

2017

## Reframing Mathematical Futures II Project: Development of a draft learning progression for algebraic reasoning

L Day

*The University of Notre Dame Australia*, [lorraine.day@nd.edu.au](mailto:lorraine.day@nd.edu.au)

M Horne

M Stephens

Follow this and additional works at: [https://researchonline.nd.edu.au/edu\\_conference](https://researchonline.nd.edu.au/edu_conference)



Part of the Education Commons

This conference paper was originally published as:

Day, L., Horne, M., & Stephens, M. (2017). Reframing Mathematical Futures II Project: Development of a draft learning progression for algebraic reasoning. *Conference of the International Group for the Psychology of Mathematics Education, 41st*.

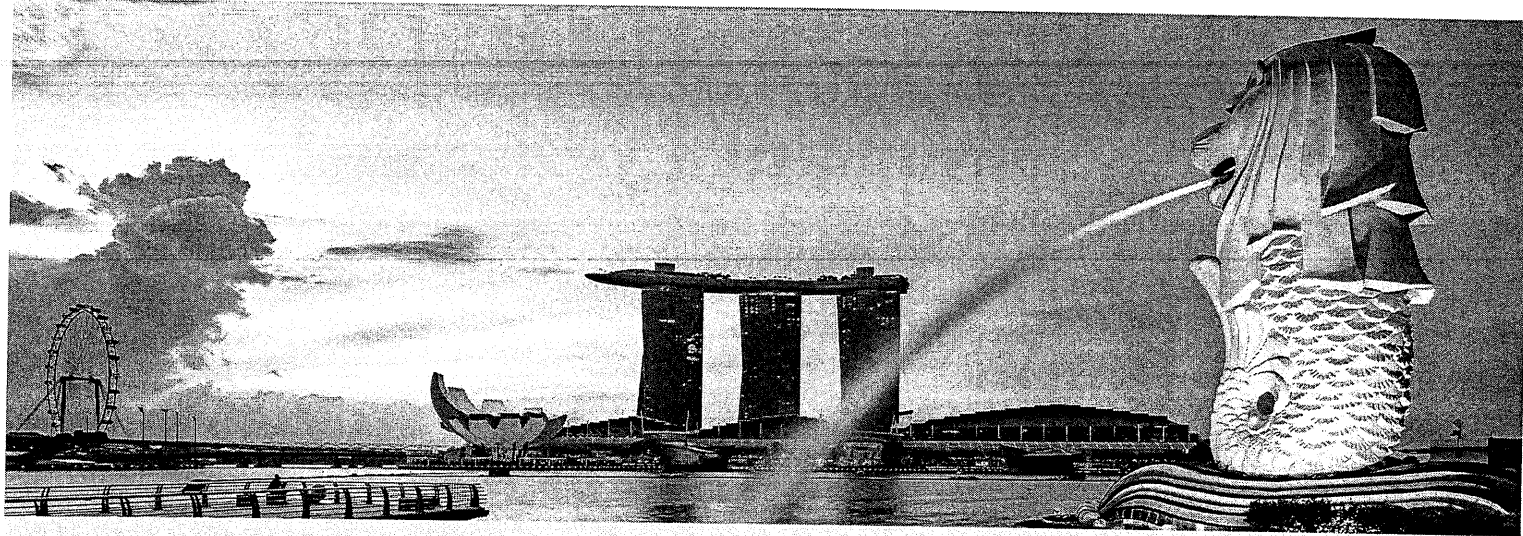
<http://www.igpme.org/index.php/publications>

Original conference paper available here:

<http://www.igpme.org/index.php/publications>

This conference paper is posted on ResearchOnline@ND at [https://researchonline.nd.edu.au/edu\\_conference/103](https://researchonline.nd.edu.au/edu_conference/103). For more information, please contact [researchonline@nd.edu.au](mailto:researchonline@nd.edu.au).





