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IN STRING – GAUGE THEORY DUALITIES

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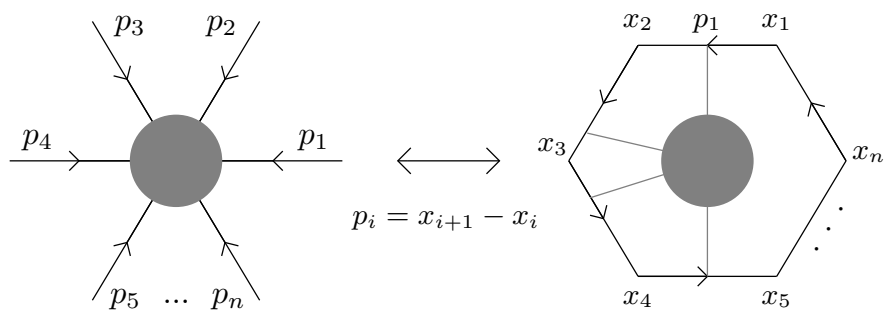
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*"Every theoretical physicist who is any good knows six or seven different theoretical representations for exactly the same physics."*¹



¹R. P. Feynman, 1965, p.168 *"The character of physical law"*, MIT Press, Cambridge, MA.

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Executive Summary

The AdS/CFT correspondence [1, 2] states a duality between string theories living on Anti-de Sitter space and supersymmetric conformal quantum field theories in Minkowski space. Beyond the relations between gauge and string theories, the correspondence has led to many new insights into the structure of supersymmetric quantum field theory itself.

In [3, 4] a strong coupling calculation of n -gluon scattering amplitudes in terms of minimal surfaces in AdS was proposed, that formally resembles the calculation of a Wilson loop over a light-like polygonal contour made out of the gluon momenta p_i . Interestingly, the duality between gluon scattering amplitudes and light-like polygonal Wilson loops was found to be true also at weak coupling in $\mathcal{N} = 4$ super Yang-Mills theory [5, 6]. In [7, 8] it was shown that the Wilson loop is governed by an anomalous conformal Ward identity that fixes the form of the expectation value for $n = 4, 5$ completely and allows for a function of conformally invariant cross ratios starting from $n = 6$. The duality persists to higher loops and legs and also makes some properties of gluon scattering amplitudes more transparent, such as the dual (super-)conformal symmetry of the amplitudes found in [9, 10]. The dual superconformal symmetry combines with the superconformal symmetry to a Yangian symmetry [11], reflecting the integrability of $\mathcal{N} = 4$ super Yang-Mills theory in the planar amplitude sector. Another interesting development is the duality [12] between amplitudes and correlation functions in a limit where adjacent operators become light-like separated.

Many of these developments have an analogue in ABJM theory [2] and it is the subject of the first part of this thesis to investigate the Wilson loop side of a possible similar duality in this theory. The main result is, that the expectation value of light-like polygonal Wilson loops vanishes at one-loop order and has a non-vanishing form at two-loop order, which is identical in its functional form to the Wilson loop in $\mathcal{N} = 4$ super Yang-Mills. We calculate the expectation value at two loops for the four-sided Wilson loop and show, that its form is governed by an anomalous conformal Ward identity. We generalise the results to n points at two-loop order and find that the expectation value of the Wilson loop is given by

$$\langle W_n \rangle_{2\text{-loop}}^{\text{ABJM}} = \left(\frac{N}{k} \right)^2 \left[-\frac{1}{2} \sum_{i=1}^n \frac{(-\hat{\mu}^2 x_{i,i+2}^2)^{2\epsilon}}{(2\epsilon)^2} + \mathcal{F}_n^{\text{WL}} + r_n + \mathcal{O}(\epsilon) \right], \quad (1)$$

where r_n is a constant with linear dependence on n and $\mathcal{F}_n^{\text{WL}}$ is the same finite part as that of the Wilson loop in $\mathcal{N} = 4$ super Yang-Mills theory at one-loop order.

These developments were presented in [13, 14, 15]. Most interestingly, it was found in [16, 17] that the four-point two-loop amplitude in ABJM theory with the tree-level part stripped off precisely agrees with this form, giving the first indication for a duality between amplitudes and Wilson loops in ABJM theory. Dual conformal and Yangian symmetry in the amplitude sector of ABJM theory [18, 19, 20] further hint at the existence of such a relation. Furthermore, non-trivial evidence for a duality between Wilson loops and correlators in ABJM theory was found at one-loop level in [21]. Recently, it was shown that, unlike the Wilson loop, the six-point amplitudes in ABJM theory do not vanish at one loop [22, 23] and we will discuss why this does not necessarily imply, that a relation between Wilson loops and amplitudes does not exist in ABJM theory.

In the second part of this thesis we calculate three-point functions of two protected operators and a twist-two operator $\hat{\mathcal{O}}_j$ with arbitrary even spin j in $\mathcal{N} = 4$ super Yang-Mills theory. In order to carry out the calculations we project the indices of the spin j operator to the light-cone and evaluate the correlator in a soft-limit where the momentum coming in at the spin j operator becomes zero. This limit largely simplifies the perturbative calculation, since all three-point diagrams effectively reduce to two-point diagrams and the dependence on the one-loop mixing matrix drops out completely. We find that the correction to the structure constant has the simple form

$$C'_{\mathcal{O}\hat{\mathcal{O}}_j}(g^2) = C'^{(0)}_{\mathcal{O}\hat{\mathcal{O}}_j} \left(1 + \frac{g^2 N}{8\pi^2} (2H_j(H_j - H_{2j}) - H_{j,2}) + \mathcal{O}(g^4) \right), \quad (2)$$

where $H_{j,m}$ are generalised harmonic sums. The result is in agreement with the extraction of the structure constant from the analysis of the operator product expansion of four-point functions of half-BPS operators [24].

Overview

The focus of this thesis lies on the gauge theory side of the AdS/CFT duality and we will in general refer to the literature instead of discussing the developments on the string theory side in order to keep the presentation more compact.

- In **chapter 1** we shortly comment on the role of supersymmetry in elementary particle physics, the question of quantum gravity and discuss some general features of the AdS/CFT correspondence in order to embed the studies of this thesis into the the context of present research.
- The following two chapters are of more general nature and may also be of interest independently of the research projects that are performed in this thesis. In **chapter 2** some well-known basic facts about conformal symmetry and conformal field theory are reviewed. We explicitly derive the transformation properties of fundamental fields under conformal transformations. Furthermore we introduce $\mathcal{N} = 4$ super Yang-Mills and ABJM theory. **Chapter 3** is devoted to the study

of the implications of conformal symmetry on correlation functions from the perspective of conformal Ward identities. Furthermore, we discuss some aspects of the renormalisation of composite operators, the operator product expansion and its relation to conformal three-point functions.

- In **chapter 4** we first discuss some techniques for the formulation and evaluation of scattering amplitudes, e.g. the spinor helicity formalism, BFCW recursions and unitarity methods. We then review dual superconformal and Yangian symmetry of scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory and the BDS ansatz for n -point gluon scattering amplitudes. In the remaining part of this chapter we discuss the duality between bosonic Wilson loops and MHV scattering amplitudes and shortly comment on more recent discoveries, such as the formulation of supersymmetric Wilson loops, the duality between superamplitudes and supercorrelators as well as the relation between form factors and periodic Wilson loops.
- In **chapter 5** we review the analogous developments in ABJM theory with a focus on a possible relation between scattering amplitudes and Wilson loops. In **chapter 6** we present the calculation of the four-point light-like Wilson loop in ABJM theory, the calculation of the anomalous conformal Ward identity that constrains the Wilson loop as well as numerical studies for the hexagonal as well as higher-point Wilson loops and present the result for general n . These calculations were partially presented in [13, 14, 15] and we give some more details, especially on the evaluation of the n -point Wilson loop.
- **Chapter 7** contains unpublished results. We introduce different limiting procedures that can be employed for the calculation of three-point functions and present the general structure of the calculation that is necessary in order to read-off the structure constant of the three-point functions of two half-BPS operators and one twist-two operator. The remaining section contain some details of the calculation.
- In **chapter 8** we draw our conclusions and comment on possible future research directions.
- We tried to delegate most explicit calculations to the Appendices and to present only the structure of calculations. Furthermore, **the Appendices A, B, C** contain our conventions, some technical details on conformal symmetry, an explicit derivation of the $\mathcal{N} = 4$ super Yang-Mills theory action from $\mathcal{N} = 1$ super Yang-Mills theory in ten dimensions and the Feynman rules. **The Appendices D, E** contain an introduction to the Mellin-Barnes technique for the evaluation of integrals and some information on the path ordering and gauge invariance of Wilson loops. In **Appendix F** we summarise our conventions and Feynman rules for Chern-Simons and ABJM theory.

Zusammenfassung

Die AdS/CFT Korrespondenz [1, 2] stellt eine Dualität zwischen Stringtheorien im Anti-de Sitter Raum und supersymmetrischen konformen Quantenfeldtheorien im Minkowskiraum dar. Über die Relationen zwischen Eich- und Stringtheorien hinaus, hat die Korrespondenz zu vielen neuen Einsichten in die Struktur der Quantenfeldtheorie selbst geführt.

In [3, 4] wurde eine Berechnung von n -Gluon-Streuamplituden durch Minimalflächen in AdS vorgeschlagen, welche formal der Berechnung einer Wilsonschleife über eine lichtartige Kontur ähnelt, die aus den Impulsen p_i der Gluonen zusammengesetzt ist. Interessanterweise wurde herausgefunden, dass die Dualität zwischen Gluon-Streuamplituden und lichtartigen Wilsonschleifen in der $\mathcal{N} = 4$ Super Yang-Mills Theorie auch bei schwacher Kopplung gilt [5, 6]. In [7, 8] wurde gezeigt, dass die Wilsonschleife durch anomale konforme Ward-Identitäten bestimmt wird, die die Form des Erwartungswertes für $n = 4, 5$ vollständig festlegen und von $n = 6$ an eine beliebige Funktion von konform invarianten Kombinationen der kinematischen Variablen zulassen. Die Dualität ist auch für höhere Ordnungen der Störungstheorie und beliebige Anzahl n von Punkten gültig und macht einige Eigenschaften der Streuamplituden, wie z.B. die duale (super-)konforme Symmetrie, die in [9, 10] gefunden wurde, transparenter. Die duale superkonforme Symmetrie lässt sich mit der superkonformen Symmetrie zu einer Yangschen Symmetrie kombinieren [11], wodurch die Integrierbarkeit der $\mathcal{N} = 4$ Super Yang-Mills Theorie im planaren Amplitudensektor reflektiert wird. Eine weitere interessante Entwicklung in diesem Kontext ist die Dualität [12] zwischen Streuamplituden und Korrelationsfunktionen, die in einem Limes betrachtet werden, für den benachbarte Operatoren lichtartig entfernt sind.

Viele dieser Entwicklungen sind analog in der ABJM Theorie [2] und es ist das Thema des ersten Teils dieser Dissertation, die Wilsonschleifenseite einer solchen möglichen Dualität zu untersuchen. Das Hauptergebnis dieser Untersuchungen ist, dass der Erwartungswert der lichtartigen polygonalen Wilsonschleifen auf Einschleifenebene verschwindet und auf Zweischleifenebene dieselbe Form hat wie die Wilsonschleife in $\mathcal{N} = 4$ Super Yang-Mills Theorie auf Einschleifenebene. Es wird der Erwartungswert auf Zweischleifenebene für die vierseitige Wilsonschleife berechnet und gezeigt, dass deren Form durch eine anomale konforme Ward-Identität bestimmt wird. Die Ergebnisse lassen sich auf den n -Punkt Fall auf Zweischleifenebene verallgemeinern, mit dem Ergebnis

$$\langle W_n \rangle_{2\text{-loop}}^{\text{ABJM}} = \left(\frac{N}{k}\right)^2 \left[-\frac{1}{2} \sum_{i=1}^n \frac{(-\hat{\mu}^2 x_{i,i+2}^2)^{2\epsilon}}{(2\epsilon)^2} + \mathcal{F}_n^{\text{WL}} + r_n + \mathcal{O}(\epsilon) \right], \quad (3)$$

wobei r_n eine Konstante ist, die linear von n abhängt und $\mathcal{F}_n^{\text{WL}}$ derselbe endliche Term wie der der Wilsonschleife in der $\mathcal{N} = 4$ super Yang-Mills Theorie auf Einschleifenebene ist.

Diese Ergebnisse wurden in [13, 14, 15] präsentiert. Interessanterweise wurde in [16, 17] gefunden, dass die Zweischleifen-Streuamplitude mit vier externen Teilchen durch den Baumgraphenanteil geteilt, genau mit dieser Form übereinstimmt und damit den ersten Hinweis auf eine mögliche Dualität zwischen Amplituden und Wilsonschleifen in der ABJM Theorie gibt. Duale konforme und Yangsche Symmetrie im Amplitudensektor [18, 19, 20] weisen weiter auf eine solche Relation hin. Weiterhin wurden nicht-triviale Anhaltspunkte für eine Dualität zwischen Korrelatoren und Wilsonschleifen auf Einschleifenebene gefunden [21]. Kürzlich wurde gezeigt, dass, entgegen der Wilsonschleife, die Streuamplituden mit sechs externen Teilchen auf Einschleifenebene nicht verschwinden [22, 23]. In der vorliegenden Dissertation wird auch kurz darauf eingegangen, warum dies nicht notwendiger Weise impliziert, dass in der ABJM Theorie keine Relation zwischen Wilsonschleifen und Amplituden existiert.

Im zweiten Teil dieser Arbeit berechnen wir Dreipunktkorrelationsfunktionen von zwei geschützten Operatoren und einem Twist-Zwei Operator $\hat{\mathcal{O}}_j$ mit beliebigem geradzahligem Spin j in der $\mathcal{N} = 4$ super Yang-Mills Theorie. Um diese Rechnungen durchzuführen, projizieren wir die Indizes des Spin j Operators auf den Lichtkegel und werten den Korrelator in einem Infrarotlimes aus, in dem der Impuls, der bei dem Spin j Operator einfließt, null wird. Dieser Limes vereinfacht die perturbativen Rechnungen enorm, da alle Dreipunkt diagramme effektiv auf Zweipunkt diagramme reduziert werden und die Abhängigkeit der Einschleifenmischungsmatrix komplett herausfällt. Die Korrektur zur Strukturkonstanten hat die einfache Form

$$C'_{\mathcal{O}\hat{\mathcal{O}}_j}(g^2) = C'^{(0)}_{\mathcal{O}\hat{\mathcal{O}}_j} \left(1 + \frac{g^2 N}{8\pi^2} (2H_j(H_j - H_{2j}) - H_{j,2}) + \mathcal{O}(g^4) \right), \quad (4)$$

hat, wobei $H_{j,m}$ verallgemeinerte harmonische Summen sind. Dieses Ergebnis stimmt mit der Strukturkonstante, die sich aus der Operatorproduktentwicklung der Vierpunktfunktionen von 1/2-BPS Operatoren extrahieren lässt, überein [24].

Chapter 1

Introduction – New Directions in Quantum Field Theory

Einstein's theory of special relativity and the theory of quantum mechanics, both developed at the beginning of the last century, induced major shifts of paradigm in physics and lead to the construction of a theory that incorporates them both, quantum field theory (QFT). The present *standard model* of elementary particle physics makes use of the concepts of QFT successfully and can be regarded as the experimentally best verified theory ever. Feynman, one of the pioneers in the development of QFT, once said:

*"Every theoretical physicist who is any good knows six or seven different theoretical representations for exactly the same physics."*¹

Besides the debatable classification of good theoretical physicists, the second part of this sentence contains the important statement that there is not the *one correct theory* that describes our world, but rather that any theory that *correctly describes* the phenomena in this world is equally valid. The content of this sentence is possibly more up-to-date than ever before, as we will argue in the following.

The experiments in elementary particle physics are described in terms of scattering amplitudes. As will be described in this thesis, there are many ways to calculate these scattering amplitudes, by at first sight completely different theoretical representations. These achievements are mainly due to the so-called *AdS/CFT correspondence* [25], that will be described in section 1.4. The correspondence relates a *string theory* in a ten-dimensional space-time to a simplified theory of elementary particles physics, the so-called maximally supersymmetric $\mathcal{N} = 4$ super Yang-Mills theory in four-dimensional space-time, which is reasonably similar to the standard model of particle physics as we will argue later on. In the spirit of Feynman's words, if we could calculate all physical observables in either of the theories, there was no way to distinguish which of them is the *correct one*. Furthermore, the duality triggered many relations between

¹R. P. Feynman, 1965, p.168 "*The character of physical law*", MIT Press, Cambridge, MA.

quantities purely on the field theory side of the duality. It turns out that scattering amplitudes are intimately related to a special type of so-called *Wilson loops* as well as *correlation functions* in a certain limit as will be described in detail in chapter 4. Using these relationships, it is even more convenient to analyse the quantum corrections to scattering amplitudes in terms of Wilson loops, providing new ways to interpret the quantum field theory itself.

Possibly, these theoretical developments could only be achieved due to the lack of important new experimental input from collider physics in the last decades, which gave theoreticians time to reorganise the theory and to get a better understanding of it. In light of the beginning "*Era of the Large Hadron Collider (LHC)*", these new theoretical developments are highly pleasing, since the discovery of new physics at the LHC requires a precise understanding of background processes of known physics. These processes are described by the theory of strong interactions called quantum chromodynamics (QCD), which is successfully described in the framework of quantum field theory. Using the standard technique of Feynman diagrams, these processes are notoriously difficult to evaluate and thus new methods are required. The status of available methods and tools can for instance be found in the introduction of the review [26].

Among the new physics that await its discovery at the LHC, there is the missing particle of the standard model, the *Higgs particle*, which is ought to be responsible for the mechanism of mass generation. In this thesis we will deal a lot with symmetries and their consequences. In particular, we will deal with *conformal symmetry*, which is introduced in chapters 2, 3 and may be, loosely spoken, interpreted as an additional *scale invariance* of the theory. One may note, that the only term in the standard model² that violates this scale invariance and thus the conformal symmetry on the classical level, is the mass term of the Higgs particle. From a conceptual point of view one may prefer a theory which is fundamentally scale invariant. Indeed, there are proposals for a classically conformal invariant standard model, with a dynamical mechanism of mass generation, see chapter 2.

1.1 Supersymmetry and Particle Physics

Another symmetry that we will encounter in this thesis is *supersymmetry*. It relates particles of different spin to each other. Apart from the hope to find the superpartners of standard model particles at the LHC, supersymmetry can be considered as a calculational tool in non-supersymmetric theories such as QCD and has already proven as useful in this endeavour.

At tree-level, gluon amplitudes in $\mathcal{N} = 4$ super Yang-Mills and QCD are identical, since these amplitudes do not depend on the particle content of the theory. Furthermore, using supersymmetric Ward identities, amplitudes with different external particles can be related to each other [27]. Also at loop-level, where the particle

²assuming massless neutrinos

content in the loop is different in QCD and supersymmetric theories, supersymmetry leads to simplifications. For instance, one can decompose pure Yang-Mills amplitudes, i.e. amplitudes A_g containing only gluon loops, into contributions of supersymmetric multiplets, see chapter 4 of [28]

$$A_g = (A_g + 4A_f + 3A_s) - 4(A_f + A_s) + A_s = A^{\mathcal{N}=4} - 4A^{\mathcal{N}=1} + A_s, \quad (1.1)$$

where the indices s, f, g refer to scalars, fermions and gluons circulating in the loop. Taking the contributions of an $\mathcal{N} = 4$ multiplet and an $\mathcal{N} = 1$ (chiral) multiplet as granted, the task of calculating the gluon contribution is reduced to computing the contribution A_s of a scalar in the loop. The supersymmetric amplitudes are simpler to analyse and, as we will see in section 4.1, certain $\mathcal{N} = 4$ SYM amplitudes are even tractable to all orders using methods of integrability.

1.2 The Hydrogen Atom of Gauge Theories

$\mathcal{N} = 4$ super Yang-Mills theory (SYM) is the maximally supersymmetric gauge theory. It resembles the standard model in that it is based on the same type of particles and interactions and both are renormalisable field theories in four-dimensional Minkowski-space. On the other hand, $\mathcal{N} = 4$ SYM exhibits a few interesting features that make it simpler. The renormalisation group β -function of $\mathcal{N} = 4$ SYM vanishes to all orders in the coupling constant and conformal symmetry is preserved at the quantum level, which is tightly linked to the supersymmetry of the theory. These properties make the model more tractable and in the *planar limit* the *integrability* of the gauge theory, see 1.5.1, allows to extract certain quantities to all orders in the coupling.

$\mathcal{N} = 4$ SYM may therefore be regarded as a testing ground for the foundations of QFT and is sometimes referred to as the *Hydrogen Atom of Gauge Theories*. Just as the understanding of the hydrogen atom was the key to the understanding of the spectra of other atoms, the full solution of $\mathcal{N} = 4$ SYM is widely believed to have important consequences also for less symmetric theories such as QCD. *Solving the theory* amounts to being able to calculate all physical quantities of interest to arbitrary order in the coupling constant or non-perturbatively. More details on $\mathcal{N} = 4$ SYM theory will be given in section 2.4.1.

1.3 The Challenge of Quantum Gravity

Another longstanding challenge in theoretical physics is the search for a viable theory of *quantum gravity*. Naive quantisation of the linearised Einstein's equations turns out to fail due to perturbative non-renormalisability in the framework of standard quantum field theory [29, 30]. An interesting alternative is the asymptotic safety scenario [31], for a review see e.g. [32]. An overview about different approaches towards the quantisation of gravity is given in the book [33], among which the *covariant quantisation* of gravity and *string theory* as well as *canonical gravity* and *loop quantum gravity* are reviewed.

Another interesting direction is the research on *composite gravity*. In a famous paper by S. Weinberg and E. Witten [34] restrictions for these types of models have been obtained. A nice review on different models of emergent gravity can be found in [35]. Models of composite gravity in terms of *spinor gravity* can be found in [36, 37].

As mentioned above, a strong competitor in the realm of quantum gravity is string theory, which describes all fundamental constituents and interactions as one-dimensional objects with vibrational excitations. String theory is advocated to have the capability of unifying all matter and forces from one concept and naturally includes massless spin-two excitations that correspond to the graviton and in this sense naturally incorporates gravity. In contrast to other approaches, string theory has a well-defined low energy limit. Similarly as mentioned above for supersymmetry, even if string theory is not realised in nature, it justifies itself by the fact, that it has led to a wealth of new discoveries within QFT, especially through the AdS/CFT correspondence and has thus led to a better understanding of quantum field theory.

Even though the research communities of these different directions claim having constructed viable theories of quantum gravity, it is fair to say, that none of them can be regarded as generally accepted nor has any of them made a quantitative prediction that has been justified or falsified.

1.4 String - Gauge Theory Dualities

A fascinating new area of research in theoretical physics has been initiated by the so-called *AdS/CFT correspondence* [1, 2], that states a duality between string theories living on *Anti-de Sitter space* (AdS) and supersymmetric conformal quantum field theories (CFTs) in the planar limit [38]. The striking feature of these dualities is that they are *strong-weak coupling dualities*, i.e. in a limit where one of the theories is strongly coupled, the other one is weakly coupled and one can thus access observables in the strongly coupled theory by calculating observables on the weakly coupled side with methods of perturbation theory. Another interesting aspect of this type of duality is, that it establishes a relation between a quantum theory that contains gravity (in terms of string theory) with a pure gauge theory without gravity. Prominent examples of such dualities are

1. $\mathcal{N} = 4$ super Yang-Mills theory \leftrightarrow Type IIB string theory on $AdS_5 \times S^5$
2. $\mathcal{N} = 6$ super Chern-Simons theory \leftrightarrow Type IIA string theory on $AdS_4 \times \mathbb{CP}^3$

The gauge theory *lives* on the $(d - 1)$ -dimensional boundary of the AdS_d space, which is conformally equivalent to $\mathbb{R}^{1,d-2}$, i.e. flat Minkowski space. Since the information of the string theory in the higher-dimensional space is conjectured to be encoded by the theory on the boundary, one also uses the term *holographic duality*.

In the following, we shall only mention a few key features, but not go into the details, since this thesis primarily deals with applications of conformal field theory in

general, even though the original motivation for most of this research is due to the correspondence. More information on the AdS/CFT correspondence can be found e.g. in the reviews [39], [40], [41] and in the recent review of AdS/CFT integrability [42].

1.4.1 Symmetry and Duality Relations

Having an exact duality means that all quantities of interest can be calculated in either of the theories. Therefore, it is necessary to set up a *dictionary* between relevant observables.

Symmetry plays a central role in the correspondence and serves for the identification of observables, i.e. for setting up the desired dictionary. The $AdS_5 \times S^5$ space-time that the string theory lives in can be embedded into flat space as

$$AdS_{d+1} = \{X \in \mathbb{R}^{d,2} | X \cdot X = -R^2\}, \quad S_{d+1} = \{Y \in \mathbb{R}^{d+2} | Y \cdot Y = R^2\}, \quad (1.2)$$

where we have chosen equal *radii* R for both spaces. By construction, the isometry group of AdS_{d+1} is $SO(d, 2)$ and $SO(d + 2)$ for S_{d+1} . Thus for $AdS_5 \times S^5$ the isometry groups are $SO(4, 2) \simeq SU(2, 2)$ and $SO(6) \simeq SU(4)$. Together with 32 fermionic directions the supergroup $PSU(2, 2|4)$ is formed. The symmetry group of the $AdS_5 \times S^5$ superspace is reflected by the same symmetry group of the $\mathcal{N} = 4$ SYM theory as we will discuss in section 2.4.1. More information on the matching of the symmetry groups can be found in [43].

String states with excitations in certain directions in the superspace are identified with composite gauge invariant operators in the QFT, which carry the corresponding charges of the global symmetry group of the field theory. It turns out that the energies of the string states can be matched with the anomalous dimensions of the corresponding operators [44, 45, 46, 47, 48]. A nice review on the matching of string energies and scaling dimensions can be found in [49].

Another outcome of the duality is an exciting relation between gluon scattering amplitudes and Wilson loops that will be described in detail in chapter 4.

The coupling g and the number of colours N of the gauge theory are related to the radius R of the $AdS_5 \times S^5$ spaces, the string coupling g_s and the (inverse) string tension α' via [1]

$$\frac{R^4}{\alpha'^2} = g^2 N, \quad 4\pi g_s = g^2. \quad (1.3)$$

The so-called *'t Hooft limit* consists in keeping the *'t Hooft coupling* $\lambda = g^2 N$ fixed while taking $N \rightarrow \infty$ [38], we are thus dealing with free ($g_s \rightarrow 0$) string theory. This limit is also called the *planar limit* because in the gauge theory, Feynman diagrams that are non-planar³ are suppressed by factors of $1/N^2$ and can be neglected. The

³Diagrams which cannot be drawn on a plane without crossing of propagators.

perturbative expansion on the gauge theory side is thus performed in powers of λ . In the *weak coupling regime* $\lambda \ll 1$ we can calculate observables in the quantum field theory with perturbation theory. However, due to the relation (1.3), this amounts to quantizing string theory in a highly curved space due to the fact that the radius R is small in string units. On the other hand, in the regime, where R is large in string units and the string energies can be calculated perturbatively in $1/R$, the 't Hooft coupling λ gets large and the regime where perturbation theory is valid on the gauge theory side is left. This is why the duality is called a *weak-strong coupling* duality. Luckily, there are overlapping regimes and due to integrability, see section 1.5.1, it is possible to evaluate the gauge theory quantities at arbitrary coupling λ .

It should be noted, that the coincidence of the global symmetry groups of the theories constrains observables, it is however far from sufficient to prove a complete duality. The obvious symmetries do not uniquely fix the value of the observables on either side of the duality.

As an example, one may consider the equivalence of amplitudes and Wilson loops purely on the gauge theory side. As we will argue in chapter 4, the expectation values of the Wilson loop are constrained by the conformal symmetry. The amplitudes and Wilson loops do also agree for the so-called *remainder function*, which is an arbitrary function of conformally invariant kinematical quantities, even though the conformal symmetry does not constrain this function. The fact that the amplitudes continue to agree with the Wilson loop shows that there is more to the amplitude Wilson loop duality than just conformal symmetry.

1.5 Solving the Theory – Integrability

As mentioned before, the strategy is to solve a simple theory, cf. section 1.2, first and then proceed to the less symmetric cases. The theory can be considered as solved when one is able to calculate the expectation value of all relevant quantities such as scattering amplitudes, correlation functions, form factors, Wilson loops and so on, exactly or to all orders in perturbation theory.

As we will see in chapter 3, correlation functions of local gauge-invariant operators $\mathcal{O}_\alpha(x)$ of the type

$$\langle \mathcal{O}_{\alpha_1}(x_1) \dots \mathcal{O}_{\alpha_n}(x_n) \rangle \quad (1.4)$$

are constrained by conformal symmetry. The space-time structure of two-point functions is completely fixed by conformal symmetry and the only relevant parameter is the (*anomalous*) *scaling dimension* $\Delta_\alpha(g)$ of the operator, that can be perturbatively calculated in orders of the coupling constant g . The space-time structure of three-point functions of operators $\mathcal{O}_\alpha(x), \mathcal{O}_\beta(x), \mathcal{O}_\gamma(x)$ is also fixed and they only depend on the scaling dimensions of the operators as well as the so-called *structure constants* $C_{\alpha\beta\gamma}(g)$.

As we will see in chapter 3, the restrictions from conformal symmetry do not uniquely fix the space-time structure of higher-point functions, but one can extract ad-

ditional information from the so-called *operator product expansion* (OPE). The OPE, together with the knowledge of the anomalous dimensions, in principle allows to determine any higher-point function and thus the scaling dimensions $\Delta_\alpha(g)$ and the structure constants $C_{\alpha\beta\gamma}(g)$ are the essential parameters that govern the behaviour of observables in the theory.

While two-point functions in $\mathcal{N} = 4$ SYM are very well understood, largely due to the existence of integrability, three-point functions are explored to a lesser extent. Chapter 7 is devoted to the investigation of the structure constants of so-called *twist-two operators* of arbitrary spin.

Also on the amplitude side substantial progress has been made. As we will describe in chapter 4, certain four- and five-point scattering amplitudes in $\mathcal{N} = 4$ SYM seem to be tractable to all orders in perturbation theory [50].

Due to the several relations between amplitudes, correlation functions and Wilson loops that we will describe in detail in chapter 4, one may expect that the exact solution of one of these objects has a strong impact on the others and it is thus certainly very reasonable to continue research on all of these objects.

1.5.1 Integrability

Even though this thesis does not explicitly deal with integrability, we shall mention some of its key features, since the understanding of two-point functions is to a large extent related to the presence of integrability in the planar limit of the gauge theory.

It was noted, that a very efficient tool for the field theory calculation of anomalous dimensions of composite operators is the *dilatation operator* [51, 52], which acts on the *chain of fields* that the composite operator is built of. It was then remarked in [53], that this operator is equivalent to the operator of the *Heisenberg spin chain*, that was solved by Bethe in 1931 [54] with an ansatz that is now being referred to as the *coordinate Bethe ansatz*. The integrability of the spin-chain is linked to the existence of higher charges that commute with the spin-chain Hamiltonian.

The generalization of the scalar sector analysed in [53] to all local operators of planar $\mathcal{N} = 4$ SYM was achieved in [55, 56]. In a certain limit, one obtains a set of algebraic equations, the so-called *asymptotic Bethe equations* [57, 58, 59], allowing for the extraction of anomalous dimensions to arbitrary order in the coupling constant. For instance the cusp or soft anomalous dimension of the twist-two operators that we will encounter in chapter 7, can be extracted to all orders [60, 61]. This would be practically impossible using usual Feynman-diagram techniques, because of the factorially growing complexity of the calculation with increasing perturbative order.

There are many indications, that integrability is lost beyond the planar limit $N \rightarrow \infty$, but it may still help to make an expansion in powers of $1/N$. It is notable, that integrable structures also appear in QCD [62, 63, 64, 65], for reviews see [66, 67, 68]. For general reviews of integrability see [69, 67, 70, 71, 72, 73, 74, 75] and especially the recent review of AdS/CFT integrability [42].

Chapter 2

Introduction to Conformal Symmetry

A *conformal field theory* is a theory which is invariant under conformal symmetry transformations at the classical level. If the conformal symmetry persists at the quantum level, the theory is a *conformal quantum field theory*. Examples for theories in which the conformal symmetry survives quantisation are the theories appearing in the AdS/CFT correspondence, i.e. $\mathcal{N} = 4$ super Yang-Mills theory and $\mathcal{N} = 6$ super Chern-Simons matter theory, see section 2.4, but up to a certain perturbative order also scalar ϕ^3 -theory in six dimensions, which is introduced in section (2.3). For these theories one can use the restrictions that conformal symmetry imposes on the expectation values of observables also at the quantum level. However, also in non-quantum-conformal theories as QCD, one can make use of conformal properties at a fixed point of the renormalisation group $\beta(g^*) = 0$. Applications of conformal symmetry in QCD are reviewed in [66].

In this and the following chapter we review some well-known facts about conformal field theory complemented by some explicit derivations in Appendix B. In section 2.1, we define conformal transformations and explicitly derive the form of infinitesimal and finite conformal transformations of the coordinates x^μ . Section 2.2 is devoted to the transformation laws of fields under conformal transformations, as well as the definition of conformal primary operators. In sections 2.3 and 2.4 we give some explicit examples of conformal and superconformal field theories.

2.1 Conformal Transformations

By definition, *conformal transformations* are coordinate transformations $x_\mu \rightarrow x'_\mu(x)$ under which the metric is invariant up to a local scale factor, i.e.

$$g'_{\mu\nu}(x') = \rho(x)g_{\mu\nu}(x). \quad (2.1)$$

Under conformal transformations angles are preserved as is easily seen in Appendix (B.2). In this thesis we will be interested in conformal transformations in flat d -

dimensional Minkowski space with metric $\eta_{\mu\nu} = \text{diag}(+, -, \dots, -)$ and specialise to this case in the following.

2.1.1 Infinitesimal Conformal Transformations

Writing an infinitesimal conformal transformation as

$$x'^{\mu} = x^{\mu} + k^{\mu}(x), \quad (2.2)$$

where every component of k^{μ} is infinitesimal, we would like to find the most general form of the *conformal Killing vectors* k^{μ} such that (2.1) is satisfied. As reviewed in detail in Appendix B.1, the most general infinitesimal transformation can be written as

$$k^{\mu} = a^{\mu} + \omega_{\mu\nu}x^{\nu} + \lambda x^{\mu} + 2(x \cdot c)x^{\mu} - c^{\mu}x^2. \quad (2.3)$$

The vector a^{μ} has d independent components and parametrises translations, $\omega_{\mu\nu} = -\omega_{\nu\mu}$ is antisymmetric and has $d(d-1)/2$ independent components parametrising Lorentz transformations. λ parametrises dilatations, and c^{μ} has d independent components and parametrises the so-called *special conformal transformations*.

2.1.2 Generators of Conformal Transformations

The effect of infinitesimal conformal transformations can be written in terms of generators G acting on the coordinates x^{μ} as

$$\delta x^{\mu} = x'^{\mu} - x^{\mu} = i \epsilon \cdot G x^{\mu}, \quad (2.4)$$

where G is any of the differential operators

$$\begin{aligned} P_{\mu} &= -i\partial_{\mu} && \text{(translations)} && (2.5) \\ M_{\mu\nu} &= i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}) && \text{(Lorentz transformations)} \\ D &= -i(x \cdot \partial) && \text{(dilatations)} \\ K_{\mu} &= -i(2x_{\mu}(x \cdot \partial) - x^2\partial_{\mu}) && \text{(special conformal transformations)} \end{aligned}$$

and the expression $\epsilon \cdot G$ symbolically represents the appropriate index contraction of the infinitesimal transformation parameters with the generators, e.g. for translations we have $\epsilon \cdot G = a_{\mu}P^{\mu}$. Then (2.3) is exactly¹ reproduced by (2.4) and (2.5). The first two generators in (2.5) satisfy the Poincaré algebra

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}), \\ [P_{\mu}, M_{\rho\sigma}] &= i(\eta_{\mu\rho}P_{\sigma} - \eta_{\mu\sigma}P_{\rho}) \end{aligned} \quad (2.6)$$

¹Due to the antisymmetry of $M_{\mu\nu}$ we define $\epsilon \cdot G = \frac{1}{2}\omega_{\mu\nu}M^{\mu\nu}$ for the Lorentztransformations.

and together with the generators for dilatations and special conformal transformations satisfy the conformal algebra²

$$\begin{aligned} [D, P_\mu] &= +i P_\mu, & [K_\mu, M_{\rho\sigma}] &= i (\eta_{\mu\rho} K_\sigma - \eta_{\mu\sigma} K_\rho), \\ [D, K_\mu] &= -i K_\mu, & [K_\mu, P_\nu] &= 2i (\eta_{\mu\nu} D - M_{\mu\nu}) \end{aligned} \quad (2.7)$$

and all other commutators vanish.

The antisymmetric operator J_{ab} with $a, b = 0, 1, \dots, d-1, d, d+1$ defined by

$$J_{\mu\nu} = M_{\mu\nu}, \quad J_{d+1,d} = D, \quad J_{\mu,d} = \frac{1}{2}(P_\mu - K_\mu), \quad J_{\mu,d+1} = \frac{1}{2}(P_\mu + K_\mu), \quad (2.8)$$

satisfies the commutation relations similar to the Lorentz algebra $so(1,3)$ in (2.6).

$$[J_{ab}, J_{cd}] = i (\eta_{ad} J_{bc} + \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac}), \quad (2.9)$$

where $\eta_{ab} = \text{diag}(+, -, -, \dots, -, +)$ if $\eta_{\mu\nu}$ is the Minkowski space metric. Therefore the conformal algebra in d -dimensional Minkowski-space is isomorphic to $so(2, d)$ and thus has $(d+1)(d+2)/2$ generators. In $d=4$ we therefore have 15 generators, corresponding to 4 translations, 6 Lorentz transformations (3 rotations and 3 boosts), 4 special conformal transformations and one dilatation.

For conformal transformations acting on a set of points x_i the generators can be generalised to

$$D = \sum_i (x_i \cdot \partial_i), \quad K^\nu = \sum_i \left(2x_i^\nu (x_i \cdot \partial_i) - x_i^2 \partial_i^\nu \right) \quad (2.10)$$

and analogously for the remaining generators. Note that for simplicity we have redefined the generators without the factor of i in (2.4) and (2.5) here, thus $\delta_{\epsilon_K} f(x_{ij}^2) = \epsilon_\mu K^\mu f(x_{ij}^2)$.

2.1.3 Finite Transformations

The finite conformal transformations can be obtained by exponentiating the infinitesimal transformations (2.5), (2.4) and are

$$\begin{aligned} x'^\mu &= x^\mu + a^\mu && \text{(translations)} && (2.11) \\ x'^\mu &= \Lambda^\mu_\nu x^\nu && \text{(Lorentz transformations)} \\ x'^\mu &= \lambda x^\mu && \text{(dilatations)} \end{aligned}$$

where Λ_μ^ν are the familiar Lorentz transformation matrices. For the special conformal transformations it is less straightforward to obtain the finite transformation

$$x'^\mu = \frac{x^\mu - c^\mu x^2}{1 - 2c \cdot x + c^2 x^2} \quad \text{(special conformal transformation)} \quad (2.12)$$

²These commutation relations can easily be derived by application of the differential operators to an arbitrary function $f(x)$ on the coordinates in the same way as the Poincaré algebra.

it is however easy to see, that it coincides infinitesimally with the version given in (2.3) by expanding (2.12) to first order in c^μ . The transformation can also be obtained by applying an inversion of the coordinates

$$I(x^\mu) = \frac{x^\mu}{x^2} \quad (2.13)$$

followed by a translation with vector $-c^\mu$ and another inversion, i.e.

$$I \circ P \circ I : \quad x^\mu \xrightarrow{I} \frac{x^\mu}{x^2} \xrightarrow{\mathbb{P}} \frac{x^\mu}{x^2} - c^\mu \xrightarrow{I} \frac{x^\mu - c^\mu x^2}{1 - 2c \cdot x + c^2 x^2}. \quad (2.14)$$

Even though the inversion itself has no infinitesimal form and thus does not belong to the Lie algebra of conformal transformations, it is helpful to check the invariance of an expression under special conformal transformations. If an expression is invariant under inversions and translations, it is consequently invariant under special conformal transformations.

2.1.4 Conformal Invariants

For instance, we can easily check that the combination of four coordinates x_i^μ

$$u_{ijkl} = \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2} \quad (2.15)$$

form *conformal invariants*. The $x_{ij}^2 := (x_i - x_j)^2$ are obviously invariant under translations and rotations and transform as

$$x_{ij}^2 \xrightarrow{I} \frac{x_{ij}^2}{x_i^2 x_j^2}, \quad x_{ij}^2 \xrightarrow{D} \lambda^2 x_{ij}^2 \quad (2.16)$$

under inversions respectively dilatations. Therefore u_{ijkl} are invariant under all conformal transformations. They can be built if we have at least four non-vanishing distances. These invariants are also called conformally invariant *anharmonic ratios* or *cross-ratios*.

2.1.5 Causality and Conformal Transformations

Note that the transformation of x^2 under special conformal transformations (2.12) is

$$x^2 \rightarrow x'^2 = \frac{x^2}{1 - 2c \cdot x + c^2 x^2}. \quad (2.17)$$

Therefore, for every vector x^μ one can find a special conformal transformation which changes a spacelike into a timelike vector and vice versa, implying a possible clash with causality, see [76] for a detailed discussion. Light-like distances are however preserved,

i.e. $x^2 = 0 \rightarrow x'^2 = 0$. We will make use of this property in the discussion of light-like Wilson loops in section 4.2.1.

The problem of causality can be avoided by restricting to infinitesimal transformations³ or by discussing the conformal properties of Green's functions in the Euclidean region. The causality problem was further discussed in [77, 78] and to avoid the conflict with local commutativity, i.e. causality, it was argued for a non-locality of the finite field transformations, reflected by the fact that instead of a true representation of the conformal group, one has in general a representation of its universal covering group [78].

2.2 Conformal Field Theory

In order to formulate a classical conformal field theory, we show how fields transform under conformal transformations following to a certain extent the original papers on infinitesimal conformal transformations of fields [79, 80, 81]. A similar derivation can be found in [82].

2.2.1 Infinitesimal Conformal Transformations of Fields

We are interested in the action of the generators of the conformal group on a field $\Phi_\alpha(x)$ of arbitrary spin-tensor structure, i.e. α represents a set of spinor and / or vector indices. By \mathbb{G} we denote the generator of an infinitesimal conformal transformation $\delta\Phi$ of the fields

$$\delta\Phi(x) = \Phi'(x) - \Phi(x) = i\epsilon \cdot \mathbb{G}_{\alpha\beta} \Phi_\beta(x) \quad (2.18)$$

and the generators $\mathbb{G} \in \{\mathbb{P}_\mu, \mathbb{K}_\mu, \mathbb{D}, \mathbb{M}_{\mu\nu}\}$ form a representation of the conformal algebra (2.7), (2.6) acting on the fields. The parameters ϵ are the same transformation parameters as in (2.4). In the following we will suppress the indices α, β but the reader should keep in mind the action of the generators on this index space.

The total change of a field $d\Phi := \Phi'(x') - \Phi(x)$ under an infinitesimal conformal transformation $x' = x + \delta x$ can be split into a transformation of the field $\delta\Phi$ and a transformation induced by the change of coordinates as

$$d\Phi(x) = \delta\Phi(x) + \delta x^\mu \partial_\mu \Phi(x) + \mathcal{O}(\delta^2), \quad (2.19)$$

which follows from $\Phi'(x + \delta x) - \Phi(x) = \Phi'(x) - \Phi(x) + \delta x^\mu \partial_\mu \Phi(x) + \mathcal{O}(\delta^2)$. To illustrate the purpose of $\delta\Phi$ imagine a scalar field $\phi(x)$ measuring the distribution of some quantity in spacetime with a local maximum at x_0 . After a coordinate transformation in which $x_0 \rightarrow x'_0$, the field must change in such a way, that it has a maximum at x'_0 , i.e. $\phi'(x'_0) = \phi(x_0)$ or $d\phi = 0$ and from (2.19) we find that the infinitesimal change in the field is $\delta\phi(x) = -\delta x^\mu \partial_\mu \phi(x)$.

³There is no continuous deformation of light-like into spacelike distances.

For a vector field $A_\mu(x)$ the situation is different, it does not only describe the absolute value of a distribution in space-time such as the scalar field, but additionally a direction in space-time. For instance under a rotation, the length of a vector is preserved just as the value of the scalar field, but its direction is changed. The change in the field δA_μ exactly measures the change of the field evaluated at the same position, i.e. $\delta A_\mu(x) = A'_\mu(x) - A_\mu(x)$.

As we will presently see, the change of the field $\delta\Phi(x)$ respectively the action of \mathbb{G} on $\Phi(x)$ can be deduced from its action on $\Phi(0)$. It is thus reasonable to classify fields according to their transformation properties under transformations which leave the origin $x^\mu = 0$ invariant, i.e. which form the *stability or isotropy subgroup* of $x = 0$ (also called the *little group*). It follows from (2.11), (2.12) that these are Lorentz transformations, dilatations and special conformal transformations. We define the action at $x = 0$ as

$$\mathbb{M}_{\mu\nu} \Phi(0) = \Sigma_{\mu\nu} \Phi(0), \quad \mathbb{D} \Phi(0) = \tilde{\Delta} \Phi(0), \quad \mathbb{K}_\mu \Phi(0) = \kappa_\mu \Phi(0), \quad (2.20)$$

where $\Sigma_{\mu\nu}$ is the spin operator associated with the field Φ with explicit representations

$$\Sigma_{\mu\nu} \phi = 0, \quad (\Sigma_{\mu\nu} \psi)_\alpha = \frac{i}{4} [\gamma_\mu, \gamma_\nu]_{\alpha\beta} \psi_\beta, \quad (\Sigma_{\mu\nu} A)_\rho = i (\eta_{\mu\rho} A_\nu - \eta_{\nu\rho} A_\mu), \quad (2.21)$$

for scalar, Dirac and vector fields. The generators $\Sigma_{\mu\nu}, \kappa_\mu, \tilde{\Delta}$ satisfy the same commutation relations as $M_{\mu\nu}, K_\mu, D$. The action of $\Sigma_{\mu\nu}$ on tensors with more general indices can be obtained by taking a sum over terms like (2.21) one for each of the indices, for more details see e.g. chapter 1.4 of [83].

According to [79], we choose a basis in index space in such a way that translations do not act on the indices⁴

$$\mathbb{P}_\mu \Phi(x) = -i \partial_\mu \Phi(x). \quad (2.22)$$

In order to evaluate the action of the other generators on $\Phi(x)$, one can use the translation operator $\Phi(x) = \exp(i\mathbb{P}_\mu x^\mu) \Phi(0)$ to write

$$\mathbb{G} \Phi(x) = e^{i\mathbb{P}\cdot x} e^{-i\mathbb{P}\cdot x} \mathbb{G} e^{i\mathbb{P}\cdot x} \Phi(0). \quad (2.23)$$

Note that the generators \mathbb{G} only act on the fields, i.e. in particular $[\mathbb{P}_\mu, x_\nu] = 0$ unlike $[P_\mu, x_\nu] = -i\eta_{\mu\nu}$. Therefore we can use

$$e^{-i\mathbb{P}\cdot x} \mathbb{G} e^{i\mathbb{P}\cdot x} = \sum_{n=0}^{\infty} \frac{i^n}{n!} x^{\mu_1} \dots x^{\mu_n} [\dots [\mathbb{G}, \mathbb{P}_{\mu_1}], \dots], \mathbb{P}_{\mu_n}. \quad (2.24)$$

The sum on the right-hand side is finite and we need at most terms with two commutators, since all higher commutators vanish due to the commutation relations of the conformal algebra. Evaluating (2.24) for the different generators, applying the result

⁴They act trivially in the field index space, i.e. $(\mathbb{P}_\mu \Phi)_\beta = P_\mu \Phi_\beta$.

to $\Phi(0)$ using (2.20) and finally translating $\Phi(0)$ to $\Phi(x)$ using the remaining factor of $\exp(i\mathbb{P} \cdot x)$ in (2.23) we find

$$\begin{aligned} \mathbb{P}_\mu \Phi(x) &= -i\partial_\mu \Phi(x), \\ \mathbb{D}\Phi(x) &= \left(ix^\mu \partial_\mu + \tilde{\Delta} \right) \Phi(x), \\ \mathbb{M}_{\mu\nu} \Phi(x) &= \left(-i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \Sigma_{\mu\nu} \right) \Phi(x), \\ \mathbb{K}_\mu \Phi(x) &= \left(\kappa_\mu - 2x_\mu \tilde{\Delta} + 2x^\nu \Sigma_{\mu\nu} - i(2x_\mu(x \cdot \partial) - x^2 \partial_\mu) \right) \Phi(x). \end{aligned} \tag{2.25}$$

The operators that we consider here satisfy $\mathbb{K}_\mu \Phi(0) = 0$, i.e. we choose $\kappa_\mu = 0$ in (2.20). In principle $\kappa_\mu \neq 0$ and nilpotent [79] is possible as well.

Since $[\mathbb{M}_{\mu\nu}, \mathbb{D}] = 0$ we have $[\Sigma_{\mu\nu}, \tilde{\Delta}] = 0$ and it follows by Schur's lemma, that any matrix that commutes with all generators $\Sigma_{\mu\nu}$ must be a multiple of the identity, and thus we write $\tilde{\Delta} = i\Delta$, where Δ is the *scaling dimension* of the field.

2.2.2 Finite Conformal Transformations of Fields

In (2.25) we have written the transformation of the fields under infinitesimal conformal transformations. The transformation law for a scalar field ϕ under finite conformal transformations $x \rightarrow x'$ reads

$$\phi'(x') = \left| \frac{\partial x'}{\partial x} \right|^{-\Delta/d} \phi(x). \tag{2.26}$$

Under a finite scaling $x \rightarrow x' = \lambda x$ (2.11) the field thus transforms as $\phi' = \lambda^{-\Delta} \phi$, which explains the name *scaling dimension*. The form of finite transformations for a general field with indices α reads

$$\Phi'_\alpha(x') = \left| \frac{\partial x'}{\partial x} \right|^{-\Delta/d} D_\alpha^\beta(R) \Phi_\beta(x), \tag{2.27}$$

where D_α^β is the appropriate representation of the Lorentz transformation

$$R_\mu^\nu = \rho^{-1/2}(x) \frac{\partial x'_\mu}{\partial x_\nu} \tag{2.28}$$

and $\rho(x)$ is the scale factor of the conformal transformation (2.1). For more information see e.g. [84, 85, 86].

2.2.3 Conformal Primary Fields and Operators

Fields transforming according to (2.26), (2.27) are called *primary fields*⁵ or *conformal primaries*. Correspondingly, a *conformal primary operator* is a local operator which

⁵More precisely, in general dimension d it is called a *quasi-primary field*. The term *primary field* is reserved for fields in $d = 2$ which have a more specific transformation law, but are at the same time *quasi-primary fields*, since they automatically satisfy the transformation laws of these.

transforms according to the above laws under conformal transformations. Recall, that we chose $\kappa_\mu = 0$ to deduce these transformation laws, i.e. the definition of a conformal primary requires

$$\mathbb{K}_\mu \mathcal{O}(0) = 0, \quad (2.29)$$

where the scaling dimension of the primary is given by $\mathbb{D} \mathcal{O}(0) = i\Delta \mathcal{O}(0)$. From the commutation relation of the conformal algebra it follows that an operator \mathcal{O}' related to a primary operator via $\mathcal{O}'(x) = \mathbb{P} \mathcal{O}(x)$ has the scaling dimension of \mathcal{O} raised by one

$$\mathbb{D} \mathcal{O}'(0) = [\mathbb{D}, \mathbb{P}] \mathcal{O}(0) + \mathbb{P}(\mathbb{D} \mathcal{O}(0)) = i(1 + \Delta) \mathbb{P} \mathcal{O}(0) = i\Delta' \mathcal{O}'(0). \quad (2.30)$$

The operator \mathcal{O}' is called a *descendant* of \mathcal{O} . Conversely, \mathbb{K} lowers the dimension of an operator by 1 and thus the conformal primaries are the operators with lowest scaling dimensions. In general it suffices to investigate the properties of conformal primaries, since relations for the descendants follow from the conformal algebra.

2.3 Classically Conformal Field Theories

A field theory is classically invariant if the action is invariant under the field transformations (2.25). Due to the commutation relations (2.7) the mass operator $M^2 = P^2$ does not commute with conformal transformations and dilatations act as

$$e^{i\alpha D} P^2 e^{-i\alpha D} = e^{2\alpha} P^2. \quad (2.31)$$

Therefore invariance under conformal transformations, either requires a continuous mass spectrum or a massless theory. We will restrict to the latter ones in this thesis. Furthermore, in [79] it was derived that all non-derivative couplings with dimensionless coupling constant as well as all couplings of Yang-Mills type are conformally invariant.

Therefore, at the classical level the only term in the standard model⁶ which is not invariant under conformal transformations is the term with the Higgs mass. An alternative for a classically conformal invariant standard model is discussed in [87, 88].

Energy momentum tensor

The Noether theorem implies that the symmetries of a field theory are reflected by the existence of conserved currents J^μ , e.g. the conserved current with respect to translations is the canonical energy momentum tensor $T_c^{\mu\nu}$. The conserved currents associated with the invariance of the action are only defined up to total divergences and can thus be redefined. In particular, it is often possible [80] to construct a symmetric, divergenceless energy momentum tensor $\theta^{\mu\nu}$, which allows to write the associated currents of all conformal transformations as

$$J^\mu = k_\nu \theta^{\mu\nu}, \quad \theta^{\mu\nu} = \theta^{\nu\mu}, \quad \partial_\mu \theta^{\mu\nu} = 0, \quad (2.32)$$

⁶assuming massless neutrinos

where the conformal Killing vectors k^μ are defined in (2.3). Using (B.4) the conservation of this current can be written as

$$\partial_\mu J^\mu = \frac{1}{2}(\partial_\mu k_\nu + \partial_\nu k_\mu)\theta^{\mu\nu} + k_\nu \partial_\mu \theta^{\mu\nu} = \frac{1}{d}(\partial \cdot k) \theta^\mu{}_\mu \quad (2.33)$$

and thus the conservation of the current – or the invariance of the action – can be attributed to the tracelessness $\theta^\mu{}_\mu = 0$ of the energy momentum tensor. This symmetry may be broken in the quantum theory and the violation of the tracelessness is called the *conformal anomaly* or *trace anomaly*.

A detailed account on this subject can be found in section 2.4 of [66]. There, it is also explained that the violation of the dilatation symmetry in massless QCD is given by $\partial_\mu J_D^\mu = \beta(g)/(2g)G_{\mu\nu}^a G^{a,\mu\nu}$ where β is the renormalisation group β -function and $G_{\mu\nu}^a$ the field strength.

Simple classically conformal field theories

As the simplest example, consider a scalar theory in d dimensions, with interaction term $g\phi^n$ where g is the coupling constant

$$S = \int d^d x (\partial_\mu \phi \partial^\mu \phi + g\phi^n). \quad (2.34)$$

The inverse length dimension $[d^d x] = -d$ of the measure must be compensated by the weight of the Lagrangian for the action to be dimensionless. Therefore, the dimension of the scalar field is $[\phi] = (d-2)/2$ and the requirement, that g is dimensionless leads to the conclusion that $n = 2d/(d-2)$. Therefore, in $d = 4$ dimensions the only allowed interaction term is ϕ^4 and in $d = 6$ it is ϕ^3 .

Scalar ϕ^3 -theory in six dimensions

The latter $(\phi^3)_6$ theory is particularly interesting, because generalised to N_f complex scalar fields ϕ_i with an interaction term with one real scalar χ

$$\mathcal{L} = \sum_{i=1}^{N_f} \partial_\mu \phi \partial^\mu \bar{\phi}_i + \frac{1}{2} \partial_\mu \chi \partial^\mu \chi + g \sum_{i=1}^{N_f} \phi_i \bar{\phi}_i \chi, \quad (2.35)$$

it has a number of features in common with QCD in four dimensions. For example, its β -function has a structure similar to QCD and for a certain number N_f of flavours [89], it can be made to vanish up a certain order in the coupling, which allows to use conformal symmetry arguments up to this order. The similarity of this theory to $\mathcal{N} = 4$ SYM will become clear in chapter 7.

Chern-Simons theory

Another example of a classically conformal field theory is Chern-Simons theory in three dimensions

$$\mathcal{S}_{\text{CS}}(A) = \frac{k}{4\pi} \int d^3 x \epsilon^{\mu\nu\rho} \text{Tr} \left(A_\mu \partial_\nu A_\rho - i \frac{2}{3} A_\mu A_\nu A_\rho \right) \quad (2.36)$$

and we will deal with the conformal properties of the theory in this thesis.

Scale without Conformal Invariance

In two dimensions scale invariance and unitarity of the theory imply conformal invariance [90]. However in $d > 2$ it is possible to have scale invariant theories which are not invariant under conformal transformations [90, 86, 91, 92, 93, 94] and thus do not allow for (2.32). The converse is however always true, conformal invariance implies scale invariance.

2.4 Superconformal Symmetry

The Poincaré algebra (2.6) can be extended to a *super-Poincaré algebra* by the introduction of fermionic supersymmetry generators Q_α^A and their conjugates $\bar{Q}_{\dot{\alpha}}^A$ which satisfy the anticommutation relations

$$\{Q_\alpha^A, \bar{Q}_{\dot{\alpha}}^B\} = 2(\sigma^\mu)_{\alpha\dot{\alpha}} P_\mu \delta^{AB}, \quad (2.37)$$

where $A = 1, \dots, \mathcal{N}$ labels the number of supersymmetry generators Q_α^A , $\sigma^\mu = (\mathbf{1}, \vec{\sigma})$ are the Pauli matrices and $\alpha, \dot{\alpha} = 1, 2$ are the $su(2)$ Lorentz indices. Additionally, we have a so-called *R-symmetry* group among which the supercharges transform. In the Lagrangian of the theory, this is reflected by a flavour symmetry of the fields. Introductions to supersymmetry can be found e.g. in [95, 96, 97, 98].

Analogously, a conformal symmetry algebra is extended to a *superconformal algebra* by the introduction of (2.37), which due to the non-trivial commutation relations of the special conformal generators K_μ and the Poincaré supercharges $Q_{\dot{\alpha}}^A$ additionally contains the so-called *conformal supercharges* S_α^A and their conjugates $\bar{S}_{\dot{\alpha}}^A$ satisfying

$$\{S_\alpha^A, \bar{S}_{\dot{\alpha}}^B\} = 2(\sigma^\mu)_{\alpha\dot{\alpha}} K_\mu \delta^{AB}. \quad (2.38)$$

Corresponding to the definition of conformal primaries in section 2.2.3, *superconformal primaries* are annihilated by the special conformal supercharges S . More information on the superconformal algebra can be found e.g. in chapter 3 of the review [40].

2.4.1 $\mathcal{N} = 4$ Super Yang-Mills Theory

As implied by its name the $\mathcal{N} = 4$ super Yang-Mills theory in $d = 4$ dimensions has four supercharges with associated *R-symmetry* algebra $su(4)$. The Lagrangian can be derived by dimensionally reducing $\mathcal{N} = 1$ super Yang-Mills theory in $D = 10$ dimensions to $d = 4$. We review this derivation in Appendix C and the resulting Lagrangian is given by⁷

⁷We omit gauge fixing and ghost terms here.

$$\begin{aligned} \mathcal{L}_{\text{SYM}} = \text{Tr} \left(-\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \phi^{AB} D^\mu \bar{\phi}_{AB} + \frac{1}{8} g^2 [\phi^{AB}, \phi^{CD}] [\bar{\phi}_{AB}, \bar{\phi}_{CD}] \right. \\ \left. + 2i \bar{\lambda}_{\dot{\alpha}A} (\sigma_\mu)^{\dot{\alpha}\beta} D^\mu \lambda_\beta^A - \sqrt{2} g \lambda^{\alpha A} [\bar{\phi}_{AB}, \lambda_\alpha^B] + \sqrt{2} g \bar{\lambda}_{\dot{\alpha}A} [\phi^{AB}, \bar{\lambda}_{\dot{\alpha}B}] \right), \end{aligned} \quad (2.39)$$

where the field content comprises four fermions $\lambda_{\alpha A}$, three independent complex scalars ϕ^{AB} which are related to their complex conjugates via $\bar{\phi}_{AB} = (\phi^{AB})^* = \frac{1}{2} \epsilon_{ABCD} \phi^{CD}$ and a gauge field A_μ with field strength and covariant derivative

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu], \quad D_\mu(\cdot) = \partial_\mu(\cdot) - ig[A_\mu, (\cdot)]. \quad (2.40)$$

All fields are matrix valued $A_\mu = A_\mu^a T^a$ and T^a are the generators of $SU(N)$ in the fundamental representation. In this formulation the $SU(4)$ R-symmetry is manifest in the indices $A, B = 1, \dots, 4$. Very often, in the literature one encounters an equivalent formulation of $\mathcal{N} = 4$ SYM, with a different definition of the scalar fields, where the matrices Σ in (C.11) are not incorporated into the definition of the scalars but remain in the Lagrangian. Then the R -symmetry is an $SO(6) \sim SU(4)$ rotation symmetry of 6 real scalars ϕ_i . In this approach the real scalar fields are usually combined to complex fields as $Z = \phi_1 + i\phi_2, W = \phi_3 + i\phi_4, X = \phi_5 + i\phi_6$. The approaches are of course completely equivalent, but the calculations slightly differ by the absorbed or non-absorbed Σ matrices.

Coupling Constant Renormalisation and β -Function

The Lagrangian obviously satisfies the restrictions of section 2.3 for classical conformal symmetry. Moreover, it turns out that the renormalisation group β -function of the theory vanishes in perturbation theory, which is believed to be true to all orders in the coupling constant [99, 100, 101, 102, 95]. For an interesting discussion on the arguments of the all-loop vanishing of the β -function we refer the reader to chapter 13.2 of [95]. Therefore, the theory remains exactly scale invariant at the quantum level. Thus also at loop-level conformal symmetry is not broken and restricts correlation functions as we will discuss in detail in section 3.

Superalgebra $su(2, 2|4)$

In $\mathcal{N} = 4$ SYM the bosonic symmetry algebra, which consists of the conformal algebra (2.9) in four dimensions $so(2, 4) \simeq su(2, 2)$ and the R -symmetry algebra $su(4)$, is lifted to the superalgebra $psu(2, 2|4)$. The algebra $psu(2, 2|4)$ plays an important role in the AdS_5/CFT_4 correspondence and a detailed account on the properties of the algebra as well as the isometries of the $AdS_5 \times S^5$ superspace can be found in [43].

For the formulation of the full superconformal symmetry algebra, it is useful to write all generators in $su(2)$ notation, e.g. the Lorentz generators are split as $\mathbb{M}_{\alpha\beta}, \bar{\mathbb{M}}_{\dot{\alpha}\dot{\beta}}$ acting on $su(2)_L$ respectively $su(2)_R$ indices.

