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A combinatorial formula for homogeneous moments

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Abstract

We establish a combinatorial formula for homogeneous moments and give some examples where it can be put to use. An application to the statistical mechanics of interacting gauged vortices is discussed.

1. *Introduction*

In this paper we shall prove the following.

THEOREM 1·1. *Suppose we are given n integrable real-valued functions* $J_1, J_2, \ldots, J_n$ *on a measure space* M*, an integer* $m \geqslant 0$*, and a non-zero constant* C*, so that*

$$\int_M (v_1 J_1 + v_2 J_2 + \cdots + v_n J_n)^{2m} = C, \tag{1·1}$$

for all $(v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ *with* ${v_1}^2 + {v_2}^2 + \cdots + {v_n}^2 = 1$*. Then there is a rational number*

$$I_{m,n} = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(m + \frac{n}{2}\right)}{\Gamma\left(m + \frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)} \tag{1·2}$$

(depending only on m and n) so that

$$\int_M \left({J_1}^2 + {J_2}^2 + \cdots + {J_n}^2\right)^m = I_{m,n}\, C. \tag{1·3}$$

We refer to quantities such as the integrals in the left-hand side of (1·1) and (1·3) as homogeneous moments (in the given functions $J_1, \ldots, J_n$). For applications, we have in mind that M should be a compact smooth manifold equipped with a volume form and that $J_1, J_2, \ldots, J_n$ should be smooth functions. A simple example may be given by taking $M = S^{n-1}$ with its usual round metric and $J_1, J_2, \ldots, J_n$ to be the coördinate functions for the standard embedding $S^{n-1} \hookrightarrow \mathbb{R}^n$. In this case, $v_1 J_1 + v_2 J_2 + \cdots + v_n J_n : S^{n-1} \to \mathbb{R}$ is simply another coördinate function on the sphere and (1·1) is evident by rotational invariance. (The constant C may always be realised by setting $v_n = 1$ and all other v_j's equal to zero: $C = \int_M {J_n}^{2m}$.) In this example, reflection symmetry directly implies that the integral on the left-hand side of (1·1) would vanish if $2m$ were replaced by an odd integer; an easy argument shows that this would also be true in the general case of our hypothesis. Thus we are not losing generality by assuming that the degree of homogeneity in (1·1) is even.

Our motivation, however, stems from a more substantial example in which M is a Kähler manifold whose structure is invariant under the action of SO(3). In this case $n=3$ and the three functions $J_1, J_2, J_3 : M \to \mathbb{R}$ are the components of the associated moment map $M \to \mathrm{so}(3)^*$. The action of SO(3) arises because M is the moduli space of gauged vortices on the round 2-sphere [**5**]. Further discussion of this case is provided at the end of this article. When $n = 3$, the rational numbers $I_{m,3}$ are, in fact, integers: $I_{m,3} = 2m+1$. That the $I_{m,n}$ are well-defined rational numbers is most easily proved without the explicit formula (1·2). Having done this in §2, we shall establish (1·2) in §3. In §4, we give a geometric setting for our hypothesis (1·1) with some examples, and conclude in §5 with a brief discussion of vortices on the 2-sphere.

2. *Combinatorics*

We may evidently extend (1·1) by homogeneity to conclude that

$$\int_M \left(v_1 J_1 + v_2 J_2 + \cdots + v_n J_n\right)^{2m} = C\left({v_1}^2 + {v_2}^2 + \cdots + {v_n}^2\right)^m, \tag{2·1}$$

for all $(v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$. Also, by simultaneously rescaling all the functions $J_1, J_2, \ldots, J_n$ we may suppose without loss of generality that $C = 1$ and our task is now to compute

$$I_{m,n} \equiv \int_M \left({J_1}^2 + {J_2}^2 + \cdots + {J_n}^2\right)^m$$

from this information. To do this we may regard (2·1) as the equality of two polynomials in the v-variables and deduce the equality of their coefficients. Examining the left-hand side of (2·1), this means that we may compute

$$\int_M {J_1}^{r_1} {J_2}^{r_2} \cdots {J_n}^{r_n} \quad \text{for all } r_1 + r_2 + \cdots + r_n = 2m.$$

Indeed, since only integers are involved in expanding the two sides of (2·1), it follows that these integrals are all rational. This is more than enough to compute $I_{m,n}$ and to conclude that these quantities are also rational.

