet and join in $S_{c,d}$ is the same as in $S_{a,b}$. Also $S_{c,d} = \{x : x \in S_{a,b},$

$c < x < d\}$. Hence it remains to show that the complement in $S_{c,d}$ is the relative complement in $S_{a,b}$ with respect to $S_{c,d}$; that is $d\,(b\,x' + a) + c = d\,x' + c$. This is immediate since $a < c < d < b$.

The next theorem gives a connection between relative boolean algebras in B.

Theorem II.5

Let $S_{a,b}$ be a relative boolean algebra of B and $x^\circ = b\,x' + a$. If $x \in S_{c,d}$, then $x^\circ \in S_{c^\circ,d^\circ}$.

Proof: Let $x \in S_{c,d}$. Then $c < d$, or by section I, $d' < x' < c'$, and $b\,d' + a < b\,x' + a < b\,c' + a$. Thus $d^\circ < x^\circ < c^\circ$ or $x^\circ \in S_{c^\circ,d^\circ}$.

Theorem II.6

Let $S_{a,b}$ be a relative boolean algebra of B and $x^\circ = b\,x' + a$. If $S_{h,k}$ is a relative boolean algebra of B and x^+ is the complement of x in $S_{h,k}$ then S_{k°,h° is a relative boolean algebra with the complement of x being $[(x^\circ)^+]^\circ$.

Proof: Since $S_{h,k} \neq \varphi$, it follows from theorem II.5 that $S_{k^\circ,h^\circ} \neq \varphi$, and thus S_{k°,h° is a relative boolean algebra with the complement of x being $k^\circ\,x' + h^\circ$. But $[(x^\circ)^+]^\circ = [k\,(b\,x' + a) + h]^\circ =$

$b\,[k\,(b\,x' + a) + h]' + a = b\,[h'\,(b\,x' + a) + k'] + a =$

$b\,h'\,x' + b\,h'\,a + b\,k' + a =$

$(b\,h'\,x + a\,x') + (b\,k' + a) = h^{\circ}\,x' + k^{\circ}.$

The function f defined by $f(x) = (((x^{+})^{\circ})^{+})^{\circ}$, where $'$ and $^{+}$ are defined as in theorem II.6, will be seen later to be a homomorphism from $S_{h,k}$ onto $S_{k^{\circ},h^{\circ}}$.

Theorem II.7

$x \in S_{a,b} \cap S_{c,d}$, where $\cap$ is the set intersection, if and only if $a + c < x < b\,d$; that is, $S_{a,b} \cap S_{c,d} = S_{a+c,\,b\,d}$.

Proof: If $x \in S_{a,b} \cap S_{c,d}$, then $x < b$ and $x < d$. Therefore $x\,(b\,d) = (x\,b)\,(x\,d) = x$ or $x < b\,d$. Also $x > a$ and $x > c$, and it follows that $x > a + c$. Hence $a + c < x < b\,d$.

If $a + c < x < b\,d$ then $a < x < b$ and $c < x < d$ or $x \in S_{a,b}$ and $x \in S_{c,d}$.

Theorem II.8

The following are equivalent if $S_{a,b} \neq \varphi$ and $S_{c,d} \neq \varphi$.

(a) $S_{a,b} \cap S_{c,d} \neq \varphi$

(b) $a + c < b\,d$

(c) $a < d$ and $c < b$

(d) $a + c \in S_{a,b} \cap S_{c,d}$

(e) $b\,d \in S_{a,b} \cap S_{c,d}$

(f) $S_{a,b} \cap S_{c,d}$ is a relative boolean algebra.

Proof: Let $S = S_{a,b} \cap S_{c,d}$.

(a) implies (b): Let $S \neq \varphi$. Then there exists $x \in S$, and by theorem II.7 $a + c < x < b\,d$. Hence $a + c < b\,d$.

(b) implies (c): Let $a + c < b\,d$. Then $0 = (a + c)(b\,d)' = (a + c)(b' + d') = a\,b' + c\,b' + a\,d' + c\,d' = c\,b' + a\,d'$. Hence $c\,b' = 0$ and $a\,d' = 0$, or $a < d$ and $c < b$.

(c) implies (d): Let $a < d$ and $c < b$. Since $a < b$ and $c < d$ it follows that $(a + c) + b\,d = (a + c + b)(a + c + d) = (b + c)(a + d) = b\,d$, or $a + c < b\,d$. By theorem II.7, $a + c \in S$.

(d) implies (e): If $a + c \in S$ then by theorem II.7 $a + c < b\,d$ and hence $b\,d \in S$ by theorem II.7.

(e) implies (f): Since $b\,d \in S$, $S \neq \varphi$, and by theorems II.7 and II.2, S is a relative boolean algebra.

(f) implies (a): Since S is a relative boolean algebra, by theorem II.2, $S \neq \varphi$.

BOOLEAN FUNCTIONS

In this section the connection between boolean functions and the sets of the form $S_{a,b}$ will be shown; namely that the range of a boolean function is a relative boolean algebra.

In this and the following sections f will denote a boolean function of n variables, that is $f(x_1,\cdots,x_n) =$

$$\sum_{(e)_B} f(e_1,\cdots,e_n)\, x_1^{e_1} \cdots x_n^{e_n}.$$

Theorem III.1

If f is a function in B such that $f(a_1,\cdots,a_n) = a$ and $f(b_1,\cdots,b_n) = b$, then there exist $c_1,\cdots,c_n$ such that $f(c_1,\cdots,c_n) = a\,c + b\,c'$ for any c an element of B, and furthermore $c_i \in S_{a_i\, b_i, a_i + b_i}$.