For further reference, we adopt the formulation of the non-trivial commutation relations of the generators of the algebra⁸ $su(2, 2|4)$ from [10]. The Lorentz generators $\mathbb{M}_{\alpha\beta}, \overline{\mathbb{M}}_{\dot{\alpha}\dot{\beta}}$ and the $su(4)$ *R-symmetry* generators \mathbb{R}^A_B act canonically on the remaining generators carrying Lorentz or $su(4)$ indices, as explicitly given in Appendix D of [70]. The dilatation \mathbb{D} acts via

$$[\mathbb{D}, \mathbb{J}] = \dim(\mathbb{J})\mathbb{J}, \quad (2.41)$$

where the non-zero dimensions of the various generators are

$$\dim(\mathbb{P}) = 1, \quad \dim(\mathbb{K}) = -1, \quad \dim(\mathbb{Q}) = \dim(\overline{\mathbb{Q}}) = \frac{1}{2}, \quad \dim(\mathbb{S}) = \dim(\overline{\mathbb{S}}) = -\frac{1}{2}.$$

The remaining non-trivial commutation relations are,

$$\begin{aligned} \{\mathbb{Q}_{\alpha A}, \overline{\mathbb{Q}}_{\dot{\alpha}}^B\} &= \delta_A^B \mathbb{P}_{\alpha\dot{\alpha}}, & \{\mathbb{S}_{\alpha A}, \overline{\mathbb{S}}_{\dot{\alpha}}^B\} &= \delta_A^B \mathbb{K}_{\alpha\dot{\alpha}}, \\ [\mathbb{P}_{\alpha\dot{\alpha}}, \mathbb{S}_{\dot{\beta}}^A] &= \epsilon_{\dot{\alpha}\dot{\beta}} \overline{\mathbb{Q}}_{\dot{\alpha}}^A, & [\mathbb{K}_{\alpha\dot{\alpha}}, \mathbb{Q}_{\beta A}] &= \epsilon_{\alpha\beta} \overline{\mathbb{S}}_{\dot{\alpha}A}, \\ [\mathbb{P}_{\alpha\dot{\alpha}}, \overline{\mathbb{S}}_{\dot{\beta}A}] &= \epsilon_{\dot{\alpha}\dot{\beta}} \mathbb{Q}_{\alpha A}, & [\mathbb{K}_{\alpha\dot{\alpha}}, \overline{\mathbb{Q}}_{\dot{\beta}A}] &= \epsilon_{\dot{\alpha}\dot{\beta}} \mathbb{S}_{\alpha A}, \\ [\mathbb{K}_{\alpha\dot{\alpha}}, \mathbb{P}^{\beta\dot{\beta}}] &= \delta_{\alpha}^{\beta} \delta_{\dot{\alpha}}^{\dot{\beta}} \mathbb{D} + \mathbb{M}_{\alpha}^{\beta} \delta_{\dot{\alpha}}^{\dot{\beta}} + \overline{\mathbb{M}}_{\dot{\alpha}}^{\dot{\beta}} \delta_{\alpha}^{\beta}, \\ \{\mathbb{Q}_{\alpha A}, \mathbb{S}_{\dot{\beta}}^B\} &= \epsilon_{\alpha\beta} \mathbb{R}^B_A + \mathbb{M}_{\alpha\beta} \delta_A^B + \epsilon_{\alpha\beta} \delta_A^B (\mathbb{D} + \mathbb{C}), \\ \{\overline{\mathbb{Q}}_{\dot{\alpha}}^A, \overline{\mathbb{S}}_{\dot{\beta}B}\} &= \epsilon_{\dot{\alpha}\dot{\beta}} \mathbb{R}^A_B + \overline{\mathbb{M}}_{\dot{\alpha}\dot{\beta}} \delta_B^A + \epsilon_{\dot{\alpha}\dot{\beta}} \delta_B^A (\mathbb{D} - \mathbb{C}), \end{aligned} \quad (2.42)$$

where \mathbb{C} is the central charge. If the central charge vanishes, the resulting algebra is $psu(2, 2|4)$. The commutations relations of the *super Lie algebra* can be summarised as

$$[j_a, j_b] = f_{ab}^c j_c, \quad (2.43)$$

where j_a is any of the generators and f_{ab}^c are the structure constants of the algebra and $[\cdot, \cdot]$ is the graded commutator. We will use this in the discussion of the *Yangian algebra* in section 4.1.8.

2.4.2 $\mathcal{N} = 6$ Super Chern-Simons Theory

The relevant field theory in the $\text{AdS}_4/\text{CFT}_3$ correspondence is a three-dimensional $\mathcal{N} = 6$ superconformal Chern-Simons theory, called ABJM theory [2]. Schematically, it reads

$$S_{\text{ABJM}} = S_{\text{CS}}(A) - S_{\text{CS}}(\hat{A}) + S_{\text{matter}},$$

where we have two copies of the Chern-Simons action (2.36) with two different $U(N)$ gauge fields A_{μ}, \hat{A}_{μ} and the fields in the matter part

$$S_{\text{matter}} = \int d^3x \text{Tr} (D_{\mu} \phi_I D^{\mu} \bar{\phi}^I) + i \text{Tr} (\bar{\psi}^I \not{D} \psi_I) + \mathcal{L}_{\text{int}} \quad (2.44)$$

⁸We omit the discussion of the hypercharge \mathcal{B} . Including it, the resulting algebras were $u(2, 2|4)$, $pu(2, 2|4)$. A detailed discussion can be found in [70].

are in the bifundamental representation and thus carry an $SU(N)$ index for each of the two gauge groups, i.e. $(\phi_I)_{ii}, (\psi_I)_{ii}$. The interaction term \mathcal{L}_{int} schematically consists of Yukawa type terms $\phi^2\psi^2$ as well as terms sextic in the scalars $(\phi)^6$. We give more details in Appendix F.3. For Chern-Simons level⁹ $k > 2$ the superconformal symmetry group is the orthosymplectic supergroup $OSp(6|4)$ its bosonic part being the R -symmetry group $SU(4) \simeq SO(6)$ and the conformal group (2.9) in three dimensions $SO(2, 3) \simeq Sp(4)$. A reformulation with manifest $SU(4)$ R -symmetry and a very clear presentation of the superconformal symmetry can be found in [103]. A detailed account on $OSp(6|4)$ can be found in [104].

The generalization of ABJM theory to different gauge groups $SU(N)_k, SU(M)_{-k}$ for the Chern-Simons actions of level k respectively $-k$ is referred to as ABJ theory [105].

⁹For $k = 1, 2$ the algebra is enhanced to $osp(8|4)$.

Chapter 3

Implications of Conformal Symmetry

Conformal symmetry imposes restrictions on observables in quantum field theory. These are due to *conformal Ward identities*, which we review in section 3.1. In section 3.2 we show how this fixes the form of two-, three- and higher point correlation functions of scalar operators and in 3.3 we generalise this to correlation functions involving operators with spin. In section 3.4 we comment on the renormalisation properties of composite operators and introduce a *conformal renormalisation scheme*.

3.1 Conformal Ward Identities

The symmetries of the quantum field theory translate into constraints on expectation values through so-called *Ward identities*. The Ward identities can be derived by considering the change of the Green's function under a transformation of the fields

$$\Phi \rightarrow \Phi' = \Phi + \delta\Phi \tag{3.1}$$

as given by (2.25). Writing the Green's function as a path integral

$$G_n = \langle \Phi_1(x_1) \dots \Phi_n(x_n) \rangle = \int \mathcal{D}\Phi e^{iS[\Phi]} \Phi_1(x_1) \dots \Phi_n(x_n) \tag{3.2}$$

and performing a change of variables $\Phi \rightarrow \Phi'$ in the functional integral, we find¹

$$\begin{aligned} \langle \Phi_1(x_1) \dots \Phi_n(x_n) \rangle &= \int \mathcal{D}\Phi' e^{iS[\Phi']} \Phi'_1(x_1) \dots \Phi'_n(x_n) \\ &= \int \mathcal{D}\Phi e^{iS[\Phi + \delta\Phi]} (\Phi_1 + \delta\Phi_1) \dots (\Phi_n + \delta\Phi_n). \end{aligned} \tag{3.3}$$

¹Note that we do not take into account the Jacobi determinant in the functional integral, since it is a c-number and does not modify the Ward identity, for more information see [106, 107, 66]. Furthermore, we suppress a normalisation factor of the path integral here.

Considering a conformal transformation (3.1), the action of a conformal field theory is classically invariant, $\delta S = 0$ and infinitesimally we get the Ward identity

$$\sum_{i=1}^n \langle \Phi_1(x_1) \dots \delta \Phi_i(x_i) \dots \Phi_n(x_n) \rangle = 0. \quad (3.4)$$

This identity imposes restrictions on the the correlation function. Considering for instance translations and Lorentz transformations, we conclude, that the correlation function must be a function of the translation and Lorentz invariant combinations $x_{ij}^2 := (x_i - x_j)^2$ of the space-time points. Using (2.25) the dilatation Ward identity reads

$$\sum_{i=1}^n (x_i \cdot \partial_i + \Delta_i) \langle \Phi_1(x_1) \dots \Phi_i(x_i) \dots \Phi_n(x_n) \rangle = 0, \quad (3.5)$$

where Δ_i is the scaling dimension of the field Φ_i . This *dilatation Ward identity* and the corresponding *special conformal Ward identity*

$$\sum_{i=1}^n \left(2x_i^\mu (x_i \cdot \partial_i + \Delta_i) + 2i \Sigma_\nu^\mu x_i^\nu - x_i^2 \partial_i^\mu \right) \langle \Phi_1(x_1) \dots \Phi_i(x_i) \dots \Phi_n(x_n) \rangle = 0 \quad (3.6)$$

constrain the space-time structure of Green's functions as we will see in the sections 3.2, 3.3.

3.1.1 Anomalous Conformal Ward Identity

If the action S is not invariant under conformal transformations, we get an *anomalous conformal Ward identity*

$$\sum_{i=1}^n \langle \Phi_1(x_1) \dots \delta \Phi_i(x_i) \dots \Phi_n(x_n) \rangle = -i \langle \delta S \Phi_1(x_1) \dots \Phi_n(x_n) \rangle, \quad (3.7)$$

where δS is the variation of the action under the transformation (3.1). This is for example the case, if the conformal symmetry of the action is broken through a regulator, e.g. in dimensional regularisation. Then, the weight of the measure $d^d x$ in $d = 4 - 2\epsilon$ does not match the weight of the Lagrangian and thus the action $S = \int d^d x \mathcal{L}$ is only invariant up to order epsilon terms under dilatations and special conformal transformations. Note that dimensional regularization preserves Poincaré invariance and thus the statement that the expectation values must be functions of x_{ij}^2 remains true.

It turns out, that these anomalous conformal Ward identities are particularly useful for the evaluation of the Wilson loop as will be explained in sections 4.2.2 and 6.4. More details on conformal Ward identities can be found in [106, 107, 108, 109] and the review [66].

3.1.2 Anomalous Dimensions

Consider a Green's function of n fields $\Phi(x)$ of classical scaling dimension Δ . Due to divergences, we have to renormalise the fields at loop-level. In a theory with vanishing β -function, the relations (3.5), (3.6) remain valid for the renormalised Green's function up to a change in the scaling dimension:

$$\Delta \rightarrow \Delta + \gamma(g). \quad (3.8)$$

It acquires an *anomalous dimension* $\gamma(g)$ which has an expansion in the coupling constant g . This can be seen as follows.

For dimensional reasons, the renormalised Green's function may be written in a form where the canonical dimension is uniquely carried by the scale μ and a dimensionless function G_n , depending on the dimensionless quantities² $\mu^2 x_{ij}^2$

$$\langle \Phi^R(x_1) \dots \Phi^R(x_n) \rangle = \mu^{n-\Delta} G_n(\mu^2 x_{ij}^2). \quad (3.9)$$

We can thus write

$$\sum_{i=1}^n (\Delta + x_i \cdot \partial_i) \langle \Phi^R(x_1) \dots \Phi^R(x_n) \rangle = \mu \frac{\partial}{\partial \mu} \langle \Phi^R(x_1) \dots \Phi^R(x_n) \rangle, \quad (3.10)$$

which is the equivalent [108, 109, 66] of the ordinary Callan-Symanzik equation

$$\left[\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} + n\gamma \right] \langle \Phi^R(x_1) \dots \Phi^R(x_n) \rangle = 0. \quad (3.11)$$

For a theory with vanishing β -function or at a fixed point of the renormalisation group $\beta(g^*) = 0$, we can thus rewrite (3.10)

$$\sum_{i=1}^n (\Delta + \gamma + x_i \cdot \partial_i) \langle \Phi^R(x_1) \dots \Phi^R(x_n) \rangle = 0. \quad (3.12)$$

Therefore, using the replacement (3.8), the dilatation Ward identity can be brought to the simple form (3.5) that it has in a free theory. The special conformal Ward identity can also be brought to the form (3.6) under the same replacement, see e.g. [66] for details.

This has the important consequence, that the restrictions on the Green's functions imposed by the free Ward-identities also apply in the interacting case, i.e. the space-time structure of the Green's functions is fixed in the same way as in the free case under the replacement (3.8). As we will see in the following sections, this uniquely fixes the structure of two and three-point functions.

²and the dimensionless coupling.

Composite operators, mixing and conformal scheme

The same statements are true for composite operators of local fields $\mathcal{O}(x) = \Phi(x)\dots\Phi(x)$. Even though the anomalous Ward-identities for the renormalised composite operators are not diagonal in the $\overline{\text{MS}}$ -scheme, i.e. under renormalisation the operators mix with other operators, one can recover conformally covariant operators in the *conformal scheme* as will be explained in section 3.4.3.

3.1.3 Protected Operators

Operators that do not get quantum corrections and consequently do not acquire anomalous dimensions are called *protected operators*.

Chiral primary operators or *BPS operators* commute with a certain number of supercharges and are protected from quantum corrections. According to the amount of supercharges that they commute with, they are called 1/2 BPS, 1/4 BPS, 1/8 BPS operators. We will make use of this type of operators in chapter 7. A detailed discussion of BPS operators can be found in section 3.4 of the review [40].

3.2 Correlation Functions of Scalar Operators

The form of two-, three- and four-point functions of conformal primary operators (2.26) was originally found by Polyakov [110]. The form of the correlators can be deduced by requiring the invariance of the correlation functions under dilatations and inversions, which by virtue of (2.14) guarantees the invariance under special conformal transformations. We give the results in the following sections and review a very simple way to derive the functional form of the correlators in detail in Appendix B.2.

3.2.1 Two-Point Functions

Using the invariance under dilatations and inversions, one can deduce that the two-point function of two scalar conformal primary operators with scaling dimensions Δ_A , Δ_B is

$$\langle \mathcal{O}_A(x_1)\mathcal{O}_B(x_2) \rangle = \delta_{AB} \frac{C_A}{(x_{12}^2)^{\Delta_A}}, \quad (3.13)$$

where C_A is a constant. We derive this result in detail in Appendix B.2.2. It is easy to see, that this two-point function indeed satisfies the Ward-identities (3.5), (3.6).

3.2.2 Three-Point Functions

We show in B.2.3 that three-point functions are also uniquely fixed to the form

$$\langle \mathcal{O}_A(x_1)\mathcal{O}_B(x_2)\mathcal{O}_C(x_3) \rangle = \frac{C_{ABC}}{|x_{12}|^{\Delta_A+\Delta_B-\Delta_C}|x_{13}|^{\Delta_A+\Delta_C-\Delta_B}|x_{23}|^{\Delta_B+\Delta_C-\Delta_A}}, \quad (3.14)$$

where C_{ABC} are called the *structure constants*. As will be explained in section 3.5 these structure constants are same constants that appear in the operator product expansion. It is easy to see, that these three-point functions indeed satisfy the Ward-identities (3.5), (3.6).

3.2.3 Higher-Point Functions

The space-time structure of four- and higher-point correlators is not uniquely fixed by conformal symmetry, since they may depend on arbitrary functions of the conformally invariant cross ratios u_{ijkl} in (2.15).

At four points we can build two of these cross ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \quad (3.15)$$

and for example the four-point function of four scalar fields has the form

$$\langle \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \rangle = \frac{1}{(x_{12}^2 x_{34}^2)^{\Delta_\phi}} \mathcal{G}(u, v), \quad (3.16)$$

where $\mathcal{G}(u, v)$ is an a priori arbitrary function of the conformal cross ratios u and v . It is possible to extract information on the structure of four- and higher-point functions using a *double operator product expansion* [111, 112, 113, 114, 24], see section 3.5 for a short introduction to the operator product expansion.

3.3 Correlation Functions of Operators with Spin

In this thesis we will also consider correlation functions with *spin* j operators³

$$\mathcal{O}_{\mu_1 \dots \mu_j}(x), \quad (3.17)$$

which are totally symmetric in the indices $\{\mu_i\}$ and traceless in the sense

$$g^{\mu_1 \mu_2} \mathcal{O}_{\mu_1 \dots \mu_j}(x) = 0. \quad (3.18)$$

Due to the total symmetry of the operator, the tracelessness applies to every index pair. The *twist* of these operators is defined as the difference of the scaling dimension Δ_j and the spin j of the operator

$$\theta = \Delta_j - j, \quad (3.19)$$

where Δ_j is the scaling dimension of the spin j operator.

³See the comments in section 3.3.4 for the notion of spin.

3.3.1 Two-Point Functions of Spin j Operators

Using the transformation law under dilatations and inversions one can show, see Appendix (B.2.4), that two-point functions of spin j operators must be of the form

$$\langle \mathcal{O}_{\mu_1 \dots \mu_j}(x_1) \mathcal{O}_{\nu_1 \dots \nu_j}(x_2) \rangle = \frac{C_j}{(x_{12}^2)^{\Delta_j}} \left(I_{\{\mu_1 \nu_1 \dots \mu_j \nu_j\}}(x_{12}) - \text{traces} \right), \quad (3.20)$$

where $\{..\}$ indicates that the right-hand side is totally symmetrised in all indices μ_k, ν_l and we subtract all traces in μ_i resp. ν_i in accord with the symmetries of the operators on the left-hand side and where $I_{\mu\nu}(x)$ is the inversion tensor

$$I_{\mu\nu}(x) = \eta_{\mu\nu} - 2 \frac{x_\mu x_\nu}{x^2}. \quad (3.21)$$

3.3.2 Three-Point Functions with Spin j Operators

Consider a three-point function involving two scalar operators $\mathcal{O}_A(x), \mathcal{O}_B(x)$ and one spin j operator $\mathcal{O}_{\mu_1 \dots \mu_j}$. Conformal symmetry restricts these correlators, see Appendix B.2.6, to the form

$$\langle \mathcal{O}_A(x_1) \mathcal{O}_B(x_2) \mathcal{O}_{\mu_1 \dots \mu_j}(x_3) \rangle = \frac{C_{ABj} \left(Y_{\mu_1 \dots \mu_j}(x_{13}, x_{23}) - \text{traces} \right)}{|x_{12}|^{\Delta_A + \Delta_B - \theta} |x_{13}|^{\Delta_A + \theta - \Delta_B} |x_{23}|^{\Delta_B + \theta - \Delta_A}}, \quad (3.22)$$

where $\theta = \Delta_j - j$ is the *twist* of the operator $\mathcal{O}_{\mu_1 \dots \mu_j}$ and

$$Y^\mu(x_{13}, x_{23}) = \frac{x_{13}^\mu}{x_{13}^2} - \frac{x_{23}^\mu}{x_{23}^2} = \frac{1}{2} \partial_{x_3}^\mu \ln \left(\frac{x_{13}^2}{x_{23}^2} \right). \quad (3.23)$$

In agreement with the operators on the left-hand side of (3.22), the right-hand side is symmetric and traceless in the indices.

Three-point functions involving two operators with spin

The reason that conformal symmetry restricts (3.22) up to the structure constant C_{ABj} is, that there is only one possible tensor structure with the correct behaviour under dilatations and inversions. When we have one spin j field $\mathcal{O}_{\mu_1 \dots \mu_j}$ a scalar operator \mathcal{O}_C and a vector operator \mathcal{O}_μ we have two possible tensor structures, which are compatible with the constraints imposed by dilatation and inversion, see e.g. [115], and thus conformal symmetry restricts the correlator to

$$\begin{aligned} \langle \mathcal{O}_{\mu_1 \dots \mu_j}(x_1) \mathcal{O}_C(x_2) \mathcal{O}_\mu(x_3) \rangle &= \frac{1}{|x_{12}|^{\theta_j + \Delta_C - \theta_1} |x_{13}|^{\theta_j + \theta_1 - \Delta_C} |x_{23}|^{\Delta_C + \theta_1 - \theta_j}} \times \\ &\left(A Y_\mu(x_{13}, x_{23}) \left(Y_{\mu_1 \dots \mu_j}(x_{12}, x_{13}) \right) \right. \\ &\left. + \frac{B}{x_{13}^2} \left[\sum_{k=1}^j I_{\mu\mu_k}(x_{13}) Y_{\mu_1 \dots \mu_{k-1}} Y_{\mu_{k+1}} \dots Y_{\mu_j}(x_{12}, x_{13}) - \text{traces} \right] \right), \end{aligned} \quad (3.24)$$

where θ_j is the twist (3.19) of the spin j operator and $\theta_1 = \Delta_{\mathcal{O}_\mu} - 1$ is analogously defined as the twist of the operator \mathcal{O}_μ and where A and B are constants. If we take a spin 2 operator $\mathcal{O}_{\mu\nu}$ instead of \mathcal{O}_μ , there are 3 possible tensor structures and thus dilatation and inversion symmetry determine the correlator only up to these three constants. In general, the higher the rank of the tensors involved, the more possible independent tensor structures that fulfill the invariance conditions arise and each of them can enter with an a priori arbitrary coefficient. For a more detailed discussion also involving correlators with fermionic fields we refer the reader to chapter III.2 of [115] or the review [116] as well as the article [117].

Correlators in Momentum Space

In general, it can be quite complicated to map expressions like (3.22) to momentum space via Fourier transformation. In this thesis, will only make use of the Fourier transformed correlators in a simplifying limit. However, if necessary, also the full position space correlator structures can be mapped to momentum space, see e.g. [118] for some useful tricks and more details.

3.3.3 Light-Cone Projection of Operators

Correlation functions of operators with spin (3.20), (3.22) contain many terms with a repeated space-time structure and therefore in some sense redundant information. Perturbative calculations can be largely simplified by contracting the indices with light-like vectors, i.e.

$$\hat{\mathcal{O}}_j = \mathcal{O}_{\mu_1 \dots \mu_j} z^{\mu_1} \dots z^{\mu_j} \quad \text{where} \quad z^2 = z_\mu z^\mu = 0. \quad (3.25)$$

The operator with the desired symmetry $\mathcal{O}_{\mu_1 \dots \mu_j}$ can be recovered by applying a second order differential operator Δ^μ

$$\Delta^\mu = \left(\frac{d}{2} - 1 + z \cdot \partial_z \right) \partial_z^\mu - \frac{1}{2} z^\mu \partial_z \cdot \partial_z \quad (3.26)$$

in the presence of the constraint $z^2 = 0$, see [119, 120] and an application in [121]. Then, the correct operator can be recovered by

$$\Delta^{\mu_1} \dots \Delta^{\mu_j} \hat{\mathcal{O}}_j \sim \mathcal{O}^{\mu_1 \dots \mu_j}. \quad (3.27)$$

Due to $[\Delta^\mu, \Delta^\nu] = 0$ and $\Delta^\mu \Delta_\mu = 0$, symmetrisation and tracelessness are automatically incorporated. For more information on the light-cone projection see e.g. [122, 123].

We denote spacetime indices that are contracted with z^μ with a *hat*, i.e.

$$\hat{x} = z_\mu x^\mu, \quad \hat{\partial} = z_\mu \frac{\partial}{\partial x_\mu} \quad \Rightarrow \quad \hat{\partial} \hat{x} = z^2 = 0. \quad (3.28)$$

As an example, consider the two-point function (3.20). The first term in (B.28) as well as all trace terms vanish⁴ and we get⁵

$$\langle \hat{\mathcal{O}}_j(x_1) \hat{\mathcal{O}}_j(x_2) \rangle = C_j (-2)^j \frac{(\hat{x}_{12})^{2j}}{(x_{12}^2)^{\Delta_j+j}}. \quad (3.29)$$

Therefore, we have only one term instead of the rapidly growing number (with increasing j) of terms in (3.20). In perturbative calculations, this simplicity is reflected by the fact that $\hat{\partial}\hat{x} = z^2 = 0$ and therefore multiple derivatives $\hat{\partial}$ acting on propagators produce only one term. This largely reduces the effort for the determination of the structure constants C_j, C_{ABj} of two-point and three-point functions with spin as we will see in a practical example in chapter 7. Analogously, the three-point functions (3.22) read

$$\langle \mathcal{O}_A(x_1) \mathcal{O}_B(x_2) \mathcal{O}_{\mu_1 \dots \mu_j}(x_3) \rangle = \frac{C_{ABj} \left(\hat{Y}(x_{13}, x_{23}) \right)^j}{|x_{12}|^{\Delta_A+\Delta_B-\theta} |x_{13}|^{\Delta_A+\theta-\Delta_B} |x_{23}|^{\Delta_B+\theta-\Delta_A}}, \quad (3.30)$$

where $|x_{ij}| = (x_{ij}^2)^{1/2}$. The generalization of (3.22), (3.24) to three operators $\hat{\mathcal{O}}_j, \hat{\mathcal{O}}_k, \hat{\mathcal{O}}_l$ of arbitrary spin can conveniently be written in the light-cone projection, see [117, 115].

3.3.4 Comment on the Notion of Spin and Higher Spin Fields

The operators (3.17) are called spin j fields due to their transformation properties under the Lorentz group and in analogy to the theory of *massive* higher spin fields introduced by Fierz and Pauli [124, 125], which are described by traceless and symmetric tensors $\phi_{\mu_1 \dots \mu_j}$ that satisfy the Klein-Gordon equation $(\partial^2 + m^2)\phi_{\mu_1 \dots \mu_j} = 0$ and are divergence-free $\partial^{\mu_1} \phi_{\mu_1 \dots \mu_j} = 0$.

Massless higher spin fields were shown to be given by tensors that satisfy the weaker statement of double tracelessness of $\phi_{\mu_1 \dots \mu_j}$ derived in [126, 127], i.e. e.g. $\phi_{\mu}^{\mu} \nu_{\dots \nu_j} = 0$. This weakened a previously assumed condition on the source to be divergence-free, while still guaranteeing that only helicities $\pm s$ are transmitted between sources. This encouraged for the search towards a theory of interacting massless higher spin fields. Higher spin theory is an active area of research, an older comprehensive review is [128], for a more recent overview see [129]. Massless [130] and massive [131] higher spin fields were also studied in *AdS* space.

3.4 Renormalisation of Composite Operators

The Callan-Symanzik equation for Green's functions of elementary fields [132, 133, 134] is derived in standard quantum field theory textbooks and we shortly review it here

⁴We take the same light-like vector for both operators $\hat{\mathcal{O}}_j, \bar{\hat{\mathcal{O}}}_j$, i.e. $z_1^\mu = z_2^\mu$.

⁵Here we have redefined C_j and absorbed the combinatorical factor from the permutations in (3.20). In general we have $j!$ terms of type $I_{\mu_i \nu_j}$ and additionally all traces that are subtracted, which do however vanish in the lightcone-projection.

and comment on the generalization for operators and operator mixing.

3.4.1 Renormalisation of Elementary Fields

In quantum field theory calculations, Green's functions

$$G_n = \langle \Phi(x_1) \dots \Phi(x_n) \rangle \quad (3.31)$$

are typically divergent at loop-level and the original (bare) fields (and couplings) are redefined as

$$\Phi^R(x) = Z_\Phi \Phi(x) \quad (3.32)$$

in order to absorb the divergences of loop calculations. These *renormalisation constants* depend on the cutoff and the renormalisation scale μ that preserves the canonical dimension of the fields and consequently the renormalised Green's function depends on the same parameters

$$G_n^R(x, \mu, g) = \langle \Phi^R(x_1) \dots \Phi^R(x_n) \rangle = (Z_\Phi)^n \langle \Phi(x_1) \dots \Phi(x_n) \rangle. \quad (3.33)$$

Since the bare Greens function G_n does not depend on the scale μ , we have

$$0 = \mu \frac{d}{d\mu} (Z_\Phi^{-n} G_n^R) = Z_\Phi^{-n} \left(-n\mu \frac{d}{d\mu} \ln Z_\Phi + \mu \frac{\partial}{\partial \mu} + \mu \frac{\partial g}{\partial \mu} \frac{\partial}{\partial g} \right) G_n^R. \quad (3.34)$$

Defining

$$\beta(g) = \mu \frac{\partial g}{\partial \mu}, \quad \gamma = -\mu \frac{d}{d\mu} \ln Z_\Phi, \quad (3.35)$$

we get the standard Callan-Symanzik renormalisation group equation

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} + n\gamma \right) G_n^R = 0, \quad (3.36)$$

where $\beta(g)$ determines the dependence of the running coupling $g(\mu)$ on the scale μ and γ is called the *anomalous dimension* of the field Φ , see the discussion in section 3.1.2.

3.4.2 Renormalisation of Composite Operators

The limit where two points in the Green's functions (3.33) approach each other may lead to additional divergences and therefore, in general, local composite operators require a renormalisation constant $Z_\mathcal{O} \neq (Z_\Phi)^2$

$$\mathcal{O}^R(x) = Z_\mathcal{O} \mathcal{O}(x) = Z_\mathcal{O} \Phi(x) \Phi(x) \quad (3.37)$$

in order to obtain finite correlation functions. The analogous renormalisation group equation with m such operators then reads

$$\left[\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} + n\gamma + m\gamma_\mathcal{O} \right] \langle \Phi^R(x_1) \dots \Phi^R(x_n) \mathcal{O}^R(y_1) \dots \mathcal{O}^R(y_m) \rangle = 0, \quad (3.38)$$

where the anomalous dimension $\gamma_{\mathcal{O}}$ of the composite operator is defined in the same way as for elementary fields by

$$\gamma(g) = -\mu \frac{d}{d\mu} \ln Z_{\mathcal{O}}. \quad (3.39)$$

Typically, *operator mixing* is present, i.e. the renormalised operators are linear combinations of other operators and we need a *renormalisation matrix* Z_{jk}

$$\mathcal{O}_i^R = \sum_j Z_{ij} \mathcal{O}_j \quad (3.40)$$

in order to define finite Green's functions. The operators \mathcal{O}_j on the right-hand side of (3.40) need to have the same quantum numbers and equal or smaller canonical dimensions as compared to the original operator. Correspondingly, we get an *anomalous dimension matrix*

$$(\gamma_{\mathcal{O}})_{ij} = -(Z_{\mathcal{O}}^{-1})_{ik} \mu \frac{d}{d\mu} (Z_{\mathcal{O}})_{kj}. \quad (3.41)$$

The renormalisation matrix can be expanded in the coupling constant as

$$Z_{jk} = \delta_{jk} + \sum_{n=1}^{\infty} g^{2n} \sum_{m=1}^n \frac{Z_{jk}^{[m](n)}}{\epsilon^m}. \quad (3.42)$$

3.4.3 Conformal Scheme

From the perspective of the Ward identities, the mixing is reflected by the appearance of the anomalous dimension matrix γ_{nm} and the so-called special conformal anomaly matrix γ_{nm}^c in the anomalous version of the Ward identities (3.5), (3.6), originating from the right-hand side of (3.7). These anomalies spoil the conclusions drawn in section 3.2, 3.3 on the form of the correlation functions of conformal operators. Formulated differently, the renormalised operators (3.42) yield finite correlation functions, but are not conformally covariant in the sense (2.27) and thus do not have conformal correlation functions.

One can however define a *conformal renormalisation scheme* by applying another finite scheme transformation [107]

$$\mathbb{O}_i = \sum_{j=0}^i B_{ij}^{-1} \mathcal{O}_j^R, \quad (3.43)$$

where the finite matrix B diagonalises the anomalous dimension matrix

$$\gamma_j(g^2) \delta_{jk} = (B^{-1} \gamma B)_{jk} \quad (3.44)$$

and has the expansion

$$B_{jk} = \delta_{jk} + \sum_{n=1}^{\infty} g^{2n} B_{jk}^{(n)}. \quad (3.45)$$

Then the full mixing matrix \mathbb{Z}_{jk} that defines conformal operators at loop-level reads

$$\mathbb{O}_j = \sum_k \mathbb{Z}_{jk} \mathcal{O}_k = \sum_k \sum_{m=k}^j B_{jm}^{-1} Z_{mk} \mathcal{O}_k. \quad (3.46)$$

Renormalisation of Twist-Two Operators

In chapter 7 we will encounter the renormalisation of so-called twist-two operators and see that the renormalisation matrix is diagonal at one loop, i.e.

$$Z_{jk}^{1} = \delta_{jk} Z_j^{(1)}. \quad (3.47)$$

At one-loop level there are however two sources of non-diagonality of the two-point functions. On the one hand, there are finite non-diagonal one-loop corrections

$$\langle \mathcal{O}_j \mathcal{O}_l \rangle^{(1)} = \text{finite} \neq 0 \quad \text{for } j \neq l \quad (3.48)$$

and on the other hand the minimal subtraction procedure (3.40), (3.42) leads to non-diagonal contributions, if the operators $\hat{\mathcal{O}}_j$ are not orthogonal at order ϵ . Then we have

$$\langle \hat{\mathcal{O}}_j^R \hat{\mathcal{O}}_l^R \rangle = \left(1 + g^2 \frac{Z_j^{(1)} + Z_l^{(1)}}{\epsilon} \right) \left(\delta_{jl} \langle \hat{\mathcal{O}}_j \hat{\mathcal{O}}_l \rangle^{(0)} + \epsilon \langle \hat{\mathcal{O}}_j \hat{\mathcal{O}}_l \rangle^{(\epsilon)} \right) + \langle \hat{\mathcal{O}}_j \hat{\mathcal{O}}_l \rangle^{(1)}, \quad (3.49)$$

where the index (ϵ) on the correlator denotes the $\mathcal{O}(\epsilon)$ part of the tree-level correlator. In order to define orthogonal operators at loop-level, we need to perform another finite renormalisation of the operators and take into account the mixing with descendants of lower dimension $\hat{\mathcal{O}}_{jk} = \hat{\partial}^{j-k} \hat{\mathcal{O}}_k$, such that the full mixing matrix \mathbb{Z}_{jk} that yields finite orthogonal operators is

$$\hat{\mathcal{O}}_j = \sum_k \mathbb{Z}_{jk} \hat{\mathcal{O}}_{jk} = \sum_k \sum_{m=k}^j B_{jm}^{-1} Z_{mk} \hat{\mathcal{O}}_{jk}, \quad \hat{\mathcal{O}}_{jk} = \hat{\partial}^{j-k} \hat{\mathcal{O}}_k. \quad (3.50)$$

The one-loop expansion of the renormalisation constant then reads

$$\mathbb{Z}_{jk} = \delta_{jk} + g^2 \underbrace{\left(-B_{jk}^{(1)} + \frac{1}{\epsilon} \delta_{jk} Z_j^{(1)} \right)}_{=Z_{jk}^{(1)}} + \mathcal{O}(g^4). \quad (3.51)$$

Thus at one loop the renormalised two-point function has the expansion

$$\langle \hat{\mathcal{O}}_i \hat{\mathcal{O}}_j \rangle = \sum_{l,k} \left(\delta_{il} + g^2 \left(-B_{il}^{(1)} + \frac{1}{\epsilon} \delta_{il} Z_i^{(1)} \right) \right) \left(\delta_{jk} + g^2 \left(-B_{jk}^{(1)} + \frac{1}{\epsilon} \delta_{jk} Z_j^{(1)} \right) \right) \langle \hat{\mathcal{O}}_{il} \hat{\mathcal{O}}_{jk} \rangle.$$

The matrix B accounts for the non-diagonal terms that arise from the one-loop diagrams and the $\mathcal{O}(\epsilon)$ expansion of the correlator and can be determined by requiring these terms to vanish.

3.5 Operator Product Expansion in Conformal Field Theory

K. G. Wilson proposed [135] that the effects of an operator product $\mathcal{O}_\alpha(x)\mathcal{O}_\beta(0)$ in the limit $x \rightarrow 0$ could be computed by replacing the product by a linear combination of local operators

$$\mathcal{O}_\alpha(x)\mathcal{O}_\beta(0) \longrightarrow \sum_{\gamma} C_{\alpha\beta\gamma}(x)\mathcal{O}_\gamma(0), \quad (3.52)$$

where the local operators \mathcal{O}_γ must have the global symmetry quantum numbers of the product $\mathcal{O}_\alpha\mathcal{O}_\beta$. The coefficient functions $C_{\alpha\beta\gamma}(x)$ are c-number valued functions and are often called *Wilson coefficients* and the corresponding operators *Wilson operators*. On dimensional grounds, in a conformal field theory, the coefficient functions $C_{\alpha\beta\gamma}$ must be of the form⁶

$$C_{\alpha\beta\gamma}(x) = \frac{C_{\alpha\beta\gamma}}{|x|^{\Delta_\alpha+\Delta_\beta-\Delta_\gamma}} \quad (3.53)$$

and therefore we can write

$$\mathcal{O}_\alpha(x_1)\mathcal{O}_\beta(x_2) \xrightarrow{x_1 \rightarrow x_2} \sum_{\gamma} \frac{C_{\alpha\beta\gamma}}{|x_{12}|^{\Delta_\alpha+\Delta_\beta-\Delta_\gamma}} \mathcal{O}_\gamma(x_2). \quad (3.54)$$

It is now easy to see why the coefficients $C_{\alpha\beta\gamma}$ are just the structure constants of the three-point functions in conformally invariant theories. Taking the limit $x_1 \rightarrow x_2$ in the conformal three-point function (3.14) yields

$$\langle \mathcal{O}_\alpha(x_1)\mathcal{O}_\beta(x_2)\mathcal{O}_\gamma(x_3) \rangle \xrightarrow{x_1 \rightarrow x_2} \frac{C_{\alpha\beta\gamma}}{|x_{12}|^{\Delta_A+\Delta_B-\Delta_C}|x_{23}|^{2\Delta_\gamma}}. \quad (3.55)$$

On the other hand, in the same limit, we can rewrite the first two operators in terms of the OPE and find

$$\begin{aligned} \langle \mathcal{O}_\alpha(x_1)\mathcal{O}_\beta(x_2)\mathcal{O}_\gamma(x_3) \rangle &\xrightarrow{x_1 \rightarrow x_2} \sum_{\delta} \frac{C_{\alpha\beta\delta}}{|x_{12}|^{\Delta_\alpha+\Delta_\beta-\Delta_\delta}} \langle \mathcal{O}_\delta(x_2)\mathcal{O}_\gamma(x_3) \rangle \\ &= \sum_{\delta} \frac{C_{\alpha\beta\delta}}{|x_{12}|^{\Delta_\alpha+\Delta_\beta-\Delta_\delta}} \frac{\delta_{\delta\gamma}}{|x_{23}|^{2\Delta_\delta}} \\ &= \frac{C_{\alpha\beta\gamma}}{|x_{12}|^{\Delta_\alpha+\Delta_\beta-\Delta_\delta}|x_{23}|^{2\Delta_\gamma}}, \end{aligned} \quad (3.56)$$

where we used the conformal two-point function (3.13) in the second line and assumed that the operators are normalised as $C_\delta = 1$. Thus the coefficient $C_{\alpha\beta\gamma}$ coincides with the structure constant in (3.55).

Above, we have restricted to a basis of scalar operators, in general however the complete set of operators also involves operators with spin (3.17). The complete operator product expansion can then be organised in terms of *Wilson operators* with increasing

⁶We restrict to scalar operators here to illustrate the general idea.

twist. The *leading* or *lowest twist operators* in $d = 4$ are *twist-two operators* which can schematically be written as

$$\mathcal{O}_{\mu_1 \dots \mu_j}(x) = D_{\mu_1} \dots D_{\mu_i} \Phi(x) D_{\mu_{i+1}} \dots D_{\mu_j} \Phi(x) + \dots \quad (3.57)$$

These composite operators transform covariantly under conformal transformations only for specific distributions of the covariant derivatives and we will be more specific on their form in chapter 7. The coefficient functions (3.53) of these operators in the OPE have to be generalised appropriately to be compatible with the form of the three-point function (3.22), see [24] for more information.

As mentioned before, using the OPE (3.56) twice, one can extract information on higher-point functions. Schematically, the four-point functions (3.16) can be written

$$\langle \mathcal{O}_\alpha(x_1) \mathcal{O}_\beta(x_2) \mathcal{O}_\gamma(x_3) \mathcal{O}_\delta(x_4) \rangle \sim \sum_\sigma \frac{C_{\alpha\beta\sigma} C_{\gamma\delta\sigma}}{|x_{12}|^{\Delta_\alpha + \Delta_\beta - \Delta_\delta} |x_{34}|^{\Delta_\gamma + \Delta_\delta - \Delta_\delta} |x_{23}|^{2\Delta_\sigma}} \cdot C_\sigma \quad (3.58)$$

In [136] it is for example shown, that a four-point two-loop correlator of 1/2-BPS operators can be completely reconstructed on the basis of the OPE and the knowledge of the one-loop anomalous dimensions for certain operators of twist 2 and 4.

The OPE, together with the knowledge of the anomalous dimensions, in principle allows to determine any higher-point function. Chapter 7 is devoted to the investigation of the coefficients of twist-two operators in the OPE.

3.5.1 OPE and Twist-Operators in QCD

Here, we would like to give an example for an application of the operator product expansion and show where *leading twist operators* in deep inelastic scattering show up, following to a large extent chapter 18 of [137]. We only sketch the ideas and refer the reader to [137] for more details.

Deep Inelastic Electron Scattering

Deep inelastic electron scattering is described by the parton model, in which the electron scatters from quarks carrying fractions of the total momentum of the proton, which are determined by parton distribution functions.

The amplitude $\mathcal{M}(ep \rightarrow eX)$ for the scattering of an electron and a proton into an electron and a final state X can be related to the forward matrix element of two quark electromagnetic currents $J^\mu(x) = \bar{q}(x)\gamma^\mu q(x)$ via the optical theorem. The total cross section averaged over initial and final electron spins then reads

$$\sigma(ep \rightarrow eX) = \frac{1}{2s} \int \frac{d^3k'}{(2\pi)^3} \frac{1}{2k'} e^4 \frac{1}{2} \sum_{\text{spins}} \bar{u}(k)\gamma_\mu u(k') \bar{u}(k')\gamma_\nu u(k) \cdot \left(\frac{1}{Q^2}\right)^2 2\text{Im}(W^{\mu\nu}),$$

where the *forward Compton amplitude* is

$$W^{\mu\nu} = i \int d^4x e^{iq \cdot x} \langle P | T \{ J^\mu(x) J^\nu(0) \} | P \rangle, \quad (3.59)$$

and $|P\rangle$ represents the proton state of momentum P^μ see [137] for details. Here we will illustrate the use of the OPE to analyse this amplitude. As mentioned before, in a massless theory the operators appearing on the right-hand side of the operator product expansion (3.52) must have the same symmetry quantum numbers as the product of operators on the left-hand side.

Dominant contributions from leading twist operators

In d dimensions the scaling dimension of a quark (fermion) is $[q(x)] = (d-1)/2$ and thus the dimension of the quark electromagnetic current is $[J^\mu] = d-1$. Thus, in the operator product expansion of the current product

$$J^\mu(x)J^\nu(0) = \sum C_i(x)\mathcal{O}_i(0) \quad (3.60)$$

an operator \mathcal{O}_i with dimension Δ_i must have a coefficient function of dimension $[C_i] = 2(d-1) - \Delta_i$. In a massless field theory, the dimension is fully carried by the x -dependence and we thus have

$$C_i(x) = c_i \left(\frac{1}{-x^2} \right)^{d-1-\Delta_i/2}, \quad (3.61)$$

where c_i is a dimensionless coefficient. The Fourier transformation in (3.59) thus yields, that the coefficient function is suppressed by a factor of

$$\left(\frac{1}{Q^2} \right)^{\Delta_i/2-1}, \quad Q^2 = -q^2, \quad (3.62)$$

for $d=4$. If we consider an operator with spin j of the form⁷

$$\mathcal{O}_j^{\mu_1 \dots \mu_j}(x) = \bar{q}(x)\gamma^{\{\mu_1}(iD^{\mu_2}) \dots (iD^{\mu_j}\}q(x) - \text{traces}, \quad (3.63)$$

we get additional kinematic factors from the matrix element of the operator

$$\langle P | \mathcal{O}_j^{\mu_1 \dots \mu_j} | P \rangle \sim P^{\mu_1} \dots P^{\mu_j} - \text{traces}. \quad (3.64)$$

Thus, the contribution of a spin j operator will be of order

$$\left(\frac{2P \cdot q}{Q^2} \right)^j \left(\frac{1}{Q^2} \right)^{(\Delta_j - j - 1)/2}. \quad (3.65)$$

The quantity $Q^2/2P \cdot q = x$ is held fixed in deep inelastic scattering. Thus, the relative size of the contribution from the OPE operators to deep inelastic scattering does not depend on the dimension but rather on the *twist* θ of the operator

$$\theta = \Delta_j - j. \quad (3.66)$$

Therefore, the *leading twist operators (smallest twist)* give the most important contributions to the OPE for deep inelastic scattering. We will consider operators of this type in chapter 7.

⁷Here $\{\dots\}$ means symmetrisation over all indices and all possible traces are subtracted.

Chapter 4

Amplitudes, Wilson Loops and Correlators in $\mathcal{N} = 4$ SYM

In this chapter we give an overview about some recent developments in quantum field theory, which have mostly been inspired by the AdS/CFT correspondence. In the following chapter we will describe some of the analogous developments in ABJM theory. In particular, we will focus on the so-called amplitude / Wilson loop duality.

In order to state the duality, we will introduce the necessary background on amplitudes in $\mathcal{N} = 4$ SYM in section 4.1 and shortly comment on the methods that are presently being used to calculate amplitudes, such as the spinor helicity formalism, BCFW recursions and unitarity cuts. In section 4.1.5 we will discuss the so-called *BDS ansatz*, which constitutes an all-loop proposal for scattering amplitudes in $\mathcal{N} = 4$ SYM based on an iterative structure in perturbative amplitude calculations and the previously known exponentiation of the IR divergences of gluon scattering amplitudes.

Unexpectedly, the scattering amplitudes exhibit a hidden symmetry called *dual conformal symmetry* distinct from the ordinary conformal symmetry of the theory. This symmetry extends to a *dual superconformal symmetry* and together with the superconformal symmetry forms a *Yangian symmetry*.

In section 4.2 we introduce the *duality between amplitudes and light-like polygonal Wilson loops* and see that the dual conformal symmetry of the amplitudes can be understood in terms of the conformal symmetry of the Wilson loop defined in a *dual configuration space*. The expectation value of the Wilson loop is governed by an *anomalous conformal Ward identity* which fixes the result of the Wilson loop and, via the duality, the result of the amplitudes for $n = 4, 5$ edges resp. gluons completely. For $n \geq 6$ particles the Ward identity admits for arbitrary functions of conformally invariant cross ratios. Interestingly, even though the BDS ansatz for scattering amplitudes breaks down starting from $n = 6$ at two loops, the duality continues to hold, thus also the part, which is not constrained by the known symmetries of the two objects, continues to agree, showing that there is more to the amplitude / Wilson loop duality than just dual conformal symmetry. We will also have a glimpse on the generalization of the duality to superamplitudes and super Wilson loops in section 4.2.3 as well as a

new *triality* between correlators, amplitudes and Wilson loops and its generalization to supersymmetric objects in section 4.3. We also comment on relations between form factors and periodic Wilson loops in section (4.4).

The duality between scattering amplitudes and Wilson loops has its origin in the strong coupling calculation of gluon amplitudes [3, 4] in terms of minimal surfaces in AdS space, which formally resembles the calculation of a Wilson loop. The analogue of the dual superconformal symmetry on the string theory side is an invariance of the free $AdS_5 \times S^5$ superstring under a combination of bosonic and fermionic T-dualities [138, 139, 140]. The T-duality exchanges the original with the dual superconformal symmetries and thus explains the presences of the dual symmetry.

In the following we focus on the weak coupling side of the duality and refer the reader to the literature for more information on the strong coupling side of the duality.

4.1 Scattering Amplitudes in $\mathcal{N} = 4$ SYM

It is possible to show [141, 142], see also the review [28], that a n -gluon tree-level amplitude can be written as

$$A_n(\{p_i, h_i, a_i\}) = \sum_{\sigma \in S_n/Z_n} \text{Tr}(T^{a_{\sigma(1)}}, \dots, T^{a_{\sigma(n)}}) A_n(\sigma(1)^{h_{\sigma(1)}}, \dots, \sigma(n)^{h_{\sigma(n)}}), \quad (4.1)$$

where p_i denote the momenta, $h_i = \pm 1$ the helicities and a_i the colour index of particle i and the sum goes over all non-cyclic permutations σ . The amplitudes A_n are called *colour ordered* or *partial amplitudes*. In order to obtain the full result for the amplitude it is thus sufficient to evaluate all colour ordered amplitudes using the simpler *colour ordered Feynman rules*, see [28]. In the following we restrict to the discussion of colour ordered amplitudes and just call them amplitudes from now on.

4.1.1 Spinor Helicity Formalism

The *spinor helicity formalism* is reviewed e.g. in [26, 28] and we will not give a detailed survey of this method, but just introduce a few terms and notations which are useful for the following discussion of amplitudes.

The basic idea of the spinor helicity formalism is to reformulate n -point scattering amplitudes A_n of n massless particles in terms of commuting, complex two-component Weyl spinors $\lambda_\alpha, \tilde{\lambda}_{\dot{\alpha}}$ which automatically satisfy the masslessness condition $p_i^2 = 0$. They can be defined by

$$p_i^{\alpha\dot{\alpha}} = p_i^\mu (\sigma_\mu)^{\alpha\dot{\alpha}} = \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}}, \quad (4.2)$$

where $\sigma^\mu = (\mathbf{1}, \vec{\sigma})$ and $\vec{\sigma}$ are the three Pauli matrices and indices are lowered and raised using the antisymmetric $\epsilon_{\alpha\beta} = -\epsilon^{\alpha\beta}$, $\epsilon_{\dot{\alpha}\dot{\beta}} = -\epsilon^{\dot{\alpha}\dot{\beta}}$, i.e. $\lambda^\alpha = \epsilon^{\alpha\beta} \lambda_\beta$. One can introduce the Lorentz invariant¹ antisymmetric contractions

$$\langle ij \rangle = (\lambda_i)^\alpha \epsilon_{\alpha\beta} (\lambda_j)^\beta = (\lambda_i)^\alpha (\lambda_j)_\alpha, \quad [ij] = (\tilde{\lambda}_i)_{\dot{\alpha}} \epsilon^{\dot{\alpha}\dot{\beta}} (\tilde{\lambda}_j)_{\dot{\beta}} = (\tilde{\lambda}_i)_{\dot{\alpha}} (\tilde{\lambda}_j)^{\dot{\alpha}}, \quad (4.3)$$

¹With respect to $SU(2)_L, SU(2)_R$

which satisfy $\langle ij \rangle [ij] = -2 p_i \cdot p_j$. Introducing suitable polarisation vectors $\epsilon_{i,\mu}^+, \epsilon_{i,\mu}^-$ for gluons of positive and negative helicity, one can evaluate gluon scattering amplitudes in the spinor helicity formalism purely in terms of (4.3).

MHV and non-MHV Amplitudes

Amplitudes with n -gluons vanish if all n or $n - 1$ gluons have the same helicity. The simplest non-trivial amplitudes are the ones where at most $n - 2$ gluons have the same helicity. These amplitudes are called *maximally helicity violating* (MHV) amplitudes for 2 negative helicities ($\overline{\text{MHV}}$ for 2 positive helicities) and have the surprisingly simple form [143, 144]

$$A_n^{\text{MHV,tree}}(i^-, j^-) = i \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle} \delta^{(4)}(\sum_i p_i), \quad (4.4)$$

where particles i, j have negative helicity and all others have positive helicity. For real momenta² the first non-trivial MHV amplitude is the amplitude with four particles. Amplitudes with $n - 3$ particles of positive helicity are called next-to-MHV (NMHV), amplitudes with $n - 2 - k$ positive helicities $N^k\text{MHV}$.

4.1.2 BCFW Recursion

Formula (4.4) can be justified recursively, see e.g. [26], by first calculating the three-point amplitude in complex kinematics and using (4.4) for $n - 1$ to show its correctness for n . The reason that this is possible is that one can factorise the n -point amplitude into a three-point and a $(n - 1)$ -point MHV amplitude by shifting the complex momenta (4.2) of two particles in a way that preserves momentum conservation and light-likeness: $p_i = \lambda_i \tilde{\lambda}_i \rightarrow (\lambda_i + z \lambda_j) \tilde{\lambda}_i$, $p_j = \lambda_j \tilde{\lambda}_j \rightarrow \lambda_j (\tilde{\lambda}_j - z \tilde{\lambda}_i)$, where z is a complex number. Choosing e.g. $i = 1, j = n$ the amplitude (4.4) becomes:

$$A_n(z) = i \frac{\langle ij \rangle^4}{(\langle 12 \rangle + z \langle n2 \rangle) \langle 23 \rangle \dots \langle n1 \rangle}. \quad (4.5)$$

The expression $A(z)/z$ has poles at $z = 0$ and $z = -\langle 12 \rangle / \langle n2 \rangle$. Therefore, by Cauchy's theorem, the vanishing³ contour integral

$$\oint \frac{dz}{2\pi i} \frac{A(z)}{z} = 0 = \sum_{z_k} \text{Res}(A(z)/z, z = z_k) \quad (4.6)$$

is equivalent to the sum over the residues of these poles. The residue at $z = 0$ is just the original amplitude, which can thus be expressed through the value of the other residue

$$A_n = A_n(z = 0) = -\text{Res}(A_n(z)/z, z = -\langle 12 \rangle / \langle n2 \rangle). \quad (4.7)$$

²For complex momenta one can also define three-particle amplitudes. For real momenta masslessness and momentum conservation forces these three momenta to be collinear and thus all scalar products $p_i \cdot p_j$ are zero.

³There are some conditions on the shifts such that the contour-integral at infinity vanishes.

Evaluating the right-hand side one finds, see [26] for more details, that the amplitude factorises into a three-point and a $(n - 1)$ -point MHV amplitude, which can be used to confirm (4.4).

The same trick can be used to construct recursions for non-MHV amplitudes and is known as the *BCFW recursion* [145, 146], however there will in general be more terms on the right-hand side, corresponding to different factorization channels. There are also BCFW type recursions for processes involving all sorts of massless and also massive fields, see e.g. the references in [26]. Furthermore, there is a supersymmetric version of the tree-level recursion [147, 148, 149] that was solved in [150]. A nice review on these methods can be found in [151].

Superamplitudes

One can also find a compact formula for the supersymmetric amplitude which contains all possible tree-level MHV amplitudes involving gluons $G^\pm(p)$, gluinos $\Gamma(p), \bar{\Gamma}(p)$ and scalars $S_{AB}(p)$. It is convenient to combine the fields of different helicity in a superstate $\Phi(\eta, p)$

$$\Phi = G^+ + \eta^A \Gamma_A + \frac{1}{2!} \eta^A \eta^B S_{AB} + \frac{1}{3!} \eta^A \eta^B \eta^C \epsilon_{ABCD} \bar{\Gamma}^D + \frac{1}{4!} \eta^A \eta^B \eta^C \eta^D \epsilon_{ABCD} G^-, \quad (4.8)$$

where η^A are anti-commuting Grassmann parameters and $A = 1\dots 4$ is the $SU(4)$ index. The superamplitude can then be written [152] as

$$\mathcal{A}_n^{\text{MHV}}(\Phi_1, \dots, \Phi_n) = i \frac{\delta^{(4)}(p) \delta^{(8)}(q)}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle}, \quad (4.9)$$

where $\delta(p) = \delta(\sum_i \lambda_i \tilde{\lambda}_i)$ is the momentum conserving delta function that also appears in (4.4) and $\delta(q) = \prod_{A,\alpha} (\sum_i \eta_i^A \lambda_i^\alpha)$ can be thought of as a super-momentum conserving delta function. A particular component amplitude is then the coefficient of the term in (4.9) that contains the powers of the Grassmann variables η_i^A corresponding to the component fields in (4.8). E.g. the MHV gluon amplitude (4.4) is the coefficient⁴ of $(\eta_i)^4 (\eta_j)^4$, since gluons i and j have negative helicity.

As mentioned before, the recursions can also be used to construct all non-MHV amplitudes [150]. The non-MHV amplitudes can be included in the expression by multiplying (4.9) with a factor $\mathcal{P}_n(\lambda_i, \tilde{\lambda}_i, \eta_i) = 1 + \mathcal{P}_n^{\text{NMHV}} + \dots$ [150] and one can extract the desired amplitude as the coefficient of the corresponding powers of η_i , exactly as in the MHV case. The recursions and their solutions [150] can conveniently be implemented into computer programs, e.g. the MATHEMATICA package GGT [153] yields compact analytical formulas for all tree-level colour ordered gauge theory amplitudes (MHV and non-MHV) with n gluons and up to four massless quark-anti-quark pairs.

⁴More correctly, we should write $(\eta_i)^4 = \eta_i^1 \eta_i^2 \eta_i^3 \eta_i^4$ since each Grassmann variable can of course appear at most once.

At loop-level the most general form of the amplitude is

$$\begin{aligned} \mathcal{A}_n(\Phi_1, \dots, \Phi_n) &= \mathcal{A}_n^{\text{MHV}} + \mathcal{A}_n^{\text{NMHV}} + \dots + \mathcal{A}_n^{\text{N}^k\text{MHV}} \\ &= i \frac{\delta^{(4)}(p)\delta^{(8)}(q)}{\langle 12 \rangle \dots \langle n1 \rangle} \left(A_{n;0}(\lambda, \tilde{\lambda}, \eta, a) + \dots + A_{n;n-4}(\lambda, \tilde{\lambda}, \eta, a) \right) \end{aligned} \quad (4.10)$$

and $A_{n;k}$ contains the coefficients $\mathcal{P}_n^{\text{N}^k\text{MHV}}$ as well as their loop corrections as a series in the coupling constant a . The loop corrections to the MHV amplitude are usually designated by $M_n \equiv A_{n;0}$ in the literature, i.e.

$$\mathcal{A}_n^{\text{MHV}}(\Phi_1, \dots, \Phi_n) = i \frac{\delta^{(4)}(p)\delta^{(8)}(q)}{\langle 12 \rangle \dots \langle n1 \rangle} M_n. \quad (4.11)$$

Superconformal invariance of the amplitudes

The tree-level amplitudes defined by (4.9) and their generalization to non-MHV amplitudes are invariant under the ordinary superconformal symmetry, i.e.

$$j_a \mathcal{A}_n = 0, \quad (4.12)$$

where $j_a \in \{p^{\alpha\dot{\alpha}}, q^{\alpha A}, \bar{q}_A^{\dot{\alpha}}, m_{\alpha\beta}, \bar{m}_{\dot{\alpha}\dot{\beta}}, r_B^A, d, s_A^\alpha, \bar{s}_\alpha^A, k_{\alpha\dot{\alpha}}\}$ are the generators of the ordinary superconformal symmetry $psu(2, 2|4)$, which satisfy the commutation relations given in section 2.4.1. An explicit representation of these generators in *on-shell superspace* $(\lambda_i^\alpha, \tilde{\lambda}_i^{\dot{\alpha}}, \eta_i^A)$ can be found in [10].

The ordinary conformal symmetry is broken at loop-level by infrared divergences and furthermore it is broken through the *holomorphic anomaly* already at tree-level due to collinear singularities [154, 155, 156, 157]. The symmetry is superficially broken at singular particle configurations where massless particles become collinear, but can be deformed to account for the holomorphic anomaly [156]. At loop-level the situation is more complicated because particles in the loop integration can become collinear with others and thus proper treatment is required [156, 158, 159].

4.1.3 Generalised Unitarity

The *state-of-the-art technique*⁵ for the evaluation of gauge theory and gravity amplitudes at loop-level is the method of *generalized unitarity*. *Unitarity methods* have a long history, see e.g. [180] for applications in the 1960s, and have become a well-developed powerful method for the calculation of loop amplitudes. Since we will not make use of this method in this thesis we refer the reader to the reviews [26, 160] and just mention some of the key concepts.

⁵Some applications and developments of the unitarity methods were listed in [160] and we recite them as further reading suggestions [161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179].

The unitarity method relies on the unitarity of the S-matrix $S = 1 + iT$, which turns into an equation for the non-trivial part of the scattering matrix T

$$SS^\dagger = 1 \quad \Rightarrow \quad TT^\dagger = -i(T - T^\dagger). \quad (4.13)$$

Since the scattering matrix has an expansion in the coupling constant, from the last equality it follows that information on the amplitude is encoded in the product of lower loop amplitudes. It is important to note, that the matrix multiplication TT^\dagger implicitly contains a sum over all possible intermediate on-shell states, i.e. also a phase space integral⁶.

The discontinuity on the left-hand side of (4.13) in a given channel⁷ of a particular loop-level amplitude can be obtained [181] by replacing two propagators that carry this invariant with a delta function

$$\frac{i}{p^2 + i\epsilon} \rightarrow 2\pi\delta^{(+)}(p^2), \quad (4.14)$$

which reduces the loop integral to a phase space integral, and the amplitude is *cut* into a product of two on-shell amplitudes exactly as stated by (4.13). One could now continue to solve the phase space integral, but the idea of the unitarity method is to turn the step (4.14) around:

For a given channel, one starts from the left-hand side of (4.13), i.e. one sews together all tree amplitudes by converting the phase space integral into a full loop integral using (4.14) backwards. The key point is, that the full amplitude is a linear combination of a restricted set⁸ of scalar basis integrals \mathcal{I}_j

$$A_n = \sum_j c_j \mathcal{I}_j, \quad (4.15)$$

where c_j are the coefficients of the integrals in the amplitude. By the above procedure of *sewing together* tree amplitudes to loop amplitudes, one can thus directly read off the coefficients c_j of the corresponding integrals. By considering a certain cut, one can thus determine all coefficients of the integrals that have the corresponding cut. Of course, by this method only those contributions which have these cuts can be detected. Amplitudes that fall into this class are called *cut constructible* and were identified in [161], e.g. $\mathcal{N} = 4$ SYM amplitudes belong to this class.

This method is more efficient than considering single Feynman diagrams, because at every step one is dealing with gauge invariant physical amplitudes and does not need to consider diagrams which cancel in the end. Furthermore, certain tree-level properties of on-shell methods can be carried over to loop-level, see e.g. [160].

⁶To see this, one can write explicit matrix indices $\sum_j T_{ij}(T^\dagger)_{jf} = -i(T - T^\dagger)_{if}$ where i, f denote a physical initial resp. final state and the sum over j is the sum over all possible physical on-shell states, i.e. all possible particles with all possible momentum configurations. This is the origin of the phase space integral.

⁷A momentum invariant, e.g. $s = (p_1 + p_2)^2$.

⁸In $\mathcal{N} = 4$ SYM the set is further reduced by requiring dual conformal invariance, see section 4.1.6.

To understand the details of this method it is instructive to review the calculation of the one-loop n -gluon MHV amplitude in $\mathcal{N} = 4$ SYM which can be found in the lectures [28].

As mentioned, by considering the cut in a certain channel, after some algebra one finds an expression $\sum_j c_j \mathcal{I}_j$ with all basis integrals \mathcal{I}_j that contain the cut⁹. Several integrals may contribute to the same cut. One can restrict the number of appearing integrals, by considering *multiple cuts* of an integral, i.e. one sews together a loop integral from more than two amplitudes. The advantage of this method of *generalised unitarity* is that one has less algebra to perform, since less integrals contribute. Cutting the maximal number of propagators, i.e. four propagators, selects a single box integral and no further algebra is required. This goes under the name *maximal generalised unitarity*, see also the review [160].

4.1.4 One-Loop Four-Point Amplitude

The simplest loop-level gluon amplitude that we can consider in $\mathcal{N} = 4$ SYM is the one-loop four-point amplitude. Calculating all Feynman diagrams or using the unitarity method described in 4.1.3 and evaluating the cuts in the s and t -channel one finds

$$A_4 = A_4^{\text{tree}} M_4, \quad (4.16)$$

where A_4^{tree} is given by (4.4)

$$M_4 = 1 - \frac{a}{2} st I_4^{(1)}(s, t) + \mathcal{O}(a^2) \quad (4.17)$$

with $a = g^2 N / (8\pi^2)$, $s = (p_1 + p_2)^2$, $t = (p_2 + p_3)^2$ and the box-integral

$$I_4^{(1)}(s, t) = \int \frac{d^d l}{(i\pi^{d/2})} \frac{1}{l^2 (p_1 + l)^2 (p_1 + p_2 + l)^2 (p_4 - l)^2} \quad (4.18)$$

is the only integral in (4.15) that contributes. The integral is infrared divergent and evaluation in dimensional regularisation with $d = 4 - 2\epsilon$ yields [50]

$$st I_4^{(1)}(s, t) = \frac{2}{\epsilon^2} (-s/\mu^2)^{-\epsilon} + \frac{2}{\epsilon^2} (-t/\mu^2)^{-\epsilon} - \ln^2\left(\frac{s}{t}\right) - \frac{4}{3}\pi^2 + \mathcal{O}(\epsilon). \quad (4.19)$$

4.1.5 BDS Ansatz for n -point All-Loop Amplitudes

Similar to (4.16), it turns out that also n -point planar MHV amplitudes can be factorised into a tree-level part and a scalar function $M_n(a)$ that carries the complete coupling constant dependence

$$A_n = A_n^{\text{tree}} M_n(a), \quad (4.20)$$

⁹It should be noted, that only integrals that contain a cut can be detected by the cutting procedure. In QCD there are rational terms, which are not detected by standard unitarity methods and require additional care.

where $M_n(a) = (1 + aM_n^{(1)} + a^2M_n^{(2)} + \dots)$. Based on an iterative structure [182, 50] for $M^{(3)}, M^{(2)}$ in terms of the one-loop scalar factor $M^{(1)}$ and an n -point one-loop calculation [162], Bern, Dixon and Smirnov (BDS) proposed the so-called *BDS-ansatz* for the all-loop n -point gluon amplitude [50]

$$M_n(a) = 1 + \sum_l a^l M_n^{(l)} = \exp \left(\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon) \right) \right), \quad (4.21)$$

where $M_n^{(1)}(l\epsilon)$ is the all-orders in ϵ function $M_n^{(1)}(\epsilon)$, that appears in the one-loop amplitude calculation with ϵ replaced by $l\epsilon$, $f^{(l)}(\epsilon) = f^{(l)}(\epsilon) + \epsilon f_1^{(l)}(\epsilon) + \epsilon^2 f_2^{(l)}(\epsilon)$, $C^{(l)}$ are constants independent of n and $E_n^{(l)}(\epsilon)$ is a constant that vanishes as $\epsilon \rightarrow 0$.