Though the preceding argument is straightforward in principle, in practise it is almost useless in establishing formulae for $I_{m,n}$. We shall rectify this deficiency in the following section. In the meantime, let us compute $I_{3,3}$ by bare hands. We are supposing that

$$\int_M (v_1 J_1 + v_2 J_2 + v_3 J_3)^6 = \left({v_1}^2 + {v_2}^2 + {v_3}^2\right)^3$$

and, by expanding as polynomials in v_1, v_2, v_3, we conclude that

$$\textstyle\int_M {J_1}^6 = \int_M {J_2}^6 = \int_M {J_3}^6 = 1$$

$$\textstyle\int_M {J_1}^4 {J_2}^2 = \int_M {J_1}^2 {J_2}^4 = \cdots = \int_M {J_2}^2 {J_4}^4 = 1/5$$

$$\textstyle\int_M {J_2}^2 {J_2}^2 {J_3}^2 = 1/15$$

(and more besides). Therefore,

$$\begin{aligned}\textstyle\int_M \left({J_1}^2 + {J_2}^2 + {J_3}^2\right)^3 &= \textstyle\int_M \left({J_1}^6 + \cdots + 3{J_1}^4{J_2}^2 + \cdots + 6{J_1}^2{J_2}^2{J_3}^2\right) \\ &= 3 \times 1 + 6 \times 3 \times 1/5 + 6 \times 1/15 = 7.\end{aligned}$$

Such naïve computations of $I_{m,n}$ rapidly get out of hand for large m and n.

3. *Calculation of $I_{m,n}$*

Given that the argument we used above for the existence of $I_{m,n}$ is purely combinatorial, it is clear that this quantity does not depend on the measure space M. Thus we can calculate $I_{m,n}$ by evaluating both the integral and the constant C in (1·3) for a specific model where the hypothesis (1·1) is satisfied. We shall do this for the first example mentioned in the Introduction, where M is S^{n-1} with metric induced from the embedding $\iota : S^{n-1} \hookrightarrow \mathbb{R}^n$ and $J_j : S^{n-1} \to [-1, 1]$ are the standard cartesian coördinates.

In this example, the integral on the left-hand side of (1·3) is just the well-known volume of S^{n-1},

$$\int_{S^{n-1}} \left({J_1}^2 + \cdots + {J_n}^2\right)^m = \int_{S^{n-1}} 1 = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}. \tag{3·1}$$

To evaluate the constant $C = \int_{S^{n-1}} {J_n}^{2m}$ on the right-hand side, we start by using cylindrical coördinates to write the euclidean metric g on $\mathbb{R}^n$ as

$$g = \mathrm{d}x^2 + \mathrm{d}r^2 + r^2\, g_{S^{n-2}}. \tag{3·2}$$

Here, $x = J_n$, $r = \sqrt{{J_1}^2 + \cdots + {J_{n-1}}^2}$ and $g_{S^{n-2}}$ denotes the metric on S^{n-2}. On $S^{n-1} \subset \mathbb{R}^n$, we have the relation

$$x^2 + r^2 = 1 \quad \Rightarrow \quad x\,\mathrm{d}x = -\,r\,\mathrm{d}r,$$

therefore (3·2) pulls back to S^{n-1} as

$$g_{S^{n-1}} = \iota^* g = \frac{\mathrm{d}x^2}{1 - x^2} + (1 - x^2)\, g_{S^{n-2}}.$$