Proof: Let $c_i = a_i\,c + b_i c'$. Since $a_i\, b_i\, c_i =$ $a_i\, b_i$ and $a_i + b_i + c_i = a_i + b_i$, it follows that $a_i\, b_i < c_i < a_i + b_i$.

Since $c_i' = (a_i\, c + b_i\, c')' = a_i'\, c + b_i'\, c'$

it follows that $c_i^{e_i} = a_i^{e_i}\, c + b_i^{e_i}\, c'$. Thus $f(c_1,\cdots,c_n) =$

$$\sum_{(e)_B} f(e_1,\cdots,e_n)\, (a_1\, c + b_1\, c')^{e_1} \cdots (a_n\, c + b_n\, c')^{e_n} =$$

$$\sum_{(e)_B} f(e_1,\cdots,e_n)\,(a_1^{e_1}\, c + b_1^{e_1}\, c')\cdots(a_n^{e_n}\, c + b_n^{e_n}\, c') =$$

$$c \sum_{(e)_B} f(e_1,\cdots,e_n)\, a_1^{e_1}\cdots a_n^{e_n} +$$

$$c' + \sum_{(e)_B} f(e_1,\cdots,e_n)\, b_1^{e_1}\cdots b_n^{e_n} =$$

$a\,c + b\,c'$. The elements $c_1,\cdots,c_n$ exhibited have the properties required in the statement of the theorem, and thus the theorem is proved.

The next two theorems are immediate results of theorem III.1.

Theorem III.2

If $f(a_1,\cdots,a_n) = a$ and $f(b_1,\cdots,b_n) = b$ then there exist $c_1,\cdots,c_n$ and $d_1,\cdots,d_n$ such that $f(c_1,\cdots,c_n) = a + b$ and $f(d_1,\cdots.d_n) = a\,b$ where $c_i,d_i \in S_{a_i\, b_i, a_i + b_i}$.

Proof: In theorem III.1 let c be equal to a then to b.

The next theorem follows by induction on theorem III.2.

Theorem III.3

If $f(p_{1i},\cdots,p_{ni}) = p_i$ for $i=1,\cdots,m$, then there exist $r_1, \cdots, r_n$ and $s_1, \cdots, s_n$ such that

$f(r_1, \cdots, r_n) = \sum_{i=1}^{m} p_i$ and $f(s_1, \cdots, s_n) = \prod_{i=1}^{m} p_i$. Further-more, $r_j, s_j \in S_{c_j, d_j}$ where $c_j = \prod_{j=1}^{m} p_{ji}$ and $d_j = \sum_{j=1}^{m} p_{ji}$.

The following theorems in this section are concerned with the restriction of the domain of f.

Theorem III.4

Let $x_i \in S_{a,b} \neq \varphi$ and f a boolean function. Then

$$f(e_1, \cdots, e_n)\, x_1^{e_1} \cdots x_n^{e_n} = f(\eta_1, \cdots, \eta_n)\, x_1^{e_1} \cdots x_n^{e_n},$$

where $\eta_i = a$ if $e_i = 0$ and $\eta_i = b$ if $e_i = 1$.

Proof: Let $x_i \in S_{a,b}$ and $\eta_i = a$ if $e_i = 0$ and $\eta_i = b$ if $e_i = 1$. Consider $\eta_i^{p_i} x_i^{e_i}$, where p_i is either 0 or 1. There are four cases to consider:

(1) if $p_i = 0$ and $e_i = 0$

(2) if $p_i = 0$ and $e_i = 1$

(3) if $p_i = 1$ and $e_i = 0$

(4) if $p_i = 1$ and $e_i = 1$.

Case (1), since $\eta_i = a$, $\eta_i^{p_i} x_i^{e_i} = a' x_i' = x_i' = x_i^{e_i}$.

Case (2), since $\eta_i = b$, $\eta_i^{p_i} x_i^{e_i} = b' x_i = 0$.

Case (3), since $\eta_i = a$, $\eta_i^{p_i} x_i^{e_i} = a\, x_i' = 0$.

Case (4), since $\eta_i = b$, $\eta_i^{p_i} x_i^{e_i} = b\, x_i = x_i = x_i^{e_i}$.

It then follows that if $p_i = e_i$ then $\eta_i^{p_i} x_i^{e_i} = x_i^{e_i}$, and if $p_i \neq e_i$ then $\eta_i^{p_i} x_i^{e_i} = 0$. Hence, if for each i, $p_i = e_i$, it follows that $\eta_1^{p_1} x_1^{e_1} \cdots \eta_n^{p_n} x_n^{e_n} = x_1^{e_1} \cdots x_n^{e_n}$, otherwise $\eta_1^{p_1} x_1^{e_1} \cdots \eta_n^{p_n} x_n^{e_n} = 0$. Thus

$f(\eta_1, \cdots, \eta_n)\, x_1^{e_1} \cdots x_n^{e_n} =$

$[\sum_{(p)_B} f(p_1, \cdots, p_n)\, \eta_1^{p_1} \cdots \eta_n^{p_n}]\, x_1^{e_1} \cdots x_n^{e_n} =$

$\sum_{(p)_B} f(p_1, \cdots, p_n)\, \eta_1^{p_1} x_1^{e_1} \cdots \eta_n^{p_n} x_n^{e_n} =$

$f(e_1, \cdots, e_n)\, x_1^{e_1} \cdots x_n^{e_n}$, since there is only one combination such that $p_i = e_i$ for each i.