Relation of the divergent part to Wilson loops

It was known before, that the infrared divergences, which in the expression above are hidden in $M_n^{(1)}(l\epsilon)$, exponentiate in a form equivalent to (4.21), i.e. that the amplitude can be written as

$$M_n = \exp(Z_n) \times \text{finite}, \quad (4.22)$$

where

$$Z_n = -\frac{1}{4} \sum_{i=1}^n \left(\frac{\Gamma_{\text{cusp}}^{(l)}}{(l\epsilon)^2} + \frac{\Gamma^{(l)}}{l\epsilon} \right) \sum_{i=1}^n \left(\frac{-s_{i,i+1}}{\mu^2} \right)^{-l\epsilon} \quad (4.23)$$

and $\Gamma_{\text{cusp}} = \sum_l a^l \Gamma_{\text{cusp}}^{(l)} = 2a - 2\zeta_2 a^2 + \mathcal{O}(a^3)$ is the *cusplike anomalous dimension*, $\Gamma^{(l)}$ is the collinear anomalous dimension and $s_{i,i+1} = (p_i + p_{i+1})^2$ are the kinematical invariants of the scattering process. The name *cusplike anomalous dimension* refers to the fact, that Wilson loops with cusps exhibit UV divergences, which cannot be removed by gauge field and coupling constant renormalisation, but need an additional renormalisation with associated anomalous dimension¹⁰ Γ_{cusp} [184, 185, 186, 187, 183].

It turned out that the anomalous dimensions of the Wilson loops play an important role in perturbative QCD, since the same universal function governs the infrared behaviour of relevant quantities in QCD [188, 189, 190, 191, 192, 193, 194]. It is known from studies of QCD, that in properly defined physical observables such as inclusive cross-sections all IR divergences cancel, however they cannot be removed from the S-matrix itself.

Finite part of the amplitudes

One of the main results of [50] is that (4.21) also implies an exponentiation of the finite part. After subtracting the divergences from $M_n(l\epsilon)$ in a suitable form based on the known exponentiation of the divergent term, the amplitude can be written as $\mathcal{M}_n = \exp(Z_n) \exp(F_n)$ or its logarithm as

$$\ln M_n = Z_n + F_n, \quad (4.24)$$

¹⁰We refer to the part of the anomalous dimension, which is proportional to the angle θ of the cusp. $\Gamma_{\text{cusp}}(a, \theta) = \theta \Gamma_{\text{cusp}}(a)$ for $\theta \gg 1$ which is true to all orders in the coupling [183].

where [50]

$$F_n^{\text{BDS}} = \frac{1}{2} \Gamma_{\text{cusp}}(a) F_n^{(1)} + C \quad (4.25)$$

and where $F_n^{(1)}$ is the finite part calculated at one-loop order and C is a constant, e.g. for $n = 4$, compare (4.19)

$$\begin{aligned} F_4^{(1)} &= \frac{1}{2} \ln^2 \left(\frac{s}{t} \right) + 4\zeta_2, \\ F_5^{(1)} &= \frac{1}{4} \sum_{i=1}^5 \left(\ln \left(\frac{s_{i,i+1}}{s_{i+3,i+4}} \right) \ln \left(\frac{s_{i+2,i+3}}{s_{i+1,i+2}} \right) + 3\zeta_2 \right). \end{aligned} \quad (4.26)$$

The remaining one-loop functions $F_n^{(1)}$ for $n \geq 6$ can be found in [162, 50].

All-loop amplitudes

The remarkable fact about (4.25) is that the coupling constant dependence is completely captured by the cusp anomalous dimension $\Gamma_{\text{cusp}}(a)$ and the full kinematical dependence on the momenta p_i is determined by the one-loop finite part $F_n^{(1)}$. Furthermore, the cusp anomalous dimension is thought to be known to all orders in the coupling constant through the BES equations [61].

Quite impressively, the BDS ansatz constitutes a proposal for an all-loop solution for n -point gluon amplitudes in $\mathcal{N} = 4$ SYM. However, it turns out, that this ansatz breaks down starting from $n = 6$ at two loops and has to be corrected [195], [196] by a remainder function R_n of conformally invariant cross ratios (2.15).

As will be discussed in section (4.2), even though the all-loop n -point BDS ansatz breaks down, a relation between light-like polygonal Wilson loops and scattering amplitudes remains valid, thus opening a new avenue for the all-loop solution of scattering amplitudes for $n \geq 6$.

4.1.6 Dual Conformal Symmetry of Amplitudes

In [9] it was remarked that the integrals appearing in planar amplitude calculations in $\mathcal{N} = 4$ SYM admit a *dual conformal symmetry*. The symmetry is also referred to as a *hidden symmetry*, because it has no direct Lagrangian origin, it is distinct from the ordinary conformal symmetry of the amplitudes (4.12).

This can be seen by rewriting the momenta in terms of dual variables x_1, \dots, x_n as

$$p_i^\mu = x_i^\mu - x_{i+1}^\mu \equiv x_{i,i+1}^\mu, \quad x_{i+n}^\mu \equiv x_i^\mu. \quad (4.27)$$

As an example, consider the loop integral that appears in the 4-point one-loop amplitude calculation (4.17)

$$stI_4^{(1)}(s, t) = \int \frac{d^d l}{(2\pi)^d} \frac{st}{l^2(p_1 + l)^2(p_1 + p_2 + l)^2(p_4 - l)^2} = \int \frac{d^d x_l}{(2\pi)^d} \frac{x_{13}^2 x_{24}^2}{x_{1l}^2 x_{2l}^2 x_{3l}^2 x_{4l}^2}, \quad (4.28)$$

where we have performed the change of variable (4.27) and shifted the loop variable¹¹ $x_l^\mu := l^\mu - x_1^\mu$. It is easy to see, that (4.28) is invariant under inversions (2.16)

$$I : \quad x_{ij}^2 \rightarrow \frac{x_{ij}^2}{x_i^2 x_j^2}, \quad d^d x_l \rightarrow \frac{d^d x_l}{(x_l^2)^d}. \quad (4.29)$$

for $d = 4$. Dilatation and translation invariance of (4.28) are trivial in x -space and thus by (2.14) equation (4.28) is invariant under dual conformal symmetry for $d = 4$. However, since the integrals suffer from infrared divergences, a regularisation needs to be introduced. Setting $d = 4 - 2\epsilon$ in dimensional regularisation explicitly breaks the dual conformal invariance, thus admitting only a *broken dual conformal invariance* of the amplitudes. The integrals are therefore also called *pseudo-conformal*.

The integrals appearing at higher orders in perturbation theory are also dual conformal invariant in the above sense, e.g. in [197] the appearing integrals were found to have this property in a four-loop calculation. Assuming dual conformal invariance to be valid to all orders, together with unitarity methods, provides a powerful mechanism for higher loop calculations.

This property was used in [170] to construct the four-point five-loop amplitude, by listing all *pseudo-conformal* integrals and determining the coefficients $0, \pm 1$ of the integrals¹² using unitarity techniques as introduced in section 4.1.3. As mentioned in [170, 178] it would be nice to have an analogous rule for the selection of integrals in the non-planar sector. One might expect such a relation between planar and non-planar integrals, since the planar and non-planar diagrams are linked by certain numerator identities [198, 199, 200, 201, 202].

In [203] a new way to regulate the IR divergences of gluon scattering amplitudes was introduced. The idea is to break the gauge group $U(N + M)$ to $U(N) \times U(1)^M$ by giving a vacuum expectation value $\sim m_i \delta_{ij}$ to the scalar fields with gauge group indices $i, j = N + 1, \dots, N + M$ into a certain direction in the $SO(6)$ internal space. This introduces heavy fields \mathcal{O}_{ia} with masses m_i and light fields \mathcal{O}_{ij} with masses $m_i - m_j$, while the fields \mathcal{O}_{ab} corresponding to the unbroken part of the gauge group ($a, b = 1, \dots, N$) remain massless. The integrals appearing in an amplitude calculation thus depend on the masses m_i . In [203] it was shown, that the integral appearing in the one-loop four-point amplitude calculation is invariant under an *extended dual conformal symmetry*

$$\hat{K}^\mu I_4^{(1)}(x_{ij}^2, m_i) = 0, \quad \hat{K}^\mu = \sum_{i=1}^4 2x_i^\mu (x_i \cdot \partial_i) - x_i^2 \partial_i^\mu + 2x_i^\mu m_i \partial_{m_i} - m_i^2 \partial_i^\mu. \quad (4.30)$$

The masses m_i can be considered as additional components in an *extended dual space* $\{x_i^\mu, m_i\}$. Since there is no need for an additional regularisation, the dual conformal symmetry is exact in the sense (4.30) at loop-level and not anomalous as in the case

¹¹Conventionally, the loop variable is designated by x_5 or x_0 in most of the literature. Here we denote it by x_l in order to avoid confusion with the second relation in (4.27).

¹²The coefficient of the integrals normalised by the tree-level amplitude.

of dimensional regularisation. More details on this regularisation can be found in the recent review [204]. The obvious conjecture is that any amplitude at any loop-level is a linear combination of integrals, which satisfy the extended dual conformal symmetry (4.30).

The appearance of this hidden dual conformal symmetry can be explained by the ordinary conformal symmetry of the Wilson loop, if one accepts the duality between amplitudes and Wilson loops, see section 4.2 for more details.

4.1.7 Dual Superconformal Symmetry

In [10] it was shown that the *hidden* dual conformal symmetry of the amplitudes in $\mathcal{N} = 4$ SYM can be extended to a *dual superconformal symmetry*. As mentioned in the introduction of this chapter, on the string side the same symmetry emerges from fermionic T-duality.

The defining relation for the dual space coordinates (4.27) in spinor helicity variables reads

$$(x_i)^{\alpha\dot{\alpha}} - (x_{i+1})^{\alpha\dot{\alpha}} = \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}} = p_i^{\alpha\dot{\alpha}} \quad (4.31)$$

and the dual variables automatically satisfy the momentum conservation constraint imposed by $\delta(p)$ in (4.9), i.e. $\sum_i \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}} = 0$. Similarly, one can introduce variables $\theta_i^{A\alpha}$ [10], which automatically satisfy the super-momentum conservation imposed by $\delta(q)$ in (4.9)

$$(\theta_i)^{A\alpha} - (\theta_{i+1})^{A\alpha} = \lambda_i^\alpha \eta_i^A = q_i^{A\alpha}. \quad (4.32)$$

The dual superconformal symmetry can be formulated by the action of generators J^a in the *full dual superspace* $(\lambda_i^\alpha, \tilde{\lambda}_i^{\dot{\alpha}}, \eta_i^A, x_i^{\alpha\dot{\alpha}}, \theta_i^{A\alpha})$. The generators J^a satisfy the commutation relations of the $psu(2, 2|4)$ algebra 2.4.1 and their explicit form is given in [10]. In [11] some of these generators were slightly reformulated to a form J'_a which leaves the amplitudes invariant

$$J'_a \mathcal{A}_n^{\text{tree}} = 0. \quad (4.33)$$

When restricting the action of the generators J'_a to the *on-shell* superspace $(\lambda_i^\alpha, \tilde{\lambda}_i^{\dot{\alpha}}, \eta_i^A)$, the generators $P_{\alpha\dot{\alpha}}, Q_{\alpha A} \in J'_a$ become trivial, the remaining generators coincide up to signs with those of j_a (4.12), except for $K'^{\alpha\dot{\alpha}}, S'^A_\alpha$. For more detailed information on dual superconformal and Yangian symmetry, see e.g. the reviews [205, 206]. At loop-level the dual conformal symmetry is broken, albeit in exactly the same manner for MHV and NMHV amplitudes, such that their ratio \mathcal{R} is an invariant [10, 207, 208, 209, 210] under dual conformal symmetry. We comment on the realization of the symmetry at loop-level in the following section.

4.1.8 Yangian Symmetry

In [11] it was shown, that in *on-shell superspace* the generators j_a and the non-trivial dual superconformal generator¹³ $K'^{\alpha\dot{\alpha}} \in J'_a$ together form the *Yangian* $Y(psu(2, 2|4))$ of the superconformal algebra $psu(2, 2|4)$.

The generators of the ordinary superconformal symmetry form the level-zero generators $j_a^{(0)} \equiv j_a$ of the Yangian algebra and satisfy the superconformal algebra

$$[j_a^{(0)}, j_b^{(0)}] = f_{ab}{}^c j_c^{(0)}, \quad (4.34)$$

where $f_{ab}{}^c$ are the structure constants of $psu(2, 2|4)$, see 2.4.1. The level-one generators can then be written in terms of the single particle level-zero generators $j_{i,a}^{(0)}$

$$j_a^{(1)} = f_a{}^{bc} \sum_{i>j}^n j_{i,b}^{(0)} j_{j,c}^{(0)}, \quad j_a = \sum_i^n j_{i,a}^{(0)} \quad (4.35)$$

and satisfy

$$[j_a^{(1)}, j_b^{(0)}] = f_{ab}{}^c j_c^{(1)}, \quad (4.36)$$

as well as the Serre relations, see [11]. It was shown in [11] that the dual superconformal generators are equivalent to these level-one generators $j_a^{(1)}$ of the Yangian. This is sufficient to construct all higher-level generators $j_a^{(l)}$ and therefore the superconformal together with the dual superconformal generators form an infinite-dimensional Yangian algebra. This translates into the following symmetry of the amplitudes

$$y \mathcal{A}_n^{\text{tree}} = 0, \quad y \in Y(psu(2, 2|4)) \quad (4.37)$$

and thus constrains the form of the amplitudes. It has been argued that the tree-level amplitudes are uniquely determined by requiring analytic properties such as the right collinear behaviour together with Yangian symmetries [157, 156].

At loop-level the symmetries are broken. In [159] superconformal anomalies are computed and it is shown that they may be cancelled through a universal one-loop deformation of the tree-level symmetry generators. The symmetry generators can be deformed in such a fashion as to render the amplitudes, defined in a dimensional regularisation scheme, invariant to one-loop order. In [211] Yangian symmetry of a certain class of light-like Wilson loops at one loop was proven and it is argued that it can be thought of as the effect of the original conformal symmetry at loop-level.

In further investigations it became clear, that dual superconformal symmetry is not a symmetry of individual amplitudes, but of the S-matrix [158, 157, 156, 159]. Equations relating the different amplitudes have been presented in [212] and it is shown that the complete planar S-matrix is fully dual supersymmetric, but this is not true for individual N^k MHV loop-amplitudes.

¹³One could also take the other non-trivial generator S' , since they are related by the commutation relations $[K', \bar{Q}] = S'$ of the algebra 2.4.1. As described in the previous section the remaining generators are trivial or coincide up to signs with those of j_a .

In [213] an all-loop equation for the \bar{Q} -symmetry acting on the BDS-subtracted planar S-matrix of $\mathcal{N} = 4$ SYM is presented, which is argued to amount for exact Yangian symmetry. In principle, these equations can be used to determine the all-loop S-matrix uniquely and to all orders in the coupling constant.

Twistor space formulation of amplitudes and Yangian symmetry

In [11] it was also shown, that the level-zero and level-one generators can be written nicely in twistor space variables [214] as first and second order differential operators. Twistor space is very useful in order to construct Yangian invariants [215, 216, 217, 218] and leads to the remarkable construction of the manifestly Yangian invariant all-loop integrand for scattering amplitudes in the planar limit of $\mathcal{N} = 4$ SYM [219, 220, 221, 222, 223, 224, 225]. Since we will not make further use of the formulation in twistor space, we refer the reader to the reviews [205, 206] and the references therein for more information.

4.2 Duality between Amplitudes and Wilson Loops

In [3, 4] Alday and Maldacena proposed a strong coupling description of planar gluon scattering amplitudes in terms of minimal surfaces in AdS space. The calculation formally resembles the strong coupling calculation of a Wilson loop's expectation value

$$\langle W_n \rangle = \langle W(C_n) \rangle = \frac{1}{N} \left\langle \text{Tr} \mathcal{P} \exp \left(i \oint_{C_n} A_\mu dz^\mu \right) \right\rangle, \quad (4.38)$$

where the light-like polygonal contour C_n is made of n points x_i related to the gluon momenta via $p_i = x_{i+1} - x_i$.

Interestingly, it was found in [5] that the relation between gluon scattering amplitudes and Wilson loops is also valid in the weak coupling regime for $n = 4$. The amplitude / Wilson loop duality states¹⁴ that up to $\mathcal{O}(\epsilon)$ terms

$$\ln M_n = \ln \langle W_n \rangle + \text{const.}, \quad (4.39)$$

where M_n is the ratio of the planar MHV amplitude and its tree-level expression as defined in (4.11). The duality relation (4.39) is illustrated in figure 4.1. As explained before, M_n was conjectured to be given to all-loop order by the BDS ansatz (4.21).

As mentioned in section 4.1.5, relations between the divergent parts were already known before [183, 226, 227], but in particular the duality (4.39) is also true for the finite parts¹⁵ of the MHV amplitudes (4.25) and the Wilson loops

$$F_n^{\text{BDS}} = F_n^{\text{WL}} + \text{const.} \quad (4.40)$$

¹⁴Upon a specific identification of the renormalisation scales and regulators in the amplitude and the Wilson loop calculation, see [5].

¹⁵Strictly speaking F_n^{BDS} , F_n^{WL} are the finite part of the logarithms of M_n and $\langle W_n \rangle$.

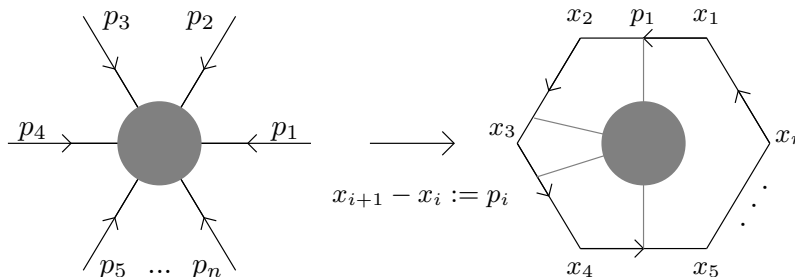


Figure 4.1: Duality between MHV scattering amplitudes A_n^{MHV} and the expectation value of Wilson loops W_n in $\mathcal{N} = 4$ super Yang-Mills.

The duality was checked for n points at one loop [6] as well as for 4, 5 and 6 points [7], [8], [228] at two loops. It was found that the expectation value of the Wilson loop is governed by an anomalous conformal Ward identity [7, 195] that completely fixes its form at 4 and 5 points and allows for an arbitrary function R_n of conformal invariants starting from 6 points, as will be reviewed in section 4.2.2. It was found that this so-called *remainder function* R_n is indeed present starting from 6 points at two-loop order and leads to a correction [195], [196] of the BDS ansatz [50] for planar gluon scattering amplitudes.

The Wilson loop is numerically under control at two loops for arbitrary n [229] and the duality was also found to hold to order epsilon at two-loops in [230]. Much effort was put into the analytical evaluation of the remainder function at 6 points [231, 232, 233] and it was possible to recast the result in a very compact form [234] using the notion of the *symbol* of a function. This technique was applied to the three-loop hexagonal Wilson loop in [235]. The remainder function was also analysed in the limit of self-crossing Wilson loops where it exhibits additional divergences [236, 237, 238].

Constraints of an operator product expansion for Wilson loops were investigated [239, 240, 241] and it was argued that this knowledge is enough to fix the full two-loop answer for arbitrary n . The duality has also been extended to tree-level amplitudes with arbitrary helicity states by introducing a suitable supersymmetric Wilson loop see section 4.2.3. Reviews on the duality between scattering amplitudes and Wilson loops can be found in [242], [243].

4.2.1 Light-Like Polygonal Wilson Loops

In order to illustrate the simplest case of the duality relation (4.39) we shortly review the one-loop calculation of the four-sided Wilson loop [5]. The contour of the n -sided polygon \mathcal{C} is given by n points x_i ($i = 1, \dots, n$) and we can parametrise each edge \mathcal{C}_i via

$$z_i^\mu(s_i) = x_i^\mu + p_i^\mu s_i, \quad p_i^\mu = x_{i+1}^\mu - x_i^\mu, \quad (4.41)$$

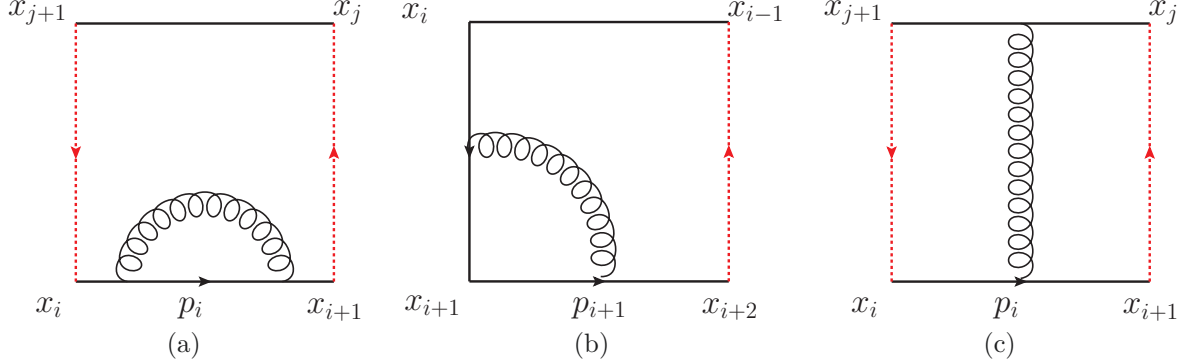


Figure 4.2: Examples for the three classes of one-loop diagrams with (a) a propagator connecting the same edge (vanishing), (b) propagator stretching between adjacent edges (divergent), (c) propagator stretching between non-adjacent edges (finite). Dotted lines represent one or more light-like distances.

where $s_i \in [0, 1]$ and $\mathcal{C} = \mathcal{C}_1 \cup \dots \cup \mathcal{C}_n$. The lowest order contribution to the expectation value of the Wilson loop (4.38), see (E.4), is

$$\langle W_n \rangle^{(1)} = \frac{i^2}{N} \text{Tr} (T^a T^b) \int_{z_i > z_j} dz_i^\mu dz_j^\nu \langle A_\mu^a(z_i) A_\nu^b(z_j) \rangle = g^2 C_F \frac{\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}} \sum_{i>j} I_{ij}, \quad (4.42)$$

where $C_F = \delta^{aa}/(2N)$, we have used the gluon propagator given in (C.33) in position-space and using (A.14), (A.15) we defined

$$\begin{aligned} I_{ij} &= \int ds_i ds_j p_i \cdot p_j (-z_i - z_j)^{2-d/2} \\ &= \frac{1}{2} \int_0^1 ds_i \int_0^1 ds_j \frac{x_{i,j+1}^2 + x_{i+1,j}^2 - x_{i,j}^2 - x_{i+1,j+1}^2}{(-x_{ij}^2 \bar{s}_i \bar{s}_j - x_{i+1,j}^2 s_i \bar{s}_j - x_{i,j+1}^2 \bar{s}_i s_j - x_{i+1,j+1}^2 s_i s_j)^{d/2-1}}. \end{aligned} \quad (4.43)$$

The sum is reduced to $i \neq j$ since for $i = j$ we have $p_i^2 = 0$ and the contribution shown in figure 4.2a vanishes. There are n divergent diagrams $I_{i+1,i}$ of the type shown in fig. 4.2b

$$I_{i+1,i} = -\frac{1}{2} (-x_{i,i+2}^2)^{2-d/2} \int_0^1 ds_i ds_{i+1} \frac{1}{(\bar{s}_i s_{i+2})^{1-\epsilon}} = -\frac{1}{2} \frac{(-x_{i,i+2}^2)^\epsilon}{\epsilon^2}$$

and one can see that the divergence arises from the cusps formed by two light-like edges, when the ends of the gluon propagator approach the cusp. It is well-known [226] that Wilson loops evaluated over light-like polygonal contours exhibit this type of UV-divergences.

Diagrams with $|i - j| \geq 1$ shown in figure 4.2c yield finite contributions. For $n = 4$, the only diagrams of this type are $I_{31} = I_{42}$ and have the result

$$I_{31} = \frac{1}{2} \int ds_1 ds_3 \frac{x_{13}^2 + x_{24}^2}{x_{13}^2 \bar{s}_1 \bar{s}_3 + x_{24}^2 s_1 s_3} = \frac{1}{4} \left(\ln^2 \left(\frac{x_{13}^2}{x_{24}^2} \right) + \pi^2 \right), \quad (4.44)$$

such that the full one-loop contribution for $n = 4$ is

$$\langle W_4 \rangle^{(1)} = \frac{g^2 N}{8\pi^2} \left[-\frac{1}{2} \sum_{i=1}^4 \frac{(-x_{i,i+2}^2 \mu^2 \pi e^{\gamma_E})^\epsilon}{\epsilon^2} + \frac{1}{2} \ln^2 \left(\frac{x_{13}^2}{x_{24}^2} \right) + \frac{\pi^2}{3} \right], \quad (4.45)$$

where we have restored the regularisation scale μ , let $C_F \rightarrow N/2$ in the planar limit and used $\Gamma(1 - \epsilon) = \exp(\epsilon \gamma_E) + \frac{1}{2} \zeta_2$. Indeed, upon a specific identification of the regularisation parameters ϵ and the scale μ^2 [5], up to a constant, this coincides with the one-loop correction to the planar scattering amplitude (4.17), (4.19) if we set $s = x_{13}^2, t = x_{24}^2$. The one-loop result was generalised to n points and reads [6]

$$\langle W_n \rangle^{(1)} = \frac{g^2 N}{8\pi^2} \left[-\frac{1}{2} \sum_{i=1}^n \frac{(-x_{i,i+2}^2 \mu^2 \pi e^{\gamma_E})^\epsilon}{\epsilon^2} + \mathcal{F}_n^{\text{WL}} \right], \quad (4.46)$$

where the finite contribution $\mathcal{F}_n^{\text{WL}}$ is equal to the finite part $F_n^{(1)}$ in the BDS conjecture [50] up to a constant, e.g. for $n = 4$ and $n = 6$

$$\begin{aligned} \mathcal{F}_4^{\text{WL}} &= \frac{1}{2} \ln^2 \left(\frac{x_{13}^2}{x_{24}^2} \right) + \text{const.}, \\ \mathcal{F}_6^{\text{WL}} &= \frac{1}{2} \sum_{i=1}^6 \left[-\ln \left(\frac{x_{i,i+2}^2}{x_{i,i+3}^2} \right) \ln \left(\frac{x_{i+1,i+3}^2}{x_{i,i+3}^2} \right) \right. \\ &\quad \left. + \frac{1}{4} \ln^2 \left(\frac{x_{i,i+3}^2}{x_{i+1,i+4}^2} \right) - \frac{1}{2} \text{Li}_2 \left(1 - \frac{x_{i,i+2}^2 x_{i+3,i+5}^2}{x_{i,i+3}^2 x_{i+2,i+5}^2} \right) \right] + \text{const.} \end{aligned} \quad (4.47)$$

4.2.2 Anomalous Conformal Ward Identities for Wilson Loops

In [7, 8] it was shown that the finite part of the expectation value is governed by an *anomalous conformal Ward identity*, which we shall shortly review here.

As we have seen in the previous section, due to the cusp divergences of the light-like Wilson loops, a regularisation needs to be introduced in order to calculate the expectation value. As mentioned earlier, dimensional regularisation explicitly breaks conformal symmetry and thus conformal Ward identities 3.1 for expectation values become anomalous. This is due to the fact that the action in $d = 4 - 2\epsilon$ dimensions is not invariant under all generators of the conformal group. While dimensional regularisation preserves invariance under translations and Lorentz transformations, the action $S_\epsilon[\Phi]$ in $d = 4 - 2\epsilon$ dimensions is not invariant under dilatations and special conformal transformations of the fields¹⁶ $\Phi \rightarrow \Phi + \delta\Phi$ as given in (2.25). The action varies as [8]

$$\delta_D S_\epsilon = \frac{2\epsilon}{\mu^{2\epsilon}} \int d^d x \mathcal{L}(x), \quad \delta_{K^\mu} S_\epsilon = \frac{4\epsilon}{\mu^{2\epsilon}} \int d^d x x^\mu \mathcal{L}(x). \quad (4.48)$$

¹⁶ Φ represents all fields that are integrated over in the path integral.

It is then easy to see how the generators of dilatations and special conformal transformations act on the expectation value of the Wilson loop

$$\langle W(\mathcal{C}_n) \rangle = \int \mathcal{D}\Phi e^{iS_\epsilon[\Phi]} \text{Tr} \left[\mathcal{P} \exp \left(i \oint_{\mathcal{C}_n} dz^\mu A_\mu(z) \right) \right]. \quad (4.49)$$

Consider a light-like polygon \mathcal{C}_n defined by n points x_i^μ satisfying $x_{i,i+1}^2 = 0$. The contour \mathcal{C}'_n that is obtained under an infinitesimal conformal transformations of the points x_i as given by (2.5), (2.4) is another light-like contour, because light-like distances trivially transform into light-like distances by (2.16). By Poincaré symmetry, the expectation values $\langle W(\mathcal{C}_n) \rangle$, $\langle W(\mathcal{C}'_n) \rangle$ are functions of the coordinates x_{ij}^2 , $x'_{ij}{}^2$ and thus the variation of these expectation values under conformal transformations is generated by (2.10), e.g. for special conformal transformation

$$\begin{aligned} \delta_{\epsilon.K} \langle W(\mathcal{C}_n) \rangle &= \langle W(\mathcal{C}_n) \rangle - \langle W(\mathcal{C}'_n) \rangle \\ &= \epsilon_\mu K^\mu \langle W(\mathcal{C}_n) \rangle \\ &= \epsilon_\mu \sum_i \left(2x_i^\mu (x_i \cdot \partial_i) - x_i^2 \partial_i^\mu \right) \langle W(\mathcal{C}_n) \rangle. \end{aligned} \quad (4.50)$$

On the other hand, we can rewrite the expectation value of the Wilson loop over the transformed contour \mathcal{C}'_n

$$\langle W(\mathcal{C}'_n) \rangle = \int \mathcal{D}\Phi e^{iS_\epsilon[\Phi]} \text{Tr} \left[\mathcal{P} \exp \left(i \oint_{\mathcal{C}'_n} dz^\mu A_\mu(z) \right) \right] \quad (4.51)$$

in terms of the original contour \mathcal{C}_n , if we evaluate it with the conformally transformed gauge field A'_μ . Rewriting $S_\epsilon[\Phi] = S_\epsilon[\Phi' - \delta\Phi'] = S_\epsilon[\Phi'] - \delta S_\epsilon[\Phi']$ and using (4.48) one can thus write

$$D \langle W(\mathcal{C}_n) \rangle = -\frac{2i\epsilon}{\mu^{2\epsilon}} \int d^d x \langle \mathcal{L}(x) W(\mathcal{C}_n) \rangle, \quad (4.52)$$

$$K^\nu \langle W(\mathcal{C}_n) \rangle = -\frac{4i\epsilon}{\mu^{2\epsilon}} \int d^d x x^\nu \langle \mathcal{L}(x) W(\mathcal{C}_n) \rangle, \quad (4.53)$$

where K^ν , D are the differential operators given by (2.10). It should be emphasised, that due to the factors of ϵ in (4.52) and (4.53), it is sufficient to evaluate only the divergent term of the correlator on the right-hand side in order to get a differential equation for the finite part of the expectation value of the Wilson loop.

The dimensionally regularised Wilson loop $\langle W(\mathcal{C}_n) \rangle$ is a dimensionless scalar function of the cusp points x_i^μ , which appear paired with the regularisation scale as $x_{ij}^2 \mu^2$. As a consequence, the expectation values satisfies

$$\left(\sum_{i=1}^n (x_i \cdot \partial_i) - \mu \frac{\partial}{\partial \mu} \right) \langle W(\mathcal{C}_n) \rangle = 0. \quad (4.54)$$

This provides a consistency condition for the right-hand side of (4.52).

The all-loop anomalous conformal Ward identity of the light-like Wilson loop in $\mathcal{N} = 4$ SYM was evaluated in [7, 8]. The special conformal Ward identity for the finite part reads

$$K^\mu F_n^{\text{WL}} = \frac{1}{2} \Gamma_{\text{cusp}}(a) \sum_{i=1}^n \ln \left(\frac{x_{i,i+2}^2}{x_{i-1,i+1}^2} \right) x_{i,i+1}^\mu. \quad (4.55)$$

Note that the dependence on the coupling constant a only enters via the cusp anomalous dimension $\Gamma_{\text{cusp}}(a)$. The solution¹⁷ of this differential equation for $n = 4, 5$ is

$$\begin{aligned} F_4^{\text{WL}} &= \frac{1}{4} \Gamma_{\text{cusp}}(a) \ln \left(\frac{x_{13}^2}{x_{24}^2} \right) + \text{const.}, \\ F_5^{\text{WL}} &= \frac{1}{8} \Gamma_{\text{cusp}}(a) \sum_{i=1}^5 \ln \left(\frac{x_{i,i+2}^2}{x_{i,i+3}^2} \right) \ln \left(\frac{x_{i+2,i+4}^2}{x_{i+1,i+3}^2} \right) + \text{const.} \end{aligned} \quad (4.56)$$

and has exactly the form¹⁸ of the finite part in the BDS conjecture (4.26), (4.25). Furthermore, one can check, that F_n^{BDS} is a solution of the Ward identity also for $n \geq 6$.

However, starting¹⁹ from $n = 6$, the functional form of F_n is not uniquely fixed by the Ward identity, but the general solution of the anomalous conformal Ward identity contains an arbitrary function $f(u_{ijkl})$ of the conformal cross ratios (2.15). This is because the cross ratios (2.15) are annihilated by all conformal generators, e.g.

$$K^\mu u_{ijkl} = K^\mu \left(\frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2} \right) = 0. \quad (4.57)$$

Remainder function

Even though the BDS ansatz contains functions of these cross ratios, it was found in [195], [196] that the BDS ansatz is not complete starting from two loops and $n = 6$, i.e.

$$F_6^{\text{BDS}} \neq F_6^{\text{MHV}}. \quad (4.58)$$

It was first noticed in [195] that the expectation value of the light-like Wilson loop at two loops for $n = 6$ does not coincide with the BDS ansatz and in [196] it was confirmed, that the two-loop $n = 6$ amplitude deviates in the same way from the BDS ansatz. Therefore, the duality between the Wilson loop and the amplitude continues to hold, i.e.

$$F_6^{\text{MHV}} = F_6^{\text{WL}} + \text{const.} \quad (4.59)$$

¹⁷Using $\mathbb{K}^\nu x_{ij}^2 = 2(x_i + x_j)^\nu x_{ij}^2$, $\mathbb{K}^\nu x_i^\mu = 2x_i^\nu x_i^\mu - x_i^2 \eta^{\mu\nu}$ it is easy to verify that these expressions are indeed solutions.

¹⁸It is easy to see the equivalence of the expressions, when rewriting $s_{i,i+1} = x_{i,i+2}^2$ and using the relation $x_{i+n} = x_i$.

¹⁹Due to the light-like separation of neighboring points $x_{i,i+1}^2$, non-trivial cross ratios can only be built starting from $n = 6$, since we need at least 4 non-vanishing distances.

For this reason, the BDS ansatz (4.21), (4.25) only constitutes an all-loop conjecture for $n = 4, 5$ particles. The deviation of the amplitude respectively the Wilson loop expectation value from the BDS ansatz is called the *remainder function*

$$R_n(u_{ijkl}) = F_n^{\text{BDS}} - F_n^{\text{WL}}, \quad (4.60)$$

which is non-trivial starting from two loops and $n = 6$. As mentioned above, much effort has been put into the numerical and analytical evaluation of this function. It is widely believed that the duality (4.59) continues to hold to all-loop order for arbitrary n . Since the evaluation of the Wilson loop is simpler – also numerically faster – than the amplitude calculation this provides a powerful tool for amplitude calculations.

4.2.3 Supersymmetric Amplitudes and Wilson Loops

The duality relation (4.39) is a relation between the expectation value of the bosonic Wilson loop (4.38) and the scalar function M_n that contains the loop corrections to the MHV amplitudes (4.11).

A natural question is, whether one can also construct a Wilson loop \mathcal{W}_n , that generates the other terms $A_{n;k}$ in (4.10) corresponding to non-MHV amplitudes, i.e. a supersymmetric object that has an expansion in η_i^A in the same way as the superamplitude (4.10).

Supersymmetric Wilson loops that reproduce the correct tree-level non-MHV amplitudes were proposed in [244, 245]. However, it was shown in [246] that the one-loop corrections to \mathcal{W}_n do not match those of the non-MHV amplitudes. As explained in [246] the mismatch is due to singularities of the Wilson loop on a light-like contour in combination with the fact that the supersymmetry algebra of $\mathcal{N} = 4$ SYM only closes *on-shell*, i.e. modulo the equations of motion of the fields. Therefore, not only the conformal symmetry but also the chiral supersymmetry is broken at loop-level, while the *dual* chiral supersymmetry of the amplitudes remains exact. The anomaly was further discussed in [247] and in [248] a regularization procedure that preserves super-Poincaré symmetry was proposed. In the following section we shortly sketch another proposal for the dual description of scattering amplitudes, which does not suffer from this supersymmetric anomaly.

4.3 Duality between Correlators and Wilson Loops

In addition to the duality between amplitudes and Wilson loops, a duality between light-like Wilson loops with n cusps and n -point correlators $G_n = \langle \mathcal{O}(x_1) \dots \mathcal{O}(x_n) \rangle$ of half-BPS protected operators was established in [249, 250, 12, 251, 252, 253]. In the limit where the positions of adjacent operators become light-like, it is found that the one-loop n -point correlator divided by its tree-level expression coincides with a light-like n -polygon Wilson loop in the adjoint representation and thus in the planar limit

via the amplitude / Wilson loop duality with the corresponding amplitude A_n

$$\lim_{x_{i,i+1}^2 \rightarrow 0} \ln(G_n/G_n^{\text{tree}}) = \ln W^{\text{adj}}(\mathcal{C}_n) = \ln \left(A_n/A_n^{\text{tree}} \right)^2. \quad (4.61)$$

Therefore one often speaks about a *triality* between correlators, amplitudes and Wilson loops. Unlike the Wilson loop / amplitude duality, see section 4.2.3, the correlator / amplitude duality has a natural generalization to supercorrelators which are dual to superamplitudes [251, 252]. An important feature of this duality is that objects are considered, which are well-defined in $d = 4$ dimensions, thus avoiding the need of regularization and the associated anomalies.

4.3.1 Hidden Permutation Symmetry in Correlation Functions

Furthermore, a new hidden symmetry of four-point correlation functions and amplitudes in $\mathcal{N} = 4$ SYM was discovered [254, 255]. The all-loop integrand of the investigated correlation functions possesses an unexpected symmetry under the exchange of the four external and all internal (integration) points. Using this symmetry, the three-loop integrand can be fixed up to a few constants, which in turn can be determined using the duality between correlators and amplitudes with the knowledge of the three-loop amplitude result, impressively illustrating the power of these dualities. As a byproduct the three-point function of two half-BPS operators and one Konishi operator at three-loop level is obtained [254]. Furthermore, using the four-point correlation function of the stress-tensor multiplets [255, 256], the five-loop anomalous dimension of the Konishi-operator was recently extracted.

4.4 Form Factors and Wilson Loops

Another duality with Wilson loops was proposed in [257]. It relates *form factors* and periodic Wilson loops, see also [258] for a strong coupling description of form factors. Form factors are expectation values of the form

$$\int d^4x e^{-iqx} \langle 1\dots n | \mathcal{O}(x) | 0 \rangle, \quad (4.62)$$

where $\langle 1\dots n |$ describes a multiparticle state with momenta p_1, \dots, p_n and $\mathcal{O}(x)$ is a gauge invariant operator. Form factors are therefore neither completely on-shell quantities such as scattering amplitudes nor completely off-shell quantities like correlation functions, but exhibit both features. Therefore, various on-shell techniques such as recursion relations and unitarity cuts can be applied. The analysis of form factors was also extended to super form factors [259] and the form factors of half-BPS operators were evaluated analytically up to two loops in [260].

Chapter 5

Amplitudes, Wilson Loops and Correlators in ABJM Theory

As mentioned in the introductory chapter, there is another AdS/CFT correspondence, which involves a three-dimensional conformal field theory that "lives" on the conformal boundary of AdS_4 space and is therefore also called AdS_4/CFT_3 correspondence.

This gauge theory is the $\mathcal{N} = 6$ ABJM theory [2], which is built upon an $SU(N) \times SU(N)$ gauge symmetry which allows for a planar $N \rightarrow \infty$ limit with $\lambda = N/k$ held fixed, where k is the common absolute value of the Chern-Simons parameters of the two $SU(N)$ subgroups. In this limit the ABJM theory is conjectured to be dual to type IIA string theory on $AdS_4 \times \mathbb{CP}_3$ [2, 261], representing an exact gauge-string duality pair very similar in nature to the well studied 4d $\mathcal{N} = 4$ super Yang Mills/ $AdS_5 \times S^5$ string duality pair.

There are many structural similarities of the 3d $\mathcal{N} = 6$ superconformal ABJM theory to $\mathcal{N} = 4$ super Yang-Mills, most notably the emergence of hidden integrability in the planar limit for the spectral problem of determining anomalous scaling dimensions of local operators [262, 263, 264, 265, 266, 267, 268]. In the following we focus on relations between amplitudes and Wilson loops. More on the AdS_4/CFT_3 duality in general and the similarities and differences to the integrable structures in $\mathcal{N} = 4$ SYM can be found in [269].

Given these insights, the question arises whether there are also similar relations between scattering amplitude, Wilson loops and correlators in ABJM theory as it is the case for $\mathcal{N} = 4$ SYM. We will shortly give an overview over developments in ABJM theory that parallel the ones in $\mathcal{N} = 4$ SYM described in the previous chapter and elaborate in more detail on the Wilson loop side in chapter 6, which constitutes a part of the research performed in this thesis.

5.1 Amplitudes in ABJM Theory

Scattering amplitudes in the ABJM theory have first been analysed in [270], where in fact more general mass deformed superconformal Chern-Simons theories with extended

supersymmetries were studied at the one-loop order. There, a vanishing result for the four-point one-loop amplitudes in the ABJM theory was found and the authors speculated whether the two-loop scattering amplitudes in $\mathcal{N} = 6$ Chern-Simons (ABJM theory) could be simply related to the one-loop $\mathcal{N} = 4$ SYM amplitudes.

In three dimensions there is no helicity degree of freedom for massless states. A spinor-helicity-like formalism for three dimensions was developed in [18, 271]. Similarly to the combination of all on-shell states into a supermultiplet (4.8) in $\mathcal{N} = 4$ SYM, one can embed the matter fields of ABJM theory into two superstates using an $su(3)$ Graßmann spinor η^A

$$\begin{aligned}\Phi(\lambda, \eta) &= \phi^A(\lambda) + \eta^A \psi_A(\lambda) + \frac{1}{2!} \epsilon_{ABC} \eta^A \eta^B \phi^C(\lambda) + \frac{1}{3!} \epsilon_{ABC} \eta^A \eta^B \eta^C \psi_4(\lambda), \\ \bar{\Phi}(\lambda, \eta) &= \bar{\psi}^A(\lambda) + \eta^A \bar{\phi}_A(\lambda) + \frac{1}{2!} \epsilon_{ABC} \eta^A \eta^B \bar{\psi}^C(\lambda) + \frac{1}{3!} \epsilon_{ABC} \eta^A \eta^B \eta^C \bar{\phi}_4(\lambda).\end{aligned}\quad (5.1)$$

Note, that the gauge fields do not appear in the superstates, since they have no on-shell degrees of freedom, Chern-Simons gauge fields in three dimensions have no freely propagating modes.

One can also define colour ordered superamplitudes and the colour decomposition requires to have Φ and $\bar{\Phi}$ fields to alternate, i.e. $A_n = A_n(\bar{\Lambda}_1, \Lambda_2, \dots, \bar{\Lambda}_{n-1}, \Lambda_n)$, where $\Lambda_i = (\eta_i, \lambda_i)$. In [270, 18] the four-point superamplitude was expressed as

$$A_4(\bar{1}, 2, \bar{3}, 4) = \frac{\delta^{(3)}(P) \delta^{(6)}(Q)}{\langle 12 \rangle \langle 23 \rangle}, \quad (5.2)$$

similar to (4.9). At tree-level, it was found that amplitudes in ABJM theory exhibit Yangian and dual superconformal symmetry [18, 272, 19, 273, 20]. A usual BCFW recursion for tree amplitudes does not work, since in three dimensions a linear shift vector as in section 4.1.2, which preserves momentum conservation and keeps the vectors light-like, is trivial. The solution is to introduce a non-linear shift [273]. Using these recursions, it was proven [273] that all tree-level amplitudes of the ABJM theory have dual superconformal symmetry.

5.2 T-Duality of $AdS_4 \times \mathbb{CP}_3$ Superstrings

As mentioned before, from the string perspective the scattering amplitude/Wilson loop duality in the AdS_5/CFT_4 system arises from a combination of bosonic and fermionic T-dualities under which the free $AdS_5 \times S^5$ superstring is self-dual [138, 139]. Hence, for the existence of an analogue duality in ABJM theory one would require a similar self-duality of the $AdS_4 \times \mathbb{CP}_3$ superstring under the combined T-dualities. The problem was analysed in [274, 275, 276, 277, 278, 279, 280] but no T-self-duality could be established so far.

5.3 Amplitudes and Wilson Loops in ABJM Theory

In [15] we calculated the expectation value of the Wilson loop operator (4.38) in the planar limit for light-like polygonal contours \mathcal{C}_n in pure Chern-Simons and ABJM theory up to two loops. The details of this computation will be given in chapter 6. As will be shown in section 6.4, conformal Ward identities force $\langle W(\mathcal{C}_n) \rangle_{1\text{-loop}}$ to depend only on conformally invariant cross ratios (2.15). At one-loop order in pure Chern-Simons and ABJM theory we find, see section 6.2, that the Wilson loop with four and six cusps vanish, leading to the conclusion that the allowed function of conformal cross ratios is trivial at six points. The conjecture that the n -point Wilson loop vanishes

$$\langle W(\mathcal{C}_n) \rangle_{1\text{-loop}}^{\text{CS}} = 0 = \langle W(\mathcal{C}_n) \rangle_{1\text{-loop}}^{\text{ABJM}} \quad (5.3)$$

was later proven analytically in [21] for arbitrary n .

Furthermore, we computed the tetragonal Wilson loop W_4 at two-loop order in pure Chern-Simons and ABJM theory. Remarkably, the result in dimensional reduction regularisation with $d = 3 - 2\epsilon$ for the Wilson loop in ABJM theory is of the same functional form as the one-loop result in $\mathcal{N} = 4$ super Yang-Mills theory. Most interestingly, it was then found by two independent approaches, using generalised unitarity methods in [16] and by a direct superspace Feynman diagram calculation in [17], that the two-loop result for four-point scattering amplitudes in ABJM theory agrees with the Wilson loop

$$M_4^{(2)} = \frac{A_4^{(2)}}{A_4^{\text{tree}}} = \langle W_4 \rangle_{2\text{-loop}}^{\text{ABJM}} + \text{const.} \quad (5.4)$$

upon a specific identification of the regularisation scales [17, 16]. This established the first non-trivial example of a Wilson loop / amplitude duality of the form (4.39) in ABJM theory. In [16] the result (5.4) has been obtained by assuming dual conformal invariance at quantum level, whereas in [17] the same result is obtained by standard Feynman diagram calculations, thus giving a proof of the validity of on-shell dual conformal invariance at loop-level.

These results were extended to the more general case of ABJ theory in [281]. Furthermore, in [17, 281] an analogue of the BDS ansatz for scattering amplitudes in ABJM theory is proposed and, using this, the form of the finite part at four loops is predicted. In order to verify this BDS-like ansatz, a four-loop computation of four-point scattering amplitudes in ABJM theory is required. Furthermore, in [282] an all-orders in ϵ identity between the four-point two-loop amplitude in ABJM and the four-point one-loop amplitude in $\mathcal{N} = 4$ SYM is derived.

In light of these findings it is natural to ask, whether the duality between Wilson loops and amplitudes in ABJM theory continues to hold beyond $n = 4$, as it does in $\mathcal{N} = 4$ super Yang-Mills. As a part of this thesis, in chapter 6 we perform numerical computations to extend our findings of [15] to the n -sided Wilson loop at two-loop

order. Interestingly, we find that the hexagonal Wilson loop at two loops also agrees with the corresponding Wilson loop in $\mathcal{N} = 4$ super Yang-Mills at one-loop order.

We perform a detailed numerical analysis for the hexagonal Wilson loop leading to a guess for the n -point case, which we numerically check also for $n > 6$ in a limited set of kinematical points, see section 6.5. Again we find, that the result agrees with the result for the Wilson loop in $\mathcal{N} = 4$ super Yang-Mills. It is thus natural to expect the result to hold for all n

$$\langle W_n \rangle_{2\text{-loop}}^{\text{ABJM}} = \left(\frac{N}{k} \right)^2 \left[-\frac{1}{2} \sum_{i=1}^n \frac{(-\mu'^2 x_{i,i+2}^2)^{2\epsilon}}{(2\epsilon)^2} + \mathcal{F}_n^{\text{WL}} + r_n \right], \quad (5.5)$$

where $\mu'^2 = \mu^2 8\pi e^{\gamma_E}$, r_n is a constant that depends linearly on n and is specified below (6.70). The details of the Wilson loop calculation in ABJM theory will be given in chapter 6.

Indeed, this is of the same form as the one-loop result (4.46) of the Wilson loop in $\mathcal{N} = 4$ SYM [6]

$$\langle W_n \rangle_{1\text{-loop}}^{\mathcal{N}=4 \text{ SYM}} = \frac{g^2 N}{8\pi^2} \left[-\frac{1}{2} \sum_{i=1}^n \frac{(-x_{i,i+2}^2 \mu^2)^\epsilon}{\epsilon^2} + \mathcal{F}_n^{\text{WL}} \right], \quad (5.6)$$

where the contribution $\mathcal{F}_n^{\text{WL}}$ in both formulas above is given by the finite part of the Wilson loop in $\mathcal{N} = 4$ SYM, see (4.47) for the expressions with $n = 4, n = 6$. We comment on the differences between these expressions in section 6.7.

Recently, also the six-point one-loop amplitudes became available [22, 23] and it was found, that they do not vanish in contrast to the Wilson loop (5.3) at one-loop order. In [22] it is shown that scattering amplitudes in ABJM theory give rise to an anomaly of the superconformal symmetry, similarly as in $\mathcal{N} = 4$ SYM theory. At tree- and one-loop level these anomalous terms can be used to determine the form of the amplitudes and it is shown that the symmetries predict a non-vanishing one-loop six-point amplitude. The prediction is confirmed by a calculation based on generalised unitarity methods. In [23] a superspace Feynman diagram calculation is used and it is argued, that the result, also for higher point functions, is in general non-vanishing.

This at least complicates a duality between Wilson loops and amplitudes in superconformal $\mathcal{N} = 6$ super Chern-Simons theory and it may very well be that there is no such duality in ABJM theory. If there was no self-T-duality in ABJM theory, there might be no reason to expect a duality between Wilson loops and scattering amplitudes. On the other hand, the amplitude / Wilson loop duality in $\mathcal{N} = 4$ SYM for the bosonic Wilson loops involved MHV amplitudes. The matching of non-MHV amplitudes in $\mathcal{N} = 4$ SYM requires the introduction of a supersymmetric Wilson loop, see section 4.2.3. The lack of helicity in three dimensions therefore complicates a map between these objects. Possibly, it is necessary to find an analog of the super-Wilson loop that matches with superamplitudes in ABJM theory.

5.4 Duality between Wilson Loops and Correlators

Furthermore, non-trivial evidence for a duality between Wilson loops and correlators in ABJM theory was found at one-loop level in [21]. Even though both quantities vanish at one loop, the equality between correlators and Wilson loops in the light-like limit is already present at the level of integrands, before showing that these expressions actually vanish. It would be very desirable to perform a two-loop computation of the correlator in order to check whether the relation between Wilson loops and correlators continues to hold.

Chapter 6

Light-like Wilson Loops in Chern-Simons and ABJM Theory

In this chapter we review the results for light-like polygonal Wilson loops in Chern-Simons and ABJM theory that were presented in [13, 14, 15]. We calculate the expectation value of the n -cusped Wilson loop operator

$$\langle W(\mathcal{C}_n) \rangle = \frac{1}{N} \langle 0 | \text{Tr} \mathcal{P} \exp \left(i \oint_{\mathcal{C}_n} A_\mu dz^\mu \right) | 0 \rangle \quad (6.1)$$

in the planar limit¹ for light-like polygonal contours \mathcal{C}_n in pure Chern-Simons and ABJM theory. Note that due to the light-like contour, there is no difference in ABJM theory between the standard loop operator (6.1) above and the 1/6 BPS supersymmetric loop operator of [283], as the terms in the exponential coupling to the scalars drop out.

6.1 Wilson Loops in Chern-Simons Theory

In 3d Chern-Simons theory, Wilson loops are the principal observables and topologically invariant with exactly known correlation functions [284] in the Euclidean (or Wick rotated) theory. This exact result is analytic in the inverse Chern-Simons parameter k and perturbative studies in a loop-expansion of the effective coupling constant $1/k$ can reproduce the exact topological and finite result to the first orders [285, 286], modulo regularisation subtleties leading to or not leading to an integer shift of k (for a review see [287]). Wilson loops in Minkowski-space with cusps and light-like segments, however, display particularly strong divergences in 4d gauge theories and seem to not have been considered in the 3d Chern-Simons literature before.

¹I.e. we take the limit $N, k \rightarrow \infty$, $\frac{N}{k}$ finite.

Wilson loops in Chern-Simons theory are usually defined with a framing procedure [284, 285, 286], which may be thought of as a widening of the Wilson line to a ribbon. This is necessary in order to define an integer twisting number of the individual loop and acts as a particular point-splitting regulator for collapsing gauge field propagators in perturbation theory, while preserving the topological structure of the theory. Here, we refrain from framing our loops as we do not encounter the problem of collapsing gauge field propagators due to the piece-wise linear structure of our loops. Moreover, the ABJM theory is not topological due to metric dependent interactions in the matter sector, so that there is no need for framing from that perspective either. Instead, we regulate our correlators by the method of dimensional reduction, which has been tested to the three-loop order in pure 3d Chern-Simons to yield a vanishing β -function and to satisfy the Slavnov-Taylor identities [288]. Here, the tensor algebra is performed in 3 dimensions to obtain scalar integrands and then the dimension of the integrations are analytically continued.

6.2 One Loop: Chern-Simons and ABJM Theory

In this section we consider the one-loop expectation value of polygonal Wilson loops with n cusps. We would like to consider kinematical configurations for which all non-zero distances satisfy $-x_{ij}^2 > 0$, such that the result for the Wilson loops will be real (In particular, this allows us to drop the $i\epsilon$ prescription of the propagators). For n odd, however, it is impossible to find vectors p_i^μ that lead to such configurations. For this reason, we will only discuss n even.

We use the metric with signature $\eta_{\mu\nu} = \text{diag}(1, -1, -1)$ and employ a similar notation as in the $\mathcal{N} = 4$ SYM case. An n -sided polygon can be defined by n points x_i ($i = 1, \dots, n$), with the edge i being the line connecting x_i and x_{i+1} . Defining

$$p_i^\mu = x_{i+1}^\mu - x_i^\mu \quad (6.2)$$

and parametrising the position z_i^μ on edge i with the parameter $s_i \in [0, 1]$ we have

$$z_i^\mu(s_i) = x_i^\mu + p_i^\mu s_i, \quad z_{ij}^\mu := z_i^\mu - z_j^\mu. \quad (6.3)$$

Furthermore, we use the notations

$$\epsilon(p, q, r) = \epsilon_{\mu\nu\rho} p^\mu q^\nu r^\rho \quad \text{and} \quad \bar{s}_i = 1 - s_i. \quad (6.4)$$

We use the Lagrangian of ABJM theory given in F.3.

At one-loop level, we only need terms quadratic in the expansion of the Wilson loop operator, and the free part of the action. Therefore, at one loop order, the expectation value of (6.1) in ABJM theory coincides² with the one in pure Chern-Simons theory.

²For each of the gauge fields A_μ, \hat{A}_μ present in ABJM theory.

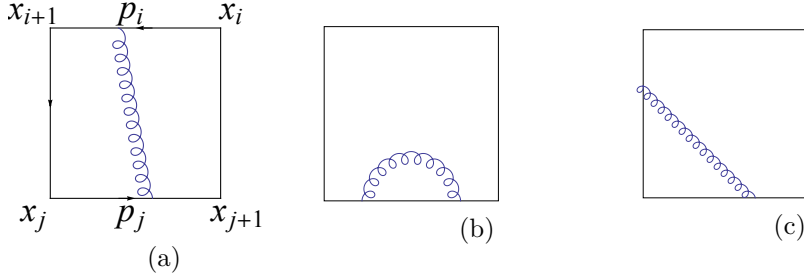


Figure 6.1: One-loop diagrams.

The expectation value at one loop is a sum over all possible diagrams where the propagator stretches between edges i and j ,

$$\begin{aligned} \langle W(\mathcal{C}_n) \rangle^{(1)} &= \frac{(i)^2}{N} \sum_{i \geq j} \int ds_i ds_j \dot{z}_i^\mu \dot{z}_j^\nu \langle (A_\mu)_{mn}(z_i) (A_\nu)_{nm}(z_j) \rangle \\ &=: -\frac{N}{k} \frac{\Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d-2}{2}}} \sum_{i \geq j} I_{ij}, \end{aligned} \quad (6.5)$$

where the domain of integration is given by $\int_0^1 ds_i \int_0^1 ds_j$ for $i \neq j$, $\int_0^1 ds_i \int_0^{s_i} ds_j$ for $i = j$ and $\dot{z}(s_i) = dz(s_i)/ds_i = p_i$ and where we have introduced a normalisation factor for later convenience. Here and throughout this chapter, we absorb the dimensional regularisation scale $(\mu^2)^\epsilon$ into k and only display it explicitly in our final results. Using the Chern-Simons propagator in the Landau gauge³

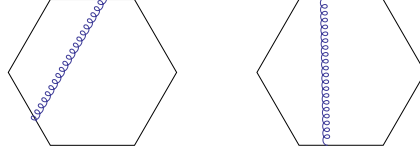
$$\langle (A_\mu)_{mn}(x) (A_\nu)_{kl}(y) \rangle = \delta_{ml} \delta_{nk} \frac{1}{k} \left(\frac{\Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d-2}{2}}} \right) \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{(-(x-y)^2)^{\frac{d}{2}}}, \quad (6.6)$$

and plugging in the expressions (6.3) for z_i , we obtain

$$I_{ij} = \int ds_i ds_j \frac{\epsilon(p_i, p_j, p_i s_i - p_j s_j + \sum_{k=j}^{i-1} p_k)}{\left(-x_{ij}^2 \bar{s}_i \bar{s}_j - x_{i+1,j}^2 s_i \bar{s}_j - x_{i,j+1}^2 \bar{s}_i s_j - x_{i+1,j+1}^2 s_i s_j \right)^{\frac{d}{2}}}, \quad (6.7)$$

where $x_{i,j}^2 = (x_i - x_j)^2$, $\bar{s}_i = 1 - s_i$ and $\epsilon(a, b, c) = \epsilon_{ijk} a^i b^j c^k$. We can immediately see that in this gauge $I_{i,i}$ and $I_{i,i+1}$ vanish due to the antisymmetry of the ϵ tensor. This corresponds to diagrams where the propagator ends on the same edge or on adjacent edges, as shown in Figures 6.1(b) and 6.1(c), respectively. Therefore we only need to keep diagrams of the type shown in Figure 6.1(a). The latter are manifestly finite in three dimensions and therefore we set $d = 3$ in the remainder of this section.

³We drop the $i\epsilon$ prescription in the propagator, since we consider kinematical configurations with $-x_{ij}^2 > 0$.

Figure 6.2: One-loop contributions $I_{i+2,i}$ and $I_{i+3,i}$ to the hexagonal Wilson loop.

6.2.1 Tetragon

As explained above, in the Landau gauge, the only non-vanishing contributions to (6.5) for the tetragon are I_{31} and I_{42} . Setting $d = 3$, they are given by

$$I_{31} = -\epsilon(p_1, p_2, p_3) \int ds_1 ds_3 \frac{1}{(-x_{13}^2 \bar{s}_1 \bar{s}_3 - x_{24}^2 s_1 s_3)^{3/2}}, \quad (6.8)$$

and

$$I_{42} = -\epsilon(p_2, p_3, p_4) \int ds_2 ds_4 \frac{1}{(-x_{24}^2 \bar{s}_2 \bar{s}_4 - x_{13}^2 s_2 s_4)^{3/2}}. \quad (6.9)$$

Taking into account that we have a closed contour, i.e. $\sum_i p_i = 0$, we can write $\epsilon(p_2, p_3, p_4) = -\epsilon(p_2, p_3, p_1) = -\epsilon(p_1, p_2, p_3)$ and thus the contributions from the two diagrams cancel each other

$$\langle W_n \rangle^{(1)} \propto \left(\begin{array}{c} \square \\ \vdots \end{array} + \begin{array}{c} \square \\ \text{---} \end{array} \right) = I_{31} + I_{42} = 0. \quad (6.10)$$

We will see in section 6.4 that this result is compatible with the restrictions imposed by conformal symmetry.

6.2.2 Hexagon and Higher Polygons

For the hexagon there are two different non-vanishing types of contributions, $I_{i+2,i}$ and $I_{i+3,i}$, as shown in Figure 6.2. The former appears in six orientations, $i = 1 \dots 6$ (with the convention that $i + 6 \equiv i$), while the latter appears in three orientations, $i = 1, 2, 3$.

Specialising the general formula (6.7) to these cases we have

$$I_{i+2,i} = \int_0^1 ds_{i+2} ds_i \frac{\epsilon(p_{i+2}, p_i, p_{i+1})}{(-\bar{s}_i \bar{s}_{i+2} x_{i,i+2}^2 - s_i \bar{s}_{i+2} x_{i,i+3}^2 - s_i s_{i+2} x_{i+1,i+3}^2)^{3/2}} \quad (6.11)$$

and

$$I_{i+3,i} = \int_0^1 ds_{i+3} ds_i \frac{\epsilon(p_{i+3}, p_i, p_{i+1} + p_{i+2})}{(-\bar{s}_i \bar{s}_{i+3} x_{i,i+3}^2 - s_i \bar{s}_{i+3} x_{i+1,i+3}^2 - \bar{s}_i s_{i+3} x_{i,i+4}^2 - s_i s_{i+3} x_{i+1,i+4}^2)^{3/2}}. \quad (6.12)$$

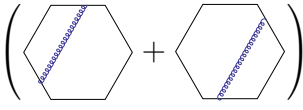
6.3. FOUR-SIDED WILSON LOOP IN CHERN-SIMONS THEORY AT TWO LOOPS 67

We checked numerically for various non-symmetric hexagon configurations that the sum over all diagrams vanishes,

$$\langle W(\mathcal{C}_6) \rangle^{(1)} \propto \sum_{i>j}^6 I_{ij} = 0. \quad (6.13)$$

It is easy to see that (6.13) is true for special configurations, as we will see presently.

