Risk Management with Copulae

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DECLARATION OF AUTHORSHIP

I hereby confirm that I have authored this master thesis independently and without use of others than the indicated sources. All passages which are literally or in general matter taken out of publications or other sources are marked as such.

Berlin, 17th August 2005,

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1. INTRODUCTION

The capital requirement from financial institutions is determined based on the amount of risk carried in their portfolios. The risk associated with a portfolio may be originated from:

1. fluctuations in the value of financial assets composing the portfolio (*market* risk),
2. fluctuations in the credibility of debtors (*credit* risk),
3. uncertainty connected with technical, personal and natural factors that may influence the portfolio value (*operational* risk).

In order to investigate the risk of a portfolio, the assets subjected to risk (*risk factors*) should be identified and the changes in the portfolio value caused by the risk factors evaluated. Specially relevant for risk management purposes are *negative changes* - the portfolio *losses*.

The *Value-at-Risk* (VaR) is a measure that quantifies the riskness of a portfolio. This measure and its accuracy are from crucial importance in determining the capital requirement from financial institutions. That is one of the reasons why increasing attention has been paid to VaR computing methods.

The losses and the probabilities associated with them (the *distribution of losses*) are necessary to describe the degree of portfolio riskness. The riskier the portfolio, the higher is the probability of losses being larger than a certain amount. Formulating in another way, the riskier the portfolio, the larger are the minimal losses for a certain probability (also called *level*). That is exactly the VaR definition: VaR is a quantile of the distribution of portfolio losses representing the minimal losses for a certain level.

Looking carefully at the distribution of losses, one verifies that large losses are influenced by simultaneous losses in risk factors. Hence, the *distribution of losses depends on joint distribution of risk factors.*

Understanding the joint distribution of risk factors is fundamental for investigating and computing the Value-at-Risk. The conventional procedure to model joint distributions of financial returns is to approximate them with *multivariate normal distributions.*

That implies, however, that the dependence structure of the returns is reduced to a fixed type. Even if the autocorrelation structure is neglected, predeterminig a multivariate normal distribution means that the following assumptions hold:

1. symmetric distribution of returns,
2. the tails of the distribution are not too heavy,
3. linear dependence.

Empirical evidence for these assumptions are barely verified and an alternative model is needed, with more flexible dependence structure and arbitrary marginal distributions. These are exactly the charactesitics of *copulae.*

Copulae are very useful for modelling and estimating multivariate distributions. The flexibilty of copulae follows basically from *Sklar's Theorem*, which tells that each joint distribution can be "decomposed" into its marginal distributions and a copula C "responsible" for the dependence structure:

$$F(x_1 \ldots, x_d) = C\{F_1(x_1), \ldots, F_d(x_d)\}$$

Two important facts for practical applications rely on this theorem:

1. the construction of multivariate distributions may be done in two independent steps: the specification of marginal distributions - not necessarily identical - and the specification of a dependence structure. Copulae "couples together" the marginal distributions into a multivariate distribution with the desired dependence structure.
2. joint distributions can be separately estimated from a sample of observations: the marginal distributions are estimated first, the dependence structure later.

The copula approach frees the modelling from the usual normality assumptions: marginal distributions with asymmetric heavy tails (typical for financial returns) can be combined together with different dependence structures, resulting in multivariate distributions (far different from the multivariate normal) that better describe the empirical characteristics of financial returns distribution.

Moreover, copulae allow for dynamical modelling and adaption to portfolios: different copulae with distinct properties can be associated to different portfolios according to their specific dependence structures. Furthermore, copulae may change as time evolves, reflecting the evolution of the dependence between financial assets. Summarizing, the Value-at-Risk estimation with copulae is more efficient and flexible than methods based on normality assumption.

2. COPULAE

This section presents the basic copulae definitions and theorems. The most important copulae, together with their standard construction and simulation methods are also discussed. For the proofs and deeper mathematical treatment refer to Joe (1997) and Nelsen (1998).

2.1 Definitions and Properties

Definition 2.1.1 (Copula):
A d-dimensional copula is a function $C : [0,1]^d \rightarrow [0,1]$ satisfying the following properties for every $u = (u_1, \ldots, u_d)^\top \in [0,1]^d$ and $j \in \{1, \ldots, d\}$:

1. if $u_j = 0$ then $C(u_1, \ldots, u_d) = 0$
2. $C(1, \ldots, 1, u_j, 1, \ldots, 1) = u_j$
3. for every $v = (v_1, \ldots, v_d)^\top \in [0,1]^d$, $v_j \leq u_j$

$$V_c(u, v) \geq 0$$

where $V_C(u, v)$ is given by

$$\sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{i_i+\ldots+i_d} C(g_{1i_1}, \ldots, g_{di_d})$$

and $g_{j1} = v_j$ and $g_{j2} = u_j$

As a consequence from the properties above, copulae are *multivariate uniform distributions*. The first and third properties state that copulae are grounded functions and that all d-dimensional boxes with vertices in $[0,1]^d$ have *non-negative C-volume*. Together they guarantee that copulae are distribution functions, while the second property tells that copulae have *uniform marginal disributions*.

Note: considering random variables $X_1, \dots, X_d$ with univariate distribution functions $F_1, \dots, F_d$ and the random variables $U_i = F_{X_i}(X_i)$, $i = 1, \dots, d$ uniform distributed in $[0, 1]$, a copula may be interpreted as *the joint distribution of the marginal distributions*.

For all $u = (u_1, \dots, u_d)^\top \in [0, 1]^d$, every copula C satisfies

$$W(u_1, \dots, u_d) \leq C(u_1, \dots, u_d) \leq M(u_1, \dots, u_d)$$

where

$$M(u_1, \dots, u_d) = \min(u_1, \dots, u_d) \tag{2.1}$$

and

$$W(u_1, \dots, u_d) = \max\left(\sum_{i=1}^{d} u_i - d + 1, 0\right) \tag{2.2}$$

$M(u_1, \dots, u_d)$ is called *Fréchet-Hoeffding upper bound* and $W(u_1, \dots, u_d)$ the *Fréchet-Hoeffding lower bound.* While M is not a copula for $d > 2$, W is a copula for all d. Besides the Fréchet-Hoeffding bounds, the product copula $\Pi(u_1, \dots, u_d)$ is also from fundamental importance. The product copula is given by:

$$\Pi(u_1, \dots, u_d) = \prod_{j=1}^{d} u_j \tag{2.3}$$

Figure 2.1 ilustrates the Fréchet-Hoeffding bounds and the product copulae.

The following theorem connects copulae with distribution functions and shows that:

- every distribution function can be "decomposed" into its marginal distribution and a (at least) one copula.
- a (unique) copula is obtained from "coupling together" every (continuous) multivariate distribution function and its marginal distributions.

Theorem 2.1.1 (Sklar's theorem): Let F be a d-dimensional distribution function with marginals $F_1 \dots, F_d$. Then there exists a copula C with

$$F(x_1, \dots, x_d) = C\{F_1(x_1), \dots, F_d(x_d)\} \tag{2.4}$$

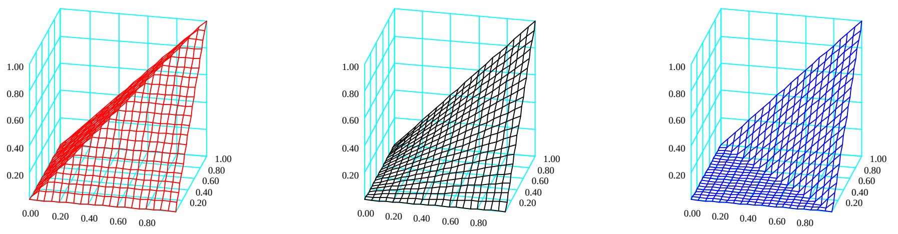

Fig. 2.1: Fréchet-Hoeffding upper bound $W(u_1, u_2)$ (left), product copula $\Pi(u_1, u_2)$ (middle), Fréchet-Hoeffding lower bound $M(u_1, u_2)$ (right).

SFEfrechet.xpl

for every $x_1, \dots, x_d \in \overline{\mathbb{R}}$. If $F_1, \dots, F_d$ are continuous, then C is unique. On the other hand, if C is a copula and $F_1, \dots, F_d$ are distribution functions, then the function F defined in (2.4) is a joint distribution function with marginals $F_1, \dots, F_d$.

Hence, for a joint distribution F with continuous marginals $F_1, \dots, F_d$ the unique copula C can be obtained from (2.4) for all $u = (u_1, \dots, u_d)^\top \in [0,1]^d$ as

$$C(u_1, \dots, u_d) = F\{F_1^{-1}(u_1), \dots, F_d^{-1}(u_d)\} \tag{2.5}$$

Definition 2.1.2 (Copula of a random variable): Let $X = (X_1, \dots, X_d)^\top$ be a random vector with distribution $X \sim F_X$ and continuous marginals $X_j \sim F_{X_j}$. The copula of X is the distribution function C_X of $u = (u_1, \dots, u_d)^\top$ where $u_j = F_{X_j}(x_j)$:

$$C_X(u_1, \dots, u_d) = F_X\{F_{X_1}^{-1}(u_1), \dots, F_{X_d}^{-1}(u_d)\} \tag{2.6}$$

For independent random variables $X_1, \ldots, X_d$ the copula of X is the product copula defined in (2.3):

$$\begin{aligned} C_X(u_1, \ldots, u_d) &= F_X(x_1, \ldots, x_d) \\ &= \prod_{j=1}^{d} F_{X_j}(x_j) \\ &= \Pi\{F_{X_1}(x_1), \ldots, F_{X_d}(x_d)\} \\ &= \Pi(u_1, \ldots, u_d) \end{aligned}$$

Note that the product copula is the same for *any* marginal distributions, i.e., it determines *the dependence structure* between the univariate variables for abitrary marginals.