Let $c = \prod_{(\eta)_{S_{a,b}}} f(\eta_1, \cdots, \eta_n)$ and $d = \sum_{(\eta)_{S_{a,b}}} f(\eta_1, \cdots, \eta_n)$, where the meet and join extend over all combinations such that either $\eta_i = a$ or $\eta_i = b$.

The next theorems show that if f is restricted to $S_{a,b} \neq \varphi$, then the range is $S_{c,d}$, where c and d are defined above.

Theorem III.5

Let f be a boolean function with a, b, c, and d defined above. Then there exist $c_1, \cdots, c_n$ and $d_1, \cdots, d_n$ such that $f(c_1, \cdots, c_n) = c$ and $f(d_1, \cdots, d_n) = d$, where $c_i, d_i \in S_{a,b}$.

Proof: Application of theorem III.3.

Theorem III.6

Let f be a boolean function with a, b, c, and d defined above. Then there exist $h_1, \cdots, h_n$ such that $f(h_1, \cdots, h_n) = h$ and $h_i \in S_{a,b}$.

Proof: Let c_i and d_i be defined as in theorem III.5. By theorem III.5 and III.1 there exist $h_1, \cdots, h_n$ such that $f(h_1, \cdots, h_n) = d\ h + c\ h' = h$ and $h_i \in S_{c_i\ d_i, c_i + d_i}$ which is a subset of $S_{a,b}$.

Theorem III.7

Let f be a boolean function and let

$c = \prod_{(\eta)_{S_{a,b}}} f(\eta_1, \cdots, \eta_n)$ and $d = \sum_{(\eta)_{S_{a,b}}} f(\eta_1, \cdots, \eta_n)$. If $x_i \in S_{a,b}$ then $f(x_1, \cdots, x_n) \in S_{c,d}$.

Proof: Let $x_i \in S_{a,b}$. Recall that

$c' = \sum_{(e)_B} [f(\eta_1, \cdots, \eta_n)]'$ and that $\eta_i = a$ if $e_i = 0$ and $\eta_i = b$ if $e_i = 1$. It then follows by theorem III.4

that: $f(x_1,\cdots,x_n) + c' =$

$$\sum_{(e)_B} f(e_1,\cdots,e_n)\, x_1^{e_1} \cdots x_n^{e_n} + c'$$

$$\sum_{(e)_B} f(\eta_1,\cdots,\eta_n)\, x_1^{e_1} \cdots x_n^{e_n} + \sum_{(e)_B} [f(\eta_1,\cdots,\eta_n)]' =$$

$$\sum_{(e)_B} x_1^{e_1} \cdots x_n^{e_n} + \sum_{(e)_B} [f(\eta_1,\cdots,\eta_n)]' = 1 + c' = 1.$$

Therefore $f(x_1,\cdots,x_n) > c.$

Also it follows that: $f(x_1,\cdots,x_n) + d =$

$$\sum_{(e)_B} f(e_1,\cdots,e_n)\, x_1^{e_1} \cdots x_n^{e_n} + d =$$

$$\sum_{(e)_B} f(\eta_1,\cdots,\eta_n)\, x_1^{e_1} \cdots x_n^{e_n} + \sum_{(e)_B} f(\eta_1,\cdots,\eta_n) =$$

$\sum_{(e)_B} f(\eta_1,\cdots,\eta_n) = d.$ Hence $f(x_1,\cdots,x_n) < d.$

Thus $f(x_1,\cdots,x_n)$ is an element of $S_{c,d}$ whenever $x_i \in S_{a,b}$. Hence the theorem is proved.

Theorem III.8

The range of a function, f, in B when the domain is restricted to $S_{a,b} \neq \varphi$ is $S_{c,d}$, where $c =$ $\prod_{(\eta)_{S_{a,b}}} f(\eta_1,\cdots,\eta_n)$ and $d = \sum_{(\eta)_{S_{a,b}}} f(\eta_1,\cdots,\eta_n)$ where the join and meet are over all combinations such that either $\eta_i = a$ or $\eta_i = b$.

Some immediate results of theorem III.8 are:

Theorem III.9

The range of a function, f, in b is $S_{h,k}$ where $h = \prod_{(e)_B} f(e_1,\cdots,e_n)$ and $k = \sum_{(e)_B} f(e_1,\cdots,e_n)$.

Theorem III.10

Let f be a boolean function whose domain is B. The range of f is B if and only if $\prod_{(e)_B} f(e_1,\cdots,e_n) = 0$ and $\sum_{(e)_B} f(e_1,\cdots,e_n) = 1$.

Another important theorem is:

Theorem III.11

If $f(0,\cdots,0) < b$ and $f(1,\cdots,1) > a$ then $f(x_1,\cdots,x_n) \in S_{a,b}$, whenever $x_i \in S_{a,b}$.

Proof: Let $x_i \in S_{a,b}$ and $f(0,\cdots,0) < b$ and $f(1,\cdots,1) > a$. It follows that $f(x_1,\cdots,x_n) = f(1,\cdots,1)\ x_1 \cdots x_n + f(x_1,\cdots,x_n) > a$.

Whenever there is an i such that $e_i \neq 0$ then

$b\ x_1^{e_1} \cdots x_n^{e_n} = x_1^{e_1} \cdots x_n^{e_n}$. Since $f(0,\cdots,0)\ b = f(0,\cdots,0)$, it follows that $b\ f(x_1,\cdots,x_n) =$

$\sum_{(e)_B} f(e_1,\cdots,e_n)\ b\ x_1^{e_1} \cdots x_n^{e_n} =$

$\sum_{(e)_B} f(e_1, \cdots, e_n)\, x_1^{e_1} \cdots x_n^{e_n} = f(x_1, \cdots, x_n)$. Thus

$f(x_1, \cdots, x_n) < b$.

