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Equivalences of smooth and continuous principal bundles with infinite-dimensional structure group

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Abstract. Let K be a Lie group, modeled on a locally convex space, and M a finite-dimensional paracompact manifold with corners. We show that each continuous principal K-bundle over M is continuously equivalent to a smooth one and that two smooth principal K-bundles over M which are continuously equivalent are also smoothly equivalent. In the concluding section, we relate our results to neighboring topics.

Key words. Infinite-dimensional Lie groups, manifolds with corners, continuous principal bundles, smooth principal bundles, equivalences of continuous and smooth principal bundles, smoothing of continuous principal bundles, smoothing of continuous bundle equivalences, non-abelian Čech cohomology, twisted K-theory.

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Introduction

This paper deals with the close interplay between continuous and smooth principal K-bundles over M, where K is a Lie group modeled on an arbitrary locally convex space (following [22]) and M a finite-dimensional paracompact manifold with corners. In this paper we give a complete proof (of a relative version) of the following theorem.

Theorem. *Each continuous principal K-bundle over M is equivalent to a smooth principal K-bundle. Moreover, two smooth principal K-bundles are continuously equivalent if and only if they are smoothly equivalent.*

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One approach to a proof of this theorem is to introduce smooth structures on classifying spaces and to smooth classifying maps. As an example, the classifying space of $K = \mathrm{GL}_n$ is isomorphic to the direct limit of the Grassmanians

$$B\,\mathrm{GL}_n \cong \mathrm{Gr}_n(\infty) := \varinjlim \mathrm{Gr}_n(k).$$

Then [10, Theorem 3.1] provides a smooth manifold structure on $B\,\mathrm{GL}_n$, and one can smooth classifying maps as in [14, Theorem 4.3.5] for the case of vector bundles or in Proposition I.13, for arbitrary finite-dimensional principal bundles. In the infinite-dimensional case, the classifying space of the diffeomorphism group $B\,\mathrm{Diff}(N)$ for a compact manifold N, which can be viewed as a nonlinear Grassmanian, can also be given a smooth structure [16, 44.21].

Smooth structures on classifying spaces are considered in [23], but only generalized de Rham cohomologies are constructed, and bundle theory is not discussed. However, a general theory for differentiable structures on classifying spaces seems to be missing. On the other hand, there exist partial answers to the above question arising from the comparison of continuous and analytic fiber bundles (cf. [12], [27] and [13]). Since these considerations use strong constraints on the structure group, e.g., its compactness in order to ensure a smooth structure on its classifying space, they cannot be used in the generality that we are aiming for.

We now describe our results in some detail. In the first section, we recall the basic facts on continuous and smooth principal bundles with a focus on the description of bundles and bundle equivalences in terms of locally trivial covers and transition functions. Furthermore, we recall briefly the concept of differential calculus and the concept of manifolds with corners that we use in this text. In the end, we outline how to prove our results for finite-dimensional structure groups by using smooth structures on classifying spaces.

The second section is exclusively devoted to the proofs of our main results and to their technical prerequisites. Lacking any smooth structure on classifying spaces in general, we have to employ totally different techniques coming from approximation results for Lie group-valued functions (cf. Proposition I.13). This enables us to smooth representatives of continuous bundles or bundle equivalences in combination with the fact that there is a large freedom of choice in the description of principal bundles by locally trivial covers and transition functions. In this way, we construct new representatives of bundles and bundle equivalences that satisfy cocycle or compatibility conditions on probably finer locally trivial covers, but which describe equivalent objects. Since this technique uses heavily the local compactness of the base manifold, there seems to be no generalization of this method to infinite-dimensional base manifolds. Eventually, we discover that the existence of smooth equivalent bundles and smooth equivalences are a feature of convexity and continuity rendering further requirements on BK unnecessary.

In the third section, we relate our results to some neighboring topics. In particular, we line out the relations to Čech cohomology and to twisted K-theory. Concrete applications of the above theorem arise, for instance, in twisted K-theory (cf. [7], [21]) and for obstruction classes of lifting gerbes (cf. [19]).

I Principal fiber bundles

In this section we provide the basic material concerning manifolds with corners and smooth and continuous principal bundles.

Definition I.1 (Continuous principal bundle). Let K be a topological group and M be a topological space. Then a *continuous principal K-bundle over M* (or shortly a *continuous principal bundle*) is a topological space P together with a continuous right action $P \times K \to P$, $(p,k) \mapsto p \cdot k$, and a map $\eta : P \to M$ such that there exists an open cover $(U_i)_{i \in I}$ of M, called a *locally trivial cover*, and homeomorphisms

$$\Omega_i : \eta^{-1}(U_i) \to U_i \times K,$$

called *local trivializations*, satisfying $\mathrm{pr}_1 \circ \Omega_i = \eta|_{\eta^{-1}(U_i)}$ and $\Omega(p \cdot k) = \Omega(p) \cdot k$. Here K acts on $U_i \times K$ by right multiplication in the second factor. We will use the calligraphic letter $\mathcal{P}$ for the tuple $(K, \eta : P \to M)$.

A *morphism of continuous bundles* or a *continuous bundle map* between two principal K-bundles $\mathcal{P}$ and $\mathcal{P}'$ over M is a continuous map $\Omega : P \to P'$ satisfying $\Omega(p \cdot k) = \Omega(p) \cdot k$. Since $P'/K \cong M \cong P/K$, it induces a map $\Omega^{\#} : M \to M$. We call Ω a *continuous bundle equivalence* if it is an isomorphism and $\Omega^{\#} = \mathrm{id}_M$.

Remark I.2 (Transition functions). If $\mathcal{P}$ is a continuous principal K-bundle over M, then the local trivializations define continuous mappings $k_{ij} : U_i \cap U_j \to K$ by

$$\Omega_i^{-1}(x,e) \cdot k_{ij}(x) = \Omega_j^{-1}(x,e) \;\text{ for all }\; x \in U_i \cap U_j, \tag{1}$$

called *transition functions*. The k_{ij} satisfy the *cocycle condition*

$$\begin{aligned} &k_{ii}(x) = e \;\text{ for all }\; x \in U_i \quad \text{and} \\ &k_{ij}(x) \cdot k_{jn}(x) \cdot k_{ni}(x) = e \;\text{ for all }\; x \in U_i \cap U_j \cap U_n. \end{aligned} \tag{2}$$

On the other hand, if $(V_i)_{i \in I}$ is an open cover and $k = (k_{ij})_{i,j \in I}$ is a collection of continuous maps $k_{ij} : V_i \cap V_j \to K$ that satisfy condition (2), then

$$P_k := \bigcup_{j \in J} \{j\} \times V_j \times K / \sim \qquad \text{with}$$
$$(j,x,k) \sim (j',x',k') \Leftrightarrow x = x' \text{ and } k_{j'j}(x) \cdot k = k'$$

defines a continuous principal K-bundle over M. Here η is given by $[i,x,k] \mapsto x$, the local trivializations by $[(i,x,k)] \mapsto (x,k)$ and the K-action by $([(i,x,k)],k') \mapsto [(i,x,kk')]$. We will write $\mathcal{P}_k$ for a bundle determined by a collection $(M, K, (V_i)_{i \in I}, (k_{ij})_{i,j \in I})$.

If k arises from the local trivializations of a given bundle $\mathcal{P}$ as in (1), then

$$\Omega : P \to P_k, \;\; p \mapsto [i, \Omega_i(p)] \;\text{ if } p \in \eta^{-1}(U_i)$$

defines a bundle equivalence between $\mathcal{P}$ and $\mathcal{P}_k$ whose inverse is given by $[i,x,k] \mapsto \Omega_i^{-1}(x,k)$.

Definition I.3 (Differential calculus on locally convex spaces; cf. [11]). Let E and F be locally convex spaces and $U \subseteq E$ be open. Then $f : U \to F$ is called *continuously differentiable* or C^1 if it is continuous, for each $v \in E$ the differential quotient

$$df(x).v := \lim_{h\to 0} \frac{1}{h}(f(x+hv) - f(x))$$

exists and the map $df : U \times E \to F$ is continuous. For $n > 1$, we recursively define

$$d^n f(x).(v_1,\ldots,v_n) := \lim_{h\to 0} \frac{1}{h}\big(d^{n-1}f(x+h).(v_1,\ldots,v_{n-1}) - d^{n-1}f(x).(v_1,\ldots,v_n)\big)$$

and say that f is C^n if $d^k f : U \times E^k \to F$ exists for all $k = 1,\ldots,n$ and is continuous. We say that f is C^∞ or *smooth* if it is C^n for all $n \in \mathbb{N}$.

Definition I.4 (Lie group). From the definition above, the notion of a *Lie group* is clear. It is a group which is a smooth manifold modeled on a locally convex space such that the group operations are smooth.

Remark I.5 (Convenient calculus). We briefly recall the basic definitions of the convenient calculus from [16]. Again, let E and F be locally convex spaces. A curve $f : \mathbb{R} \to E$ is called smooth if it is smooth in the sense of Definition I.3. Then the c^∞-topology on E is the final topology induced from all smooth curves $f \in C^\infty(\mathbb{R}, E)$. If E is a Fréchet space, then the c^∞-topology is again a locally convex vector topology which coincides with the original topology [16, Theorem 4.11]. If $U \subseteq E$ is c^∞-open, then $f : U \to F$ is said to be of class C^∞ or smooth if

$$f_*(C^\infty(\mathbb{R}, U)) \subseteq C^\infty(\mathbb{R}, F),$$

i.e. if f maps smooth curves to smooth curves. The chain rule [9, Proposition 1.15] implies that each smooth map in the sense of Definition I.3 is smooth in the convenient sense. On the other hand, [16, Theorem 12.8] implies that on a Fréchet space a smooth map in the convenient sense is smooth in the sense of Definition I.3. Hence for Fréchet spaces, the two notions coincide.