# **PME 41**

## **Proceedings**

Singapore | July 17-22, 2017

---

Of the 41st Conference of the International Group for the  
Psychology of Mathematics Education

---

**Volume 2**

**Editors | Berinderjeet Kaur, Weng Kin Ho, Tin Lam Toh, Ban Heng Choy**



41<sup>st</sup> PME Annual Conference  
17-22 July 2017  
Singapore

# Proceedings of the 41<sup>st</sup> Conference of the International Group for the Psychology of Mathematics Education

Editors

Berinderjeet Kaur

Weng Kin Ho

Tin Lam Toh

Ban Heng Choy

Volume 2

PME 41, Singapore, 17-22 July 2017



41<sup>st</sup> PME Annual Conference  
17-22 July 2017  
Singapore

---

Cite as:

Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). (2017). Proceedings of the 41<sup>st</sup> Conference of the International Group for the Psychology of Mathematics Education (Vol. 2). Singapore: PME.

Website: <http://math.nie.edu.sg/pme41>

Proceedings are also available via <http://www.igpme.org>

Copyrights © 2017 left to the authors  
All rights reserved

ISBN: 978-981-11-3742-6

Cover design: Mohamed Fadzli Ibrahim, National Institute of Education, Singapore.  
Printed by Welon Printing Co. Pte Ltd

# REFRAMING MATHEMATICAL FUTURES II PROJECT: DEVELOPMENT OF A DRAFT LEARNING PROGRESSION FOR ALGEBRAIC REASONING

Lorraine Day<sup>1</sup>, Marj Horne<sup>2</sup> and Max Stephens<sup>3</sup>

<sup>1</sup>The University of Notre Dame Australia, <sup>2</sup>RMIT University, <sup>3</sup>The University of Melbourne

*Curriculum documents make a clear distinction between algebraic skills and algebraic reasoning, where the development of the former is far more readily articulated than the latter. While there are many studies of algebraic reasoning, these are usually topic specific and/or highly contextual. What are the big ideas of algebraic reasoning and is it possible to map their learning trajectory? This paper reports on the preliminary phase of a large national study in Australia which is designed to move beyond the hypothetical and to provide an evidence-based foundation for a learning progression. Using rich assessment tasks designed for middle years students of mathematics, this paper reports on the method of analysis used and some preliminary findings.*

## INTRODUCTION

This research is situated within the Reframing Mathematical Futures II (RMFII) Project (2014-2017) which is funded by the Australian Government through the Australian Mathematics and Science Partnership Projects. This competitive grant Project followed on from the Reframing Mathematical Futures (RMF) Priority Project (2013) that aimed to improve multiplicative thinking and proportional reasoning in Years 7-10 using the Scaffolding Numeracy in the Middle Years (SNMY) resources (Siemon et al., 2006). All participating schools in the RMFII Project also participated in the RMF Project, although some did this after having joined the second Project. RMFII is aimed at building a sustainable, evidence-based, integrated learning and teaching resource to support the development of mathematical reasoning in Years 7-10. The Australian Curriculum: Mathematics (Australian Curriculum, Assessment and Reporting Authority (ACARA), 2016) consists of three content strands (Number and Algebra, Measurement and Geometry, and Statistics and Probability) and four proficiency strands (Understanding, Fluency, Problem Solving and Reasoning). Three areas of mathematical reasoning, aligned to the content strands of the Australian Curriculum: Mathematics, were identified to be investigated. These areas were Algebraic Reasoning, Spatial Reasoning, and Statistical and Probabilistic Reasoning. This paper addresses the component of the Project that aims to identify and map the 'big ideas' in algebraic reasoning. For the purpose of the Reframing Mathematical Futures II Project algebraic reasoning encompasses:

- *Core knowledge* needed to recognise, interpret, represent and analyse algebraic situations and the relationships and connections between them;
- *Ability to apply* that knowledge in unfamiliar situations to prove that something is true or false, solve problems, generate and test conjectures, make and defend generalisations; and
- *A capacity to explain and communicate* reasoning and solution strategies in multiple ways.