The next theorem shows that copulae are *invariant under monotone increasing transformations.* This property is very useful for obtaining copula families in subsequent sections.

Theorem 2.1.2 (Invariance under monotone increasing transformations):
Let $X = (X_1, \ldots, X_d)^\top$ be a random vector with continuous marginals and copula C_X and $T_1, \ldots, T_d$ be strictly increasing functions on $RanX_1, \ldots, RanX_d$. Let $Y = (Y_1, \ldots, Y_d)^\top$, $Y_i = T_i(X_i)$ be a random vector with copula C_Y. Then $C_X = C_Y$ almost everywhere.

A d-dimensional random variable determines a copula through its joint and marginal distributions. Moreover, monotone increasing transformations on the random variable do not affect the copula. These are the main ideas used to obtain the Gaussian copula: the random variable $X = (X_1, \ldots, X_d)^\top$ with multivariate normal distribution and copula C_X is monotone increasingly tranformed into the standardised variable $Z = (Z_1, \ldots, Z_d)^\top$, $Z_j \sim N(0, 1)$. The copula of the random variable Z is C_X.

For *absolute continuous* copula, there exists a *copula densitiy.* Copula densities are essential for estimation procedures, as seen in chapter3.

Definition 2.1.3 (Copula density): For an absolutely continuous copula C, the *copula density* is defined as

$$c(u_1, \ldots, u_d) = \frac{\partial^d C(u_1, \ldots, u_d)}{\partial u_1 \ldots \partial u_d} \tag{2.7}$$

Given a random variable $X = (X_1, \dots, X_d)^\top$, with absolute continuous distribution function F and copula C_X, the density c_X is obtained differentiating (2.6):

$$c_X(u_1, \dots, u_d) = \frac{f\{F_{X_1}^{-1}(u_1), \dots, F_{X_d}^{-1}(u_d)\}}{\prod_{j=1}^{d} f_j\{F_{X_j}^{-1}(u_j)\}} \tag{2.8}$$

where f is the joint density of F_X and f_j the density of F_{X_j}. The density from the copula of X can be determined from its joint density and inverse marginal distributions.

2.2 Gaussian Copula

The Gaussian copula represents the *dependence structure* from the multivariate normal distribution, that means, *normal* marginal distributions combined with Gaussian copula form multivariate normal distributions.

The combination of *non-normal* marginal distributions results in *meta-Gaussian* distributions, i.e., distributions where *only* the dependence structure is Gaussian (an example is plotted in the upper left part from figure 2.4).

To obtain the Gaussian copula, let $X = (X_1, \dots, X_d)^\top \sim N_d(\mu, \Sigma)$ with $X_j \sim N(\mu_j, \sigma_j)$ and $\sigma_j = \Sigma_{jj}$ for $j = 1, \dots, d$. From Sklar's Theorem there exists a copula C_X such that:

$$F_X(x_1, \dots, x_d) = C_X\{F_{X_1}(x_1), \dots, F_{X_d}(x_d)\}$$

where F_{X_i} is the distribution function of X_i and F_X the distribution function of X.

Let $Y_j = T_j(X_j)$, where $T(x)$ is the transformation

$$T_j(x) = \frac{x - \mu_j}{\sigma_j}$$

Then $Y_j \sim N(0, 1)$ and $Y = (Y_1, \dots, Y_d)^\top \sim N_d(0, \Psi)$ where Ψ is the correlation matrix associated with Σ. Moreover, there exists a copula C_Ψ^{Ga}, called *Gaussian copula*, such that:

$$F_Y(x_1, \dots, x_d) = C_\Psi^{Ga}\{\Phi(x_1), \dots, \Phi(x_d)\} \tag{2.9}$$

where Φ is the distribution function of Y_j and F_Y the distribution function of Y. An explicit expression for the Gaussian copula is obtained rewriting (2.9) with $u_j = \Phi(x_j)$:

$$\begin{aligned} C_{\Psi}^{Ga}(u_1, \ldots, u_d) &= F_Y\{\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)\} \\ &= \int_{-\infty}^{\Phi^{-1}(u_1)} \cdots \int_{-\infty}^{\Phi^{-1}(u_d)} 2\pi^{-\frac{d}{2}} \mid \Psi \mid^{-\frac{1}{2}} e^{(-\frac{1}{2}r^{\top}\Psi^{-1}r)} dr_1 \ldots dr_d \end{aligned}$$

where $r = (r_1, \ldots, r_d)^{\top}$.

As $T(x)$ is increasing it follows from theorem 2.1.2 that

$$C_X = C_{\Psi}^{Ga}$$

Thus, *any* multivariate normal distribution can be constructed from its marginal distributions and the Gaussian copula C_{Ψ}^{Ga} with the desired correlation matrix Ψ.

Remark 2.1: If $\Psi = \mathcal{I}_d$ the Gaussian copula becomes the product copula as

$$\begin{aligned} C_{\mathcal{I}_d}^{Ga}(u_1, \ldots, u_d) &= \int_{-\infty}^{\Phi^{-1}(u_1)} \cdots \int_{-\infty}^{\Phi^{-1}(u_d)} 2\pi^{-\frac{d}{2}} e^{(-\frac{1}{2}\sum_{j=1}^{d} r_j^2)} dr_1 \ldots dr_d \\ &= \int_{-\infty}^{\Phi^{-1}(u_1)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}r_1^2} dr_1 \ldots \int_{-\infty}^{\Phi^{-1}(u_d)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}r_d^2} dr_d \\ &= \Phi\{\Phi^{-1}(u_1)\} \ldots \Phi\{\Phi^{-1}(u_d)\} \\ &= \Pi(u_1, \ldots, u_d) \end{aligned}$$

The *density of the Gaussian copula* (figure 2.2) is obtained by differentiating (2.9),

$$\mid 2\pi\Psi \mid^{-\frac{1}{2}} \exp\left(-\frac{1}{2}x^{\top}\Psi^{-1}x\right) = c_{\Psi}^{Ga}\{\Phi(x_1), \ldots, \Phi(x_d)\} \prod_{j=1}^{d} 2\pi^{-\frac{1}{2}} \exp\left(-\frac{1}{2}x_j^2\right)$$

rearranging terms and defining $\zeta_j = \Phi^{-1}(u_j)$, $\zeta = (\zeta_1, \ldots, \zeta_d)^{\top}$:

$$c_{\Psi}^{Ga}(u_1, \ldots, u_d) = \mid \Psi \mid^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\zeta^{\top}(\Psi^{-1} - \mathcal{I}_d)\zeta\right\} \tag{2.10}$$

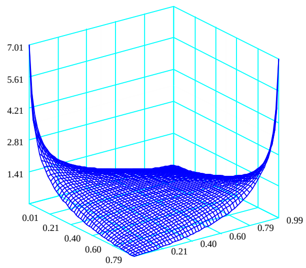

Fig. 2.2: Density of Gaussian copula, $c^{Ga}_{\Psi}(u_1, u_2)$, $\psi_{12} = 0.5$.

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2.3 Student's t-Copula

The t-copula, containing the dependence structure form the multivariate t-distribution, may be obtained in a similar way.

Let $X = (X_1, \ldots, X_d)^\top \sim t_d(\nu, \mu, \Sigma)$ and $Y = (Y_1, \ldots, Y_d)^\top \sim t_d(\nu, 0, \Psi)$ where Ψ is the correlation matrix associated with Σ. The unique copula from Y is the *Student's t-copula* $C^t_{\nu,\Psi}$. Moreover, it follows from theorem 2.1.2 that $C_X = C^t_{\nu,\Psi}$.

For $u = (u1, \ldots, u_d)^\top \in [0,1]^d$, the *Student's t-copula* is given by

$$C^t_{\nu,\Psi}(u_1, \ldots, u_d) = t_{\nu,\Psi}\{t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_d)\}$$

where t_ν^{-1} is the quantile function from the univariate t-distribution and $t_{\nu,\Psi}$ the distribution function of Y.

The *density of the t-copula* (figure 2.3) is given by

$$c^t_{\nu,\Psi}(u_1, \ldots, u_d) = \frac{t_{\nu,\Psi}\{t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_d)\}}{\prod_{j=1}^d t_{\nu,\Psi}\{t_\nu^{-1}(u_j)\}} \tag{2.11}$$

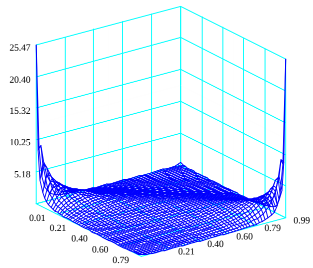

Fig. 2.3: Density of t-copula, $c^t_{\nu,\Psi}(u_1, u_2)$, $\psi_{12} = 0.2$, $\nu = 3$.