Remark I.6 (Manifold with corners). A *d-dimensional manifold with corners* is a paracompact topological space such that each point has a neighborhood that is homeomorphic to an open subset of

$$\mathbb{R}^d_+ = \{(x_1,\ldots,x_d) \in \mathbb{R}^d : x_i \geq 0 \text{ for all } i = 1,\ldots,d\}$$

and such that the corresponding coordinate changes are smooth (cf. [29], [20]). The crucial point here is the notion of smoothness for non-open domains. We define a map $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ to be smooth if for each $x \in A$, there exists a neighborhood U_x of x which is open in $\mathbb{R}^n$, and a smooth map $f_x : U_x \to \mathbb{R}^m$ such that $f_x\big|_{A\cap U_x} = f\big|_{A\cap U_x}$.

Remark I.7 (Paracompact spaces). We recall some basic facts from general topology. If X is a topological space, then a collection of subsets $(U_i)_{i\in I}$ of X is called *locally finite* if each $x \in X$ has a neighborhood that has non-empty intersection with only finitely many U_i, and X is called *paracompact* if each open cover has a locally finite refinement. If X is the union of countably many compact subsets, then it is called *σ-compact*, and if each open cover has a countable subcover, it is called *Lindelöf*.

Now let M be a finite-dimensional manifold with corners, which is in particular locally compact and locally connected. For these spaces, [6, Theorems XI.7.2+3] imply that M is paracompact if and only if each component is σ-compact, equivalently, Lindelöf. Furthermore, [6, Theorem VIII.2.2] implies that M is normal in each of these cases.

Definition I.8 (Smooth principal bundle). If K is a Lie group and M is a smooth manifold with corners, then a continuous principal K-bundle over M is called a *smooth principal K-bundle over M* (or shortly a *smooth principal bundle*) if the transition functions from Remark I.2 are smooth for some choice of local trivializations.

Remark I.9 (Smooth structure on smooth principal bundles). If $\mathcal{P}$ is a smooth principal bundle, then we define on P the structure of a smooth manifold with corners by requiring the local trivializations

$$\Omega_i : \eta^{-1}(U_i) \to U_i \times K$$

that define the smooth transition functions from Definition I.8 to be diffeomorphisms. This actually defines a smooth structure on P, since it is covered by $(\eta^{-1}(U_i))_{i\in I}$ and since the coordinate changes

$$(U_i \cap U_j) \times K \to (U_i \cap U_j) \times K, \quad (x,k) \mapsto \Omega_j(\Omega_i^{-1}(x,k)) = (x, k_{ij}(x)\cdot k)$$

are smooth because the k_{ij} are assumed to be smooth. A continuous bundle map between smooth principal bundles is called a *morphism of smooth principal bundles* (or a *smooth bundle map*) if it is smooth with respect to the smooth structure on the bundles just described.

Remark I.10 (Bundle equivalences). If $\mathcal{P}$ and $\mathcal{P}'$ are two principal K-bundles over M, then there exists an open cover $(U_i)_{i\in I}$ of M such that we have local trivializations

$$\Omega_i : \eta^{-1}(U_i) \to U_i \times K,$$
$$\Omega_i' : \eta'^{-1}(U_i) \to U_i \times K$$

for $\mathcal{P}$ and $\mathcal{P}'$. In fact, if $(V_j)_{j\in J}$ and $(V'_{j'})_{j'\in J'}$ are locally trivial covers of M (for $\mathcal{P}$ and for $\mathcal{P}'$, respectively), then

$$(V_j \cap V_{j'})_{(j,j')\in J\times J'}$$

is simultaneously a locally trivial cover for both $\mathcal{P}$ and $\mathcal{P}'$, and the local trivializations are given by restricting the original ones.

If $\mathcal{P}_k$ and $\mathcal{P}_{k'}$ are given by transition functions k_{ij} and k'_{ij} with respect to the same open cover $(U_i)_{i\in I}$ (i.e., $k_{ij} : U_i \cap U_j \to K$ and $k'_{ij} : U_i \cap U_j \to K$), then a bundle equivalence $\Omega : P_k \to P_{k'}$ defines for each $i \in I$ a continuous map

$$\varphi_i : U_i \times K \to K \;\text{ by }\; \Omega([(i,x,k)]) = [(i,x,\varphi_i(x,k))]. \tag{3}$$

Furthermore, we have $\varphi_i(x,k) = \varphi_i(x,e)\cdot k$, since Ω satisfies $\Omega(p\cdot k) = \Omega(p)\cdot k$. Setting $f_i(x) := \varphi_i(x,e)$, we thus obtain continuous maps $f_i : U_i \to K$ satisfying the *compatibility condition*

$$f_j(x) = k'_{ji}(x)\cdot f_i(x)\cdot k_{ij}(x) \;\text{ for all }\; x \in U_i \cap U_j, \tag{4}$$

since $[(i,x,k)] = [(j,x,k_{ji}(x)k)]$ has to be mapped to the same element of $P_{k'}$ by Ω. On the other hand, if for each $i \in I$ we have continuous maps $f_i : U_i \to K$ satisfying (4), then

$$P_k \ni [(i,x,k)] \mapsto [(i,x,f_i(x)\cdot k)] \in P_{k'}$$

defines a bundle equivalence between $\mathcal{P}_k$ and $\mathcal{P}_{k'}$ which covers the identity on M.

If $\mathcal{P}_k$ and $\mathcal{P}_{k'}$ are smooth and the maps k_{ij} and k'_{ij} are smooth, then it follows directly from (3) that a bundle equivalence described by continuous maps $f_i : U_i \to K$ is smooth if and only if these maps are smooth.

Lemma I.11 (Smooth and continuous homotopies coincide; [28, Corollary 12], [17]). *Let M be a finite-dimensional manifold with corners and N be a smooth manifold modeled on a locally convex space. If $f : M \to N$ is continuous, then there exists a continuous map $F : [0,1] \times M \to N$ such that $F(0,x) = f(x)$ and $F(1,\,\cdot\,) : M \to N$ is smooth. Furthermore, if $f,g : M \to N$ are smooth and there exists a continuous homotopy between f and g, then there exists a smooth homotopy between f and g.*

Lemma I.12 (Smooth structures on classifying spaces). *If K is a compact Lie group, then it has a smooth classifying bundle $EK \to BK$ (cf.* [15, Chapter 4.11]*), which is in general infinite-dimensional.*

Proof. Let $\mathrm{O}_k \subseteq \mathrm{GL}_k(\mathbb{R})$ denote the orthogonal group. If k is sufficiently large, then we may identify K with a subgroup of O_k, and from [26, Theorem 19.6] we get the following formulae:

$$EK = \varinjlim \mathrm{O}_n / (\mathrm{O}_{n-k} \times \mathrm{id}_{\mathbb{R}^k}),$$
$$BK = \varinjlim \mathrm{O}_n / (\mathrm{O}_{n-k} \times K).$$

Thus EK and BK are smooth manifolds by [10, Theorem 3.1], and since the action of K is smooth, it follows that $EK \to BK$ is a smooth K-principal bundle. □

Proposition I.13 (Smoothing finite-dimensional principal bundles). *If $\mathcal{P}$ is a continuous principal K-bundle over M, K is a finite-dimensional Lie group and M is a finite-dimensional manifold with corners, then there exists a smooth bundle which is continuously equivalent to $\mathcal{P}$. Moreover, two smooth principal K-bundles over M are smoothly equivalent if and only if they are continuously equivalent.*

Proof. Let C be a maximal compact subgroup of K. Since K/C is contractible, there exists a C-reduction of $\mathcal{P}$, i.e., we may choose a locally trivial open cover $(U_i)_{i\in I}$ with transition functions k_{ij} that take values in C. They define a continuous principal C-bundle which is given by a classifying map $f : M \to BC$.

By Lemma I.11, f is homotopic to some smooth map $\widetilde{f} : M \to BC$ which in turn determines a smooth principal C-bundle $\widetilde{\mathcal{P}}$ over M given by smooth transition functions $\widetilde{k}_{ij}$. Furthermore, since f and $\widetilde{f}$ are homotopic, $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ are equivalent, and we thus have a continuous bundle equivalence given by continuous mappings $f_i : U_i \to K$. The claim follows if we regard k_{ij}, $\widetilde{k}_{ij}$ and f_i as mappings into K.

Since smooth bundles yield smooth classifying maps and smooth homotopies of classifying maps yield smooth bundle equivalences (all the constructions in the topological setting depend only on partitions of unity which we can assume to be smooth here), the second claim is also immediate. □

Remark I.14 (On the previous proof). The previous proof can also be obtained without the need of passing to the direct limit in Proposition I.12, because $\mathrm{O}_{n+k} / \mathrm{O}_n$ is already a universal bundle if $\dim(M) \leq n$ (cf. [26, Remark 19.7]).

II Equivalences of smooth and continuous bundles

In this section, we prove the two main results of this paper. We start with the description of two important tools: a proposition for smoothing continuous maps and a lemma for fading out continuous functions. Then we provide some technical data for the proofs, namely covers of the finite-dimensional paracompact base manifold with corners and suitable identity neighborhoods in the Lie group. On this basis, we finally prove our claims after outlining the underlying ideas in Remark II.10.

Remark II.1 (Topology on $C(X,G)$). If X is a Hausdorff space and G is a topological group, then $C(X,G)_c$ denotes the topological group of continuous functions with respect to pointwise multiplication and the topology of compact convergence. A basic open identity neighborhood in this topology is given by

$$\lfloor C, W \rfloor := \{f \in C(X,G) : f(C) \subseteq W\}$$

for a compact subset $C \subseteq X$ and an open identity neighborhood $W \subseteq G$.