In these coordinates, the volume form on S^{n-1} can then be written as

$$\mathrm{dvol}_{S^{n-1}} = \sqrt{\det\left(g_{S^{n-1}}\right)}\, \mathrm{d}x \wedge \mathrm{dvol}_{S^{n-2}} = (1 - x^2)^{(n-3)/2}\, \mathrm{d}x \wedge \mathrm{dvol}_{S^{n-2}}. \tag{3·3}$$

We observe in passing that the case $n = 3$ is special, as (3·3) then gives an identification of the volume forms on S^2 and on the cylinder in $\mathbb{R}^3$ defined by $r = 1$ and $|x| \leqslant 1$; this is the celebrated hat-box theorem of Archimedes, of which our equation (3·3) may be regarded as a generalisation to arbitrary dimensions. The constant on the right-hand side of (1·3) can now be evaluated as

$$\begin{aligned}
\int_{S^{n-1}} {J_n}^{2m} &= \int_{[-1,1]\times S^{n-2}} x^{2m}\left(1 - x^2\right)^{(n-3)/2}\, \mathrm{d}x \wedge \mathrm{dvol}_{S^{n-2}} \\
&= \int_{S^{n-2}} 1 \times \int_0^1 t^{m-1/2}(1 - t)^{(n-3)/2}\, \mathrm{d}t \\
&= \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})}\, B\left(m + \frac{1}{2}, \frac{n-1}{2}\right) \\
&= \frac{2\pi^{(n-1)/2}\, \Gamma(m + \frac{1}{2})}{\Gamma\left(m + \frac{n}{2}\right)},
\end{aligned} \tag{3·4}$$

where B denotes Euler's Beta-function [**1**]. Finally, we obtain $I_{m,n}$ as a quotient of (3·1) and (3·4), which yields (1·2) using $\Gamma(1/2) = \sqrt{\pi}$.

We would like to make two remarks on the formula (1·2):

(A) Notice that (1·2) gives $I_{0,n} = 1$, and so it follows from the recursion relation $\Gamma(z+1) = z\,\Gamma(z)$ that $I_{m,n}$ is obviously a rational number for all $m \in \mathbb{N}$. More explicitly, we can write

$$I_{m,n} = \begin{cases} \dfrac{2^{2m-1}\left(m+\frac{n}{2}-1\right)!(m-1)!}{(2m-1)!\left(\frac{n}{2}-1\right)!} & \text{if } n \text{ is even,} \\[2ex] \dfrac{(2m+n-2)!(m-1)!\left(\frac{n-3}{2}\right)!}{2(2m-1)!(n-2)!\left(m+\frac{n-3}{2}\right)!} & \text{if } n \text{ is odd.} \end{cases}$$

(B) Our result for $I_{m,n}$ as expressed in equation (1·2) turns out to be a value of Gauß's hypergeometric function,

$$I_{m,n} = {}_2F_1\left(2m, n-1; m+\tfrac{n}{2}; \tfrac{1}{2}\right),$$

which comes as a consequence of Gauß's second summation theorem (cf. [**9**, p. 32]). The property $I_{m,n} \in \mathbb{Q}$ provides examples of the curious fact that the hypergeometric function sometimes assumes rational values when its argument and parameters are rational. This behaviour is general for the geometric series, but it is not understood under which circumstances it generalises to hypergeometric series.

4. *Geometry and examples*

We shall now describe a geometric setup where the hypothesis (1·1) naturally arises, and which inspires generalisations of Theorem 1·1. Let us suppose that V is a real inner product space and that a compact Lie group G has an orthogonal represention on V, $\rho : G \to \mathrm{O}(V)$. Typically, we take as M a smooth manifold where G acts, equipped with an invariant volume form, which may in turn be induced by a riemannian or symplectic structure on M. We also assume that $J : M \to V$ is a G-equivariant mapping. At this point, symmetry considerations will severely restrict the integrals