Therefore $f(x_1, \cdots, x_n) \in S_{a,b}$.

BOOLEAN FUNCTIONS IN A RELATIVE BOOLEAN ALGEBRA

In this section it will be understood that $\eta_i = a$ whenever $e_i = 0$ and $\eta_i = b$ whenever $e_i = 1$. This section will consider the relationship between functions in B and functions in a relative boolean algebra of B.

Let $S_{a,b}$ be a relative boolean algebra of B and $x^{o} = b\,x' + a$.

A useful theorem is:

Theorem IV.1

If $x_i \in S_{a,b}$ then $b\,x_1^{e_1} \cdots x_n^{e_n} + a = x_1^{\eta_1} \cdots x_n^{\eta_n}$, where $x_i^{\eta_i} = x_i$ if $\eta_i = b$ and $x_i^{\eta_i} = x^{o}$ if $\eta_i = a$.

Proof: Let $x_i \in S_{a,b}$. If $e_i = 0$ for some i, then it follows that there exists a permutation, $i_1, i_2, \cdots, i_j, i_{j+1}, \cdots, i_n$, of the first n positive integers such that $e^{i_k} = 1$ if $1 \leq k \leq j$ and $e^{i_k} = 0$ if $j+1 \leq k \leq n$. Hence: $b\,x_1^{e_1} \cdots x_n^{e_n} + a =$

$$b\,x_{i_1} \cdots x_{i_j}\, x'_{i_{j+1}} \cdots x'_{i_n} + a\,x_{i_1} \cdots x_{i_j} =$$

$$x_{i_1} \cdots x_{i_j} (b\,x'_{i_{j+1}} + a) \cdots (b\,x'_{i_j} + a) =$$

$$x_{i_1} \cdots x_{i_j}\, x^{o}_{i_{j+1}} \cdots x^{o}_{i_n} = x_1^{\eta_1} \cdots x_n^{\eta_n}.$$

If $e_i = 1$ for all i, then $b\, x_1^{e_1} \cdots x_n^{e_n} =$

$x_1^{\eta_1} \cdots x_n^{\eta_n}$.

Hence in all cases the statement is true.

An alternate statement of theorem IV.1 follows.

Theorem IV.1a

If $x_i \in S_{a,b}$ then $a'\, b\, x_1^{e_1} \cdots x_n^{e_n} = a'\, x_1^{\eta_1} \cdots x_n^{\eta_n}$.

A function, g, of n variables of a relative boolean algebra, $S_{a,b}$, of B will have the form $g(x_1, \cdots, x_n) =$

$\sum_{(\eta)_{S_{a,b}}} g(\eta_1, \cdots, \eta_n)\, x_1^{\eta_1} \cdots x_n^{\eta_n}$, where the join extends over all combinations such that either $\eta_i = a$ or $\eta_i = b$.

$x^b = x$ and $x^a = x^\circ = b\, x' + a$, the complement in $S_{a,b}$.

The next theorem connects functions in B and functions in $S_{a,b}$.

If f is a function in B, then the statement "f is a function in $S_{a,b}$" means there exists a g in $S_{a,b}$ such that $f(x_1, \cdots, x_n) = g(x_1, \cdots, x_n)$ whenever $x_i \in S_{a,b}$.

Theorem IV.2

If f is a function in B, and $x_i \in S_{a,b}$ implies that $f(x_1, \cdots, x_n) \in S_{a,b}$, then f is a function in $S_{a,b}$ when the domain of the variables is restricted to $S_{a,b}$.

Proof: Since $x_i \in S_{a,b}$ and $f(x_1, \cdots, x_n) \in S_{a,b}$, it follows from theorem III.4 and IV.1 that $f(x_1, \cdots, x_n) =$

$$f(x_1, \cdots, x_n) + a = \sum_{(e)_B} f(\eta_1, \cdots, \eta_n)\, x_1^{\eta_1} \cdots x_n^{\eta_n} + a =$$

$$\sum_{(e)_B} f(\eta_1, \cdots, \eta_n)\,(b\, x_1^{\eta_1} \cdots x_n^{\eta_n} + a) + a =$$

$$\sum_{(\eta)_{S_{a,b}}} f(\eta_1, \cdots, \eta_n)\,(x_1^{\eta_1} \cdots x_n^{\eta_n} + a) + a =$$

$$\sum_{(\eta)_{S_{a,b}}} f(\eta_1, \cdots, \eta_n)\, x_1^{\eta_1} \cdots x_n^{\eta_n} + a =$$

$$\sum_{(\eta)_{S_{a,b}}} f(\eta_1, \cdots, \eta_n)\, x_1^{\eta_1} \cdots x_n^{\eta_n}.$$

Thus $f(x_1, \cdots, x_n)$ has the form of a function in $S_{a,b}$ whenever $x_i \in S_{a,b}$.

The importance of theorem IV.2 is not obvious from its statement. One result is that if a boolean albebra B is imbedded as a relative boolean algebra in another boolean algebra $\bar{B}$, that is, the given algebra

B is a relative boolean algebra of a larger boolean algebra $\bar{B}$, no new functions from B into B are formed by rules with coefficients from $\bar{B}$.

