RELATIONS BETWEEN THE CHOW MOTIVE
AND THE NONCOMMUTATIVE MOTIVE
OF A SMOOTH PROJECTIVE VARIETY

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ABSTRACT. In this note we relate the notions of Lefschetz type, decomposability, and isomorphism, on Chow motives with the notions of unit type, decomposability, and isomorphism, on noncommutative motives. Examples, counter-examples, and applications are also described.

1. INTRODUCTION

Let k be a base field and R a commutative ring of coefficients.

Chow motives. In the early sixties Grothendieck envisioned the existence of a "universal" cohomology theory of schemes. Among several conjectures and developments, a contravariant $\otimes$-functor

$$M(-)_R : \mathrm{SmProj}(k)^{\mathrm{op}} \longrightarrow \mathsf{Chow}(k)_R$$

from smooth projective k-schemes to *Chow motives* (with R coefficients) was constructed. Intuitively speaking, $\mathsf{Chow}(k)_R$ encodes all the geometric/arithmetic information about smooth projective k-schemes and acts as a gateway between algebraic geometry and the assortment of the numerous Weil cohomology theories such as de Rham, Betti, l-adic, crystalline, etc; see [1, 11, 24].

Noncommutative motives. A *differential graded (=dg) category* $\mathcal{A}$ is a category enriched over complexes of k-vector spaces; see §3.2. Every (dg) k-algebra A gives naturally rise to a dg category $\underline{A}$ with a single object and (dg) k-algebra of endomorphisms A. Another source of examples is provided by k-schemes since the category of perfect complexes $\mathrm{perf}(X)$ of every smooth projective k-scheme X admits a unique dg enhancement $\mathrm{perf}_{\mathsf{dg}}(X)$; see [23]. All the classical invariants such as algebraic K-theory, cyclic homology, and topological Hochschild homology, extend naturally from k-algebras (and from k-schemes) to dg categories. In order to study all these invariants simultaneously the notion of additive invariant was introduced in [36]. Roughly speaking, a functor $E : \mathrm{dgcat}(k) \to \mathsf{D}$ from the category of dg categories towards an additive category is called *additive* if it inverts Morita equivalences and sends semi-orthogonal decompositions to direct sums. Thanks to the work [5, 17, 18, 30, 34, 35, 37, 38], all the above mentioned invariants are additive. In [36] the universal additive invariant was also constructed

$$(1.1) \qquad\qquad U(-)_R : \mathrm{dgcat}(k) \longrightarrow \mathsf{Hmo}_0(k)_R .$$

Given any R-linear additive category D, there is an induced equivalence of categories

$$(1.2) \qquad U(-)_R^* : \mathrm{Fun}_{\mathsf{add}}(\mathsf{Hmo}_0(k)_R, \mathsf{D}) \xrightarrow{\sim} \mathrm{Fun}_{\mathsf{A}}(\mathrm{dgcat}(k), \mathsf{D}),$$

where the left-hand side denotes the category of additive functors and the right-hand side the category of additive invariants. Because of this universal property, which is reminiscent from motives, $\mathsf{Hmo}_0(k)_R$ is called the category of *noncommutative motives*. The tensor product of k-algebras extends also naturally to dg categories giving rise to a symmetric monoidal structure $-\otimes-$ on $\mathrm{dgcat}(k)$ which descends to $\mathsf{Hmo}_0(k)_R$ making the above functor (1.1) symmetric monoidal.

Motivating questions. Let X be a smooth projective k-scheme. In order to study it we can proceed in two distinct directions. On one direction we can associate to X its Chow motive $M(X)_R$. On another direction we can associate to X the noncommutative motive $U(\mathrm{perf}_{\mathsf{dg}}(X))_R$. Note that while $M(X)_R$ encodes all the information about the numerous Weil cohomology theories of X, $U(\mathrm{perf}_{\mathsf{dg}}(X))_R$ encodes all the information about the different additive invariants of $\mathrm{perf}_{\mathsf{dg}}(X)$.

Let $\mathbf{L} \in \mathsf{Chow}(k)_R$ be the Lefschetz motive and $\mathbf{1} := U(\underline{k})_R$ the $\otimes$-unit of $\mathsf{Hmo}_0(k)_R$. Following [10], a Chow motive is called of *Lefschetz R-type* if it is isomorphic to $\mathbf{L}^{\otimes l_1} \oplus \cdots \oplus \mathbf{L}^{\otimes l_m}$ for some choice of non-negative integers $l_1, \ldots, l_m$. In the same vein, a noncommutative motive in $\mathsf{Hmo}_0(k)_R$ is called of *unit R-type* if it is isomorphic to $\oplus_{i=1}^m \mathbf{1}$ for a certain non-negative integer m. The following implication was proved in [10, §4] (assuming that $\mathbb{Z} \subseteq R$):

$$(1.3) \qquad M(X)_R \text{ Lefschtez } R\text{-type} \Rightarrow U(\mathrm{perf}_{\mathsf{dg}}(X))_R \text{ unit } R\text{-type}.$$

In the particular case where $R = \mathbb{Q}$, this implication becomes an equivalence

$$(1.4) \qquad M(X)_{\mathbb{Q}} \text{ Lefschetz } \mathbb{Q}\text{-type} \Leftrightarrow U(\mathrm{perf}_{\mathsf{dg}}(X))_{\mathbb{Q}} \text{ unit } \mathbb{Q}\text{-type};$$

see [25, §1]. Hence, it is natural to ask the following:

Question A: Does the above implication (1.3) *admits a partial converse ?*

Recall that an object in an additive category is called indecomposable if its only non-trivial idempotent endomorphisms are $\pm$ the identity; otherwise it is called decomposable. Our second motivating question is the following:

Question B: What is the relation between the (in)decomposability of $M(X)_R$ and the (in)decomposability of $U(\mathrm{perf}_{\mathsf{dg}}(X))_R$?

Let X and Y be smooth projective k-schemes. Another motivating question is:

Question C: Does the following implication holds

$$(1.5) \qquad M(X)_R \simeq M(Y)_R \Rightarrow U(\mathrm{perf}_{\mathsf{dg}}(X))_R \simeq U(\mathrm{perf}_{\mathsf{dg}}(Y))_R \text{ ?}$$

How about its converse ?

In this note we provide precise answers to these questions; consult Corollaries 2.4 and 2.12 for applications.