Consider the configuration where opposite edges are anti-parallel, i.e. $p_i = -p_{i+3}$. From (6.12) we see that $I_{i,i+3} = 0$ due to the antisymmetry of the ϵ tensor. Furthermore, taking into account that for this configuration we have $x_{i,i+2}^2 = x_{i+3,i+5}^2$, it is easy to see from equation (6.11) that the integrands of $I_{i,i+2}$ and $I_{i+3,i+5}$ are the same. Finally, using $\sum_i p_i = 0$ one can see that the Levi-Civita symbols produce a differing sign, such that

$$I_{i,i+2} + I_{i+3,i+5} = \left(\text{Hexagon 1} + \text{Hexagon 2} \right)_{p_i = -p_{i+3}} = 0, \quad (6.14)$$


i.e. the contributions coming from those diagrams cancel pairwise, and we arrive at equation (6.13), in the specific anti-parallel kinematical configuration $p_i = -p_{i+3}$.

The conjecture that all n -cusped Wilson loops vanish at one-loop order in Chern-Simons theory was indeed proven in [21].

6.3 Four-Sided Wilson Loop in Chern-Simons Theory at Two Loops

In this section we calculate the two-loop contributions to the tetragonal light-like Wilson loop in pure Chern Simons theory. The results are consistent with the anomalous conformal Ward identity to be discussed in section 6.4.

Expanding the Wilson loop to quartic order, see (E.4), and performing Wick contractions leads to the topologies shown in Figure 6.3. We are taking the planar limit and therefore drop all non-planar graphs. Moreover, all diagrams where one propagator is connected to a single edge or adjacent edges vanish in our gauge for the same reason as at the one-loop order and are not displayed.

6.3.1 Ladder Diagrams

Let us begin by computing diagrams of ladder topology as shown in Figure 6.7a. There are two different orientations of this diagram, and it is easy to see that they give the same contribution. Taking into account this factor of 2, the contribution of the ladder diagrams is

$$\langle W_n \rangle_{\text{ladder}}^{(2)} = 2 \left(\frac{N}{k} \right)^2 \left(\frac{\Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d-2}{2}}} \right)^2 I_{\text{ladder}}(x_{13}^2, x_{24}^2), \quad (6.15)$$

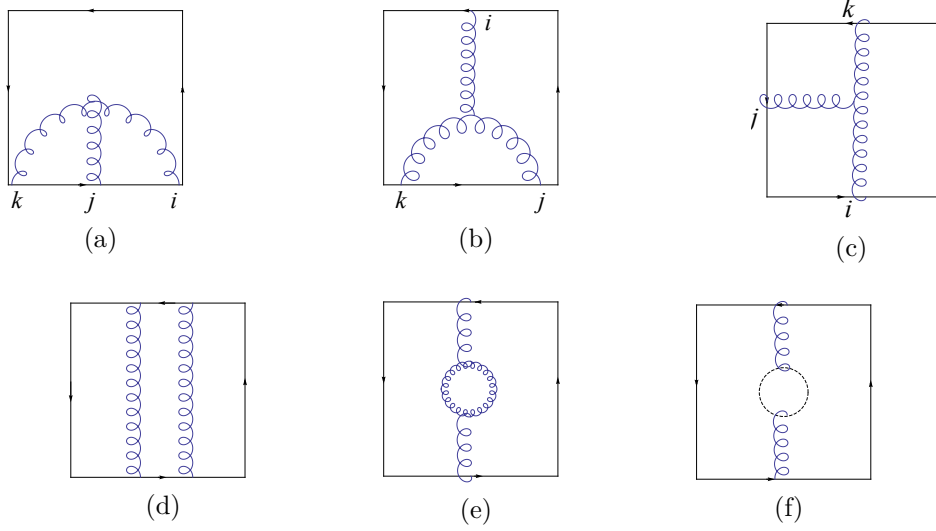


Figure 6.3: Planar two-loop topologies appearing in the polygonal Wilson loop in CS theory. Diagrams where one propagator is connected to a single edge or to two adjacent edges vanish in our gauge and are not displayed.

where

$$I_{\text{ladder}}(x_{13}^2, x_{24}^2) = \int ds_{i,j,k,l} \frac{\epsilon(\dot{z}_i, \dot{z}_l, z_i - z_l)}{[-(z_i - z_l)^2]^{\frac{d}{2}}} \frac{\epsilon(\dot{z}_j, \dot{z}_k, z_j - z_k)}{[-(z_j - z_k)^2]^{\frac{d}{2}}}. \quad (6.16)$$

The integral is finite and may be calculated for $d = 3$

$$I_{\text{ladder}}(x_{13}^2, x_{24}^2) = \frac{1}{4} \int_0^1 ds_i \int_0^{s_i} ds_j \int_0^1 ds_k \int_0^{s_k} ds_l \frac{x_{13}^2 x_{24}^2 (x_{13}^2 + x_{24}^2)}{[x_{13}^2 \bar{s}_i \bar{s}_l + x_{24}^2 s_i s_l]^{\frac{3}{2}} [x_{13}^2 \bar{s}_j \bar{s}_k + x_{24}^2 s_j s_k]^{\frac{3}{2}}} + O(\epsilon).$$

We computed this integral by first carrying out some of the parameter integrals and then deriving a differential equation for it, which could be solved. The result is remarkably simple,

$$I_{\text{ladder}}(x_{13}^2, x_{24}^2) = \frac{1}{2} \left[\ln^2 \left(\frac{x_{13}^2}{x_{24}^2} \right) + \pi^2 \right] + O(\epsilon). \quad (6.17)$$

Including the prefactors and dropping $O(\epsilon)$ terms, the contribution to the Wilson loop is

$$\langle W_4 \rangle_{\text{ladder}}^{(2)} = \left(\frac{N}{k} \right)^2 \frac{1}{4} \left[\ln^2 \left(\frac{x_{13}^2}{x_{24}^2} \right) + \pi^2 \right]. \quad (6.18)$$

6.3.2 Vertex Diagrams

The diagrams with one three-gluon vertex shown in Figures 6.3a , 6.3b and 6.7b are obtained by contracting the cubic term in the expansion of the Wilson loop in (E.4)

6.3. FOUR-SIDED WILSON LOOP IN CHERN-SIMONS THEORY AT TWO LOOPS⁶⁹

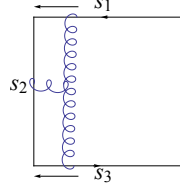


Figure 6.4: The divergence in the vertex diagram arises from the integration region where $s_1 \rightarrow 1$, $s_3 \rightarrow 0$ (and $\beta_1 \rightarrow 0$, $\beta_3 \rightarrow 0$), see equation (6.22).

with the interaction term of the Lagrangian,

$$\langle W_n \rangle_{\text{vertex}}^{(2)} = \left(\frac{N}{k}\right)^2 \frac{i}{2\pi} \left(\frac{\Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d-2}{2}}}\right)^3 \sum_{i>j>k} I_{ijk}, \quad (6.19)$$

where

$$I_{ijk} = - \int dz_i^\mu dz_j^\nu dz_k^\rho \epsilon^{\alpha\beta\gamma} \epsilon_{\mu\alpha\sigma} \epsilon_{\nu\beta\lambda} \epsilon_{\rho\gamma\tau} \int d^d w \frac{(w-z_i)^\sigma (w-z_j)^\lambda (w-z_k)^\tau}{|w-z_i|^d |w-z_j|^d |w-z_k|^d}, \quad (6.20)$$

and $|z_i| = (-z_i^2)^{\frac{1}{2}}$. Here the indices of I_{ijk} refer to the edges of the Wilson loop that the propagators attach to. The expression can be shown to be antisymmetric under the exchange of any two indices, and therefore the only non-vanishing contributions are the ones for $i \neq j \neq k$. As a consequence, topologies 6.3a and 6.3b can be discarded.

Specialising to the tetragon, we have four contributions which are symmetric under $x_{13}^2 \leftrightarrow x_{24}^2$ and thus it is sufficient to compute one of them

$$\begin{aligned} I_{321} &= \int d^d w \int_0^1 ds_{1,2,3} \frac{\epsilon(p_2, p_3, w) \epsilon(p_2, p_1, w)}{|w|^d |w-z_{12}|^d |w-z_{32}|^d} \\ &= \frac{i\pi^{\frac{d}{2}} \Gamma(d-1)}{8 \Gamma\left(\frac{d}{2}\right)^3} x_{13}^2 x_{24}^2 \int_0^1 d^3 s_{1,2,3} d^3 \beta_{1,2,3} (\beta_1 \beta_2 \beta_3)^{\frac{d-2}{2}} \delta\left(\sum_{i=1}^3 \beta_i - 1\right) \times \\ &\quad \times \left(\frac{1}{\Delta^{d-1}} - 2 \frac{(d-1)}{\Delta^d} \beta_1 \beta_3 \bar{s}_1 s_3 (x_{13}^2 + x_{24}^2)\right), \end{aligned} \quad (6.21)$$

where the second and third line is obtained by introducing Feynman parameters in the standard way and integrating over w . More details may be found in Appendix G.1. Δ is given by

$$\begin{aligned} \Delta &= -\beta_1 \beta_2 z_{12}^2 - \beta_2 \beta_3 z_{23}^2 - \beta_1 \beta_3 z_{13}^2 \\ &= -x_{13}^2 \beta_1 \bar{s}_1 (\beta_3 \bar{s}_3 + \beta_2 s_2) - x_{24}^2 \beta_3 s_3 (\beta_2 \bar{s}_2 + \beta_1 s_1). \end{aligned} \quad (6.22)$$

One might naively think that this diagram should give a finite answer due to the antisymmetry of the ϵ tensors. The result would indeed be finite in the case of smooth

contours [285] or contours with a single cusp. However, the presence of two cusps gives rise to a region in the integration space of Feynman parameters where the first summand in the third line of (6.21) induces a divergent contribution. The relevant region of Feynman parameters is $s_1 \rightarrow 1, s_3 \rightarrow 0$ (and $\beta_1 \rightarrow 0, \beta_3 \rightarrow 0$), see equation (6.22), and is illustrated in Figure 6.4. Due to the presence of three independent vectors p_1^μ, p_2^μ and p_3^μ the ϵ tensors do not suppress this region. We find that this term produces a $1/\epsilon$ pole in dimensional reduction. The second summand in the third line of (6.21) is finite.

The divergent and finite pieces can be separated using Mellin-Barnes techniques. The details can be found in Appendix G.1. We have not computed the coefficients of the ϵ^0 terms analytically (the coefficient of the pole-term is evaluated analytically in the Appendix G.5.1), but we have good numerical evidence that they give the following result:

$$I_{321} = \frac{i\pi^{\frac{d}{2}+1} \Gamma(d-1)}{8 \Gamma\left(\frac{d}{2}\right)^3} \left[2 \ln(2) \frac{(-x_{13}^2)^{2\epsilon} + (-x_{24}^2)^{2\epsilon}}{\epsilon} + \ln^2\left(\frac{x_{13}^2}{x_{24}^2}\right) + a_6 + \mathcal{O}(\epsilon) \right], \quad (6.23)$$

where $a_6 = 8.354242685 \pm 2 \cdot 10^{-9}$, see (G.26) and below for an analytical guess of this constant. Taking into account all prefactors and restoring the regularisation scale, $k \rightarrow \mu^{-2\epsilon} k$, we can write the result, up to terms of order ϵ , as

$$\langle W_4 \rangle_{\text{vertex}}^{(2)} = -\frac{1}{4} \left(\frac{N}{k}\right)^2 \left[\ln(2) \sum_{i=1}^4 \frac{(-x_{i,i+2}^2 \tilde{\mu}^2)^{2\epsilon}}{\epsilon} + \ln^2\left(\frac{x_{13}^2}{x_{24}^2}\right) + a_6 - 8 \ln(2) \right], \quad (6.24)$$

where $\tilde{\mu}^2 = \mu^2 \pi e^{\gamma_E}$.

6.3.3 Gauge Field and Ghost Loops

It is well known [288] that in the dimensional reduction (DRED) scheme the gauge field loop diagrams shown in Figure 6.3e exactly cancel against the ghost loop diagrams shown in Figure 6.3f :

$$\langle W_n \rangle_{\text{gluon loop}}^{(2)} = -\langle W_n \rangle_{\text{ghost loop}}^{(2)}. \quad (6.25)$$

Details of this cancellation can be found in Appendix G.2.

6.3.4 Result for the Two-Loop Tetragon in CS Theory

Summing up the results (6.18), (6.24) and (6.25) for the tetragon, interestingly, the $\ln^2(x_{13}^2/x_{24}^2)$ terms in the two-gauge-field diagram and the vertex diagram exactly cancel and we obtain

$$\langle W_n \rangle^{(2)} = -\left(\frac{N}{k}\right)^2 \frac{1}{4} \left[\ln(2) \sum_{i=1}^4 \frac{(-x_{i,i+2}^2 \tilde{\mu}^2)^{2\epsilon}}{\epsilon} + a_6 - 8 \ln(2) - \pi^2 \right]. \quad (6.26)$$

where $\tilde{\mu}^2 = \mu^2 \pi e^{\gamma_E}$ and we recall that $a_6 = 8.354242685 \pm 2 \cdot 10^{-9}$. As we will see in the next section, the cancellation observed here that led to the finite part of (6.26) being a constant is in fact a consequence of the (broken) conformal symmetry of the Wilson loops under consideration.

6.4 Anomalous Conformal Ward Identities

The structure of the above results can be understood from conformal symmetry, by deriving anomalous conformal Ward identities for the Wilson loops analogously to section 4.2.2.

If the Wilson loop $\langle W_n \rangle$ were well-defined in $d = 3$ dimensional Minkowski space it would enjoy the conformal invariance of the underlying gauge theory and we would conclude that $\langle W(\mathcal{C}) \rangle = \langle W(\mathcal{C}') \rangle$. This is indeed the case at one loop order, see section 6.2. However, as we have seen in section 6.3, starting from two loops, divergences force us to introduce a regularisation and calculate in $d = 3 - 2\epsilon$ dimensions, thereby breaking the conformal invariance of the action. The latter leads to an anomalous term in the conformal Ward identities for the Wilson loops

$$D \langle W_n \rangle = -\frac{2i\epsilon}{\mu^{2\epsilon}} \int d^d x \langle \mathcal{L}(x) W_n \rangle, \quad (6.27)$$

$$K^\nu \langle W_n \rangle = -\frac{4i\epsilon}{\mu^{2\epsilon}} \int d^d x x^\nu \langle \mathcal{L}(x) W_n \rangle, \quad (6.28)$$

for dilatations and special conformal transformations. Here the operators on the left-hand sides act in the canonical way on the coordinates of the cusp points,

$$D = \sum_i (x_i \cdot \partial_i), \quad K^\nu = \sum_i (2x_i^\nu (x_i \cdot \partial_i) - x_i^2 \partial_i^\nu). \quad (6.29)$$

We emphasise that thanks to the factor of ϵ on the r.h.s. of (6.27) and (6.28) it is sufficient to know the divergent part of the integrals appearing on the r.h.s. of those equations in order to obtain information about the finite part of $\langle W_n \rangle$.

6.4.1 One-Loop Insertions

At order N/k we have a contribution from the contraction of the kinetic part of the Lagrangian insertion with the second order expansion of the Wilson loop operator, shown in fig. 6.5,

$$\langle \mathcal{L}(x) W_n \rangle^{(1)} = \langle \mathcal{L}_{\text{kin}}(x) W_n \rangle^{(1)} = \frac{(i)^2}{N} \int_{z_i > z_j} dz_{i,j}^{\mu,\nu} \epsilon^{\alpha\beta\gamma} \langle \text{Tr}(A_\alpha \partial_\beta A_\gamma)(x) \text{Tr}(A_\mu A_\nu) \rangle^{(1)}. \quad (6.30)$$

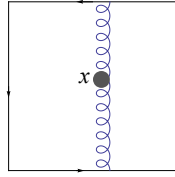


Figure 6.5: Lagrangian insertion contributing to the Ward identities at one loop.

The direct calculation of the right-hand sides of (6.27) and (6.28), yields a vanishing result as $\epsilon \rightarrow 0$. Thus we have

$$D\langle W_n \rangle^{(1)} = O(\epsilon), \quad \text{and} \quad K^\nu \langle W_n \rangle^{(1)} = O(\epsilon), \quad (6.31)$$

in other words the conformal symmetry is unbroken for $\epsilon = 0$. As a consequence, the expectation value of the Wilson loop is constrained to be a function of conformally invariant variables. Starting from the Lorentz invariants x_{ij}^2 the most general conformal invariants are the cross-ratios

$$u_{ijkl} := \frac{x_{ij}^2 x_{kl}^2}{x_{il}^2 x_{jk}^2}. \quad (6.32)$$

In our case where neighbouring points are light-like separated, $x_{i,i+1}^2 = 0$, non-vanishing cross-ratios can only be written down starting from $n = 6$. The special conformal Ward identities (6.31) then imply that $\langle W_n \rangle^{(1)}$ is given by a function of conformal cross-ratios,

$$\langle W_n \rangle^{(1)} = g_n(u_{ijkl}), \quad \langle W_n \rangle^{(1)} = \text{const.} \quad (6.33)$$

Since there are no non-vanishing conformal cross-ratios at four points, $\langle W_n \rangle^{(1)}$ must be a constant.

Let us now compare against the results of our one-loop computation of section 6.2. There, the constant on the r.h.s. of the second equation in (6.33) was found to be zero for the tetragon. Moreover, analytical investigations of certain symmetric contours and numerical investigations for non-symmetric contours show that $g_6(u_{ijkl})$ is zero for the hexagon. It was proven in [21] for arbitrary n that g_n is zero.

6.4.2 Two-loop Insertions

At two loops there are several diagrams that contribute to the insertion of the Lagrangian into the Wilson loop, $\langle \mathcal{L}(x)W_n \rangle$, that correspond to the kinetic term, the gauge field vertex, the ghost kinetic term and the ghost vertex in $\mathcal{L}(x)$. Those diagrams are shown in Figure 6.6. We do not display diagrams that vanish for kinematical reasons as at one-loop level.

Just as at one-loop level, only diagrams giving rise to divergent integrals will contribute to the anomalous Ward identities.

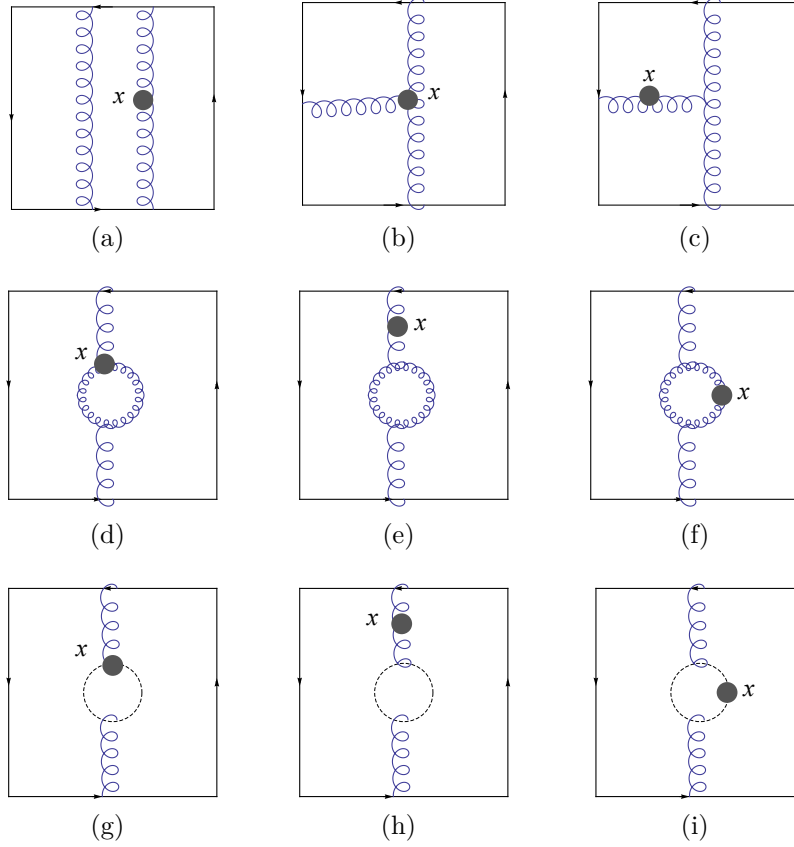


Figure 6.6: Two-loop contributions to the anomalous conformal Ward identity. The grey point in the pictures denotes the point x of the Lagrangian insertion in $\langle \mathcal{L}(x)W_n \rangle$. The diagrams of the second and third line cancel pairwise.

Insertion into the ladder diagram

Let us consider the insertion of the kinetic term of the action into the ladder diagram as shown in Figure 6.6a. For the dilatation Ward identity these are exactly the two-gluon diagrams calculated above, which are finite. For the special conformal Ward identity the integration is slightly more complicated, but the finiteness is easy to check for all contributions. Thus, this diagram does not contribute to the anomalous Ward identities.

Insertion of the interaction term

Next, we can contract the cubic order expansion of the Wilson loop with the vertex term of the Lagrangian insertion, as shown in Figure 6.6b,

$$\langle \mathcal{L}(x)W_n \rangle_{(b)}^{(2)} = \langle \mathcal{L}_{\text{int}}(x) \frac{1}{N} (i)^3 \int d^3 z_{i,j,k}^{\mu,\nu\rho} \text{Tr} (A_\mu A_\nu A_\rho) \rangle. \quad (6.34)$$

For the dilatation Ward identity, we trivially have

$$\int d^d x \langle \mathcal{L}(x) W_4 \rangle_{(b)}^{(2)} = \frac{1}{i} \langle W_n \rangle_{\text{vertex}}^{(2)} = i \left(\frac{N}{k} \right)^2 \frac{\ln(2)}{\epsilon} + \mathcal{O}(\epsilon^0), \quad (6.35)$$

which is just the vertex diagram that was calculated in (G.56), up to a factor of i .

The contribution to the special conformal Ward identity is more complicated. We have

$$\int d^d x x^\nu \langle \mathcal{L}(x) W_n \rangle_{(b)}^{(2)} = \left(\frac{N}{k} \right)^2 \frac{1}{2\pi} \left(\frac{\Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d-2}{2}}} \right)^3 \sum_{i>j>k} I'_{ijk} \quad (6.36)$$

with

$$I'_{321} = \int_0^1 ds_{1,2,3} \int d^d x (x + z_2)^\nu \frac{\epsilon(p_2, p_3, x) \epsilon(p_2, p_1, x)}{|x|^d |x - z_{12}|^d |x - z_{32}|^d} \quad (6.37)$$

$$= \frac{2\pi i \ln(2)}{\epsilon} (x_2 + x_3)^\nu + \mathcal{O}(\epsilon^0), \quad (6.38)$$

where the coefficient $\ln(2)$ was computed numerically to 10 relevant digits. The reason the pole arises was discussed in section 6.3.2.

Details of this calculation can be found in Appendix G.3.1. Summing up all four contributions $I'_{321}, I'_{421}, I'_{432}, I'_{431}$ we arrive at

$$\int d^d x x^\nu \langle \mathcal{L}(x) W_n \rangle_{(b)}^{(2)} = \frac{i}{\epsilon} \left(\frac{N}{k} \right)^2 \frac{\ln(2)}{4} \sum_{i=1}^4 x_i^\nu. \quad (6.39)$$

Insertion of the kinetic term into the vertex diagram

Furthermore, we can contract one gauge field of the kinetic term of the insertion with the Wilson loop and the other one with the 3-gauge-field vertex, leading to a diagram of the type displayed in Figure 6.6c,

$$\begin{aligned} \langle \mathcal{L}(x) W_n \rangle_{(c)}^{(2)} &= \langle \mathcal{L}_{\text{kin}}(x) \frac{1}{N} \oint_{z_i > z_j > z_k} dz_{i,j,k}^{\mu,\nu,\rho} \text{Tr} (A_\mu A_\nu A_\rho) \left(i \int d^d w \mathcal{L}_{\text{int}} \right) \rangle \quad (6.40) \\ &= \left(\frac{k}{4\pi} \right)^2 (i)^3 \frac{2}{3} \frac{1}{N} \int d^d x \int d^d w \oint dz_{i,j,k}^{\mu,\nu,\rho} \\ &\quad \epsilon^{\alpha\beta\gamma} \epsilon^{\delta\sigma\tau} \langle \text{Tr} (A_\alpha \partial_\beta A_\gamma)(x) \text{Tr} (A_\mu A_\nu A_\rho) \text{Tr} (A_\delta A_\sigma A_\tau)(w) \rangle. \end{aligned}$$

Let us Wick-contract the kinetic term with $A_\nu(z_j)$ (the two other contractions are discussed below.) We obtain

$$\left(\frac{N}{k} \right)^2 \frac{i}{8\pi^2} \left(\frac{\Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d-2}{2}}} \right)^4 \int d^d w \oint dz_{i,j,k}^{\mu\nu\rho} \epsilon^{\delta\sigma\tau} I_{\nu\sigma} G_{\mu\tau}(z_i - w) G_{\rho\delta}(z_k - w),$$

where $G_{\mu\nu} = \epsilon_{\mu\nu\rho}(x-y)^\rho / (-(x-y)^2)^{\frac{d}{2}}$ and where

$$I_{\nu\sigma}(x-z_j, x-w) = \epsilon^{\alpha\beta\gamma} \left[G_{\alpha\nu}(x-z_j) \partial_\beta^{(x)} G_{\gamma\sigma}(x-w) + G_{\alpha\sigma}(x-w) \partial_\beta^{(x)} G_{\gamma\nu}(x-z_j) \right]. \quad (6.41)$$

The only dependence on the insertion point x is in $I_{\nu\sigma}$. For the dilatation Ward identity the integral $\int d^d x I_{\nu\sigma}$ can easily be computed (for details see Appendix G.3.2) and effectively gives a propagator such that we have

$$\int d^d x \langle \mathcal{L}(x) W_n \rangle_{(c)}^{(2)} = 3i \langle W_n \rangle^{\text{vertex}} = -3i \left(\frac{N}{k} \right)^2 \frac{\ln(2)}{\epsilon} + O(\epsilon^0), \quad (6.42)$$

where a factor of 3 was included since the insertion can be in any of the 3 propagators of the vertex diagram and thus we get three times the same contribution.

The contribution to the special conformal Ward identity is more complicated, since the integration $\int d^d x x^\nu I_{\nu\sigma}$ does not just yield a propagator. Performing the calculation, we find

$$\int d^d x x^\nu \langle \mathcal{L}(x) W_n \rangle_{(c)}^{(2)} = -\frac{i}{\epsilon} \left(\frac{N}{k} \right)^2 \frac{3}{4} \ln(2) \sum_{i=1}^4 x_i^\nu + O(\epsilon^0), \quad (6.43)$$

where the coefficient $\ln(2)$ was computed numerically (details can be found in Appendix G.3.2).

Insertions with gauge field and ghost loops

The gauge-field-ghost-insertions in Figure 6.6 cancel pairwise. For the dilatation Ward identity the insertions of the three gauge-field and the gauge-field-ghost vertices as shown in diagrams 6.6d, 6.6g are identical to the gauge field and ghost loop diagrams (6.25) and thus cancel. It is not necessary to perform the integration over the insertion point to see how the cancellation occurs and thus the contributions to the special conformal Ward identity cancel as well (for details on the cancellation see G.2).

Insertions of the kinetic term into the gauge field propagator as shown in diagrams 6.6e, 6.6h cancel as well, since the insertion is the same for both diagrams and thus the algebraic relations responsible for the cancellation in (6.25) remain unchanged.

Inserting gauge field respectively ghost kinetic terms into the propagators inside the loop in diagrams 6.6f, 6.6i produce slightly more complicated expressions. Nevertheless they cancel as well as can be seen in a straightforward calculation. For the dilatation Ward identity the cancellation can be seen in an even simpler way by noticing that the integration over the insertion point x effectively yields a gauge field respectively ghost propagator. Thus the diagrams are identical to the ones in (6.25) and cancel.

6.4.3 Anomalous Ward Identities and Generalisation to Higher Polygons

Summing up the divergent contributions of (6.35) (6.42), and inserting them into the dilatation Ward identity (6.27), we obtain

$$D \langle W_n \rangle^{(2)} = - \left(\frac{N}{k} \right)^2 \ln(2) \left(\sum_{i=1}^4 1 \right) + \mathcal{O}(\epsilon), \quad (6.44)$$

where we have written the factor 4 as $(\sum_{i=1}^4 1)$ to emphasise its origin from the sum of four vertex-type diagrams. Note that only the divergent part of the vertex-diagram was required here. Summing up (6.39) and (6.43), and inserting them into the special conformal Ward identity (6.28), we obtain

$$K^\nu \langle W_n \rangle^{(2)} = -2 \left(\frac{N}{k} \right)^2 \ln(2) \sum_{i=1}^4 x_i^\nu + \mathcal{O}(\epsilon). \quad (6.45)$$

Let us now explain how these equations can be generalised from $n = 4$ to arbitrary n . In our two-loop computation, we found that the only diagrams contributing to (6.44) and (6.45) are those producing poles in ϵ . The mechanism for how these poles are generated was described in section 6.3.2, see in particular Figure 6.4. It is clear that for $n > 4$ cusps, the same type of vertex diagram will produce the divergent terms. Although those diagrams will depend on one further kinematical variable w.r.t. the four-point case, this dependence cannot change the (leading) UV pole ϵ^{-1} of the diagrams. Since there are n diagrams of this type at n points, we expect

$$D \langle W_n \rangle^{(2)} = - \left(\frac{N}{k} \right)^2 \ln(2) \left(\sum_{i=1}^n 1 \right) + \mathcal{O}(\epsilon), \quad (6.46)$$

and

$$K^\nu \langle W_n \rangle^{(2)} = -2 \left(\frac{N}{k} \right)^2 \ln(2) \sum_{i=1}^n x_i^\nu + \mathcal{O}(\epsilon). \quad (6.47)$$

We will now proceed to discuss the solution of these Ward identities and compare them to the result of the two-loop computation of the tetragon Wilson loop in section 6.3.4.

6.4.4 Solution to the Anomalous Conformal Ward Identities

Using $D(x_{ij}^2) = 2x_{ij}^2$ it is clear that the most general solution to the dilatation Ward identity (6.46) is

$$\langle W_n \rangle^{(2)} = - \left(\frac{N}{k} \right)^2 \left[\frac{\ln(2)}{4} \sum_{i=1}^n \frac{(-x_{i,i+2}^2 \tilde{\mu}^2)^{2\epsilon}}{\epsilon} + f_n \left(\frac{x_{ij}^2}{x_{kl}^2} \right) \right] + \mathcal{O}(\epsilon), \quad (6.48)$$

where f is an arbitrary function of dimensionless variables and we recall that $\tilde{\mu}^2 = \mu^2 \pi e^{\gamma_E}$. Of course, this is exactly what we expect from (4.54).

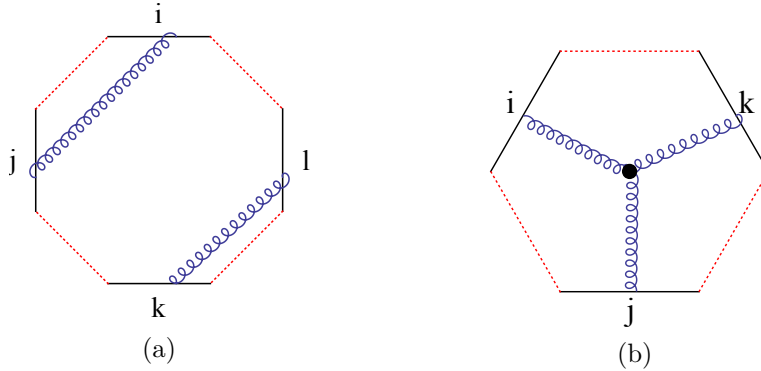


Figure 6.7: 6.7a Two-gluon diagrams I_{ijkl} given in (6.52), 6.7b vertex diagrams I_{ijk} given in (6.54). Dashed lines represent one or more light-like distances.

The result for the special conformal Ward identity is more interesting. Plugging (6.48) into the special conformal Ward identity (6.45) and using $\mathbb{K}^\nu \ln(x_{kl}^2) = 2(x_k + x_l)^\nu$, it is easy to see that the function f_n is allowed to depend on conformally invariant cross-ratios only, i.e. $f_n(x_{ij}^2/x_{kl}^2) = g(u_{abcd})$. Therefore, we finally have

$$\langle W_n \rangle_{2\text{-loop}}^{(\text{CS})} = -\frac{1}{4} \left(\frac{N}{k} \right)^2 \left[2 \ln(2) \sum_{i=1}^n \frac{(-x_{i,i+2}^2 \tilde{\mu}^2)^{2\epsilon}}{2\epsilon} + g_n(u_{abcd}) + \mathcal{O}(\epsilon) \right]. \quad (6.49)$$

In the four-point case, there are no non-vanishing cross-ratios, and therefore in particular g_4 must be a constant. This is in agreement with (6.26) and thus represents an independent check of the direct perturbative computation, including its finite part (recall that deriving the Ward identity does not rely on the finite parts of the direct perturbative computation). So, even though the result for the vertex diagram (6.24) was obtained numerically, its functional form is an analytical result, since we know the analytical expression for the ladder diagram and the sum of vertex and ladder diagram through the solution to the anomalous conformal Ward identity.

6.5 N-Sided Wilson Loops in CS Theory at Two Loops

As mentioned in the previous section, the solution (6.49) of the Ward identity for the light-like polygonal Wilson loop in pure Chern-Simons theory contains a function g_n that depends only on conformally invariant cross-ratios $u_{abcd} = (x_{ab}^2 x_{cd}^2)/(x_{ad}^2 x_{cb}^2)$ which can be constructed starting from $n = 6$. In this section we review the investigations [13], that lead to the conclusion that g_n is a trivial function at two loops.

As in the four-point case, at two loops there are two types of contributions to the Wilson loop in pure Chern-Simons theory, one from two-gluon diagrams $\langle W_n \rangle^{\text{two-gluon}} =$

$\sum I_{ijkl}$ and another one from diagrams involving a three-gluon vertex $\langle W_n \rangle^{\text{vertex}} = \sum I_{ijk}$ such that

$$\langle W_n \rangle_{2\text{-loop}}^{\text{CS}} = \langle W_n \rangle^{\text{two-gluon}} + \langle W_n \rangle^{\text{vertex}}. \quad (6.50)$$

The indices i, j, k, l denote the edges that the propagators attach to, see figure 6.7 and their expressions are given below in (6.52), (6.53). Contributions from gauge- and ghost-loops cancel in dimensional reduction regularisation [288] as explained in 6.3.3.

Two-Gluon Diagrams

The contributions from the two-gluon diagrams are all finite and can be written as

$$\langle W_n \rangle^{\text{two-gluon}} = \frac{1}{4} \left(\frac{N}{k} \right)^2 \sum_{i \geq j \geq k \geq l} (I_{ij,kl} + I_{il,jk}), \quad (6.51)$$

where

$$I_{ij,kl} = \int ds_{i,j,k,l} \frac{\epsilon(\dot{z}_i, \dot{z}_j, z_i - z_j)}{(-(z_i - z_j)^2)^{3/2}} \frac{\epsilon(\dot{z}_k, \dot{z}_l, z_k - z_l)}{(-(z_k - z_l)^2)^{3/2}} \quad (6.52)$$

and the integration boundaries haven to be chosen according to the path ordering, such that $z(s_i) > z(s_j) > z(s_k) > z(s_l)$. The diagram vanishes due to the epsilon tensor contractions if the gluon propagator of at least one of the index pairs connects the same or adjacent edges for the same reason as in the four-point case. For $n = 4$ only the diagram (6.16) appears.

Vertex Diagrams

We write the generalization of the vertex diagrams (6.20) as

$$\begin{aligned} \langle W_n \rangle^{\text{vertex}} &= -\frac{1}{4} \left(\frac{N}{k} \right)^2 \pi^{3-d} \Gamma(d-1) \frac{1}{4\pi} \sum_{i>j>k} (I_{ijk}^{(a)} + I_{ijk}^{(b)}) \\ &= -\frac{1}{4} \left(\frac{N}{k} \right)^2 (\pi e^{\gamma_E})^{2\epsilon} (1-2\epsilon) \frac{1}{4\pi} \sum_{i>j>k} (I_{ijk}^{(a)} + I_{ijk}^{(b)}) + \mathcal{O}(\epsilon). \end{aligned} \quad (6.53)$$

The indices i, j, k indicate the edges the gluon-propagators connect to, see figure 6.7b, and

$$\begin{aligned} I_{ijk}^{(a)} &= \int ds_{i,j,k}, d[\beta]_3 \epsilon^{\alpha\beta\gamma} \epsilon_{\mu\alpha\sigma} \epsilon_{\nu\beta\lambda} \epsilon_{\rho\gamma\tau} p_i^\mu p_j^\nu p_k^\rho \\ &\quad \left[\partial_j^\lambda \Delta \partial_i^\sigma \partial_k^\tau \Delta + \partial_i^\sigma \Delta \partial_j^\lambda \partial_k^\tau \Delta + \partial_k^\tau \Delta \partial_i^\sigma \partial_j^\lambda \Delta \right] \Delta^{1-d}, \\ I_{ijk}^{(b)} &= \int ds_{i,j,k}, d[\beta]_3 \epsilon^{\alpha\beta\gamma} \epsilon_{\mu\alpha\sigma} \epsilon_{\nu\beta\lambda} \epsilon_{\rho\gamma\tau} p_i^\mu p_j^\nu p_k^\rho \\ &\quad \left[\partial_i^\sigma \Delta \partial_j^\lambda \Delta \partial_k^\tau \Delta \right] \Delta^{-d} (1-d), \end{aligned} \quad (6.54)$$

where $\partial_i^\mu = \partial/\partial z_{i,\mu}$ and

$$\int d[\beta]_3 = \int_0^1 d\beta_{i,j,k} \delta\left(\sum_m \beta_m - 1\right) (\beta_i \beta_j \beta_k)^{d/2-2},$$

$$\Delta = -z_{ij}^2 \beta_i \beta_j - z_{ik}^2 \beta_i \beta_k - z_{kj}^2 \beta_k \beta_j.$$

The above expressions are obtained by introducing Feynman parameters in the standard way and integrating over w , exactly as in the four-point case in section 6.3.2. The details are given in Appendix G.4.

As in the four-point case, the vertex diagrams are divergent in the region of integration where all three propagators approach the same edge (all diagrams with more than one propagator on the same edge vanish identically due to the antisymmetry of the Levi-Civita symbol), and we split them up as in (G.92)

$$\langle W_n \rangle^{\text{vertex}} = \langle W_n \rangle^{\text{div}} + \langle W_n \rangle^{\text{finite}}. \quad (6.55)$$

The divergent part is calculated analytically in Appendix G.5.1 and reads

$$\langle W_n \rangle^{\text{div}} = - \left(\frac{N}{k}\right)^2 \left(\frac{\ln(2)}{2} \sum_{i=1}^n \frac{(-x_{i,i+2}^2 \tilde{\mu}^2)^{2\epsilon}}{2\epsilon}\right).$$

Thus, the function g_n in (6.49) is given by

$$g_n(u_{abcd}) = \langle W_n \rangle^{\text{two-gluon}} + \langle W_n \rangle^{\text{finite}}. \quad (6.56)$$

We evaluate these contributions using a Mathematica program that generates all n -point diagrams, performs the index-contractions and numerically integrates the diagrams for randomly generated kinematical configurations, see Appendix G.6.

Hexagonal Wilson loop

For the hexagonal Wilson loop we evaluated the contributions in (6.56) for a large set of conformally equivalent and conformally non-equivalent kinematical configurations.

Conformally equivalent configurations must yield the same result, since, by the anomalous conformal Ward identity, the expectation value is constrained to the form (6.49), and thus the function g_n depends only on conformally invariant quantities.

It turns out, that even for kinematical configurations which are not conformally equivalent, the unknown function yields the same constant⁴

$$g_6(u_{abcd}) = c_6 - 12 \ln(2), \quad c_6 = 5.57 \pm 0.05. \quad (6.57)$$

In figure 6.8 we show the results for the two-gluon and vertex contributions for a continuously deformed kinematical configuration, generated as explained in app. G.6, in

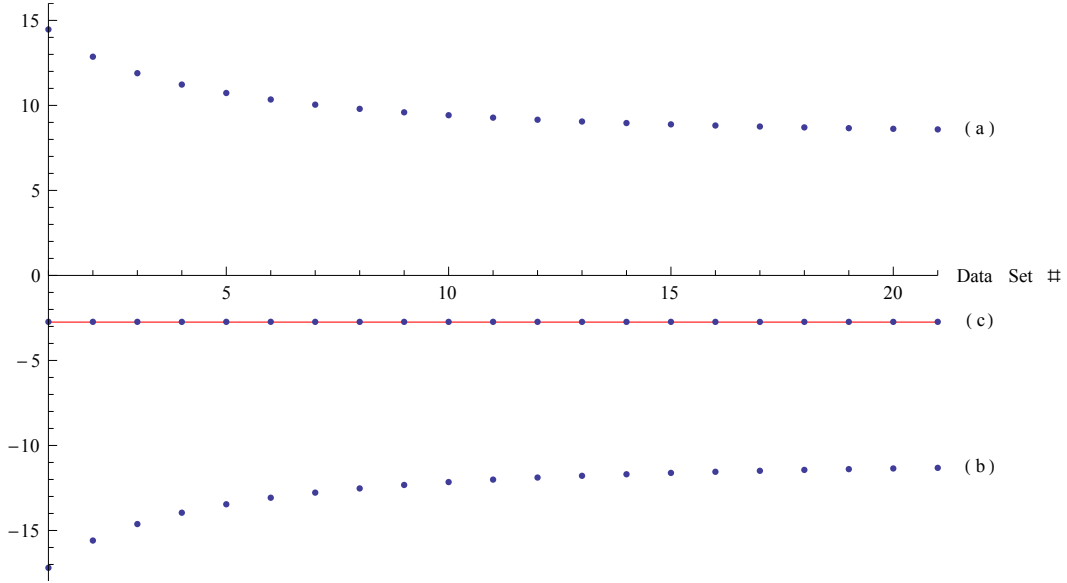


Figure 6.8: Numerical results for the hexagon $n = 6$: (a) $= \langle W_n \rangle^{\text{two-gluon}}$, (b) $= \langle W_n \rangle^{\text{finite}}$ and their sum (c) $= g_n(u_{abcd})$, see (6.56), for the kinematical points given in G.6. We omit the common factor $\frac{1}{4} \left(\frac{N}{k} \right)^2$.

order to illustrate, how the different contributions vary while their sum remains constant.

Generalization to n Cusps

It turns out that also for $n > 6$ the function of conformal cross ratios is just a constant, i.e.

$$g_n(u_{abcd}) = c_n - 2n \ln(2). \quad (6.58)$$

In figure 6.9 we show the dependence of the numerical constant c_n on the number of cusps n up to $n = 14$. Clearly, the constant depends linearly on the number of cusps n . It seems reasonable to assume that this dependence holds for all n , i.e.

$$c_n = a + b \cdot n, \quad (6.59)$$

which is the line shown in fig. 6.9 with the parameters⁵ $a = 6.6 \pm 0.1$, $b = -2.028 \pm 0.025$.

⁴The analytical term with $\ln(2)$ in (6.57),(6.58) arises from the multiplication of the analytically known divergent term with an $\mathcal{O}(\epsilon)$ expansion of the prefactor, see (6.53).

⁵We determine the constants in (6.59) from the results at $n = 4$ and $n = 6$, since here we have the smallest number of integrals and thus the best numerical result. At n -points we have to evaluate $2\binom{n}{3} + 2\binom{n}{4}$ two-gluon (6.51) and vertex integrals (G.94).

Thus, we expect the Chern-Simons contribution to the n -point Wilson loop to be

$$\langle W_n \rangle^{\text{CS}} = -\frac{1}{4} \left(\frac{N}{k} \right)^2 \left[2 \ln(2) \sum_{i=1}^n \frac{(-x_{i,i+2}^2 \tilde{\mu}^2)^{2\epsilon}}{2\epsilon} + g_n \right], \quad (6.60)$$

where g_n is the constant given by (6.58) and (6.59).

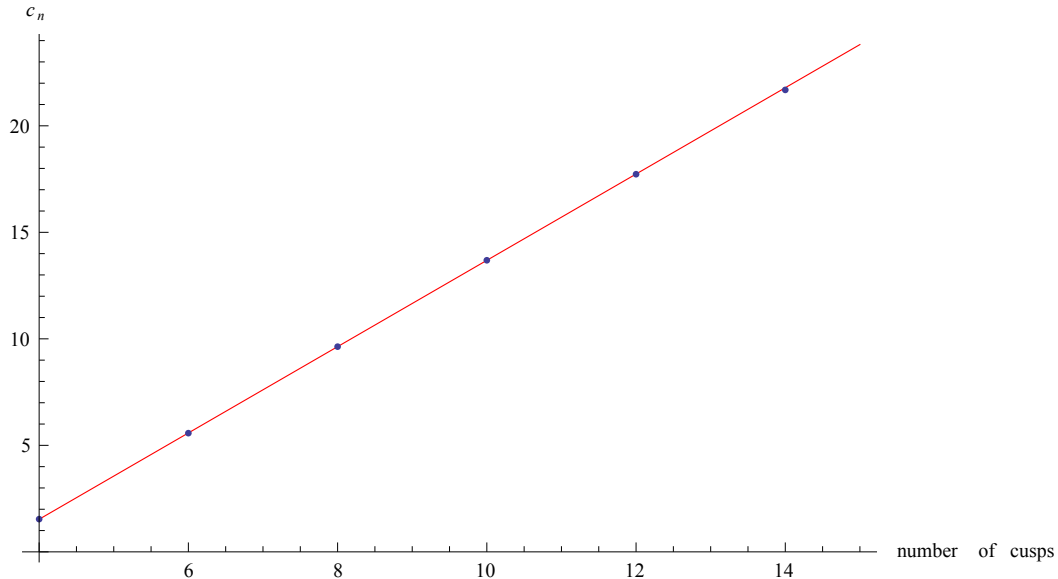


Figure 6.9: This figure shows the dependence of the constant c_n (6.59) on the number of cusps n of the Wilson loop.

6.6 N-Sided Wilson Loops in ABJM Theory at Two Loops

Here we explain how the results are modified in ABJM theory. We use the Wilson loop operator proposed in [283]

$$\langle W(A, \hat{A}) \rangle = \frac{1}{2N} \left\langle \text{Tr} \mathcal{P} \exp \left(i \oint_{\mathcal{C}} A_{\mu} dz^{\mu} \right) + \hat{\text{Tr}} \mathcal{P} \exp \left(i \oint_{\mathcal{C}} \hat{A}_{\mu} dz^{\mu} \right) \right\rangle. \quad (6.61)$$

Note that the sign(s) in the exponent(s) in (6.61) are correlated to corresponding signs in the Lagrangian by the requirement of gauge invariance, see Appendix E.

6.6.1 Gauge Field Contributions

In ABJM theory there is a second copy of the gauge field \hat{A}_{μ} with opposite sign in the Lagrangian (F.12). Up to a sign, the gauge field contributions for both gauge groups

are identical at one loop,

$$\langle W \rangle_A^{(1)} = -\langle W \rangle_{\hat{A}}^{(1)}, \quad (6.62)$$

due to the different sign of the propagator for the second gauge field, $\langle A_\mu A_\nu \rangle = -\langle \hat{A}_\mu \hat{A}_\nu \rangle$. Thus, at one loop the diagrams cancel. This does not differ from the result of pure Chern-Simons theory, since the expectation value at one loop vanishes, as we found in section 6.2 for $n = 4, 6$.

At two loops, however, the sign has no effect, since the two-gluon diagram contains an even number of propagators and in the vertex diagram we have to take into account the sign of the interaction term as well. Therefore, the two-loop diagrams are identical

$$\langle W \rangle_A^{(2)} = \langle W \rangle_{\hat{A}}^{(2)}. \quad (6.63)$$

Thus, up to two loops, the expectation value for pure gauge field contributions is the same in ABJM theory and Chern-Simons theory

$$\langle W(A, \hat{A}) \rangle_{\text{gauge fields}} = \langle W(A) \rangle_{\text{CS}}. \quad (6.64)$$

6.6.2 Matter Contributions

In pure Chern-Simons theory the one-loop correction to the gauge field propagator is zero, since the contributions of gauge fields and ghosts exactly cancel against each other, see (6.25).

In ABJM theory we have to take into account fermionic and bosonic matter in the loop. This gauge field self energy has been calculated in [289, 264, 283] and the corrected propagator reads⁶

$$G_{\mu\nu}^{(1)}(x) = \left(\frac{2\pi}{k}\right)^2 \frac{N\delta_I^I}{8} \frac{\Gamma(1 - \frac{d}{2})\Gamma(\frac{d}{2})^2}{\Gamma(d-1)\pi^d} \left(\frac{\Gamma(d-2)}{\Gamma(2 - \frac{d}{2})} \frac{\eta_{\mu\nu}}{(-x^2)^{d-2}} - \partial_\mu \partial_\nu \left(\frac{\Gamma(d-3)}{\Gamma(3 - \frac{d}{2})} \frac{1}{4} \frac{1}{(-x^2)^{d-3}} \right) \right), \quad (6.65)$$

for details see Appendix F.4. We can drop the derivative term in (6.65) (it would not contribute to the gauge-invariant Wilson loop) and instead use the propagator

$$G_{\mu\nu}^{(1)}(x) = -\frac{1}{N} \left(\frac{N}{k}\right)^2 \pi^{2-d} \Gamma\left(\frac{d}{2} - 1\right)^2 \frac{\eta_{\mu\nu}}{(-x^2)^{d-2}}, \quad (6.66)$$

which up to two small differences is the tree-level $\mathcal{N} = 4$ SYM gluon propagator. The first difference is a trivial prefactor, and the second is that since we are at two loops, the power of $1/x^2$ is $1 - 2\epsilon$ here, as opposed to $1 - \epsilon$ in the one-loop computation in

⁶Recall that we absorbed the regularisation scale into the coupling constant: $k \rightarrow \mu^{-2\epsilon}k$

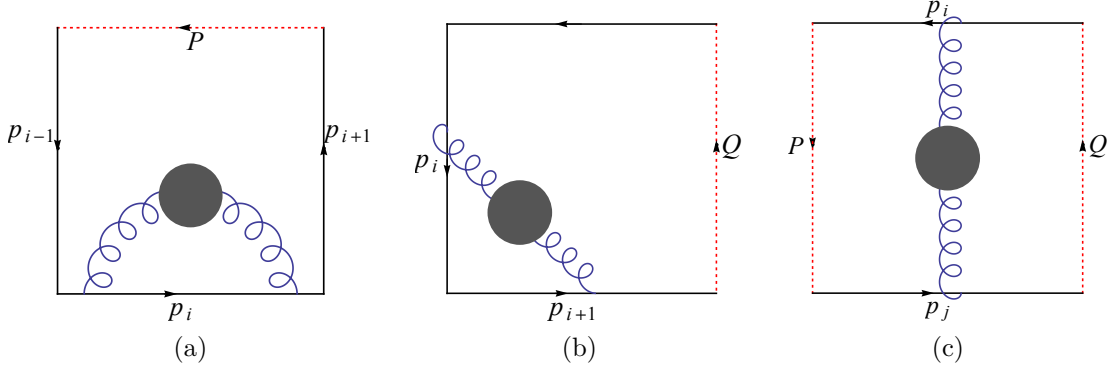


Figure 6.10: Examples for the three classes of diagrams involving the gauge field self energy: diagrams with (a) a propagator connecting the same edge (vanishing), (b) propagator stretching between adjacent edges (divergent), (c) propagator stretching between non-adjacent edges (finite). These diagrams have the same structure as the 1-loop diagrams in $\mathcal{N} = 4$ SYM. Dashed lines labelled with P and Q represent one or more light-like distances.

$\mathcal{N} = 4$ SYM. Thus it is clear that the results will be very similar to the expectation value of the Wilson loop in $\mathcal{N} = 4$ SYM. The corresponding calculation in $\mathcal{N} = 4$ SYM was carried out in [5, 6] and we briefly summarise the results.

As in $\mathcal{N} = 4$ SYM we have three classes of diagrams shown in figure 6.10. Diagram 6.10a vanishes due to the light-likeness of the edges, whereas 6.10b yields a divergent, and 6.10c yields a finite contribution. We have

$$\begin{aligned}
\langle W_n \rangle_{\text{matter}}^{(2)} &= \frac{i^2}{N} \text{Tr} \int_{z_i > z_j} dz_i^\mu dz_j^\nu \langle A_\mu A_\nu \rangle^{(1)} & (6.67) \\
&= -N \sum_{i > j} \int ds_i ds_j p_i^\mu p_j^\nu G_{\mu\nu}^{(1)}(z_i - z_j) \\
&= \left(\frac{N}{k}\right)^2 \left((4\pi e^{\gamma_E})^{2\epsilon} + \frac{\pi^2}{2} \epsilon^2 + \mathcal{O}(\epsilon^3) \right) \sum_{i > j} I_{ij},
\end{aligned}$$

where

$$\begin{aligned}
I_{ij} &= \int ds_i ds_j p_i \cdot p_j (-z_i - z_j)^{2-d} & (6.68) \\
&= \frac{1}{2} \int_0^1 ds_i \int_0^1 ds_j \frac{x_{i,j+1}^2 + x_{i+1,j}^2 - x_{i,j}^2 - x_{i+1,j+1}^2}{(-x_{ij}^2 \bar{s}_i \bar{s}_j - x_{i+1,j}^2 s_i \bar{s}_j - x_{i,j+1}^2 \bar{s}_i s_j - x_{i+1,j+1}^2 s_i s_j)^{d-2}}.
\end{aligned}$$

There are n divergent diagrams $I_{i+1,i}$ of the type shown in fig. 6.10b

$$I_{i+1,i} = -\frac{1}{8} \frac{(-x_{i,i+2}^2)^{2\epsilon}}{\epsilon^2}$$

and the finite diagrams I_{ij} with $|i - j| \geq 1$, see fig. 6.10c, were solved in [6]

$$I_{ij} = \frac{1}{2} \left(-\text{Li}_2 \left(1 - ax_{i,j}^2 \right) - \text{Li}_2 \left(1 - ax_{i+1,j+1}^2 \right) + \text{Li}_2 \left(1 - ax_{i,j+1}^2 \right) + \text{Li}_2 \left(1 - ax_{i+1,j}^2 \right) \right),$$

where

$$a = \frac{x_{ij}^2 + x_{i+1,j+1}^2 - x_{i,j+1}^2 - x_{i+1,j}^2}{x_{ij}^2 x_{i+1,j+1}^2 - x_{i,j+1}^2 x_{i+1,j}^2}.$$

The sum over all finite diagrams $\sum_{i>j+1}^n I_{ij} = \mathcal{F}_n^{\text{WL}}$ is related to the well-known finite part of the BDS conjecture [50] via $\mathcal{F}_n^{\text{WL}} = \mathcal{F}_n^{\text{BDS}} + \text{const.}$, explicitly spelled out in (4.47) for $n = 4$ and $n = 6$. Then, the full matter part reads

$$\langle W_n \rangle_{2\text{-loop}}^{\text{matter}} = -\frac{1}{4} \left(\frac{N}{k} \right)^2 \left[\sum_{i=1}^n \frac{(-x_{i,i+2}^2 \mu^2 4\pi e^{\gamma_E})^{2\epsilon}}{2\epsilon^2} - 4\mathcal{F}_n^{\text{WL}} + \frac{n}{4}\pi^2 \right]. \quad (6.69)$$

where we have restored the regularisation scale μ .

6.7 Equivalence of Wilson Loops in $\mathcal{N} = 4$ SYM and ABJM Theory

Taking into account the Chern-Simons result (6.60), the full result in ABJM theory can be written as

$$\langle W_n \rangle_{2\text{-loop}}^{\text{ABJM}} = \left(\frac{N}{k} \right)^2 \left[-\frac{1}{2} \sum_{i=1}^n \frac{(-x_{i,i+2}^2 \mu'^2)^{2\epsilon}}{(2\epsilon)^2} + \mathcal{F}_n^{\text{WL}} + r_n \right],$$

where $\mu'^2 = \mu^2 8\pi e^{\gamma_E}$, $r_n = -(\frac{n}{4}\pi^2 + c_n - 2n \ln(2) - 5n \ln^2(2))/4$ and c_n is the numerical constant given by (6.59).

Indeed, this is of the same form as the one-loop result of the Wilson loop in $\mathcal{N} = 4$ SYM [6]

$$\langle W_n \rangle_{1\text{-loop}}^{\mathcal{N}=4 \text{ SYM}} = \frac{g^2 N}{8\pi^2} \left[-\frac{1}{2} \sum_{i=1}^n \frac{(-x_{i,i+2}^2 \mu^2)^\epsilon}{\epsilon^2} + \mathcal{F}_n^{\text{WL}} \right], \quad (6.70)$$

up to a few small differences. The first difference is a trivial prefactor, which is due to the different definitions of the coupling constant in the two theories. Secondly, since we are at two loops in ABJM theory, in the divergent part we have 2ϵ instead of ϵ in the one-loop calculation in $\mathcal{N} = 4$ SYM. The finite pieces are exactly the same up to the n -dependent constant r_n .

Remark on the analytical guess for the four-point result

We would like to remark, that in [15] for $n = 4$ we found an analytical guess for the numerical constant a_6 in the vertex diagram, see (G.26), which leads to the final result

$$\langle W_4 \rangle_{2\text{-loop}}^{\text{ABJM}} = \left(\frac{N}{k} \right)^2 \left[-\frac{1}{2} \sum_{i=1}^4 \frac{(-x_{i,i+2}^2 \mu^2)^{2\epsilon}}{(2\epsilon)^2} + \frac{1}{2} \ln^2 \left(\frac{x_{13}^2}{x_{24}^2} \right) + \frac{2}{3} \pi^2 + 3 \ln^2(2) - 2 \ln(2) \right].$$

Note that the $\ln(2)$ term has a different transcendentality than the other terms. It stems from the vertex diagrams of the pure CS part (6.26) of the Wilson loop. We carefully rechecked the appearance of this term, since from the experience in $\mathcal{N} = 4$ SYM one might expect only terms of homogenous transcendentality, but conclude that – given our regularization procedure (dimensional reduction) – it is correct. Furthermore, it is interesting to note, that up to this $\ln(2)$ term, the constant exactly agrees with the constant determined in the four-point amplitude calculation in [17].

Chapter 7

Three-Point Functions of Twist-Two Operators in $\mathcal{N} = 4$ SYM

As mentioned in chapter 1.5, two-point functions in $\mathcal{N} = 4$ SYM are very well understood, largely due to the existence of integrability, while three-point functions are explored to a lesser extent. Progress for three-point functions, comparable to the one that was made for the two-point functions with the help of integrability, would have important implications for all higher-point functions, via the operator product expansion as explained in section 3.5. The investigation of structure constants can thus be seen as the next step towards the full solution of the theory in the sense of section 1.5. In general, the structure constants of three-point functions, see section 3.2, receive radiative corrections

$$C_{\alpha\beta\gamma}(\lambda) = C_{\alpha\beta\gamma}^{(0)} + \lambda C_{\alpha\beta\gamma}^{(1)} + \dots \quad (7.1)$$

and like for two-point functions, there are non-renormalisation theorems such that the three-point correlators of half-BPS operators do not get quantum corrections [290, 291, 292, 293, 294]. We will explicitly see such cancellations in a special case of the three-point functions that we consider in this chapter.

Direct computations of three-point functions in $\mathcal{N} = 4$ SYM theory have been performed in [295, 296, 297, 298, 299, 300, 301, 302, 303]. In [304] a large number of three-point functions involving scalar primary operators of up to and including length five is considered.

The role of integrability for three-point function calculations was first addressed in [299, 298, 300]. and applications of integrability methods can be found in [305, 306, 307, 308] at tree-level and in [309, 308] for loop-level three-point functions of scalar single trace operators. It does indeed turn out, that three-point functions can be studied efficiently using integrability.

The question of a weak-strong coupling matching was addressed in [306, 310, 311, 312, 313]. In [312] for instance it is found that the correlator of three BMN states, each with two impurities and with a non-zero momentum, completely agrees with the string-theoretical calculation of the 3-string vertex matrix elements.

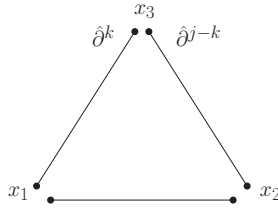


Figure 7.1: Three-point function of two BPS and one twist-two spin j operator at tree-level. The derivatives acting on x_3 , indicated with $\hat{\partial}^k$, $\hat{\partial}^{j-k}$, are distributed in a certain way as explained in section 7.2.

Here, we calculate the structure constants of three-point correlators with two protected operators $\mathcal{O}, \tilde{\mathcal{O}}$ and a twist-two operator $\hat{\mathcal{O}}_j$ of arbitrary even spin j in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory at one loop. We perform a special infrared limit on the correlator in order to simplify the calculations. In this limit we do not rely on any knowledge of the one-loop mixing matrix, since it drops out completely. Furthermore, in this limit the three-point calculation is effectively reduced to calculating two-point functions. We find that the normalisation invariant structure constant has the simple form (7.34)

$$C'_{\mathcal{O}\tilde{\mathcal{O}}_j}(g^2) = C'^{(0)}_{\mathcal{O}\tilde{\mathcal{O}}_j} \left(1 + \frac{g^2 N}{8\pi^2} (2H_j(H_j - H_{2j}) - H_{j,2}) + \mathcal{O}(g^4) \right), \quad (7.2)$$

where $H_{j,m}$ are generalised harmonic sums

$$H_{j,m} = \sum_{r=1}^j \frac{1}{r^m}, \quad H_j = \sum_{r=1}^j \frac{1}{r}, \quad H_{j,2} = \sum_{r=1}^j \frac{1}{r^2}. \quad (7.3)$$

The structure constant can also be deduced from the double OPE of four-point functions of half-BPS operators and can be found in equation (5.27) of [24], which agrees with our direct calculation and thus confirms the validity of the operator product expansion. The method that we use here however, can also be applied to more general operators than the ones considered in [24].

The explicit calculations are rather technical and we shift most of them to the appendix. In the following sections we give an outline of the calculation. In section 7.3.1 we give a plug-in formula for the structure constant and specify the location of the corresponding contributions in the appendix.

7.1 Limits on Three-Point Functions

The calculation of the structure constant $C_{\mathcal{O}\tilde{\mathcal{O}}_j}$ can be largely simplified by considering appropriate limits of the correlator. In contrast to two-point functions, we can take several limits or integrate out parts of the space-time structure of the three-point correlator without trivializing its structure. Taking the limits corresponds to a projection onto a certain component of the 2^j terms in the correlator structure (3.30).

7.1.1 Limit $x_2 \rightarrow \infty$

Multiplying by appropriate factors of x_{12}^2, x_{23}^2 and taking the limit $x_2 \rightarrow \infty$ in (3.30) we obtain

$$\lim_{x_2 \rightarrow \infty} |x_{12}|^{\Delta_1 + \Delta_2 - \theta} |x_{23}|^{\theta + \Delta_2 - \Delta_1} \langle \mathcal{O} \tilde{\mathcal{O}} \hat{\mathcal{O}}_j \rangle = C_{\mathcal{O} \tilde{\mathcal{O}} \mathcal{O}_j} \frac{(\hat{x}_{13})^j}{|x_{13}|^{j + \theta + \Delta_1 - \Delta_2}}. \quad (7.4)$$

Taking the same limit in the evaluation of the correlator on the left-hand side, only very few terms in the Feynman-diagram calculation remain and the structure constant can easily be read off. We use this limit in order to calculate the tree-level structure constant in Appendix H.8 for general dimension. For the one-loop calculation it is however more appropriate to calculate the diagrams in the limit described in the next section.

7.1.2 Collinear Limit of the Three-Point Correlator

Another possibility is to take the momentum P going in at $\hat{\mathcal{O}}_j$ to zero, i.e. $P = p_1 + p_2 \rightarrow 0$. This limit can be implemented in position space by integrating the three-point function over x_3 , which can be seen as follows.