Four Phases of the Project were identified (in each of the three areas of mathematical reasoning):

1. Develop draft learning progressions from the research literature;
2. Develop, trial and validate assessment tasks;
3. Use the results to develop formative Learning and Assessment Frameworks (LAFs) and accompanying resources to support teaching and assessment;
4. Trial the above with partner schools, and evaluate in terms of student learning and shifts in teacher knowledge.

This paper will concentrate on the first and second Phases given above, that is, on the development of the draft learning progression (DLP) and the development, trialling and validation of the assessment tasks.

## **DEVELOPMENT OF DRAFT LEARNING PROGRESSION**

The idea of developing a draft learning progression built on Simon's (1995) suggestion of constructing hypothetical learning trajectories as mini-theories of student learning. This was seen as a useful place to begin, as learning trajectories assist teachers to see where on the continuum students are and hence provide a starting point for teaching (Siemon, Izard, Breed, & Virgona, 2006). It should be noted here that there was discussion around the nomenclature of the construct. It was decided that the term "learning progression" would be more clearly understood by teachers in Australia. The distinctions between learning progressions and learning trajectories made by Ellis, Weber and Lockwood (2014) were not considered, as there was no intention to get tied up with semantics.

Although there has been much debate about the meaning and use of the terms learning progressions and learning trajectories, there are common elements of the varied interpretations and it is these commonalities that were used as the focus. One of the common elements is that learning takes place over time and effective teaching involves recognising where the learners are in their learning journey as a starting point to design challenging yet achievable learning experiences to support the students' progress. The second commonality is that learning progressions or trajectories are based on hypothesised pathways derived from a synthesis of relevant literature, the design and trialling of learning activities aimed at progressing learning within the hypothesised framework, and evaluation methods to assess where learners are on their journey.

In Australia, learning progressions have tended to take the form of learning and assessment frameworks such as the LAF developed and validated as part of the Scaffolding Numeracy in the Middle Years Project (Siemon et al., 2006). RMFII was designed along similar lines. By providing teachers with such a framework they are supported to recognise and understand students' learning needs, know what learning aspects should be targeted and how to assist students in their mathematical learning (Siemon et al., 2006). It is expected that by identifying and explaining the 'big ideas' involved in algebraic reasoning, as well as working with teachers to recognise and interpret student learning needs, will assist to improve learning outcomes for students in Years 7-10.

The process of developing the DLP for algebraic reasoning began with a comprehensive review of the literature about algebraic concept development and about learning trajectories and progressions. In this way, it was hoped to identify possible structures as well as for looking for what might sit within those structures. The first draft of the DLP was a synthesis of the research literature which was arbitrarily divided into eight zones of increasingly complex ideas and strategies. Although as researchers who actively work against pre-conceptions of what may be found so as not to influence what was found in the literature, inevitably when designing a DLP prior knowledge was used to group the ideas and strategies.

Once the first draft was in place, a thematic analysis was carried out to determine the 'big ideas' that were emerging. Five themes were identified: Pattern and Sequence; Generalisation; Function; Equivalence; and Equation Solving. There was a discussion about whether Equation Solving was part of the 'big idea' of Equivalence and it was decided to continue to separate them at that stage. The first draft was then examined to consolidate and condense the key ideas and then organised under the five 'big ideas'. This became the second draft of the DLP. An example of the Generalisation 'big idea' is provided in Table 1.