Next it is of interest to examine the extension of a boolean function, that is, given a function g in $S_{a,b}$ is there a function f in B such that whenever $x_i \in S_{a,b}$ then $f(x_1,\cdots,x_n) = g(x_1,\cdots,x_n)$? It will be shown that a necessary and sufficient condition that a function g in $S_{a,b}$ and a function f in B have the property that $f(x_1,\cdots,x_n) = g(x_1,\cdots,x_n)$, $x_i \in S_{a,b}$, is that the following conditions hold:

(i) $a' \, g(\eta_1,\cdots,\eta_n) < f(e_1,\cdots,e_n) < g(\eta_1,\cdots,\eta_n) + b'$

(ii) $g(b,\cdots,b) < f(1,\cdots,1) < g(b,\cdots,b) + b'$

(iii) $a' \, g(a,\cdots,a) < f(0,\cdots,0) < g(a,\cdots,a)$.

Theorem IV.3

Let g be a function in $S_{a,b}$, and f a function in B, such that $x_i \in S_{a,b}$ implies $g(x_1,\cdots,x_n) = f(x_1,\cdots,x_n)$. Then $a' \, g(\eta_1,\cdots,\eta_n) < f(e_1,\cdots,e_n) < g(\eta_1,\cdots,\eta_n) + b'$.

Proof: By theorem IV.1a, $a' \, g(\eta_1,\cdots,\eta_n) = a' \, b \, g(\eta_1,\cdots,\eta_n) = a' \, b \, f(\eta_1,\cdots,\eta_n) =$

$$\sum_{(\bar{e})_B} f(\bar{e}_1,\cdots,\bar{e}_n) \, a' \, b \, \eta_1^{\bar{e}_1} \cdots \eta_n^{\bar{e}_n} =$$

$\sum\limits_{(\bar{e})_B} f(\bar{e}_1,\cdots,\bar{e}_n)\ a'\ \eta_1^{\bar{\eta}_1} \cdots \eta_n^{\bar{\eta}_n}$, where $\bar{\eta}_i = a$ whenever $\bar{e}_i = 0$ and $\bar{\eta}_i = b$ whenever $\bar{e}_i = 1$. If $\bar{\eta}_i \neq \eta_i$, then $\eta_i^{\bar{\eta}_i} = a$ and $\eta_i^{\bar{\eta}_i}\ a' = 0$. Also if $\bar{\eta}_i = \eta_i$ then $\eta_i^{\bar{\eta}_i} = b$ and $\bar{e}_i = e_i$. Thus $a'\ g(\eta_1,\cdots,\eta_n) = f(e_1,\cdots,e_n)\ a'\ b$. Thus $a'\ g(\eta_1,\cdots,\eta_n) < f(e_1,\cdots,e_n)$.

Again, by theorem IV.1, $g(\eta_1,\cdots,\eta_n) + b' =$

$b\ g(\eta_1,\cdots,\eta_n) + a + b' =$

$\sum\limits_{(\bar{e})_B} f(\bar{e}_1,\cdots,\bar{e}_n)\ b\ \eta_1^{\bar{e}_1} \cdots \eta_n^{\bar{e}_n} + a + b' =$

$\sum\limits_{(\bar{e})_B} f(\bar{e}_1,\cdots,\bar{e}_n)\ \eta_1^{\bar{\eta}_1} \cdots \eta_n^{\bar{\eta}_n} + a + b'$, where $\bar{\eta}_i = a$ whenever $\bar{e}_i = 0$ and $\bar{\eta}_i = b$ whenever $\bar{e}_i = 1$. Since $a < b$ and $\bar{\eta}_i \neq \eta_i$ implies $\eta^{\bar{\eta}_i} = a$, it follows that whenever i exists such that $\eta_i \neq \bar{\eta}_i$ then $\eta_1^{\bar{\eta}_1} \cdots \eta_n^{\bar{\eta}_n} = a$, and $f(\bar{e}_1,\cdots,\bar{e}_n)\ \eta_1^{\bar{\eta}_1} \cdots \eta_n^{\bar{\eta}_n} + a = f(\bar{e}_1,\cdots,\bar{e}_n)\ a + a = a$. Thus, since $\eta^{\bar{\eta}_i} = b$ whenever $\eta_i = \bar{\eta}_i$, $g(\eta_1,\cdots,\eta_n) + b' = f(e_1,\cdots,e_n)\ b + a + b' = f(e_1,\cdots,e_n) + a + b'$. Hence $g(\eta_1,\cdots,\eta_n) + b' > f(e_1,\cdots,e_n)$.

Thus the theorem has been established.

Theorem IV.4

Let g be a function in $S_{a,b}$ and f a function in B, such that $g(x_1,\cdots,x_n) = f(x_1,\cdots,x_n)$ whenever $x_i \in S_{a,b}$. Then $f(1,\cdots,1) > g(b,\cdots,b)$ and $f(0,\cdots,0) < g(a,\cdots,a)$.

Proof: By definition of a boolean function $f(b,\cdots,b) = f(1,\cdots,1)\, b + f(0,\cdots,0)\, b'$. Since $f(b,\cdots,b) = g(b,\cdots,b) \in S_{a,b}$, $f(b,\cdots,b) < b$. Therefore $0 = b'\, f(b,\cdots,b) = f(0,\cdots,0)\, b'$.

Hence $g(b,\cdots,b) = f(b,\cdots,b) = f(1,\cdots,1)\, b$, and $g(b,\cdots,b) < f(1,\cdots,1)$.

By theorem IV.3, $f(0,\cdots,0)\,[g(a,\cdots,a) + b'] = f(0,\cdots,0)$. From above, $f(0,\cdots,0)\, b' = 0$. Hence $f(0,\cdots,0) = f(0,\cdots,0)\,[g(a,\cdots,a) + b'] = f(0,\cdots,0)\, g(a,\cdots,a)$, or $f(0,\cdots,0) < g(a,\cdots,a)$.