Proposition II.2 (Smoothing). *Let M be a finite-dimensional manifold with corners, K a Lie group modeled on a locally convex space and $f \in C(M,K)$. If $A \subseteq M$ is closed and $U \subseteq M$ is open such that f is smooth on a neighborhood of $A \setminus U$, then each open neighborhood O of f in $C(M,K)_c$ contains a map g which is smooth on a neighborhood of A and equals f on $M \setminus U$.*

Proof. This is [28, Corollary 12], see also [24, Theorem A.3.3] or [14, Theorem 2.5]. □

Remark II.3 (Centered chart, convex subset). Let K be a Lie group modeled on a locally convex topological vector space E. A chart $\varphi : W \to \varphi(W) \subseteq E$ with $e \in W$ and $\varphi(e) = 0$ is called a *centered chart*. A subset L of W is called *φ-convex* if it is identified with a convex subset $\varphi(L)$ in E. If W itself is φ-convex, we speak of a *convex centered chart*.

It is clear that every open identity neighborhood in K contains a φ-convex open neighborhood for some centered chart φ, because we can pull back any convex open neighborhood that is small enough from the underlying locally convex vector space.

Lemma II.4 (Fading-out). *Let M be a finite-dimensional manifold with corners, A and B be closed subsets satisfying $B \subseteq A^0$, $\varphi : W \to \varphi(W)$ be a convex centered chart of a Lie group K modeled on a locally convex space, and $f : A \to W$ be a continuous function. Then there is a continuous function $F : M \to W \subseteq K$ that coincides with f on an open neighborhood of B and is the identity on an open neighborhood of $M \setminus A^0$. Moreover, F can be chosen in a way that if $W' \subseteq W$ is another φ-convex set containing e, then $f(x) \in W'$ implies $F(x) \in W'$ for each $x \in A$, and if f is smooth on an open set $U \subseteq A$, then F is also smooth on U.*

Proof. Since M is paracompact and the closed sets $M \setminus A^0$ and B are disjoint by assumption, there exists a smooth map $\lambda : M \to [0,1]$ such that λ is identically 1 on a neighborhood of B and is identically 0 on a neighborhood of $M \setminus A^0$ (see [14, Theorem 2.1]). Since $\varphi(W)$ is a convex zero neighborhood in E, we have $[0,1] \cdot \varphi(W) \subseteq \varphi(W)$. We use this to define the continuous function

$$f_\lambda : A \to W, \quad x \mapsto \varphi^{-1}(\lambda(x) \cdot \varphi(f(x))),$$

that coincides, by the choice of λ, with f on $M \setminus \operatorname{supp}(1-\lambda) \subseteq B$ and is identically e on $M \setminus \operatorname{supp}(\lambda) \supseteq M \setminus A^0$. So we may extend f_λ to the continuous function

$$F : M \to W, \quad x \mapsto \begin{cases} f_\lambda(x), & \text{if } x \in A \\ e, & \text{if } x \in M \setminus A^0 \end{cases}$$

that satisfies all requirements. □

Lemma II.5 (Squeezing-in manifolds with corners). *Let W be an open neighborhood of a point x in $\mathbb{R}^d_+$ (cf. Remark* I.6*) and $C \subseteq W$ be a compact set containing x. Then there exists an open set V satisfying $x \in C \subseteq V \subseteq \overline{V} \subseteq W$ whose closure $\overline{V}$ is a compact manifold with corners.*

Proof. For every $x = (x_1, \ldots, x_d) \in C$, there is an $\varepsilon_x > 0$ such that

$$B(x, \varepsilon_x) := [x_1 - \varepsilon_x, x_1 + \varepsilon_x] \times \cdots \times [x_d - \varepsilon_x, x_d + \varepsilon_x] \cap \mathbb{R}^d_+ \qquad (5)$$

is contained in W. The interiors $V_x := B(x, \varepsilon_x)^0$ in $\mathbb{R}^d_+$ form an open cover of the compact set C, of which we may choose a finite subcollection $(V_{x_i})_{i=1,\ldots,m}$ covering C. The union $V := \bigcup_{i=1}^m V_{x_i}$ satisfies all requirements. In particular, $\overline{V}$ is a compact manifold with corners, because it is a finite union of cubes whose sides are orthogonal to the coordinate axes. □

Proposition II.6 (Nested covers). *Let M be a connected finite-dimensional manifold with corners and $(U_j)_{j\in J}$ be an open cover of M. Then there exist countable open covers $\big(U_i^{[\infty]}\big)_{i\in\mathbb{N}}$ and $\big(U_i^{[0]}\big)_{i\in\mathbb{N}}$ of M such that $\overline{U}_i^{[\infty]} := \overline{U_i^{[\infty]}}$ and $\overline{U}_i^{[0]} := \overline{U_i^{[0]}}$ are compact manifolds with corners, $\overline{U}_i^{[\infty]} \subseteq U_i^{[0]}$ for all $i \in \mathbb{N}$, and such that even the cover $\big(\overline{U}_i^{[0]}\big)_{i\in\mathbb{N}}$ of M by compact sets is locally finite and subordinate to $(U_j)_{j\in J}$.*

In this situation, let L be any countable subset of the open interval $(0,\infty)$. Then for every $\lambda \in L$, there exists a countable, locally finite cover $\big(U_i^{[\lambda]}\big)_{i\in\mathbb{N}}$ of M by open sets whose closures are compact manifolds with corners such that $\overline{U}_i^{[\lambda]} \subseteq U_i^{[\mu]}$ holds whenever $0 \le \mu < \lambda \le \infty$.

Proof. For every $x \in M$, we have $x \in U_{j(x)}$ for some $j(x) \in J$. Let (U_x, φ_x) be a chart of M around x such that $\overline{U_x} \subseteq U_{j(x)}$. We can even find an open neighborhood V_x of x whose closure $\overline{V_x}$ is compact and contained in U_x. Since M is paracompact, the open cover $(V_x)_{x\in M}$ has a locally finite subordinated cover $(V_i)_{i\in I}$, where $V_i \subseteq V_x$ and $\overline{V_i} \subseteq \overline{V_x} \subseteq U_x$ for suitable $x = x(i)$. Since M is also Lindelöf, we may assume that $I = \mathbb{N}$.

To find suitable covers $U_i^{[\infty]}$ and $U_i^{[0]}$, we are going to enlarge the sets V_i so carefully in two steps that the resulting covers remain locally finite. More precisely, $U_i^{[\infty]}$ and $U_i^{[0]}$ will be defined inductively so that even the family $(V_k^i)_{k\in\mathbb{N}}$ with

$$V_k^i := \begin{cases} \overline{U}_k^{[0]} & \text{for } k \le i \\ V_k & \text{for } k > i \end{cases}$$

is still a locally finite cover of M for every $i \in \mathbb{N}_0$. We already know this for $i = 0$, because $V_k^0 = V_k$ for all $k \in \mathbb{N}$. For $i > 0$, we proceed by induction.

For every point $y \in \overline{V_i}$, there is an open neighborhood $V_{i,y}$ of y inside $U_{x(i)}$ whose intersection with just finitely many V_j^{i-1} is non-empty. Under the chart $\varphi_{x(i)}$, this neighborhood $V_{i,y}$ is mapped to an open neighborhood of $\varphi_{x(i)}(y)$ in the modeling space $\mathbb{R}_+^d$ of M. There exist real numbers $\varepsilon_0(y) > \varepsilon_\infty(y) > 0$ such that the cubes $B(y, \varepsilon_\infty(y))$ and $B(y, \varepsilon_0(y))$ introduced in (5) are compact neighborhoods of $\varphi_{x(i)}(y)$ contained in $\varphi_{x(i)}(V_{i,y})$. Since $\overline{V_i}$ is compact, it is covered by finitely many sets $V_{i,y}$, say by $(V_{i,y})_{y\in Y}$ for a finite subset Y of $\overline{V_i}$. We define the open sets

$$U_i^{[\infty]} := \bigcup_{y\in Y} \varphi_{x(i)}^{-1}\big(B(y, \varepsilon_\infty(y))^0\big) \quad \text{and} \quad U_i^{[0]} := \bigcup_{y\in Y} \varphi_{x(i)}^{-1}\big(B(y, \varepsilon_0(y))^0\big),$$

whose closures are compact manifolds with corners, because each is a finite union of cubes under the chart $\varphi_{x(i)}$. On the one hand, the construction guarantees

$$V_i \subseteq U_i^{[\infty]} \subseteq \overline{U}_i^{[\infty]} \subseteq U_i^{[0]} \subseteq \overline{U}_i^{[0]} \subseteq \bigcup_{y\in Y} V_{i,y} \subseteq U_{x(i)}.$$

On the other hand, the cover $\big(V_k^i\big)_{k\in\mathbb{N}}$ is locally finite, because it differs from the locally finite cover $\big(V_k^{i-1}\big)_{k\in\mathbb{N}}$ in the single set $V_i^i = \overline{U}_i^{[0]}$.