The Fourier transformed correlator is

$$\begin{aligned} \langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle &= \int \frac{d^d p_1}{(2\pi)^d} \frac{d^d p_2}{(2\pi)^d} \frac{d^d P}{(2\pi)^d} e^{i(p_1 x_1 + p_2 x_2 + P x_3)} \\ &\langle \mathcal{O}_1(p_1) \mathcal{O}_2(p_2) \mathcal{O}_3(P) \rangle (2\pi)^d \delta(p_1 + p_2 + P), \end{aligned} \quad (7.5)$$

where the momentum conserving delta function is due to translation invariance of the correlator. Integrating this expression over x_3 leads to

$$\begin{aligned} &\int d^d x_3 \langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle \\ &= \int \frac{d^d p_1}{(2\pi)^d} \frac{d^d p_2}{(2\pi)^d} e^{i(p_1 x_1 + p_2 x_2)} (2\pi)^d \delta(p_1 + p_2) \langle \mathcal{O}_1(p_1) \mathcal{O}_2(p_2) \mathcal{O}_3(-p_1 - p_2) \rangle \end{aligned} \quad (7.6)$$

$$= \int \frac{d^d p_1}{(2\pi)^d} e^{i p_1 x_{12}} \langle \mathcal{O}_1(p_1) \mathcal{O}_2(-p_1) \mathcal{O}_3(0) \rangle, \quad (7.7)$$

where we used $\int d^d x e^{i(p_1 + p_2)x_3} = (2\pi)^d \delta(p_1 + p_2)$. Thus, integration of the correlator over x_3 is equivalent to setting $P = 0$ in momentum space, see figure 7.2. Taking this limit largely simplifies the perturbative calculations that we will perform in this chapter.

7.2 Definition of the Operators

We use the mostly minus metric and the conventions as given in Appendix A and adapt the notation from [121]. The action of $\mathcal{N} = 4$ SYM is given in (C.21). We choose the

$$\int d^d x_3 \quad \begin{array}{c} \bullet \\ x_3 \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ x_1 \quad x_2 \end{array} = \int \frac{d^d p_1}{(2\pi)^d} e^{ip_1 x_{12}} \quad \begin{array}{c} \bullet \\ P = p_1 + p_2 = 0 \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ p_1 \quad p_2 \end{array} \quad (7.8)$$

Figure 7.2: Taking the limit $P = 0$ in momentum space corresponds to integrating the correlator over x_3 .

following BPS-operators

$$\mathcal{O}(x) = \text{Tr} \left(\bar{\phi}_{12}(x) \bar{\phi}_{13}(x) \right), \quad \tilde{\mathcal{O}}(x) = \text{Tr} \left(\bar{\phi}_{12}(x) \phi^{13}(x) \right). \quad (7.9)$$

where $\bar{\phi}_{AB} = \bar{\phi}_{AB}^a T^a$ and we normalise the generators T^a in the fundamental representation according to $\text{Tr} (T^a T^b) = \delta^{ab}/2$. The twist-two operator $\hat{\mathcal{O}}_j$ projected to the light-cone, see also section 3.3.3, is given by

$$\hat{\mathcal{O}}_j(x) = \sum_{k=0}^j a_{jk}^{1/2} \text{Tr} \left(\hat{D}^k \phi^{12}(x) \hat{D}^{j-k} \phi^{12}(x) \right), \quad (7.10)$$

where $\hat{D} = D^\mu z_\mu$ is the light-cone projected covariant derivative ($z^2 = 0$) and $a_{jk}^{1/2}$ are numerical coefficients, that are related to the *Gegenbauer polynomials* as explained in the following section. Due to the property $a_{j,j-k} = (-1)^j a_{j,k}$ we consider j even, since the operator vanishes otherwise. As we will explicitly see in section 7.4.1, these operators have conformal two-point functions (3.29)

$$\langle \hat{\mathcal{O}}_j(x_1) \hat{\mathcal{O}}_k(x_2) \rangle = \delta_{jk} C_j 2^{2j} \frac{(\hat{x}_{12})^{2j}}{(-x_{12}^2)^{2j+\theta}} \quad (j \text{ even}), \quad (7.11)$$

where $\theta = \Delta_j - j$ is the twist (dimension minus spin) of the operators. For the twist-two operators considered here $\theta = 2$. As reviewed in section 3.3.3, conformal symmetry constrains this type of three-point functions to the form

$$\langle \mathcal{O}(x_1) \tilde{\mathcal{O}}(x_2) \hat{\mathcal{O}}_j(x_3) \rangle = C_{\mathcal{O}\tilde{\mathcal{O}}_j}(g^2) \frac{(\hat{Y}_{12,3})^j}{|x_{12}|^{\Delta_1+\Delta_2-\theta} |x_{13}|^{\Delta_1+\theta-\Delta_2} |x_{23}|^{\Delta_2+\theta-\Delta_1}}, \quad (7.12)$$

where $|x_{ij}| = (-x_{ij}^2)^{1/2}$ and

$$\hat{Y}_{12;3} = Y^\mu(x_{13}, x_{23}) z_\mu, \quad Y_{12,3}^\mu = \left(\frac{x_{13}^\mu}{x_{13}^2} - \frac{x_{23}^\mu}{x_{23}^2} \right), \quad z^2 = 0. \quad (7.13)$$

We will calculate the one-loop correction to the structure constants $C_{\mathcal{O}\tilde{\mathcal{O}}_j}(g^2)$ and normalise them in a way suitable for the operator product expansion with the result (7.2).

7.2.1 Operator in Terms of Gegenbauer Polynomials

At tree-level, the gauge fields do not enter the calculations, and therefore the covariant derivatives in (7.10) can be reduced to ordinary derivatives

$$\hat{\mathcal{O}}_j^{\text{tree}} = \sum_{k=0}^j a_{jk}^{1/2} \text{Tr} (\hat{\partial}^k \phi^{12} \hat{\partial}^{j-k} \phi^{12}). \quad (7.14)$$

The numerical coefficients a_{jk}^ν are related to the so-called *Gegenbauer polynomials* $C_j^\nu(x)$ such that

$$\sum_k a_{jk}^\nu x^k y^{j-k} = (x+y)^j C_j^\nu \left(\frac{x-y}{x+y} \right) \quad (7.15)$$

and $\nu = h - 3/2$, where $h = d/2$. We list some properties on Gegenbauer polynomials in Appendix H.1. Therefore, we can rewrite the operators in the bi-local form [314, 315]

$$\hat{\mathcal{O}}_j^{\text{tree}} = \underbrace{(\hat{\partial}_a + \hat{\partial}_b)^j C_j^{1/2} \left(\frac{\hat{\partial}_a - \hat{\partial}_b}{\hat{\partial}_a + \hat{\partial}_b} \right)}_{=: D_{ab}^j} \text{Tr} (\phi^{12}(x_a) \phi^{12}(x_b)) \Big|_{x_a=x_b}. \quad (7.16)$$

As we will see in the remainder of this section, the formulation in terms of Gegenbauer polynomials is very useful in practical calculations, since this allows to use properties such as the orthogonality of Gegenbauer polynomials. In section 7.4.1 we show how the tree-level two-point function can be calculated for general index ν of the Gegenbauer polynomials.

7.3 Structure of the Calculation and Results

In the conformal scheme of section 3.4.3, the form (7.11),(7.12) of the two- and three-point functions of the renormalised operators is preserved at loop-level under the replacement of the classical dimensions of the operators (7.9), (7.10) with their anomalous dimensions, i.e.

$$\theta(g^2) = \Delta_j - j = \Delta_j^{(0)} + \gamma_j(g^2) - j = d - 2 + g^2 \gamma_j^{(1)} + \mathcal{O}(g^4) \quad (7.17)$$

and the dimension of the protected operators $\Delta_{\mathcal{O}} = \Delta_{\bar{\mathcal{O}}} = d - 2$ remains fixed¹. The renormalised twist-two operators mix with descendants of lower spin operators as

$$\hat{\mathcal{O}}_j = \sum_k \mathbb{Z}_{jk} \hat{\partial}^{j-k} \hat{\mathcal{O}}_k. \quad (7.18)$$

¹The (anomalous) dimension of the operators is defined at $\epsilon = 0$. We keep writing d here and in the following however, because the arguments are also valid for scalar $(\phi^3)_6$ theory in $d = 6$; integer d is however required.

As we will see in section 7.4 at one loop \mathbb{Z}_{jk} is given through

$$\mathbb{Z}_{jk} = \delta_{jk} + g^2 \left(-B_{jk}^{(1)} + \frac{1}{\epsilon} \delta_{jk} Z_j^{(1)} \right) + \mathcal{O}(g^4). \quad (7.19)$$

where $B_{jk}^{(1)}$ is the finite one-loop mixing matrix specified in (7.47) and it has only entries for $k < j$ and $Z_j^{(1)}$ is the renormalisation constant given in (7.39). Therefore, the renormalised three-point function of two BPS and one twist-operator reads

$$\langle \mathcal{O}(x_1) \tilde{\mathcal{O}}(x_2) \hat{\mathcal{O}}_j(x_3) \rangle = C_{\mathcal{O}\tilde{\mathcal{O}}_j}(g^2) \frac{(\hat{Y}_{12,3})^j}{((-x_{13}^2)(-x_{23}^2))^{\theta/2} (-x_{12}^2)^{\Delta_{\mathcal{O}} - \theta/2}}. \quad (7.20)$$

Applying the limit described in section 7.1.2 and therefore integrating over x_3 , see Appendix H.4, yields

$$\int d^d x_3 \frac{(\hat{Y}_{12,3})^j}{((-x_{13}^2)(-x_{23}^2))^{\theta/2}} = N(g^2, d) \frac{(\hat{x}_{12})^j}{(-x_{12}^2)^{\theta - d/2 + j}},$$

where the normalisation factor is

$$N(g^2, d) = -i \frac{\Gamma(\theta - d/2 + j) \Gamma((d - \theta)/2)^2 \Gamma(j + (\theta - 1)/2) 2^{\theta + 2j - 2}}{\Gamma(d - \theta) \Gamma\left(j + \frac{\theta}{2}\right) \Gamma(j + \theta - 1) \pi^{\frac{1}{2} - \frac{d}{2}}}. \quad (7.21)$$

As can be seen from (7.17), this function has an expansion² in g and for $d = 4$ we explicitly get

$$\begin{aligned} N(g^2) &= N^{(0)} + g^2 N^{(1)} \\ &= -i \frac{\Gamma(2j) 2\pi^2}{\Gamma(j+1)^2} \left(1 + \frac{\gamma_j(g^2)}{2} \left(-\frac{1}{j} - 2H_j + 2H_{2j-1} + 2 \right) \right) + \mathcal{O}(g^4), \end{aligned} \quad (7.22)$$

where the anomalous dimension of the twist-two operators will be given in section 7.4.3. By calculating the left-hand side of the integrated three-point function

$$\int d^d x_3 \langle \mathcal{O} \tilde{\mathcal{O}} \hat{\mathcal{O}}_j \rangle = N(g^2) \left(C_{\mathcal{O}\tilde{\mathcal{O}}_j}^{(0)} + g^2 C_{\mathcal{O}\tilde{\mathcal{O}}_j}^{(1)} \right) \frac{(\hat{x}_{12})^j}{(-x_{12}^2)^{j + d - 3 + \gamma_j(g^2)/2}} \quad (7.23)$$

in terms of Feynman diagrams, we can thus easily read off the structure constants. In the following section we explain how to read off the one-loop structure constant from the perturbative calculations.

7.3.1 The One-Loop Structure Constant

We calculate the correlator in the limit described in section 7.1.2 which is equivalent to the integration of the correlator over x_3 and thus all the descendants drop out, since we can write

$$\int d^d x_3 \langle \mathcal{O} \tilde{\mathcal{O}} \hat{\mathcal{O}}_j \rangle = \sum_k \mathbb{Z}_{jk} \int d^d x_3 \hat{\delta}_3^{j-k} \langle \mathcal{O} \tilde{\mathcal{O}} \hat{\mathcal{O}}_k \rangle = \sum_k \mathbb{Z}_{jk} \delta_{jk} \int d^d x_3 \langle \mathcal{O} \tilde{\mathcal{O}} \hat{\mathcal{O}}_k \rangle. \quad (7.24)$$

²It does not have an expansion in ϵ , since the renormalised three-point function is defined at $\epsilon = 0$.

Thus from (7.19) one can see that the contributions of order g^2 to the renormalised three-point function in terms of bare Feynman diagrams are

$$\langle \mathcal{O} \tilde{\mathcal{O}} \hat{\mathcal{O}}_j \rangle \Big|_{g^2} = \lim_{\epsilon \rightarrow 0} \left(\langle \mathcal{O} \tilde{\mathcal{O}} \hat{\mathcal{O}}_j \rangle^{(1)} + g^2 \langle \mathcal{O} \tilde{\mathcal{O}} \hat{\mathcal{O}}_j \rangle^{(0,\epsilon)} \frac{Z_j^{(1)}}{\epsilon} \right), \quad (7.25)$$

where $\langle \mathcal{O} \tilde{\mathcal{O}} \hat{\mathcal{O}}_j \rangle^{(0,\epsilon)}$ denotes the tree-level correlator calculated in $d = 4 - 2\epsilon$ dimensions, including its $\mathcal{O}(\epsilon)$ expansion. Therefore, our calculation does not rely on the knowledge of the mixing matrix B .

In order to read off the finite one-loop structure constant, we can combine the known structure in the conformal scheme (7.23) and the result of the explicit one-loop calculation (7.25) and solve for $C_{\mathcal{O} \tilde{\mathcal{O}}_j}^{(1)}$

$$\begin{aligned} C_{\mathcal{O} \tilde{\mathcal{O}}_j}^{(1)} &= \lim_{\epsilon \rightarrow 0} \left[\int \langle \mathcal{O} \tilde{\mathcal{O}} \hat{\mathcal{O}}_j \rangle^{(1)} + \int \langle \mathcal{O} \tilde{\mathcal{O}} \hat{\mathcal{O}}_j \rangle^{(0,\epsilon)} \frac{Z_j^{(1)}}{\epsilon} g^2 \right] / \left(N^{(0)} \frac{(\hat{x})^j}{(-x^2)^{3+j}} \right) \\ &\quad - g^2 \frac{N^{(1)}}{N^{(0)}} C_{\mathcal{O} \tilde{\mathcal{O}}_j}^{(0)}, \end{aligned} \quad (7.26)$$

where the division through the space-time part is to be understood symbolically. In the following, we calculate all these contributions and (7.26) becomes a simple plug-in formula. The first term is given³ through (7.85), the second term is given by (7.55) and (7.39) and here we have to take into account the expansion of all these expressions to finite order. The constants $N^{(0)}, N^{(1)}$ were given in (7.22). The tree-level structure constant is given in (7.56).

7.3.2 Normalisation Invariant Structure Constants

In section 3.5 we have shown the equivalence of the coefficients in the operator product expansion and the structure constants of three-point functions. As mentioned there, we assumed that the operators are normalised such that the two-point functions have $C_j = 1$. The operators (7.16) in terms of Gegenbauer polynomials are not normalised to one, but as we will see in section 7.4.4 are normalised as (7.40)

$$\langle \hat{\mathcal{O}}_j(x_1) \hat{\mathcal{O}}_j(x_2) \rangle = \delta_{jk} \left(C_j^{(0)} + g^2 C_j^{(1)} \right) 2^{2j} \frac{(\hat{x}_{12})^{2j}}{(-x_{12}^2)^{2j+\theta}}, \quad (7.27)$$

where $C_j^{(0)}, C_j^{(1)}$ are explicitly calculated in section 7.4.

As can be seen from (7.20), (7.11), under rescalings of the operators

$$\hat{\mathcal{O}}_j \rightarrow \lambda_j \hat{\mathcal{O}}_j, \quad \mathcal{O} \rightarrow \lambda \mathcal{O}, \quad \tilde{\mathcal{O}} \rightarrow \tilde{\lambda} \tilde{\mathcal{O}}, \quad (7.28)$$

³We have calculated it in momentum space and thus an additional Fourier transformation using (7.86) for $d = 4$ is necessary.

the structure constant and the normalisation of the two-point function changes as

$$C_{12j}(g^2) \rightarrow \lambda \tilde{\lambda} \lambda_j C_{12j}(g^2), \quad C_j \rightarrow \lambda_j^2 C_j, \quad C \rightarrow \lambda^2 C, \quad \tilde{C} \rightarrow \tilde{\lambda}^2 \tilde{C}.$$

Clearly, the ratio

$$C'_{\mathcal{O}\tilde{\mathcal{O}}_j} = \frac{C_{\mathcal{O}\tilde{\mathcal{O}}_j}}{\sqrt{C\tilde{C}C_j}} \quad (7.29)$$

is invariant under the rescalings (7.28). For operators with two-point functions that are normalised to one, the factor in the denominator of (7.29) is equal to one and it is thus this structure constant which is relevant in the operator product expansion. In other words, $C'_{\mathcal{O}\tilde{\mathcal{O}}_j}$ is the structure constant for operators which have two-point functions that are normalised to one. In our case, the BPS operators (7.9) are normalised to $C = \tilde{C} = 2^{-6}/\pi^4$ and do not get quantum corrections and thus $C'_{\mathcal{O}\tilde{\mathcal{O}}_j}$ is related to $C_{\mathcal{O}\tilde{\mathcal{O}}_j}$ through

$$C'_{\mathcal{O}\tilde{\mathcal{O}}_j} = \frac{C_{\mathcal{O}\tilde{\mathcal{O}}_j}(g^2)}{\sqrt{C_j(g^2)C}} = \frac{C_{\mathcal{O}\tilde{\mathcal{O}}_j}^{(0)}}{\sqrt{C_j^{(0)}C}} \left(1 + g^2 \left(\frac{C_{\mathcal{O}\tilde{\mathcal{O}}_j}^{(1)}}{C_{\mathcal{O}\tilde{\mathcal{O}}_j}^{(0)}} - \frac{1}{2} \frac{C_j^{(1)}}{C_j^{(0)}} \right) \right) = C'_{\mathcal{O}\tilde{\mathcal{O}}_j}{}^{(0)} + g^2 C'_{\mathcal{O}\tilde{\mathcal{O}}_j}{}^{(1)}. \quad (7.30)$$

The two-point structure constants are calculated in section 7.4.4.

Result for the structure constants

Putting together (7.30) using (7.41), (7.43) for the normalisation of the two-point functions as well as (7.56), (7.26) for the structure constants of the three-point functions, we find that the normalisation invariant structure constant is⁴

$$C'_{\mathcal{O}\tilde{\mathcal{O}}_j}{}^{(1)}/C'_{\mathcal{O}\tilde{\mathcal{O}}_j}{}^{(0)} = \frac{g^2 N}{8\pi^2} \left(2H(j)^2 - 2H(j)H(2j) - \sum_{r=1}^j \frac{1}{r^2} \right). \quad (7.31)$$

In the large j limit we find

$$\lim_{j \rightarrow \infty} C'_{\mathcal{O}\tilde{\mathcal{O}}_j}{}^{(1)}/C'_{\mathcal{O}\tilde{\mathcal{O}}_j}{}^{(0)} = \frac{g^2 N}{8\pi^2} \left(-2 \ln(2) (\ln(j) + \gamma_E) - \frac{\pi^2}{6} \right). \quad (7.32)$$

The one-loop correction thus scales with $\ln(j)$ for large j , as shown in figure 7.3. This result consists of the contributions from the two-point functions

$$\frac{1}{2} C_j^{(1)}/C_j^{(0)} = \frac{g^2 N}{8\pi^2} (3H(j)^2 - 2H(j)H(2j)) \quad (7.33)$$

⁴We have put all contributions into a Mathematica file and verified the result (7.34). One could proceed and solve some of the finite sums that remain in the solutions of the integrals or prove them via induction from the known result.

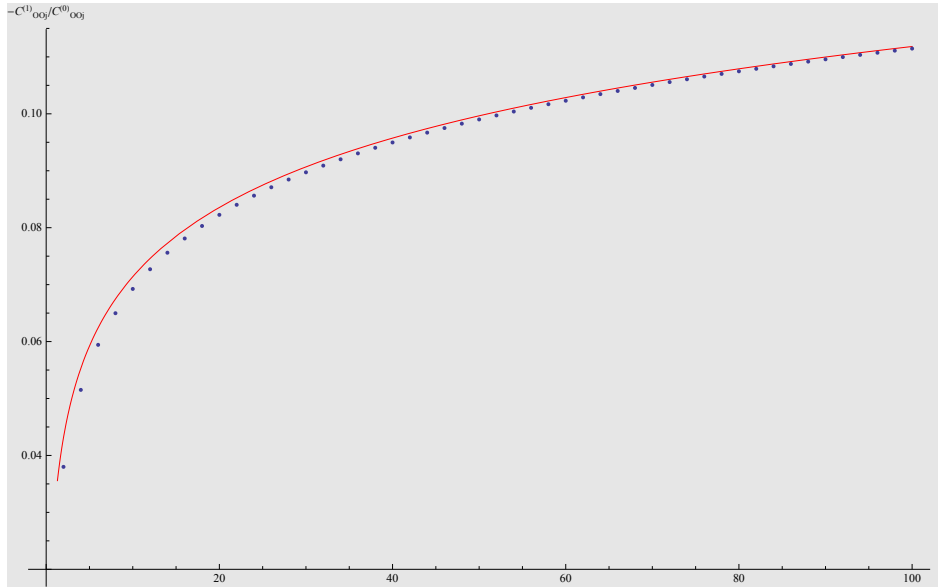


Figure 7.3: Here we show the dependence of the ratio of the one-loop structure constant and its tree-level value on the spin j . The points are given by (7.34) and the curve represents the large j limit (7.32). Note, that for graphical reasons we plot the negative value of the ratio here.

and the contribution from the three-point functions

$$C_{\mathcal{O}\tilde{\mathcal{O}}_j}^{(1)}/C_{\mathcal{O}\tilde{\mathcal{O}}_j}^{(0)} = \frac{g^2 N}{8\pi^2} \left(5H(j)^2 - 4H(j)H(2j) - \sum_{r=1}^j \frac{1}{r^2} \right), \quad (7.34)$$

which both scale as $\ln(j)^2 + \dots$ for $j \rightarrow \infty$ where the ellipses stand for subleading terms.

7.4 Two-Point Functions of Twist-Two Operators

Here, we briefly list the results for the two-point function calculations of the operators (7.16). The calculations are rather tedious and are given in detail in the appendix. We will give more details on the three-point calculation in the following section, since the focus of this chapter is on three-point functions.

7.4.1 Tree-Level Two-Point Functions of Twist-Two Operators

In Appendix H.5 we calculate the tree-level two-point function of the operators (7.16) for scalar fields in general⁵ dimension d using the Schwinger parametrisation of the scalar propagators and some properties of the Gegenbauer polynomials. The result for $d = 4$ is given in (H.52) and reads

$$\langle \hat{\mathcal{O}}_j \hat{\mathcal{O}}_k \rangle^{d=4} = \delta_{jk} g^4 \delta^{aa} \frac{2^{2j-1}}{(4\pi^2)^2} \Gamma(2j+1) \frac{(\hat{x}_{12})^{2j}}{(-x_{12}^2)^{2j+2}}, \quad (7.35)$$

where $\delta^{aa} = N^2 - 1$ is the dimension of the gauge group $SU(N)$.

7.4.2 Bare One-Loop Two-Point Function

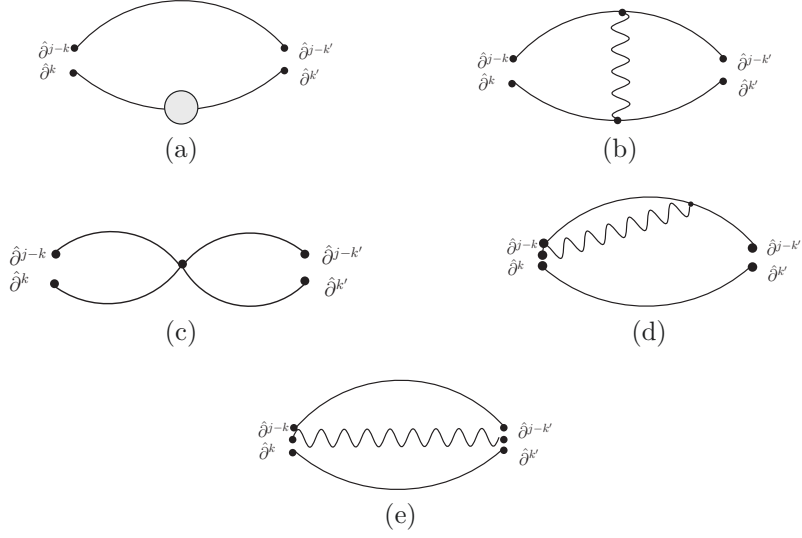
The one-loop corrections to the two-point function are shown in figure 7.4. The self-energy diagram 7.4a can be calculated in a similar way using the orthogonality properties of the Gegenbauer polynomials. For the other diagrams it is however more convenient to use the representations (7.14) of the operators. For the calculation of the *kite diagram* in figure 7.4b we use the integration by parts (IBP) technique and the four-scalar diagram in figure 7.4c can be shown to cancel exactly against a contribution from the kite diagram. This cancellation can be understood, by investigating the decomposition of the kite diagram into scalar basis integrals as shown in figure 7.5. For the diagrams in figures 7.4d, 7.4e we need the representation with one gauge field in the operator (7.79). The diagram in figure 7.4e vanishes in the light-cone projection, since it contains a gluon propagator contracted with the light-like vectors z_μ and thus $\langle A_\mu A_\nu \rangle z^\mu z^\nu = 0$ in our gauge.

Adding up all non-vanishing contributions we thus find

$$\begin{aligned} \langle \hat{\mathcal{O}}_j \hat{\mathcal{O}}_j \rangle^{(1)} &= \langle \mathcal{O}_j \hat{\mathcal{O}}_j \rangle^{(7.4a)} + \langle \mathcal{O}_j \hat{\mathcal{O}}_j \rangle^{(7.4b+7.4c)} + \langle \mathcal{O}_j \hat{\mathcal{O}}_j \rangle^{(7.4d)} \\ &= (\text{H.69}) + (\text{H.79}) + (\text{H.82}) \end{aligned} \quad (7.36)$$

and the explicitly solved integrals can be found in the appendix in the equations specified in the last line of (7.36)

⁵We do however require d to be integer. Choosing $d = 4 - 2\epsilon$ leads to $\mathcal{O}(\epsilon)$ corrections and the unrenormalised correlator also gets non-diagonal terms.

Figure 7.4: One-loop corrections to the two-point function of $\langle \hat{\mathcal{O}}_j \hat{\mathcal{O}}_j \rangle$.

7.4.3 Anomalous Dimension of Twist-Two Operators

We can extract the divergent contributions of (7.36) and find the following contributions (H.70), (H.76), (H.84) to the anomalous dimension

$$\gamma_j^{(7.4a)} = \frac{g^2 N}{4\pi^2}, \quad \gamma_j^{(7.4b)} = -\frac{g^2 N}{4\pi^2} \frac{1}{(j+1)}, \quad \gamma_j^{(7.4d)} = \frac{g^2 N}{4\pi^2} \left(2H_j + \frac{1}{(j+1)} - 1 \right). \quad (7.37)$$

Adding up the contributions we get the well-known result [316, 317] for the leading order contribution to the anomalous dimension

$$\gamma_j = \left(\frac{g^2 N}{8\pi^2} \right) 4H_j + \mathcal{O}(g^4), \quad H_j = \sum_{k=1}^j \frac{1}{k}. \quad (7.38)$$

In order to avoid terms of $\log(\pi)$ and γ_E in the final results, we define the renormalisation constant in the $\overline{\text{MS}}$ -scheme (3.40), (3.42)

$$Z_j^{(1)} = \frac{H_j}{4\pi^h} e^{\epsilon \gamma_E}, \quad H_j = \sum_{k=1}^j \frac{1}{k}. \quad (7.39)$$

7.4.4 Renormalised Two-Point Function

The renormalised two-point function has the form

$$\langle \hat{\mathcal{O}}_j(x_1) \hat{\mathcal{O}}_k(x_2) \rangle = \delta_{jk} C_j(g^2) \frac{(\hat{I}_{12})^j}{(-x_{12}^2)^{j+\theta}} = \delta_{jk} \left(C_j^{(0)} + g^2 C_j^{(1)} \right) 2^{2j} \frac{(\hat{x}_{12})^{2j}}{(-x_{12}^2)^{j+\theta}}. \quad (7.40)$$

The tree-level normalisation constant $C_j^{(0)}$ can be read off from (7.35) and we have

$$C_j^{(0)} = g^4 \delta^{aa} \frac{\Gamma(2j+1)}{2^5 \pi^4}. \quad (7.41)$$

Reading off the finite one-loop normalisation constant $C_j^{(1)}$ is a little more complicated. The renormalised two-point function for $j = k$ reads

$$\langle \hat{\mathcal{O}}_j^R(x_1) \hat{\mathcal{O}}_j^R(x_2) \rangle = \langle \hat{\mathcal{O}}_j \hat{\mathcal{O}}_j \rangle^{(0,\epsilon)} \left(1 + \frac{(-\mu^2 x^2)^\epsilon}{\epsilon} g^2 2Z_j^{(1)} \right) + \langle \hat{\mathcal{O}}_j \hat{\mathcal{O}}_j \rangle^{(1)}. \quad (7.42)$$

The $\mathcal{O}(\epsilon^0)$ expansion of the tree-level correlator multiplied with the renormalisation constant exactly cancels the pole term from the one loop diagrams, whereas its $\mathcal{O}(\epsilon)$ expansion multiplied with the pole term yields another finite contribution to the normalisation constant. Thus, the one-loop normalisation is

$$C_j^{(1)} = \lim_{\epsilon \rightarrow 0} \left(g^2 \langle \hat{\mathcal{O}}_j \hat{\mathcal{O}}_j \rangle^{(0,\epsilon)} \frac{2Z_j^{(1)}}{\epsilon} + \langle \hat{\mathcal{O}}_j \hat{\mathcal{O}}_j \rangle^{(1)} \right) / \left(2^{2j} \frac{\hat{x}^{2j}}{(-x^2)^{j+\theta}} \right), \quad (7.43)$$

where the division by the space-time structure is to be understood symbolically. The diagonal tree-level correlator including its $\mathcal{O}(\epsilon)$ expansion is given in H.6.2 and the one-loop contribution is given in (7.36).

7.4.5 Mixing Matrix

The renormalised two-point function (7.42) in the $\overline{\text{MS}}$ -scheme yields finite but non-diagonal correlators.

$$\langle \hat{\mathcal{O}}_j^R(x_1) \hat{\mathcal{O}}_k^R(x_2) \rangle \neq 0 \quad \text{for } j \neq k. \quad (7.44)$$

As explained in section 3.4.3, in order to have a diagonal basis of renormalised twist-two operators

$$\langle \hat{\mathcal{O}}_j(x_1) \hat{\mathcal{O}}_k(x_2) \rangle = \delta_{jk} \langle \hat{\mathcal{O}}_j(x_1) \hat{\mathcal{O}}_j(x_2) \rangle, \quad (7.45)$$

it is necessary to subtract the finite non-diagonal terms by introducing another finite renormalisation matrix B .

$$\mathbb{Z}_{jk} = \delta_{jk} + g^2 \left(-B_{jk}^{(1)} + \frac{1}{\epsilon} \delta_{jk} Z_j^{(1)} \right) + \mathcal{O}(g^4). \quad (7.46)$$

The result is well-known and was redetermined in a very simple way in [121] in the same conventions that we use and reads

$$B_{jk}^{(1)} = 2d_{jk}^{1/2} \left(H_j + H_k + H_{\frac{j-k}{2}} - H_{\frac{j+k}{2}} - 2H_{j-k} \right), \quad (7.47)$$

where $d_{jk}^{1/2}$ is related to the expansion of the Gegenbauer polynomials C_j^ν with respect to the index ν and is given in (H.8). One can determine the mixing matrix by requiring the above orthogonality, i.e. by calculating the finite non-diagonal ($j \neq k$) terms of the one-loop correlator. We do not go into the details, since the mixing matrix drops out in our limit.

7.4.6 Protected Operator for $j = 0$

For $j = 0$ the one-loop corrections cancel and thus the operator is protected and has anomalous dimension zero, this can be seen as follows. For $j = 0$ there are no covariant derivatives and the operator (7.10) is

$$\hat{\mathcal{O}}_{j=0}(x) = \text{Tr} \left(\phi^{12}(x) \phi^{12}(x) \right). \quad (7.48)$$

Therefore, the diagrams shown in figures 7.4d, 7.4e are absent. To understand the cancellation among the remaining diagrams, it is instructive to consider the decomposition of the kite diagram into scalar basis integrals, see H.7.2, which is diagrammatically represented in figure 7.5. The last four integrals in figure 7.5 are all identical and cancel with the self-energy diagrams. The second diagram cancels with the four-scalar diagram (this is also true for $j \neq 0$) and the only remaining contribution is the first diagram shown in figure 7.5 which is finite in momentum space and yields the result

$$\langle \mathcal{O}_{j=0}(p) \bar{\mathcal{O}}_{j=0}(-p) \rangle^{(1)} = -\frac{1}{(4\pi)^4} \frac{ig^6 N \delta^{aa}}{(-p^2)^{4-d}} 6\zeta(3) + \mathcal{O}(\epsilon). \quad (7.49)$$

The Fourier transformation to configuration space yields a factor of $1/\Gamma(4-d) \propto \epsilon$ and thus up to order epsilon terms the correlator does not get quantum corrections

$$\langle \mathcal{O}_{j=0}(x_1) \mathcal{O}_{j=0}(x_2) \rangle^{(1)} = \mathcal{O}(\epsilon). \quad (7.50)$$

Thus, the anomalous dimension of the operator (7.48) is zero: $\gamma_{j=0} = 0$. For an operator with two different flavours like the ones we chose in (7.9), e.g.

$$\mathcal{O}(x) = \text{Tr} (\bar{\phi}_{12}(x) \bar{\phi}_{13}(x)), \quad (7.51)$$

we have the same type of diagrams as for (7.48), but slightly different overall combinatorics. Due to the fact that the flavours are distinct, we get a factor of $1/2$ as compared to (7.49).

7.5 Tree-Level Three-Point Functions

As explained in section 7.1 we can calculate the three-point function in different limits in order to simplify the calculations. Even though the tree-level calculation is slightly simpler in the limit given in 7.1.1, see the calculation in Appendix H.8, we calculate the tree-level three-point function in the limit $p_1 + p_2 \rightarrow 0$ given in section

Figure 7.5: Due to the momenta in the numerator, that originate from the gluon-vertices, the kite diagram is decomposed into these scalar integrals.

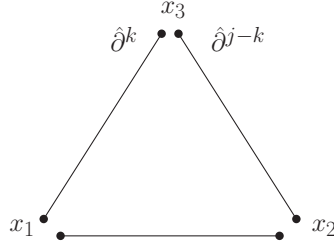


Figure 7.6: 3-point function at tree-level

7.1.2. This is, because the limit is better suited for the calculation at one-loop level and we would like to illustrate the method at tree-level.

In momentum space, the three-point function with the twist operator in the representation (7.16) reads

$$\langle \mathcal{O}(p_1) \tilde{\mathcal{O}}(p_2) \hat{\mathcal{O}}_j \rangle = \frac{1}{8} 2^{i^{3+j}} g^6 \delta^{aa} \int \frac{d^d k}{(2\pi)^d} \frac{(\hat{p}_1 + \hat{p}_2)^j C_j^{1/2} \left(\frac{2\hat{k} - \hat{p}_1 - \hat{p}_2}{\hat{p}_1 + \hat{p}_2} \right)}{k^2 (p_1 - k)^2 (p_1 + p_2 - k)^2}, \quad (7.52)$$

where the factor of $\frac{1}{8}$ is due to the traces over the gauge group and the factor of 2 is due to the two different possible contractions with $\hat{\mathcal{O}}_j$.

Now we take the limit $p_1 + p_2 \rightarrow 0$ in momentum space. Then, due to the factor $(p_1 + p_2)^j$ only the term with the highest power, i.e. j , in the Gegenbauer polynomial can survive. The corresponding coefficient is given in (H.6) and reads

$$c_{jj}^{1/2} = 2^{1-j} \frac{\Gamma(2j)}{\Gamma(j)\Gamma(j+1)} \quad \text{where} \quad C_j^{1/2}(x) = \sum_{k=0}^j c_{jk}^{1/2} x^k. \quad (7.53)$$

Thus, in this limit we get

$$\begin{aligned} \langle \mathcal{O}(p) \tilde{\mathcal{O}}(-p) \hat{\mathcal{O}}_j(0) \rangle &= g^6 \delta^{aa} 2^{j-2} i^{3+j} c_{jj}^{1/2} \int \frac{d^d k}{(2\pi)^d} \frac{(\hat{k})^j}{k^4 (p+k)^2} \\ &= -2^{j-2} g^6 \delta^{aa} i^{3+j} c_{jj}^{1/2} b_j(2, 1) \frac{\hat{p}^j}{(-p^2)^{3-h}}, \end{aligned} \quad (7.54)$$

where the bubble integrals $b_j(\alpha_1, \alpha_2)$ are defined in (A.2). Fourier transformation to x -space using (A.16) and insertion of the coefficients yields

$$\begin{aligned} \int d^d x_3 \langle \mathcal{O} \tilde{\mathcal{O}} \hat{\mathcal{O}}_j \rangle &= \int \frac{d^d p}{(2\pi)^d} \langle \mathcal{O}(p) \tilde{\mathcal{O}}(-p) \hat{\mathcal{O}}_j(0) \rangle e^{-ipx_{12}} \\ &= -i g^6 \delta^{aa} \frac{2^{d-7+j}}{\pi^{\frac{d}{2}}} \frac{\Gamma(2j)\Gamma(h-1)\Gamma(j-2+h)}{\Gamma(j)\Gamma(j+1)} \frac{(\hat{x}_{12})^j}{(-x_{12}^2)^{2h-3+j}}. \end{aligned} \quad (7.55)$$

Comparing with (7.22), (7.23) we can easily read off the tree-level structure constant for $d = 4$

$$C_{\mathcal{O}\tilde{\mathcal{O}}_j}^{(0)} = g^6 \delta^{aa} \frac{2^{j-8} \Gamma(j+1)}{\pi^6}. \quad (7.56)$$

Note that the three-point function for a scalar theory in general dimension d has the same expression (7.54) with $c_{jj}^{1/2}$ replaced with c_{jj}^ν and $\nu = d/2 - 3/2$. The corresponding result has also been calculated in the other limit described in section 7.1.1 in Appendix H.8.

7.6 One-Loop Three-Point Functions

It is advantageous to write down the full diagrams, group the contributions to see cancellations that are independent of the limit and only then take the limit $p_1 + p_2 \rightarrow 0$. In this way we avoid possible ambiguities in taking the limit for diagram 7.7g, because it cancels with a contribution of diagram 7.7d as we will see in the next section. We will make use of the fact that we consider j even such that factors like $(-1)^j$ are not displayed.

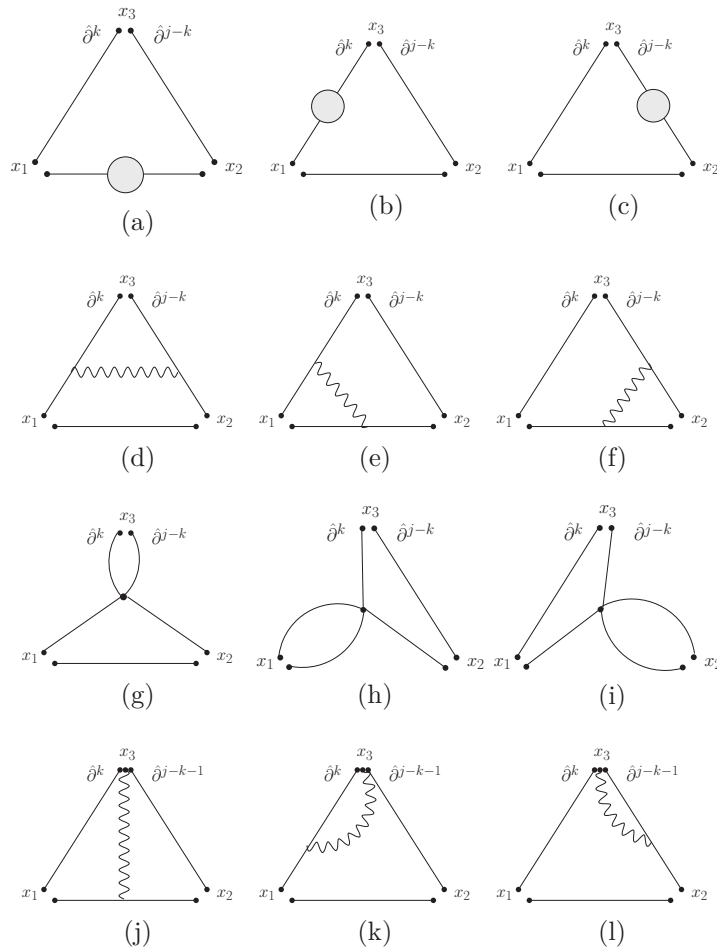


Figure 7.7: Feynman diagrams contributing to the three-point function at one loop. For j even the diagrams in the second and third column are identical.

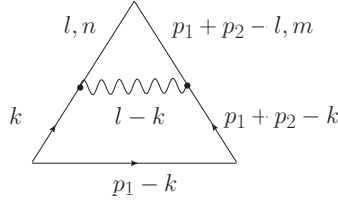


Figure 7.8: Momentum routing for diagram 7.7d to obtain (7.57), the labels m and n designate the power of the momenta in the numerator of the expression, i.e. $(\hat{l})^n$ and $(\hat{p}_1 + \hat{p}_2 - \hat{l})^m$.

7.6.1 Cancellations between Diagrams 7.7d and 7.7g

Choosing the momentum routing as shown in figure 7.8, diagram 7.7d reads

$$\begin{aligned}
\langle \mathcal{O}(p_1) \tilde{\mathcal{O}}(p_2) \hat{\mathcal{O}}_j \rangle^{7.7d} &= \tag{7.57} \\
&= \frac{1}{8} 4 \frac{i^2}{2!} i^{6+j} (i^2 f^{abc} f^{abc}) \int d^d \tilde{k}, \tilde{l} (\hat{p}_1 + \hat{p}_2)^j C_j^{1/2} \left(\frac{2\hat{l} - (\hat{p}_1 + \hat{p}_2)}{\hat{p}_1 + \hat{p}_2} \right) \\
&\quad \frac{(k+l)^\mu (-\eta_{\mu\nu}) (2P - k - l)^\nu}{k^2 (P - k)^2 (l - k)^2 l^2 (P - l)^2 (p_1 - k)^2} \\
&= \frac{1}{4} g^8 N \delta^{aa} i^{j+2} \int d^d \tilde{k}, \tilde{l} \frac{\hat{C}_j(p_1 + p_2, l)}{(p_1 + k)^2} \left(-2P^2 i(1, 1, 1, 1, 1, P) - i(1, 1, 0, 1, 1, P) \right. \\
&\quad \left. + i(0, 1, 1, 1, 1, P) + i(1, 0, 1, 1, 1, P) + i(1, 1, 1, 0, 1, P) + i(1, 1, 1, 1, 0, P) \right),
\end{aligned}$$

where we have used $(l+k) \cdot (l+k+2P) = -2P^2 - (l-k)^2 + (P+l)^2 + (P+k)^2 + k^2 + l^2$, shifted $k \rightarrow -k$, $l \rightarrow -l$ and introduced the notation

$$i(a_1, a_2, a_3, a_4, a_5, P) := \frac{1}{k^{2a_1} (P+k)^{2a_2} (k-l)^{2a_3} l^{2a_4} (P+l)^{2a_5}}, \tag{7.58}$$

$$\hat{C}_j(p_1 + p_2, l) := (\hat{p}_1 + \hat{p}_2)^j C_j^{1/2} \left(\frac{2\hat{l} - (\hat{p}_1 + \hat{p}_2)}{\hat{p}_1 + \hat{p}_2} \right),$$

in order to get a more compact notation. The four-scalar diagram 7.7g reads

$$\langle \mathcal{O}(p_1) \tilde{\mathcal{O}}(p_2) \hat{\mathcal{O}}_j \rangle^{7.7g} = \frac{1}{4} g^8 N \delta^{aa} i^{j+2} \int d^d \tilde{k}, \tilde{l} \frac{\hat{C}_j(p_1 + p_2, l)}{(p_1 + k)^2} i(1, 1, 0, 1, 1, P) \tag{7.59}$$

and thus exactly cancels the second term in (7.57).

7.6.2 Diagrams 7.7b, 7.7c

Using the 1-loop corrected propagator (C.44) the self-energy diagram 7.7b multiplied with 1/2 reads

$$\frac{1}{2} \langle \mathcal{O} \tilde{\mathcal{O}} \hat{\mathcal{O}}_j \rangle^{7.7b} = \frac{g^8}{4} N \delta^{aa} i^{2+j} b_0(1, 1) \int d^d \tilde{l} \frac{\hat{C}_j(p_1 + p_2, l)}{(-l^2)^{3-d/2} (p_1 + l)^2 (p_1 + p_2 + l)^2}. \tag{7.60}$$



Figure 7.9: Momentum routing for diagram 7.7e to obtain (7.62), the labels m and n designate the power of the momenta in the numerator of the expression, i.e. $(\hat{l})^n$ and $(\hat{p}_1 + \hat{p}_2 - \hat{l})^m$.

We can obtain the self-energy diagram 7.7c by exchanging $p_1 \leftrightarrow p_2$

$$\frac{1}{2} \langle \mathcal{O} \tilde{\mathcal{O}} \hat{\mathcal{O}}_j \rangle^{7.7c} = \frac{g^8}{4} N \delta^{aa} i^{2+j} b_0(1, 1) \int d^d \tilde{l} \frac{\hat{C}_j(p_1 + p_2, l)}{(-l^2)^{3-d/2} (p_2 + l)^2 (p_1 + p_2 + l)^2}. \quad (7.61)$$

We will proceed with the analysis of these diagrams in section 7.6.6.

7.6.3 Cancellations between diagrams 7.7b, 7.7a, 7.7h and 7.7e

The reduction of diagram 7.7h cancels the contributions from half of the self-energy diagrams and the four-scalar interaction term such that only finite integrals remain. Since we consider j even, the analysis is the same for 7.7f and the corresponding self-energy and four-scalar diagrams.

We use the momentum routing shown in figure 7.9 to obtain

$$\begin{aligned} \langle \mathcal{O}(p_1) \tilde{\mathcal{O}}(p_2) \hat{\mathcal{O}}_j \rangle^{7.7e} &= \quad (7.62) \\ &= \frac{1}{8} 4 \frac{(i)^2}{2!} i^j (i^2 f^{abd} f^{abd}) i^6 \int d^d \tilde{k}, \tilde{l} (\hat{p}_1 + \hat{p}_2)^j C_j^{1/2} \left(\frac{2\hat{l} - (\hat{p}_1 + \hat{p}_2)}{\hat{p}_1 + \hat{p}_2} \right) \\ &\quad \frac{(k+l)^\mu (-\eta_{\mu\nu}) (2p_1 - k - l)^\nu}{k^2 (p_1 - k)^2 (k - l)^2 l^2 (p_1 - l)^2 (p_1 + p_2 - l)^2} \\ &= \frac{1}{4} i^{2+j} g^8 N \delta^{aa} \int d^d \tilde{k}, \tilde{l} \frac{\hat{C}_j(p_1 + p_2, l)}{(p_1 + p_2 + l)^2} i(1, 1, 1, 1, 1, p_1) (l+k) \cdot (l+k+2p_1) \\ &= \frac{1}{4} i^{2+j} g^8 N \delta^{aa} \int d^d \tilde{k}, \tilde{l} \frac{\hat{C}_j(p_1 + p_2, l)}{(p_1 + p_2 + l)^2} \left(-2p_1^2 i(1, 1, 1, 1, 1, p_1) - i(1, 1, 0, 1, 1, p_1) \right. \\ &\quad \left. + i(0, 1, 1, 1, 1, p_1) + i(1, 0, 1, 1, 1, p_1) + i(1, 1, 1, 0, 1, p_1) + i(1, 1, 1, 1, 0, p_1) \right), \end{aligned}$$

where we used $(l+k) \cdot (l+k+2p_1) = -2p_1^2 - (l-k)^2 + k^2 + (p_1+k)^2 + l^2 + (p_1+l)^2$ in the last line as well as the abbreviations (7.58). The origin of this decomposition is illustrated in figure 7.10. As we will see, the second, third and fourth term in the brackets cancels against contributions from the self-energy and four-scalar interaction

diagrams.

The four-scalar interaction diagram 7.7h is

$$\begin{aligned} \langle \mathcal{O}(p_1) \tilde{\mathcal{O}}(p_2) \hat{\mathcal{O}}_j \rangle^{7.7h} &= \\ &= \frac{1}{8} 2 \frac{i}{1!} \left(-\frac{1}{g^2} (f^{abc} f^{acb} + 0) \right) i^5 i^j \int d^d \tilde{k}, \tilde{l} \hat{C}_j(p_1 + p_2, l) \frac{i(1, 1, 0, 1, 1, p_1)}{(p_1 + p_2 + l)^2} \\ &= \frac{1}{4} i^{2+j} g^8 N \delta^{aa} \int d^d \tilde{k}, \tilde{l} \hat{C}_j(p_1 + p_2, l) \frac{i(1, 1, 0, 1, 1, p_1)}{(p_1 + p_2 + l)^2} \end{aligned}$$

and cancels exactly against the second contribution in the brackets in the last line of (7.62).

We have calculated the self-energy diagram 7.7b in (7.60) and now see that half of this term cancels with the fourth contribution in the brackets in the last line of (7.62) by either solving the integral in (7.62) or *undoing* the integral from the one-loop corrected propagator. Choosing the latter way and using therefore

$$\int d^d \tilde{k} \frac{1}{(-k^2)(-l-k)^2(-l^2)} = \frac{b_0(1, 1)}{(-l^2)^{3-d/2}}, \quad (7.63)$$

we can thus rewrite (7.60) as

$$\frac{1}{2} \langle \mathcal{O}(p_1) \tilde{\mathcal{O}}(p_2) \hat{\mathcal{O}}_j \rangle^{7.7b} = -\frac{1}{4} i^{2+j} g^8 N \delta^{aa} \int d^d \tilde{k}, \tilde{l} \hat{C}_j(p_1 + p_2, l) \frac{i(1, 0, 1, 1, 1, p_1)}{(p_1 + p_2 + l)^2} \quad (7.64)$$

and see that this term indeed cancels with the fourth contribution in the brackets in the last line of (7.62). We repeat the same steps for the self-energy diagram 7.7a multiplied with 1/2 and find

$$\frac{1}{2} \langle \mathcal{O}(p_1) \tilde{\mathcal{O}}(p_2) \hat{\mathcal{O}}_j \rangle^{7.7a} = -\frac{1}{4} i^{2+j} g^8 N \delta^{aa} \int d^d \tilde{k}, \tilde{l} \hat{C}_j(p_1 + p_2, l) \frac{i(0, 1, 1, 1, 1, p_1)}{(p_1 + p_2 + l)^2}, \quad (7.65)$$

which thus cancels exactly with the third contribution in the brackets in the last line of (7.62). Thus the only terms that remain in (7.62) are the first and the last two terms, i.e. we can write

$$\begin{aligned} \langle \mathcal{O} \tilde{\mathcal{O}} \hat{\mathcal{O}}_j \rangle^{7.7e} + \langle \mathcal{O} \tilde{\mathcal{O}} \hat{\mathcal{O}}_j \rangle^{7.7h} + \frac{1}{2} \langle \mathcal{O} \tilde{\mathcal{O}} \hat{\mathcal{O}}_j \rangle^{7.7b} + \frac{1}{2} \langle \mathcal{O} \tilde{\mathcal{O}} \hat{\mathcal{O}}_j \rangle^{7.7a} \\ = \frac{1}{4} i^{2+j} g^8 N \delta^{aa} \int d^d \tilde{k}, \tilde{l} \frac{\hat{C}_j(p_1 + p_2, l)}{(p_1 + p_2 + l)^2} \\ \left(-2p_1^2 i(1, 1, 1, 1, 1, p_1) + i(1, 1, 1, 0, 1, p_1) + i(1, 1, 1, 1, 0, p_1) \right). \end{aligned} \quad (7.66)$$

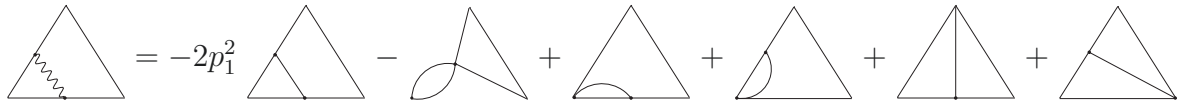


Figure 7.10: Due to the momenta from the gluon vertices, the integrand decomposes into simpler integrals, which cancel with the self-energy and four-scalar interaction terms. Note, that this has no effect on the other numerator momenta.

7.6.4 Cancellations between Diagrams 7.7a, 7.7c, 7.7i and 7.7f

The same analysis as in 7.6.3 can be repeated for diagrams 7.7a, 7.7c, 7.7i and 7.7f which have the same expressions under the exchange $p_1 \leftrightarrow p_2$ and thus we find

$$\begin{aligned} & \langle \mathcal{O}\tilde{\mathcal{O}}\hat{\mathcal{O}}_j \rangle^{7.7i} + \langle \mathcal{O}\tilde{\mathcal{O}}\hat{\mathcal{O}}_j \rangle^{7.7f} + \frac{1}{2} \langle \mathcal{O}\tilde{\mathcal{O}}\hat{\mathcal{O}}_j \rangle^{7.7a} + \frac{1}{2} \langle \mathcal{O}\tilde{\mathcal{O}}\hat{\mathcal{O}}_j \rangle^{7.7c} \\ &= \frac{1}{4} i^{2+j} g^8 N \delta^{aa} \int d^d \tilde{k}, \tilde{l} \frac{\hat{C}_j(p_1 + p_2, l)}{(p_1 + p_2 + l)^2} \\ & \quad \left(-2p_2^2 i(1, 1, 1, 1, 1, p_2) + i(1, 1, 1, 0, 1, p_2) + i(1, 1, 1, 1, 0, p_2) \right). \end{aligned} \quad (7.67)$$

7.6.5 Protected Three-Point Function for $j = 0$

For the special case $j = 0$, all contributions cancel, since then the operator at x_3 is also a protected operator, see section 7.4.6. In this case diagrams 7.7k, 7.7l, 7.7j do not appear, since there is no covariant derivative from the operator at x_3 and furthermore $\hat{C}_{j=0} = 1$. Furthermore, for $j = 0$, the third and fifth term in (7.57) exactly cancel with the self-energy diagrams. If we rewrite 7.7b in terms of (7.64) and correspondingly for 7.7c the cancellation for $j = 0$ is trivial. Then, the remaining contributions are

$$\begin{aligned} & \langle \mathcal{O}\tilde{\mathcal{O}}\hat{\mathcal{O}}_{j=0} \rangle^{(1)} \\ &= \frac{1}{4} (i)^2 g^8 N \delta^{aa} \int d^d \tilde{k}, \tilde{l} \left(\frac{-2P^2 i(1, 1, 1, 1, 1, P)}{(p_1 + k)^2} + \frac{i(0, 1, 1, 1, 1, P)}{(p_1 + k)^2} + \frac{i(1, 0, 1, 1, 1, P)}{(p_1 + k)^2} \right) \\ & \quad + \left(\frac{-2p_1^2 i(1, 1, 1, 1, 1, p_1)}{(p_1 + p_2 + l)^2} + \frac{i(1, 1, 1, 0, 1, p_1)}{(p_1 + p_2 + l)^2} + \frac{i(1, 1, 1, 1, 0, p_1)}{(p_1 + p_2 + l)^2} + (p_1 \leftrightarrow p_2) \right). \end{aligned}$$

From the definition (7.58) one easily verifies

$$\int d^d \tilde{k}, \tilde{l} \frac{i(1, 0, 1, 1, 1, P)}{(p_1 + k)^2} = \int d^d \tilde{k}, \tilde{l} \frac{i(1, 1, 1, 1, 0, p_1)}{(p_1 + p_2 + l)^2}, \quad (7.68)$$

$$\int d^d \tilde{k}, \tilde{l} \frac{i(0, 1, 1, 1, 1, P)}{(p_1 + k)^2} = \int d^d \tilde{k}, \tilde{l} \frac{i(1, 1, 1, 1, 0, p_2)}{(p_1 + p_2 + l)^2}, \quad (7.69)$$

$$\int d^d \tilde{k}, \tilde{l} \frac{i(1, 1, 1, 0, 1, p_1)}{(p_1 + p_2 + l)^2} = \int d^d \tilde{k}, \tilde{l} \frac{i(1, 1, 1, 0, 1, p_2)}{(p_1 + p_2 + l)^2}. \quad (7.70)$$

The second equality can be obtained by changing variables $k \rightarrow -k - p_1 - p_2$, $l \rightarrow -l - p_1 - p_2$ and the last equality by the same change of variables and an additional renaming $k \leftrightarrow l$.

Now we can use the relation shown in figure 7.11 that was derived in [318] and all diagrams cancel, e.g.

$$\int d^d \tilde{k}, \tilde{l} \left(\frac{-2P^2 i(1, 1, 1, 1, 1, P)}{(p_1 + k)^2} + 2 \frac{i(1, 1, 1, 0, 1, p_2)}{(p_1 + p_2 + l)^2} \right) = 0 \quad (7.71)$$

and thus all one-loop contributions cancel

$$\langle \mathcal{O} \tilde{\mathcal{O}} \hat{\mathcal{O}}_{j=0} \rangle^{(1)} = 0. \quad (7.72)$$

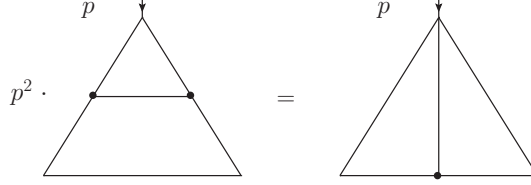


Figure 7.11: Relation for scalar three-point integrals derived in [318]. It would be desirable to derive a modified identity for diagrams with numerator momenta.

7.6.6 Implementation of the Limit $p_1 + p_2 \rightarrow 0$

We can now safely take the limit $p_1 + p_2 \rightarrow 0$, since the only diagram that could lead to ambiguities in this limit is the four-scalar diagram 7.7g which has been cancelled by a contribution from 7.7d. For the same reason as in (7.5) we only need one term from the Gegenbauer polynomials, since

$$\lim_{p_1+p_2 \rightarrow 0} \hat{C}_j(p_1 + p_2, l) = c_{jj}^{1/2} (2\hat{l})^j, \quad (7.73)$$

where $c_{jj}^{1/2}$ is the coefficient with the highest power of the Gegenbauer polynomial that was given in (7.53). Furthermore, we have

$$\frac{i(a_1, a_2, a_3, a_4, a_5, P \rightarrow 0)}{(k + p_1)^2} = i(a_1 + a_2, 1, a_3, a_4 + a_5, 0, p_1) \quad (7.74)$$

and the integrals turn into simple two-point bubble integrals that are easily solved, see Appendix A.2. Denoting from now on $p_1 = p$ we find

$$\begin{aligned} \langle \mathcal{O} \tilde{\mathcal{O}} \hat{\mathcal{O}}_j \rangle^{7.7d} + \langle \mathcal{O} \tilde{\mathcal{O}} \hat{\mathcal{O}}_j \rangle^{7.7g} &= \\ &= -c_{jj}^{1/2} 2^{j-1} i^{2+j} g^8 N \delta^{aa} b_j(4 - \frac{d}{2}, 1) (b_j(2, 1) + b_j(1, 1)) \frac{\hat{p}^j}{(-p^2)^{5-d}}. \end{aligned} \quad (7.75)$$

The self-energy diagram multiplied with one half (7.60) becomes

$$\frac{1}{2} \langle \mathcal{O} \tilde{\mathcal{O}} \hat{\mathcal{O}}_j \rangle^{7.7b} = c_{jj}^{1/2} i^{2+j} g^8 N \delta^{aa} 2^{j-2} b_0(1, 1) b_j(4 - \frac{d}{2}, 1) \frac{\hat{p}^j}{(-p^2)^{5-d}}. \quad (7.76)$$

Since $p_1 = -p_2 = p$ and j even we find the same result for the other remaining half self-energy diagram

$$\frac{1}{2} \langle \mathcal{O} \tilde{\mathcal{O}} \hat{\mathcal{O}}_j \rangle^{7.7c} = c_{jj}^{1/2} i^{2+j} g^8 N \delta^{aa} 2^{j-2} b_0(1, 1) b_j(4 - \frac{d}{2}, 1) \frac{\hat{p}^j}{(-p^2)^{5-d}}. \quad (7.77)$$

Applying the limit to the diagrams in (7.66) we find

$$\begin{aligned} & \langle \mathcal{O}\tilde{\mathcal{O}}\hat{\mathcal{O}}_j \rangle^{7.7e} + \langle \mathcal{O}\tilde{\mathcal{O}}\hat{\mathcal{O}}_j \rangle^{7.7h} + \frac{1}{2} \langle \mathcal{O}\tilde{\mathcal{O}}\hat{\mathcal{O}}_j \rangle^{7.7b} + \frac{1}{2} \langle \mathcal{O}\tilde{\mathcal{O}}\hat{\mathcal{O}}_j \rangle^{7.7a} \\ &= i^{2+j} g^8 N \delta^{aa} 2^{j-2} c_{jj}^{1/2} (2c_{0j}(1, 1, 1, 2, 1) + c_{0j}(1, 1, 1, 1, 1) + c_{0j}(1, 1, 1, 2, 0)) \frac{\hat{p}^j}{(-p^2)^{5-d}}. \end{aligned} \quad (7.78)$$

The integrals $c_{nm}(a_1, a_2, a_3, a_4, a_5)$ are defined in (H.12) and solved in Appendix H.2 using the IBP technique. All these integrals are finite. Since j is even we get the same contribution for $p_2 = -p$ when taking the limit in (7.67).

7.6.7 Diagrams with One Covariant Derivative

For $j \neq 0$ we have additional diagrams with exactly one gauge field from the covariant derivative in $\hat{\mathcal{O}}_j$. We derive the form of the operator with exactly one gauge field in Appendix H.3 and find that it can be written as

$$\hat{\mathcal{O}}_j^A = \frac{1}{2} f^{abc} (1 + (-1)^j) \left(\sum_{k=1}^j a_{jk}^{1/2} \sum_{m=1}^k \binom{k}{m} \hat{\partial}^{m-1} \hat{A}^a \hat{\partial}^{k-m} \phi^{12,b} \hat{\partial}^{j-k} \phi^{12,c} \right). \quad (7.79)$$

The diagrams are shown in figure 7.7j, 7.7k, 7.7l. We will directly calculate these diagrams in the limit $p_1 + p_2 \rightarrow 0$.

7.6.8 Diagrams 7.7k and 7.7l

Diagram 7.7k and 7.7l have the same expression under the exchange $p_1 \leftrightarrow p_2$ and in the limit $p_1 = -p_2$ they yield identical contributions, since we consider j even. We start with diagram 7.7k, which arises from two different contractions with the operator with one gauge field (H.43) for j even.

$$\hat{\mathcal{O}}_j = f^{abc} \left(\sum_{k=1}^j a_{jk}^{1/2} \sum_{m=1}^k \binom{k}{m} \hat{\partial}^{m-1} \hat{A}^a \hat{\partial}^{k-m} \phi^{12,b} \hat{\partial}^{j-k} \phi^{12,c} \right). \quad (7.80)$$

We thus get

$$\begin{aligned} & \langle \mathcal{O}(p) \tilde{\mathcal{O}}(-p) \hat{\mathcal{O}}_j(0) \rangle^{7.7k} = \\ &= i^{6+j} \frac{g^8 N \delta^{aa}}{4} \sum_{k=1}^j a_{jk} \sum_{m=1}^k \binom{k}{m} \int d^d \tilde{k}, \tilde{l} \frac{(\hat{k} + \hat{l}) \left((\hat{l} - \hat{k})^{m-1} (-\hat{l})^{k-m} \hat{k}^{j-k} + (k \leftrightarrow l) \right)}{k^4 (p+k)^2 (l-k)^2 l^2} \\ &= i^{6+j} \frac{g^8 N \delta^{aa}}{4} \sum_{k=1}^j a_{jk} \sum_{m=1, n=0}^{k, m-1} \binom{k}{m} \binom{m-1}{n} (-1)^{n+k-m} \\ & \quad \int d^d \tilde{k}, \tilde{l} \frac{\left(\hat{l}^{k-n} \hat{k}^{j-k+n} + \hat{l}^{k-n-1} \hat{k}^{j-k+n+1} + (k \leftrightarrow l) \right)}{-(-k^2)^2 (-(p+k)^2) (-(l-k))^2 (-l^2)}. \end{aligned}$$

The integrals are simple two-point integrals

$$\langle \mathcal{O}(p) \tilde{\mathcal{O}}(-p) \hat{\mathcal{O}}_j(0) \rangle^{7.7k} = -i^{6+j} \frac{g^8 N \delta^{aa}}{4} b_j(4 - d/2, 1) \frac{\hat{p}^j}{(-p^2)^{5-d}} S^{7.7k}[j], \quad (7.81)$$

where

$$S^{7.7k}[j] = \sum_{k=1}^j a_{jk} \sum_{m=1}^k \sum_{n=0}^{m-1} \binom{k}{m} \binom{m-1}{n} (-1)^m \quad (7.82)$$

$$[b_{k-n}(1, 1) - b_{k-n-1}(1, 1) + b_{j-k+n}(1, 1) - b_{j-k+n+1}(1, 1)].$$

We can solve at least the sum over n using the relation (H.40).