2. Statements of results

Our first main result, which answers Question A, is the following:

Theorem 2.1. *Let X be an irreducible smooth projective k-scheme of dimension d. Assume that $\mathbb{Z} \subseteq R$ and that every finitely generated projective $R[1/(2d)!]$-module is free (e.g. R a principal ideal domain). Assume also that $U(\mathrm{perf}_{\mathsf{dg}}(X))_R \simeq \oplus_{i=1}^m \mathbf{1}$*

for a certain non-negative integer m. Under these assumptions, there is a choice of integers (up to permutation) $l_1, \ldots, l_m \in \{0, \ldots, d\}$ giving rise to an isomorphism

$$M(X)_{R[1/(2d)!]} \simeq \mathbf{L}^{\otimes l_1} \oplus \cdots \oplus \mathbf{L}^{\otimes l_m} .$$

Intuitively speaking, Theorem 2.1 shows that the converse of the above implication (1.3) holds as soon as one inverts the integer $(2d)!$ (or equivalently its prime factors). By combining this result with (1.3), one obtains a refinement of (1.4):

Corollary 2.2. *Given X and R as in Theorem 2.1, we have the equivalence*

$$M(X)_{R[1/(2d)!]} \text{ Lefschetz-type} \Leftrightarrow U(\mathrm{perf}_{\mathsf{dg}}(X))_{R[1/(2d)!]} \text{ unit-type} .$$

In the particular case where X is a curve C and $R = \mathbb{Z}$, Corollary 2.2 reduces to

$$M(C)_{\mathbb{Z}[1/2]} \text{ Lefschetz } \mathbb{Z}[1/2]\text{-type} \Leftrightarrow U(\mathrm{perf}_{\mathsf{dg}}(C))_{\mathbb{Z}[1/2]} \text{ unit } \mathbb{Z}[1/2]\text{-type} .$$

Moreover, since the prime factors of 4! are $\{2, 3\}$, one has

$$M(S)_{\mathbb{Z}[1/6]} \text{ Lefschetz } \mathbb{Z}[1/6]\text{-type} \Leftrightarrow U(\mathrm{perf}_{\mathsf{dg}}(S))_{\mathbb{Z}[1/6]} \text{ unit } \mathbb{Z}[1/6]\text{-type}$$

for every surface S. As the following proposition shows, the (strict) converse of implication (1.3) is false !

Proposition 2.3. *(see §5) Let q be a non-singular quadratic form and Q_q the associated smooth projective quadric. Assume that q is even dimensional, anisotropic, and with trivial discriminant and trivial Clifford invariant. Under these assumptions, $M(Q_q)_{\mathbb{Z}}$ is not of Lefschetz $\mathbb{Z}$-type while $U(\mathrm{perf}_{\mathsf{dg}}(Q_q))_{\mathbb{Z}}$ is of unit $\mathbb{Z}$-type (and hence of unit R-type for every commutative ring R).*

Proposition 2.3 applies to all 3-fold Pfister forms and to all elements of the third power of the fundamental ideal $I(k)$ of the Witt ring $W(k)$; see Example 5.4.

As an application of Theorem 2.1, we obtain the following sharpening of the main result of [25]; recall from *loc. cit.* that the isomorphism (2.5) below was obtained only with rational coefficients.

Corollary 2.4. *Let X be an irreducible smooth projective k-scheme of dimension d. Assume that $\mathrm{perf}(X)$ admits a full exceptional collection $(\mathcal{E}_1, \ldots, \mathcal{E}_m)$ of length m. Under these assumptions, there is a choice of integers (up to permutation) $l_1, \ldots, l_m \in \{0, \ldots, d\}$ giving rise to an isomorphism*

$$(2.5) \qquad M(X)_{\mathbb{Z}[1/(2d)!]} \simeq \mathbf{L}^{\otimes l_1} \oplus \cdots \oplus \mathbf{L}^{\otimes l_m} .$$

Thanks to the work [3, 12, 15, 20, 27], Corollary 2.4 applies to projective spaces and rational surfaces (in the case of an arbitrary base field k), and to smooth quadric hypersurfaces, Grassmannians, flag varieties, Fano threefolds with vanishing odd cohomology, and toric varieties (in the case where $k = \mathbb{C}$). Conjecturally, it applies also to all the homogeneous spaces of the form G/P, with P a parabolic subgroup of a semisimple algebraic group G; see [21].

Our second main result, which partially answers Question B, is the following:

Theorem 2.6. *Let X be an irreducible smooth projective k-scheme of dimension d. Under the assumption $\mathbb{Z} \subseteq R$, the following implication holds:*

$$(2.7) \qquad M(X)_{R[1/(2d)!]} \text{ decomposable} \Rightarrow U(\mathrm{perf}_{\mathsf{dg}}(X))_{R[1/(2d)!]} \text{ decomposable} .$$

It is unclear to the authors if the (strict) converse of (2.7) also holds. As the following proposition shows, if one does not invert the dimension of X, this is false !

Proposition 2.8. *Let A be a central simple k-algebra of degree $\sqrt{\dim(A)} = d$ and $X = \mathrm{SB}(A)$ the associated Severi-Brauer variety.*

(i) *For every commutative ring R one has the following motivic decomposition*

$$(2.9) \qquad U(\mathrm{perf}_{\mathsf{dg}}(X))_R \simeq \mathbf{1} \oplus U(\underline{A})_R \oplus U(\underline{A})_R^{\otimes 2} \oplus \cdots \oplus U(\underline{A})_R^{\otimes d-1} \, .$$

In particular, the noncommutative motive $U(\mathrm{perf}_{\mathsf{dg}}(X))_R$ is decomposable.

(ii) *(Karpenko) When A is moreover a division algebra and $d = p^s$ for a certain prime p and integer $s \geq 1$, then the Chow motive $M(X)_{\mathbb{Z}}$ (and even $M(X)_{\mathbb{Z}/p\mathbb{Z}}$) is indecomposable.*

Proposition 2.8 shows that the decomposition (2.9) is "truly noncommutative".

Given a smooth projective k-scheme X and an integer l, let us write $M(X)_R(l)$ instead of $M(X)_R \otimes \mathbf{L}^{\otimes l}$. Our third main result, which in particular answers Question C, is the following:

Theorem 2.10. *Let $\{X_i\}_{1 \leq i \leq n}$ (resp. $\{Y_j\}_{1 \leq j \leq m}$) be irreducible smooth projective k-schemes of dimension d_i (resp. d_j), $d := \max\{d_i, d_j \,|\, i, j\}$, and $\{l_i\}_{1 \leq i \leq n}$ (resp. $\{l_j\}_{1 \leq j \leq m}$) arbitrary integers. Assume that $\mathbb{Z} \subseteq R$ and $1/(2d)! \in R$. Under these assumptions, we have the following implication*

$$\oplus_i M(X_i)_R(l_i) \simeq \oplus_j M(Y_j)_R(l_j) \Rightarrow \oplus_i U(\mathrm{perf}_{\mathsf{dg}}(X_i))_R \simeq \oplus_j U(\mathrm{perf}_{\mathsf{dg}}(Y_j))_R \, .$$

It is unclear to the authors if the (strict) converse of Theorem 2.10 also holds. As the following (counter-)example shows, this is false in general !

Example 2.11. The Chow motives $M(X)_{\mathbb{Z}}$ and $M(\widehat{X})_{\mathbb{Z}}$ of an abelian variety X and of its dual $\widehat{X}$ are in general not isomorphic. However, thanks to the work [26], we have $U(\mathrm{perf}_{\mathsf{dg}}(X))_R \simeq U(\mathrm{perf}_{\mathsf{dg}}(\widehat{X}))_R$ for every commutative ring R.

Finally, by combining Theorem 2.10 with (1.2), we obtain the application:

Corollary 2.12. *Let X (resp. Y) be an irreducible smooth projective k-scheme of dimension d_X (resp. d_Y), and $d := \max\{d_X, d_Y\}$. Assume that $\mathbb{Z} \subseteq R$ and $1/(2d)! \in R$. Under these assumptions, we have the following implication*

$$M(X)_R \simeq M(Y)_R \Rightarrow E(X) \simeq E(Y)$$

for every additive invariant E with values in a R-linear category.