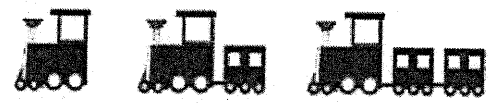
Zone	Generalisation	Sources
1	Explain a generalisation of a simple physical situation.	Carpenter, Franke, & Levi (2003); Panorkou, Maloney, & Confrey (2013); Perso (2003); Schliemann, Carraher, & Brizuela (2007); Watson (2009).
2	Explore and conjecture about patterns in the structure of number, identifying numbers that change and numbers that can vary.	Blanton, & Kaput (2011); Carraher, Schliemann, Bruzella, & Earnest (2006); Mason (2008); Miller, & Warren (2012); Panorkou, Maloney, & Confrey (2013); Perso (2003); Warren, Miller, & Cooper (2011).
3	Explain generalisations by telling stories in words, with materials and using symbols.	Blanton, & Kaput (2003); Mason (2008); Miller, & Warren (2012); Panorkou, Maloney, & Confrey (2013); Perso (2003); Tierney, & Monk (2008); Warren, Miller, & Cooper (2011); Wilkie (2015).
4	Explain generalisations using symbols and explore relationships using technology.	Carpenter, Franke, & Levi (2003); Panorkou, Maloney, & Confrey (2013); Perso (2003); Stacey, & MacGregor (2001); Wilkie (2015).
5	Follow, compare and explain rules for linking successive terms in a	Kaput (1998); Kaput, Blanton, & Moreno (2008); Knuth, Alibali, McNeil, Weinberg, & Stephens (2005); Panorkou,

- sequence or pair quantities using one or two operations.
  - 6 Use and interpret basic algebraic conventions for representing situations involving a variable quantity.
  - 7 Use and interpret algebraic conventions for representing generality and relationships between variables and establish equivalence using the distributive property and inverses of addition and multiplication.
  - 8 Combine facility with symbolic representation and understanding of algebraic concepts to represent and explain mathematical situations.
- Maloney, & Confrey (2013); Perso (2003); Swafford, & Langrall (2000); Tierney, & Monk (2008).  
 Kieran, & Sfard (1998); Perso (2003); Stacey, & MacGregor (2000), Wilkie (2015); Yerushalmy (2000).  
 Panorkou, Maloney, & Confrey (2013); Perso (2003).  
 Panorkou, Maloney, & Confrey (2013); Perso (2003); Yerushalmy (2000).

Table 1: The ‘big idea’ of Generalisation from the second draft learning progression.

The DLP was then used to select, modify and design a range of rich algebraic tasks which were trialled with 1550 students from Years 7-10 providing valid responses. The tasks that were designed contained some items that addressed one of the ‘big ideas’ while others addressed several of the ‘big ideas’ in a single task. Two assessment forms were designed containing only algebraic reasoning, two that included items of both algebraic and statistical reasoning and another two that included both algebraic and spatial reasoning. There were common items across all of the forms. Each of the assessment forms also included one of the validated extended tasks that had been used in the RMF Project, as a benchmark item. An example of a task, Trains (ATRNS) is shown in Figure 1.

**5. Trains**



toy train size 1      toy train size 2      toy train size 3

The engine of the train has 8 wheels, 4 on each side, and each carriage has 6 wheels, 3 on each side.

The table shows the number of wheels on each train:

Train size	1	2	3	4	5	6
Number of wheels	8	14				

**a** ATRNS1: Fill in the table to show the number of wheels for the trains size 3, 4, 5 and 6.

**b** ATRNS2: The largest train set in the toy shop is size 15. How many wheels does the size 15 have? Explain your reasoning, using as much mathematics as you can.

**c** ATRNS3: Bert says his train has 60 wheels. Can Bert be correct? Explain your reasoning, using as much mathematics as you can.