7.6.9 Diagram 7.7j

Diagram 7.7j is calculated in a very similar way and the two possible contractions yield

$$\langle \mathcal{O}(p) \tilde{\mathcal{O}}(-p) \hat{\mathcal{O}}_j(0) \rangle^{7.7j} = i^{6+j} \frac{g^8 N \delta^{aa}}{4} \sum_{k=1}^j a_{jk} \sum_{m=1}^k \binom{k}{m} (-1)^{k-m} \quad (7.83)$$

$$\int d^d \tilde{k}, \tilde{l} \frac{(2\hat{p} + \hat{k} + \hat{l}) \left((\hat{k} - \hat{l})^{m-1} \hat{k}^{k-m} \hat{l}^{j-k} + (k \leftrightarrow l) \right)}{k^2(p+k)^2(l-k)^2 l^2(p+l)^2}.$$

Introducing another binomial sum for rewriting $(k-l)^{m-1}$ and noting that due to $f_{nm}(1, 1, 1, 1, 1)$ the term with $(k \leftrightarrow l)$ yields the same contribution we have

$$\langle \mathcal{O}(p) \tilde{\mathcal{O}}(-p) \hat{\mathcal{O}}_j(0) \rangle^{7.7j} = i^{6+j} \frac{g^8 N \delta^{aa}}{2} \sum_{k=1}^j a_{jk} \sum_{m=1}^k \sum_{n=0}^{m-1} \binom{k}{m} \binom{m-1}{n} (-1)^{k-m+n}$$

$$\left(2\hat{p} f_{k-1-n, j-k+n} + f_{k-n, j-k+n} + f_{k-1-n, j-k+n+1} \right) \quad (7.84)$$

and all integrals $f_{nm}(1, 1, 1, 1, 1)$ are finite and were solved in H.2.2.

7.6.10 Full Bare Three-Point Function

Taking into account the exact cancellations between diagrams, as well as the fact that diagrams with $p_1 \leftrightarrow p_2$ are identical in the limit $p_1 \rightarrow -p_2$ for j even, the remaining bare one-loop contribution to the three-point function is given by

$$\langle \mathcal{O} \tilde{\mathcal{O}} \hat{\mathcal{O}}_j \rangle^{(1)} = \sum_{\alpha=a..l} \langle \mathcal{O} \tilde{\mathcal{O}} \hat{\mathcal{O}}_j \rangle^{7.7\alpha} \quad (7.85)$$

$$= 2 \times (7.76) + 2 \times (7.81) + (7.84) + (7.75) + 2 \times (7.78).$$

We have calculated all diagrams in momentum space. In order to read off the structure constant we Fourier transform the expression to position space using

$$\int \frac{d^d p}{(2\pi)^d} \frac{\hat{p}^j}{(-p^2)^{5-d}} e^{-ipx} = i \frac{\Gamma(3d/2 - 5 + j)}{\Gamma(5 - d)} \frac{(2i)^j}{4^{5-d} \pi^{d/2}} \frac{\hat{x}^j}{(-x^2)^{3h-5+j}} \quad (7.86)$$

We simply plug in all these equations into a Mathematica file and find that the result for the normalisation invariant structure constant is given by (7.34).

7.6.11 Anomalous Dimension from Three-Point Function

As a check of the relative factors and signs between the diagrams we can read off the anomalous dimension from the three-point function. It is interesting to note that even though all diagrams in 7.7 except the one in 7.7j are divergent, the divergences *localised* at the BPS operators exactly cancel due to the cancellations described in sections 7.6.3, 7.6.4. This is what one would expect, since the only non-BPS operator is $\hat{\mathcal{O}}_j$ and the divergences should be *localised* at this operator. It turns out that we get the following contributions to the anomalous dimension

$$\gamma_j^{(7.7b+7.7c)/2} = \frac{g^2 N}{4\pi^2}, \quad \gamma_j^{7.7d+7.7g} = \frac{g^2 N}{4\pi^2} \left(-\frac{1}{j+1} \right), \quad \gamma_j^{7.7k+7.7l} = \frac{g^2 N}{4\pi^2} \left(2H_j + \frac{1}{j+1} - 1 \right),$$

very similar to the two-point case in section 7.4.3. The sum of all contributions correctly yields the first order of the anomalous dimension

$$\gamma_j = \left(\frac{g^2 N}{4\pi^2} \right) 2H_j + \mathcal{O}(g^4). \quad (7.87)$$

Some more details can be found in Appendix H.9.

Chapter 8

Conclusions and Outlook

In the first part of this thesis, we have analysed light-like polygonal Wilson loops in Chern-Simons and ABJM theory and found remarkable similarities to the Wilson loop in $\mathcal{N} = 4$ super Yang-Mills. We found that the two-loop expectation value of the n -sided light-like Wilson loop in ABJM theory (5.5) has the same functional form as the n -sided light-like polygonal Wilson loop in $\mathcal{N} = 4$ super Yang-Mills theory at one-loop order (5.6). Furthermore, for $n = 4$, the expectation value precisely agrees with the form of the expectation value of the two-loop calculation of four-point scattering amplitudes in ABJM theory. We have shown, that the form of the Wilson loop is fixed by anomalous conformal Ward identities. Starting from $n = 6$, an arbitrary *remainder function* of the conformally invariant cross ratios could potentially contribute. Using numerical analysis we have found however, that this function is a trivial constant. We have also checked that this statement is true for $n = 8, 10, 12, 14$ and expect it to hold for all n . This fact is decisive for the equality to Wilson loops in $\mathcal{N} = 4$ super Yang-Mills theory, where the remainder function at one loop is also trivial. Interestingly, this follows a similar pattern in the spectrum of planar anomalous dimensions persisting to all orders in the coupling.

Furthermore, recent one-loop six-point amplitude calculations in ABJM theory [22, 23] have revealed non-vanishing results in contrast to the vanishing of the Wilson loop in ABJM theory. Together with the fact, that no T-self-duality of the type IIA string on $AdS_4 \times \mathbb{CP}^3$ could be found so far, this might indicate that there is no amplitude / Wilson loop duality in ABJM theory.

On the other hand, in the $\mathcal{N} = 4$ SYM duality the bosonic Wilson loop matches the tree-level stripped MHV amplitudes and for the matching of non-MHV amplitudes the introduction of a supersymmetric Wilson loop is necessary. Due to the lack of helicity in the three-dimensional ABJM theory, one may thus suspect, that a suitable supersymmetric Wilson loop is necessary in order to find a relation between the Wilson loop and higher-point amplitudes. It would be interesting, whether one can find a Wilson loop that matches the six-point amplitudes at one loop.

In this context one should also keep in mind, that non-trivial evidence for a duality between Wilson loops and correlators in ABJM theory was found at one-loop level in

[21].

Furthermore, it would be desirable to extend the knowledge on amplitudes in ABJM theory to higher loops and legs. It would also be interesting to perform a four-point amplitude or Wilson loop computation at four loops, to check whether the conjectured BDS-like ansatz [17] for the four-point amplitude in ABJM theory indeed holds. Additionally, it would be very interesting to investigate the symmetry of the integrands found in [254] for $\mathcal{N} = 4$ SYM also in ABJM theory and possibly make use of this in order to determine higher-loop correlation functions. The possible duality between correlators and Wilson loops in ABJM theory has only been tested at one-loop order and it would certainly be interesting to compare a two-loop calculation of correlators to the Wilson loop result.

Furthermore, it is fair to say, that it has not yet been fully understood, why the duality between amplitudes and Wilson loops in $\mathcal{N} = 4$ SYM works at all. It would be interesting to see, what makes the integrands identical and whether there are additional hidden symmetries that fully fix the perturbative calculations.

In the second part of this thesis we have calculated three-point functions of two half-BPS operators and one spin j operator. It turns out that the one-loop corrections (7.2) to the structure constants have a simple result in terms of generalised harmonic sums.

For the calculation we have used a special infrared limit, where the momentum P going in at the twist-two operator $\hat{\mathcal{O}}_j$ is zero. One simplification in this limit is, that the three-point calculation is effectively reduced to the calculation of two-point functions. Furthermore, a large simplification occurs due to the fact, that the one-loop mixing matrix drops out completely in this limit. we have shown that the result matches a previous result in the literature, which is obtained from an application of the operator product expansion on four-point correlators and thus independently confirm this result by our direct calculation. Furthermore, our procedure can easily be applied to the calculation of other structure constants, e.g. the calculation of three-point functions involving two operators with spin. It would also be interesting to apply integrability methods to the calculation of three-point functions of operators with spin.

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Appendix A

Notation and Conventions

Transposition and conjugation

1. $(A)^*$ denotes the complex conjugate of A
2. $(A)^T$ denotes the transposition of A , i.e. $(A_{ij})^T = A_{ji}$
3. $(A)^\dagger$ is the hermitian conjugate of A , i.e. $(A)^\dagger = ((A)^*)^T$

Metric-Convention

For the calculations¹ throughout this thesis, we will use the mostly minus metric of d -dimensional Minkowski space

$$\eta_{\mu\nu} = \text{diag}(1, -1, -1, \dots, -1) \quad (\text{A.1})$$

and write Lorentz invariants k^2, x^2 in integrals in such a way that the integrals are real in the *euclidean (space-like) regime*, i.e. they are real for $-k^2 > 0, -x^2 > 0$. Therefore, the x-space propagators in explicit calculations are $1/(-x^2 + i\epsilon)^a$, but we will suppress the $i\epsilon$ prescription. Furthermore, we will frequently rewrite $h = d/2$ where d is the dimension of space-time.

Derivatives

For derivatives with respect to x^μ , respectively several points x_i^μ we will use the notation

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \quad \text{and} \quad \partial_{i,\mu} = \frac{\partial}{\partial x_i^\mu}. \quad (\text{A.2})$$

¹For simplicity of notation we will however just write $1/x^2$ in the introductory chapters.

Expectation values

We denote the expectation value of operators by

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle := \langle 0 | T(\mathcal{O}_1 \dots \mathcal{O}_n) | 0 \rangle = Z^{-1} \int \mathcal{D}\Phi e^{iS[\Phi]} \mathcal{O}_1 \dots \mathcal{O}_n, \quad (\text{A.3})$$

where $Z = \int \mathcal{D}\Phi e^{iS[\Phi]}$.

Normalisation of generators

We normalise the generators T^a ($a = 1 \dots N^2 - 1$) of the $SU(N)$ algebra

$$[T^a, T^b] = i f^{abc} T^c \quad (\text{A.4})$$

in the fundamental representation as

$$\text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab} \quad (\text{A.5})$$

and we will abbreviate the number of generators as $\delta^{aa} = N^2 - 1$ in the calculations. The generators satisfy the relation

$$\sum_{c,d} f^{acd} f^{bcd} = N \delta^{ab}. \quad (\text{A.6})$$

The latter relation is independent of the representation and can easily be derived in the adjoint representation in the standard normalisation

$$(T_A^a)^{bc} = i f^{abc}, \quad (\text{A.7})$$

where we have

$$\text{Tr}(T_A^a T_A^b) = N \delta^{ab}. \quad (\text{A.8})$$

One can thus easily derive the relation (A.6)

$$\sum_{c,d} f^{acd} f^{bcd} = - \sum_{c,d} f^{acd} f^{bdc} = \sum_{c,d} (T_A^a)^{cd} (T_A^b)^{dc} = \text{Tr}(T_A^a T_A^b) = N \delta^{ab}. \quad (\text{A.9})$$

Definition of Polygons

An n -sided polygon can be defined by n points x_i ($i = 1, \dots, n$), with the edge i being the line connecting x_i and x_{i+1} . Defining

$$p_i^\mu = x_{i+1}^\mu - x_i^\mu \quad (\text{A.10})$$

and paramtrising the position z_i^μ on edge i with the parameter $s_i \in [0, 1]$ we have

$$z_i^\mu(s_i) = x_i^\mu + p_i^\mu s_i. \quad (\text{A.11})$$

Furthermore, we use the notation

$$\epsilon(p, q, r) = \epsilon_{\mu\nu\rho} p^\mu q^\nu r^\rho \quad \text{and} \quad \bar{s}_i = 1 - s_i. \quad (\text{A.12})$$

One can easily check that with the definition $x_{ij}^\mu = x_i^\mu - x_j^\mu$

$$2x_{ij} \cdot x_{mn} = x_{in}^2 + x_{jm}^2 - x_{im}^2 - x_{jn}^2. \quad (\text{A.13})$$

Using (A.13) we can rewrite the scalar products

$$2p_i \cdot p_j = x_{i,j+1}^2 + x_{i+1,j}^2 - x_{i,j}^2 - x_{i+1,j+1}^2. \quad (\text{A.14})$$

Furthermore, using the definition (A.11), one can easily show that

$$(z_i - z_j)^2 = x_{ij}^2 \bar{s}_i \bar{s}_j + x_{i+1,j}^2 s_i \bar{s}_j + x_{i,j+1}^2 \bar{s}_i s_j + x_{i+1,j+1}^2 s_i s_j. \quad (\text{A.15})$$

A.1 Fourier Transformation

In Minkowski-space with signature $(+ - - - \dots)$ we have

$$\int \frac{d^d p}{(2\pi)^d} \frac{e^{-ip \cdot x}}{(-p^2 - i\epsilon)^k} = \frac{\Gamma(\frac{d}{2} - k)}{\Gamma(k)} \frac{1}{4^k \pi^{\frac{d}{2}}} \frac{i}{(-x^2 + i\epsilon)^{\frac{d}{2} - k}}. \quad (\text{A.16})$$

Note, that this can be derived from the Euclidean Fourier-transform by Wick-rotating. The sign of the Wick-rotation depends on the location of the poles. Since the sign of $i\epsilon$ changes after the Fourier-transform, for the transformation back from x-space to p-space, we have to Wick-rotate into the other direction, getting an overall sign.

$$\int d^d x \frac{e^{+ip \cdot x}}{(-x^2 + i\epsilon)^k} = \frac{\Gamma(\frac{d}{2} - k)}{\Gamma(k)} \frac{1}{4^k \pi^{\frac{d}{2}}} \frac{(-i)}{(-p^2 - i\epsilon)^{\frac{d}{2} - k}}. \quad (\text{A.17})$$

This yields the identity when transforming and transforming back.

A.2 Bubble Integrals

We define the scalar bubble integral as

$$B_0(\alpha_1, \alpha_2) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(-k^2)^{\alpha_1} (-(p+k)^2)^{\alpha_2}} = b_0(\alpha_1, \alpha_2) \frac{1}{(-p^2)^{\alpha_1 + \alpha_2 - h}}, \quad (\text{A.18})$$

where $h = d/2$ and

$$b_0(\alpha_1, \alpha_2) = \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(\frac{d}{2} - \alpha_1) \Gamma(\frac{d}{2} - \alpha_2) \Gamma(\alpha_1 + \alpha_2 - \frac{d}{2})}{\Gamma(d - \alpha_1 - \alpha_2) \Gamma(\alpha_1) \Gamma(\alpha_2)}. \quad (\text{A.19})$$

Furthermore, the bubble integral with momenta in the numerator is

$$\begin{aligned} B_n(\alpha_1, \alpha_2) &= \int \frac{d^d k}{(2\pi)^d} \frac{\hat{k}^n}{(-k^2 - i\epsilon)^{\alpha_1} (-(p+k)^2 - i\epsilon)^{\alpha_2}} \\ &= b_n(\alpha_1, \alpha_2) \frac{\hat{p}^n}{(-p^2 - i\epsilon)^{\alpha_1 + \alpha_2 - \frac{d}{2}}}, \end{aligned} \quad (\text{A.20})$$

where

$$b_n(\alpha_1, \alpha_2) = i \frac{(-1)^n}{(4\pi)^{d/2}} \frac{\Gamma(\frac{d}{2} + n - \alpha_1) \Gamma(\frac{d}{2} - \alpha_2) \Gamma(\alpha_1 + \alpha_2 - \frac{d}{2})}{\Gamma(d + n - \alpha_1 - \alpha_2) \Gamma(\alpha_1) \Gamma(\alpha_2)}. \quad (\text{A.21})$$

For the analogous integrals in x-space, one has to Wick-rotate into the other direction, since the poles are located differently in the complex plane, thus giving the integral

$$\int \frac{d^d x}{(2\pi)^d} \frac{\hat{x}^n}{(-x^2 + i\epsilon)^{\alpha_1} (-(x+y)^2 + i\epsilon)^{\alpha_2}} = -b_n(\alpha_1, \alpha_2) \frac{\hat{y}^n}{(-y^2 + i\epsilon)^{\alpha_1 + \alpha_2 - \frac{d}{2}}}. \quad (\text{A.22})$$

Appendix B

Technical Details on Conformal Symmetry

B.1 Derivation of Conformal Killing Vectors

Here, we present some details on the derivation of infinitesimal conformal transformations, roughly following chapter four of the book [82]. Under conformal transformations, by definition, the metric transforms as

$$g'_{\mu\nu}(x') = \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} g_{\rho\sigma}(x') = \rho(x) g_{\mu\nu}(x). \quad (\text{B.1})$$

Angles between two tangent vectors $u^{\mu}(x), v^{\mu}(x)$ defined by

$$\cos \theta = \frac{u \cdot v}{|u||v|} = \frac{g_{\alpha\beta}(x) u^{\alpha} v^{\beta}}{\sqrt{g_{\alpha\beta}(x) u^{\alpha} u^{\beta}} \sqrt{g_{\alpha\beta}(x) v^{\alpha} v^{\beta}}} \quad (\text{B.2})$$

are thus preserved under conformal transformations. Inserting (2.2) into (B.1) and specialising to flat Minkowski space with metric $\eta_{\mu\nu}$, the terms linear in k^{μ} read

$$\partial_{\mu} k_{\nu} + \partial_{\nu} k_{\mu} = \omega(x) \eta_{\mu\nu}, \quad (\text{B.3})$$

where we have written $\rho(x) = 1 + \omega(x)$. Taking the trace of (B.3) solving for $\omega(x) = 2/d \partial^{\alpha} k_{\alpha}$ and reinserting it into (B.3) leads to the *conformal killing equation* in flat space

$$\partial_{\mu} k_{\nu} + \partial_{\nu} k_{\mu} = \frac{2}{d} \eta_{\mu\nu} \partial^{\alpha} k_{\alpha}. \quad (\text{B.4})$$

For $d = 1$ this equation does not lead to any restrictions on $k_{\mu}(x)$ and thus every smooth transformation is conformal. The case $d = 2$ is special, since then (B.4) has an infinite number of solutions and the conformal group is infinite-dimensional. In this thesis we will only explicitly make use of conformal transformations for $d \geq 3$ and thus

specialise to this case from now on. The derivative of (B.4) in the following linear combination with permuted indices leads to the second order differential equation

$$(\eta_{\mu\rho}\partial_\nu + \eta_{\nu\rho}\partial_\mu - \eta_{\mu\nu}\partial_\rho)(\partial \cdot k) = d\partial_\mu\partial_\nu k_\rho. \quad (\text{B.5})$$

where we have written $(\partial^\alpha k_\alpha) = (\partial \cdot k)$. Contraction of (B.5) with ∂^ρ yields

$$(2-d)\partial_\mu\partial_\nu(\partial \cdot k) = \eta_{\mu\nu}\partial^2(\partial \cdot k). \quad (\text{B.6})$$

Contracting this equation with $\eta^{\mu\nu}$ yields

$$2(1-d)\partial^2(\partial \cdot k) = 0. \quad (\text{B.7})$$

Combining (B.6) and (B.7) we find $\partial_\mu\partial_\nu(\partial \cdot k) = 0$ which is thus at most linear in x^μ . Combining this fact with (B.5) it is clear that $\partial_\mu\partial_\nu k_\rho(x) = \text{constant}$ and thus k_ρ is at most quadratic in x^μ , i.e. we can write

$$k_\mu(x) = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho. \quad (\text{B.8})$$

Inserting this ansatz into the first (B.3) respectively second (B.5) order differential equations for k^μ yields constraints for these coefficients. Each order in x^μ can be investigated separately. There are no constraints on a^μ which has thus d independent components. Inserting the first order term in x^μ into (B.4) we get

$$b_{\nu\mu} + b_{\mu\nu} = \frac{2}{d}\eta_{\mu\nu}b^\rho{}_\rho \quad (\text{B.9})$$

and thus a combination of a trace term proportional to the metric and an antisymmetric part $\omega_{\nu\mu} = -\omega_{\mu\nu}$ satisfies this equation, i.e.

$$b_{\mu\nu} = \lambda\eta_{\mu\nu} + \omega_{\mu\nu}. \quad (\text{B.10})$$

The first term describes an infinitesimal dilatation

$$x'^\mu = x^\mu + \lambda x^\mu \quad (\text{B.11})$$

and has one free parameter λ and the second term generates infinitesimal rotations (Lorentz transformations)

$$x'^\mu = x^\mu + \omega_{\mu\nu}x^\nu \quad (\text{B.12})$$

and has $d(d-1)/2$ free parameters. Inserting the second order term of k^μ into (B.5) we find

$$c_{\mu\nu\rho} = \eta_{\nu\mu}c_\rho + \eta_{\rho\mu}c_\nu - \eta_{\nu\rho}c_\mu \quad \text{with} \quad c_\mu = \frac{1}{d}c^\lambda{}_{\lambda\mu}, \quad (\text{B.13})$$

which generates infinitesimal special conformal transformations

$$x'^\mu = x^\mu + 2(x \cdot c)x^\mu - c^\mu x^2 \quad (\text{B.14})$$

and c^μ has d independent components that parametrise the transformation. All these transformations form the most general vector $k^\mu(x)$ given in (2.3).

B.2 Correlators in Conformal Field Theory

The form of the correlation functions can be found by analysing the behaviour of the fields in the correlation functions under conformal transformations. To achieve this, it is useful to represent the change of a conformal primary operator $\mathcal{O}(x)$ by the action of unitary operators U in the state space. Assuming the invariance of the vacuum under these transformations $U|0\rangle = |0\rangle$, we can deduce the following relation between the Green's functions of the transformed and the original operators

$$\begin{aligned} \langle 0|\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)|0\rangle &= \langle 0|(U^{-1}U)\mathcal{O}_1(x_1)(U^{-1}U)\dots(U^{-1}U)\mathcal{O}_n(x_n)(U^{-1}U)|0\rangle \\ &= \langle 0|\mathcal{O}'_1(x_1)\dots\mathcal{O}'_n(x_n)|0\rangle. \end{aligned} \quad (\text{B.15})$$

B.2.1 Transformation of Scalar Operators

We use the following transformation properties of scalar conformal primary operators

$$\mathcal{O}'(x) = U_D\mathcal{O}(x)U_D^{-1} = \lambda^\Delta\mathcal{O}(\lambda x), \quad (\text{B.16})$$

$$\mathcal{O}'(x) = U_I\mathcal{O}(x)U_I^{-1} = \frac{1}{(x^2)^\Delta}\mathcal{O}(I(x)), \quad (\text{B.17})$$

under dilatations respectively inversions. The right-hand side of (B.16) is obtained from the generalization of (2.26) to scalar conformal primary operators $\mathcal{O}(x)$ and the transformation under inversions (B.17) can be deduced from (2.26) and the fact that special conformal transformations can be represented as inversions followed by a translation and another inversion (2.14).

B.2.2 Two-Point Functions of Scalar Operators

Dilatations (B.16) for the two-point function in (B.18) yield

$$\langle 0|\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)|0\rangle = \lambda^{\Delta_1+\Delta_2}\langle 0|\mathcal{O}_1(\lambda x_1)\mathcal{O}_2(\lambda x_2)|0\rangle, \quad (\text{B.18})$$

which requires

$$\langle 0|\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)|0\rangle = \frac{c}{(x_{12}^2)^{(\Delta_1+\Delta_2)/2}}, \quad (\text{B.19})$$

where c is a constant. Inversions (B.17) lead to

$$\langle 0|\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)|0\rangle = \frac{1}{(x_1^2)^{\Delta_1}}\frac{1}{(x_2^2)^{\Delta_2}}\langle 0|\mathcal{O}_1(I(x_1))\mathcal{O}_2(I(x_2))|0\rangle. \quad (\text{B.20})$$

Using (B.19) and the inversion of distances (2.16) we thus find

$$\langle 0|\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)|0\rangle = \frac{1}{(x_1^2)^{\Delta_1}}\frac{1}{(x_2^2)^{\Delta_2}}\frac{c(x_1^2x_2^2)^{(\Delta_1+\Delta_2)/2}}{(x_{12}^2)^{(\Delta_1+\Delta_2)/2}}, \quad (\text{B.21})$$

which can only be satisfied for $\Delta_1 = \Delta_2$, otherwise the correlation function has to vanish. Thus we find

$$\langle 0 | \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) | 0 \rangle = \frac{c \delta_{\Delta_1 \Delta_2}}{(x_{12}^2)^{\Delta_1}}. \quad (\text{B.22})$$

It is easy to see, that this two-point function indeed satisfies the Ward-identities (3.5), (3.6).

B.2.3 Three-Point Functions

Dilatations (B.16) for the three-point function in (B.18) yield

$$\langle 0 | \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) | 0 \rangle = \lambda^{\Delta_1 + \Delta_2 + \Delta_3} \langle 0 | \mathcal{O}_1(\lambda x_1) \mathcal{O}_2(\lambda x_2) \mathcal{O}_3(\lambda x_3) | 0 \rangle, \quad (\text{B.23})$$

which requires

$$\langle 0 | \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) | 0 \rangle = \frac{c}{(x_{12}^2)^a (x_{13}^2)^b (x_{23}^2)^c}, \quad (\text{B.24})$$

where c is a constant and $a + b + c = (\Delta_1 + \Delta_2 + \Delta_3)/2$. Inversions (B.17) lead to

$$\langle 0 | \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) | 0 \rangle = \frac{1}{(x_1^2)^{\Delta_1}} \frac{1}{(x_2^2)^{\Delta_2}} \frac{1}{(x_3^2)^{\Delta_3}} \langle 0 | \mathcal{O}_1(I(x_1)) \mathcal{O}_2(I(x_2)) \mathcal{O}_3(I(x_3)) | 0 \rangle$$

Using (B.24) and the inversion of distances (2.16) we thus find

$$\langle 0 | \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) | 0 \rangle = \frac{1}{(x_1^2)^{\Delta_1}} \frac{1}{(x_2^2)^{\Delta_2}} \frac{1}{(x_3^2)^{\Delta_3}} \frac{c (x_1^2)^{a+b} (x_2^2)^{a+c} (x_3^2)^{b+c}}{(x_{12}^2)^a (x_{13}^2)^b (x_{23}^2)^c}, \quad (\text{B.25})$$

which yields $2a = \Delta_1 + \Delta_2 - \Delta_3$, $2b = \Delta_1 + \Delta_3 - \Delta_2$, $2c = \Delta_2 + \Delta_3 - \Delta_1$ and thus we find

$$\langle 0 | \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) | 0 \rangle = \frac{c}{|x_{12}|^{\Delta_1 + \Delta_2 - \Delta_3} |x_{13}|^{\Delta_1 + \Delta_3 - \Delta_2} |x_{23}|^{\Delta_2 + \Delta_3 - \Delta_1}}. \quad (\text{B.26})$$

It is easy to see, that this two-point function indeed satisfies the Ward-identities (3.5), (3.6).

B.2.4 Transformation of Operators with Spin

In the same way as for scalar operators (B.16), one can derive the transformation properties for primary conformal operators with spin

$$\mathcal{O}'_{\mu_1 \dots \mu_j}(x) = U_D \mathcal{O}_{\mu_1 \dots \mu_j}(x) U_D^{-1} = \lambda^{\Delta_j} \mathcal{O}_{\mu_1 \dots \mu_j}(\lambda x), \quad (\text{B.27})$$

$$\mathcal{O}'_{\mu_1 \dots \mu_j}(x) = U_I \mathcal{O}_{\mu_1 \dots \mu_j}(x) U_I^{-1} = \frac{1}{(x^2)^{\Delta_j}} I_{\mu_1 \nu_1}(x) \dots I_{\mu_j \nu_j}(x) \mathcal{O}_{\nu_1 \dots \nu_j}(I(x)),$$

under dilatations respectively inversions. Here, $I_{\mu\nu}(x)$ is the inversion tensor

$$I_{\mu\nu}(x) = \eta_{\mu\nu} - 2 \frac{x_\mu x_\nu}{x^2}, \quad (\text{B.28})$$

which satisfies the property

$$I_{\mu\sigma}(x_1) I^{\sigma\tau}(I(x_{12})) I_{\tau\nu}(x_2) = I_{\mu\nu}(x_{12}) \quad (\text{B.29})$$

and can be used for explicitly working out the transformation properties below.

B.2.5 Two-Point Functions of Operators with Spin

The transformations (B.27) lead to the conformal symmetry conditions

$$\begin{aligned} \langle \mathcal{O}_{\mu_1 \dots \mu_j}(x_1) \mathcal{O}_{\nu_1 \dots \nu_j}(x_2) \rangle &= \lambda^{2\Delta_j} \langle \mathcal{O}_{\mu_1 \dots \mu_j}(\lambda x_1) \mathcal{O}_{\nu_1 \dots \nu_j}(\lambda x_2) \rangle, \\ \langle \mathcal{O}_{\mu_1 \dots \mu_j}(x_1) \mathcal{O}_{\nu_1 \dots \nu_j}(x_2) \rangle &= \frac{1}{(x_1^2)^{\Delta_j}} \frac{1}{(x_2^2)^{\Delta_j}} I_{\mu_1 \sigma_1}(x_1) \dots I_{\mu_j \sigma_j}(x_1) \times \\ &\quad \times I_{\nu_1 \tau_1}(x_2) \dots I_{\nu_j \tau_j}(x_2) \langle \mathcal{O}_{\sigma_1 \dots \sigma_j}(I(x_1)) \mathcal{O}_{\tau_1 \dots \tau_j}(I(x_2)) \rangle. \end{aligned} \quad (\text{B.30})$$

The solution for a traceless, totally symmetric operator as introduced in (3.3) is

$$\langle \mathcal{O}_{\mu_1 \dots \mu_j}(x_1) \mathcal{O}_{\nu_1 \dots \nu_j}(x_2) \rangle = \frac{C_j}{(x_{12}^2)^{\Delta_j}} \left(I_{\{\mu_1 \nu_1 \dots \mu_j \nu_j\}}(x_{12}) - \text{traces} \right), \quad (\text{B.31})$$

where C_j is a constant, $\{..\}$ indicates that the right-hand side is totally symmetrised in all indices μ_k, ν_l and we subtract all traces in μ_i resp. ν_i in accord with the symmetries of the operators on the left-hand side.

B.2.6 Three-Point Functions of Operators with Spin

The transformations (B.27) lead to the conformal symmetry conditions

$$\begin{aligned} \langle \mathcal{O}_A(x_1) \mathcal{O}_B(x_2) \mathcal{O}_{\mu_1 \dots \mu_j}(x_3) \rangle &= \lambda^{\Delta_A + \Delta_B + \Delta_j} \langle \mathcal{O}_A(\lambda x_1) \mathcal{O}_B(\lambda x_2) \mathcal{O}_{\mu_1 \dots \mu_j}(\lambda x_3) \rangle, \\ \langle \mathcal{O}_A(x_1) \mathcal{O}_B(x_2) \mathcal{O}_{\mu_1 \dots \mu_j}(x_3) \rangle &= \frac{I_{\mu_1 \nu_1}(x_3) \dots I_{\mu_j \nu_j}(x_3)}{(x_1^2)^{\Delta_A} (x_2^2)^{\Delta_B} (x_3^2)^{\Delta_j}} \langle \mathcal{O}_A(I(x_1)) \mathcal{O}_B(I(x_2)) \mathcal{O}_{\nu_1 \dots \nu_j}(I(x_3)) \rangle. \end{aligned} \quad (\text{B.32})$$

The solution of these constraints is

$$\langle \mathcal{O}_A(x_1) \mathcal{O}_B(x_2) \mathcal{O}_{\mu_1 \dots \mu_j}(x_3) \rangle = \frac{C_{ABj} \left(Y_{\mu_1 \dots \mu_j}(x_{13}, x_{23}) - \text{traces} \right)}{|x_{12}|^{\Delta_A + \Delta_B - \theta} |x_{13}|^{\Delta_A + \theta - \Delta_B} |x_{23}|^{\Delta_B + \theta - \Delta_A}}, \quad (\text{B.33})$$

where C_{ABj} is a constant, $\theta = \Delta_j - j$ is the *twist* of the operator $\mathcal{O}_{\mu_1 \dots \mu_j}$ and

$$Y^\mu(x_{13}, x_{23}) = \frac{x_{13}^\mu}{x_{13}^2} - \frac{x_{23}^\mu}{x_{23}^2} = \frac{1}{2} \partial_{x_3}^\mu \ln \left(\frac{x_{13}^2}{x_{23}^2} \right). \quad (\text{B.34})$$

For three-point functions with more operators with spin we can have several separately invariant space-time structures and the general solution is a linear combination of these with a priori arbitrary coefficients for each term.

Appendix C

$\mathcal{N} = 4$ SYM from Dimensional Reduction

In this section we describe how to obtain $\mathcal{N} = 4$ Super Yang-Mills theory in four dimensions from dimensional reduction of $\mathcal{N} = 1$ SYM in ten dimensions. We will closely follow the presentation and notation of [122], but give a few more details.

C.1 $\mathcal{N} = 1$ Super Yang-Mills Theory in Ten Dimensions

We start from the $\mathcal{N} = 1$ Super Yang-Mills action in $D = 10$ with the *mostly minus* metric $g_{MN} = \text{diag}(+, -, \dots, -)$

$$S = \int d^D x \text{Tr} \left(-\frac{1}{2} F_{MN} F^{MN} + i\bar{\psi} \Gamma^M D_M \psi \right), \quad (\text{C.1})$$

where all fields are matrix valued in $\text{SU}(N)$, i.e. $\psi = \psi^a T^a$, $A_M = A_M^a T^a$ and

$$\begin{aligned} F_{MN} &= \partial_M A_N - \partial_N A_M - ig[A_M, A_N], \\ D_M(\cdot) &= \partial_M(\cdot) - ig[A_M, (\cdot)]. \end{aligned} \quad (\text{C.2})$$

For this action to be supersymmetric, it is necessary that ψ be a Majorana-Weyl spinor, i.e. it satisfies¹ the Weyl condition

$$\Gamma\psi = \psi, \quad \Gamma = i\Gamma^0 \dots \Gamma^{D-1} \quad (\text{C.3})$$

and the Majorana condition

$$\psi^T C_{10} = \psi^\dagger \Gamma^0 = \bar{\psi}, \quad (\text{C.4})$$

¹These two conditions reduce the number of fermionic degrees of freedom from 32 to 8, which equals the degrees of freedom of the massless gauge field in ten dimensions. For a linear realization of supersymmetry without auxiliary fields it is required that the bosonic gauge field carries the same number of degrees of freedom as the fermionic field.

where the ten-dimensional Dirac matrices Γ^M and the charge conjugation matrix C_{10} are constructed from a four-dimensional and a six-dimensional representation of the Clifford algebra, the explicit representation is given in section C.2.2. The Majorana-Weyl conditions are satisfied if the spinor has the form

$$\psi = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \lambda_\alpha^A \\ 0 \end{pmatrix}}_{4 \text{ left-handed Weyl spinors}} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \bar{\lambda}_{\dot{\alpha}, A} \end{pmatrix}}_{4 \text{ right-handed Weyl spinors}}, \quad (\text{C.5})$$

where $A = 1 \dots 4$ denotes four left-handed respectively right-handed Weyl spinors and where $(\lambda_\alpha^A)^* = \bar{\lambda}_{\dot{\alpha}}^A$ and $(\lambda^{\alpha A})^* = \bar{\lambda}^{\dot{\alpha} A}$. Weyl-spinor indices are raised and lowered using the antisymmetric tensor $\epsilon^{\alpha\beta} = -\epsilon_{\alpha\beta}$ with components $\epsilon^{12} = \epsilon_{12} = -\epsilon_{i\dot{2}} = -\epsilon^{i\dot{2}} = 1$

$$\lambda^\alpha = \epsilon^{\alpha\beta} \lambda_\beta, \quad \bar{\lambda}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}} \lambda^{\dot{\beta}}, \quad \lambda_\alpha = \lambda^\beta \epsilon_{\beta\alpha}, \quad \bar{\lambda}^{\dot{\alpha}} = \bar{\lambda}_{\dot{\beta}} \epsilon^{\dot{\beta}\dot{\alpha}}. \quad (\text{C.6})$$

Note that $(\epsilon_{\dot{\alpha}\dot{\beta}})^* = -\epsilon_{\alpha\beta}$.

One can show that the action is invariant under the supersymmetry transformations

$$\delta A_M = -i\bar{\xi}\Gamma_M\psi, \quad \delta\psi = \frac{i}{2}\Gamma_{MN}F^{MN}\xi, \quad (\text{C.7})$$

where $\Gamma_{MN} = \frac{i}{2}(\Gamma_M\Gamma_N - \Gamma_N\Gamma_M)$.

C.2 Reduction to $\mathcal{N} = 4$ SYM in Four Dimensions

Here, we follow the procedure of dimensional reduction given in [122]. We split up the ten-dimensional coordinates labelled by capital latin letters $M = 0, \dots, 9$ into greek indices $\mu = 0, 1, 2, 3$ and small latin indices $m = 4, \dots, 9$. Now, we let the fields only depend on the coordinates x_μ , i.e

$$\partial_m A_M(x_\mu) = 0, \quad \partial_m \psi(x_\mu) = 0, \quad m = 4, \dots, 9. \quad (\text{C.8})$$

The remaining non-vanishing components are then

$$-\frac{1}{2}F_{MN}F^{MN} = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - D_\mu A_a D^\mu A^a + \frac{1}{2}g^2[A_a, A_b][A^a, A^b] \quad (\text{C.9})$$

$$i\bar{\psi}\Gamma_M D^M \psi = i\bar{\psi}\Gamma_\mu D^\mu \psi + g\bar{\psi}\Gamma_a[A^a, \psi]. \quad (\text{C.10})$$

We can now rewrite these terms using the form of the Majorana-Weyl spinor (C.5) and the representation of the Γ matrices in terms of four- and six-dimensional Dirac matrices as given in section C.2.2.

We define a complex scalar field, which is related to the six real components A^a of the gauge field via

$$\phi^{AB} = \frac{1}{\sqrt{2}}\Sigma^{a, AB}A^a, \quad \bar{\phi}_{AB} = (\phi^{AB})^* = \frac{1}{2}\epsilon_{ABCD}\phi^{CD} = \frac{1}{\sqrt{2}}\bar{\Sigma}_{AB}^a A^a, \quad (\text{C.11})$$

where $\Sigma^a = (\bar{\Sigma}^a)^*$ is defined in C.2.2 and satisfies $\Sigma^a \bar{\Sigma}^b + \Sigma^b \bar{\Sigma}^a = -2\delta^{ab}$. The above defined scalar field is antisymmetric in $A \leftrightarrow B$. Thus we have²

$$\begin{aligned} [\phi^{AB}, \phi^{CD}][\bar{\phi}_{AB}, \bar{\phi}_{CD}] &= \frac{1}{4}[A^a, A^b][A^c, A^d]\Sigma^{a,AB}\Sigma^{b,CD}\bar{\Sigma}_{AB}^c\bar{\Sigma}_{CD}^d \\ &= \frac{1}{16}[A^a, A^b][A^c, A^d]\text{Tr}(\Sigma^a\bar{\Sigma}^c + \Sigma^c\bar{\Sigma}^a)\text{Tr}(\Sigma^b\bar{\Sigma}^d + \Sigma^d\bar{\Sigma}^b) \\ &= \frac{1}{4}[A^a, A^b][A^c, A^d]\delta^{ac}\delta^{bd}\text{Tr}(I_4)\text{Tr}(I_4) \\ &= 4[A_a, A_b][A^a, A^b]. \end{aligned} \quad (\text{C.12})$$

Note, that due to the metric for the components $a = 4, \dots, 6$ we have $A_a = -A^a$. An analogous calculation leads to

$$\begin{aligned} D_\mu\phi^{AB}D^\mu\bar{\phi}_{AB} &= \frac{1}{2}D_\mu A_a D^\mu A_b \Sigma^{a,AB}\bar{\Sigma}_{AB}^b = -\frac{1}{4}D_\mu A_a D^\mu A_b (-2\delta^{ab})\text{Tr}(I_4) \\ &= -2D_\mu A_a D^\mu A^a. \end{aligned} \quad (\text{C.13})$$

In order to simplify the fermionic terms we note that $\Gamma^a = (\gamma^5 \otimes \hat{\gamma}^a)$ and $\Gamma^0 = (\gamma^0 \otimes I_8)$ and thus

$$\Gamma^0\Gamma^a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \Sigma^{a,AB} \\ \bar{\Sigma}_{AB}^a & 0 \end{pmatrix}. \quad (\text{C.14})$$

Then

$$\begin{aligned} \bar{\psi}\Gamma^a[A_a, \psi] &= -\psi^\dagger\Gamma^0\Gamma^a[A^a, \psi] \\ &= -\begin{pmatrix} (\lambda_\alpha^A)^* \\ (\bar{\lambda}_A^{\dot{\alpha}})^* \end{pmatrix}^T \begin{pmatrix} 0 & \Sigma^{a,AB} \\ \bar{\Sigma}_{AB}^a & 0 \end{pmatrix} \begin{pmatrix} [A^a, \lambda_\alpha^B] \\ -[A^a, \bar{\lambda}_B^{\dot{\alpha}}] \end{pmatrix} \\ &= +\sqrt{2}\bar{\lambda}_{\dot{\alpha},A}[\phi^{AB}, \bar{\lambda}_B^{\dot{\alpha}}] - \sqrt{2}\lambda^{\alpha,A}[\bar{\phi}_{AB}, \lambda_\alpha^B]. \end{aligned} \quad (\text{C.15})$$

The last term is

$$\begin{aligned} i\bar{\psi}\Gamma^\mu D_\mu\psi &= i(\bar{\lambda}_A^{\dot{\alpha}}, \lambda^{A,\alpha}) \begin{pmatrix} (\sigma^\mu)^{\dot{\alpha}\beta} & 0 \\ 0 & (\bar{\sigma}^\mu)_{\alpha\dot{\beta}} \end{pmatrix} \begin{pmatrix} D_\mu\lambda_\beta^A \\ D_\mu\bar{\lambda}_A^{\dot{\beta}} \end{pmatrix} \\ &= i\bar{\lambda}^{\dot{\alpha},A}(\sigma^\mu)^{\dot{\alpha}\beta}D_\mu\lambda_\beta^A + i\lambda^{A,\alpha}(\bar{\sigma}^\mu)_{\alpha\dot{\beta}}D_\mu\bar{\lambda}_A^{\dot{\beta}} \\ &= 2i\bar{\lambda}_{\dot{\alpha},A}(\sigma^\mu)^{\dot{\alpha}\beta}D_\mu\lambda_\beta^A, \end{aligned} \quad (\text{C.16})$$

where we have used the identity $(\bar{\sigma}^\mu)_{\alpha\dot{\beta}} = \epsilon_{\dot{\beta}\gamma}(\sigma^\mu)^{\dot{\gamma}\delta}\epsilon_{\delta\alpha}$ and the rules for raising and lowering Weyl spinor indices in the last line.

²Note that a, b, c, d are the euclidean space-time indices, not the $SU(N)$ group indices. Furthermore, we use cyclic invariance which is due to the trace over the group indices, which is not shown explicitly here.

Putting everything together we find the maximally supersymmetric gauge theory $\mathcal{N} = 4$ SYM

$$S = \int d^d x \operatorname{Tr} \left(-\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \phi^{AB} D^\mu \bar{\phi}_{AB} + \frac{1}{8} g^2 [\phi^{AB}, \phi^{CD}] [\bar{\phi}_{AB}, \bar{\phi}_{CD}] \right. \\ \left. + 2i \bar{\lambda}_{\dot{\alpha}A} (\sigma_\mu)^{\dot{\alpha}\beta} D^\mu \lambda_\beta^A - \sqrt{2} g \lambda^{\alpha A} [\bar{\phi}_{AB}, \lambda_\alpha^B] + \sqrt{2} g \bar{\lambda}_{\dot{\alpha}A} [\phi^{AB}, \bar{\lambda}_{\dot{\alpha}B}^{\dot{\alpha}}] \right), \quad (\text{C.17})$$

where $d = 4$ and the remaining volume integrals $V = \int dx_4 \dots dx_9$ have been absorbed into a redefinition of all fields $\Phi \rightarrow V^{1/2} \Phi$ and the coupling constant $g \rightarrow V^{-1/2} g$. Then, the fields have classical dimension $[A_\mu] = [\phi^{AB}] = d/2 - 1$ and $[\lambda] = (d - 1)/2$ and the coupling constant is dimensionless $[g] = 0$.

The Lagrangian is invariant under the supersymmetry transformations that follow from the reduction of the supersymmetry transformations (C.7) in $D = 10$ to $d = 4$

$$\begin{aligned} \delta A^\mu &= -i \xi^\alpha{}^A \bar{\sigma}^\mu_{\alpha\dot{\beta}} \bar{\lambda}_A^{\dot{\beta}} - i \bar{\xi}_{\dot{\alpha}A} \sigma^\mu{}^{\dot{\alpha}\beta} \lambda_\beta^A, \\ \delta \phi^{AB} &= -i \sqrt{2} \left\{ \xi^\alpha{}^A \lambda_\alpha^B - \xi^\alpha{}^B \lambda_\alpha^A - \varepsilon^{ABCD} \bar{\xi}_{\dot{\alpha}C} \bar{\lambda}_{\dot{\alpha}D}^{\dot{\alpha}} \right\}, \\ \delta \lambda_\alpha^A &= \frac{i}{2} F_{\mu\nu} \sigma^{\mu\nu}{}_{\alpha\beta} \xi_\beta^A - \sqrt{2} \left(D_\mu \phi^{AB} \right) \bar{\sigma}^\mu_{\alpha\dot{\beta}} \bar{\xi}_B^{\dot{\beta}} + i g [\phi^{AB}, \bar{\phi}_{BC}] \xi_\alpha^C, \\ \delta \bar{\lambda}_{\dot{\alpha}A} &= \frac{i}{2} F_{\mu\nu} \bar{\sigma}^{\mu\nu\dot{\alpha}\beta} \bar{\xi}_{\dot{\beta}A} + \sqrt{2} \left(D_\mu \bar{\phi}_{AB} \right) \sigma^\mu{}^{\dot{\alpha}\beta} \xi_\beta^B + i g [\bar{\phi}_{AB}, \phi^{BC}] \bar{\xi}_{\dot{\alpha}C}^{\dot{\alpha}}. \end{aligned} \quad (\text{C.18})$$

The theory is said to be the maximally supersymmetric *gauge theory*, since any theory with $\mathcal{N} > 4$ would necessarily contain fields of spin $j > 1$.

C.2.1 Action in $SU(N)$ Components

Using $[T^a, T^b] = i f^{abc} T^c$ and taking the generators to be normalised as $\operatorname{Tr} T^a T^b = \frac{1}{2} \delta^{ab}$ we can take the trace and rewrite the action as

$$S = \int d^d x \left(-\frac{1}{4} F_{MN}^a F^{MN,a} + \frac{i}{2} \bar{\psi}^a \Gamma^M D_M \psi^a \right) \quad (\text{C.19})$$

with

$$\begin{aligned} F_{MN}^a &= \partial_M A_N^a - \partial_N A_M^a + g f^{abc} A_M^b A_N^c, \\ D_M \psi^a &= \partial_M \psi^a + g f^{abc} A_M^b \psi^c. \end{aligned} \quad (\text{C.20})$$

Taking the trace and rescaling the fields $\Phi \rightarrow \frac{1}{g} \Phi$ we get from (C.17)

$$\begin{aligned} S &= \frac{1}{g^2} \int d^d x \left(-\frac{1}{4} F_{\mu\nu}^a F^{a,\mu\nu} + \frac{1}{4} D_\mu \phi^{a,AB} D^\mu \bar{\phi}_{AB}^a \right. \\ &\quad - \frac{1}{16} f^{abc} f^{ade} \phi^{b,AB} \phi^{c,CD} \bar{\phi}_{AB}^d \bar{\phi}_{CD}^e + i \bar{\lambda}_{\dot{\alpha}A}^a (\sigma_\mu)^{\dot{\alpha}\beta} D^\mu \lambda_\beta^{a,A} \\ &\quad \left. - \frac{i}{\sqrt{2}} f^{abc} \lambda^{a,\alpha A} \bar{\phi}_{AB}^b \lambda_\alpha^{c,B} + \frac{i}{\sqrt{2}} f^{abc} \bar{\lambda}_{A,\dot{\alpha}}^a \phi^{b,AB} \bar{\lambda}_B^{c,\dot{\alpha}} \right). \end{aligned} \quad (\text{C.21})$$

Taking into account that $\phi^{12} = -\phi^{21}$ and $\bar{\phi}_{34} = \phi^{12}$, the kinetic term for ϕ^{12} reads $\partial_\mu \phi^{12} \partial^\mu \bar{\phi}_{12}$ such that we have the standard propagator i/p^2 for a complex scalar field. Summation over all repeated indices is implied and we do not display ghost-fields and gauge terms. The field strength and covariant derivative read

$$\begin{aligned} F_{\mu\nu}^a &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c, \\ D_\mu \psi^a &= \partial_\mu \psi^a + f^{abc} A_\mu^b \psi^c. \end{aligned} \quad (\text{C.22})$$

C.2.2 Dirac Matrices in $D = 10$

We construct the ten-dimensional 32×32 Dirac matrices in terms of a representation of four- and six-dimensional Clifford algebra which are explicitly given in the following sections. From these lower dimensional representations we build up the ten-dimensional Dirac matrices in the following way

$$\Gamma^M = (\gamma^\mu \otimes I_8, \gamma^5 \otimes \hat{\gamma}^a), \quad \mu = 0, 1, 2, 3 \quad a = 4, \dots, 9. \quad (\text{C.23})$$

They satisfy the Clifford algebra $\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}$ with metric $\eta^{MN} = \text{diag}(+ - \dots -)$. The chiral and charge conjugation matrix are

$$\Gamma = \gamma^5 \otimes \hat{\gamma}^7, \quad C_{10} = C_4 \otimes C_6, \quad (\text{C.24})$$

where the four- and six-dimensional chiral and charge conjugation matrices are defined in the following.

Four-Dimensional Clifford Algebra

We take the four-dimensional matrices in the Weyl (chiral) representation

$$\gamma^\mu = \begin{pmatrix} 0 & \bar{\sigma}^{\mu, \alpha\beta} \\ \sigma_{\dot{\alpha}\beta}^\mu & 0 \end{pmatrix}, \quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \quad \eta^{\mu\nu} = \text{diag}(+ - - -), \quad (\text{C.25})$$

where $\sigma^\mu = (1, \vec{\sigma})$, $\bar{\sigma}^\mu = (1, -\vec{\sigma})$ and $\vec{\sigma}$ comprise the three standard Pauli matrices. In this representation we have

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C_4 = i\gamma^2\gamma^0 = \begin{pmatrix} -\epsilon^{\alpha\beta} & 0 \\ 0 & -\epsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix}, \quad (\text{C.26})$$

where $\epsilon^{\alpha\beta} = -\epsilon_{\dot{\alpha}\dot{\beta}}$ and $\epsilon^{12} = \epsilon_{12} = -\epsilon_{i\dot{2}} = -\epsilon^{\dot{1}2} = 1$.

Six-Dimensional Clifford Algebra

Furthermore, we take the Dirac matrices in six-dimensional Euclidean space to be

$$\hat{\gamma}^a = \begin{pmatrix} 0 & \Sigma^{a, AB} \\ \bar{\Sigma}_{AB}^a & 0 \end{pmatrix}, \quad \{\hat{\gamma}^a, \hat{\gamma}^b\} = -2\delta^{ab}, \quad \delta^{ab} = \text{diag}(+\dots+), \quad (\text{C.27})$$

where the 4×4 matrices Σ^a are

$$\Sigma^a = (\eta^1, \eta^2, \eta^3, i\bar{\eta}^1, i\bar{\eta}^2, i\bar{\eta}^3), \quad (\text{C.28})$$

$$\bar{\Sigma}^a = (\Sigma^a)^*, \quad (\text{C.29})$$

$$(\eta^i)_{AB} = \epsilon_{iAB} + \delta_{iA}\delta_{4B} - \delta_{iB}\delta_{4A}, \quad (\text{C.30})$$

$$(\bar{\eta}^i)_{AB} = \epsilon_{iAB} - \delta_{iA}\delta_{4B} + \delta_{iB}\delta_{4A}, \quad (\text{C.31})$$

where ϵ^{iAB} is defined to be zero if $A, B > 3$. Furthermore, the chiral and charge conjugation matrices are defined as

$$\hat{\gamma}^7 = i\hat{\gamma}^1\hat{\gamma}^2\hat{\gamma}^3\hat{\gamma}^4\hat{\gamma}^5\hat{\gamma}^6 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C_6 = \hat{\gamma}^1\hat{\gamma}^2\hat{\gamma}^3 = \begin{pmatrix} 0 & \delta_A^B \\ \delta_B^A & 0 \end{pmatrix}. \quad (\text{C.32})$$

C.3 Feynman Rules of $\mathcal{N} = 4$ SYM

The gauge field propagator in Feynman gauge, the complex scalar field and the fermion propagators following from (C.21) are

$$\mu, a \text{ } \sim \text{ } \nu, b = \langle A_\mu^a(p) A_\nu^b(-p) \rangle = -\frac{i\delta^{ab} g_{\mu\nu}}{p^2} g^2, \quad (\text{C.33})$$

$$a, AB \text{ } \sim \text{ } b, CD = \langle \phi^{a,AB}(p) \bar{\phi}_{b,CD}(-p) \rangle = \frac{i\delta^{ab}}{p^2} g^2 (\delta_C^A \delta_D^B - \delta_D^A \delta_C^B),$$

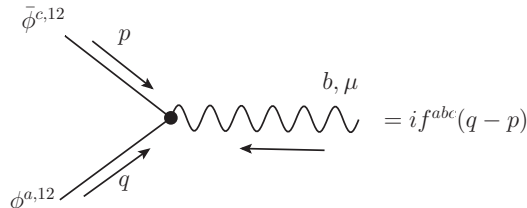
$$\alpha, a, A \text{ } \sim \text{ } \beta, b, B = \langle \lambda_\alpha^{aA}(p) \bar{\lambda}_{\beta,B}^b(-p) \rangle = \frac{i\delta^{ab} p_\mu (\bar{\sigma}^\mu)_{\alpha\beta}}{p^2} g^2 \delta_B^A.$$

These propagators can easily be Fourier transformed to position space by using (A.16), e.g. the scalar propagator is

$$\langle \phi^{a,12}(x_1) \bar{\phi}_{12}^b(x_2) \rangle = g^2 \delta^{ab} \int \frac{d^d p}{(2\pi)^d} \frac{i}{p^2} e^{-ipx_{12}} = g^2 \frac{\Gamma(h-1)}{4\pi^h} \frac{\delta^{ab}}{(-x_{12}^2)^{h-1}}, \quad (\text{C.34})$$

where $h = d/2$.

Furthermore, we list the Feynman rules for some interaction terms that will be of use in the calculations. From the scalar-scalar-gluon interaction term $\frac{1}{4} D_\mu \phi^{a,AB} D^\mu \bar{\phi}_{a,AB}$ we get



$$= i f^{abc} (q-p)^\mu \quad (\text{C.35})$$

in momentum space. The interaction-term of four scalars is

$$-\frac{1}{16g^2} f^{abc} f^{ade} \phi^{b,AB} \phi^{c,CD} \bar{\phi}_{d,AB} \bar{\phi}_{e,CD}. \quad (\text{C.36})$$

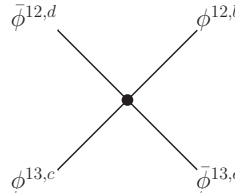
Taking into account that $\bar{\phi}_{12} = \phi^{34}$ we find the interaction term with four scalars of the same flavour

$$-\frac{1}{2g^2} f^{abc} f^{ade} \phi^{b,12} \phi^{e,12} \bar{\phi}_{c,12} \bar{\phi}_{d,12}. \quad (\text{C.37})$$

However, taking two different flavours, e.g. $\phi^{12} = \bar{\phi}_{34}$, $\phi^{13} = -\bar{\phi}_{24}$ we get

$$-\frac{1}{g^2} (f^{abc} f^{ade} + f^{adc} f^{abe}) \phi^{b,12} \phi^{c,13} \bar{\phi}_{d,12} \bar{\phi}_{e,13}. \quad (\text{C.38})$$

There are two terms since the exchange of $\{A, B\} \leftrightarrow \{C, D\}$ does not yield identical terms if we choose different flavours.



$$= -\frac{1}{g^2} (f^{abc} f^{ade} + f^{adc} f^{abe}). \quad (\text{C.39})$$

C.4 One-Loop Corrections to the Scalar Propagator

Here, we review the calculation of one-loop corrections to the scalar propagator

$$\langle \phi^{a,12}(p) \bar{\phi}_{12}^b(-p) \rangle = \frac{i\delta^{ab}}{p^2}. \quad (\text{C.40})$$

For the correction from the gluon loop we get a symmetry factor of two, since there are two vertices

$$\begin{aligned} \langle \phi^{a,12} \bar{\phi}_{12}^b \rangle_{\text{gluons}}^{(1)} &= 2 \frac{(i)^2 (i)^4}{2! p^4} f^{cda} f^{bcd} (i)^2 \int \frac{d^d l}{(2\pi)^d} \frac{(l-2p)_\mu (-\eta^{\mu\nu}) (l-2p)_\nu}{l^2 (l-p)^2} \\ &= f^{acd} f^{bcd} \frac{1}{p^4} \int \frac{d^d l}{(2\pi)^d} \frac{(l-2p)^2}{l^2 (l-p)^2} \\ &= 2N \delta^{ab} \frac{1}{p^2} \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2 (l-p)^2} \\ &= -\frac{2N \delta^{ab}}{(-p^2)^{3-\frac{d}{2}}} b_0(1, 1) \end{aligned} \quad (\text{C.41})$$

and where we used $f^{acd} f^{bcd} = N \delta^{ab}$. Furthermore, we have to take into account the corrections from the fermion loop. The relevant part of the interaction Lagrangian is

$$\mathcal{L}_{int} = \frac{1}{g^2} \frac{i}{\sqrt{2}} f^{abc} \left(\bar{\lambda}_{\dot{\alpha}}^{a,A} \phi^{b,AB} \bar{\lambda}^{c,\dot{\alpha}B} - \lambda^{a,\alpha A} \bar{\phi}^{b,AB} \lambda_{\alpha}^{c,B} \right). \quad (\text{C.42})$$

There are four terms in each vertex, two with ϕ^{12} and two with $\bar{\phi}_{12}$. We get a symmetry factor of 2 for the vertices and always need both terms in (C.42). Another factor of 2 is due to $\bar{\phi}_{34} = \phi^{12}$

$$\begin{aligned}
\langle \phi^{a,12} \bar{\phi}_{12}^b \rangle_{\text{fermions}}^{(1)} &= -4 \frac{i^2}{2!} \left(\frac{i}{\sqrt{2}} \right)^2 f^{def} f^{ghi} \delta^{ae} \delta^{hb} \frac{(i)^2}{p^4} \int \frac{d^d l}{(2\pi)^d} \\
&\quad \epsilon^{\alpha\beta} \epsilon^{\dot{\beta}\dot{\alpha}} \left(\langle \lambda_\beta^{d,1} \bar{\lambda}_{\dot{\alpha}}^{g,1} \rangle \langle \lambda_\alpha^{f,2} \bar{\lambda}_{\dot{\beta}}^{i,2} \rangle + \langle \lambda_\beta^{d,1} \bar{\lambda}_{\dot{\beta}}^{i,1} \rangle \langle \lambda_\alpha^{f,2} \bar{\lambda}_{\dot{\alpha}}^{g,2} \rangle \right. \\
&\quad \left. + \langle \lambda_\beta^{d,2} \bar{\lambda}_{\dot{\beta}}^{i,2} \rangle \langle \lambda_\alpha^{f,1} \bar{\lambda}_{\dot{\alpha}}^{g,1} \rangle + \langle \lambda_\beta^{d,2} \bar{\lambda}_{\dot{\alpha}}^{g,2} \rangle \langle \lambda_\alpha^{f,1} \bar{\lambda}_{\dot{\beta}}^{i,1} \rangle \right) \\
&= -2g^4 f^{daf} f^{gbi} \frac{1}{p^4} \int \frac{d^d l}{(2\pi)^d} \frac{p_\mu (p-l)_\nu}{l^2 (p-l)^2} \\
&\quad \epsilon^{\alpha\beta} \epsilon^{\dot{\beta}\dot{\alpha}} \left(\delta^{dg} \delta^{fi} \bar{\sigma}_{\beta\dot{\alpha}}^\mu \bar{\sigma}_{\alpha\dot{\beta}}^\nu + \delta^{di} \delta^{fg} \bar{\sigma}_{\beta\dot{\beta}}^\mu \bar{\sigma}_{\alpha\dot{\alpha}}^\nu \right) \\
&= -2g^4 f^{afd} f^{bfd} 2 \text{Tr}(\sigma^\mu \bar{\sigma}^\nu) \frac{1}{p^4} \int \frac{d^d l}{(2\pi)^d} \frac{p_\mu (p-l)_\nu}{l^2 (p-l)^2} \\
&= -4g^4 N \delta^{ab} 2 \frac{1}{p^4} \int \frac{d^d l}{(2\pi)^d} \frac{p \cdot (p-l)}{l^2 (p-l)^2} = 4g^4 N \frac{\delta^{ab}}{(-p^2)^{3-\frac{d}{2}}} b_0(1,1).
\end{aligned} \tag{C.43}$$

Thus, the one-loop correction to the scalar propagator reads

$$\langle \phi^{a,12}(p) \bar{\phi}_{12}^b(-p) \rangle^{(1)} = \frac{2N \delta^{ab}}{(-p^2)^{3-\frac{d}{2}}} b_0(1,1). \tag{C.44}$$

Fourier transformation yields

$$\langle \phi^{a,12}(x_1) \bar{\phi}_{12}^b(x_2) \rangle^{(1)} = -g^4 \frac{N \delta^{ab}}{(-x_{12}^2)^{2h-3}} \frac{1}{2} \frac{1}{(4\pi^h)^2} \frac{\Gamma(2h-3)\Gamma(h-1)^2\Gamma(2-h)}{\Gamma(2h-2)\Gamma(3-h)}, \tag{C.45}$$

which has also been calculated in [319] with a different metric.

Appendix D

Introduction to the Mellin-Barnes Technique

In the calculation of Feynman diagrams one often has to deal with integrals of the type

$$\int_0^1 ds_1 \dots ds_n \frac{1}{(X_1(s_1, \dots, s_n) + \dots + X_m(s_1, \dots, s_n))^\lambda}, \quad (\text{D.1})$$

where X_m are products of the integrations variables s_i and $\bar{s}_i = 1 - s_i$. There is a possibility to simplify the integrations over the parameters s_i for the price of introducing integrations in the complex plane. Depending on how many terms the sum in the denominator has, this technique may be very useful.