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With $\zeta_j = t_\nu^{-1}(u_j)$ the density of the t-copula can be expressed as:

$$c^t_{\nu,\Psi}(u_1, \dots, u_d) = \mid \Psi \mid^{-\frac{1}{2}} \frac{\Gamma(\frac{\nu+d}{2}) \left\{\Gamma(\frac{\nu}{2})\right\}^{d-1} \left(1 + \frac{1}{\nu}\zeta^\top \Psi^{-1}\zeta\right)^{-\frac{\nu+d}{2}}}{\left\{\Gamma(\frac{\nu+1}{2})\right\}^d \prod_{j=1}^d \left(1 + \frac{1}{\nu}\zeta_j^2\right)^{-\frac{\nu+1}{2}}} \tag{2.12}$$

2.4 Archimedean Copulae

Definition 2.4.1: Let $\phi : [0, 1] \rightarrow [0, \infty]$ be a continuous, strictly decreasing function with $\phi(1) = 0$. The *pseudo inverse* of ϕ is the function $\phi^{[-1]}$ such that

$$\phi^{[-1]} = \begin{cases} \phi^{-1}(t), & 0 \leq t \leq \phi(0) \\ 0, & \phi(0) \leq t \leq \infty \end{cases}$$

Theorem 2.4.1: Let $\phi : [0, 1] \rightarrow [0, \infty]$ be a convex, strictly decreasing continuous function with $\phi(1) = 0$. Then the function $C : [0, 1]^2 \rightarrow [0, 1]$

$$C(u_1, u_2) = \phi^{[-1]}\{\phi(u_1) + \phi(u_2)\} \tag{2.13}$$

is a copula.

Copulae of the form (2.13) are called *Archimedean copulae* and the functions ϕ their *generators*. If in addition $\phi(0) = \infty$, ϕ is called a *strict generator* and $\phi^{[-1]} = \phi^{-1}$.

2.4.1 Example: Gumbel copula

The function $\phi(t) = (-\ln t)^{\theta}$, $\theta \in [1, \infty)$ is convex, strictly decreasing and continuous in $[0, 1]$ with $\phi(0) = \infty$ and $\phi(1) = 0$, thus it is a strict generator and $\phi^{-1}(t) = e^{-t\theta^{-1}}$. The function $C : [0, 1]^2 \to [0, 1]$

$$C(u_1, u_2) = e^{-\{(-\ln u_1)^{\theta} + (-\ln u_2)^{\theta}\}^{\theta^{-1}}}$$

is the *Gumbel* copula . For $\theta = 1$ we obtain the product copula: $C(u_1, u_2) = \Pi(u_1, u_2)$, for $\theta \to \infty$ we obtain the Fréchet-Hoeffding upper bound:

$$C_{\theta}(u_1, u_2) \stackrel{\theta \to \infty}{\longrightarrow} \min(u_1, u_2)$$

2.5 Multivariate Archimedean Copulae

The next theorem generalizes the concepts of archimedean copulae for the d-dimensional case. Detailed treatment and proofs can be found in Nelsen (1998).

Definition 2.5.1: A function $f(t)$ is completely monotonic in an interval $[a, b]$ if for $t \in [a, b]$ and $k \in \mathbb{N}$ it satisfies

$$(-1)^k \frac{d^k}{dt^k} f(t) \geq 0$$

Theorem 2.5.1: Let ϕ be a strict generator. The function $C^d : [0, 1]^d \to [0, 1]$

$$C^d(u_1, \ldots, u_d) = \phi^{-1}\{\phi(u_1) + \ldots + \phi(u_d)\}$$

is a copula for all $d \geq 2$ if and only if ϕ^{-1} is completely monotonic in $[0, \infty)$.

Some d-dimensional archimedean copulae are presented below. For more generators, copula families and respective properties, see Joe (1997).

1. **Frank copula**, $0 < \theta \leq \infty$

$$C_\theta(u_1, \ldots, u_d) = -\frac{1}{\theta} \ln \left\{ 1 + \frac{\prod_{j=1}^{d} (e^{-\theta u_j} - 1)}{(e^{-\theta} - 1)^{d-1}} \right\}$$

2. **Gumbel copula**, $1 \leq \theta \leq \infty$

$$C_\theta(u_1, \ldots, u_d) = \exp \left[- \left\{ \sum_{j=1}^{d} (-\ln u_j)^\theta \right\}^{\theta^{-1}} \right]$$

3. **Ali-Mikhail-Haq copula**, $-1 \leq \theta < 1$

$$C_\theta(u_1, \ldots, u_d) = \frac{\prod_{j=1}^{d} u_j}{1 - \theta \left(\prod_{j=1}^{d} 1 - u_j \right)}$$

4. **Clayton copula**, $\theta > 0$

$$C_\theta(u_1, \ldots, u_d) = \left\{ \left(\sum_{j=1}^{d} u_j^{-\theta} \right) - d + 1 \right\}^{-\frac{1}{\theta}}$$

where the density of the Clayton copula is given by

$$c_\theta(u_1, \ldots, u_d) = \prod_{j=1}^{d} \{1 + (j-1)\theta\} u_j^{-(\theta+1)} \left(\sum_{j=1}^{d} u_j^{-\theta} - d + 1 \right)^{-(\theta^{-1}+d)}$$

2.6 Distributions Constructed with Copulae

Joint distributions with different dependence between the marginal distributions can be easily constructed with copulae. As an example, the standard normal and t_3 marginal distributions are be coupled with 4 distinct copulae C to build the joint distribtuion F given by

$$F(x_1, x_2) = C\{\Phi(x_1), t_3(x_2)\}$$

The density function of F is

$$f(x_1, x_2) = c\{\Phi(x_1), t_3(x_2)\}\varphi(x_1)f_{t,3}(x_2)$$

where $\varphi(x)$ is the density function from the standard normal distribution and $f_{t,3}(x)$ from the t-distribution with 3 degrees of freedom. The contour plots from $f(x_1, x_2)$ are shown in figure 2.4 for the respective copula choices.

2.7 Monte Carlo Simulation

The simulation from d pseudo random variables with joint distribution defined by a copula C and d marginal distributions F_n, $n = 1, \ldots, d$, may follow different techniques. A standard method for archimedean copulae is the conditional distribution method, shortly described in the sequence. For more details and different methods, see Bouyé (2000), Devroye (1986) and Embrechts (2005).

Defining the copulae n-dimensional marginal distribution C_n for $n = 2, \ldots, d-1$ as

$$C_n(u_1, \ldots, u_n) = C(u_1, \ldots, u_n, 1, \ldots, 1)$$

and the derivative of C_n with respect to the first $n-1$ arguments as

$$c^n_{n-1}(u_1, \ldots, u_n) = \frac{\partial^{n-1} C_n(u_1, \ldots, u_n)}{\partial u_1, \ldots, \partial u_{n-1}}$$

the probability $P(U_n \leq u_n, U_1 = u_1, \ldots, U_{n-1} = u_{n-1})$ can be written as

$$\lim_{\Delta u_1, \ldots, \Delta u_{n-1} \to 0} \frac{C_n(u_1 + \Delta u_1, \ldots, u_{n-1} + \Delta u_{n-1}, u_n) - C_n(u_1, \ldots, u_n)}{\Delta u_1, \ldots, \Delta u_{n-1}} =$$

$$= c^n_{n-1}(u_1, \ldots, u_n)$$

Thus, the conditional probability $\Lambda(u_n)$ is a function of the ratio of deriva-

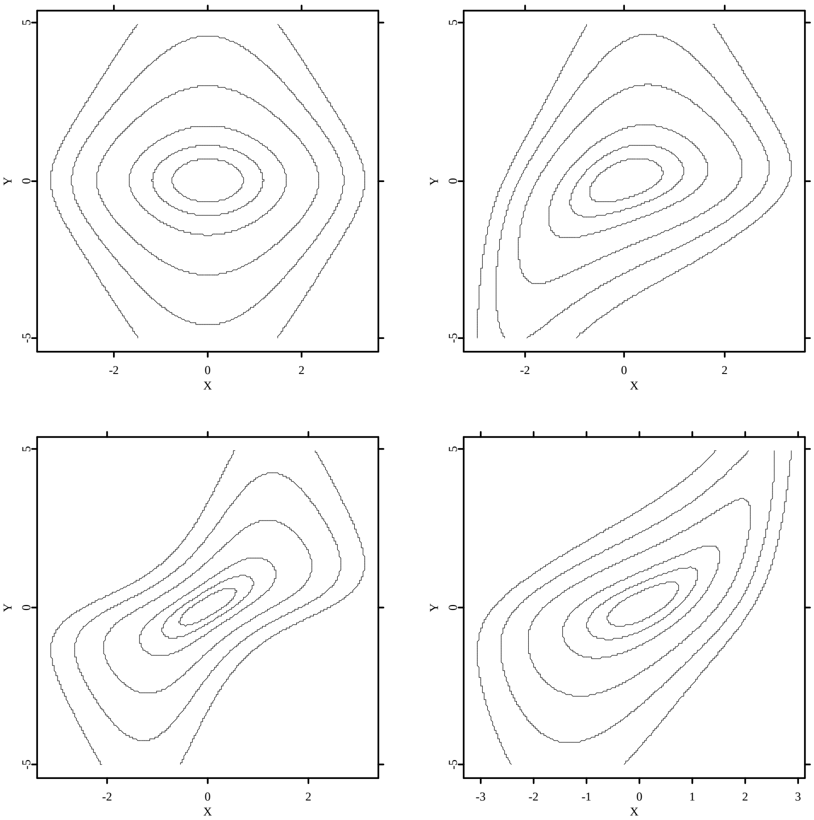

Fig. 2.4: Pdf contour plots, $F(x_1, x_2) = C\{\Phi(x_1), t_3(x_2)\}$ with (clockwise) Gaussian ($\rho = 0$), Clayton ($\theta = 0.9$), Frank ($\theta = 8$) and Gumbel ($\theta = 2$) copulae.