**5. ATRNS1**

SCORE	DESCRIPTION
0	No response or irrelevant response
1	All least two entries correct
2	Table completed correctly [10, 16, 22, 30]

**ATRNS2**

SCORE	DESCRIPTION
0	No response or irrelevant response
1	Correct response (15) with no explanation/working or incorrect response with working or show some understanding of pattern or incorrect with working no show major calculation work
2	Correct response with an explanation that reflects the use of an additive strategy (e.g., goes up by 6 or continues table to a train size of 15)
3	Correct response with an explanation of a multiplicative approach expressed in words or as a rule (e.g., you need to times 15 by 2 and add 2 or 15 x 2 + 2)

**ATRNS3**

SCORE	DESCRIPTION
0	No response or irrelevant response
1	Correct response (No) but with no explanation
2	Correct response with reasoning to support conclusion (e.g., 60 - 2 is 58 and 58 is not divisible by 6 or a size 3 train would have 58 wheels and a size 10 train would have 62 wheels so you can't have a train with 60 wheels)

Figure 1. Trains question with rubric.



The task was designed to allow students fairly easy access at the start but to require explanation of reasoning in the latter parts of the question. The rubrics were designed to value algebraic reasoning over correct answers being provided with no explanation. In the ATRNS2 task a student scores a 1 if incorrect but with reasoning showing some understanding of the pattern. A score of 3, however, required a multiplicative understanding of the relationship with appropriate explanation, which may be in words, symbols or a combination.

## RESULTS

Using Rasch analysis of actual student responses to these three ATRNS items, it was possible to rank the assessment items into eight zones. For example, in Table 2 below, ATRNS1.2 refers to the item ATRNS1 with an achieved score of two points. Each of the seven scores given in Table 2 is then matched with its associated Rasch zone.

ATRNS1.1	ATRNS1.2	ATRNS2.1	ATRNS2.2	ATRNS2.3	ATRNS3.1	ATRNS3.2
Zone 1	Zone 1	Zone 3	Zone 4	Zone 6	Zone 2	Zone 4

Table 2: Results of Rasch analysis on the above items

Responses to the seven assessment scales shown in Table 2 were scaled using Rasch analysis. For this group of items, Rasch scales range from Zone 1 to Zone 6. Completing some of the pattern in the table was the easiest task at Zone 1. Providing a correct answer to item a) with either no or a descriptive (e.g. I counted) explanation or using an additive strategy was the easiest to achieve at Zone 1. Whereas extending the pattern to a larger train (ATRNS2.1) scaled at Zone 3 and giving the correct answer to the larger train, as well as providing a mathematical explanation using a multiplicative strategy, was more difficult for students and was scaled at Zone 6. The sections of the rubrics that required elaborated explanations involving algebraic reasoning, that is ATRNS2.3 and ATRNS3.2, were the most difficult for students being scaled at Zone 6 and Zone 4 respectively. It is noticeable that it is when the students need to explain or provide reasons or even to give partial reasons for their answers that they have the most difficulty. Extending the experimental sample to include older students in the middle years may change these scores, but, even at this preliminary stage of data analysis, it is clear that many students lack confidence or experience when asked to provide explanations for their thinking. The challenge for teachers is to give more careful attention to supporting students' development and articulation of mathematical reasoning.

When analysing the data from the Rasch analysis and mapping it back to the DLP, it was decided that the distinction made between Equivalence and Equation Solving was unnecessary as was the distinction made between Pattern and Sequence and Function, as the Pattern work appeared to overlap with the lower echelons of the Function 'big ideas'. As a result, the original five 'big ideas' were collapsed into three 'big ideas', those of Pattern and Function, Generalisation and Equivalence. Relating the data from the Rasch analysis for this question back to the DLP suggests that rather than the

students explaining the simple patterns at the lowest level they are really only identifying the pattern and for these students the explanations start much later in zones 3 and 4. This indicates the need for the teaching of algebraic reasoning and not just algebraic procedure.