In the following we shall prove the integral representation

$$\frac{1}{(X + Y)^\lambda} = \frac{1}{2\pi i} \frac{1}{\Gamma(\lambda)} \int_{\beta - i\infty}^{\beta + i\infty} \left(\frac{Y^z}{X^{\lambda+z}} \Gamma(\lambda + z) \Gamma(-z) \right) dz, \quad (\text{D.2})$$

where $-\text{Re}(\lambda) < \beta < 0$. Note that the poles of $\Gamma(-z)$ are located at $z_0 = 0, 1, 2, \dots, n$ and $\Gamma(\lambda + z)$ has poles at $z_0 = -(\lambda), -(\lambda + 1), \dots, -(\lambda + n)$ where $n \in N$.

- We first consider the case $Y < X$:

By expanding the left-hand side of (D.2) as a Taylor series, one can rewrite

$$\begin{aligned} (X + Y)^{-\lambda} &= X^{-\lambda} \left(1 + \frac{Y}{X} \right)^{-\lambda} \\ &= X^{-\lambda} \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)} \lambda(\lambda+1)\dots(\lambda+n-1) \left(\frac{Y}{X} \right)^n \\ &= X^{-\lambda} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\lambda+n)}{\Gamma(n+1)\Gamma(\lambda)} \left(\frac{Y}{X} \right)^n. \end{aligned} \quad (\text{D.3})$$

One can evaluate the right-hand side of (D.2) by using of the residue theorem. Since $Y < X$ we can close the contour to the right at $z \rightarrow +\infty$ without adding any contributions to the integral. Note that we get an extra minus sign by reversing

the orientation of the integration to the mathematically positive direction. Since $\beta < 0$ we enclose the simple poles of $\Gamma(-z)$ at $z_0 = 0, 1, 2, \dots, n$.

By repeated use of

$$\Gamma(-z) = \frac{1}{-z} \Gamma(-z + 1),$$

it is easy to see that the residue of $\Gamma(-z)$ at $z_0 = n \in N$ is

$$\text{Res}(z_0 = n, \Gamma(-z)) = -\frac{(-1)^n}{n!} = -\frac{(-1)^n}{\Gamma(n+1)}. \quad (\text{D.4})$$

Then we have

$$\begin{aligned} & \oint_{\text{Re}(z) \geq \beta} dz \underbrace{\frac{1}{2\pi i} \frac{1}{\Gamma(\lambda)} \left(\frac{Y^z}{X^{\lambda+z}} \Gamma(\lambda+z) \Gamma(-z) \right)}_{f(z)} \quad (\text{D.5}) \\ &= -2\pi i \sum_{n=0}^{\infty} \text{Res}(z_0 = n, f(z)) \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)\Gamma(n+1)} \frac{Y^n}{X^{\lambda+n}}. \end{aligned}$$

Note, that the extra minus sign in the second line is due to reversing the orientation of the integration contour to the mathematically *positive* sense. Comparing this to (D.3) we find that (D.2) is true for $Y < X$.

- Now we consider the case $Y > X$:

In this case we can close the contour to the left and the orientation of the integration path has mathematically positive sense automatically. Then we have poles at $z_0 = -(\lambda + n)$ for $n \in N$, which yield the residue

$$\text{Res}(z_0 = -(\lambda + n), \Gamma(\lambda + z)) = \text{Res}(z_0 = -n, \Gamma(z)) = \frac{(-1)^n}{n!}. \quad (\text{D.6})$$

Taylor expansion in X/Y therefore yields same result as the integral.

Remarks

Sometimes it is necessary to pick the real part of the integration contour differently, therefore we pick up different poles. Taking $0 < \beta < 1$ for example yields

$$\frac{1}{(X+Y)^\lambda} - \frac{1}{X^\lambda} = \frac{1}{2\pi i} \frac{1}{\Gamma(\lambda)} \int_{\beta-i\infty}^{\beta+i\infty} \left(\frac{Y^z}{X^{\lambda+z}} \Gamma(\lambda+z) \Gamma(-z) \right) dz. \quad (\text{D.7})$$

Appendix E

The Wilson Loop Operator

The Wilson loop operator defined by

$$W(\mathcal{C}) = \frac{1}{N} \text{Tr} \mathcal{P} \exp \left(i \oint_{\mathcal{C}} A_{\mu} dz^{\mu} \right) \quad (\text{E.1})$$

is invariant under $SU(N)$ gauge transformations

$$A_{\mu} \rightarrow A'_{\mu} = g(x) (A_{\mu} + i\partial_{\mu}) g^{-1}(x), \quad g(x) \in SU(N), \quad (\text{E.2})$$

if the path ordering¹ is defined as

$$\begin{aligned} \mathcal{P} (A_{\mu}(z(t))A_{\nu}(z(t'))) &= A_{\mu}(z(t))A_{\nu}(z(t')) & \text{for } t > t', \\ \mathcal{P} (A_{\mu}(z(t))A_{\nu}(z(t'))) &= A_{\nu}(z(t'))A_{\mu}(z(t)) & \text{for } t' > t \end{aligned} \quad (\text{E.3})$$

and $s \in (0, 1)$ parametrises the path along the curve \mathcal{C} . The path ordered exponential in the Wilson loop operator then has the expansion

$$\begin{aligned} \mathcal{P} \exp \left(i \oint_{\mathcal{C}} A_{\mu} dz^{\mu} \right) &= \mathbf{1} + i \oint_{\mathcal{C}} dz_i^{\mu} A_{\mu} + (i)^2 \oint_{\mathcal{C}} dz_i^{\mu} \int^{z_i} dz_j^{\nu} A_{\mu} A_{\nu} \\ &+ (i)^3 \oint_{\mathcal{C}} dz_i^{\mu} \int^{z_i} dz_j^{\nu} \int^{z_j} dz_k^{\rho} A_{\mu} A_{\nu} A_{\rho} \\ &+ (i)^4 \oint_{\mathcal{C}} dz_1^{\mu} \int^{z_1} dz_j^{\nu} \int^{z_j} dz_k^{\rho} \int^{z_k} dz_l^{\sigma} A_{\mu} A_{\nu} A_{\rho} A_{\sigma} + \dots \end{aligned} \quad (\text{E.4})$$

The sign in the exponent of the Wilson line

$$W(x, y) = \mathcal{P} \exp \left(i \int_y^x A_{\mu} dz^{\mu} \right) \quad (\text{E.5})$$

is fixed by requiring the transformation property $W(x, y) \rightarrow g(x)W(x, y)g^{-1}(y)$ under local gauge transformations (E.2) of A_{μ} .

¹There are different conventions on the path ordering. They are equivalent, it is however important to choose the sign in the Wilson loop such that it is gauge invariant. Reversing the sign in the exponent of the Wilson loop does not just reverse the integration contour, due to the path ordering.

Appendix F

$\mathcal{N} = 6$ Super Chern-Simons (ABJM) theory

F.1 Cherns-Simons Theory

The Chern-Simons action

$$S_{\text{CS}} = \frac{k}{4\pi} \int d^d x \epsilon^{\mu\nu\rho} \text{Tr} \left(A_\mu \partial_\nu A_\rho - \frac{2}{3} i A_\mu A_\nu A_\rho \right) \quad (\text{F.1})$$

is invariant under $SU(N)$ gauge transformations

$$A_\mu \rightarrow A'_\mu = g(x) (A_\mu + i\partial_\mu) g^{-1}(x), \quad g(x) \in SU(N). \quad (\text{F.2})$$

More precisely, the action is invariant under infinitesimal transformations $g(x) = 1 + i\alpha(x)$ and transforms like

$$S \rightarrow S' + (2\pi k)\delta S, \quad (\text{F.3})$$

where

$$\delta S = -\frac{1}{24\pi^2} \int d^d x \epsilon^{\mu\nu\rho} \text{Tr} \left((\partial_\mu g^{-1}) g (\partial_\nu g^{-1}) g (\partial_\rho g^{-1}) g \right) \quad (\text{F.4})$$

under finite transformations. Since δS takes integer values, $\exp(iS)$ is invariant under large gauge transformations for $k \in \mathbb{N}$.

Quantising the theory with the standard Fadeev-Popov procedure yields the gauge fixing and ghost action

$$\mathcal{S}_{\text{g.f.}} = \frac{k}{4\pi} \int d^d x \text{Tr} \left(\frac{1}{\xi} (\partial^\mu A_\mu)^2 + \bar{c} (\partial^\mu D_\mu) c \right), \quad (\text{F.5})$$

where $D_\mu c = \partial_\mu c + i[A_\mu, c]$. In Landau gauge ($\xi = 0$) the gauge field propagator reads

$$\langle (A_\mu)_{ij}(x) (A_\nu)_{kl}(y) \rangle = \delta_{il} \delta_{jk} \frac{1}{k} \left(\frac{\Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d-2}{2}}} \right) \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{(-(x-y)^2)^{\frac{d}{2}}}, \quad (\text{F.6})$$

where we have rescaled the coupling constant $k \rightarrow \mu^{-2\epsilon} k$ and restore the dependence on the regularisation scale μ only in the final results. The ghost propagator is

$$\langle c(x)\bar{c}(y) \rangle = \delta_{il}\delta_{jk} \frac{1}{k} \left(\frac{\Gamma\left(\frac{d}{2} - 1\right)}{\pi^{\frac{d-2}{2}}} \right) \frac{1}{(-(x-y)^2)^{\frac{d}{2}-1}}. \quad (\text{F.7})$$

Note that the gauge field propagator is related to the ghost propagator by

$$\langle (A_\mu)_{ij}(x) (A_\nu)_{kl}(y) \rangle = \frac{1}{2} \epsilon_{\mu\nu\rho} \partial^\rho \langle c(x)\bar{c}(y) \rangle, \quad (\text{F.8})$$

which can be used to see the cancellation of gauge field and ghost loop contributions to the one-loop gauge field propagator in a simple way.

F.2 Other Conventions

The different conventions found in the literature on Chern-Simons theory deserve a short comment. One can show, that

$$\mathcal{S} = \frac{k}{4\pi} \int d^d x \text{Tr} \left(A_\mu \partial_\nu A_\rho - \frac{2}{3} s A_\mu A_\nu A_\rho \right) \epsilon^{\mu\nu\rho}, \quad (\text{F.9})$$

$$W(\mathcal{C}) = \frac{1}{N} \text{Tr} \mathcal{P} \exp \left(s \oint_{\mathcal{C}} A_\mu dz^\mu \right) \quad (\text{F.10})$$

are invariant (in the sense mentioned above) under gauge transformations

$$A_\mu \rightarrow A'_\mu = g(x) \left(A_\mu - \frac{1}{s} \partial_\mu \right) g^{-1}(x), \quad (\text{F.11})$$

where s is a real or imaginary parameter. The sign in the Wilson loop and the Lagrangian are thus correlated through gauge invariance, see ???. Taking a hermitian gauge field $(A_\mu)^\dagger = A_\mu$ we can choose $s = i$, which corresponds to the choice used throughout this document.

All other conventions found in the literature can be obtained by rescaling the gauge field $A_\mu \rightarrow -A_\mu$, $A_\mu \rightarrow iA_\mu$ etc. Note however, that this changes factors in the Lagrangian, the gauge transformation, the covariant derivative and the Wilson loop. A sign difference in the Wilson loop only may also be due to a different definition of the path ordering (E.3).

F.3 Lagrangian of ABJM Theory

The action of ABJM theory is

$$\mathcal{S}_{\text{ABJM}} = \mathcal{S}_{\text{CS}} + \mathcal{S}_{\text{g.f.}} + \hat{\mathcal{S}}_{\text{CS}} + \hat{\mathcal{S}}_{\text{g.f.}} + \mathcal{S}_{\text{matter}}, \quad (\text{F.12})$$

where $\hat{\mathcal{S}}$ is obtained from \mathcal{S} by replacing A_μ with the gauge field in the anti-fundamental representation \hat{A}_μ and letting $k \rightarrow -k$. Explicitly, we have

$$\begin{aligned} S_{\text{CS}} + \hat{S}_{\text{CS}} &= \frac{k}{4\pi} \int d^3x \varepsilon^{\mu\nu\rho} \left[\text{Tr} (A_\mu \partial_\nu A_\rho - \frac{2}{3} i A_\mu A_\nu A_\rho) - \text{Tr} (\hat{A}_\mu \partial_\nu \hat{A}_\rho - \frac{2}{3} i \hat{A}_\mu \hat{A}_\nu \hat{A}_\rho) \right], \\ S_{\text{gf}} + \hat{S}_{\text{gf}} &= \frac{k}{4\pi} \int d^3x \left[\frac{1}{\xi} \text{Tr} (\partial_\mu A^\mu)^2 - \text{Tr} (\partial_\mu \bar{c} D_\mu c) - \frac{1}{\xi} \text{Tr} (\partial_\mu \hat{A}^\mu)^2 + \text{Tr} (\partial_\mu \bar{\hat{c}} D_\mu \hat{c}) \right], \\ S_{\text{matter}} &= \int d^3x \left[\text{Tr} (D_\mu C_I D^\mu \bar{C}^I) + i \text{Tr} (\bar{\psi}^I \not{D} \psi_I) \right] + S_{\text{int}}. \end{aligned} \quad (\text{F.13})$$

The field content consists of two $U(N)$ gauge fields $(A_\mu)_{ij}$ and $(\hat{A}_\mu)_{\hat{i}\hat{j}}$, the complex fields $(C_I)_{i\hat{i}}$ and $(\bar{C}^I)_{\hat{i}i}$ as well as the fermions $(\psi_I)_{i\hat{i}}$ and $(\bar{\psi}^I)_{\hat{i}i}$ in the $(\mathbf{N}, \bar{\mathbf{N}})$ and $(\bar{\mathbf{N}}, \mathbf{N})$ of $U(N)$ respectively, $I = 1, 2, 3, 4$ is the $SU(4)_R$ index. We employ the covariant gauge fixing function $\partial_\mu A^\mu$ for both gauge fields and have two sets of ghosts (\bar{c}, c) and $(\bar{\hat{c}}, \hat{c})$. S_{int} are the sextic scalar potential and $\Psi^2 C^2$ Yukawa type potentials spelled out in [2] or in components in [261]. The covariant derivative is given by $D_\mu c = \partial_\mu c + i[A_\mu, \cdot]$. It's action on Ψ^I, C^I is given in [283] and not needed here. The Lagrangian has been recast into a manifestly $SU(4)$ R-Symmetry invariant form and can be found in [103, 20].

F.4 One-loop Gauge Field Propagator in ABJM Theory

Here we calculate the one-loop correction to the gauge field propagator in d dimensions, see also [283]. We have fermionic and bosonic contributions in the loop and thus

$$G_{\mu\nu}^{(1)}(p) = G_{\mu\nu}^{(F,1)}(p) + G_{\mu\nu}^{(B,1)}(p), \quad (\text{F.14})$$

where

$$G_{\mu\nu}^{(1)}(p) = \left(\frac{2\pi}{k} \right)^2 \frac{\epsilon_{\mu\rho\kappa} p^\kappa}{p^2} \left(\Pi_{\rho\lambda}^{(B)}(p) + \Pi_{\rho\lambda}^{(F)}(p) \right) \frac{\epsilon_{\lambda\nu\delta} p^\delta}{p^2} \quad (\text{F.15})$$

and

$$\begin{aligned} \Pi_{\mu\nu}^{(B)}(p) &= +N \delta_I^J \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{(2k+p)_\mu (2k+p)_\nu}{k^2 (p+k)^2}, \\ \Pi_{\mu\nu}^{(F)}(p) &= -N \delta_I^J \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{\text{Tr} (\gamma_\mu (\not{p} + \not{k}) \gamma_\nu \not{k})}{k^2 (p+k)^2}. \end{aligned} \quad (\text{F.16})$$

We use the DRED scheme for Dirac matrix operations as well as for Levi-Civita tensor contractions, i.e. we work in strictly $d = 3$ to obtain scalar integrands and only then continue the loop momenta to d -dimensional space to perform the integrals in d dimensions. This scheme has been shown to respect the Slavnov-Taylor identities up to two loop order in [288]. Then we have

$$\text{Tr} (\gamma_\mu (\not{p} + \not{k}) \gamma_\nu \not{k}) = 2 (-\eta_{\mu\nu} (p+k) \cdot k + 2k_\mu k_\nu + p_\mu k_\nu + p_\mu k_\nu). \quad (\text{F.17})$$

The last two terms can be dropped, since they vanish when contracted with (F.15). The same is true for terms proportional to p_μ, p_ν in the bosonic term. Summing up all remaining terms we get

$$+ N\delta_I^I \mu^{2\epsilon} 2\eta_{\mu\nu} \int \frac{d^d k}{(2\pi)^d} \frac{k \cdot (p+k)}{k^2 (p+k)^2}. \quad (\text{F.18})$$

Introducing Feynman parameters, we have

$$+ N\delta_I^I \mu^{2\epsilon} 2\eta_{\mu\nu} \int_0^1 d\alpha \int \frac{d^d k}{(2\pi)^d} \frac{k \cdot (p+k)}{[(k+\bar{\alpha}p)^2 - \Delta]^2}, \quad (\text{F.19})$$

where $\Delta = -\alpha\bar{\alpha}p^2$. Then, we shift $k = l - \bar{\alpha}p$ and drop terms linear in l_μ

$$+ N\delta_I^I \mu^{2\epsilon} 2\eta_{\mu\nu} \int_0^1 d\alpha \int \frac{d^d l}{(2\pi)^d} \frac{l^2 - \alpha\bar{\alpha}p^2}{[l^2 - \Delta]^2}. \quad (\text{F.20})$$

Using the standard integrals

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^2}{[l^2 - \Delta]^2} = -\frac{i}{(4\pi)^{\frac{d}{2}}} \frac{d\Gamma(1 - \frac{d}{2})}{2(\Delta)^{1 - \frac{d}{2}}}, \quad \int \frac{d^d l}{(2\pi)^d} \frac{1}{[l^2 - \Delta]^2} = \frac{i}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(2 - \frac{d}{2})}{(\Delta)^{2 - \frac{d}{2}}} \quad (\text{F.21})$$

and $\epsilon_{\lambda\kappa\mu}\epsilon_{\lambda\nu\delta} = \eta_{\kappa\nu}\eta_{\mu\delta} - \eta_{\kappa\delta}\eta_{\mu\nu}$ we get

$$G_{\mu\nu}^{(1)}(p) = \left(N\delta_I^I \mu^{2\epsilon} 2 \frac{(-i)}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(1 - \frac{d}{2})\Gamma(\frac{d}{2})^2}{\Gamma(d-1)} \right) \left(\frac{2\pi}{k} \right)^2 \frac{1}{(-p^2)^{3 - \frac{d}{2}}} (p_\mu p_\nu - \eta_{\mu\nu} p^2). \quad (\text{F.22})$$

Fourier transformation (A.16) of (F.22) leads to

$$\begin{aligned} G_{\mu\nu}^{(1)}(x) &= \mu^{2\epsilon} \int \frac{d^d p}{(2\pi)^d} G_{\mu\nu}^{(1)}(p) e^{-ipx} \quad (\text{F.23}) \\ &= \left(\frac{2\pi}{k} \right)^2 \frac{N\delta_I^I}{8} \frac{\Gamma(1 - \frac{d}{2})\Gamma(\frac{d}{2})^2}{\Gamma(d-1)} \frac{(\mu^{2\epsilon})^2}{\pi^d} \left(\frac{\Gamma(d-2)}{\Gamma(2 - \frac{d}{2})} \frac{\eta_{\mu\nu}}{(-x^2)^{d-2}} - \partial_\mu \partial_\nu \left(\frac{\Gamma(d-3)}{\Gamma(3 - \frac{d}{2})} \frac{1}{4} \frac{1}{(-x^2)^{d-3}} \right) \right). \end{aligned}$$

Appendix G

Details of the Wilson Loop Calculations

G.1 Vertex Diagram for the Tetragon

Using $\epsilon(p_1, p_2, z_{12}) = \epsilon(p_3, p_2, z_{32}) = 0$ the first line in (6.21) can be rewritten as

$$I_{321} = \int d^3 s_{1,2,3} \epsilon(p_1, p_2, \partial_{z_1}) \epsilon(p_3, p_2, \partial_{z_3}) \int d^d w \frac{(d-2)^{-2}}{|w|^d |w - z_{12}|^{d-2} |w - z_{32}|^{d-2}}. \quad (\text{G.1})$$

We begin by introducing Feynman parameters,

$$\int d^d w \frac{1}{|w|^d |w - z_{12}|^{d-2} |w - z_{32}|^{d-2}} = \int [d\beta]_3 \int d^d w \frac{1}{(-(w - \beta_1 z_{12} - \beta_3 z_{32})^2 + \Delta)^{(3d-4)/2}}, \quad (\text{G.2})$$

where

$$\int [d\beta]_3 = \int_0^1 d\beta_1 d\beta_2 d\beta_3 (\beta_1 \beta_2 \beta_3)^{(d-2)/2-1} \beta_2 \delta\left(\sum_i \beta_i - 1\right) \frac{\Gamma(\frac{3d}{2} - 2)}{\Gamma(\frac{d}{2}) \Gamma(\frac{d}{2} - 1)^2} \quad (\text{G.3})$$

and

$$\Delta = 2\beta_1 \beta_3 (z_{12} \cdot z_{32}) - z_{12}^2 \beta_1 \bar{\beta}_1 - z_{32}^2 \beta_3 \bar{\beta}_3. \quad (\text{G.4})$$

Shifting the integration contour $w \rightarrow l = w - \beta_1 z_{12} - \beta_3 z_{32}$ we have a standard integral

$$\int d^d l \frac{1}{[l^2 - \Delta]^n} = (-1)^n i \pi^{d/2} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{d}{2}}. \quad (\text{G.5})$$

Thus we get

$$I_{321} = \frac{c_1}{(d-2)^2} \int [d\beta]_3 d^3 s_{1,2,3} \epsilon(p_1, p_2, \partial_{z_1}) \epsilon(p_3, p_2, \partial_{z_3}) \frac{1}{\Delta^{d-2}}, \quad (\text{G.6})$$

where $c_1 = i\pi^{\frac{d}{2}}\Gamma(d-2)/\Gamma(3d/2-2)$. Evaluating the action of the derivatives and abbreviating $x_{13}^2 = s, x_{24}^2 = t$ we obtain

$$I_{321} = c_2 st \int_0^1 d^3 s_{1,2,3} d^3 \beta_{1,2,3} (\beta_1 \beta_2 \beta_3)^{(d-2)/2} \delta\left(\sum_i \beta_i - 1\right) \left(\frac{1}{\Delta^{d-1}} - 2\frac{(d-1)}{\Delta^d} \beta_1 \beta_3 \bar{s}_1 s_3 (s+t)\right), \quad (\text{G.7})$$

where $c_2 = i\pi^{d/2}\Gamma(d-1)/(8\Gamma^3(d/2))$ and both terms are separately symmetric under $s \leftrightarrow t$. Performing the change of variables

$$\beta_1 = xy, \quad \beta_2 = \bar{x}y, \quad \beta_3 = \bar{y}, \quad \sum_i \beta_i = 1 \quad x, y \in [0, 1] \quad (\text{G.8})$$

with Jacobian y we can rewrite (G.37) in a form where all integrations range from 0 to 1,

$$I_A = c_2 st \int_0^1 d^3 s dx dy \frac{(x\bar{x}\bar{y})^{\frac{d-2}{2}}}{\Delta_y^{d-1}}, \quad (\text{G.9})$$

$$I_B = -2(d-1)c_2 st(s+t) \int_0^1 d^3 s dx dy \frac{(x\bar{y})^{\frac{d}{2}} \bar{x}^{\frac{d-2}{2}} \bar{s}_1 s_3}{\Delta_y^d},$$

where

$$\Delta_y = -(sx\bar{s}_1(\bar{s}_3\bar{y} + s_2\bar{x}y) + t\bar{y}s_3(xs_1 + \bar{s}_2\bar{x})) \quad (\text{G.10})$$

and $I_{321} = I_A + I_B$. The integral I_A is divergent as $\bar{s}_1, s_3 \rightarrow 0$, see also fig. 6.4

G.1.1 Numerical Evaluation using the Mellin-Barnes Method

In this section, we switch from the Feynman parametrisation in equation (G.9) to a Mellin-Barnes representation, as the latter is very convenient to perform a systematic expansion in ϵ . An introduction to the Mellin Barnes technique can be found in section D, see also [320] for a more complete discussion.

In the first step the sum in the denominator is transformed into an integral over a product of terms. Since the denominator in (G.37) consists of a sum of four terms, we will introduce 3 Mellin parameters z_1, z_2, z_3 . By repeated use of the Mellin-Barnes representation

$$\frac{1}{(X+Y)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \frac{Y^z}{X^{\lambda+z}} \Gamma(z+\lambda)\Gamma(-z)dz, \quad (\text{G.11})$$

where $-\text{Re}(\lambda) < \beta < 0$, one obtains

$$\frac{(2\pi i)^3 \Gamma(\lambda)}{(a+b+c+d)^\lambda} = \int dz_{1,2,3} a^{z_1} b^{z_2} c^{z_3} d^{-\lambda-z_1-z_2-z_3} \Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3)\Gamma(\lambda+z_1+z_2+z_3), \quad (\text{G.12})$$

where the real parts β_i of the integration contour have to be chosen such that the arguments in all Γ functions have positive real part. Applying (G.12) to the denominator of I_A (G.9), we can rewrite I_A as

$$I_A = \frac{c_2}{\Gamma(d-1)} \int d\tilde{z}_{1,2,3} \Gamma(-z_1) \Gamma(-z_2) \Gamma(-z_3) \Gamma(d-1+z_1+z_2+z_3) (-s)^{z_1+z_2+1} (-t)^{-d-z_1-z_2+2} \int d^3 s dx dy s_1^{z_3} \bar{s}_1^{z_1+z_2} s_2^{z_2} \bar{s}_2^{-z_1-z_2-z_3-d+1} s_3^{-z_1-z_2-d+1} \bar{s}_3^{z_1} x^{z_1+z_2+z_3+d/2-1} \bar{x}^{-z_1-z_3-d/2} y^{z_2} \bar{y}^{-z_2-d/2}, \quad (\text{G.13})$$

where $d\tilde{z} = (2\pi i)^{-1} dz$. The integrals over s_1, s_2, s_3, x, y can be carried out using

$$\int_0^1 s_i^{a-1} (1-s_i)^{b-1} dt = \Gamma(a)\Gamma(b)/\Gamma(a+b), \quad (\text{G.14})$$

and we arrive at

$$I_A = \frac{c_2}{\Gamma(d-1)} \int d\tilde{z}_{1,2,3} (-s)^{z_1+z_2+1} (-t)^{-d-z_1-z_2+2} \Gamma(-z_1) \Gamma(-z_2) \Gamma(-z_3) \times \Gamma(z_1+1) \Gamma(z_2+1) \Gamma(z_3+1) \Gamma(d-1+z_1+z_2+z_3) \Gamma(-d/2-z_2+1) \Gamma(z_1+z_2+1) \times \Gamma(-d-z_1-z_2+2) \Gamma(-d/2-z_1-z_3+1) \Gamma(-d-z_1-z_2-z_3+2) \Gamma(d/2+z_1+z_2+z_3) \times [\Gamma(2-d/2) \Gamma(-d-z_2+3) \Gamma(-d-z_1-z_3+3) \Gamma(z_1+z_2+z_3+2)]^{-1}. \quad (\text{G.15})$$

Recall that we investigate the kinematical region where $s, t < 0$. One can see that this integral is divergent as $\epsilon \rightarrow 0$ by noticing that for $\epsilon = 0$ it is impossible to choose the integration contours such that all poles of $\Gamma(\dots + z_1)$ are to the left of the integration contour and all poles of $\Gamma(\dots - z_1)$ are to the right of the contour.¹ The reason is that the poles of $\Gamma(z_1+z_2+1)$ and $\Gamma(-d-z_1-z_2-z_3+2) = \Gamma(-z_1-z_2-1+2\epsilon)$ “glue together” at $z_1 = -z_2 - 1$ for $\epsilon = 0$. However, one can find allowed contours for $\epsilon \neq 0$.

By shifting the contour left to the pole at $z_1 = -z_2 - 1$ we pick up a residue. The factor of $\Gamma(-z_1-z_2-1+2\epsilon)$ evaluated at the residue, results in a divergent factor of $\Gamma(2\epsilon)$. The remaining integral over the shifted contour yields a finite contribution.

The steps of shifting contours and taking residues have been automatized in [321] and we used this package to systematically extract the pole terms. Applying this procedure to (G.15) and expanding in ϵ yields 3 integrals:

$$I_A^{(1)} = c_2 \int \frac{dz_1}{2\pi i} \left(\frac{1}{\epsilon} + 2 \log(-s) - g_1(z_1) \right) f_1(z_1) + \mathcal{O}(\epsilon), \quad (\text{G.16})$$

$$I_A^{(2)} = c_2 \int \frac{dz_1 dz_3}{(2\pi i)^2} \left(\frac{1}{\epsilon} + 2 \log(-s) - g_2(z_1, z_3) \right) f_2(z_1, z_3) + \mathcal{O}(\epsilon), \quad (\text{G.17})$$

$$I_A^{(3)} = c_2 \int \frac{dz_1 dz_2 dz_3}{(2\pi i)^3} \left(\frac{s}{t} \right)^{1+z_1+z_2} f_3(z_1, z_2, z_3). \quad (\text{G.18})$$

We do not specify the values of the real parts β_i as well as the functions f_i, g_i here, which are lengthy expressions of products of Γ functions and can be obtained automatically

¹This is necessary in order for the previous steps to be well-defined.

by expanding (G.15) with the help of [321]. Adding up the divergent part of (G.16) and (G.17) we get

$$I_{\text{div}}^{\text{vertex}} = \frac{a_1}{\epsilon} c_2, \quad (\text{G.19})$$

where by numerical integration one finds

$$a_1 = \int \frac{dz_1}{2\pi i} f_1(z_1) + \int \frac{dz_1 dz_3}{(2\pi i)^2} f_2(z_1, z_3) = 8.710344 \pm 10^{-6} \approx 4\pi \ln(2) = 8.710344361 \dots \quad (\text{G.20})$$

accurately approximated by our analytic guess. Further numerical evaluation of the finite part of $I_A^{(1)}$, $I_A^{(2)}$, $I_A^{(3)}$ yields

$$I_{A,\text{finite}} = c_2 \left(a_1 \ln(-s) + a_1 \ln(-t) + a_2 \ln^2 \left(\frac{s}{t} \right) + a_4 \right). \quad (\text{G.21})$$

where a_1 has the same value as above and

$$a_2 = -0.84 \pm 0.01 \quad a_4 = -14.375216465 \pm 10^{-9}. \quad (\text{G.22})$$

Numerical analysis for I_B suggests

$$I_B = c_2 \left(a_3 \ln^2 \left(\frac{s}{t} \right) + a_5 \right), \quad (\text{G.23})$$

where

$$a_3 = 3.97 \pm 0.01, \quad a_5 = 40.620843911 \pm 10^{-9}. \quad (\text{G.24})$$

Adding up a_2 , a_3 we obtain

$$a_2 + a_3 = 3.136 \pm 0.02 \approx \pi. \quad (\text{G.25})$$

Thus suggesting

$$\begin{aligned} I_A + I_B &= c_2 \left(\frac{a_1}{\epsilon} + a_1 (\ln(-s) + \ln(-t)) + (a_2 + a_3) \ln \left(\frac{s}{t} \right) + (a_4 + a_5) + \mathcal{O}(\epsilon) \right) \\ &\approx c_2 \pi \left(\frac{4 \ln(2)}{\epsilon} + 4 \ln(2) (\ln(-s) + \ln(-t)) + \ln^2 \left(\frac{s}{t} \right) + a_6 + \mathcal{O}(\epsilon) \right), \end{aligned}$$

where

$$a_6 = (a_4 + a_5)/\pi = 8.354242685 \pm 2 \cdot 10^{-9}. \quad (\text{G.26})$$

The constant fits the value $a_6 \approx -\frac{2}{3}\pi^2 + 16 \ln(2) + 8 \ln^2(2) = 8.35424273\dots$. The result can be rewritten in the form

$$I_A + I_B \approx c_2 \pi \left(2 \ln(2) \frac{((-s)^{2\epsilon} + (-t)^{2\epsilon})}{\epsilon} + \ln^2 \left(\frac{s}{t} \right) + a_6 + \mathcal{O}(\epsilon) \right). \quad (\text{G.27})$$

G.2 Gauge Field and Ghost Loops

It is well known [288], that the contributions of ghost and gauge field loops to the gauge field self energy cancel. We briefly review the cancellation of the gauge field and ghost loop corrections in the Wilson loop, since from this it is easy to see, how the cancellation for the insertions in the conformal Ward identities takes place.

G.2.1 Gauge Field Loop

The gauge field loop-diagram arises at second order in perturbation theory

$$\begin{aligned}
\langle W_n \rangle^{\text{gluon-loop}} & \quad (G.28) \\
&= \frac{1}{N} \langle \text{Tr} \left(- \oint dz_i^\mu dz_j^\nu A_\mu A_\nu \right) \left(-\frac{1}{2!} \right) \left(\frac{k}{4\pi} \int d^d x \epsilon^{\alpha\beta\gamma} \text{Tr} \left(\frac{2}{3} i A_\alpha A_\beta A_\gamma \right) \right)^2 \rangle \\
&= c_8 \left(\frac{2}{3} \right)^2 \oint dz_i^\mu dz_j^\nu \int d^d x d^d y \epsilon^{\alpha\beta\gamma} \epsilon^{\delta\sigma\tau} \langle \text{Tr} (A_\mu A_\nu) \text{Tr} (A_\alpha A_\beta A_\gamma)(x) \text{Tr} (A_\delta A_\sigma A_\tau)(y) \rangle,
\end{aligned}$$

where

$$c_8 = -\frac{1}{N} \frac{1}{2} \left(\frac{k}{4\pi} \right)^2. \quad (G.29)$$

Taking into account that A_μ, A_ν give 3 identical contractions with one of the vertex terms and that we can contract them either with the x - or y -dependent vertex, we get a symmetry factor of $3 \cdot 3 \cdot 2$. The remaining contractions of the gauge fields are dictated by taking into account only planar diagrams. Thus we get

$$\langle W_n \rangle^{\text{gluon-loop}} = 8c_8 \oint dz_i^\mu dz_j^\nu \int d^d x d^d y \epsilon^{\alpha\beta\gamma} \epsilon^{\delta\sigma\tau} \langle A_\mu A_\alpha \rangle \langle A_\nu A_\delta \rangle \langle A_\beta A_\tau \rangle \langle A_\gamma A_\sigma \rangle. \quad (G.30)$$

To proceed, we recall the relation (F.8) between gauge field and ghost propagator and write

$$\begin{aligned}
\epsilon^{\alpha\beta\gamma} \epsilon^{\delta\sigma\tau} \langle A_\beta A_\tau \rangle \langle A_\gamma A_\sigma \rangle &= \frac{1}{4} \epsilon^{\alpha\beta\gamma} \epsilon^{\delta\sigma\tau} \epsilon_{\beta\tau\kappa} \epsilon_{\gamma\sigma\rho} \partial_x^\rho \langle c(x) \bar{c}(y) \rangle \partial_x^\kappa \langle c(x) \bar{c}(y) \rangle \\
&= \frac{1}{2} \partial_x^\rho \langle c(x) \bar{c}(y) \rangle \partial_y^\kappa \langle c(x) \bar{c}(y) \rangle, \quad (G.31)
\end{aligned}$$

where we used $\epsilon^{\alpha\beta\gamma} \epsilon^{\delta\sigma\tau} \epsilon_{\beta\tau\kappa} \epsilon_{\gamma\sigma\rho} = -(\eta_\kappa^\alpha \eta_\rho^\delta + \eta_\rho^\alpha \eta_\kappa^\delta)$ and $\partial_x F(x-y) = -\partial_y F(x-y)$ in the last step.

G.2.2 Ghost Loop

The ghost loop diagram arises from contraction of the second order perturbation theory expansion of the gauge-field-ghost vertex term

$$\begin{aligned} \langle W_n \rangle^{\text{ghost loop}} &= \frac{1}{N} \langle \text{Tr} \left(- \oint dz_i^\mu dz_j^\nu A_\mu A_\nu \right) \left(-\frac{1}{2!} \right) \left(\frac{k}{4\pi} \int d^d x \text{Tr} (\partial^\mu \bar{c} i[A_\mu, c]) \right)^2 \rangle \\ &= 2c_8 \oint dz_i^\mu dz_j^\nu \int d^d x d^d y \langle \text{Tr} (A_\mu A_\nu) \text{Tr} (\partial_x^\rho \bar{c} A_\rho c) \text{Tr} (\partial_y^\sigma \bar{c} A_\sigma c) \rangle, \end{aligned} \quad (\text{G.32})$$

where c_8 is the same as defined above and the factor of 2 is due to the fact that the evaluation of the first line yields two identical planar diagrams that are kept and two identical non-planar diagrams that we drop. Contracting A_μ either with the x - or y -dependent vertex, we get a symmetry factor of 2. There is only one way for the remaining contractions and thus we get

$$\langle W_n \rangle^{\text{ghost loop}} = -4c_8 \oint dz_i^\mu dz_j^\nu \int d^d x d^d y \langle A_\mu A_\sigma \rangle \langle A_\nu A_\rho \rangle \partial_x^\rho \langle c(y) \bar{c}(x) \rangle \partial_y^\sigma \langle c(x) \bar{c}(y) \rangle, \quad (\text{G.33})$$

where a factor of -1 due to the anti-commuting ghost fields in the loop was taken into account. Summing up (G.30) and (G.33) we get

$$\langle W_n \rangle^{\text{gauge field loop}} + \langle W_n \rangle^{\text{ghost loop}} = 0. \quad (\text{G.34})$$

The same relation (F.8) can be used to show the vanishing for the dilatation and special conformal Ward identities.

G.3 Conformal Ward identity

G.3.1 Insertion of the Interaction Term

We can rewrite (6.37) as

$$I'_{321} = \frac{1}{(d-2)^2} \int d^3 s_{1,2,3} \epsilon(p_1, p_2, \partial_{z_1}) \epsilon(p_3, p_2, \partial_{z_2}) \int d^d x \frac{(x+z_2)^\nu}{|x|^d |x-z_{12}|^{d-2} |x-z_{32}|^{d-2}}. \quad (\text{G.35})$$

Introducing Feynman parameters, changing the integration variable to $l = x - \beta_1 z_{12} - \beta_3 z_{32}$, using the same notation as in app. G.1, integrating over l and evaluating the action of the derivatives yields

$$\begin{aligned} I'_{321} &= \frac{c_1}{(d-2)^2} \int d^3 s_{1,2,3} \int d[\beta]_3 \left((\beta_1 z_1 + \beta_2 z_2 + \beta_3 z_3)^\nu \epsilon(p_1, p_2, \partial_{z_1}) \epsilon(p_3, p_2, \partial_{z_2}) \frac{1}{\Delta^{d-2}} \right. \\ &\quad \left. + \epsilon(p_1, p_2, p_3) 2\beta_1 \beta_3 \epsilon_{\alpha\beta}{}^\nu p_2^\beta (\beta_1 p_1^\alpha \bar{s}_1 + \beta_3 p_3^\alpha s_3) \frac{(2-d)}{\Delta^{d-1}} \right), \end{aligned} \quad (\text{G.36})$$

The last term can be shown to be finite and the first term is very similar to the vertex diagram. Evaluation of the derivatives as in G.1 yields

$$I'_{321} = c_2 st \int_0^1 d^3 s_{1,2,3} d^3 \beta_{1,2,3} (\beta_1 \beta_2 \beta_3)^{(d-2)/2} \delta\left(\sum_i \beta_i - 1\right) \quad (\text{G.37})$$

$$(\beta_1 z_1 + \beta_2 z_2 + \beta_3 z_3)^\nu \left(\frac{1}{\Delta^{d-1}} - 2 \frac{(d-1)}{\Delta^d} \beta_1 \beta_3 \bar{s}_1 s_3 (s+t) \right) + \text{finite}.$$

It can be shown, e.g. using the Mellin Barnes technique as in G.1, that all divergent contributions are due to the first term. We have the following divergent contributions:

$$st \int d^3 s d[\beta]_3 \beta_1 \beta_2 \beta_3 \left(\frac{\beta_1 z_1}{\Delta^{d-1}} \right) = \frac{1}{\epsilon} a x_2^\nu + \mathcal{O}(\epsilon^0), \quad (\text{G.38})$$

$$st \int d^3 s d[\beta]_3 \beta_1 \beta_2 \beta_3 \left(\frac{\beta_2 z_2}{\Delta^{d-1}} \right) = \frac{1}{\epsilon} b (x_2^\nu + x_3^\nu) + \mathcal{O}(\epsilon^0),$$

$$st \int d^3 s d[\beta]_3 \beta_1 \beta_2 \beta_3 \left(\frac{\beta_3 z_3}{\Delta^{d-1}} \right) = \frac{1}{\epsilon} (a x_3^\nu) + \mathcal{O}(\epsilon^0).$$

Numerical evaluation yields

$$a = 1.8562 \pm 0.0001, \quad b = 2.4989 \pm 0.0001. \quad (\text{G.39})$$

To good accuracy we find

$$a + b = 4.35517 \pm 0.0002 \approx 2\pi \ln(2) = 4.35517\dots \quad (\text{G.40})$$

Summarising I'_{321} then reads

$$I'_{321} = \frac{c_2}{\epsilon} (a + b) (x_2 + x_3)^\nu + \mathcal{O}(\epsilon^0) \approx \frac{2\pi i \ln(2)}{\epsilon} (x_2 + x_3)^\nu + \mathcal{O}(\epsilon^0). \quad (\text{G.41})$$

G.3.2 Insertion of the Kinetic Term into the Vertex Diagram

For the kinetic insertion into the vertex diagram we have (6.40)

$$\langle \mathcal{L}(x) W_4 \rangle_{(c)}^{(2)} = \underbrace{\left(\frac{N}{k} \right)^2 \frac{i}{8\pi^2} \left(\frac{\Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d-2}{2}}} \right)^4}_{=: c_3} \int d^d w \oint dz_{i,j,k}^{\mu\nu\rho} \epsilon^{\delta\sigma\tau} I_{\nu\sigma} G_{\mu\tau}(z_i - w) G_{\rho\delta}(z_k - w) \quad (\text{G.42})$$

$$+ \text{cyclic}(\mu, \nu, \rho; z_i, z_j, z_k),$$

where $G_{\mu\nu}(x - y) = \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{(-(x-y)^2)^{\frac{d}{2}}}$ and

$$I_{\nu\sigma}(x - z_j, x - w) = \epsilon^{\alpha\beta\gamma} \left(G_{\alpha\nu}(x - z_j) \partial_\beta^{(x)} G_{\gamma\sigma}(x - w) + G_{\alpha\sigma}(x - w) \partial_\beta^{(x)} G_{\gamma\nu}(x - z_j) \right). \quad (\text{G.43})$$

and the two other contractions are contained in $\text{cyclic}(\mu, \nu, \rho; z_i, z_j, z_k)$. For the dilatation Ward identity the integration over x can be performed by introducing two Feynman parameters. The result simply yields a propagator

$$\int d^d x I_{\nu\sigma} = -\frac{4\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \epsilon_{\nu\sigma\varphi} \frac{(z_j - w)^\varphi}{(-(z_j - w)^2)^{\frac{d}{2}}} = -\frac{4\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} G_{\nu\sigma}(z_j - w), \quad (\text{G.44})$$

In the case of the special conformal Ward identity the integration is a little more involved. The integral over $d^d x$ can be solved by introducing Feynman parameters and after some algebra one finds

$$\begin{aligned} \int d^d x x^\lambda \langle \mathcal{L}(x) W_4 \rangle_{(c)}^{(2)} &= 2c_3 c_4 \oint dz_{i,j,k}^{\mu,\nu,\rho} \int d^d w \epsilon^{\delta\sigma\tau} \epsilon^{\alpha\beta\gamma} \epsilon_{\alpha\nu\xi} \epsilon_{\gamma\sigma\varphi} \\ &\partial^\xi \left((2\eta^{\varphi\lambda} \partial_\beta + \eta_\beta^\lambda \partial^\varphi) \left(\frac{1}{((z_j - w)^2)^{\frac{d}{2}-2}} \right) - \partial^\varphi \partial_\beta \left(\frac{(z_j + w)^\lambda}{((z_j - w)^2)^{\frac{d}{2}-2}} \right) \right) \\ &G_{\rho\delta} G_{\mu\tau} + \text{cyclic}(\mu, \nu, \rho; z_i, z_j, z_k), \end{aligned} \quad (\text{G.45})$$

where all derivatives are taken with respect to z_j and c_4 is a constant obtained through integration over $d^d x$. Inserting the propagators we can write this in a form convenient to solve the integral over $d^d w$

$$2c_3 c_4 \oint dz_{i,j,k}^{\mu,\nu,\rho} \epsilon^{\delta\sigma\tau} \epsilon^{\alpha\beta\gamma} \epsilon_{\alpha\nu\xi} \epsilon_{\gamma\sigma\varphi} \epsilon_{\rho\delta\chi} \epsilon_{\mu\tau\theta} \partial_i^\theta \partial_j^\xi \partial_k^\chi \left((2\eta^{\varphi\lambda} \partial_{j,\beta} + \eta_\beta^\lambda \partial_j^\varphi) J_j - \partial_j^\varphi \partial_{\beta,j} J_j^\lambda \right) + \text{cyclic}, \quad (\text{G.46})$$

where the integrals read

$$\begin{aligned} J_j &= \frac{1}{(2-d)^2} \int d^d w \frac{1}{((z_j - w)^2)^{\frac{d}{2}-2} ((z_i - w)^2)^{\frac{d}{2}-1} ((z_k - w)^2)^{\frac{d}{2}-1}}, \\ J_j^\lambda &= \frac{1}{(2-d)^2} \int d^d w \frac{(z_j + w)^\lambda}{((z_j - w)^2)^{\frac{d}{2}-2} ((z_i - w)^2)^{\frac{d}{2}-1} ((z_k - w)^2)^{\frac{d}{2}-1}}. \end{aligned} \quad (\text{G.47})$$

The integrations over w can be performed by introducing three Feynman parameters β_i and we get

$$J_j = c_5 \int d[\beta]_{3,j} \left(\frac{1}{\Delta} \right)^{d-4}, \quad J_j^\lambda = c_5 \int d[\beta]_{3,j} (z_j + \sum_i \beta_i z_i)^\lambda \left(\frac{1}{\Delta} \right)^{d-4}, \quad (\text{G.48})$$

where

$$\int d[\beta]_{3,j} = \int_0^1 d\beta_i d\beta_j d\beta_k \delta\left(\sum_i \beta_i - 1\right) (\beta_i \beta_j \beta_k)^{\frac{d}{2}-2} \beta_j^{-1} \quad (\text{G.49})$$

and c_5 is a constant obtained by integrating over w , the product $c_3 c_4 c_5$ is explicitly given below. The expression for Δ is the same as in (G.62) For the cyclic permutations we can use the same expression, replacing the measure with $d[\beta]_{3,i}$ respectively $d[\beta]_{3,k}$,

i.e. exchanging β_j^{-1} with β_i^{-1} respectively β_k^{-1} in (G.49).

All three contributions can then be written as

$$\begin{aligned}
2c_3c_4c_5 \oint dz_{i,j,k}^{\mu,\nu,\rho} \epsilon^{\delta\sigma\tau} \epsilon^{\alpha\beta\gamma} \epsilon_{\gamma\sigma\varphi} & \quad (G.50) \\
& \left(\epsilon_{\alpha\mu\xi} \epsilon_{\nu\delta\chi} \epsilon_{\rho\tau\theta} \partial_k^\theta \partial_i^\xi \partial_j^\chi \left((2\eta^{\varphi\lambda} \partial_{i,\beta} + \eta_\beta^\lambda \partial_i^\varphi) J_i - \partial_i^\varphi \partial_{\beta,i} J_i^\lambda \right) \right. \\
& \quad \epsilon_{\alpha\nu\xi} \epsilon_{\rho\delta\chi} \epsilon_{\mu\tau\theta} \partial_i^\theta \partial_j^\xi \partial_k^\chi \left((2\eta^{\varphi\lambda} \partial_{j,\beta} + \eta_\beta^\lambda \partial_j^\varphi) J_j - \partial_j^\varphi \partial_{\beta,j} J_j^\lambda \right) \\
& \quad \left. \epsilon_{\alpha\rho\xi} \epsilon_{\mu\delta\chi} \epsilon_{\nu\tau\theta} \partial_j^\theta \partial_k^\xi \partial_i^\chi \left((2\eta^{\varphi\lambda} \partial_{k,\beta} + \eta_\beta^\lambda \partial_k^\varphi) J_k - \partial_k^\varphi \partial_{\beta,k} J_k^\lambda \right) \right),
\end{aligned}$$

where

$$2c_3c_4c_5 = i \frac{\pi^{2-d}}{128} \left(\frac{N}{k} \right)^2 \Gamma(d-4). \quad (G.51)$$

We can evaluate the derivatives and contractions with the computer and find that the non-vanishing contributions have the structure

$$\begin{aligned}
& \int d^d x x^\lambda \langle \mathcal{L}(x) W_4 \rangle^{\text{vertex-insertion}} & (G.52) \\
& = \left(\frac{N}{k} \right)^2 \sum_{i>j>k} \int_0^1 ds_{i,j,k} \int d[\beta]_3 \left(I_{ijk,-d-1}^\lambda + I_{ijk,-d}^\lambda + I_{ijk,-d+1}^\lambda \right)
\end{aligned}$$

where $I_{ijk,p}$ are lengthy terms proportional to $1/\Delta^p$.

For the conformal Ward identity we are only interested in the divergent part of the above quantities, which can be automatically extracted with the Mellin-Barnes technique. We find that all terms vanish except for $i \neq j \neq k$. Specialising to the case $i = 3, j = 2, k = 1$ we find

$$\begin{aligned}
\int I_{312,-d-1}^\lambda & = \mathcal{O}(\epsilon^0), & (G.53) \\
\int I_{312,-d}^\lambda & = \frac{i}{\epsilon} (a_1 x_2^\lambda + a_2 x_3^\lambda) + \mathcal{O}(\epsilon^0), \\
\int I_{321,-d+1}^\lambda & = \frac{i}{\epsilon} (b_1 x_2^\lambda + b_2 x_3^\lambda) + \mathcal{O}(\epsilon^0).
\end{aligned}$$

Numerical evaluation of the integrals yields

$$\begin{aligned}
a_1 & = 0.3465735 \pm 10^{-6} \approx \frac{1}{2} \ln(2) = 0.3465735\dots, & (G.54) \\
a_2 & = 0.3465735 \pm 8 \cdot 10^{-7} \approx \frac{1}{2} \ln(2) = 0.3465735\dots, \\
b_1 & = -0.8664339 \pm 14 \cdot 10^{-7} \approx -\frac{5}{4} \ln(2) = -0.8664339\dots, \\
b_2 & = -0.8664339 \pm 10^{-6} \approx -\frac{5}{4} \ln(2) = -0.8664339\dots.
\end{aligned}$$

Adding up the results, summing over all four diagrams and taking into account the corresponding prefactors we get

$$\int d^d x x^\lambda \langle \mathcal{L}(x) W_4 \rangle_{(c)}^{(2)} \approx -i \frac{3 \ln(2)}{4 \epsilon} \left(\sum_i x_i^\lambda \right) + \mathcal{O}(\epsilon^0). \quad (\text{G.55})$$

G.4 Vertex Diagram for Higher Polygons

We start from (G.56)

$$\langle W_n \rangle_{\text{vertex}}^{(2)} = \left(\frac{N}{k} \right)^2 \frac{i}{2\pi} \left(\frac{\Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d-2}{2}}} \right)^3 \sum_{i>j>k} I_{ijk}, \quad (\text{G.56})$$

where

$$I_{ijk} = \frac{1}{(d-2)^3} \int dz_{i,j,k}^{\mu,\nu,\rho} \epsilon^{\alpha\beta\gamma} \epsilon_{\mu\alpha\sigma} \epsilon_{\nu\beta\lambda} \epsilon_{\rho\gamma\tau} \partial_i^\sigma \partial_j^\lambda \partial_k^\tau (J), \quad (\text{G.57})$$

and

$$J = \int d^d w \frac{1}{((-w^2)(-(w-z_{ij})^2)(-(w-z_{kj})^2))^{d/2-1}}. \quad (\text{G.58})$$

We begin by introducing Feynman parameters,

$$J = \int d^d w \frac{1}{(|w||w-z_{ij}||w-z_{kj}|)^{d-2}} = \int [d\beta]_3 \int d^d w \frac{1}{(-(w-\beta_1 z_{ij}-\beta_3 z_{kj})^2 + \Delta)^{3d/2-3}}, \quad (\text{G.59})$$

where $|w| = (-w^2)^{1/2}$ and

$$\int [d\beta]_3 = \int_0^1 d\beta_i d\beta_j d\beta_k (\beta_i \beta_j \beta_k)^{(d-2)/2-1} \delta\left(\sum_i \beta_i - 1\right) \frac{\Gamma\left(\frac{3d}{2} - 3\right)}{\Gamma\left(\frac{d}{2} - 1\right)^3} \quad (\text{G.60})$$

and

$$\Delta = 2\beta_i \beta_k (z_{ij} \cdot z_{kj}) - z_{ij}^2 \beta_i \bar{\beta}_i - z_{kj}^2 \beta_k \bar{\beta}_k \quad (\text{G.61})$$

or when taking into account the delta function

$$\Delta = -z_{ij}^2 \beta_i \beta_j - z_{ik}^2 \beta_i \beta_k - z_{kj}^2 \beta_k \beta_j. \quad (\text{G.62})$$

Shifting the integration contour $w \rightarrow l = w - \beta_1 z_{ij} - \beta_3 z_{kj}$ we have a standard integral

$$\int d^d l \frac{1}{[l^2 - \Delta]^n} = (-1)^n i \pi^{d/2} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta} \right)^{n - \frac{d}{2}} \quad (\text{G.63})$$

and thus

$$J = \int [d\beta]_3 i \pi^{d/2} \frac{\Gamma(d-3)}{\Gamma(3d/2-3)} \frac{1}{\Delta^{d-3}}. \quad (\text{G.64})$$

Since only second and first derivatives $\partial_i^\sigma \partial_k^\tau \Delta = 2\eta^{\tau\sigma} \beta_i \beta_k$ and $\partial_i^\sigma \Delta = -2(z_{ij}^\sigma \beta_j + z_{ik}^\sigma \beta_k) \beta_i$ are non-vanishing we have

$$\begin{aligned} \partial_i^\sigma \partial_j^\lambda \partial_k^\tau \Delta^{3-d} &= (3-d)(2-d) \Delta^{1-d} \left[\partial_j^\lambda \Delta \partial_i^\sigma \partial_k^\tau \Delta + 2 \text{ terms} \right] \\ &+ (3-d)(2-d)(1-d) \Delta^{-d} \partial_i^\sigma \Delta \partial_j^\lambda \Delta \partial_k^\tau \Delta. \end{aligned} \quad (\text{G.65})$$

We thus split up the integral as

$$I_{ijk} = c \left(I_{ijk}^{(a)} + I_{ijk}^{(b)} \right), \quad (\text{G.66})$$

where

$$\begin{aligned} I_{ijk}^{(a)} &= \int d[s, \beta]_6 \epsilon^{\alpha\beta\gamma} \epsilon_{\mu\alpha\sigma} \epsilon_{\nu\beta\lambda} \epsilon_{\rho\gamma\tau} p_i^\mu p_j^\nu p_k^\rho \left[\partial_j^\lambda \Delta \partial_i^\sigma \partial_k^\tau \Delta + 2 \text{ terms} \right] \Delta^{1-d}, \\ I_{ijk}^{(b)} &= \int d[s, \beta]_6 \epsilon^{\alpha\beta\gamma} \epsilon_{\mu\alpha\sigma} \epsilon_{\nu\beta\lambda} \epsilon_{\rho\gamma\tau} p_i^\mu p_j^\nu p_k^\rho \left[\partial_i^\sigma \Delta \partial_j^\lambda \Delta \partial_k^\tau \Delta \right] \Delta^{-d} (1-d) \end{aligned} \quad (\text{G.67})$$

and

$$\int d[s, \beta]_6 = \int_0^1 ds_{i,j,k} \int_0^1 d\beta_{i,j,k} \delta\left(\sum_m \beta_m - 1\right) (\beta_i \beta_j \beta_k)^{d/2-2}, \quad c = \frac{i\pi^{d/2} \Gamma(d-1)}{8 \Gamma(\frac{d}{2})^3}. \quad (\text{G.68})$$

Inserting this into (G.56) yields

$$\langle W_n \rangle_{\text{vertex}}^{(2)} = -\frac{1}{4} \left(\frac{N}{k} \right)^2 \pi^{3-d} \Gamma(d-1) \frac{1}{4\pi} \sum_{i>j>k} \left(I_{ijk}^{(a)} + I_{ijk}^{(b)} \right). \quad (\text{G.69})$$

For $n = 4$ the expressions (G.67) are identical to (G.9). For $n > 4$ the structure of the denominator Δ involves more terms and thus for an analogous evaluation of the finite parts of the divergent diagrams using Mellin-Barnes representations as in G.1.1 we would need more Mellin-Barnes parameters. Therefore we will evaluate the divergent integrals in a different manner in the following section, which is better suited for a numerical evaluation of the finite part and as a side effect allows for the analytical evaluation of the pole term.

G.5 Simplifications of the Divergent Vertex Diagram

As in the four-point case, diagrams with $i = k + 2, j = k + 1, k$ are divergent. It is sufficient to evaluate one of these diagrams, since the others have the same structure and can be obtained by shifting the index k . We choose $i = 3, j = 2, k = 1$ in the following. After performing the variable change $\beta_i = xy, \beta_j = \bar{x}y, \beta_k = \bar{y}$ (where $\bar{x} = 1 - x, \bar{y} = 1 - y$) we get

$$\begin{aligned} I_{321}^{(a)} &= x_{13}^2 x_{24}^2 \int_0^1 ds_{i,j,k} dx dy \\ &\frac{(\bar{x}x\bar{y})^{d/2-1}}{(-s_j \bar{s}_k \bar{x}y x_{13}^2 - x(\bar{y}(\bar{s}_i \bar{s}_k x_{13}^2 + s_i(\bar{s}_k x_{14}^2 + s_k x_{24}^2)) + s_i \bar{s}_j \bar{x}y x_{24}^2))^{d-1}}. \end{aligned} \quad (\text{G.70})$$

The diagram is divergent as $s_i \rightarrow 0, s_k \rightarrow 1$ and it is convenient to introduce new variables $s_i = r \sin \theta, \bar{s}_k = r \cos \theta$ such that

$$I_{321}^{(a)} = \int_0^1 ds_j dx dy (\bar{x} x \bar{y})^{d/2-1} \left(\int_0^{\pi/4} d\theta \int_0^{1/\cos \theta} r dr + \int_{\pi/4}^{\pi/2} d\theta \int_0^{1/\sin \theta} r dr \right) \quad (\text{G.71})$$

$$\frac{r^{1-d} x_{13}^2 x_{24}^2}{(-x_{13}^2 \cos \theta \bar{y}(x + s_j \bar{x}) - x_{24}^2 \sin \theta x(\bar{s}_j \bar{x} y + \bar{y}) + r x \bar{y} \cos \theta \sin \theta (x_{13}^2 + x_{24}^2 - x_{14}^2))^{d-1}}.$$

The integrand has the structure

$$f(r) = \frac{r^{2-d}}{[a + br]^{d-1}} = \frac{r^{2-d}}{a^{d-1}} + \sum_{n=1}^{\infty} \frac{f^{(n)}}{n!} \Big|_{r=0} r^{n+2-d}, \quad (\text{G.72})$$

where $a + br > 0$ everywhere except at the boundary. For the following analysis we use the short-hand notations

$$a = e \cos \theta + f \sin \theta, \quad b = \cos \theta \sin \theta g, \quad (\text{G.73})$$

where

$$e = -x_{13}^2 \bar{y}(x + s_j \bar{x}), \quad f = -x_{24}^2 x(\bar{y} + y \bar{s}_j \bar{x}), \quad g = -x \bar{y}(-x_{13}^2 - x_{24}^2 + x_{14}^2). \quad (\text{G.74})$$

G.5.1 Analytical Result for the Pole Term

For extracting the leading pole we can discard terms of order r in the denominator, they contribute to the finite part of the integral. Thus we integrate only the first term in (G.72) and use

$$\int r^{2-d} = \frac{1}{3-d} r^{3-d} \Big|_0^{1/\cos \theta} = \frac{1}{2\epsilon} + \mathcal{O}(\epsilon^0). \quad (\text{G.75})$$

Setting $d = 3$ in the remaining finite integrals we thus have

$$I_{321}^{(a,div)} = \frac{1}{2\epsilon} \int_0^1 ds_j dx dy (\bar{x} x \bar{y})^{1/2} \quad (\text{G.76})$$

$$\int_0^{\pi/2} d\theta \frac{x_{13}^2 x_{24}^2}{(\sin \theta (-x_{13}^2 x(\bar{y} + s_j \bar{x} y)) + \cos \theta (-x_{24}^2 \bar{y}(x + \bar{s}_j \bar{x})))^2} + \mathcal{O}(\epsilon^0).$$

The integral over θ can be solved

$$\int_0^{\pi/2} d\theta \frac{1}{(e \sin \theta + f \cos \theta)^2} = \frac{1}{ef}, \quad (\text{G.77})$$

such that we get

$$I_{321}^{(a,div)} = \frac{1}{2\epsilon} \int_0^1 ds_j dx dy \frac{(\bar{x} x \bar{y})^{1/2}}{x(\bar{y} + s_2 \bar{x} y) \bar{y}(x + \bar{s}_2 \bar{x})} + \mathcal{O}(\epsilon^0). \quad (\text{G.78})$$

Integration over x and y yields

$$I_{321}^{(a,div)} = \frac{\pi}{\epsilon} \int_0^1 ds_j \frac{1}{s_j} \left(\frac{1}{\sqrt{1-s_j}} - \frac{1}{\sqrt{s_j+1}} \right) = \frac{4\pi \ln 2}{\epsilon} + \mathcal{O}(\epsilon^0). \quad (\text{G.79})$$

G.5.2 Finite Terms of I_A

There are two types of finite contributions that we have to take into account: one comes from taking into account the second term in (G.72) and the other one from considering the $\mathcal{O}(\epsilon)$ expansions in (G.71) multiplied with the divergence (G.75) as well as the $\mathcal{O}(\epsilon^0)$ expansion of (G.75).