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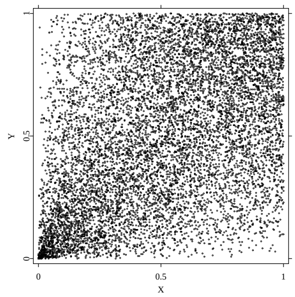

Fig. 2.5: Monte Carlo sample of 10.000 realizations of pseudo random variable with uniform marginals in $[0, 1]$ and dependence structure given by Clayton copula, $\theta = 0.79$.

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tives:

$$\begin{aligned}
\Lambda(u_n) &= P(U_n \leq u_n \mid U_1 = u_1, \dots, U_{n-1} = u_{n-1}) \\
&= \frac{P(U_n \leq u_n, U_1 = u_1, \dots, U_{n-1} = u_{n-1})}{P(U_1 = u_1, \dots, U_{n-1} = u_{n-1})} \\
&= \frac{c_{n-1}^{n}(u_1, \dots, u_n)}{c_{n-1}^{n-1}(u_1, \dots, u_{n-1})}
\end{aligned}$$

The generation of d pseudo random numbers with given marginal distributions F_n, $n = 1, \dots, d$ and dependence structure given by the copula C follows the steps:

1. generate pseudo random numbers $v_1, \dots, v_d$ independent and uniformly distributed in $[0, 1]$.

2. for $n = 1, \dots, d$ generate the pseudo random numbers as $u_n = \Lambda^{-1}(v_n)$. The pseudo random numbers $u_1, \dots, u_d$ have uniform marginal distributions in $[0, 1]$ and dependence structure given by the copula C.

3. set $x_n = F_n^{-1}(u_n)$. The pseudo random numbers $x_1, \dots, x_d$ are distributed with the desired marginal distributions and dependence structure.

If C is the Gaussian copula, the simulation follows:

1. generate pseudo random numbers $v_1, \ldots, v_d$ distributed as $N(0, \Psi)$

2. set $u_n = \Phi(v_n)$, $n = 1, \ldots, d$. The pseudo random numbers $u = (u_1, \ldots, u_d)$ have uniform marginal distributions in $[0, 1]$ and dependence structure given by C_Ψ^{Ga}.

3. set $x_n = F_n^{-1}(u_n)$. The pseudo random numbers $x_1, \ldots, x_d$ are distributed with the desired marginal distributions and dependence structure.

If the marginal distributions are normal, the pseudo random numbers are multivariate normal distributed. Otherwise their distribution is called *Meta-Gaussian* distribution.

If C is the t-copula, the simulation follows:

1. generate pseudo random numbers $v_1, \ldots, v_d$ distributed as $t_d(\nu, 0, \Psi)$

2. set $u_n = t_\nu(v_n)$, $n = 1, \ldots, d$ where t_ν is the univariate t distribution with ν degrees of freedom. The pseudo random numbers $u = (u_1, \ldots, u_d)$ have uniform marginal distributions in $[0, 1]$ and dependence structure given by $C_{\nu,\Psi}^t$.

3. set $x_n = F_n^{-1}(u_n)$. The pseudo random numbers $x_1, \ldots, x_d$ are distributed with the desired marginal distributions and dependence structure.

If the marginal distributions are t_ν, the pseudo random numbers are multivariate t distributed. Otherwise their distribution is called *Meta-t* distribution.

Repeating one of the procedures above T times yields a Monte Carlo sample $\{x_{n,t}\}_{t=1}^T$, for $n = 1, \ldots, d$ of a random variable distributed as desired.

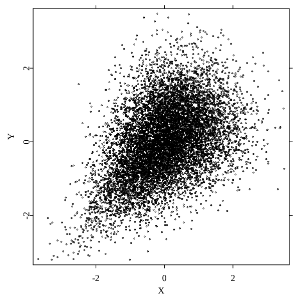

Fig. 2.6: Monte Carlo sample of 10.000 realizations of pseudo random variable with standard normal marginals and dependence structure given by Clayton copula with $\theta = 0.79$.

SFEclaytonMC.xpl

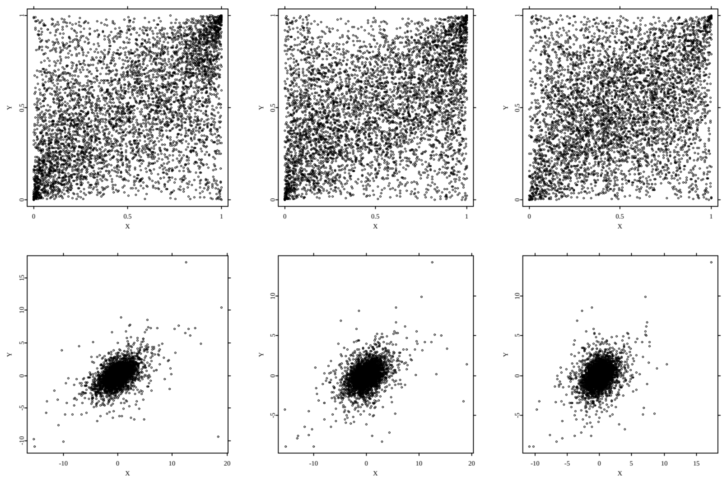

Fig. 2.7: Scatterplots of Monte Carlo sample (5.000 realizations) of pseudo random variable $X = (X_1, X_2, X_3)^\top$ with uniform (above) and t_3 marginal distributions (below). Dependence structure given by t-copula with $\nu = 3$ and $\psi_{i,j} = 0.5$, $i, j = 1, 2, 3$, $i \neq j$.

SFEtMC.xpl

3. COPULAE ESTIMATION

Let X be a d-dimensional random variable with parametric univariate marginal distributions $F_{X_j}(x_j;\delta_j)$, $j = 1,\dots,d$. Further let a copula belong to a parametric family $\mathcal{C} = \{C_\theta, \theta \in \Theta\}$. From Sklar's Theorem the distribution of X can be expressed as

$$F_X(x_1,\dots,x_d) = C\{F_{X_1}(x_1;\delta_1),\dots,F_{X_d}(x_d;\delta_d);\theta\}$$

and its density as

$$f(x_1,\dots,x_d;\delta_1,\dots,\delta_d,\theta) = c\{F_{X_1}(x_1;\delta_1),\dots,F_{X_d}(x_d;\delta_d);\theta\}\prod_{j=1}^{d} f_j(x_j;\delta_j)$$

where

$$c(u_1,\dots,u_d) = \frac{\partial^d C(u_1,\dots,u_d)}{\partial u_1\dots\partial u_d}$$

For a sample of observations $\{x_t\}_{t=1}^T$, $x_t = (x_{1,t},\dots,x_{d,t})^\top$ and a vector of parameters $\alpha = (\delta_1,\dots,\delta_d,\theta)^\top \in \mathbb{R}^{k+1}$ the likelihood function is given by

$$L(\alpha;x_1,\dots,x_T) = \prod_{t=1}^{T} f(x_{1,t},\dots,x_{d,t};\delta_1,\dots,\delta_d,\theta)$$

and the log-likelihood function by

$$\ell(\alpha;x_1,\dots,x_T) = \sum_{t=1}^{T}\ln c\{F_{X_1}(x_{1,t};\delta_1),\dots,F_{X_d}(x_{d,t};\delta_d);\theta\} + \sum_{t=1}^{T}\sum_{j=1}^{d}\ln f_j(x_{j,t};\delta_j)$$

The vector of parameters $\alpha = (\delta_1,\dots,\delta_d,\theta)^\top$ contains d parameters δ_j from the marginals and the copula parameter θ. All these parameters can be estimated *in one step*. For practical applications, however, a two steps estimation procedure is more efficient.

3.1 Maximum Likelihood Estimation

In the Maximum Likelihood estimation method (also called *full maximum likelihood*), the vector of parameters α is estimated in one single step through

$$\tilde{\alpha}_{FML} = \arg\max_{\alpha} \ell(\alpha)$$

The estimates $\tilde{\alpha}_{FML} = (\tilde{\delta}_1, \ldots, \tilde{\delta}_d, \tilde{\theta})^\top$ solve

$$(\partial\ell/\partial\delta_1, \ldots, \partial\ell/\partial\delta_d, \partial\ell/\partial\theta) = 0$$

3.2 IFM - Inference for Margins

In the IFM (*inference for margins*) method, the parameters δ_j from the marginal distributions are estimated in the first step and used to estimate the dependece parameter θ in the second step:

1. for $j = 1, \ldots, d$ the log-likelihood function for each of the marginal distributions are

$$\ell_j(\delta_j) = \sum_{t=1}^{T} \ln f_j(x_{j,t}; \delta_j)$$

and the estimated parameters

$$\hat{\delta}_j = \arg\max_{\delta} \ell_j(\delta_j)$$

2. the *pseudo log-likelihood* function

$$\ell(\theta, \hat{\delta}_1, \ldots, \hat{\delta}_d) = \sum_{t=1}^{T} \ln c\{F_{X_1}(x_{1,t}; \hat{\delta}_1), \ldots, F_{X_d}(x_{d,t}; \hat{\delta}_d); \theta\}$$

is maximized over θ to get the dependence parameter estimate $\hat{\theta}$.