A closer comparison of the zones from the Rasch analysis with the zones of the DLP shows some similarity and some difference. ATRNS1.1 and 1.2 required students to identify and complete, at least partially, a number pattern related to a real situation. This fits within the DLP zone 1 “explain a generalisation of a simple physical situation”. The second part of the question required the students to extrapolate the pattern to data beyond the figures provided in the table, which meant they needed to generalise and apply it. Doing this at a purely numerical level fitted into zone 3 while explaining it partially or additively equated to zone 4, which in the DLP was “explain generalisations using symbols”. The more sophisticated explanation involving multiplicative thinking was at zone 6 of the Rasch model, although it is a closer match to zone 5 of the DLP. This indicates that the Rasch data supports the DLP, at least to some extent, at the lower levels, but further data is needed for the higher zones.

### **Limitations of the Rasch analysis data**

Although there were 1563 students in the database for algebraic reasoning, only 1550 provided valid responses. One of the limitations of using Rasch analysis is that it relies on student responses to assessment items. What was seen from the data was that the items that students perceived to be more difficult were often not attempted which meant that the more challenging algebraic reasoning assessment items were not able to be ranked. More trialling will be necessary, perhaps with older students in the Years 7-10 range, in order to incorporate the more challenging types of assessment items within the eight Rasch zones. As a result it would be expected that some of the data presented here would change zones once the upper zones include more challenging algebraic reasoning items.

### **CONCLUSION**

The development of the DLP involved several stages. The first was an extensive review of the literature on algebraic reasoning and on learning progressions to identify both possible structures for the DLP and what might fit within those structures. Following the literature review a thematic analysis was carried out to identify the ‘big ideas’ that were emerging. Five ‘big ideas’ were identified and the DLP was structured around these headings. Appropriate algebraic reasoning tasks were found, modified or designed based on the DLP and then sent to schools all around Australia for trialling. Once the trial data were received and a Rasch analysis was applied, it was seen that the five ‘big ideas’ could reasonably be collapsed into three ‘big ideas’, those of Pattern and Function, Generalisation and Equivalence. It would appear that more extensive trialling of items that students perceived as difficult will be necessary in order to tease out the upper areas of the DLP. The results indicate a need for the teaching of algebraic reasoning and the encouragement for students to give explanations of their thinking.

As classrooms include more discussion and reasoning the results of such a Rasch analysis might move closer to the DLP which was initially proposed but at the moment there is a great need for targeted teaching of algebraic reasoning.

## References

- Australian Curriculum, Assessment and Reporting Authority (ACARA). (2016). *The Australian curriculum: Mathematics*. Retrieved from <http://www.australiancurriculum.edu.au/mathematics/structure>
- Blanton, M., & Kaput, J. (2011). Functional thinking as a route into algebra in the elementary grades. In J. Cai & E. Knuth (Eds.), *Early algebraization: Advances in mathematics education* (pp. 5-23). Doi: 10.1007/978-3-642-17735-4\_2
- Carpenter, T., Franke, M., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in the elementary school*. Portsmouth, NH: Heinemann.
- Carraher, D., Schliemann, A., Brizuela, B., & Earnest, D. (2006). Arithmetic and algebra in early mathematics education. *Journal for Research in Mathematics Education*, 37(2), 87-115.
- Ellis, A., Weber, E., & Lockwood, E. (2014). The case for learning trajectories research. In S. Oesterle, P. Liljedahl, C. Nicol, & D. Allan (Eds.), *Proc. joint meeting of PME 38 and PME-NA 36* (Vol. 3, pp. 1-8). Vancouver, Canada: PME.
- Kaput, J. (1998). Transforming algebra from an engine of inequity to an engine of mathematical power by “algebrafying” the K-12 curriculum. In S. Fennel (Ed.), *The nature and role of algebra in the K-14 curriculum: Proceedings of a National Symposium* (pp. 25-26). Washington, DC: National Research Council, National Academy Press.
- Kaput, J., Blanton, M., & Moreno, L. (2008). Algebra from a symbolization point of view. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 19-56). New York, NY: Routledge.
- Kieran, C., & Sfard, A. (1999). Seeing through symbols: The case of equivalent expressions. *Focus on Learning Problems in Mathematics*, 21(1), 1-17.
- Knuth, E., Alibali, M., McNeil, N., Weinberg, A., & Stephens, A. (2005). Middle school students’ understanding of core algebraic concepts: Equality and variable. *ZDM International Reviews on Mathematics Education*, 37, 68-76.
- Mason, J. (2008). Making use of children’s powers to produce algebraic thinking. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 57-94). New York, NY: Routledge.
- Miller, J., & Warren, E. (2012). An exploration into growing patterns with young Australian Indigenous students. In J. Dindyal, L. Chang, & S. Ng (Eds.), *Mathematics education: Expanding horizons*, Proceedings of the 35<sup>th</sup> Annual Conference of the Mathematics Education Group of Australasia (pp. 505-512). Singapore: MERGA.
- Panorkou, N., Maloney, A., & Confery, J. (2013). A learning trajectory for early equations and expressions for the common core standards. In N. Martinez, & A. Castro (Eds.), *Proc. 35<sup>th</sup> annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 417-424). Chicago, IL: University of Illinois at Chicago.
- Perso, T. (2003). *Everything you want to know about algebra outcomes for your class, K-9*. Perth, WA: Mathematical Association of Western Australia.