3. Preliminaries

3.1. **Orbit categories.** Let $\mathcal{C}$ be an additive symmetric monoidal category and $\mathcal{O} \in \mathcal{C}$ a $\otimes$-invertible object. Recall from [33, §7] that the *orbit category* $\mathcal{C}/_{-\otimes\mathcal{O}}$ has the same objects as $\mathcal{C}$ and morphisms

$$\mathrm{Hom}_{\mathcal{C}/_{-\otimes\mathcal{O}}}(a, b) := \oplus_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(a, b \otimes \mathcal{O}^{\otimes j}) \, .$$

Given objects a, b and c and morphisms

$$\underline{f} = \{f_j\}_{j \in \mathbb{Z}} \in \oplus_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(a, b \otimes \mathcal{O}^{\otimes j}) \quad \underline{g} = \{g_k\}_{k \in \mathbb{Z}} \in \oplus_{k \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(b, c \otimes \mathcal{O}^{\otimes k}) \, ,$$

the lth-component of the composition $\underline{g} \circ \underline{f}$ is given by the finite sum $\sum_j ((g_{l-j} \otimes \mathcal{O}^{\otimes j}) \circ f_j)$. We obtain in this way an additive category $\mathcal{C}/_{-\otimes\mathcal{O}}$ and a canonical additive projection functor $\pi : \mathcal{C} \to \mathcal{C}/_{-\otimes\mathcal{O}}$. Note that π comes equipped with a natural isomorphism $\pi \circ (-\otimes\mathcal{O}) \overset{\sim}{\Rightarrow} \pi$ and that it is universal among all such functors. Note also that this construction is functorial: given any other symmetric monoidal

category $\mathcal{D}$, a $\otimes$-invertible object $\mathcal{O}' \in \mathcal{D}$, and an additive $\otimes$-functor $\mathcal{C} \to \mathcal{D}$ which sends $\mathcal{O}$ to $\mathcal{O}'$, we obtain an induced additive functor $\mathcal{C}/_{-\otimes\mathcal{O}} \to \mathcal{D}/_{-\otimes\mathcal{O}'}$.

3.2. **Dg categories.** Let $\mathcal{C}(k)$ be the category of complexes of k-vector spaces. A *differential graded (=dg) category* $\mathcal{A}$ is a category enriched over $\mathcal{C}(k)$ (morphisms sets $\mathcal{A}(x, y)$ are complexes) in such a way that composition fulfills the Leibniz rule $d(f \circ g) = d(f) \circ g + (-1)^{\deg(f)} f \circ d(g)$. A *dg functor* $F : \mathcal{A} \to \mathcal{B}$ is a functor enriched over $\mathcal{C}(k)$; consult [16]. In what follows we will write $\mathrm{dgcat}(k)$ for the category of (small) dg categories and dg functors.

Let $\mathcal{A}$ be a dg category. The *opposite* dg category $\mathcal{A}^{\mathrm{op}}$ has the same objects as $\mathcal{A}$ and complexes of morphisms given by $\mathcal{A}^{\mathrm{op}}(x, y) := \mathcal{A}(y, x)$. A *right $\mathcal{A}$-module M* is a dg functor $M : \mathcal{A}^{\mathrm{op}} \to \mathcal{C}_{\mathsf{dg}}(k)$ with values in the dg category $\mathcal{C}_{\mathsf{dg}}(k)$ of complexes of k-vector spaces. Let us denote by $\mathcal{C}(\mathcal{A})$ the category of right $\mathcal{A}$-modules; see [16, §2.3]. Recall from [16, §3.2] that the *derived category $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$* is the localization of $\mathcal{C}(\mathcal{A})$ with respect to the class of objectwise quasi-isomorphisms. A dg functor $F : \mathcal{A} \to \mathcal{B}$ is called a *Morita equivalence* if the restriction of scalars functor $\mathcal{D}(\mathcal{B}) \xrightarrow{\sim} \mathcal{D}(\mathcal{A})$ is an equivalence of (triangulated) categories; see [16, §4.6].

The *tensor product $\mathcal{A} \otimes \mathcal{B}$* of two dg categories $\mathcal{A}$ and $\mathcal{B}$ is defined as follows: the set of objects is the cartesian product of the sets of objects of $\mathcal{A}$ and $\mathcal{B}$ and the complexes of morphisms are given by $(\mathcal{A} \otimes \mathcal{B})((x, z), (y, w)) := \mathcal{A}(x, y) \otimes \mathcal{B}(z, w)$.

3.3. K_0-**motives.** Recall from [8, §5.2][9, §5.2] the construction of the category $\mathsf{KM}(k)_R$ of K_0-*motives*. As explained in *loc. cit.*, its objects are the smooth projective k-schemes, its morphisms are given by

$$\mathrm{Hom}_{\mathsf{KM}(k)_R}(X, Y) := K_0(X \times Y)_R = K_0(X \times Y)_{\mathbb{Z}} \otimes_{\mathbb{Z}} R,$$

and its symmetric monoidal structure is induced by the product of k-schemes. Furthermore, $\mathsf{KM}(k)_R$ comes equipped with a canonical (contravariant) $\otimes$-functor

$$(3.1) \qquad M_0(-)_R : \mathrm{SmProj}(k)^{\mathrm{op}} \longrightarrow \mathsf{KM}(k)_R \qquad X \mapsto X$$

that sends a morphism $f : X \to Y$ in $\mathrm{SmProj}(k)$ to the class $[\mathcal{O}_{\Gamma_f^t}] \in K_0(Y \times X)_R$ of the transpose Γ_f^t of the graph $\Gamma_f := \{(x, f(x)) \mid x \in X\} \subset X \times Y$ of f.

Notation 3.2. Given a finite family $X_1, \ldots, X_n$ of irreducible smooth projective k-schemes of dimensions $d_1, \ldots, d_n$, let us denote by $(X_1, \ldots, X_n)_R$ the full subcategory of $\mathsf{KM}(k)_R$ consisting of the objects $\{M_0(X_i)_R \mid 1 \le i \le n\}$. Its closure (inside $\mathsf{KM}(k)_R$) under finite direct sums will be denoted by $(X_1, \ldots, X_n)_R^{\oplus}$.

As explained in [25, §4.4], there is a well-defined R-linear additive *fully faithful* $\otimes$-functor θ making the following diagram commute

$$(3.3) \qquad \begin{array}{ccc} \mathrm{SmProj}(k)^{\mathrm{op}} & \xrightarrow{\ \mathrm{perf}_{\mathsf{dg}}(-)\ } & \mathrm{dgcat}(k) \\ {\scriptstyle M_0(-)_R}\downarrow & & \downarrow{\scriptstyle U(-)_R} \\ \mathsf{KM}(k)_R & \xrightarrow[\ \theta\ ]{} & \mathsf{Hmo}_0(k)_R\,. \end{array}$$

Intuitively speaking, the category of K_0-motives embeds fully faithfully into the category of noncommutative motives.

4. Proof of Theorem 2.1

Recall that by assumption $\mathbb{Z} \subseteq R$. Let us then denote by $R[1/\mathbb{Z}]$ the localization of R at $\mathbb{Z}\backslash\{0\}$.