$$\Phi_m(v) := \int_M \langle v, J\rangle^{2m} \tag{4·1}$$

with $v \in V$. In fact, for all $g \in G$,

$$\Phi_m(\rho(g)v) = \int_M \langle \rho(g)v, J\rangle^{2m} = \int_M \langle v, \rho(g^{-1})J\rangle^{2m} = \Phi_m(v),$$

and this implies that $\Phi(v)$ is an invariant polynomial restricted by Weyl's classical invariant theory [**11**]. Because G is represented by orthogonal transformations, $||v||^{2m}$ is one of the invariant polynomials of degree $2m$; however, these ingredients are still not enough to enforce (1·1). In the following, we discuss three realisations of this general setup that lead to condition (1·1) being satisfied, or to more general conditions that still allow us to use the arguments in sections 2 and 3 to determine $\int_M \langle J, J\rangle^{2m}$ by combinatorial means.

Example 4·1. Let $G=\mathrm{SO}(V)$, ρ be the defining representation, and $J\colon M\hookrightarrow V$ be the inclusion of any invariant measurable set. Then necessarily $\Phi_m(v)=C\,||v||^{2m}$. A particular case is when M is a single $\mathrm{SO}(V)$-orbit embedded in V, which yields the example that we used in the calculations of Section 3.

Example 4·2. Let $G=\mathrm{SO}(4)$, $V=\mathrm{so}(4)\cong\bigwedge^2\mathbb{R}^4$ and ρ be the adjoint representation. The ring of invariants is freely generated by two polynomials of degree two [**2**], the squared norm $||v||^2$ and the pfaffian $\mathrm{Pf}(v)$. Thus we can write

$$\Phi_m(v)=\sum_{j=0}^{m}C_j\,||v||^{2(m-j)}\,\mathrm{Pf}(v)^j$$

for suitable real constants C_j. Our hypothesis in the form (2·1) will hold at least for v in the hypersurface defined by $\mathrm{Pf}(v)=0$, and the argument in section 2 will still lead to a combinatorial formula for the homogeneous moments in (1·3). Whether (2·1) holds more generally depends on the choice of M and J. For instance, one can show that, for $m=1$, (2·1) holds for all $v\in V$ if and only if, say,

$$\int_M J_{12}\,J_{34}=0, \tag{4·2}$$

where the indices refer to the standard basis of $\bigwedge^2\mathbb{R}^4$. An interesting example is when M is the 4-manifold of simple 2-vectors in $\bigwedge^2\mathbb{R}^4$ of unit norm, the adjoint orbit of SO(4) given by the algebraic equations $\mathrm{Pf}(v)=0$ and $||v||^2=1$, and $J\colon M\hookrightarrow\bigwedge^2\mathbb{R}^4$ is the inclusion; an easy check shows that (4·2) is satisfied, hence our hypothesis (1·1) holds true in this case.

Example 4·3. We take M to be a symplectic manifold with moment map $J\colon M\to\mathrm{g}^*$; ρ will be the coadjoint representation, and we can use the Killing form to identify g with g^*. Again, Φ_m is restricted to be a linear combination of the G-invariants in $\mathrm{Sym}^{2m}(\mathrm{g})$ and $||v||^{2m}$ is one of them. In particular, if $G=\mathrm{SO}(3)$, then any invariant must be a scalar multiple of $||v||^{2m}$ and our hypothesis (1·1) must hold. A particular case of this situation that illustrates the usefulness of our formula (1·3) will be discussed in the next section.

5. *Interacting vortices on a* 2*-sphere*

In this section, we describe an application of Theorem 1·1 to a natural setting where our hypothesis holds true. Let N be a positive integer. We consider the measure space M to be the moduli space $\mathcal{M}_N$ of N-vortices on a 2-sphere of radius $R>\sqrt{N}$ [**5**]. This is just $\mathbb{CP}^N$ as a complex manifold, but equipped with a Kähler structure ω_{L^2} induced from a gauge-theoretic version of the L^2 norm on the space of fields. The associated Kähler metric $g_{r\bar{s}}$ encodes information about the physics of vortices at low energies; for example, its geodesic flow gives a good approximation to the slow dynamics of the abelian Higgs model at critical coupling [**10**]. For $N>1$, this Kähler structure is distinct from the Fubini–Study structure ω_{FS} on $\mathbb{CP}^N$, although it has been argued that [**3**]

$$\omega_{L^2}=2\pi\big(R^2-N\big)\omega_{\mathrm{FS}}+o\big(R^2-N\big)$$

as $R^2\searrow N$.