Theorem IV.25

Let g be a function in $S_{a,b}$ and f a function in B. A necessary and sufficient condition that f be the same as g whenever the domain of f is restricted to $S_{a,b}$ is that the following conditions be satisfied:

(i) $a'\, g(\eta_1,\cdots,\eta_n) < f(e_1,\cdots,e_n) < g(\eta_1,\cdots,\eta_n) + b'$

(ii) $g(b,\cdots,b) < f(1,\cdots,1) < g(b,\cdots,b) + b'$

(iii) $a'\, g(a,\cdots,a) < f(0,\cdots,0) < g(a,\cdots,a)$.

Proof: The necessity follows from theorems IV.3 and IV.4.

Assume the conditions hold. Since $f(1,\cdots,1) >$ $g(b,\cdots,b) > a$ and $f(0,\cdots,0) < g(a,\cdots,a) < b$, it follows by theorem III.11 that $f(x_1,\cdots,x_n) \in S_{a,b}$, whenever $x_i \in S_{a,b}$. Thus $f(x_1,\cdots,x_n) = f(x_1,\cdots,x_n) + a =$

$$\sum_{(e)_B} f(e_1,\cdots,e_n)\, x_1^{e_1} \cdots x_n^{e_n} + a >$$

$$\sum_{(e)_B} a'\, g(\eta_1,\cdots,\eta_n)\, x_1^{e_1} \cdots x_n^{e_n} + a =$$

$$\sum_{(e)_B} a'\, g(\eta_1,\cdots,\eta_n)\, b\, x_1^{e_1} \cdots x_n^{e_n} + a.$$ By theorem IV.1,

$$f(x_1,\cdots,x_n) > \sum_{(\eta)_{S_{a,b}}} a'\, g(\eta_1,\cdots,\eta_n)\, x_1^{\eta_1} \cdots x_n^{\eta_n} + a =$$

$$a'\, g(x_1,\cdots,x_n) + a = g(x_1,\cdots,x_n).$$

Also $f(x_1,\cdots,x_n) = b\, f(x_1,\cdots,x_n) + a$ whenever $x_i \in S_{a,b}$ from above. Thus, by theorem IV.1, $f(x_1,\cdots,x_n) =$

$$a + b \sum_{(e)_B} f(e_1,\cdots,e_n)\, x_1^{e_1} \cdots x_n^{e_n} <$$

$$a + b \sum_{(e)_B} [g(\eta_1,\cdots,\eta_n) + b']\, x_1^{e_1} \cdots x_n^{e_n} =$$

$$a + \sum_{(e)_B} g(\eta_1,\cdots,\eta_n)\, [b\, x_1^{e_1} \cdots x_n^{e_n} + a] =$$

$$a + \sum_{(\eta)_{S_{a,b}}} g(\eta_1,\cdots,\eta_n)\, x_1^{\eta_1} \cdots x_n^{\eta_n} = a + g(x_1,\cdots,x_n).$$

Thus $f(x_1,\cdots,x_n) < g(x_1,\cdots,x_n)$.

Therefore $f(x_1,\cdots,x_n) = g(x_1,\cdots,x_n)$ if $x_i \in S_{a,b}$.

FUNCTIONS BETWEEN RELATIVE BOOLEAN ALGEBRAS

A homomorphism between boolean algebras B and $\overline{B}$ is a function, h, of one variable such that:

(1) $h(p + q) = h(p) + h(q)$

(2) $h(p\, q) = h(p)\, h(q)$

(3) $h(p') = [h(p)]'$

whenever p and q are elements of B.

If h is a homomorphism then $h(0) = h(p\, p') = h(p)\, [h(p)]' = 0$ and $h(1) = h(p + p') = h(p) + [h(p)]' = 1$. Hence the distinguished elements are preserved by the mapping h.

A boolean homomorphism is a homomorphism which can be expressed as a boolean function.

This section deals with functions between relative boolean algebras of B where the functions are boolean functions.

Let $S_{a,b}$ and $S_{c,d}$ be relative boolean algebras of B. Let h be a boolean function of one variable in B whose domain is $S_{a,b}$ and whose range is $S_{c,d}$. Let $h(x + y) = h(x) + h(y)$ and $h(x\, y) = h(x)\, h(y)$, whenever x and y are elements of $S_{a,b}$. For all $q \in S_{a,b}$, $h(q) = h(q + a) = h(q) + h(a)$. Since the range of h is $S_{c,d}$, if $p \in S_{c,d}$ then there exists a $q \in S_{a,b}$ such that $h(q) = p$. Hence $p = p + h(a)$ for all p in $S_{c,d}$. Since the distinguished elements are unique and $h(a)$ satisfies axiom 3, $h(a) = c$. Similarly it follows that $h(b) = d$. Let $x^\circ = b\,x' + a$ and $x^+ = d\,x' + c$, the relative complements of $S_{a,b}$ and $S_{c,d}$ respectively. Also $h(x^\circ) + h(x) = h(x + x^\circ) = h(b) = d$ and $h(x^\circ)\,h(x) = h(x^\circ\,x) = h(a) = c$, thus $h(x^\circ) = [h(x)]^+$. Therefore if h is a boolean function with domain $S_{a,b}$ and range $S_{c,d}$ and h preserves meet and join, then h is a boolean homomorphism.

If h is a boolean homomorphism with domain $S_{a,b}$ and $h(a) = c$, $h(b) = d$, then by theorem III.8 the range of h is $S_{c,d}$.