Proposition 4.1. *There exists a well-defined additive functor Ψ making the following diagram commute*

$$(4.2) \qquad \begin{array}{ccc} \mathrm{SmProj}(k)^{\mathrm{op}} & =\!\!=\!\!=\!\!=\!\!= & \mathrm{SmProj}(k)^{\mathrm{op}} \\ {\scriptstyle M(-)_{R[1/\mathbb{Z}]}} \downarrow & & \downarrow {\scriptstyle M_0(-)_R} \\ \mathsf{Chow}(k)_{R[1/\mathbb{Z}]} & & \\ {\scriptstyle \pi} \downarrow & & \\ \mathsf{Chow}(k)_{R[1/\mathbb{Z}]}/_{-\otimes R[1/\mathbb{Z}](1)} & \underset{\Psi}{\longleftarrow} & \mathsf{KM}(k)_R \, , \end{array}$$

where $R[1/\mathbb{Z}](1) \in \mathsf{Chow}(k)_{R[1/\mathbb{Z}]}$ denotes the Tate motive.

Proof. Let X and Y be irreducible smooth projective k-schemes of dimensions d_X and d_Y. As explained in [1, §4], given $j \in \mathbb{Z}$, one has a canonical isomorphism

$$\mathrm{Hom}_{\mathsf{Chow}(k)_{R[1/\mathbb{Z}]}}(M(X)_{R[1/\mathbb{Z}]}, M(Y)_{R[1/\mathbb{Z}]} \otimes R[1/\mathbb{Z}](1)^{\otimes j}) \simeq CH^{d_X+j}(X \times Y)_{R[1/\mathbb{Z}]} \, ,$$

where $CH^{d_X+j}(X \times Y)_{R[1/\mathbb{Z}]}$ denotes the $R[1/\mathbb{Z}]$-linear Chow group of algebraic cycles of codimension $d_X + j$ on $X \times Y$ modulo rational equivalence. By definition of the orbit category category, the $R[1/\mathbb{Z}]$-module

$$\mathrm{Hom}_{\mathsf{Chow}(k)_{R[1/\mathbb{Z}]}/_{-\otimes R[1/\mathbb{Z}](1)}}(\pi(M(X)_{R[1/\mathbb{Z}]}), \pi(M(Y)_{R[1/\mathbb{Z}]}))$$

identifies with

$$\oplus_{j\in\mathbb{Z}} CH^{d_X+j}(X \times Y)_{R[1/\mathbb{Z}]} = \oplus_{i=0}^{d_X+d_Y} CH^i(X \times Y)_{R[1/\mathbb{Z}]} \, .$$

This shows us that the category $\mathsf{Chow}(k)_{R[1/\mathbb{Z}]}/_{-\otimes R[1/\mathbb{Z}](1)}$ agrees with the category $\mathbf{CHM}_{kR[1/\mathbb{Z}]}$ of *all* correspondences considered at [9, page 3128]. On the other hand, recall from §3.3 that

$$\mathrm{Hom}_{\mathsf{KM}(k)_R}(M_0(X)_R, M_0(Y)_R) = K_0(X \times Y)_R \, .$$

The searched functor Ψ is defined on objects by sending $M_0(X)_R$ to $\pi(M(X)_{R[1/\mathbb{Z}]})$. On morphisms is defined by the following assignment

$$K_0(X \times Y)_R \longrightarrow \oplus_{i=0}^{d_X+d_Y} CH^i(X \times Y)_{R[1/\mathbb{Z}]} \qquad \alpha \mapsto ch(\alpha) \cdot \pi_Y^*(\mathrm{Td}(Y)) \, ,$$

where $ch(-)$ denotes the Chern character, $\mathrm{Td}(Y)$ the Todd class of Y and π_Y the projection $X \times Y \to Y$ morphism. As explained at [9, page 3128], it follows from the Grothendieck-Riemann-Roch theorem that the above assignments give rise to a well-defined additive functor Ψ. The fact that the diagram (4.2) commutes follows also from the Grothendieck-Riemann-Roch theorem; see [9, page 3129]. $\qquad \square$

Consider the following diagram of additive functors

$$\mathsf{Chow}(k)_R \to \cdots \to \mathsf{Chow}(k)_{R[1/n!]} \to \mathsf{Chow}(k)_{R[1/(n+1)!]} \to \cdots \to \mathsf{Chow}(k)_{R[1/\mathbb{Z}]} \, .$$

Since the Tate motive $R(1)$ is mapped to itself, the functoriality of $(-)/_{-\otimes R(1)}$ gives rise to the following diagram

$$\cdots\cdots\cdots \to \mathsf{Chow}(k)_{R[1/n!]}/_{-\otimes R(1)} \to \cdots \to \mathsf{Chow}(k)_{R[1/\mathbb{Z}]}/_{-\otimes R[1/\mathbb{Z}](1)} \, .$$

Proposition 4.3. *Given a finite family of irreducible smooth projective k-schemes $X_1, \ldots, X_n$ of dimensions $d_1, \ldots, d_n$, the composition (see Notation 3.2)*

$$(X_1, \ldots, X_n)_R \subset \mathsf{KM}(k)_R \xrightarrow{\Psi} \mathsf{Chow}(k)_{R[1/\mathbb{Z}]}/_{-\otimes R[1/\mathbb{Z}](1)}$$

factors through the functor

$$(\mathsf{Chow}(k)_{R[1/(2d)!]}/_{-\otimes R(1)})^\natural \longrightarrow \mathsf{Chow}(k)_{R[1/\mathbb{Z}]}/_{-\otimes R[1/\mathbb{Z}](1)}$$

where $d := \max\{d_1, \ldots, d_n\}$.

Proof. Let X_r and X_s be any two k-schemes in $\{X_1, \ldots, X_n\}$. From the construction of the functor Ψ (see the proof of Proposition 4.1), it is clear that it suffices to show that the homomorphism

$$K_0(X_r \times X_s)_R \longrightarrow \oplus_{i=0}^{d_r+d_s} CH^i(X_r \times X_s)_{R[1/\mathbb{Z}]} \qquad \alpha \mapsto ch(\alpha) \cdot \pi_{X_s}^*(\mathrm{Td}(X_s))$$

factors through

$$\oplus_{i=0}^{d_r+d_s} CH^i(X_r \times X_s)_{R[1/(2d)!]} \longrightarrow \oplus_{i=0}^{d_r+d_s} CH^i(X_r \times X_s)_{R[1/\mathbb{Z}]}\,.$$

Recall from [28, §1] that the Chern character $ch(\alpha)$ is given by $\sum_{m\geq 0} \frac{\mathfrak{S}_m(\alpha)}{m!}$ where $\mathfrak{S}_m(\alpha)$ is a polynomial with integral coefficients in the Chern classes $c_i(\alpha) \in CH^i(X_r \times X_s)_{\mathbb{Z}}$. Moreover, since $X_r \times X_s$ is of dimension $d_r + d_s$, one has $\mathfrak{S}_m(\alpha) = 0$ for $m > d_r + d_s$. Hence, in order to show that $ch(\alpha)$ belongs to $\oplus_{i=0}^{d_r+d_s} CH^i(X_r \times X_s)_{R[1/(2d)!]}$, it suffices to show that the numbers $\{m! \,|\, 0 \leq m \leq d_r + d_s\}$ are invertible in $R[1/(2d)!]$. This follows from Lemma 4.5(i) below since $d_r + d_s \leq 2d$. Now, recall again from [28, §1] that the Todd class $\mathrm{Td}(X_s)$ is given by $\sum_{m\geq 0} \frac{\mathfrak{D}_m}{T_m}$. Here, $\mathfrak{D}_m$ is a polynomial with integral coefficients in the Chern classes and T_m is the product $\prod_p p^{[\frac{m}{p-1}]}$ taken over all the prime numbers; the symbol $[-]$ stands for the integral part. Since X_s is of dimension d_s, one has also $\mathfrak{D}_m = 0$ for $m > d_s$. Moreover, we have a well-defined ring homomorphism

$$(4.4) \qquad \pi_{X_s}^* : \oplus_{i=0}^{d_s} CH^i(X_s)_{R[1/(2d)!]} \longrightarrow \oplus_{i=0}^{d_r+d_s} CH^i(X_r \times X_s)_{R[1/(2d)!]}\,.$$