- Schliemann, A., Carraher, D., & Brizuela, B. (2007). *Bringing out the algebraic character of arithmetic: From children's ideas to classroom practice*. Hillsdale, NJ: Lawrence Erlbaum Associates.
- Siemon, D., Izard, J., Breed, M., & Virgona, J. (2006). The derivation of learning assessment framework for multiplicative thinking. In J. Novotna, H. Moraova, M. Kratka, & N. Stehlikova (Eds.) *Proc 30<sup>th</sup> Conf. of Int. Group for the Psychology of Mathematics Education*, (Vol. 5, pp. 113-120). Prague, Czech Republic: PME.
- Siemon, D., Izard, J., Stephens, M., Dole, S., Breed, M., & Virgona, J. (2006). *Scaffolding numeracy in the middle years (SNMY) resources*. Retrieved from <http://www.education.vic.gov.au/school/teachers/teachingresources/discipline/maths/assessment/Pages/resourcelibrary.aspx#2>
- Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, 26(2), 114-145.
- Stacey, K., & MacGregor, M. (2001). Curriculum reform and approaches to algebra. In R. Sutherland, T. Rojano, A. Bell, & R. Lins (Eds.), *Perspectives on school algebra* (pp. 141-153). Dordrecht, The Netherlands: Kluwer.
- Swafford, J., & Langrall, C. (2000). Grade 6 students' preinstructional use of equations to describe and represent problem situations. *Journal for Research in Mathematics Education*, 31(1), 89-112.
- Tierney, C., & Monk, S. (2008). Children's reasoning about change over time. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 185-200). New York, NY: Routledge.
- Watson, A. (2009). *Key understandings in mathematics learning, Paper 6: Algebraic reasoning*. London, UK: Nuffield Foundation.
- Warren, E., Miller, J., & Cooper, T. (2011). An exploration of young students' ability to generalise function tasks. In P. Sullivan, & M. Goos (Eds.) *Mathematics: Traditions and (new) practices*. Proceedings of 34<sup>th</sup> Annual Conference of the Mathematics Education Research Group of Australasia (pp. 752-759). Alice Springs, NT: MERGA.
- Wilkie, K. (2015). Exploring early secondary students' algebraic generalisation in geometric contexts. In K. Beswick, T. Muir, & J. Wells (Eds.), *Proc. 39<sup>th</sup> Conf. of Int. Group for the Psychology of Mathematics Education*, (Vol. 4, pp. 297-304). Hobart, Australia: PME.
- Yerushalmy, M. (2000). Problem solving strategies and mathematical resources: A longitudinal view on problem solving in a functions based approach to algebra. *Educational Studies in Mathematics*, 43, 125-147.