The estimates $\hat{\alpha}_{IFM} = (\hat{\delta}_1, \ldots, \hat{\delta}_d, \hat{\theta})^\top$ solve

$$(\partial\ell_1/\partial\delta_1, \ldots, \partial\ell_d/\partial\delta_d, \partial\ell/\partial\theta) = 0$$

3.3 CML - Canonical Maximum Likelihood

In the CML (*canonical maximum likelihood*) method, the univariate marginal distributions are estimated through the empirical distribution function $\hat{F}$. For $j = 1, \ldots, d$

$$\hat{F}_j(x) = \frac{1}{T+1} \sum_{t=1}^{T} I(x_{j,t} \leq x)$$

The *pseudo log-likelihood* function is

$$\ell(\theta) = \sum_{t=1}^{T} \ln c\{\hat{F}_1(x_{1,t}), \ldots, \hat{F}_d(x_{d,t}); \theta\}$$

and the copula parameter estimator $\hat{\theta}_{CML}$ is given by

$$\hat{\theta}_{CML} = \arg\max_{\theta} \ell(\theta)$$

Notice that the first step of the IMF and CML methods estimates the marginal distributions. After marginals are estimated, a *pseudo sample* $\{u_t\}$ of observations transformed in the unit d-cube is obtained and used in the *copula* estimation.

3.4 Gaussian Copula Estimation

From a pseudo sample $\{u_t\}_{t=1}^{T}$ where $u = (u_1, \ldots, u_d)^\top \in [0,1]^d$, the density of the Gaussian copula is given by

$$c_{\Psi}^{Ga}(u_1, \ldots, u_d) = |\ \Psi\ |^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\zeta^\top(\Psi^{-1} - \mathcal{I}_d)\zeta\right\}$$

and the pseudo log-likelihood function by

$$\ell(\Psi; u_{1,t}, \ldots, u_{d,t}) = -\frac{T}{2} \ln |\ \Psi\ | - \frac{1}{2} \sum_{t=1}^{T} \zeta_t^\top(\Psi^{-1} - \mathcal{I}_d)\zeta_t$$

where $\zeta_t = (\zeta_{1,t}, \ldots, \zeta_{d,t})^\top$ and $\zeta_{j,t} = \Phi^{-1}(u_{j,t})$.

The maximum-likelihood estimator for Ψ is

$$\hat{\Psi} = \arg\max_{\Psi \in \mathcal{P}} \ell(\Psi)$$

where $\mathcal{P}$ is the set of all lower-triangular matrices with one in the diagonal. The maximization is feasible but very slow for high dimensions, see Embrechts (2005). An approximate solution can be obtained using the ML estimator for the covariance matrix Σ as

$$\hat{\Sigma} = \arg\max_{\Sigma} \ell(\Sigma)$$

The estimator is then

$$\hat{\Sigma} = \frac{1}{T} \sum_{t=1}^{T} \zeta_t^\top \zeta_t$$

and defining

$$\Lambda = \text{diag}(\hat{\Sigma}_{ii})$$

we obtain

$$\hat{\Psi} = \Lambda^{-1} \hat{\Sigma} \Lambda^{-1}$$

3.5 Student's t-Copula Estimation

One possible estimation method for the Student's t-copula is based on the estimation from Kendall's tau with method of moments, as in Embrechts (2005). For a pseudo sample $\{u_t\}_{t=1}^T$ where $u = (u_1, \dots, u_d)^\top \in [0,1]^d$, the Kendall's tau coefficient for each pair of observations $i, j = 1, \dots, d$ is given by

$$\hat{\rho}_\tau(u_i, u_j) = \binom{T}{2}^{-1} \sum_{1 \le t_1 \le t_2 \le T} sign(u_{i,t_1} - u_{i,t_2})(u_{j,t_1} - u_{j,t_2})$$

Each element from the correlation matrix Ψ is estimated as

$$\hat{\psi}_{ij} = \sin\left\{\frac{\pi}{2} \hat{\rho}_\tau(u_i, u_j)\right\}$$

The parameter ν is estimated through maximum-likelihood with the estimated matrix $\hat{\Psi}$ held fixed. In this case the pseudo log-likelihood function is given by

$$\ell(\nu; u_{1,t}, \dots, u_{d,t}) = \sum_{t=1}^{T} \ln\{c^t_{\nu,\hat{\Psi}}(u_{1,t}, \dots, u_{d,t})\}$$

where $c^t_{\nu,\hat{\Psi}}(u_{1,t},\ldots,u_{d,t})$ is defined in equation 2.11. The estimator for the number of degrees of freedom is then

$$\hat{\nu} = \arg\max_{\nu \in \mathbb{N}_+} \ell(\nu)$$

4. VALUE-AT-RISK AND COPULAE

This section introduces the main assumptions and steps necessary for estimating the VaR from a linear portfolio using copulae. Static and time-varying methods as well as their VaR performance evaluation through backtesting are described in the sequence.

4.1 Value-at-Risk

Let $w = (w_1, \ldots, w_d)^\top \in \mathbb{R}^d$ a portfolio of positions on d assets and $S_t = (S_{1,t}, \ldots, S_{d,t})^\top$ be a non-negative random vector representing the prices of the assets at t, where t is a time index. The value V_t of the portfolio w is given by

$$V_t = \sum_{j=1}^{d} w_j S_{j,t} \tag{4.1}$$

and the random variable

$$L_{t+\tau} = (V_{t+\tau} - V_t) \tag{4.2}$$

also called *profit and loss (P&L) function*, expresses the change in the portfolio value between τ periods.

Defining the *log-returns* $X_{t+\tau}$ in τ periods as $X_{t+\tau} = \ln S_{t+\tau} - \ln S_t$ and considering $\tau = 1$, equation (4.2) can be written as

$$L_{t+1} = \sum_{j=1}^{d} w_j S_{j,t}(e^{X_{j,t+1}} - 1) \tag{4.3}$$

The distribution function from L, dropping the time index, is given by

$$F_L(x) = P(L \le x) \tag{4.4}$$

The *Value-at-Risk* at level α from a portfolio w is defined as the α-quantile from F_L:

$$VaR(\alpha) = F_L^{-1}(\alpha) \tag{4.5}$$

It follows from equations 4.3 and 4.4 that F_L depends on the d-dimensional distribution of log-returns F_X. In general, the *loss distribution* F_L depends on a random process representing the *risk factors* influencing the P&L from a portfolio. In the present case log-returns are a suitable risk factor choice. Thus, modelling their distribution is essential to obtain the quantiles from F_L.

A log-returns process $\{X_t\}$ can be modelled as

$$X_{j,t} = \mu_{j,t} + \sigma_{j,t}\varepsilon_{j,t}$$

where $\varepsilon_t = (\varepsilon_{1,t}, \dots, \varepsilon_{d,t})^\top$ are standardised *i.i.d.* innovations with $E[\varepsilon_{j,t}] = 0$ and $E[\varepsilon_{j,t}^2] = 1$ for $j = 1, \dots, d$, $\mathcal{I}_t$ is the available information at time t,

$$\mu_{j,t} = E[X_{j,t} \mid \mathcal{I}_{t-1}]$$

is the conditional mean given $\mathcal{I}_{t-1}$ and

$$\sigma_{j,t}^2 = E[(X_{j,t} - \mu_{j,t})^2 \mid \mathcal{I}_{t-1}]$$

is the conditional variance given $\mathcal{I}_{t-1}$. The innovations $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d)^\top$ have joint distribution F_ε and ε_j has continuous marginal distributions F_j, $j = 1, \dots, d$.

4.2 VaR estimation with Copulae

The innovations ε have distribution function described by

$$F_\varepsilon(\varepsilon_1, \dots, \varepsilon_d) = C_\theta\{F_1(\varepsilon_1), \dots, F_d(\varepsilon_d)\} \tag{4.6}$$

where C_θ is a copula belonging to a parametric family $\mathcal{C} = \{C_\theta, \theta \in \Theta\}$. To obtain the Value-at-Risk in this set up, the dependence parameter and distribution function from residuals are estimated from a sample of log-returns and used to generate P&L Monte Carlo samples. Their quantiles at different levels are the estimators for the Value-at-Risk. The whole procedure is summarized below.

For a portfolio w on d assets and a sample $\{x_{j,t}\}_{t=1}^{T}$, $j = 1, \dots, d$ of log-returns, the Value-at-Risk at level $1-\alpha$ is estimated according to the following steps:

1. estimation of residuals $\hat{\varepsilon}_t$
2. specification and estimation of marginal distributions $F_j(\hat{\varepsilon}_j)$
3. specification of a parametric copula family $\mathcal{C}$ and estimation of dependence parameter θ
4. generation of Monte Carlo sample of innovations ε and losses L
5. estimation of $\widehat{VaR}(1-\alpha)$, the empirical $(1-\alpha)$-quantile from L.