Therefore, in order to show that $\mathrm{Td}(X_s)$ belongs to $\oplus_{i=0}^{d_s} CH^i(X_s)_{R[1/(2d)!]}$ (which by (4.4) implies that $\pi_{X_s}^*(\mathrm{Td}(X_s))$ belongs to $\oplus_{i=0}^{d_r+d_s} CH^i(X_r \times X_s)_{R[1/(2d)!]}$), it suffices to show that the numbers $\{T_m \,|\, 0 \leq m \leq d_s\}$ are invertible in $R[1/(2d)!]$. Since $d_s \leq d$ this follows from Lemma 4.5(ii) below. $\qquad \square$

Lemma 4.5. *Let d be a non-negative integer.*
  (i) *If $m \leq 2d$ then $m!$ is invertible in $R[1/(2d)!]$.*
  (ii) *If $m \leq d$ then $T_m := \prod_p p^{[\frac{m}{p-1}]}$ is invertible in $R[1/(2d)!]$.*

Proof. Item (i) follows simply from the fact that $m!$ is a factor of $(2d)!$. In what concerns item (ii) note first that the case $d = 0$ is trivial since $T_0 = 1$. Let us then assume that $d \geq 1$ and $m \geq 1$. Note that $[\frac{m}{p-1}] \neq 0$ if and only if $p - 1 < m$. Hence, the prime factors of T_m are the prime numbers p' such that $p' \leq m+1 \leq d+1 \leq 2d$; in the last inequality we use the assumption $d \geq 1$. By item (i) this then implies that T_m is invertible in $R[1/(2d)!]$. $\qquad \square$

We now have all the ingredients needed for the conclusion of the proof of Theorem 2.1. Recall that by hypothesis $U(\mathrm{perf}_{\mathsf{dg}}(X))_R \simeq \oplus_{i=1}^m \mathbf{1}$ for a certain non-negative integer m. Since $\mathbf{1} = U(\underline{k})_R \simeq U(\mathrm{perf}_{\mathsf{dg}}(\mathrm{spec}(k)))_R$ one concludes from the commutativity of diagram (3.3) and from the additiveness and fully faithfulness

of θ that $M_0(X)_R \simeq \oplus_{i=1}^m M_0(\mathrm{spec}(k))_R$. Recall also that by construction the orbit category $\mathsf{Chow}(k)_{R[1/(2d)!]}/_{-\otimes R(1)}$ is additive. Hence, by extending the functor Ψ to finite direct sums (see Notation 3.2) one obtains a well-defined additive functor

$$(4.6) \qquad \Psi^{\oplus} : (X_1, \ldots, X_n)_R^{\oplus} \longrightarrow \mathsf{Chow}(k)_{R[1/(2d)!]}/_{-\otimes R(1)}.$$

Note that the above isomorphism $M_0(X)_R \simeq \oplus_{i=1}^m M_0(\mathrm{spec}(k))_R$ belongs to the category $(X, \mathrm{spec}(k))^{\oplus}$. By applying to it the above functor (4.6) one then obtains

$$(4.7) \qquad \pi(M(X)_{R[1/(2d)!]}) \simeq \oplus_{i=1}^m \pi(M(\mathrm{spec}(k))_{R[1/(2d)!]}).$$

Consequently, using the equalities

$$\begin{array}{rcl} \oplus_{i=1}^m (\pi(M(\mathrm{spec}(k))_{R[1/(2d)!]})) & = & \pi(\oplus_{i=1}^m M(\mathrm{spec}(k))_{R[1/(2d)!]}) \\ (\oplus_{i=1}^m M(\mathrm{spec}(k))_{R[1/(2d)!]}) \otimes R(j) & = & \oplus_{i=1}^m R(j), \end{array}$$

there exist morphisms in the orbit category

$$\underline{f} = \{f_j\}_{j \in \mathbb{Z}} \in \oplus_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{Chow}(k)_{R[1/(2d)!]}}(M(X), \oplus_{i=1}^m R(j))$$

$$\underline{g} = \{g_k\}_{k \in \mathbb{Z}} \in \oplus_{k \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{Chow}(k)_{R[1/(2d)!]}}(\oplus_{i=1}^m M(\mathrm{spec}(k)), M(X) \otimes R(k))$$

verifying the equalities $\underline{g} \circ \underline{f} = \mathrm{id} = \underline{f} \circ \underline{g}$; note that we have removed some subscripts in order to simplify the exposition. As explained in [1, §4], one has

$$\mathrm{Hom}_{\mathsf{Chow}(k)_{R[1/(2d)!]}}(M(X), \oplus_{i=1}^m R(j)) \simeq \oplus_{i=1}^m CH^{d+j}(X)_{R[1/(2d)!]}$$

and also the isomorphism

$$\mathrm{Hom}_{\mathsf{Chow}(k)_{R[1/(2d)!]}}(\oplus_{i=1}^m M(\mathrm{spec}(k)), M(X) \otimes R(k)) \simeq \oplus_{i=1}^m CH^k(X)_{R[1/(2d)!]}.$$

As a consequence, $f_j = 0$ for $j \neq \{-d, \ldots, 0\}$ and $g_k = 0$ for $k \neq \{0, \ldots, d\}$. The sets of morphisms $\{f_{-l} \mid 0 \leq l \leq d\}$ and $\{g_l \otimes R(1)^{\otimes(-l)} \mid 0 \leq l \leq d\}$ give then rise to well-defined morphisms

$$\begin{array}{rcl} \alpha : M(X)_{R[1/(2d)!]} & \longrightarrow & \oplus_{l=0}^d \oplus_{i=1}^m R(1)^{\otimes(-l)} \\ \beta : \oplus_{l=0}^d \oplus_{i=1}^m R(1)^{\otimes(-l)} & \longrightarrow & M(X)_{R[1/(2d)!]} \end{array}$$

in $\mathsf{Chow}(k)_{R[1/(2d)!]}$. The composition $\beta \circ \alpha$ agrees with the 0th-component of the composition $\underline{g} \circ \underline{f} = \mathrm{id}$, i.e. it agrees with the identity of $M(X)_{R[1/(2d)!]}$. Since $R(1)^{\otimes(-l)} = \mathbf{L}^{\otimes l}$ we conclude then that $M(X)_{R[1/(2d)!]}$ is a direct factor of the Chow motive $\oplus_{l=0}^d \oplus_{i=1}^m \mathbf{L}^{\otimes l} \in \mathsf{Chow}(k)_{R[1/(2d)!]}$. By definition of the Lefschetz motive $\mathbf{L}$ we have the following equalities