For a proof of the second claim, we fix an enumeration $\lambda_1, \lambda_2, \ldots$ of L for an inductive construction of the covers. Then for any $n \geq 1$ and $i \in \mathbb{N}$, we apply Lemma II.5 to $C := \varphi_i\big(\overline{U}_i^{[\overline{\lambda}]}\big)$ and $W := \varphi_i\big(U_i^{[\underline{\lambda}]}\big)$, where $\overline{\lambda}$ (respectively $\underline{\lambda}$) is the smallest (respectively largest) element of $\lambda_1, \ldots, \lambda_{n-1}$ larger than (respectively smaller than) λ_n for $n > 1$ and ∞ (respectively 0) for $n = 1$. We get open sets $U_i^{[\lambda_n]}$ such that the condition $\overline{U}_i^{[\lambda]} \subseteq U_i^{[\mu]}$ holds whenever $0 \leq \mu < \lambda \leq \infty$ are elements in $\{\lambda_1, \ldots, \lambda_n\}$, and eventually in L. This completes the proof. □

Remark II.7 (Locally finite covers by compact sets). If $\big(\overline{U}_i\big)_{i\in I}$ is a locally finite cover of M by compact sets, then for fixed $i \in I$, the intersection $\overline{U}_i \cap \overline{U}_j$ is non-empty for only finitely many $j \in I$. Indeed, for every $x \in \overline{U}_i$, there is an open neighborhood U_x of x such that $I_x := \{j \in I : U_x \cap \overline{U}_j \neq \emptyset\}$ is finite. Since $\overline{U}_i$ is compact, it is covered by finitely many of these sets, say by $U_{x_1}, \ldots, U_{x_n}$. Then $J := I_{x_1} \cup \cdots \cup I_{x_n}$ is the finite set of indices $j \in J$ such that $\overline{U}_i \cap \overline{U}_j$ is non-empty, proving the claim.

Remark II.8 (Intersections). From now on, multiple lower indices on subsets of M always indicate intersections, namely $U_{1\ldots r} := U_1 \cap \ldots \cap U_r$.

Lemma II.9 (Suitable identity neighborhoods). *Let M be a finite-dimensional manifold with corners that is covered locally finitely by countably many compact sets $\big(\overline{U}_i\big)_{i\in\mathbb{N}}$. Moreover, let $k_{ij} : \overline{U}_{ij} \to K$ be continuous functions into a Lie group K so that $k_{ij} = k_{ji}^{-1}$ holds for all $i, j \in \mathbb{N}$. Then for any convex centered chart $\varphi : W \to \varphi(W)$ of K, each sequence of open unit neighborhoods $(W'_j)_{j\in\mathbb{N}}$ with $W'_j \subseteq W$ and each $\alpha \in \mathbb{N}$, there are φ-convex open identity neighborhoods $W^\alpha_{ij} \subseteq W$ in K for indices $i < j$ and $W^\alpha_j \subseteq W'_j$ for $j \in \mathbb{N}$ that satisfy*

$$k_{ji}(x) \cdot (W^\alpha_{ij})^{-1} \cdot W^\alpha_i \cdot k_{ij}(x) \subseteq W^\alpha_j \ \text{ for all } \ x \in \overline{U}_{ij\alpha} \text{ and } i < j, \tag{6}$$

$$k_{ji}(x) \cdot (W^\alpha_{ij})^{-1} \cdot W^\alpha_{in} \cdot k_{ij}(x) \subseteq W^\alpha_{jn} \ \text{ for all } \ x \in \overline{U}_{ijn\alpha} \text{ and } i < j < n \tag{7}$$

Proof. Initially, we set $W^\alpha_i := W'_i$ for all i, respectively $W^\alpha_{ij} := W$ for all $i < j$, disregarding the conditions (6) and (7). These sets are shrinked later so that they satisfy (6) and (7).

Our first goal is to satisfy (6). We note that the condition in (6) becomes trivial if $\overline{U}_{j\alpha}$ is empty, because this implies $\overline{U}_{ij\alpha} = \emptyset$. So we need to consider at most finitely many conditions (6) corresponding to the finitely many $j \in \mathbb{N}$ such that $\overline{U}_{j\alpha} \neq \emptyset$, and we deal with those inductively in decreasing order of j, starting with the maximal such index.

For fixed j and all $i < j$ with $\overline{U}_{ij\alpha} \neq \emptyset$, we describe below how to make the φ-convex identity neighborhoods W^α_{ij} and W^α_i on the left hand side smaller so that the corresponding conditions (6) are satisfied. Making W^α_{ij} and W^α_i smaller does not compromise any conditions on $W^\alpha_{ij'}$ and $W^\alpha_{j'}$ for $j' > j$ that we guaranteed before, because these sets can only appear on the left hand side of such conditions.

To satisfy condition (6) for given $i < j$ and W^α_j, we note that the function

$$\varphi_{ij} : \overline{U}_{ij\alpha} \times K \times K \to K, \quad (x, k, k') \mapsto k_{ji}(x) \cdot k^{-1} \cdot k' \cdot k_{ij}(x)$$

is continuous and maps all the points (x, e, e) for $x \in \overline{U}_{ij\alpha}$ to the identity e in K. Hence we may choose open neighborhoods U_x of x and φ-convex open identity neighborhoods $W_x \subseteq W_{ij}^\alpha$ and $W'_x \subseteq W_i^\alpha$ such that $\varphi_{ij}(U_x \times W_x \times W'_x) \subseteq W_j^\alpha$. Since $\overline{U}_{ij\alpha}$ is compact, it is covered by finitely many U_x, say by $(U_x)_{x\in F}$ for a finite set $F \subseteq \overline{U}_{ij\alpha}$. Then we replace W_{ij}^α and W_i^α by their subsets $\bigcap_{x\in F} W_x$ and $\bigcap_{x\in F} W'_x$, respectively, which are φ-convex open identity neighborhoods such that $\varphi_{ij}(\overline{U}_{ij\alpha} \times W_{ij}^\alpha \times W_i^\alpha) \subseteq W_j^\alpha$, in other words, (6) is satisfied

Our second goal is to make the sets W_{ij}^α also satisfy (7), which is non-trivial for the finitely many triples $(i, j, n) \in \mathbb{N}^3$ with $i < j < n$ that satisfy $\overline{U}_{ijn\alpha} \neq \emptyset$. We can argue as above, except for a slightly more complicated order of processing the sets W_{jn}^α on the right hand side. Namely, we define the following total order

$$(i, j) < (i', j') \quad :\Leftrightarrow \quad j < j' \text{ or } (j = j' \text{ and } i < i') \tag{8}$$

on pairs of real numbers, in particular on pairs of indices (i, j) in $\mathbb{N} \times \mathbb{N}$ with $i < j$. Note that this guarantees $(i, j) < (j, n)$ and $(i, n) < (j, n)$ whenever i, j, n are as in condition (7). We process the pairs (j, n) with $\overline{U}_{ijn\alpha} \neq \emptyset$ for some i in descending order, starting with the maximal such pair. At each step, we fix W_{jn}^α and make W_{ij}^α and W_{in}^α smaller for all relevant $i < j$ so that (7) is satisfied. This does not violate any conditions (6) or (7) that we guaranteed earlier in the process, because W_{ij}^α and W_{in}^α can only appear on the left hand side of such conditions. For the choice of the smaller identity neighborhoods, we use the continuous function

$$\varphi_{ijn} : \overline{U}_{ijn\alpha} \times K \times K \to K, \quad (x, k, k') \mapsto k_{ji}(x) \cdot k^{-1} \cdot k' \cdot k_{ij}(x)$$

and the compactness of $\overline{U}_{ijn\alpha}$ and argue as before. We thus accomplish our second goal. □

Remark II.10 (Outline of the proofs). Although the proofs of our main results are quite technical, the underlying ideas are easy to explain. The following two theorems require us to construct principal bundles and/or equivalences between them, and we always do so locally on countable covers of the base manifold by induction. In these constructions, every new transition function (respectively, every new local representative of an equivalence)

- is already determined by cocycle conditions (respectively, by compatibility conditions) on a subset of its domain,
- from which it will be "faded out" to a continuous function on the whole domain
- and smoothed, if necessary.

In each such step, we need a safety margin to modify the functions without compromising previous achievements too much, and these safety margins are the nested open covers provided by Proposition II.6. In order to "fade out" appropriately, we need to make sure that the values of the corresponding functions stay in certain identity neighborhoods of the structure group. This is achieved with the data from Lemma II.9.

Theorem II.11 (Smoothing continuous principal bundles). *Let K be a Lie group modeled on a locally convex space, M be a finite-dimensional connected paracompact manifold with corners and $\mathcal{P}$ be a continuous principal K-bundle over M. If $C \subseteq M$ is closed* *and the restriction of $\mathcal{P}$ to some open neighborhood of C is smooth, then there exists an open neighborhood T of C such that the restriction $\mathcal{P}|_T$ extends to a smooth principal K-bundle $\widetilde{\mathcal{P}}$ over M, in the sense that $\mathcal{P}|_T$ is a K-invariant open subset of $\widetilde{\mathcal{P}}$ and the K-action and bundle projection from $\mathcal{P}|_T$ extend to the ones on $\widetilde{\mathcal{P}}$. Furthermore, there exists a continuous bundle equivalence $\Omega : P \to \widetilde{P}$, which restricts to the identity on $\eta^{-1}(T)$.*

Proof. We assume that the continuous bundle $\mathcal{P}$ is given by $\mathcal{P}_k$ as in Remark I.2, where $(U_j)_{j\in J}$ is a locally trivial cover of M and $k_{ij} : U_{ij} \to K$ are continuous transition functions that satisfy the cocycle condition $k_{ij} \cdot k_{jn} = k_{in}$ pointwise on U_{ijn}. That $\mathcal{P}$ is smooth on a neighborhood of C implies that there exists an open neighborhood S of C such that the restriction of each k_{ij} to $U_{ij} \cap S$ is smooth. In fact, let S' be an open neighborhood of C such that $\mathcal{P}|_{S'}$ is smooth. Since M is normal (see Remark I.7), we find open sets $S, T \subseteq M$ that satisfy $C \subseteq T \subseteq \overline{T} \subseteq S \subseteq \overline{S} \subseteq S'$, which we fix from now on. In addition, there exists a locally trivial cover of S', together with local trivializations, such that the resulting transition functions are smooth. Restricting the continuous transition functions of an arbitrary locally trivial cover to the complement of $\overline{S}$, adding the smooth ones and the ones induced by the cocycle condition yields the desired collection of transition functions.