Theorem V.1

Let $S_{a,b}$ and $S_{c,d}$ be relative boolean algebras of B. The following are equivalent:

(i) $a\,c = a\,d$ and $b'\,c = b'\,d$

(ii) there exists a boolean homomorphism between $S_{a,b}$ and $S_{c,d}$

(iii) there exists a boolean function, h, in B whose domain is $S_{a,b}$ and range is $S_{c,d}$.

Proof:

(i) implies (ii): Consider the function f discribed by $f(x) = d\,x + c$, x $S_{a,b}$. Since $f(a) = d\,a + c = a\,c + c = c$ and $f(b) = db + c = b\,d + b'\,c + c = d\,b + b'\,d + c = d + c = d$, it follcws from theorem III.8

that the range of f is $S_{c,d}$. Since $f(x + y) = d(x + y) + c = (d x + c) + (d x + c) = f(x) + f(y)$ and $f(x y) = d x y + c = (d x + c)(d y + c) = f(x) f(y)$ the binary operations are preserved, and f is a boolean honomorphism from $S_{a,b}$ to $S_{c,d}$.

(ii) implies (iii): Since all boolean homomorphisms have as a range a relative boolean algebra it follows that there exists a boolean function in B with domain $S_{a,b}$ and $S_{c,d}$.

(iii) implies (i): Let f be a function with domain $S_{a,b}$ and range $S_{c,d}$. Then by theorem III.8 $f(a) f(b) = c$ and $f(a) + f(b) = d$ or $c = f(1) a + f(0) b' + f(1) f(0) a' b$ and $d = f(1) b + f(0) a'$. Thus $c a = f(1) a = a [f(1) b + f(0) a'] = a d$ and $c b' = f(0) b' = b' [f(1) b + f(0) a'] = b' d$.

Note in theorem V.1 that if there is any boolean function with domain $S_{a,b}$ and range $S_{c,d}$, then there exists a boolean homomorphism with that domain and range. Furthermore the constants a, b, c, and d completely determine the existence of a homomorphism.

If $S_{c,d}$ is any relative boolean algebra of B, then since $S_{0,1} = B$ and $0 c = 0 d$ and $1' c = 1' d$, there is a boolean homomorphism from B to $S_{c,d}$, namely, $f(x) = d x + c$.

Theorem V.2

Let $S_{a,b}$ and $S_{c,d}$ be relative boolean algebras of B. $S_{c,d}$ is a homomorphic image of $S_{a,b}$, that is there exists a boolean homomorphism from $S_{a,b}$ to $S_{c,d}$, if and only if there exists a relative boolean algebra, $S_{k,h}$, of B with complementation $^{\circ}$, such that $a^{\circ} = d$ and $b^{\circ} = c$.

Proof: Let $S_{c,d}$ be a homomorphic image of $S_{a,b}$. Then by the previous theorem, $a\ c = a\ d$ and $b'\ c = b'\ d$. Thus $d\ a' + c = d\ a' + a\ c + c = d\ a' + d\ a + c = d + c = d$ and $d\ b' + c = c\ b' + c = c$. Hence if $^{\circ}$ is the relative complement with respect to $S_{c,d}$, then $a^{\circ} = d$ and $b^{\circ} = c$, as was to be shown.

Let $S_{a,b}$, $S_{c,d}$, and $S_{h,k}$ be relative boolean algebras of B. Furthermore let $a^{\circ} = k\ a' + h = d$ and $b^{\circ} = k\ b' + h = c$. Consider $f(x) = x^{\circ} = k\ x' + h$. Since $f(a) + f(b) = d$ and $f(a)\ f(b) = c$ it follows from theorem III.8 that f has a range $S_{c,d}$ when the domain of f is $S_{a,b}$. By theorem V.1, $S_{c,d}$ is a homomorphic image of $S_{a,b}$.

Let $S_{a,b}$ and $S_{h,k}$ be relative boolean algebras in B and x° be the relative complement of x with respect to $S_{h,k}$. By theorems II.6 and V.2, $S_{b^{\circ},a^{\circ}}$ is a homomorphic image of $S_{a,b}$.

There may be many equations for boolean functions in B which are homomorphisms with domain $S_{a,b}$ and range $S_{c,d}$, but the next theorem shows that if f and g are boolean homomorphisms from $S_{a,b}$ to $S_{c,d}$ then $f(x) = g(x)$ whenever $x \in S_{a,b}$.

Theorem V.3

Let $S_{c,d}$ and $S_{a,b}$ be relative boolean algebras B. If f is a boolean homomorphism with domain $S_{a,b}$ and range $S_{c,d}$ then $f(x) = d\,x + c$ whenever $x \in S_{a,b}$.

Proof: Since f is a homomorphism from $S_{a,b}$ to $S_{c,d}$, $f(b) = f(1)\,b + f(0)\,b' = d$ or $f(1)\,b = d\,b$. Also $f(a) = f(1)\,a + f(0)\,a' = c$ or $f(0)\,a' = c\,a'$. Let $x \in S_{a,b}$. Then $f(x) = f(1)\,x + f(0)\,x' =$ $f(1)\,b\,x + f(0)\,a'\,x' = d\,b\,x + c\,a'\,x' = d\,x + c\,x' =$ $d\,x + c$.

If $S_{a,b}$ and $S_{c,d}$ are relative boolean algebras of B, and there is a boolean homomorphism from $S_{a,b}$ to $S_{c,d}$, then the function f, $f(x) = d\,x + c$ is a boolean homomorphism, as was shown in the proof of theorem V.1.