$$(4.8) \qquad \mathrm{Hom}_{\mathsf{Chow}(k)_{R[1/(2d)!]}}(\mathbf{L}^{\otimes p}, \mathbf{L}^{\otimes q}) = \delta_{pq} \cdot R[1/(2d)!] \qquad p, q \geq 0,$$

where δ_{pq} stands for the Kronecker symbol. This implies that $M(X)_{R[1/(2d)!]}$ decomposes into a direct sum (indexed by l) of direct factors of $\oplus_{i=1}^m \mathbf{L}^{\otimes l}$. Note that a direct factor of $\oplus_{i=1}^m \mathbf{L}^{\otimes l}$ is the same data as an idempotent element of $\mathrm{End}(\oplus_{i=1}^m \mathbf{L}^{\otimes l})$. Thanks to (4.8) we have the following isomorphism

$$\mathrm{End}(\oplus_{i=1}^m \mathbf{L}^{\otimes l}) \simeq M_{m \times m}(R[1/(2d)!]),$$

where the right-hand side stands for $m \times m$ matrices with coefficients in $R[1/(2d)!]$. Hence, a direct factor of $\oplus_{i=1}^m \mathbf{L}^{\otimes l}$ is the same data as an idempotent element of $M_{m \times m}(R[1/(2d)!])$, i.e. a finitely projective projective $R[1/(2d)!]$-module. Since by hypothesis all these modules are free we then conclude that the only direct factors of $\oplus_{i=1}^m \mathbf{L}^{\otimes l}$ are its subsums. As a consequence, $M(X)_{R[1/(2d)!]}$ is isomorphic to a subsum of $\oplus_{l=0}^d \oplus_{i=1}^m \mathbf{L}^{\otimes l}$ indexed by a subset S of $\{0, \ldots, d\} \times \{1, \ldots, m\}$. By

construction of the orbit category we have $\pi(\mathbf{L}^{\otimes l}) \simeq \pi(M(\mathrm{spec}(k))_{\mathbb{Z}[1/(2d)!]})$ for every $l \geq 0$. Therefore, since the above direct sum (4.7) contains m terms we conclude that the cardinality of S is also m. This means that there is a choice of integers (up to permutation) $l_1, \ldots, l_m \in \{0, \ldots, d\}$ giving rise to an isomorphism

$$M(X)_{R[1/(2d)!]} \simeq \mathbf{L}^{\otimes l_1} \oplus \cdots \oplus \mathbf{L}^{\otimes l_m}$$

in $\mathsf{Chow}(k)_{R[1/(2d)!]}$. This achieves the proof.

5. Quadratic forms and associated quadrics

Recall from [22] the basics about quadratic forms. In this section, k will be a field of characteristic $\neq 2$ and V a finite dimensional k-vector space.

Definition 5.1. Let (V, q) be a quadratic form and B_q the associated bilinear pairing. The dimension of q is by definition the dimension of V.
 (i) The form (V, q) is called *non-singular* if the assignment $x \mapsto B_q(-, x)$ gives rise to an isomorphism $V \xrightarrow{\sim} V^*$; see [22, §I.1].
 (ii) The form (V, q) is called *anisotropic* if the equality $q(x) = 0$ holds only when $x = 0$; see [22, §I.3]. Note that when k is algebraically closed, any isotropic form has dimension 1.
 (iii) Given two quadratic forms (V_1, q_1) and (V_2, q_2), the *orthogonal sum* $(V_1 \oplus V_2, q_1 \perp q_2)$ is the quadratic form defined by the map $(q_1 \perp q_2)(x_1, x_2) = q_1(x_1) + q_2(x_2)$ [22, §I.2]. The *tensor product* $(V_1 \otimes V_2, q_1 \otimes q_2)$ is the quadratic form defined by the map $(q_1 \otimes q_2)(v_1 \otimes v_2) := q_1(v_1) \cdot q_2(v_2)$; see [22, §I.6].
 (iv) The *determinant* of q is defined as $d(q) := \det(M_q) \cdot (k^*)^2$, where M_q is the matrix of the bilinear form B_q and k^* is the multiplicative group $k \setminus \{0\}$. The determinant of q is then an element of $k^*/(k^*)^2$ which is well-defined up to isometry; see [22, §I.1]. The *signed determinant* of q is defined as $d_\pm(q) := (-1)^{\frac{n(n-1)}{2}} d(q)$, where n is the dimension of q; see [22, §II.2].
 (v) The *discriminant extension* k_q defined by q is the degree 2 quadratic extension $k(\sqrt{d})$ of the base field k (with $d := d_\pm(q)$).

Every quadratic form q gives rise to a $\mathbb{Z}/2\mathbb{Z}$-graded Clifford algebra $\mathcal{C}(q)$; see [22, §V.1]. The even part $\mathcal{C}_0(q)$ of $\mathcal{C}(q)$ is called the *even Clifford algebra of q*. Suppose q is nonsingular. When q is odd dimensional, $\mathcal{C}_0(q)$ is a central simple k-algebra. On the other hand, when q is even dimensional, $\mathcal{C}_0(q)$ is a central simple k_q-algebra, and we have the following two cases: (i) whenever $d_\pm(q)$ is not a square in k, the even Clifford algebra $\mathcal{C}_0(q)$ is a central simple k_q-algebra; (ii) whenever $d_\pm(q)$ is a square in k (that is, $k_q = k \times k$), the even Clifford algebra $\mathcal{C}_0(q)$ is the product of two isomorphic central simple k-algebras. In any case, we get a well defined central simple algebra, i.e. an Azumaya algebra. We denote by β_q such a central simple algebra and call it the *Clifford invariant* of q. The following definitions are not standard, but follow automatically from [22, §V.2].

Definition 5.2. Let (V, q) be a non-singular quadratic form over k.
 (i) The form (V, q) has *trivial discriminant* if k_q splits, *i.e.* if $k_q = k \oplus k$. Equivalently, (V, q) has trivial discriminant if $d_\pm(q) = 1 \in k^*/(k^*)^2$.
 (ii) The form (V, q) has *trivial Clifford invariant* if $\beta_q = 0$ in the Brauer group.

Remark 5.3. An even dimensional quadratic form q has trivial discriminant and trivial Clifford invariant if and only if $\mathcal{C}_0(q) = M_r(k) \times M_r(k)$, where $r := 2^{\dim(q)-2}$,

and $M_r(k)$ denotes the algebra of $r \times r$ matrices over k; see the chart at [22, page 111]. In particular, $\mathcal{C}_0$ is Morita equivalent to $k \times k$.

As explained in [22, §II.1], the isometry classes of anisotropic quadratic forms over k from the *Witt ring* $W(k)$, whose sum (resp. product) is induced by the orthogonal sum (resp. tensor product) of quadratic forms; see Definition 5.1(iii). The classes of the even dimensional anisotropic quadratic forms give rise to the so called *fundamental ideal* $I(k) \subset W(k)$; see [22, §II].

If q is a quadratic form whose isometry class lies in $I^3(k)$, then q is anisotropic and has trivial discriminant and trivial Clifford invariant; see [22, Cor. 3.4]. As proved in [22, Thm. 6.11], the converse is also true. Hence, we deduce that a non-singular quadratic form q satisfies the assumptions of Proposition 2.3 if and only if its isometry class belongs to $I^3(k)$. In particular, there is no such quadratic forms if $I^3(k) = 0$ (e.g. if k is algebraically closed or finite; see [22, §XI.6]).