Proposition II.6 yields open covers $\big(U_i^{[\infty]}\big)_{i\in\mathbb{N}}$ and $\big(U_i^{[0]}\big)_{i\in\mathbb{N}}$ of M subordinate to $(U_j)_{j\in J}$ with $\overline{U}_i^{[\infty]} \subseteq U_i^{[0]}$ for all $i \in \mathbb{N}$. For every $i \in \mathbb{N}$, we denote by U_i an open set of the cover $(U_j)_{j\in J}$ that contains $\overline{U}_i^{[0]}$ and observe that $(U_i)_{i\in\mathbb{N}}$ is still a locally trivial open cover of M. In our construction, we need open covers not only for pairs $(j, n) \in \mathbb{N} \times \mathbb{N}$ with $j < n$, but also for pairs $(j - 1/3, n)$, $(j - 2/3, n)$ in-between and (n, n) to enable continuous extensions and smoothing. The function

$$\lambda : \Big\{(j, n) \in \frac{1}{3}\mathbb{N}_0 \times \mathbb{N} : j \le n\Big\} \to [0, \infty), \quad \lambda(j, n) = \frac{n(n-1)}{2} + j,$$

is tailored to map the pairs $(0, 1), (1, 1), (1, 2), (2, 2), (1, 3), (2, 3), (3, 3), (1, 4), \ldots$ to the integers $0, 1, 2, \ldots$, respectively, and the other pairs in-between. If we apply the second part of Proposition II.6 to the countable subset $L := (\operatorname{im}\lambda) \setminus \{0\}$ of $(0, \infty)$, we get open sets $U_i^{[jn]} := U_i^{[\lambda(j,n)]}$ for all pairs (j, n) in the domain of λ such that $\big(\overline{U}_i^{[jn]}\big)_{i\in\mathbb{N}}$ are again locally finite covers. We note that $(j, n) < (j', n')$ in the sense of (8) implies $\overline{U}_i^{[j'n']} \subseteq U_i^{[jn]}$.

Let $\varphi : W \to \varphi(W)$ be an arbitrary convex centered chart of K and consider the countable compact cover $\big(\overline{U}_i^{[0]}\big)_{i\in\mathbb{N}}$ of M and the restrictions $k_{ij}|_{\overline{U}_{ij}^{[0]}}$ of the continuous transition functions to the corresponding intersections. Then Lemma II.9, applied to the sequence of open unit neighborhoods which is constantly W, yields open φ-convex identity neighborhoods W_{ij}^{α} and W_i^{α} with the corresponding properties.

Our first goal is the construction of smooth maps $\widetilde{k}_{ij} : U_{ij}^{[0]} \to K$ that satisfy the cocycle condition on the open cover $\big(U_i^{[\infty]}\big)_{i\in\mathbb{N}}$ of M, which uniquely determines a smooth principal K-bundle $\mathcal{P}_{\widetilde{k}}$ by Remarks I.2 and I.9. Furthermore, we shall construct $\widetilde{k}_{ij}$ in a way that guarantees

$$\widetilde{k}_{ij}\Big|_{U_{ij}^{[0]}\cap T} = k_{ij}\Big|_{U_{ij}^{[0]}\cap T}$$

ensuring that we may view $\mathcal{P}_{\widetilde{k}}|_T$ as a subset of $\mathcal{P}_k$. These maps $\widetilde{k}_{ij}$ will be constructed step-by-step in increasing order with respect to (8), starting with the minimal index $(1,2)$. At all times during the construction, the conditions

(a) $\widetilde{k}_{jn} = \widetilde{k}_{ji}\cdot\widetilde{k}_{in}$ pointwise on $\overline{U}_{ijn}^{[jn]}$ for all $i<j<n$ in $\mathbb{N}$,
(b) $\big(\widetilde{k}_{jn}\cdot k_{nj}\big)\big(\overline{U}_{jn\alpha}^{[jn]}\big)\subseteq W_{jn}^{\alpha}$ for all $j<n$ and α in $\mathbb{N}$ and
(c) $\widetilde{k}_{jn}\big|_{U_{jn}^{[0]}\cap T} = k_{jn}\big|_{U_{jn}^{[0]}\cap T}$ for all $j,n\in\mathbb{N}$

will be satisfied whenever all $\widetilde{k}_{ij}$ involved have already been constructed. We are now going to construct the smooth maps $\widetilde{k}_{jn}$ for indices $j<n$ in $\mathbb{N}$ (and implicitly $\widetilde{k}_{nj}$ as $\widetilde{k}_{nj}(x) := \widetilde{k}_{jn}(x)^{-1}$), assuming that this has already been done for pairs of indices smaller than (j,n).

- To satisfy all relevant cocycle conditions, we start with

$$\widetilde{k}'_{jn} : \bigcup_{i<j}\overline{U}_{ijn}^{[j-1,n]} \to K,\quad \widetilde{k}'_{jn}(x) := \widetilde{k}_{ji}(x)\cdot\widetilde{k}_{in}(x)\ \text{ for } x\in\overline{U}_{ijn}^{[j-1,n]}.$$

  This function is well-defined, because the cocycle conditions (a) for lower indices assert that for any indices $i'<i<j$ and any point $x\in\overline{U}_{i'jn}^{[j-1,n]}\cap\overline{U}_{ijn}^{[j-1,n]}$, we have

$$\widetilde{k}_{ji'}(x)\cdot\widetilde{k}_{i'n}(x) = \widetilde{k}_{ji'}(x)\cdot\widetilde{k}_{i'i}(x)\cdot\widetilde{k}_{ii'}(x)\cdot\widetilde{k}_{in}(x) = \widetilde{k}_{ji}(x)\cdot\widetilde{k}_{in}(x),$$

  because $\overline{U}_{i'ijn}^{[j-1,n]}$ is contained in both $\overline{U}_{i'ij}^{[ij]}$ and $\overline{U}_{i'in}^{[in]}$. Furthermore, $\widetilde{k}'_{jn}$ coincides with k_{jn} on $\bigcup_{i<j}\overline{U}_{ijn}^{[j-1,n]}\cap T$ as $\widetilde{k}'_{jn}(x) = \widetilde{k}_{ji}(x)\cdot\widetilde{k}_{in}(x) = k_{ji}(x)\cdot k_{in}(x) = k_{jn}(x)$.
- Next, we want to extend the map $\widetilde{k}'_{jn}$ on $\bigcup_{i<j}\overline{U}_{ijn}^{[j-1,n]}$ to a continuous map k'_{jn} on $U_{jn}^{[0]}$ without compromising the cocycle conditions too much. To do this, we consider the function $\varphi_{jn} := \widetilde{k}'_{jn}\cdot k_{nj} : \bigcup_{i<j}\overline{U}_{ijn}^{[j-1,n]}\to K$. For all $i<j$, $\alpha\in\mathbb{N}$ and $x\in\overline{U}_{ijn\alpha}^{[j-1,n]}$, conditions (b) above and (7) of Lemma II.9 imply

$$\begin{aligned}\varphi_{jn}(x) = (\widetilde{k}'_{jn}k_{nj})(x) &= k_{ji}(x)\cdot\big(\underbrace{(\widetilde{k}_{ij}\cdot k_{ji})(x)}_{\in W_{ij}^{\alpha}}\big)^{-1}\cdot\underbrace{(\widetilde{k}_{in}\cdot k_{ni})(x)}_{\in W_{in}^{\alpha}}\cdot k_{ij}(x)\\ &\in k_{ji}(x)\cdot(W_{ij}^{\alpha})^{-1}\cdot W_{in}^{\alpha}\cdot k_{ij}(x)\subseteq W_{jn}^{\alpha},\end{aligned}$$

  because $\overline{U}_{ijn\alpha}^{[j-1,n]}$ is contained in both $\overline{U}_{ij\alpha}^{[ij]}$ and $\overline{U}_{in\alpha}^{[in]}$. Since the values of φ_{jn} are contained in particular in the identity neighborhood W, we may apply Lemma II.4 to $M := U_{jn}^{[0]}$ and its subsets $A := \bigcup_{i<j}\overline{U}_{ijn}^{[j-1,n]}$ and $B := \bigcup_{i<j}\overline{U}_{ijn}^{[j-2/3,n]}$. This yields a continuous function $\Phi_{jn} : U_{jn}^{[0]}\to W$ that coincides with φ_{jn} on a neighborhood of B, is the identity on a neighborhood of $U_{jn}^{[0]}\setminus A^0$ and satisfies $\Phi_{jn}(x)\in W_{jn}^{\alpha}$

for all $x \in \overline{U}_{jn\alpha}^{[j-1,n]}$. We define $k'_{jn} : U_{jn}^{[0]} \to K$ by $k'_{jn} := \Phi_{jn} k_{jn}$ and note that k'_{jn} coincides with the smooth function $\widetilde{k}'_{jn}$ on a neighborhood of B and with k_{jn} on a neighborhood of $U_{jn}^{[0]} \setminus A^0$. In addition, Φ_{jn} is the identity on $A \cap T$, because $\widetilde{k}'_{jn}$ and k_{jn} coincide there. Furthermore, by the last conclusion from Lemma II.4, Φ_{jn} is smooth on $U_{jn}^{[0]} \cap S$, because $\widetilde{k}'_{jn}$ and k_{nj} are smooth on $A^0 \cap S$ and Φ_{jn} is the identity on an open neighborhood of $U_{jn}^{[0]} \setminus A^0$. Consequently, k'_{jn} coincides with k_{jn} on $U_{jn}^{[0]} \cap T$ and is smooth on $U_{jn}^{[0]} \cap S$, which is an open neighborhood of $U_{jn}^{[0]} \cap \overline{T}$.