4.3 Time-varying Copulae and Backtesting

The application of the (*static*) procedure described above on different subsets of a sample $\{x_{j,t}\}_{t=1}^{T}$ delivers a sequence of fitted dependence parameters for a copula family. Hence the denomination *time-varying copulae.*

Using subsets of size w scrolled on the sample (i.e., a *moving window* of size w),

$$\{x_t\}_{t=s-w+1}^{s}$$

for $s = w, \dots, T$, the procedure above generates the time series $\{\widehat{VaR}_t\}_{t=w}^{T}$ of Value-at-Risk and $\{\hat{\theta}_t\}_{t=w}^{T}$ dependence parameters estimates.

Backtesting is used to evaluate the performance of the specified copula family $\mathcal{C}$. The estimated values for the VaR are compared with the true realizations $\{l_t\}$ of the P&L function, an *exceedance* occuring for each l_t smaller than $\widehat{VaR}_t(1-\alpha)$. The ratio of the number of exceedances to the number of observations gives the *exceedances ratio* $\hat{\alpha}$:

$$\hat{\alpha} = \frac{1}{T-w}\sum_{t=w}^{T} I\{l_t < \widehat{VaR}_t(1-\alpha)\}$$

5. EMPIRICAL RESULTS

The estimation methods described in the preceeding section are used on two exchange rates portfolio, the first composed of 2 positions, the second of 5 positions. Different copulae are used in static and dynamic set up and their VaR performance is compared based on backtesting.

5.1 2-dimensional Exchange Rate Portfolio

In this section, the Value-at-Risk of portfolios on exchange rates (DEM/USD and GBP/USD from 01.12.1979 to 01.04.1994) is computed using different copulae. Assuming the log-returns $\{X_{j,t}\}$ follow a GARCH(1,1) process we have

$$X_{j,t} = \mu_{j,t} + \sigma_{j,t}\varepsilon_{j,t}$$

where

$$\sigma^2_{j,t} = \omega_j + \alpha_j \sigma^2_{j,t-1} + \beta_j (X_{j,t-1} - \mu_{j,t-1})^2$$

and $\omega > 0$, $\alpha_j \geq 0$, $\beta_j \geq 0$, $\alpha_j + \beta_j < 1$.

The fit of a GARCH(1,1) model to the sample of log returns $\{x_t\}_{t=1}^T$, $x_t = (x_{1,t}, x_{2,t})^\top$, $T = 3718$, gives the estimates $\hat{\omega}_j$, $\hat{\alpha}_j$ and $\hat{\beta}_j$, as in table 5.1, and empirical residuals $\{\hat{\varepsilon}_t\}_{t=1}^T$, where $\hat{\varepsilon}_t = (\hat{\varepsilon}_{1,t}, \hat{\varepsilon}_{2,t})^\top$. The scatterplot of epirical residuals is depicted in figure 5.1.

	$\hat{\omega}_j$	$\hat{\alpha}_j$	$\hat{\beta}_j$
$j = 1$	0.00	0.07	0.89
$j = 2$	0.00	0.05	0.93

Tab. 5.1: GARCH(1,1) parameters, 2-dimensional portfolio

	$\hat{\mu}_j$	$\hat{\sigma}_j$
$j = 1$	0.0081	0.9987
$j = 2$	0.1887	0.9991

Tab. 5.2: Parameters from marginal distributions

Copula	$\hat{\theta}$
Gaussian	0.767
Clayton	1.861
Gumbel	2.283

Tab. 5.3: Dependence parameter for different static copulae.

The marginal distributions are specified as normal, i.e., $\hat{\varepsilon}_j \sim N(\hat{\mu}_j, \hat{\sigma}_j)$ with parameters $\hat{\delta}_j = (\hat{\mu}_j, \hat{\sigma}_j)$ estimated from the data as in table 5.2.

Static Copulae

The dependence parameters are estimated (table 5.3) for different copula families (Gaussian, Clayton and Gumbel). Variuos portfolios are used to generate the P&L samples and the estimated Value-at-Risk for each of them are in table 5.4.

Time-varying Copulae

In the dynamic approach, the empirical residuals are sampled in moving windows with fixed size $w = 250$, $\{\hat{\varepsilon}_t\}_{t=s-w+1}^{s}$, for $s = w, \dots, T$. The time series from estimated dependence parameters for each copula family are in figure 5.4.

The same portfolio compositions as in the static case are used to generate P&L samples. The series of estimated Value-at-Risk and the P&L function for selected portfolios are plotted in figure 5.5, 5.6. and 5.7. Backtesting results for each copula, portfolio and quantiles at levels $1 - \alpha$ ofr $\alpha_1 = 0.05$, $\alpha_2 = 0.01$, $\alpha_3 = 0.005$ and $\alpha_4 = 0.001$ are in tables 5.5, 5.6 and 5.7.

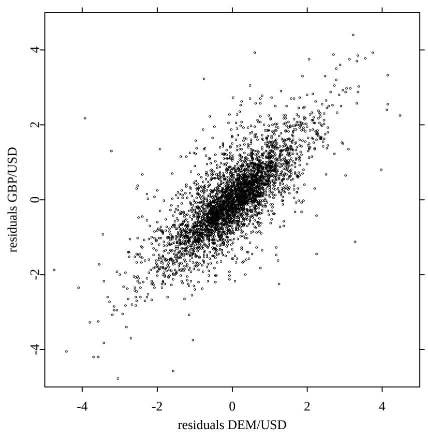

Fig. 5.1: Scatterplot from residuals $\hat{\varepsilon}_1$ and $\hat{\varepsilon}_2$.

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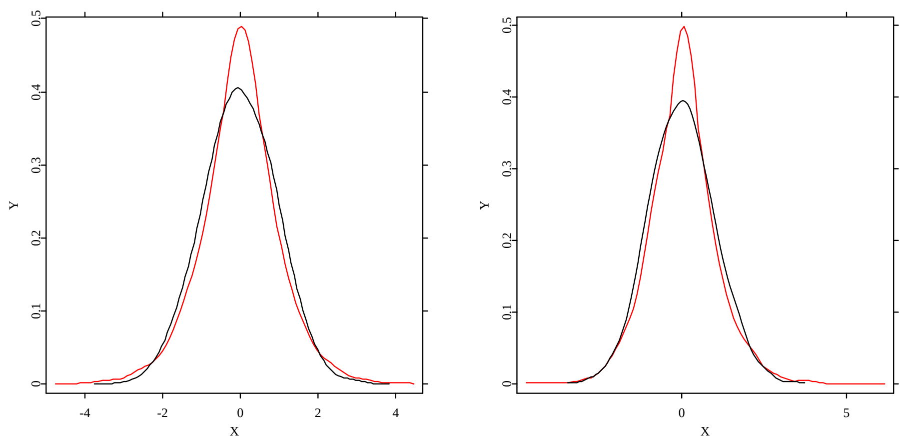

Fig. 5.2: Kernel density estimator of the residuals (red) and of the normal density (black) from DEM/USD (left) and GBP/USD (right). Quartic kernel, $\hat{h} = 2.78\hat{\sigma}n^{-0.2}$.

SFEresDens.xpl

Portfolio	$\alpha(\times 10^2)$ 5	1	0.5	0.1
$(1,1)$	-0.030	-0.042	-0.046	-0.055
	-0.026	-0.038	-0.042	-0.049
	-0.026	-0.043	-0.051	-0.078
$(1,2)$	-0.031	-0.044	-0.049	-0.058
	-0.029	-0.043	-0.048	-0.056
	-0.028	-0.048	-0.056	-0.089
$(1,3)$	-0.033	-0.046	-0.052	-0.062
	-0.033	-0.048	-0.054	-0.064
	-0.031	-0.053	-0.062	-0.099
$(2,1)$	-0.058	-0.083	-0.091	-0.109
	-0.049	-0.071	-0.079	-0.093
	-0.049	-0.082	-0.097	-0.147
$(2,3)$	-0.061	-0.086	-0.095	-0.113
	-0.056	-0.081	-0.091	-0.106
	-0.054	-0.090	-0.108	-0.168
$(3,2)$	-0.061	-0.086	-0.095	-0.113
	-0.075	-0.109	-0.122	-0.143
	-0.074	-0.125	-0.149	-0.226
$(-1,1)$	-0.027	-0.039	-0.043	-0.052
	-0.026	-0.031	-0.034	-0.041
	-0.020	-0.028	-0.031	-0.037
$(-1,2)$	-0.026	-0.037	-0.040	-0.050
	-0.020	-0.029	-0.034	-0.040
	-0.017	-0.024	-0.026	-0.030
$(-1,3)$	-0.025	-0.035	-0.039	-0.048
	-0.019	-0.029	-0.032	-0.040
	-0.015	-0.021	-0.023	-0.025
$(-2,1)$	-0.056	-0.080	-0.088	-0.106
	-0.044	-0.064	-0.070	-0.084
	-0.043	-0.062	-0.069	-0.082
$(-2,3)$	-0.054	-0.075	-0.083	-0.102
	-0.042	-0.061	-0.068	-0.081
	-0.037	-0.052	-0.058	-0.069
$(-3,2)$	-0.084	-0.118	-0.132	-0.159
	-0.066	-0.096	-0.105	-0.125
	-0.063	-0.090	-0.100	-0.119

Tab. 5.4: $\widehat{VaR}(1-\alpha)$ for different portfolios and α values (static copulae). For each portfolio estimated with Gaussian, (first), Clayton (second) and Gumbel copula (third row).