Theorem V.3 does not give the form of the boolean function in B, but if B is the domain then there is a unique boolean function which is a homomorphism from

B to $S_{c,d}$. Theorem V.4 follows from theorem V.3

Theorem V.4

If B is the domain and $S_{c,d}$ is the range of a boolean homomorphism f then $f(x) = d\ x + c$. Furthermore f is the only boolean homomorphism from B onto $S_{c,d}$.

Let $S_{a,b}$ and $S_{c,d}$ be relative boolean algebras in B and $^{\circ}$, $^{+}$ be the complementation operations in $S_{a,b}$ and $S_{c,d}$ respectively. If $S_{c,d}$ is the homomorphic image of $S_{a,b}$, then the function f defined by $f(x) = (x^{\circ})^{+}$, will be shown to be a boolean homomorphism from $S_{a,b}$ to $S_{c,d}$. $f(x) = (x^{\circ})^{+} = (x\ b' + a)^{+} =$ $d\ (b\ x' + a)' + c = d\ a'\ x + d\ b' + c = d\ a'\ x + c =$ $d\ a'\ x + c\ a\ x + c$. By theorem V.1 $c\ a = d\ a$ thus $f(x) = d\ a'\ x + d\ a\ x + c = d\ x + c$, and therefore f is a boolean homomorphism.

Let $S_{a,b}$ and $S_{h,k}$ be relative boolean algebras of B and $x^{\circ} = b\ x' + a$, $x^{+} = k\ x' + h$. On page 14 it was stated that $f(x) = (((x^{+})^{\circ})^{+})^{\circ}$ is a homomorphism; this will now be shown. By theorem II.6, $S_{k^{\circ},h^{\circ}}$ is a relative boolean algebra with the complement of x being $((x^{\circ})^{+})^{\circ}$. Thus, from above, $f(x) = (((x^{+})^{\circ})^{+})^{\circ}$ is a boolean homomorphism from $S_{h,k}$ to $S_{k^{\circ},h^{\circ}}$.

RELATIVE BOOLEAN ALGEBRAS AND SUB-BOOLEAN ALGEBRAS

A sub-boolean algebra of a boolean algebra B is a subset of B which is closed under meet, join and complementation of B.

Let S be a sub-boolean algebra of B. Let $x \in S$ then $x' \in S$ and $x + x' = 1 \in S$, also $x\,x' = 0 \in S$. Thus S is a boolean algebra with distinguished elements 0, 1.

This section deals with the relationship between relative boolean algebras and sub-boolean algebras.

Let SRBA mean " sub-boolean algebra of a relative boolean algebra ", and RSBA mean " relative boolean algebra of a sub-boolean algebra ".

It will be shown that if S is a SRBA of B then S is a RSBA of B and if S is a RSBA of B then S is a SRBA of B, where B is a boolean algebra.

Theorem VI.1

If $\bar{S}$ is a sub-boolean algebra of B and $\bar{S}_{a,b}$ is a relative boolean algebra of $\bar{S}$, then $\bar{S}_{a,b}$ is a sub-boolean algebra of the relative boolean algebra $S_{a,b}$ of B.

That is, a RSBA is a SRBA.

Proof: Since $\bar{S}_{a,b} = \{x : a, b, x \in \bar{S}, \text{ and } a < x < b\}$, it follows that $\bar{S}_{a,b}$ is a subset of $S_{a,b}$. The complement in the RSBA, $\bar{S}_{a,b}$, is defined by $b\,x' + a$, the same as in $S_{a,b}$. Also the meet and join in both $\bar{S}_{a,b}$ and $S_{a,b}$ are the same as in B and thus the same. Hence $\bar{S}_{a,b}$ is a sub-boolean algebra of $S_{a,b}$.

Theorem VI.2

Let $S_{a,b}$ be a relative boolean algebra of B and S a sub-boolean algebra of $S_{a,b}$. Then S is a relative boolean algebra of some sub-boolean algebra of B. That is, a SRBA is a RSBA.

Proof: Let $\bar{S} = \{x : b\,x + a \in S\}$. If $x, y \in \bar{S}$ then $x + y$ and $x\,y$ are elements of $\bar{S}$ since $b\,(x + y) + a = (b\,x + a)\,(b\,y + a)$ and $b\,x\,y + a = (b\,x + a)\,(b\,y + a)$ are elements of S. Let $x \in \bar{S}$. Then $b\,x' + a$ is the complement of $b\,x + a$ with respect to $S_{a,b}$. Thus $b\,x' + a \in S$ and $x' \in \bar{S}$. Hence $\bar{S}$ is a sub-algebra of B. Note that, since $b\,a + a \in S$ and $b\,b + a \in S$, then a and b are elements of $\bar{S}$.

Let $\bar{S}_{a,b} = \{x : x \in \bar{S},\ a < x < b\}$. $\bar{S}_{a,b}$ is a relative boolean albebra of $\bar{S}$. If $x \in \bar{S}_{a,b} \subset \bar{S}$, then $x = b\,x + a$, and by definition of $\bar{S}$, $b\,x + a \in S$. Hence $x \in S$ and $\bar{S}_{a,b} \subset S$. If $x \in S \subset S_{a,b}$, then $a < x < b$ and $x = a\,x + b$. Thus $x \in \bar{S}_{a,b}$. Therefore $S = \bar{S}_{a,b}$. Since the complement of x in $\bar{S}$ and S is $b\,x' + a$, and since the meet and join are the same as in B, the boolean algebras $\bar{S}_{a,b}$ and S are identical. Therefore S is a RSBA.