- We finally get the smooth map $\widetilde{k}_{jn} : U_{jn}^{[0]} \to K$ that we are looking for if we apply Proposition II.2 to the function k'_{jn} on $M := A := U_{jn}^{[0]}$, to the open complement U of $\bigcup_{i<j} \overline{U}_{ijn}^{[j-1/3,n]} \cup (U_{jn}^{[0]} \cap \overline{T})$ in M, and to the neighborhood

$$O_{jn} := \left(\bigcap_{\alpha \in \mathbb{N}} \left\lfloor \overline{U}_{jn\alpha}^{[jn]}, W_{jn}^{\alpha} \right\rfloor \right) \cdot k_{jn}$$

  of both k_{jn} and k'_{jn}, where $k'_{jn} \in O_{jn}$ follows from firstly $\Phi_{jn}(x) \in W_{jn}^{\alpha}$ and secondly $k'_{jn}(x) = \Phi_{jn}(x) \cdot k_{jn}(x) \in W_{jn}^{\alpha} \cdot k_{jn}(x)$ for all $x \in \overline{U}_{jn\alpha}^{[jn]}$. Note that O_{jn} is really open, because Remark II.7 asserts that just finitely many of the sets $\overline{U}_{jn\alpha}^{[jn]}$ for fixed $\alpha \in \mathbb{N}$ are non-empty and may influence the intersection.
  By the choice of U, the result $\widetilde{k}_{jn}$ coincides with both k'_{jn} and $\widetilde{k}'_{jn}$ on $\bigcup_{i<j} \overline{U}_{ijn}^{[jn]}$, so it satisfies the cocycle conditions (a). It also satisfies (b) by the choice of O_{jn} and, furthermore, (c), because $\widetilde{k}_{jn}$ coincides with k'_{jn} on $U_{jn}^{[0]} \cap \overline{T}$ by the choice of U and k'_{jn} coincides with k_{jn} on $U_{jn}^{[0]} \cap T$.

This concludes the construction of the smooth principal K-bundle $\mathcal{P}_{\widetilde{k}}$. Since

$$\widetilde{k}_{jn}\Big|_{U_{jn}^{[\infty]} \cap T} = k_{jn}\Big|_{U_{jn}^{[\infty]} \cap T},$$

we may view $\mathcal{P}_{\widetilde{k}}|_{\eta^{-1}(T)}$ as the subset $\mathcal{P}_k|_{\eta^{-1}(T)}$ of $\mathcal{P}_k$. We use the same covers of M and identity neighborhoods in K for the construction of continuous functions $f_i : \overline{U}_i^{[0]} \to K$ such that

(d) $f_n = \widetilde{k}_{nj} \cdot f_j \cdot k_{jn}$ pointwise on $\overline{U}_{jn}^{[nn]}$ for all $j < n$ in $\mathbb{N}$,
(e) $f_n\big(\overline{U}_{n\alpha}^{[0]}\big) \subseteq W_n^{\alpha}$ for all $\alpha, n \in \mathbb{N}$ and
(f) $f_n\big(\overline{U}_n^{[0]} \cap T\big) = \{e\}$ for all $n \in \mathbb{N}$.

Then Remark I.10 tells us that the restriction of the maps f_i to the sets $U_i^{[\infty]}$ of the open cover is the local description of a bundle equivalence $\Omega : P_k \to P_{\widetilde{k}}$. Indeed, all the sets $\overline{U}_{jn}^{[nn]}$ of condition (d) contain the corresponding sets $U_{jn}^{[\infty]}$ of the open cover. Furthermore, condition (f) implies that the restriction of Ω to $\eta^{-1}(T)$ is the identity.

We start with the constant function $f_1 \equiv e$, which clearly satisfies conditions (e) and (f). Then we construct f_n for $n > 1$ inductively as follows:

- To satisfy condition (d), we start with

$$f'_n : \bigcup_{j<n} \overline{U}_{jn}^{[jn]} \to K, \quad f'_n(x) = \widetilde{k}_{nj}(x) \cdot f_j(x) \cdot k_{jn}(x) \text{ for } x \in \overline{U}_{jn}^{[jn]}.$$

  This continuous function is well-defined, since the conditions (d) for f_j on $\overline{U}_{j'jn}^{[jn]} \subseteq \overline{U}_{j'j}^{[jj]}$ and (a) for $j' < j < n$ on $\overline{U}_{j'jn}^{[jn]}$ guarantee that

$$\begin{aligned}\widetilde{k}_{nj}(x) \cdot f_j(x) \cdot k_{jn}(x) &= \widetilde{k}_{nj}(x) \cdot \widetilde{k}_{jj'}(x) \cdot f_{j'}(x) \cdot k_{j'j}(x) \cdot k_{jn}(x) \\ &= \widetilde{k}_{nj'}(x) \cdot f_{j'}(x) \cdot k_{j'n}(x)\end{aligned}$$

  holds for all $x \in \overline{U}_{j'jn}^{[jn]}$. In addition, f'_n is the identity on $\bigcup_{j<n} \overline{U}_{jn}^{[jn]} \cap T$ by conditions (c) and (f).
- To apply Lemma II.4, we need to know something about the values of f'_n. For arbitrary $\alpha \in \mathbb{N}$ and $x \in \overline{U}_{jn\alpha}^{[jn]}$, conditions (b), (e), and (6) of Lemma II.9 imply

$$\begin{aligned}f'_n(x) &= \widetilde{k}_{nj}(x) \cdot f_j(x) \cdot k_{jn}(x) = k_{nj}(x) \cdot \big(\widetilde{k}_{jn}(x) \cdot k_{nj}(x)\big)^{-1} \cdot f_j(x) \cdot k_{jn}(x) \\ &\in k_{nj}(x) \cdot \big(W_{jn}^{\alpha}\big)^{-1} \cdot W_j^{\alpha} \cdot k_{jn}(x) \subseteq W_n^{\alpha},\end{aligned}$$

  so that the values of f'_n are, altogether, contained in the identity neighborhood W of K. If we apply Lemma II.4 to $M := \overline{U}_n^{[0]}$, to f'_n on $A := \bigcup_{j<n} \overline{U}_{jn}^{[jn]}$ and to the smaller set $B := \bigcup_{j<n} \overline{U}_{jn}^{[nn]}$, then we get a continuous function $f_n : \overline{U}_n^{[0]} \to W$, which satisfies the conditions (d), (e) and (f).

This concludes the construction of the bundle equivalence. □

Theorem II.12 (Smoothing continuous bundle equivalences). *Let $\mathcal{P}$ and $\mathcal{P}'$ be smooth principal K-bundles over the finite-dimensional connected paracompact manifold with corners M and let $\Omega : P \to P'$ be a continuous bundle equivalence. If $C \subseteq M$ is closed and Ω is smooth on an open neighborhood of $\eta^{-1}(C)$, then there exists an open neighborhood T of C and a smooth bundle equivalence $\widetilde{\Omega} : P \to P'$ with*

$$\Omega\big|_{\eta^{-1}(T)} = \widetilde{\Omega}\big|_{\eta^{-1}(T)}.$$

Proof. Let $(U_j)_{j\in J}$ be an open cover of M that is locally trivial for both bundles $\mathcal{P}$ and $\mathcal{P}'$. The Proposition II.6 yields locally finite open covers $\big(U_i^{[\lambda]}\big)_{i\in\mathbb{N}}$ of M for every $\lambda \in \{0,\infty\} \cup \big(1 + \frac{1}{3}\mathbb{N}\big)$ such that the closures $\overline{U}_i^{[\lambda]}$ are compact manifolds with corners and

$$\begin{aligned}\overline{U}_i^{[\infty]} &\subseteq U_i^{[j+1]} \subseteq \overline{U}_i^{[j+1]} \subseteq U_i^{[j+2/3]} \subseteq \overline{U}_i^{[j+2/3]} \subseteq U_i^{[j+1/3]} \\ &\subseteq \overline{U}_i^{[j+1/3]} \subseteq U_i^{[j]} \subseteq U_i^{[0]} \subseteq \overline{U}_i^{[0]} \subseteq U_i\end{aligned}$$

holds for all $i, j \in \mathbb{N}$, where U_i denotes a suitable set of the cover $(U_j)_{j\in J}$ for every $i \in \mathbb{N}$. According to Remarks I.2 and I.9, we may then describe the smooth bundles

$\mathcal{P}$ and $\mathcal{P}'$ by smooth transition functions $k = (k_{ij})_{i,j\in\mathbb{N}}$ and $k' = (k'_{ij})_{i,j\in\mathbb{N}}$ on the open cover $(U_i)_{i\in\mathbb{N}}$, equivalently, by their restrictions to any open cover $\big(U_i^{[\lambda]}\big)_{i\in\mathbb{N}}$ from above. In these local descriptions of the bundles, the bundle equivalence Ω can, as in Remark I.10, be seen as a family $f_i : U_i \to K$ of continuous maps for $i \in \mathbb{N}$ that satisfy

$$f_j(x) = k'_{ji}(x) \cdot f_i(x) \cdot k_{ij}(x) \text{ for all } i, j \in \mathbb{N} \text{ and } x \in U_{ij}. \tag{9}$$

Let $S \subseteq M$ be an open neighborhood of C such that $\Omega|_S$ is smooth. This means that the restriction of each f_i to $U_i \cap S$ is smooth. In addition, there exists an open neighborhood T of C with $\overline{T} \subseteq S$ since M is normal (see Remark I.7).