There is a local description of ω_{L^2}, which we briefly recall here [**7**, **8**]. This uses the fact that $\mathbb{CP}^N\cong\mathrm{Sym}^N(S^2):=(S^2)^N/\mathrm{S}_N$. We denote by $\Delta\subset(S^2)^N$ the set of fixed points of elements of S_N, and let z be a complex stereographic coördinate on an open set $U\subset S^2$. Then in terms of the natural coördinates $(z_1, \ldots, z_N)$ for $U^N \subset (S^2)^N$, which we may interpret as giving the positions of N individual vortex cores,

$$\omega_{L^2} \equiv \frac{\mathrm{i}}{2} \sum_{r,s=1}^{N} g_{r\bar{s}}\, \mathrm{d}z_r \wedge \mathrm{d}\bar{z}_s = \mathrm{i} \sum_{r,s=1}^{N} \left(\frac{R^2 \delta_{rs}}{\left(1 + |z_r|^2\right)^2} + \frac{\partial b_r}{\partial \bar{z}_s} \right) \mathrm{d}z_r \wedge \mathrm{d}\bar{z}_s. \tag{5·1}$$

The functions $b_r(z_1, \ldots, z_N)$ are defined on $U^N - \Delta$ in terms of a solution to an elliptic PDE reminiscent of the Liouville equation. They satisfy

$$b_r(\ldots, z_r, \ldots, z_s, \ldots) = b_s(\ldots, z_s, \ldots, z_r, \ldots),$$

therefore local quantities like $g_{r\bar{s}}$ in (5·1) descend to the moduli space. Although the b_r are not known explicitly, some statements about them (and the metric) can be made using the symmetry of the problem. For example, the fact that SO(3) acts on S^2 by isometries implies the relations [**6**]

$$\sum_{r=1}^{N} (z_r b_r - \bar{z}_r \bar{b}_r) = 0 \quad \text{and} \quad \sum_{r=1}^{N} \left(2z_r + {z_r}^2\, b_r + \bar{b}_r\right) = 0,$$

which in turn can be used to show that (5·1) is preserved by the induced action of SO(3) on $\mathcal{M}_N$. Thus the Liouville measure associated to ω_{L^2} is SO(3)-invariant. In addition, there exists a moment map $J = (J_1, J_2, J_3)\colon \mathcal{M}_N \to \mathrm{so}(3)^*$, for which our hypothesis (1·1) is obviously satisfied. Its components can be calculated as [**6**]

$$J_1 = 2\pi \sum_{r=1}^{N} \left(R^2 \frac{z_r + \bar{z}_r}{1 + |z_r|^2} + \frac{1}{2}(b_r + \bar{b}_r) \right),$$

$$J_2 = -2\pi \mathrm{i} \sum_{r=1}^{N} \left(R^2 \frac{z_r - \bar{z}_r}{1 + |z_r|^2} - \frac{1}{2}(b_r - \bar{b}_r) \right),$$

$$J_3 = 2\pi \sum_{r=1}^{N} \left(R^2 \frac{1 - |z_r|^2}{1 + |z_r|^2} - (z_r b_r + 1) \right),$$

and can be interpreted as angular momenta along the three coördinate axes of an ambient $\mathbb{R}^3$.