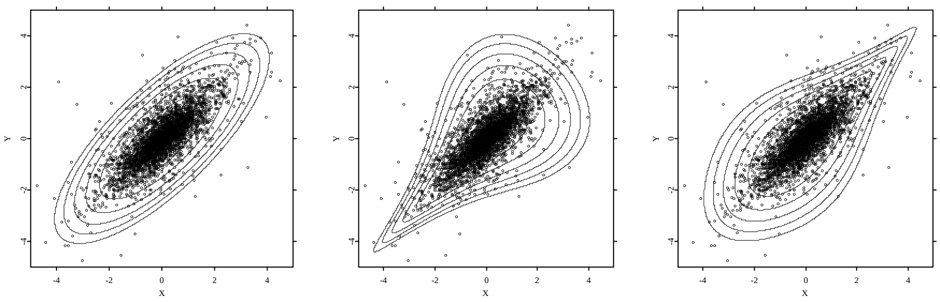

Fig. 5.3: Residuals $\hat{\varepsilon}$ and fitted copulae: Gaussian ($\hat{\rho} = 0.76898$), Clayton ($\hat{\theta} = 1.8611$), Gumbel ($\hat{\theta} = 2.2833$).

SFEstaticCop.xpl

Portfolio	$\alpha(\times 10^2)$ 5	1	0.5	0.1
$(1,1)$	4.81	1.58	1.00	0.37
$(1,2)$	4.61	1.41	0.92	0.34
$(1,3)$	4.75	1.41	0.95	0.37
$(2,1)$	5.07	1.81	1.03	0.43
$(2,3)$	4.61	1.44	0.92	0.34
$(3,2)$	4.98	1.64	1.03	0.43
$(-1,1)$	3.51	0.72	0.34	0.14
$(-1,2)$	1.84	0.37	0.23	0.11
$(-1,3)$	1.96	0.46	0.23	0.11
$(-2,1)$	4.18	1.06	0.72	0.20
$(-2,3)$	2.76	0.43	0.17	0.14
$(-3,2)$	3.83	0.89	0.57	0.17
avg	3.91	1.10	0.68	0.27
std.dev.	1.15	0.52	0.35	0.12

Tab. 5.5: Clayton copula, exceedances ratio $\hat{\alpha}(\times 10^2)$ for different portfolios.

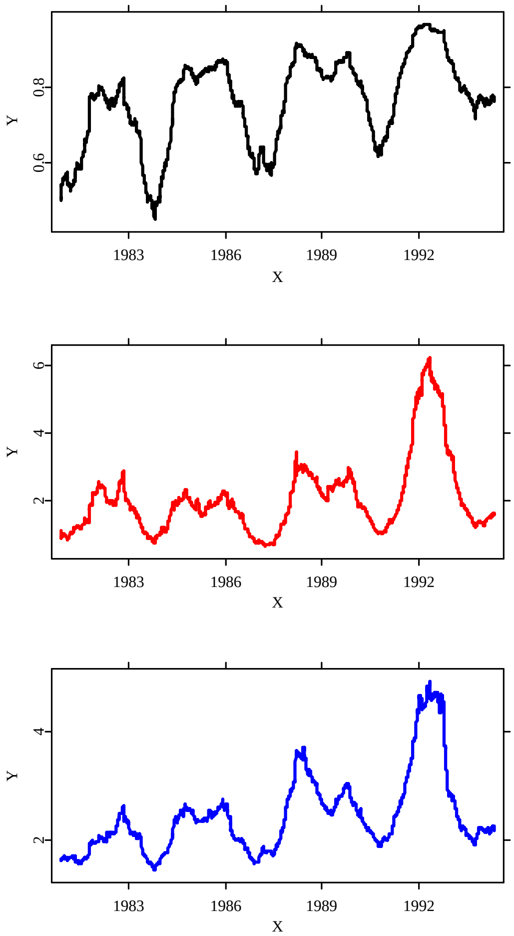

Fig. 5.4: Dependence parameter $\hat{\theta}$, estimated using IFM method, Gaussian (black), Clayton (red) and Gumbel (blue) copulae, moving window ($w = 250$).

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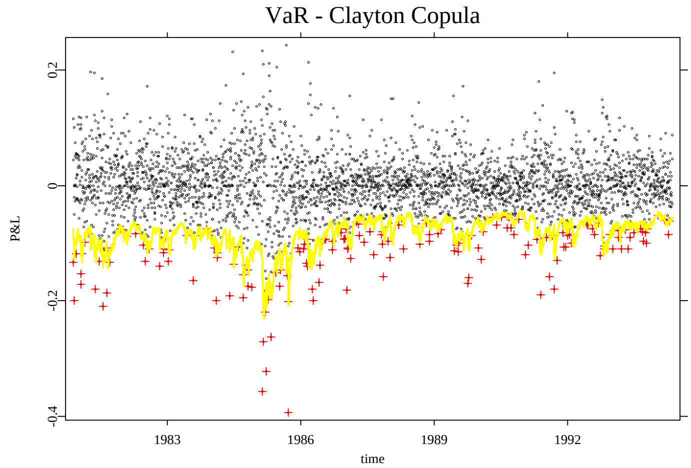

Fig. 5.5: $\widehat{VaR}(1-\alpha)$ (yellow), P&L (black) and exceedances (red), $\alpha = 0.05$, $\hat{\alpha} = 0.04987$, $w = (3,2)^\top$. P&L samples generated with Clayton copula.

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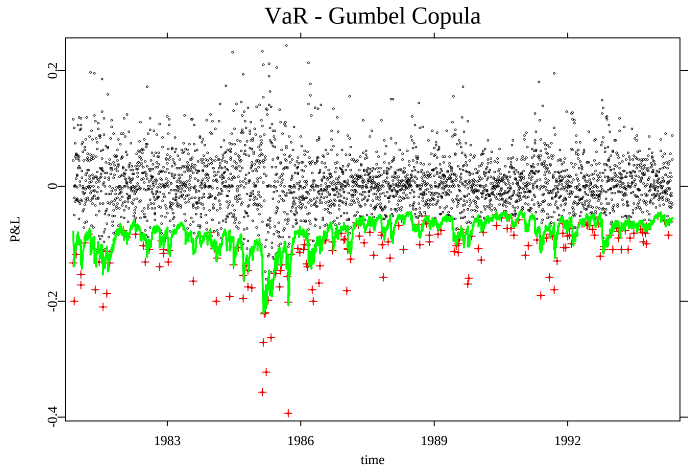

Fig. 5.6: $\widehat{VaR}(1-\alpha)$ (green), P&L (black) and exceedances (red), $\alpha = 0.05$, $\hat{\alpha} = 0.0521$, $w = (3,2)^\top$. P&L samples generated with Gumbel copula.

SFEgumbelSIM2ptv.xpl

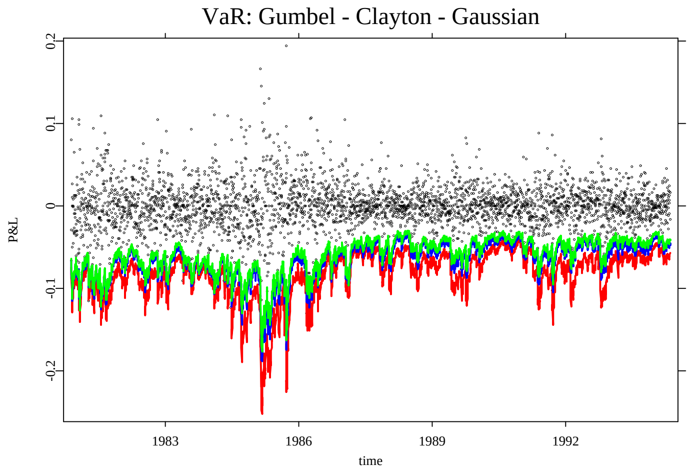

Fig. 5.7: $\widehat{VaR}(1-\alpha)$ and P&L (black), $\alpha = 0.01$, estimated with Gumbel copula (green), $\hat{\alpha} = 0.0167$, Clayton copula (blue), $\hat{\alpha} = 0.0106$, and Gaussian copula (red), $\hat{\alpha} = 0.0034$, $w = (-2, 1)^\top$.

SFEClayGumbGauss.xpl

	$\alpha(\times 10^2)$			
Portfolio	5	1	0.5	0.1
$(1, 1)$	5.21	1.09	0.43	0.09
$(1, 2)$	5.16	1.03	0.43	0.09
$(1, 3)$	4.92	0.98	0.49	0.09
$(2, 1)$	5.21	1.03	0.49	0.12
$(2, 3)$	5.16	1.00	0.49	0.09
$(3, 2)$	5.21	1.06	0.46	0.12
$(-1, 1)$	5.21	1.90	1.33	0.58
$(-1, 2)$	5.96	1.67	1.04	0.46
$(-1, 3)$	4.64	1.09	0.52	0.26
$(-2, 1)$	5.10	1.67	1.12	0.52
$(-2, 3)$	5.53	2.07	1.30	0.55
$(-3, 2)$	5.01	1.72	1.15	0.52
avg.	5.20	1.36	0.77	0.29
std.dev.	0.32	0.41	0.38	0.22

Tab. 5.6: Gumbel copula, exceedances ratio $\hat{\alpha}(\times 10^2)$ for different portfolios.