We shall inductively construct smooth maps $\widetilde{f}_i : \overline{U}_i^{[0]} \to K$ such that

(a) $\widetilde{f}_j = k'_{ji} \cdot \widetilde{f}_i \cdot k_{ij}$ pointwise on $\overline{U}_{ij}^{[j]}$ for all $i < j$ in $\mathbb{N}$,
(b) $\big(\widetilde{f}_i \cdot f_i^{-1}\big)\big(\overline{U}_{i\alpha}^{[i]}\big) \subseteq W_i^\alpha$ for all $i, \alpha \in \mathbb{N}$ and
(c) $\widetilde{f}_i\big|_{\overline{U}_i^{[0]}\cap T} = f_i\big|_{\overline{U}_i^{[0]}\cap T}$ for all $i \in \mathbb{N}$

are satisfied at each step, where the W_i^α are φ-convex identity neighborhoods provided by Lemma II.9 that we apply to the countable compact cover $\big(\overline{U}_i^{[0]}\big)_{i\in\mathbb{N}}$, to the transition functions k'_{ij}, to a convex centered chart $\varphi : W \to \varphi(W)$ of K and to the sequence of unit neighborhoods which is constantly W (we do not need the W_{ij}^α in this proof). These maps $\widetilde{f}_i$ describe a smooth bundle equivalence between $\mathcal{P}$ and $\mathcal{P}'$ when restricted to the open cover $\big(U_i^{[\infty]}\big)_{i\in\mathbb{N}}$, because (a) asserts that $\widetilde{f}_j = k'_{ji} \cdot \widetilde{f}'_i \cdot k_{ij}$ is satisfied on $U_{ij}^{[\infty]}$ for all $i < j$, in particular.

To construct the smooth function $\widetilde{f}_1 : \overline{U}_1^{[0]} \to K$, we apply Proposition II.2 to the continuous map $f := f_1$ on $M := \overline{U}_1^{[0]}$, the closed set $A := \overline{U}_1^{[0]} \cap \overline{T}$, the open set $U := \overline{U}_1^{[0]} \backslash \overline{T}$ and to the open neighborhood

$$O_1 := \bigcap_{\alpha\in\mathbb{N}} \Big\lfloor \overline{U}_{1\alpha}^{[0]}, W_1^\alpha \Big\rfloor \cdot f_1$$

of f_1, which is indeed open, since only finitely many $\overline{U}_{1\alpha}^{[0]}$ are non-empty by Remark II.7. By construction, $\widetilde{f}_1$ satisfies (b) and (c). To construct the smooth function $\widetilde{f}_j : \overline{U}_j^{[0]} \to K$ inductively for $j > 1$, we need the usual three steps:

- In order to satisfy (b) in the end, we define a continuous map

$$\widetilde{f}'_j : \bigcup_{i<j} \overline{U}_{ij}^{[j-1]} \to K, \quad \widetilde{f}'_j(x) := k'_{ji}(x) \cdot \widetilde{f}_i(x) \cdot k_{ij}(x) \text{ for } x \in \overline{U}_{ij}^{[j-1]}.$$

  If x is an element of both $\overline{U}_{ij}^{[j-1]}$ and $\overline{U}_{i'j}^{[j-1]}$ for $i' < i < j$, condition (a) for $j - 1$ and the cocycle conditions of both k and k' assert that the so-defined values for $\widetilde{f}'_j(x)$ agree. Furthermore, the compatibility condition together with condition (c) ensure that $\widetilde{f}'_j$ coincides with f_j on $\bigcup_{i<j} \overline{U}_{ij}^{[j-1]} \cap T$.

- This definition of $\widetilde{f}'_j$, along with (9) and property (6) in Lemma II.9 asserts that

$$\begin{aligned}\varphi_j(x) := \widetilde{f}'_j(x) \cdot f_j(x)^{-1} &= k'_{ji}(x) \cdot \widetilde{f}_i(x) \cdot k_{ij}(x) \cdot f_j(x)^{-1} \\ &= k'_{ji}(x) \cdot \underbrace{\widetilde{f}_i(x) \cdot f_i(x)^{-1}}_{\in W_i^\alpha} \cdot k'_{ij}(x) \in W_j^\alpha\end{aligned}$$

  holds for all $x \in \overline{U}_{ij\alpha}^{[j-1]}$, $i < j$ and α in $\mathbb{N}$. So we may apply Lemma II.4 to $A := \bigcup_{i<j} \overline{U}_{ij}^{[j-1]}$ and $B := \bigcup_{i<j} \overline{U}_{ij}^{[j-2/3]}$ to fade out φ_j to a continuous map Φ_j on $M := \overline{U}_j^{[0]}$. Then Φ_j coincides with φ_i on B and maps $\overline{U}_{j\alpha}^{[j]}$ into W_j^α. Since $\widetilde{f}'_j$ coincides with f_j on $A \cap T$, φ_j and, consequently, Φ_j is the identity on $(A \cap T) \cup (\overline{U}_j^{[0]} \backslash A)$. Furthermore, φ_j is smooth on $A^0 \cap S$, because so are all its constituents, and on a neighborhood of $\overline{U}_j^{[0]} \setminus A^0$, because it is the identity there. By the last conclusion of Lemma II.4, Φ_j is smooth on $\overline{U}_j^{[0]} \cap S$.
- Accordingly, $\Phi_j \cdot f_j$ is an element of the open (due to Remark II.7) neighborhood

$$O_j := \bigcap_{\alpha \in \mathbb{N}} \left\lfloor \overline{U}_{j\alpha}^{[j]}, W_j^\alpha \right\rfloor \cdot f_j$$

  of f_j and is smooth on $\bigcup_{i<j} U_{ij}^{[j-2/3]}$ and on $\overline{U}_j^{[0]} \cap S$. If we apply Proposition II.2 to $M := A := \overline{U}_j^{[0]}$, $U := M \setminus \left(\overline{T} \cup \bigcup_{i<j} \overline{U}_{ij}^{[j-1/3]}\right)$, O_j, and to $f := \Phi_j \cdot f_j$, then we obtain a smooth map $\widetilde{f}_j : \overline{U}_j^{[0]} \to K$.

The map $\widetilde{f}_j$ satisfies (a), because so does $\widetilde{f}'_j$, with which it coincides on $\bigcup_{i<j} \overline{U}_{ij}^{[j]}$. Moreover, (b) is satisfied due to the choice of O_j. So is (c), because Φ_j is the identity on $\overline{U}_j^{[0]} \cap T$ and $\Phi_j \cdot f_j$ remains unchanged on $M \setminus U \supseteq \overline{U}_j^{[0]} \cap T$ in the last step. This concludes the construction. □

Corollary II.13 (Equivalences of smooth and continuous bundles). *Let K be a Lie group modeled on a locally convex space and M be a finite-dimensional connected manifold with corners. Then each continuous principal K-bundle over M is continuously equivalent to a smooth principal bundle. Moreover, two smooth principal K-bundles over M are smoothly equivalent if and only if they are continuously equivalent.*

Proof. The first statement is Theorem II.11, applied to $C = \emptyset$. In the same way, the second assertion follows from Theorem II.12. □

III Related topics

In this section, we explain the relations of the results of the preceding section to the problem of extending bundles from submanifolds, to non-abelian Čech cohomology and to twisted K-theory. One encounters the first situation, e.g., in the construction of topological field theories. While non-abelian Čech cohomology is only an equivalent sheaf-theoretic framework for the problem, we show in the end how applications arise in twisted K-theory.

Remark III.1 (Abelian Čech cohomology). Let M be a paracompact topological space with an open cover $\mathcal{U} = (U_i)_{i\in I}$ and A be an abelian topological group. Then for $n \geq 0$, an *n-cochain* f is a collection of continuous functions $f_{i_1\ldots i_{n+1}} : U_{i_1\ldots i_{n+1}} \to A$, and we denote the set of n-cochains by $C^n(\mathcal{U}, A)$ and set it to $\{0\}$ if $n < 0$. We then define the boundary operator

$$\delta_n : C^n(\mathcal{U}, A) \to C^{n+1}(\mathcal{U}, A),\ \ \delta(f)_{i_0 i_1\ldots i_{n+1}} = \sum_{k=0}^{n}(-1)^k f_{i_0\ldots\widehat{i_k}\ldots i_{n+1}},$$

where $\widehat{i_k}$ means that we omit the index i_k. Then $\delta_{n+1}\circ\delta_n = 0$, and we define

$$\check{H}^n_c(\mathcal{U}, A) := \ker(\delta_n)/\operatorname{im}(\delta_{n-1}) \quad \text{and} \quad \check{H}^n_c(M, A) := \varinjlim \check{H}^n_c(\mathcal{U}, A). \tag{10}$$

The group $\check{H}^1(M, A)$ is the *n-th continuous Čech cohomology*. If, in addition, M is a smooth manifold with or without corners and A is a Lie group, then the same construction with smooth instead of continuous functions leads to the corresponding *n-th smooth Čech cohomology*.

Theorem III.2 (Isomorphism for abelian Čech cohomology). *Let M be a finite-dimensional connected manifold and A be an abelian locally convex Lie group, then for each $n \in \mathbb{N}$, the canonical map $\iota : \check{H}^n_s(M, A) \to \check{H}^n_c(M, A)$ defines an isomorphism of abelian groups.*

Proof. It clearly suffices to show that ι is bijective, so take some $[(f_i)_{i\in\mathbb{N}^n}]$, defining an element of $H^n_c(M, A)$. Then $[(f_i)_{i\in\mathbb{N}^n}]$ is in some $H^n_c(\mathcal{U}, A)$ and as before, we may assume that $\mathcal{U}$ is a countable cover of M. Choosing a bijection $\lambda : \mathbb{N}^n \to \mathbb{N}^+$ induces a total order on $\mathbb{N}^n$. From Proposition II.6, we obtain corresponding open covers $\mathcal{U}^{[i]}$ for each $i \in \mathbb{N}^n \cup \{0, \infty\}$ such that $\overline{U}^{[\infty]}_i \subseteq U^{[k]}_i \subseteq \overline{U}^{[k]}_i \subseteq U^{[k']}_i \subseteq \overline{U}^{[k']}_i \subseteq U^{[0]}_i$ if $k' < k$. In addition, we choose an arbitrary φ-convex neighborhood W in A. We set the stage for the induction by defining $\widetilde{f}_i$ for the least $n-1$ elements $i_1, \ldots, i_{n-1}$ of $\mathbb{N}^n$ to be smooth functions $\widetilde{f}_i \in C^\infty(U_i, A)$ with $(\widetilde{f}_i \cdot f_i^{-1})(U_i) \subseteq W$ (cf. Proposition II.2).