The metric on $\mathcal{M}_N$ has been used to study the statistical mechanics of a gas of vortices in the abelian Higgs model at critical coupling, both in the noninteracting case where the net forces experienced by the vortices are zero [**4**] and in presence of a background potential [**7**]. A more physically interesting situation would be the case where inter-vortex interactions are introduced. At the level of the dynamics on the moduli space, these would be described by a potential $V\colon \mathcal{M}_N \to \mathbb{R}$ invariant under the action of SO(3). The simplest nontrivial potential with this property is just (a multiple of) the square of the moment map,

$$V = \mu^2 ||J||^2 = \mu^2\left({J_1}^2 + {J_2}^2 + {J_3}^2\right), \tag{5·2}$$

where μ^2 is a positive coupling constant. From the point of view of the dynamics of several particles, one can expect this potential to be repulsive; this simply means that if $N-2$ vortex positions are kept fixed, $\int_{\mathcal{M}_N} V$ increases as the remaining two vortex cores come closer together on the 2-sphere. To justify this, we consider the case where $N = 2$ and the vortices are close to each other and symmetrically positioned at

$$z_1 = \sigma e^{\mathrm{i}\theta}, \quad z_2 = -z_1$$

where $0 < \sigma < 1$ and $0 \leqslant \theta < 2\pi$. Rotational symmetry implies that [**8**]

$$b_1 = b(\sigma)e^{-\mathrm{i}\theta}, \quad b_2 = -b(\sigma)e^{-\mathrm{i}\theta}$$

for some real function b. One can follow an argument similar to the one in Appendix C of [**8**] to obtain the estimate

$$b(\sigma) = \frac{1}{\sigma} - 2R^2\sigma + o(\sigma^2) \tag{5·3}$$

as $\sigma \searrow 0$. The formulae above imply that, in this configuration, the only nonzero component of the moment map is

$$J_3(\sigma) = 4\pi\left(R^2\frac{1-\sigma^2}{1+\sigma^2} - \sigma b(\sigma) - 1\right).$$

The estimate (5·3) then yields

$$\frac{\partial\left({J_3}^2\right)}{\partial\sigma} = -2^8\pi^2\left(R^2-1\right)\sigma + o(\sigma^2)$$

when $\sigma \searrow 0$, and the coefficient in the leading term is always negative for $R > \sqrt{2}$. Thus the potential (5·2) is creating a force that will locally contribute to push two approaching vortices apart. An attractive potential can be obtained from (5·2) by reversing the sign of the coupling constant.

In the model for interacting vortex dynamics corresponding to the potential (5·2), the partition function is given by (see [**7**] for details)

$$\begin{aligned} Z &= \frac{1}{(2\pi\hbar)^{2N}}\int_{\mathrm{T}^*\mathcal{M}_N} \exp\left(-\left(\frac{1}{2\pi}\sum_{r,s} g^{r\bar{s}} w_r \bar{w}_s + \mu^2 ||J||^2\right)/T\right)\frac{\omega_{\mathrm{can}}^{2N}}{(2N)!} \\ &= \left(\frac{T}{2\hbar^2}\right)^N \int_{\mathcal{M}_N} \exp\left(-\frac{\mu^2}{T}\,||J||^2\right)\frac{\omega_{L^2}^N}{N!}. \end{aligned}$$

Here, $w_r = \pi\sum_{s=1}^N g_{r\bar{s}}\dot{\bar{z}}_s$ denote canonical momenta to the moduli z_r,

$$\omega_{\mathrm{can}} = \frac{1}{2}\sum_{r=1}^N (\mathrm{d}z_r \wedge \mathrm{d}\bar{w}_r + \mathrm{d}\bar{z}_r \wedge \mathrm{d}w_r)$$

is the canonical symplectic form on the phase space $\mathrm{T}^*\mathcal{M}_N$, $2\pi\hbar$ is Planck's constant and T denotes the absolute temperature. We have indicated explicitly the Liouville measures on $(\mathrm{T}^*\mathcal{M}_N, \omega_{\mathrm{can}})$ and $(\mathcal{M}_N, \omega_{L^2})$ for clarity. It turns out that Theorem 1·1 can be used to find a closed expression for this partition function. In fact, after expanding the exponential, we can organise the remaining integral as a sum of integrals of powers of a single component of the moment map by making use of (1·3) with $n = 3$:

$$Z = \left(\frac{T}{2\hbar^2}\right)^N \sum_{m=0}^{\infty}\frac{(-1)^m\mu^{2m}}{m!\,T^m}(2m+1)\int_{\mathcal{M}_N} {J_3}^{2m}.$$