Example 5.4. (3-fold Pfister forms) In order to describe quadratic forms in the powers of the fundamental ideal $I(k)$, one considers *Pfister forms*. The isometry class of the 2-dimensional quadratic form $x^2 + ay^2$ is denoted by $\langle 1, a \rangle$ and called a *1-fold Pfister form*. An *n-fold Pfister form* is the isometry class of a $2n$-dimensional quadratic form $\langle 1, a_1 \rangle \otimes \ldots \otimes \langle 1, a_n \rangle$. The key property of Pfister forms is that whenever k is a function field, the ideal $I^n(k)$ is additively generated by the n-fold Pfister forms; see [22, §X.1, Prop. 1.2]. Hence, whenever k is a function field, 3-fold Pfister forms satisfy assumptions of Proposition 2.3

Proof of Proposition 2.3. The fact that the Chow motive $M(Q_q)_{\mathbb{Z}}$ is *not* of Lefschetz $\mathbb{Z}$-type was proved in [32]; see also [7, §XVII]. In what concerns $U(\mathrm{perf}_{\mathsf{dg}}(Q_q))_{\mathbb{Z}}$, recall from [2, 19] that we have the following semi-orthogonal decomposition

$$\mathrm{perf}(Q_q) = \langle \mathrm{perf}(\mathcal{C}_0(q)), \mathcal{O}(-d+1), \ldots, \mathcal{O} \rangle.$$

As proved in [25, §5], semi-orthogonal decompositions become direct sums in the category of noncommutative Chow motives. Since $\mathrm{perf}_{\mathsf{dg}}(\mathcal{C}_0(q))$ is Morita equivalent to $\mathcal{C}_0(q)$ one then obtains the following motivic decomposition

$$(5.5) \qquad U(\mathrm{perf}_{\mathsf{dg}}(Q_q))_{\mathbb{Z}} \simeq U(\underline{\mathcal{C}_0(q)})_{\mathbb{Z}} \oplus \mathbf{1}^{\oplus n}.$$

Using the above Remark 5.3 one has moreover

$$(5.6) \qquad U(\underline{\mathcal{C}_0(q)})_{\mathbb{Z}} \oplus \mathbf{1}^{\oplus n} \simeq \mathbf{1}^{\oplus 2} \oplus \mathbf{1}^{\oplus n}.$$

By combining (5.5)-(5.6) one concludes that $U(\mathrm{perf}_{\mathsf{dg}}(Q_q))_{\mathbb{Z}}$ is of unit $\mathbb{Z}$-type and so the proof is finished.

6. Proof of Theorem 2.6

Let $n \geq 1$. Following [31, page 498], let us denote by $\mathcal{S}_n$ the category of those abelian groups G which verify the following two conditions:
 (i) there exist an integer m such that $mg = 0$ for all $g \in G$.
 (ii) if p is a prime factor of m then $p = 2$ or $p < n$.
As explained in *loc. cit.*, $\mathcal{S}_n$ is a Serre subcategory of the category of all abelian groups. We start with the following "arithmetic" result:

Lemma 6.1. *Given any two abelian groups G and H, the following holds:*
 (i) *Assume that $G = H$ modulo $\mathcal{S}_1$ or modulo $\mathcal{S}_2$. Then, $G_{\mathbb{Z}[1/2]} \simeq H_{\mathbb{Z}[1/2]}$.*
 (ii) *Assume that $G = H$ modulo $\mathcal{S}_n$ with $n \geq 3$. Then, $G_{\mathbb{Z}[1/(n-1)!]} \simeq H_{\mathbb{Z}[1/(n-1)!]}$.*

Proof. In the cases where $n = 1, 2$ the integer m is always a power of 2. Hence, if one inverts 2 one inverts also m. In the remaining cases the prime factors of m are always $\leq n - 1$. Hence, if one inverts $(n-1)!$ one inverts all the prime factors of m and consequently m itself. $\qquad\square$

Proposition 6.2. *Let X be an irreducible smooth projective k-scheme of dimension d_X. Under the assumption $\mathbb{Z} \subseteq R$, the following holds:*

(i) *The Todd class $\mathrm{Td}(X)$ is invertible in the Chow ring $\oplus_{i=0}^{d_X} CH^i(X)_{R[1/(2d_X)!]}$.*

(ii) *The Chern character induces an isomorphism*

$$(6.3) \qquad K_0(X)_{R[1/(2d_X)!]} \xrightarrow{\sim} \oplus_{i=0}^{d_X} CH^i(X)_{R[1/(2d_X)!]} \qquad \alpha \mapsto ch(\alpha) \, .$$

Proof. Recall from the proof of Proposition 4.3 that the Todd class $\mathrm{Td}(X) \in \oplus_{i=0}^{d_X} CH^i(X)_{R[1/d_X!]}$ is given by $\sum_{m \geq 0} \frac{\mathfrak{D}_m}{T_m}$, where $\mathfrak{D}_m$ is a polynomial with integral coefficients in the Chern classes. Moreover, $\mathfrak{D}_m = 0$ for $m > d_X$. Since by definition $\mathfrak{D}_0 = T_0 = 1$ one then observes that $\mathrm{Td}(X)$ is invertible in $\oplus_{i=0}^{d_X} CH^i(X)_{R[1/d_X!]}$ and consequently in $\oplus_{i=0}^{d_X} CH^i(X)_{R[1/(2d_X)!]}$. This proves item (i).

Let us now prove item (ii). The case $d_X = 0$ is clear and so we assume that $d \geq 1$. As proved at [31, page 52], the Chern character combined with the Gersten-Quillen spectral sequence give rise to the following equality

$$K_0(X)_{\mathbb{Z}} = \oplus_{i=0}^{d_X} E_2^{i,-i}(X) \qquad \text{modulo } \mathcal{S}_{d_X} \, .$$

Moreover, as proved at [29, Prop. 5.14], we have the identifications

$$E_2^{i,-i}(X) \simeq CH^i(X)_{\mathbb{Z}} \qquad 0 \leq i \leq d_X \, .$$

Using Lemma 6.1 one obtains then the following isomorphisms:

$$(6.4) \qquad\qquad K_0(X)_{\mathbb{Z}[1/2]} \simeq \oplus_{i=0}^{d_X} CH^i(X)_{\mathbb{Z}[1/2]} \qquad d_X = 1, 2$$

$$(6.5) \qquad K_0(X)_{\mathbb{Z}[1/(d_X-1)!]} \simeq \oplus_{i=0}^{d_X} CH^i(X)_{\mathbb{Z}[1/(d_X-1)!]} \qquad d_X \geq 3 \, .$$

The searched isomorphisms (6.3) can now be obtained by tensoring (6.4)-(6.5) with $R[1/(2d_X)!]$. $\qquad\square$

Let $X_1, \ldots, X_n$ be a finite family of irreducible smooth projective k-schemes of dimensions $d_1, \ldots, d_n$. Recall from §4 the construction of the functor

$$\Psi^{\oplus} : (X_1, \ldots, X_n)_R^{\oplus} \longrightarrow \mathsf{Chow}(k)_{R[1/(2d)!]}\big/{-\otimes R(1)} \, ,$$

where $d := \max\{d_1, \ldots, d_n\}$.