Defining $\widetilde{f}_k$ inductively on $\bigcup_{i<k}(U_i \cap U_k)$ by the cocycle condition, fading it out appropriately to U_k and smoothing it out, we obtain $\widetilde{f}_k \in C^\infty(U_k, A)$, satisfying $(\widetilde{f}_k \cdot f_k^{-1})(U_k) \subseteq W$ and satisfying the cocycle condition on $(U^{[k]}_i)_{i\in\mathbb{N}}$. In the end, this yields a cocycle $[\widetilde{f}_i]$ in $\check{H}^n_s(\mathcal{U}^{[\infty]}, A)$, which is equivalent to $[(f_i|_{U^{[\infty]}_i})_{i\in\mathbb{N}^n}]$. This yields the surjectivity and the injectivity may be achieved analogously. □

Remark III.3 (On the previous proof). Note we have no occurrence of Lemma II.9 in the abelian case, as the adjoint action, occurring implicitly in the cocycle condition, is trivial in this case. Thus we can achieve that $\widetilde{f}_k \cdot f_k^{-1}$ always has values in one fixed identity neighborhood, simplifying the proof significantly.

Remark III.4 (Non-abelian Čech cohomology; cf. [4, Section 12] and [8, 3.2.3]). If $n = 0, 1$, then we can perform a similar construction as in the previous remark in the case of a not necessarily abelian group K. The definition of an *n-cochain* is the same as in the commutative case, but we run into problems when writing down the boundary operator δ. However, we may define $\delta_0(f)_{ij} = f_i \cdot f_j^{-1}$, $\delta_1(k)_{ijl} = k_{ij} \cdot k_{jl} \cdot k_{li}$ and call the elements of $\delta_1^{-1}(\{e\})$ 2-*cocycles* (or *cocycles*, for short).

The way to circumvent difficulties for $n = 1$ is the observation that even in the non-abelian case, $C_c^1(\mathcal{U}, K)$ acts on cocycles by $(f_i, k_{ij}) \mapsto f_i \cdot k_{ij} \cdot f_j^{-1}$. Thus we define two cocycles k_{ij} and k'_{ij} to be equivalent if $k'_{ij} = f_i \cdot k_{ij} \cdot f_j^{-1}$ on U_{ij} for some $f_i \in C^1(\mathcal{U}, K)$, and by $\check{H}_c^1(\mathcal{U}, K)$ the equivalence classes (or the orbit space) of this action. Then $\check{H}_c^1(\mathcal{U}, K)$ is not a group, but we may nevertheless take the direct limit

$$\check{H}_c^1(M, K) := \lim_{\rightarrow} \ \check{H}_c^1(\mathcal{U}, K)$$

of sets and define it to be the 1*st* (*non-abelian*) *continuous Čech cohomology* of M with coefficients in K. A representing space of $\check{H}_c^1(M, K)$ would then be the set of equivalence classes of continuous principal K-bundles over M.

Again, if M is a smooth manifold with corners and K is a Lie group, we can adopt this construction to define the 1*st* (*non-abelian*) *smooth Čech cohomology* $\check{H}_s^1(M, K)$.

Theorem III.5 (Isomorphism for non-abelian Čech cohomology). *If M is a finite-dimensional connected manifold with corners and K is a Lie group modeled on a locally convex space, then the canonical map $\iota : \check{H}_s^1(M, K) \to \check{H}_c^1(M, K)$ is a bijection.*

Proof. We identify smooth and continuous principal bundles with Čech 1-cocycles and smooth and continuous bundle equivalences with Čech 0-cochains as in Remark I.10. For each open cover $\mathcal{U}$ of M, we have the canonical map $\check{H}_s^1(\mathcal{U}, K) \to \check{H}_c^1(\mathcal{U}, K)$. Now each cocycle $k_{ij} : U_{ij} \to K$ defines a principal bundle $\mathcal{P}$ with locally trivial cover $\mathcal{U}$. We may assume by Corollary II.13 that $\mathcal{P}$ is continuously equivalent to a smooth principal bundle $\widetilde{\mathcal{P}}$, and thus that $\mathcal{U}$ is also a locally trivial covering for $\widetilde{\mathcal{P}}$. This shows that the map is surjective, and the injectivity follows from Corollary II.13 in the same way. Accordingly, the map induced on the direct limit is a bijection. □

Remark III.6 (The projective unitary group). Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space and denote by $\mathrm{U}(\mathcal{H})$ the group of unitary operators. If we equip $\mathrm{U}(\mathcal{H})$ with the norm topology, then the exponential series, restricted to skew-self-adjoint operators $L(\mathrm{U}(\mathcal{H}))$, induces a Banach–Lie group structure on $\mathrm{U}(\mathcal{H})$ (cf. [22, Example 1.1]). Then $\mathrm{U}(1) \cong Z(U(\mathcal{H}))$, and it can also be shown that $\mathrm{PU}(\mathcal{H}) := \mathrm{U}(\mathcal{H})/\,\mathrm{U}(1)$ is a Lie group modeled on $L(\mathrm{U}(\mathcal{H}))/i\mathbb{R}$.

Remark III.7 (Eilenberg–MacLane spaces). If X is a topological space with trivial n-th homotopy group $\pi_n(X)$ for all but one $n \in \mathbb{N}$, then it is called an *Eilenberg–MacLane space* $K(n, \pi_n(X))$. Since $\mathrm{U}(1)$ is a $K(1, \mathbb{Z})$, the long exact homotopy sequence [2, Theorem VII.6.7] shows that $\mathrm{PU}(\mathcal{H})$ is a $K(2, \mathbb{Z})$, since $\mathrm{U}(\mathcal{H})$ is contractible [18, Theorem 3]. By the same argument, the classifying space $B\,\mathrm{PU}(\mathcal{H})$ is a $K(3, \mathbb{Z})$, since its total space $E\,\mathrm{PU}(\mathcal{H})$ is contractible. Thus

$$\check{H}^3(M, \mathbb{Z}) \cong [M, B\,\mathrm{PU}(\mathcal{H})] \cong \check{H}^1_c(M, \mathrm{PU}(\mathcal{H}))$$

by [2, Corollary VII.13.16]. The representing class $[\mathcal{P}]$ in $\check{H}^3(M, \mathbb{Z})$ is called the *Dixmier–Douady class* of $\mathcal{P}$ (cf. [3], [5]). It describes the obstruction of $\mathcal{P}$ to be the projectivization of an (automatically trivial) principal $\mathrm{U}(\mathcal{H})$-bundle.

Corollary III.8 (Smoothing $\mathrm{PU}(\mathcal{H})$-bundles). *If M is a finite-dimensional connected manifold with corners, then*

$$\check{H}^3(M, \mathbb{Z}) \cong \check{H}^1_c(M, \mathrm{PU}(\mathcal{H})) \cong \check{H}^1_s(M, \mathrm{PU}(\mathcal{H})).$$

Remark III.9 (Twisted K-theory; cf. [25, Section 2], [1]). The Dixmier–Douady class of a principal $\mathrm{PU}(\mathcal{H})$-bundle over M induces a twisting of the K-theory of M in the following manner. For any paracompact space, the K-theory $K^0(M)$ is defined to be the Grothendieck group of the monoid of equivalence classes of finite-dimensional complex vector bundles over X, where addition and multiplication is defined by taking direct sums and tensor products of vector bundles [15]. Furthermore, the space of Fredholm operators $\mathrm{Fred}(\mathcal{H})$ is a representing space for K-theory, i.e. $K^0(M) \cong [M, \mathrm{Fred}(\mathcal{H})]$, where $[\cdot, \cdot]$ denotes homotopy classes of continuous maps. Since $\mathrm{PU}(\mathcal{H})$ acts (continuously) on $\mathrm{Fred}(\mathcal{H})$ by conjugation, we can form the associated vector bundle $\mathcal{P}_{\mathrm{Fred}(\mathcal{H})} := \mathrm{Fred}(\mathcal{H}) \times_{\mathrm{PU}(\mathcal{H})} \mathcal{P}$. Then the homotopy classes of sections $[M, P_{\mathrm{Fred}(\mathcal{H})}]$ (or equivalently, the equivariant homotopy classes of equivariant maps $[P_{\mathrm{Fred}(\mathcal{H})}, \mathrm{Fred}(\mathcal{H})]^{\mathrm{PU}(\mathcal{H})}$) define the *twisted K-theory* $K_{\mathcal{P}}(M)$. Now Corollary II.13 implies that we may assume $\mathcal{P}$ to be smooth. Since the action of $\mathrm{PU}(\mathcal{H})$ on $\mathrm{Fred}(\mathcal{H})$ is locally given by conjugation, it is smooth (recall that $\mathrm{Fred}(\mathcal{H})$ is an open subset of $B(\mathcal{H})$, giving it a natural manifold structure), whence is $\mathcal{P}_{\mathrm{Fred}(\mathcal{H})}$. Due to Lemma I.11, we may, in the computation of $K_{\mathcal{P}}(M)$, restrict our attention to smooth sections and smooth homotopies.

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