Now we invoke the following result of [**7**], which is obtained as an application of the Duistermaat–Heckman formula to the circle action generated by J_3 on the symplectic manifold $(\mathcal{M}_N, \omega_{L^2})$:

$$\int_{\mathcal{M}_N} {J_3}^{2m} = \sum_{j=0}^N \frac{(-1)^{N-j}(2m)!}{j!(N-j)!(N+2m)!}(2\pi(R^2-N)(2j-N))^{N+2m}.$$

The partition function can then be expressed as

$$Z = \left(\frac{T}{2\hbar^2}\right)^N \sum_{j=0}^{N} \sum_{m=0}^{\infty} \frac{(-1)^{j+m}(2m+1)!}{j!(N-j)!m!(N+2m)!} \left(\frac{\mu^2}{T}\right)^m \left(4\pi(R^2-N)\left(\frac{N}{2}-j\right)\right)^{N+2m}$$

$$= \frac{1}{N!}\left(\frac{\tilde{A}T}{2\hbar^2}\right)^N \sum_{j=0}^{N} \frac{(-1)^j}{j!(N-j)!}\left(\frac{N}{2}-j\right)^N {}_2F_2\left(1, \frac{3}{2}; \frac{N+1}{2}, \frac{N+,2}{2}; -\frac{(\mu\tilde{A})^2}{T}\left(\frac{N}{2}-j\right)^2\right)$$

in terms of the generalised hypergeometric function ${}_2F_2$ [**9**], and where we introduced the area available for N vortices on the sphere

$$\tilde{A} := 4\pi(R^2-N).$$

It is still a challenging problem to understand the physics determined by Z — in particular, it would be interesting to obtain an equation of state for this system (in some approximation) and analyse whether it allows for phase transitions.

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REFERENCES

[**1**] M. ABRAMOWITZ and I. A. STEGUN. *Handbook of Mathematical Functions* (National Bureau of Standards, 1965).
[**2**] M. AUDIN. Spinning tops: A course in integrable systems. *Camb. Stud. Adv. Math.* **51** (1996).
[**3**] J. M. BAPTISTA and N. S. MANTON. The dynamics of vortices on S^2 near the Bradlow limit. *J. Math. Phys.* **44** (2003), 3495–3508, `hep-th/0208001`.
[**4**] N. S. MANTON. Statistical mechanics of vortices. *Nucl. Phys. B* **400 [FS]** (1993). 624–632.
[**5**] N. S. MANTON and P. M. SUTCLIFFE. Topological solitons. *Camb. Monogr. Math. Phys.* (2004).
[**6**] N. M. ROMÃO. Quantum Chern–Simons vortices on a sphere. *J. Math. Phys.* **42** (2001), 3445–3469, `hep-th/0010277`.
[**7**] N. M. ROMÃO. Gauged vortices in a background. *J. Phys. A: Math. Gen.* **38** (2005), 9127–9144, `hep-th/0503014`.
[**8**] T. M. SAMOLS: Vortex scattering. *Commun. Math. Phys.* **145** (1992), 149–180.
[**9**] L. J. SLATER. *Generalised Hypergeometric Functions* (Cambridge University Press, 1966).
[**10**] D. STUART. Dynamics of Abelian Higgs vortices in the near Bogomolny regime. *Commun. Math. Phys.* **159** (1994), 51–91.
[**11**] H. WEYL. *The Classical Groups* (Princeton University Press, 1939).