Portfolio	$\alpha(\times 10^2)$ 5	1	0.5	0.1
$(1,1)$	3.72	1.09	0.66	0.23
$(1,2)$	5.13	1.64	1.21	0.52
$(1,3)$	6.14	1.96	1.55	0.75
$(2,1)$	3.29	0.78	0.58	0.14
$(2,3)$	4.32	1.47	0.92	0.43
$(3,2)$	3.34	0.86	0.63	0.20
$(-1,1)$	1.28	0.23	0.14	0.09
$(-1,2)$	0.84	0.17	0.12	0.01
$(-1,3)$	1.04	0.32	0.20	0.01
$(-2,1)$	1.99	0.35	0.17	0.09
$(-2,3)$	0.98	0.23	0.14	0.09
$(-3,2)$	1.76	0.32	0.14	0.09
avg.	2.81	0.80	0.54	0.23
std.dev.	1.75	0.63	0.48	0.21

Tab. 5.7: Gaussian copula, exceedances ratio $\hat{\alpha}(\times 10^2)$ for different portfolios.

5.2 5-dimensional Exchange Rate Portfolio

In this section, the Value-at-Risk of exchange rate portfolios composed of 5 positions (USD value of GBP, FRF, CHF, DEM and AUD from 04.01.1994 to 15.08.1997) is computed using time-varying Clayton copula.

The fit of a GARCH(1,1) model to the sample of log returns $\{x_t\}_{t=1}^T$, $x_t = (x_{1,t}, \dots, x_{5,t})^\top$, $T = 907$, gives the estimates $\hat{\omega}_j$, $\hat{\alpha}_j$ and $\hat{\beta}_j$, as in table 5.2, and empirical residuals $\{\hat{\varepsilon}_t\}_{t=1}^T$, where $\hat{\varepsilon}_t = (\hat{\varepsilon}_{1,t}, \dots, \hat{\varepsilon}_{5,t})^\top$, as in upper right part of figure 5.8. The marginal distributions are specified as normal, $\hat{\varepsilon}_j \sim N(\hat{\mu}_j, \hat{\sigma}_j)$, the estimated parameters $\hat{\delta}_j = (\hat{\mu}_j, \hat{\sigma}_j)$ are in table 5.9.

The estimated Value-at-Risk at level $1 - \alpha$ together with the P&L function are plotted in figure 5.9. Backtesting results for each portfolio for $\alpha_1 = 0.05$, $\alpha_2 = 0.01$, $\alpha_3 = 0.005$ and $\alpha_4 = 0.001$ are in table 5.10.

	$\hat{\omega}$	$\hat{\alpha}$	$\hat{\beta}$
x_1	0.000	0.043	0.931
x_2	0.000	0.047	0.932
x_3	0.000	0.099	0.813
x_4	0.000	0.043	0.940
x_5	0.000	0.002	0.000

Tab. 5.8: GARCH(1,1) parameters, 5-dimensional portfolio

	$\hat{\mu}(\times 10^2)$	$\hat{\sigma}$
$j = 1$	2.52	1.00
$j = 2$	-0.46	0.99
$j = 3$	-0.36	1.00
$j = 4$	-0.86	1.00
$j = 5$	2.28	1.00

Tab. 5.9: Parameters from marginal distributions.

	$\alpha(\times 10^2)$			
Portfolio	5	1	0.5	0.1
$(1, 1, 1, 1, 1)$	5.02	0.61	0.47	0.15
$(1, 2, 3, 2, 1)$	5.78	0.91	0.47	0.47
$(1, 3, 1, 2, 3)$	3.96	0.47	0.47	0.30
$(2, 1, 2, 3, 1)$	5.33	0.91	0.61	0.47
$(2, 1, 3, 2, 1)$	5.63	0.91	0.47	0.47
$(2, 3, 1, 1, 2)$	3.96	0.76	0.61	0.15
$(2, 3, 3, 2, 1)$	5.78	0.91	0.47	0.47
$(3, 1, 2, 1, 3)$	3.96	0.76	0.61	0.15
$(3, 1, 2, 2, 2)$	4.87	0.76	0.61	0.15
$(3, 2, 3, 2, 3)$	4.57	0.61	0.61	0.15
avg.	4.79	0.75	0.50	0.38
std.dev.	0.77	0.15	0.07	0.08

Tab. 5.10: Clayton copula, exceedances ratio $\hat{\alpha}$ for different portfolios.

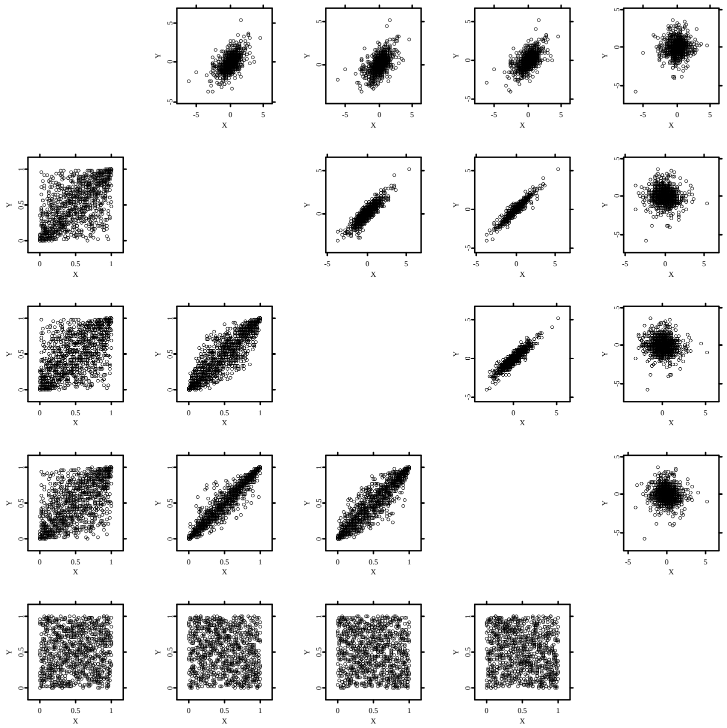

Fig. 5.8: Scatterplots from GARCH residulas (upper triangular) and from residuals mapped on unit square by the cdf (lower triangular).

SFE5dim.xpl

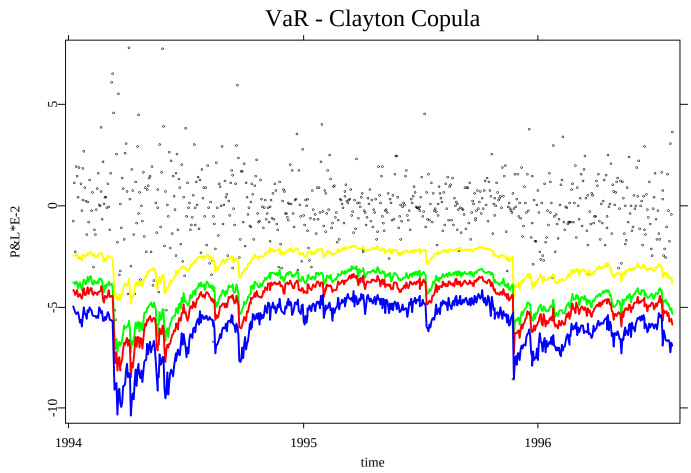

Fig. 5.9: $\widehat{VaR}(1-\alpha)$ and P&L (black), estimated with Clayton copula, $\alpha_1 = 0.05$ (yellow), $\alpha_2 = 0.01$ (green), $\alpha_3 = 0.005$ (red) and $\alpha_4 = 0.001$ (blue), $w = (3,1,2,1,3)^\top$.

SFE5dim.xpl

5.3 Conclusion

Three different copulae - Gumbel, Clayton and Gaussian - were used to estimate the Value-at-Risk from the 2-dimensional portfolio (DEM/USD, GBP/USD). From the time series of estimated dependence parameters, we verify that the dependence structure is represented in similar form with all copula families, as in figure 5.4.

Using backtest results to compare the performance in the VaR estimation, we remark that in average the Clayton and Gaussian copulae *overestimated* the VaR. In terms of capital requirement, a financial institution computing VaR with those copulae would be requested to keep *more* capital aside than necessary to guarantee the desired confidence level.

The estimation with Gumbel copula, on another side, produced results close to the desired level. Gumbel copulae seems to represent specific data dependence structures (like lower tail dependencies, relevant to explain simultaneous losses) better than Gaussian and Clayton copulae (it is well known that Gaussian copula does not present any tail dependencies).

Hence, the choice of the best copula for VaR estimation based on backtesting performance depends on the dependence structure of the data set used and should be investigated case by case. The theory for copula model selection tests is developed in Chen (2004) for static set up. That may be the first step to develop dynamic model selection tests, where after testing in a window for the best copula, the Value-at-Risk is estimated with the choosen copula family.

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