Proposition 6.6. *The induced $R[1/(2d)!]$-linear functor*

$$\Psi^{\oplus} : (X_1, \ldots, X_n)_{R[1/(2d)!]}^{\oplus} \longrightarrow \mathsf{Chow}(k)_{R[1/(2d)!]}\big/{-\otimes R(1)}$$

is fully faithful.

Proof. Let X_r and X_s be any two k-schemes in $\{X_1, \ldots, X_n\}$. From the construction of $\Psi^{\oplus}$ it is clear that it suffices to show that the homomorphism

$$(6.7) \qquad \begin{array}{ccc} K_0(X_r \times X_s)_{R[1/(2d)!]} & \longrightarrow & \oplus_{i=0}^{d_r+d_s} CH^i(X_r \times X_s)_{R[1/(2d)!]} \\ \alpha & \mapsto & ch(\alpha) \cdot \pi_{X_s}^*(\mathrm{Td}(X_s)) \end{array}$$

is an isomorphism. Thanks to Proposition 6.2(i) (applied to $X = X_s$) the Todd class $\mathrm{Td}(X_s)$ is an invertible element of $\oplus_{i=0}^{d_s} CH^i(X_s)_{R[1/(2d_s)!]}$ and hence of the Chow ring $\oplus_{i=0}^{d_s} CH^i(X_s)_{R[1/(2d)!]}$. Moreover, since

$$\pi_{X_s}^* : \oplus_{i=0}^{d_s} CH^i(X_s)_{R[1/(2d)!]} \longrightarrow \oplus_{i=0}^{d_r+d_s} CH^i(X_r \times X_s)_{R[1/(2d)!]}$$

is a ring homomorphism we conclude that $\pi_{X_s}^*(\mathrm{Td}(X_s))$ is an invertible element of $\oplus_{i=0}^{d_r+d_s} CH^i(X_r \times X_s)_{R[1/(2d)!]}$. Therefore, in order to prove that (6.7) is an isomorphism it suffices to show that the induced Chern character homomorphism

$$K_0(X_r \times X_s)_{R[1/(2d)!]} \longrightarrow \oplus_{i=0}^{d_r+d_s} CH^i(X_r \times X_s)_{R[1/(2d)!]} \quad \alpha \mapsto ch(\alpha)$$

is an isomorphism. This follows now from Proposition 6.2(ii) above applied to $X = X_r \times X_s$. $\qquad\square$

We now have all the ingredients needed for the conclusion of the proof of Theorem 2.6. Recall that by hypothesis X is an irreducible smooth projective k-scheme of dimension d. By combining the commutativity of diagram (3.3) with the fully faithfulness of the functor θ one obtains an $R[1/(2d)!]$-algebra isomorphism

$$\mathrm{End}(U(\mathrm{perf}_{\mathsf{dg}}(X))_{R[1/(2d)!]}) \simeq \mathrm{End}(M_0(X)_{R[1/(2d)!]}) .$$

Thanks to the fully faithfulness of the functor $\Psi^\oplus$ of Proposition 6.6 (applied to the category $(X)^\oplus_{R[1/(2d)!]}$) and the commutativity of diagram (4.2), one has moreover

$$\mathrm{End}(M_0(X)_{R[1/(2d)!]}) \simeq \mathrm{End}(\Psi^\oplus(M_0(X)_{R[1/(2d)!]})) \simeq \mathrm{End}(\pi(M(X)_{R[1/(2d)!]})) .$$

Now, recall that by construction the projection functor

$$\pi : \mathsf{Chow}(k)_{R[1/(2d)!]} \longrightarrow \mathsf{Chow}(k)_{R[1/(2d)!]}/_{-\otimes R(1)}$$

is faithful. Consequently, one obtains the following inclusion of $R[1/(2d)!]$-algebras

$$\mathrm{End}(M(X)_{R[1/(2d)!]}) \hookrightarrow \mathrm{End}(U(\mathrm{perf}_{\mathsf{dg}}(X))_{R[1/(2d)!]}) .$$

This automatically gives rise to the searched implication (2.7).

7. Proof of Proposition 2.8

Item (ii) was proved in [13, Thm. 2.2.1] for $M(X)_{\mathbb{Z}}$. The same result holds for $M(X)_{\mathbb{Z}/p\mathbb{Z}}$; see [14, Cor. 2.22]. Let us now show item (i). Recall from [4] that we have the following semi-orthogonal decomposition

$$\mathrm{perf}(X) = \langle \mathrm{perf}(k), \mathrm{perf}(A), \mathrm{perf}(A^{\otimes 2}), \dots, \mathrm{perf}(A^{\otimes d-1}) \rangle .$$

As proved in [25, §5], semi-orthogonal decompositions become direct sums in the category of noncommutative motives. Since $\mathrm{perf}_{\mathsf{dg}}(A^{\otimes i})$ is Morita equivalent to $\underline{A}^{\otimes i}$, one then obtains the following motivic decomposition

$$(7.1) \qquad U(\mathrm{perf}_{\mathsf{dg}}(X))_R \simeq U(\underline{k})_R \oplus U(\underline{A})_R \oplus U(\underline{A}^{\otimes 2})_R \oplus \dots \oplus U(\underline{A}^{\otimes d-1})_R .$$

Finally, since the functor $U(-)_R$ is symmetric monoidal, (7.1) identifies with (2.9).

Remark 7.2. Item (ii) holds also for $M(X)_{\mathbb{Z}/p^n\mathbb{Z}}$ and hence on the p-adic integers; see [6, Rmq. 2.3].

8. Proof of Theorem 2.10

Note first that, by combining the commutativity of diagram (3.3) with the fully faithfulness of the functor θ, it suffices to prove the implication

$$(8.1) \quad \oplus_{i=1}^n M(X_i)_R(l_i) \simeq \oplus_{j=1}^m M(Y_j)_R(l_j) \Rightarrow \oplus_{i=1}^n M_0(X_i)_R \simeq \oplus_{j=1}^m M_0(Y_j)_R .$$

As explained in §3.1, the projection functor $\pi : \mathsf{Chow}(k)_R \to \mathsf{Chow}(k)_R/_{-\otimes R(1)}$ is additive and moreover sends $M(X_i)_R(l_i)$ to $\pi(M(X_i)_R)$ (up to isomorphism). Hence, the left-hand side of (8.1) gives rise to an isomorphism

$$(8.2) \qquad\qquad \oplus_{i=1}^{n} \pi(M(X_i)_R) \simeq \oplus_{j=1}^{m} \pi(M(Y_j)_R) \,.$$

Since by hypothesis $1/(2d)! \in R$, Proposition 6.6 furnish us a fully faithful functor

$$\Psi^{\oplus} : (X_1, \ldots, X_n, Y_1, \ldots, Y_m)_R^{\oplus} \longrightarrow \mathsf{Chow}(k)_R/_{-\otimes R(1)} \,.$$

Using the commutativity of diagram (4.2), one observes that

$$\Psi^{\oplus}(\oplus_{i=1}^{n} M_0(X_i)_R) \simeq \oplus_{i=1}^{n} \pi(M(X_i))_R \quad \Psi^{\oplus}(\oplus_{j=1}^{m} M_0(Y_j)_R) \simeq \oplus_{j=1}^{m} \pi(M(Y_j))_R \,.$$

By combining these isomorphisms with (8.2), one obtains then the right-hand side of (8.1). This achieves the proof.

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