Order ϵ expansions

Considering the ϵ expansion of the terms that are multiplied with the pole term, instead of (G.75) we thus get

$$\begin{aligned} I_{321}^{(a,1)} &= x_{13}^2 x_{24}^2 \int (x\bar{x}\bar{y})^{1/2-\epsilon} \frac{1}{2\epsilon} \left(\int_0^{\pi/4} \left(\frac{1}{\cos\theta} \right)^{2\epsilon} + \int_{\pi/4}^{\pi/2} \left(\frac{1}{\sin\theta} \right)^{2\epsilon} \right) \frac{1}{a^{2-2\epsilon}} \\ &= I_{321}^{(a,div)} + \text{(I)} + \text{(II)} + \text{(III)}, \end{aligned} \quad (\text{G.80})$$

where $I_{321}^{(a,div)}$ is given in (G.79) and

$$\begin{aligned} \text{(I)} &= -\frac{1}{2} x_{13}^2 x_{24}^2 \int ds_j dx dy \log(x\bar{x}\bar{y}) (x\bar{x}\bar{y})^{1/2} \int_0^{\pi/2} d\theta \frac{1}{a^2}, \\ \text{(II)} &= x_{13}^2 x_{24}^2 \int ds_j dx dy (x\bar{x}\bar{y})^{1/2} \int_0^{\pi/2} d\theta \frac{\log a}{a^2}, \\ \text{(III)} &= -x_{13}^2 x_{24}^2 \int ds_j dx dy (x\bar{x}\bar{y})^{1/2} \left(\int_0^{\pi/4} d\theta \frac{\log \cos \theta}{a^2} + \int_{\pi/4}^{\pi/2} d\theta \frac{\log \cos \theta}{a^2} \right) \end{aligned} \quad (\text{G.81})$$

and where a is given in (G.73). Integration over θ yields

$$\text{(I)} = -\frac{1}{2} x_{13}^2 x_{24}^2 \int ds_j dx dy \log(x\bar{x}\bar{y}) (x\bar{x}\bar{y})^{1/2} \frac{1}{ef} \quad (\text{G.82})$$

$$\text{(II)} + \text{(III)} = x_{13}^2 x_{24}^2 \int ds_j dx dy (x\bar{x}\bar{y})^{1/2} \left(\frac{1}{ef} + \frac{\log\left(\frac{ef}{e+f}\right)}{ef} \right). \quad (\text{G.83})$$

Finite term

The finite contribution from the second term in (G.72) can be integrated over r and resummed. We have

$$\begin{aligned} \int_0^{1/\cos\theta} dr \sum_{n=1}^{\infty} \frac{f^{(n)}}{n!} \Big|_{r=0} r^{n-1} &= \int_0^{1/\cos\theta} dr \sum_{n=1}^{\infty} (-1)^n (n+1) r^{n-1} \\ &= \sum_{n=1}^{\infty} (-1)^n \left(1 + \frac{1}{n}\right) r^n \Big|_0^{1/\cos\theta} \left(\frac{b}{a}\right)^n \frac{1}{a^2} \\ &= \frac{1}{a^2} \sum_{n=1}^{\infty} (-1)^n \left(1 + \frac{1}{n}\right) \left(\frac{b/\cos\theta}{a}\right)^n \\ &= -\left(\frac{1}{a^2}\right) \left(\frac{b/\cos\theta}{b/\cos\theta + a} + \log\left(1 + \frac{b/\cos\theta}{a}\right)\right). \end{aligned} \quad (\text{G.85})$$

We used that $-1 < (b/\cos\theta)/a < 0$. An analagous term results from integration with upper boundary $1/\sin\theta$. Both terms can then be integrated over θ and we find

$$I_{321}^{(a,2)} = \int ds_j dx dy (x\bar{x}\bar{y})^{1/2} \frac{\log\left(\frac{e+f}{e+f+g}\right)}{ef}, \quad (\text{G.86})$$

where $b = \cos\theta \sin\theta g$ and $g = -x\bar{y}(-x_{13}^2 - x_{24}^2 + x_{14}^2)$. We always have $g < 0$, but $e + f + g > 0$, thus $|e + f| > |g|$.

Sum of all terms

Adding up (G.80) and (G.86) we get

$$\begin{aligned} & I_{321}^{(a,1)} + I_{321}^{(a,2)} \\ &= \frac{4\pi \ln(2)}{\epsilon} + (\text{I}) + \int ds_j dx dy \frac{(x\bar{x}\bar{y})^{1/2}}{e'f'} \left(1 + \log\left(\frac{ef}{e+f}\right) + \log\left(\frac{e+f}{e+f+g}\right) \right) \\ &= \frac{4\pi \ln(2)}{\epsilon} + (\text{I}) + 4\pi \ln(2) \left(\ln(-x_{13}^2) + \ln(-x_{24}^2) \right) \\ &+ \int ds_j dx dy \frac{(x\bar{x}\bar{y})^{1/2}}{e'f'} \left(1 + \ln(e'f') - \frac{1}{2} \ln\left(e' \frac{x_{13}^2}{x_{24}^2} + f' + \frac{g}{x_{24}^2}\right) - \frac{1}{2} \ln\left(e' + f' \frac{x_{24}^2}{x_{13}^2} + \frac{g}{x_{13}^2}\right) \right), \end{aligned}$$

where we used (G.75) and defined $e = -x_{13}^2 e'$, $f = -x_{24}^2 f'$. The first integral is a constant and the remaining two integrals depend on all three distances.

$$I_{321}^{(a)} = 2\pi \ln(2) \left(\frac{(-x_{13}^2)^{2\epsilon}}{\epsilon} + \frac{(-x_{24}^2)^{2\epsilon}}{\epsilon} \right) + I_{321}^{(a,f)} \quad (\text{G.87})$$

and where

$$\begin{aligned} I_{321}^{(a,f)} &= -\frac{1}{2} \int_0^1 ds_j dx dy \frac{(x\bar{x}\bar{y})^{1/2}}{e'f'} \left(\ln\left(e' \frac{x_{13}^2}{x_{24}^2} + f' + \frac{g}{x_{24}^2}\right) + \ln\left(e' + f' \frac{x_{24}^2}{x_{13}^2} + \frac{g}{x_{13}^2}\right) \right) \\ &+ \int_0^1 ds_j dx dy \frac{(x\bar{x}\bar{y})^{1/2}}{e'f'} (1 + \ln(x\bar{x}\bar{y}) + \ln(e'f')). \end{aligned} \quad (\text{G.88})$$

Note, that the last term is a constant independent of x_{ij}^2 . We can easily solve some of the integrals in the last term by using (G.76)-(G.79), it is however sufficient to use this form for the numerical integration.

G.5.3 Finite Integral I_B

We have

$$\begin{aligned} I_{321}^{(b)} &= -4x_{13}^2 x_{24}^2 (x_{13}^2 + x_{24}^2 - x_{14}^2) \int dx dy ds_j (x\bar{y})^{3/2} \bar{x}^{1/2} \\ &\left(\int_0^{\pi/4} d\theta dr \frac{r^3 \cos\theta \sin\theta}{r^3 (a+rb)^3} + \int_{\pi/4}^{\pi/2} \dots \right). \end{aligned} \quad (\text{G.89})$$

Expanding in a Taylor series in r as for the divergent integral, the second line of the previous formula reads

$$\begin{aligned}
& \left(\int_0^{\pi/4} d\theta \int_0^{1/\cos\theta} dr \right) \sum_{n=0}^{\infty} (-1)^n \frac{r^n}{n!} \frac{\Gamma(d+n)}{\Gamma(d)} \left(\frac{b}{a} \right)^n \frac{1}{a^3} \cos\theta \sin\theta + \int_{\pi/4}^{\pi/2} \dots \quad (\text{G.90}) \\
&= \int_0^{\pi/4} \frac{1}{2} \frac{\sin\theta}{a^3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{b/\cos\theta}{a} \right)^n (n+2) + \int_{\pi/4}^{\pi/2} \dots \\
&= \int_0^{\pi/4} \frac{1}{2} \frac{\sin\theta}{a^2} \left(\frac{a}{(a+b/\cos\theta)^2} + \frac{1}{(a+b/\cos\theta)} \right) + \int_{\pi/4}^{\pi/2} \dots \\
&= \frac{1}{2e(e+f)(e+f+g)} + \frac{1}{2f(e+f)(e+f+g)} = \frac{1}{2ef(e+f+g)}
\end{aligned}$$

and thus

$$\begin{aligned}
I_{321}^{(b)} &= -2x_{13}^2 x_{24}^2 (x_{13}^2 + x_{24}^2 - x_{14}^2) \int dx dy ds_j (x\bar{y})^{3/2} \bar{x}^{1/2} \frac{1}{ef(e+f+g)} \quad (\text{G.91}) \\
&= 2 \int_0^1 dx dy ds_j (x\bar{y})^{3/2} \bar{x}^{1/2} \frac{(x_{14}^2 - x_{13}^2 - x_{24}^2)}{e'f'(e+f+g)}.
\end{aligned}$$

G.5.4 Full Vertex Integral for Numerical Evaluation

Using (G.87) and (G.91) the vertex integrals (6.53) can thus be split up into a divergent and a finite piece

$$\langle W_n \rangle^{\text{vertex}} = \langle W_n \rangle^{\text{div}} + \langle W_n \rangle^{\text{finite}}, \quad (\text{G.92})$$

where the divergent piece is

$$\langle W_n \rangle^{\text{div}} = - \left(\frac{N}{k} \right)^2 \left(\frac{\ln(2)}{2} \sum_{i=1}^n \frac{(-x_{i,i+2}^2 \mu^2 \pi e^{\gamma_E})^{2\epsilon}}{2\epsilon} \right). \quad (\text{G.93})$$

The remaining finite piece is

$$\langle W_n \rangle^{\text{finite}} = - \frac{1}{4} \left(\frac{N}{k} \right)^2 \left(\frac{1}{4\pi} \sum_{i>j>k} \left(I_{ijk}^{(a,f)} + I_{ijk}^{(b)} \right) - n 2 \ln(2) \right), \quad (\text{G.94})$$

where $I_{ijk}^{(a,f)}$, $I_{ijk}^{(b)}$ are given by (G.88), (G.91) for the case $i = k+2, j = k+1, k$. All other cases are treated purely numerically starting from (G.67). In particular I_{ijk}^a is not divergent in this case and $I_{ijk}^{(a,f)} = I_{ijk}^{(a)}$ in these cases.

These are the expressions that are used for the numerical evaluation of the finite parts of the divergent diagrams.

G.6 Generation of Kinematical Configurations

A set of n light-like vectors p_i^μ satisfying momentum conservation, i.e. $p_i^2 = 0$ and $\sum p_i^\mu = 0$, can easily be generated by choosing the p_i^0 components of $n - 3$ vectors and the angle θ_i between $p_i^1 = p_i^0 \cos \theta_i$, $p_i^2 = p_i^0 \sin \theta_i$. The remaining components are then fixed. For n even it is possible to choose configurations, where all non-light-like distances x_{ij}^2 are space-like. For the numerical evaluation we make use of this type of configurations, such that all integrals are real.

The configurations used for the results shown in fig. 6.8 are obtained by continuously deforming two angles θ_i , leading to conformally non-equivalent kinematical configurations. We use the angles

$$\theta_i(a) := \pi \left\{ \frac{16}{9}a, \frac{13}{9}, \frac{5}{3}a, \frac{13}{8}, \frac{19}{14}, 1 \right\} \quad (\text{G.95})$$

and choose $p_i^{\mu=0} = \{1, -3, 4\}$, the remaining components are then fixed. The parameter a is chosen between $a = 1$ and $a = 1.2$ in steps of 0.01.

Appendix H

Details of the Three-Point Function Calculation

H.1 Some Properties of Gegenbauer Polynomials

The Gegenbauer polynomials $C_j^\nu(x)$ are polynomials of degree j in the variable x and are solutions of the Gegenbauer differential equation, which is of second order and depends on the parameter ν . The Gegenbauer polynomials satisfy the orthogonality relation

$$\int_{-1}^1 (1-y^2)^{\nu-\frac{1}{2}} C_j^\nu(y) C_k^\nu(y) dy = \delta_{jk} 2^{1-2\nu} \pi \frac{\Gamma(j+2\nu)}{\Gamma^2(\nu)\Gamma(j+1)(j+\nu)} = N(j,\nu)\delta_{jk}. \quad (\text{H.1})$$

One can directly find a representation for the coefficients related to the Gegenbauer polynomials by

$$(\hat{\partial}_a + \hat{\partial}_b)^j C_j^\nu \left(\frac{\hat{\partial}_a - \hat{\partial}_b}{\hat{\partial}_a + \hat{\partial}_b} \right) = \sum_{k=0}^j a_{jk}^\nu \hat{\partial}_a^k \hat{\partial}_b^{j-k} = (\hat{\partial}_a + \hat{\partial}_b)^j \sum_{k=0}^j c_{jk}^\nu \left(\frac{\hat{\partial}_a - \hat{\partial}_a}{\hat{\partial}_a + \hat{\partial}_b} \right)^k \quad (\text{H.2})$$

and for $\nu = 1/2, 3/2$ we have

$$\begin{aligned} a_{jk}^{1/2} &= (-1)^k \binom{j}{k} \binom{j}{k}, \\ a_{jk}^{3/2} &= \frac{1}{2} (-1)^k (k+1) \binom{j+1}{k+1} \binom{j+2}{k+1}, \end{aligned} \quad (\text{H.3})$$

clearly satisfying $a_{jk} = (-1)^j a_{j,j-k}$. Of course the coefficient a_{jj} coincides with the one determined before. Useful identities are

$$\sum_{k=0}^j (-1)^{j-k} a_{jk}^{3/2} = \frac{2(2j+1)\Gamma(2j)}{\Gamma(j)\Gamma(j+1)}, \quad (\text{H.4})$$

$$\sum_{k=0}^j (-1)^{j-k} a_{jk}^{1/2} = \frac{2\Gamma(2j)}{\Gamma(j)\Gamma(j+1)}. \quad (\text{H.5})$$

The coefficient c_{jj}^λ in the expansion $C_j^\nu(x) = \sum_k x^k c_{jk}^\nu$ is easy to deduce from a derivative identity of the Gegenbauer polynomials

$$\frac{d}{dx} C_n^\lambda(x) = 2\lambda C_{n-1}^{\lambda+1}, \quad C_0^\lambda = 1.$$

This leads to

$$\frac{d^j}{dx^j} C_j^\lambda(x) = 2^j \frac{\Gamma(\lambda+j)}{\Gamma(\lambda)} \Rightarrow c_{jj}^\lambda = 2^j \frac{\Gamma(\lambda+j)}{\Gamma(\lambda)\Gamma(j+1)}. \quad (\text{H.6})$$

H.1.1 Expansion of Gegenbauer Polynomials

The Gegenbauer Polynomials can be expanded with respect to their index using

$$\frac{\partial}{\partial \rho} C_j^{\nu+\rho}(x)|_{\rho=0} = -2 \sum_{k=0}^j d_{jk}^\nu C_k^\nu(x) \quad (\text{H.7})$$

with coefficients

$$d_{jk}^\nu = -(1 + (-1)^{j-k}) \frac{k+\nu}{(j+k+2\nu)(j-k)} \quad \text{for } j > k, \quad (\text{H.8})$$

$$d_{jj}^\nu = -1/2 \frac{\Gamma(\nu)}{\Gamma(\nu+j)} \frac{\partial}{\partial \rho} \left(\frac{\Gamma(\nu+\rho+j)}{\Gamma(\nu+\rho)} \right)_{\rho=0} = \frac{1}{2} (\psi(\lambda) - \psi(j+\lambda)).$$

In $d = 6, 4$ we get

$$d_{jj}^{3/2} = -\frac{1}{2} H_{j+\frac{1}{2}} - \log(2) + 1 = \frac{1}{2} (H_j - 2H_{2j+1} + 2), \quad (\text{H.9})$$

$$d_{jj}^{1/2} = -\frac{1}{2} H_{j-\frac{1}{2}} - \log(2) = \frac{1}{2} (H_{j-1} - 2H_{2j-1}),$$

where $H_{j+1/2}$ are harmonic numbers. One can reformulate the above equation using

$$H_{j+1/2} = 2H_{2j+1} - H_j - 2\ln 2. \quad (\text{H.10})$$

H.2 Basis Integrals Solved by the IBP Method

We define the following set of integrals¹

$$f_{mn}(a_1, a_2, a_3, a_4, a_5) = \int \frac{d^d k, l}{(2\pi)^{2d}} \frac{(\hat{k})^m (\hat{l})^n}{k^{2a_1} (p+k)^{2a_2} (l-k)^{2a_3} l^{2a_4} (p+l)^{2a_5}}, \quad (\text{H.11})$$

where $\hat{k} = z^\mu k_\mu$ is the contraction with a light-like vector $z^2 = 0$. The integral is symmetric under the simultaneous exchange $(m, a_1, a_2) \leftrightarrow (n, a_4, a_5)$. In particular in this thesis we need $f_{mn}(1, 1, 1, 1, 1)$ and $f_{j0}(2, 1, 1, 1, 1)$. Since the momentum dependence is the same everywhere we define c_{mn} by stripping off the equal coefficients

$$f_{mn}(a_1, a_2, a_3, a_4, a_5) = c_{mn}(a_1, a_2, a_3, a_4, a_5) \frac{(\hat{p})^{m+n}}{(-p^2)^{\sum_i a_i - d}} \frac{1}{(4\pi)^d}. \quad (\text{H.12})$$

H.2.1 Bubble Integrals

The integrals (H.11) with one argument set to zero are easy to solve and are the building blocks of $f_{mn}(1, 1, 1, 1, 1)$ after using the IBP method. We have

$$f_{mn}(a_1, a_2, 0, a_4, a_5) = (-1)^{\sum_i a_i} b_m(a_1, a_2) b_n(a_4, a_5) \frac{(\hat{p})^{m+n}}{(-p^2)^{\sum_i a_i - d}} \quad (\text{H.13})$$

and

$$f_{mn}(a_1, a_2, a_3, a_4, 0) = (-1)^n (-1)^{\sum_i a_i} b_n(a_4, a_3) b_{m+n}(a_1 + a_3 + a_4 - \frac{d}{2}, a_2) \frac{(\hat{p})^{m+n}}{(-p^2)^{\sum_i a_i - d}}.$$

By the symmetry of the integral (H.11) $(m, a_1, a_2) \leftrightarrow (n, a_4, a_5)$ we also know

$$f_{mn}(a_1, 0, a_3, a_4, a_5) = f_{nm}(a_4, a_5, a_3, a_1, 0). \quad (\text{H.14})$$

Furthermore, we have

$$f_{mn}(a_1, a_2, a_3, 0, a_5) = (-1)^{\sum_i a_i} \frac{(\hat{p})^{n+m}}{(-p^2)^{\sum_i a_i - d}} \sum_{j=0}^n \sum_{i=0}^{n-j} \binom{n}{j} \binom{n-j}{i} (-1)^n b_{n-j}(a_5, a_3) b_{m+n-i-j}(a_1, a_2 + a_3 + a_5 - \frac{d}{2}) \quad (\text{H.15})$$

and $f_{mn}(0, a_2, a_3, a_4, a_5)$ is related to this integral by the symmetry $(m, a_1, a_2) \leftrightarrow (n, a_4, a_5)$. We can perform the sum over i by using (H.40) and find

$$f_{mn}(a_1, a_2, a_3, 0, a_5) = \frac{(\hat{p})^{n+m}}{(-p^2)^{\sum_i a_i - d}} \frac{\Gamma(a_1 - m) \Gamma(a_1 + a_2 + a_3 + a_5 - d)}{\Gamma(a_1) \Gamma(a_1 + a_2 + a_3 + a_5 - d - m)} \sum_{j=0}^n \binom{n}{j} (-1)^{m+n-j} b_j(a_5, a_3) b_j(a_2 + a_3 + a_5 - \frac{d}{2}, a_1 - m). \quad (\text{H.16})$$

The sum can be performed using Mathematica in terms of generalised hypergeometric functions

¹These integrals were considered before in [322].

H.2.2 Integrals $f_{mn}(1, 1, 1, 1, 1)$

First we would like to solve the integral $f_{mn}(1, 1, 1, 1, 1)$, given through the definition

$$f_{mn}(a_1, a_2, a_3, a_4, a_5) = \int d^d k, l \underbrace{\frac{(\hat{k})^m (\hat{l})^n}{k^{2a_1} (p+k)^{2a_2} (k-l)^{2a_3} l^{2a_4} (p+l)^{2a_5}}}_{\hat{i}_{mn}(\dots)}. \quad (\text{H.17})$$

The integral is symmetric under simultaneous exchange of $\{m, a_1, a_2\} \leftrightarrow \{n, a_4, a_5\}$ and can be solved by using the IBP method, i.e. using the fact that - with suitable boundary conditions - the divergence of the integral vanishes.

$$\int \frac{\partial}{\partial k^\mu} (v^\mu \hat{i}_{mn}(1, 1, 1, 1, 1)) = 0. \quad (\text{H.18})$$

Evaluating the differentiation explicitly we find

$$0 = \left(\frac{\partial}{\partial k^\mu} v^\mu \right) f_{mn} + \int \hat{i}_{mn}(1, 1, 1, 1, 1) \left(-\frac{2k \cdot v}{k^2} - \frac{2(p+k) \cdot v}{(p+k)^2} + \frac{2(l-k) \cdot v}{(l-k)^2} \right) + m \int \hat{v} \hat{i}_{m-1, n}(1, 1, 1, 1, 1).$$

Choosing $v^\mu = k^\mu, z^\mu, p^\mu$ etc. one finds several relations between the integrals with different indices. These relations contain simpler integrals, that can be explicitly solved. The last term provides recursions in m, n .

In the following, we will use the convention, that all arguments of f_{mn} are 1 if not explicitly specified, i.e. $f_{mn} := f_{mn}(1, 1, 1, 1, 1)$.

We start by choosing $v^\mu = (l-k)^\mu$, which yields the equation

$$0 = -d f_{mn} + \int \hat{i}_{mn} \left(-\frac{2k \cdot (l-k)}{k^2} - \frac{2(p+k) \cdot (l-k)}{(p+k)^2} + 2 \right) - m f_{mn} + m f_{m-1, n+1}. \quad (\text{H.19})$$

The term in the brackets can be rewritten by simply using

$$\begin{aligned} -2k \cdot (l-k) &= (l-k)^2 + k^2 - l^2, \\ -2(p+k) \cdot (l-k) &= (l-k)^2 + (k+p)^2 - (l+p)^2 \end{aligned} \quad (\text{H.20})$$

and thus we get we can write this as

$$(d+m-4)f_{mn} = \text{bubbles}(m, n) + m f_{m-1, n+1}, \quad (\text{H.21})$$

where we abbreviate the known integrals as

$$\begin{aligned} \text{bubbles}(m, n) &= f_{mn}(2, 1, 0, 1, 1) - f_{mn}(2, 1, 1, 0, 1) \\ &+ f_{mn}(1, 2, 0, 1, 1) - f_{mn}(1, 2, 1, 1, 0). \end{aligned} \quad (\text{H.22})$$

We can actually solve for all integrals by writing down (H.21) for $m = n + 1$ and $n = m - 1$, i.e.

$$(d + (n + 1) - 4)f_{n+1,m-1} = \text{bubbles}(n + 1, m - 1) + (n + 1)f_{nm} \quad (\text{H.23})$$

and using $f_{mn}(1, 1, 1, 1, 1) = f_{nm}(1, 1, 1, 1, 1)$. Solving (H.23) for $f_{n+1,m-1}$ and inserting it into (H.21) we get f_{nm} in terms of known integrals:

$$f_{mn}(1, 1, 1, 1, 1) = \frac{(n + d - 3)\text{bubbles}(m, n) + m\text{bubbles}(n + 1, m - 1)}{(d - 4)(d - 3 + m + n)}. \quad (\text{H.24})$$

Note, that the coefficient on the left-hand side is of order ϵ in $d = 4 - 2\epsilon$. All appearing integrals were solved in section H.2.1.

H.2.3 $f_{00}(1, 1, 1, 1, 1)$ in $d = 4$

The recursion has been solved in (H.24) and $b(m, n)$ is given in (H.22). For $n = m = 0$ we have

$$f_{00}(1, 1, 1, 1, 1) = \frac{\text{bubbles}(0, 0)}{(d - 4)}. \quad (\text{H.25})$$

The bubbles then read

$$\text{bubbles}(0, 0) = 2b_0(1, 1)(b_0(3 - d/2, 2) - b_0(2, 1)) \quad (\text{H.26})$$

and we would now like to investigate the case $d = 4 - 2\epsilon$. Furthermore, we have

$$(b_0(3 - d/2, 2) - b_0(2, 1)) = -\frac{i}{(4\pi)^2}6\zeta(3)\epsilon^2 + \mathcal{O}(\epsilon^3) \quad (\text{H.27})$$

and thus

$$\text{bubbles}(0, 0) = \frac{2}{(4\pi)^4}6\zeta(3)\epsilon + \mathcal{O}(\epsilon^2), \quad (\text{H.28})$$

which leads to

$$f_{00}(1, 1, 1, 1, 1) = -\frac{1}{(4\pi)^4}6\zeta(3) + \mathcal{O}(\epsilon). \quad (\text{H.29})$$

H.2.4 Integral $f_{nm}(2, 1, 1, 1, 1)$

For the three-point function calculation we need the integral

$$f_{j,0}(2, 1, 1, 1, 1) = \int \frac{d^d k, l}{(2\pi)^{2d}} \frac{(\hat{k})^j}{k^4(p+k)^2(l-k)^2l^2(p+l)^2}. \quad (\text{H.30})$$

Using the IBP method, just as in section H.2.2 for $f_{mn}(1, 1, 1, 1, 1)$, we find

$$(d - 5 + m)f_{m,n}(2, 1, 1, 1, 1) = mf_{m-1,n+1}(2, 1, 1, 1, 1) + \text{bubbles}_2(m, n), \quad (\text{H.31})$$

where

$$\begin{aligned} \text{bubbles}_2(m, n) &= 2f_{mn}(3, 1, 0, 1, 1) - 2f_{mn}(3, 1, 1, 0, 1) \\ &\quad + f_{mn}(2, 2, 0, 1, 1) - f_{mn}(2, 2, 1, 1, 0) \end{aligned} \quad (\text{H.32})$$

and all these integrals are solved in the following section. Thus we can obtain f_{mn} recursively from (H.31), the recursion ends at $f_{0,m+n}$ and

$$f_{0,m+n}(2, 1, 1, 1, 1) = \frac{1}{d-5} \text{bubbles}_2(0, m+n). \quad (\text{H.33})$$

Note that the recursion for $f_{1,j}$ takes the form

$$f_{1,j-1}(2, 1, 1, 1, 1) = (f_{0,j}(2, 1, 1, 1, 1) + \text{bubbles}_2(1, j-1)) / (-2\epsilon). \quad (\text{H.34})$$

H.2.5 A Useful Relation between Euler Beta-Functions

For the Euler Beta-function defined as

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad (\text{H.35})$$

a useful relation can be derived

$$\sum_{n=0}^j \binom{j}{n} (-1)^n B(n+x, y) = B(j+y, x). \quad (\text{H.36})$$

Proof by induction $j \rightarrow j+1$

It is easy to show that

$$B(y+j+1, x) = B(y+j, x) - \sum_{n=0}^j (-1)^n \binom{j}{n} B(x+n+1, y). \quad (\text{H.37})$$

Using (H.36) for the first term on the right-hand side of (H.37) we get

$$\begin{aligned} &\sum_{n=0}^j (-1)^n \binom{j}{n} B(x+n, y) - \sum_{n=0}^j (-1)^n \binom{j}{n} B(x+n+1, y) \\ &= \sum_{n=0}^j (-1)^n \binom{j}{n} B(x+n, y) - \sum_{n=1}^{j+1} (-1)^{n-1} \binom{j}{n-1} B(x+n, y) \\ &= \sum_{n=1}^j (-1)^n \left(\binom{j}{n} + \binom{j}{n-1} \right) B(x+n, y) \\ &\quad + \binom{j+1}{0} B(x, y) + (-1)^{j+1} \binom{j+1}{j+1} B(x+j+1, y) \\ &= \sum_{n=0}^{j+1} (-1)^n \binom{j+1}{n} B(x+n, y) \end{aligned} \quad (\text{H.38})$$

and thus we have shown that the formula is valid for $j + 1$ by using it for j . This completes the proof by induction².

Application to bubble integrals

We can use (H.36) for simplifying sums as

$$(-1)^j \sum_{n=0}^j \binom{j}{n} b_n(x, y) = b_j(y, x) \quad (\text{H.39})$$

or by letting $x \rightarrow x - m$, which is equivalent to $b_n \rightarrow (-1)^m b_{n+m}$ we get

$$(-1)^{j+m} \sum_{n=0}^j \binom{j}{n} b_{n+m}(x, y) = b_j(y, x - m) \frac{\Gamma(x - m)\Gamma(x + y - d/2)}{\Gamma(x)\Gamma(x + y - m - d/2)}. \quad (\text{H.40})$$

H.3 Form of Twist-Two Operators with Exactly One Gauge Field

Here we deduce a form of the operator (7.10) with exactly one covariant derivative. The gauge field can appear in any of the derivatives in (7.10) and therefore

$$\begin{aligned} \hat{\mathcal{O}}_j^A = & \sum_{k=1}^{j-1} a_{jk}^{1/2} \text{Tr} \left(\sum_{m=1}^k \hat{\partial}^{m-1} \left(-i[\hat{A}, \hat{\partial}^{k-m} \phi^{12}] \right) \hat{\partial}^{j-k} \phi^{12} \right. \\ & \left. + \sum_{m=1}^{j-k} \hat{\partial}^k \phi^{12} \hat{\partial}^{m-1} \left(-i[\hat{A}, \hat{\partial}^{j-k-m} \phi^{12}] \right) \right) \\ & + (a_{j0}^{1/2} + a_{jj}^{1/2}) \sum_{m=0}^{j-1} \text{Tr} \left(\hat{\partial}^m \left(-i[\hat{A}, \hat{\partial}^{j-m-1} \phi^{12}] \right) \phi^{12} \right). \end{aligned} \quad (\text{H.41})$$

Taking the trace we can write this as

$$\begin{aligned} \hat{\mathcal{O}}_j^A = & \frac{1}{2} f^{abc} \sum_{k=1}^{j-1} a_{jk}^{1/2} \left(\sum_{m=1}^k \hat{\partial}^{m-1} \left(\hat{A}^a \hat{\partial}^{k-m} \phi^{12,b} \right) \hat{\partial}^{j-k} \phi^{12,c} \right. \\ & \left. + \sum_{m=1}^{j-k} \hat{\partial}^{m-1} \left(\hat{A}^a \hat{\partial}^{j-k-m} \phi^{12,b} \right) \hat{\partial}^k \phi^{12,c} \right) \\ & + \frac{1}{2} f^{abc} (a_{j0}^{1/2} + a_{jj}^{1/2}) \sum_{m=0}^{j-1} \hat{\partial}^m \left(\hat{A}^a \hat{\partial}^{j-m-1} \phi^{12,b} \right) \phi^{12,c}. \end{aligned} \quad (\text{H.42})$$

Using $a_{jk}^{1/2} = (-1)^j a_{j,j-k}^{1/2}$ we find

$$\hat{\mathcal{O}}_j^A = \frac{1}{2} f^{abc} \left(1 + (-1)^j \right) \left(\sum_{k=1}^j a_{jk}^{1/2} \sum_{m=1}^k \hat{\partial}^{m-1} \left(\hat{A}^a \hat{\partial}^{k-m} \phi^{12,b} \right) \hat{\partial}^{j-k} \phi^{12,c} \right)$$

²I am grateful to Martin Heinze for a discussion on this subject and suggesting the idea for working out the last and most important steps to the proof by induction.

and rewriting the derivative $\hat{\partial}^{m-1}$ as a binomial sum

$$\hat{\mathcal{O}}_j^A = \frac{1}{2} f^{abc} (1 + (-1)^j) \left(\sum_{k=1}^j a_{jk}^{1/2} \sum_{m=1}^k \sum_{i=0}^{m-1} \binom{m-1}{i} \hat{\partial}^i \hat{A}^a \hat{\partial}^{k-1-i} \phi^{12,b} \hat{\partial}^{j-k} \phi^{12,c} \right).$$

We can furthermore rewrite the summation

$$\sum_{m=1}^k \sum_{i=0}^{m-1} \binom{m-1}{i} \hat{\partial}^i \hat{A}^a \hat{\partial}^{k-1-i} \phi^{12,b} = \sum_{m=1}^k \binom{k}{m} \hat{\partial}^{m-1} \hat{A}^a \hat{\partial}^{k-m} \phi^{12,b},$$

such that we get

$$\hat{\mathcal{O}}_j^A = \frac{1}{2} f^{abc} (1 + (-1)^j) \left(\sum_{k=1}^j a_{jk}^{1/2} \sum_{m=1}^k \binom{k}{m} \hat{\partial}^{m-1} \hat{A}^a \hat{\partial}^{k-m} \phi^{12,b} \hat{\partial}^{j-k} \phi^{12,c} \right). \quad (\text{H.43})$$

This is the form of the operator, that we will insert in the perturbative calculations.

H.4 Integral over Conformal Structure of the Three-Point Correlator

We assume a general exponent θ and j even, to solve the integral

$$\begin{aligned} & \int d^d x_3 \frac{(\hat{Y}_{12,3})^j}{((-x_{13}^2)(-x_{23}^2))^{\theta/2}} \quad (\text{H.44}) \\ &= \sum_{k=0}^j \binom{j}{k} (-1)^k \int d^d x_3 \frac{(\hat{x}_{13})^k (\hat{x}_{23})^{j-k}}{(-x_{13}^2)^{\theta/2+k} (-x_{23}^2)^{\theta/2+j-k}} \\ &= 2^{-j} \sum_k \binom{j}{k} (-1)^k \frac{\Gamma(\theta/2)\Gamma(\theta/2)}{\Gamma(\theta/2+k)\Gamma(\theta/2+j-k)} \hat{\partial}_1^k \hat{\partial}_2^{j-k} \int d^d x_3 \frac{1}{((-x_{13}^2)(-x_{23}^2))^{\theta/2}} \\ &= -i 2^{-j} \sum_k \binom{j}{k} (-1)^k \frac{\Gamma((d-\theta)/2)^2 \Gamma(\theta-d/2) \pi^{d/2}}{\Gamma(\theta/2+k)\Gamma(\theta/2+j-k)\Gamma(d-\theta)} \hat{\partial}_1^k \hat{\partial}_2^{j-k} \frac{1}{(-x_{12}^2)^{\theta-d/2}} \\ &= -i \frac{(\hat{x}_{12})^j \Gamma(\theta-d/2+j)}{(-x_{12}^2)^{\theta-d/2+j}} \frac{\Gamma((d-\theta)/2)^2}{\Gamma(d-\theta) \pi^{-\frac{d}{2}}} \sum_k \binom{j}{k} \frac{1}{\Gamma(\theta/2+k)\Gamma(\theta/2+j-k)} \\ &= -i \frac{(\hat{x}_{12})^j}{(-x_{12}^2)^{\theta-d/2+j}} \frac{\Gamma(\theta-d/2+j)\Gamma((d-\theta)/2)^2 \Gamma\left(\frac{1}{2}(2j+\theta-1)\right)}{\Gamma(d-\theta)\Gamma\left(j+\frac{\theta}{2}\right)\Gamma(j+\theta-1)} \frac{2^{\theta+2j-2}}{\pi^{\frac{1}{2}-\frac{d}{2}}}, \end{aligned}$$

where we used the x -space version (A.22) of the bubble integral.

H.5 Details of the Tree-Level Two-Point Function Calculation

In the following, we suppress the gauge group indices and the trace over the gauge group, such that the calculation is also valid for other scalar theories in general dimen-

sion, e.g. for $(\phi^3)_{d=6}$ theory. Furthermore, we take general flavours $I \neq J$. I.e. the operator under consideration is

$$\hat{\mathcal{O}}_j^{\text{tree}} = (\hat{\partial}_a + \hat{\partial}_b)^j C_j^\nu \left(\frac{\hat{\partial}_a - \hat{\partial}_b}{\hat{\partial}_a + \hat{\partial}_b} \right) (\phi^I(x_a) \bar{\phi}^J(x_b)) \Big|_{x_a=x_b}. \quad (\text{H.45})$$

We will take into account the gauge group factor at the end of the calculation when specializing to the operators (7.16) in $\mathcal{N} = 4$ SYM, where the corresponding flavours are $I = \{12\}, J = \{34\}$.

The two-point functions of the operators (7.16) can straightforwardly be evaluated using the Schwinger parametrisation³ of the scalar propagators (C.34)

$$\langle \phi_I(x_1) \bar{\phi}^J(x_2) \rangle = g^2 \delta^{IJ} \frac{\Gamma(h-1)}{4\pi^h} \frac{1}{(-x_{12}^2)^{h-1}} = \delta^{IJ} \frac{1}{4\pi^h} \int_0^\infty d\alpha \alpha^{h-2} \exp(-\alpha(-x_{12}^2)), \quad (\text{H.46})$$

where I and J denote the flavours of the fields.

$$\begin{aligned} \langle \hat{\mathcal{O}}_j \hat{\mathcal{O}}_k \rangle &= (\hat{\partial}_a + \hat{\partial}_b)^j C_j^\nu \left(\frac{\hat{\partial}_a - \hat{\partial}_b}{\hat{\partial}_a + \hat{\partial}_b} \right) (\hat{\partial}_c + \hat{\partial}_d)^k C_k^\nu \left(\frac{\hat{\partial}_c - \hat{\partial}_d}{\hat{\partial}_c + \hat{\partial}_d} \right) \langle (\phi_a^I \bar{\phi}_b^J) (\bar{\phi}_c^I \phi_d^J) \rangle \Big|_{a=b,c=d} \\ &= (\hat{\partial}_a + \hat{\partial}_b)^j C_j^\nu \left(\frac{\hat{\partial}_a - \hat{\partial}_b}{\hat{\partial}_a + \hat{\partial}_b} \right) (\hat{\partial}_c + \hat{\partial}_d)^k C_k^\nu \left(\frac{\hat{\partial}_c - \hat{\partial}_d}{\hat{\partial}_c + \hat{\partial}_d} \right) \frac{1}{(4\pi^h)^2} \frac{g^4 \Gamma(h-1)^2}{((-x_{ac}^2)(-x_{bd}^2))^{h-1}} \Big|_{a=b,c=d}, \end{aligned}$$

where $\phi_a \equiv \phi(x_a)$. Due to $\hat{\partial}_a \hat{x}_{ab} = z^2 = 0$, we have

$$\begin{aligned} (\hat{\partial}_a)^k \exp(\alpha x_{ac}^2) &= 2^k (\alpha)^k (\hat{x}_{ac})^k \exp(\alpha x_{ac}^2), \\ (\hat{\partial}_c)^k \exp(\alpha x_{ac}^2) &= 2^k (-\alpha)^k (\hat{x}_{ac})^k \exp(\alpha x_{ac}^2). \end{aligned} \quad (\text{H.47})$$

Thus we have

$$\begin{aligned} \langle \hat{\mathcal{O}}_j \hat{\mathcal{O}}_k \rangle &= g^4 (-1)^k (2)^{j+k} (\hat{x}_{12})^{j+k} \int \frac{d\alpha d\beta}{(4\pi^h)^2} (\alpha\beta)^{h-2} (\alpha + \beta)^{j+k} \\ &\quad \times C_j^\nu \left(\frac{\alpha - \beta}{\alpha + \beta} \right) C_k^\nu \left(\frac{\alpha - \beta}{\alpha + \beta} \right) \exp((\alpha + \beta)x_{12}^2) \end{aligned} \quad (\text{H.48})$$

Substituting $y = \frac{\alpha - \beta}{\alpha + \beta}$ and $\alpha + \beta = z$ (Jacobian = $\frac{z}{2}$), we get

$$\begin{aligned} \langle \hat{\mathcal{O}}_j \hat{\mathcal{O}}_k \rangle &= g^4 (-1)^k (2)^{j+k} 2^{3-2h} \frac{(\hat{x}_{12})^{j+k}}{(4\pi^h)^2} \\ &\quad \int_0^\infty dz z^{2h-4+j+k+1} \exp(zx_{12}^2) \int_{-1}^1 (1-y^2)^{h-3/2-\frac{1}{2}} C_j^\nu(y) C_k^\nu(y). \end{aligned} \quad (\text{H.49})$$

³Formally, this is valid in the euclidean regime $x_{12}^2 < 0$.

Using (H.46) and (H.1) we thus get

$$\langle \hat{\mathcal{O}}_j \hat{\mathcal{O}}_k \rangle = g^4 \delta_{jk} 2^{2j+3-2h+1-2\nu} \frac{1}{(4\pi^h)^2} \frac{\pi}{\Gamma^2(\nu)} \frac{\Gamma(2j+2\nu+1)\Gamma(j+2\nu)}{\Gamma(j+1)(j+\nu)} \frac{(\hat{x}_{12})^{2j}}{(-x_{12}^2)^{2j+2\nu+1}} \quad (\text{H.50})$$

for integer⁴ d , such that $\nu = (d-3)/2$. For $d = 2h = 6$ and $I \neq J$

$$\langle \hat{\mathcal{O}}_j \hat{\mathcal{O}}_k \rangle^{d=6} = g^4 \delta_{jk} (-1)^k \frac{2^{2j-2}}{(4\pi^3)^2} (j+1)(j+2)\Gamma(2j+3) \frac{(\hat{x}_{12})^{2j}}{(-x_{12}^2)^{2j+4}}. \quad (\text{H.51})$$

Specialising to $\mathcal{N} = 4$ SYM in $d = 2h = 4$, we have to take into account a factor of $\delta^{aa}/4$ from the traces $\text{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$ and another factor of 2, since we have two possible identical (for j even) contractions (due to $\bar{\phi}_{34} = \phi^{12}$)

$$\langle \hat{\mathcal{O}}_j \hat{\mathcal{O}}_k \rangle^{d=4} = g^4 \delta_{jk} \delta^{aa} \frac{2^{2j-1}}{(4\pi^2)^2} \Gamma(2j+1) \frac{(\hat{x}_{12})^{2j}}{(-x_{12}^2)^{2j+2}}. \quad (\text{H.52})$$

These results were derived in a similar way in [121]. We generalise this calculation in section H.6.1.

H.6 Integrals with Gegenbauer Polynomials

H.6.1 Two-Point Function with Two Different Propagators

The following formula is useful, when evaluating the contractions of two (possibly different) propagators G_τ, G_σ . We would like to evaluate

$$I_{\sigma,\tau}^{i,j} = D_{ab}^i D_{cd}^j (G_\tau(x_{ac})G_\sigma(x_{bd}) + G_\sigma(x_{ac})G_\tau(x_{bd})), \quad (\text{H.53})$$

where

$$D_{ab}^i = (\hat{\partial}_a + \hat{\partial}_b)^i C_i^\nu \left(\frac{\hat{\partial}_a - \hat{\partial}_b}{\hat{\partial}_a + \hat{\partial}_b} \right) \quad (\text{H.54})$$

parametrises the operators in the Gegenbauer representation. The expression arises in the contribution to the two-point function when using general propagators in the Schwinger representation

$$\begin{aligned} G_\tau(x_{ac}) &= g_\tau \frac{\Gamma(\tau)}{(-x_{ac}^2)^\tau} = g_\tau \int_0^\infty d\alpha \alpha^{\tau-1} \exp(\alpha x_{ac}^2), \\ G_\sigma(x_{ac}) &= g_\sigma \frac{\Gamma(\sigma)}{(-x_{ac}^2)^\sigma} = g_\sigma \int_0^\infty d\beta \beta^{\sigma-1} \exp(\beta x_{ac}^2). \end{aligned} \quad (\text{H.55})$$

⁴As we will see later in $d = 4 - 2\epsilon$ the correlators defined in terms of Gegenbauer polynomials have non-vanishing non-diagonal contributions.

Then we can write

$$I_{\sigma,\tau}^{i,j} = g_\tau g_\sigma 2^{i+j} (-1)^j \hat{x}_{12}^{i+j} \int d\alpha d\beta (\alpha + \beta)^{i+j} \left(\alpha^{\tau-1} \beta^{\sigma-1} + \alpha^{\sigma-1} \beta^{\tau-1} \right) \times C_i^\nu C_j^\nu \left(\frac{\alpha - \beta}{\alpha + \beta} \right) \exp((\alpha + \beta)x_{12}^2). \quad (\text{H.56})$$

Using $C_j^\nu(-x) = (-1)^j C_j^\nu(x)$ and exchanging $\alpha \leftrightarrow \beta$ in the second expression in the brackets we find

$$I_{\sigma,\tau}^{i,j} = g_\tau g_\sigma 2^{i+j} (-1)^j \hat{x}_{12}^{i+j} (1 + (-1)^{i+j}) \int d\alpha d\beta (\alpha + \beta)^{i+j} \alpha^{\tau-1} \beta^{\sigma-1} \times C_i^\nu C_j^\nu \left(\frac{\alpha - \beta}{\alpha + \beta} \right) \exp((\alpha + \beta)x_{12}^2). \quad (\text{H.57})$$

Changing variables to $z = \alpha + \beta$ and $y = (\alpha - \beta)/(\alpha + \beta)$ with Jacobian $z/2$ we find

$$I_{\sigma,\tau}^{i,j} = g_\tau g_\sigma 2^{i+j+1-\tau-\sigma} (-1)^j \hat{x}_{12}^{i+j} (1 + (-1)^{i+j}) \int_0^\infty dz z^{i+j-1+\tau+\sigma} \exp(zx_{12}^2) \int_{-1}^1 dy (1+y)^{\tau-1} (1-y)^{\sigma-1} C_i^\nu C_j^\nu(y). \quad (\text{H.58})$$

Using the Schwinger parametrisation backwards leads to

$$I_{\sigma,\tau}^{i,j} = g_\tau g_\sigma 2^{i+j+1-\tau-\sigma} (-1)^j (1 + (-1)^{i+j}) \frac{\hat{x}_{12}^{i+j} \Gamma(i+j+\tau+\sigma)}{(-x_{12}^2)^{i+j+\tau+\sigma}} \int_{-1}^1 dy (1+y)^{\tau-1} (1-y)^{\sigma-1} C_i^\nu(y) C_j^\nu(y). \quad (\text{H.59})$$

H.6.2 Two-Point Function at Tree-Level including $\mathcal{O}(\epsilon)$ Terms

We can extract the tree-level two-point function by using section H.6.1 for two identical propagators with $\sigma = \tau = h - 1 = 2 - \epsilon$ as well as $g_\tau = g_\sigma = 1/4\pi^h$. The trace normalisation yields an additional factor of $1/4$. Then we have (H.59)

$$\langle \hat{\mathcal{O}}_i(x_1) \bar{\hat{\mathcal{O}}}_j(x_2) \rangle = \frac{g^4 \delta^{aa}}{4} \frac{(-1)^i}{(4\pi^h)^2} 2^{i+j+3-d} (1 + (-1)^{i+j}) \frac{\hat{x}_{12}^{i+j} \Gamma(i+j+2h-2)}{(-x_{12}^2)^{i+j+2h-2}} \int_{-1}^1 dy (1-y^2)^{1-\epsilon} C_i^\nu(y) C_j^\nu(y). \quad (\text{H.60})$$

The integral is solved in H.6.3 and we have

$$\int_{-1}^1 dy (1-y^2)^{1-\epsilon} C_i^\nu(y) C_j^\nu(y) = \delta_{ij} N(j) - \epsilon I_{ij}, \quad (\text{H.61})$$

where $N(j)$ is given in (H.1), I_{ij} in (H.67) and the tree-level contribution thus yields (H.52), since $I_{ij} = 0$ for $i+j = \text{uneven}$.

Diagonal contribution

For $i = j$ we have

$$\langle \hat{\mathcal{O}}_j(x_1) \bar{\hat{\mathcal{O}}}_j(x_2) \rangle = g^4 \delta^{aa} \frac{2^{2j-2}}{(2\pi)^d} \Gamma(2j + 2h - 2) \frac{\hat{x}_{12}^{i+j}}{(-x_{12}^2)^{i+j+2h-2}} \left(N(j, 1/2) - \epsilon I_{jj}^{1/2} \right), \quad (\text{H.62})$$

where $N(j, 1/2) = 2/(2j + 1)$ is given in (H.64) and $I_{jj}^{1/2}$ in (H.68). Expanding the Γ function we get

$$\begin{aligned} \langle \hat{\mathcal{O}}_j(x_1) \bar{\hat{\mathcal{O}}}_j(x_2) \rangle &= g^4 \delta^{aa} \frac{2^{2j-1}}{(2\pi)^d} \Gamma(2j + 1) \frac{(\hat{x}_{12})^{i+j}}{(-x_{12}^2)^{i+j+2h-2}} \quad (\text{H.63}) \\ &\left(1 + 2\epsilon \left(2H_{2j-1} - H_{2j+1} - H_j - H_{j-1} + \frac{1}{2j+1} - \log(2) + \gamma_E \right) \right). \end{aligned}$$

H.6.3 Integral I_{ij}

The orthogonality relation of the Gegenbauer polynomials with arbitrary $\nu > -1/2$ is

$$\int_{-1}^1 (1 - y^2)^{\nu - \frac{1}{2}} C_j^\nu(y) C_k^\nu(y) = \delta_{jk} 2^{1-2\nu} \pi \frac{\Gamma(j + 2\nu)}{\Gamma^2(\nu) \Gamma(j + 1) (j + \nu)} = \delta_{jk} N(j, \nu). \quad (\text{H.64})$$

However, we calculate the correlators in $d = 6 - 2\epsilon$ dimensions, but with fixed index $\nu \equiv 3/2$ of the Gegenbauer polynomials. Thus in the evaluation of the tree-level correlator with Schwinger parameters we get the expression

$$\begin{aligned} \int_{-1}^1 (1 - y^2)^{\nu' - \epsilon - \frac{1}{2}} C_j^{\nu'}(y) C_k^{\nu'}(y) &= \int_{-1}^1 (1 - y^2)^{\nu' - 1/2} C_j^{\nu'}(y) C_k^{\nu'}(y) \quad (\text{H.65}) \\ &- \epsilon \int_{-1}^1 (1 - y^2)^{\nu' - 1/2} \log(1 - y^2) C_j^{\nu'}(y) C_k^{\nu'}(y). \end{aligned}$$

which is non-diagonal at order $\mathcal{O}(\epsilon)$. This integral is easy to evaluate by using the expansion of the Gegenbauer polynomials w.r.t. their index

$$C_j^{\nu - \epsilon}(x) = C_j^\nu(x) + \epsilon \left(2 \sum_{k=0}^j d_{jk}^\nu C_k^\nu(x) \right) \quad (\text{H.66})$$

with d_{jk}^ν as given in (H.8), also including a d_{jj} term.

Explicit solution for $d = 4$

Then, by using the exact expression (H.64) for $\nu = 1/2 - \epsilon$, expanding the indices in $C_j^{1/2-\epsilon}$ w.r.t. ϵ and comparing the order ϵ terms we find

$$\begin{aligned} I_{ij}^{1/2} &= \int_{-1}^1 dy \log(1-y^2) C_i^{1/2} C_j^{1/2} = \\ &= 2 \left(d_{ij}^{1/2} N(j, \nu = \frac{1}{2}) + d_{ji}^{1/2} N(k, \nu = \frac{1}{2}) \right) - \delta_{ij} N(j, \nu = \frac{1}{2} - \epsilon) \Big|_{\epsilon} \\ &= 4 \left(d_{ij}^{1/2} \frac{1}{(2j+1)} + d_{ji}^{3/2} \frac{1}{(2i+1)} \right) - \delta_{ij} \frac{2^{2\epsilon} \pi}{\Gamma^2(1/2 - \epsilon)} \frac{\Gamma(j+1-2\epsilon)}{\Gamma(j+1)(j+1/2-\epsilon)} \Big|_{\epsilon}. \end{aligned} \quad (\text{H.67})$$

Also note that $d_{ij} = 0$ for $j > i$, such that only both terms are present when $i = j$, then we have

$$\begin{aligned} I_{jj}^{1/2} &= -\frac{4}{(2j+1)} \left(H_{j-\frac{1}{2}} - H_j + \frac{1}{2j+1} + \log(2) \right) \\ &= -2N(j, 1/2) \left(-H_{j-1} - H_j + 2H_{2j-1} + \frac{1}{2j+1} - \log(2) \right). \end{aligned} \quad (\text{H.68})$$

Note that here we also had to take into account the $d_{jj}^{1/2}$ terms and in the last line we used the identity (H.10).

H.7 Two-Point Correlator at One Loop

H.7.1 Diagram 7.4a

For the self-energy diagrams we do not have to take into account the gauge fields from the covariant derivative and can thus start again from the tree-level operator (7.16). Taking into account a factor of $1/4$ from the trace over the gauge group as well as the fact that we have two identical contractions (for $i + j$ even, otherwise the correlator vanishes anyways) we find

$$\begin{aligned} \langle \hat{\mathcal{O}}_i(x_1) \hat{\mathcal{O}}_j(x_2) \rangle^{(7.4a)} &= 2 \frac{1}{4} D_{ab}^i D_{cd}^j (G_{\sigma}(x_{ac}) G_{\tau}(x_{bd}) + G_{\tau}(x_{ac}) G_{\sigma}(x_{bd})) \Big|_{x_a=x_b, x_c=x_d} \\ &= \frac{1}{2} \delta^{aa} I_{\sigma, \tau}^{i, j}(x_{12}), \end{aligned}$$

where $I_{\sigma, \tau}^{i, j}$ is given in (H.59) and here we have $\sigma = 2h - 3$, $\tau = h - 1$ and

$$g_{\sigma} = -\frac{1}{2} \frac{g^4 N}{(4\pi^h)^2} \frac{\Gamma(h-1)^2 \Gamma(2-h)}{\Gamma(2h-2) \Gamma(3-h)}, \quad g_{\tau} = \frac{g^2}{4\pi^h}.$$

Then for $i = j$ the full correlator reads

$$\langle \hat{\mathcal{O}}_j(x_1) \hat{\mathcal{O}}_j(x_2) \rangle^{(7.4a)} = g^6 N \delta^{aa} g_{\tau} g_{\sigma} 2^{2j-3h+5} \Gamma(2j+3h-4) \left(N(j, 1/2) - \frac{3}{2} \epsilon I_{jj}^{1/2} \right) \frac{(\hat{x}_{12})^{2j}}{(-x_{12}^2)^{2j+3h-4}}.$$

Expanding the prefactor in ϵ we find

$$\begin{aligned} \langle \hat{\mathcal{O}}_j(x_1) \hat{\mathcal{O}}_j(x_2) \rangle^{(7.4a)} &= \frac{g^6 N \delta^{aa} 2^{2j-2} \Gamma(2j+2)}{(2\pi)^{3h}} \left(-\frac{1}{\epsilon} - 2 + 3H_{2j+1} - 3\gamma_E \right) \times \quad (\text{H.69}) \\ &\times \left(N(j, 1/2) - \frac{3}{2} \epsilon I_{jj}^{1/2} \right) \frac{(\hat{x}_{12})^{2j}}{(-x_{12}^2)^{2j+3h-4}}, \end{aligned}$$

where $N(j, 1/2) = 2/(2j+1)$ is given in (H.64) and $I_{jj}^{1/2}$ in (H.68).

Divergent part

Taking into account the normalisation $N(j, 1/2)$, see (H.1), the leading divergent term reads

$$\begin{aligned} \langle \hat{\mathcal{O}}_j(x_1) \hat{\mathcal{O}}_j(x_2) \rangle^{(7.4a)} &= -g^6 N \delta^{aa} \frac{1}{\epsilon} \frac{2^{2j-7}}{\pi^6} \Gamma(2j+1) \frac{(\hat{x}_{12})^{2j}}{(-x_{12}^2)^{2j+2}} \\ &= -g^2 N \frac{1}{\epsilon} \frac{1}{4\pi^2} \langle \hat{\mathcal{O}}_j \hat{\mathcal{O}}_j \rangle^{(0)} + \mathcal{O}(\epsilon^0), \end{aligned}$$

where we have taken into account the tree-level correlator (H.52) in the last equality. Thus the contribution to the anomalous dimension is

$$\gamma_j^{(7.4a)} = \frac{g^2 N}{4\pi^2}. \quad (\text{H.70})$$

H.7.2 Diagram 7.4b

Using the Feynman rules for the scalar gluon vertex (C.35), taking into account that there are four possible contractions as well as the colour factors

$$\text{Tr}(T^a T^b) \text{Tr}(T^c T^d) f^{ace} f^{bde} = \frac{1}{4} N \delta^{aa},$$

letting $l \rightarrow -l$, $k \rightarrow -k$ and introducing two binomial sums, we find

$$\begin{aligned} \langle \hat{\mathcal{O}}_j(p) \bar{\hat{\mathcal{O}}}_j(-p) \rangle^{(7.4b)} &= i(i)^{2j} \frac{g^6}{2} N \delta^{aa} \sum_{k, k'=0}^j a_{jk}^{1/2} a_{jk'}^{1/2} \sum_{n=0}^k \sum_{m=0}^{k'} \binom{k}{n} \binom{k'}{m} (-1)^{-k-k'} \\ &(\hat{p})^{n+m} \int \frac{d^d k, l}{(2\pi)^{2d}} \frac{(\hat{k})^{j-n} (\hat{l})^{j-m} (l+k+2p) \cdot (l+k)}{k^2 (p+k)^2 (l-k)^2 l^2 (l+p)^2}. \quad (\text{H.71}) \end{aligned}$$

Using the basic integral (H.11) and $(l+k+2p) \cdot (l+k) = -2p^2 - (l-k)^2 + k^2 + (k+p)^2 + l^2 + (l+p)^2$ we can rewrite this as

$$\begin{aligned} \langle \hat{\mathcal{O}}_j(p) \bar{\hat{\mathcal{O}}}_j(-p) \rangle^{(7.4b)} &= i(i)^{2j} \frac{g^6}{2} N \delta^{aa} \sum_{k, k'=0}^j a_{jk}^{1/2} a_{jk'}^{1/2} \sum_{n=0}^k \sum_{m=0}^{k'} \binom{k}{n} \binom{k'}{m} (-1)^{-k-k'} \\ &\frac{1}{(4\pi)^d} \frac{(\hat{p})^{2j}}{(p^2)^{4-d}} \left(-2c_{j-n, j-m}(1, 1, 1, 1, 1) - c_{j-n, j-m}(1, 1, 0, 1, 1) \right. \\ &\quad + c_{j-n, j-m}(0, 1, 1, 1, 1) + c_{j-n, j-m}(1, 0, 1, 1, 1) \\ &\quad \left. + c_{j-n, j-m}(1, 1, 1, 0, 1) + c_{j-n, j-m}(1, 1, 1, 1, 0) \right). \end{aligned}$$

These integrals have been solved in Appendix H.2. We define⁵

$$S^{7.4b}[j] := \sum_{k,k'=0}^j a_{jk}^{1/2} a_{jk'}^{1/2} \sum_{n=0}^k \sum_{m=0}^{k'} \binom{k}{n} \binom{k'}{m} (-1)^{-k-k'} \quad (\text{H.72})$$

$$\left(-c_{j-n,j-m}(1, 1, 0, 1, 1) + c_{j-n,j-m}(0, 1, 1, 1, 1) + c_{j-n,j-m}(1, 0, 1, 1, 1) \right. \\ \left. + c_{j-n,j-m}(1, 1, 1, 0, 1) + c_{j-n,j-m}(1, 1, 1, 1, 0) \right). \quad (\text{H.73})$$

Extracting only the $1/\epsilon^2$ term, we find (the sum over the first term appears to be zero and would cancel with the four-scalar diagram anyways)

$$S^{7.4b}[j] = -\frac{1}{\epsilon^2} \frac{2}{(2j+1)(j+1)} + \mathcal{O}\left(\frac{1}{\epsilon}\right) \quad (\text{H.74})$$

and thus have

$$\langle \hat{\mathcal{O}}_j(p) \bar{\hat{\mathcal{O}}}_j(-p) \rangle^{(7.4b)} = -i(i)^{2j} g^6 \frac{2N\delta^{aa}}{\pi^4} \frac{(\hat{p})^{2j}}{(p^2)^{4-d}} \frac{1}{\epsilon^2} \left(\frac{2^{-9}}{(2j+1)(j+1)} \right).$$

Fourier transformation (A.16) to position space yields, using $1/\Gamma(4-d) = 2\epsilon + \mathcal{O}(\epsilon^2)$,

$$\langle \hat{\mathcal{O}}_j(x_1) \bar{\hat{\mathcal{O}}}_j(x_2) \rangle^{(7.4b)} = g^6 N \delta^{aa} \frac{1}{\epsilon} \frac{2^{2j-7}}{\pi^6} \Gamma(3h-4+j) \frac{(\hat{x}_{12})^{2j}}{(x_{12}^2)^{3h-4+j}} \quad (\text{H.75}) \\ = g^6 N \delta^{aa} \frac{1}{\epsilon} \frac{\Gamma(2j+1)}{(j+1)} \frac{2^{2j-3}}{(4\pi^2)^2} \frac{1}{\pi^2} \frac{(\hat{x}_{12})^{2j}}{(x_{12}^2)^{2+2j}} \\ = \left(\frac{1}{\epsilon} \frac{g^2 N}{4\pi^2} \frac{1}{(j+1)} \right) \langle \hat{\mathcal{O}}_j \bar{\hat{\mathcal{O}}}_j \rangle^{(0)},$$

where the tree-level correlator (H.52) was used. Thus, the contribution to the anomalous dimension is

$$\gamma_j^{(7.4b)} = -\frac{g^2 N}{4\pi^2} \frac{1}{(j+1)}. \quad (\text{H.76})$$

H.7.3 Diagram 7.4c

After introducing a binomial sum, the four-scalar diagram reads

$$\langle \mathcal{O}_{j=0}(p) \bar{\mathcal{O}}_{j=0}(-p) \rangle^{7.4c} \quad (\text{H.77}) \\ = (g^2)^4 (i)^4 \frac{i}{1!} \left(-\frac{1}{2g^2} f^{gba} f^{gab} \right) i^{2j} \left(\int d^d l \frac{\hat{p}^k (\hat{p} - \hat{l})^{j-k}}{l^2 (p-l)^2} \right)^2 \\ = i g^6 \frac{N \delta^{aa}}{2} \frac{\hat{p}^{2j}}{(p^2)^{4-d}} \frac{1}{(4\pi)^d} \sum_{k=0}^j a_{jk} a_{jk'} (-1)^{-k-k'} \sum_{n,m}^{k,k'} \binom{k}{n} \binom{k'}{m} c_{j-n,j-m}(1, 1, 0, 1, 1).$$

The sum can be solved using (H.40). The diagram does however cancel with the corresponding term in the kite diagram, so no further investigation is necessary.

⁵ $c_{nm}(1, 1, 1, 1, 1)$ is $\mathcal{O}(1)$ and can be neglected completely for this analysis, since the Fourier transformation yields another factor of ϵ such that the x-space expression is $\mathcal{O}(\epsilon)$.

H.7.4 Cancellation of Four-Scalar Interaction Diagram

Taking into account the cancellation of the four-scalar diagram and the fact, that the first term in (H.72) is of order ϵ after Fourier transformation and can be completely neglected, we have

$$\langle \hat{\mathcal{O}}_j(p) \bar{\hat{\mathcal{O}}}_j(-p) \rangle^{7.4b+7.4c} = i(i)^{2j} \frac{g^6}{2} N \delta^{aa} \frac{1}{(4\pi)^d} \frac{(\hat{p})^{2j}}{(p^2)^{4-d}} S[j]^{7.4b+7.4c},$$

where we defined

$$\begin{aligned} S[j]^{7.4b+7.4c} : &= \sum_{k,k'=0}^j a_{jk}^{1/2} a_{jk'}^{1/2} \sum_{n=0}^k \sum_{m=0}^{k'} \binom{k}{n} \binom{k'}{m} (-1)^{-k-k'} \\ &\quad \left(c_{j-n,j-m}(0, 1, 1, 1, 1) + c_{j-n,j-m}(1, 0, 1, 1, 1) \right. \\ &\quad \left. + c_{j-n,j-m}(1, 1, 1, 0, 1) + c_{j-n,j-m}(1, 1, 1, 1, 0) \right). \end{aligned} \quad (\text{H.78})$$

Fourier transformation yields

$$\langle \hat{\mathcal{O}}_j(p) \bar{\hat{\mathcal{O}}}_j(-p) \rangle^{7.4b+7.4c} = -g^6 N \delta^{aa} 2^{2j-9} \frac{1}{\pi^{3h}} \frac{\Gamma\left(\frac{3d}{2} + 2j - 4\right)}{\Gamma(4-d)} \frac{(\hat{x}_{12})^{2j}}{(-x_{12}^2)^{3d/2-4+2j}} S[j]^{7.4b+7.4c}. \quad (\text{H.79})$$

H.7.5 Diagram 7.4d

We have also calculated this diagram in x -space as a comparison, which is actually somewhat simpler, since we can perform the integral in x -space. In momentum space the diagram reads

$$\langle \hat{\mathcal{O}}_j(p) \bar{\hat{\mathcal{O}}}_j(-p) \rangle^{(7.4d)} = 2 \langle \hat{\mathcal{O}}_j^A(p) \bar{\hat{\mathcal{O}}}_j^{\text{free}}(-p) \rangle = g^6 2i^{2j+1} N \delta^{aa} \frac{\hat{p}^{2j}}{(-p^2)^{4-d}} S^{(7.4d)}[j], \quad (\text{H.80})$$

where

$$\begin{aligned} S^{(7.4d)}[j] &= \sum_{k=1, k'=0}^j \sum_{m=1}^k \sum_{n=0}^{m-1} \binom{k}{m} \binom{m-1}{n} a_{jk}^{1/2} a_{jk'}^{1/2} \\ &\quad (-1)^{m+k'} \left(b_{j-k+n} - b_{j-k+n+1} \right) \sum_p^{k+k'-m} \binom{k+k'-m}{p} b_{2j-p}(3-h, 1) \\ &\quad - (-1)^{m+k'} \left(b_{k-m+n} - b_{k-m+n+1} \right) \sum_p^{2j-k-k'-m} \binom{2j-k-k'-m}{p} b_{2j-p}(3-h, 1). \end{aligned} \quad (\text{H.81})$$

We can solve the last binomial sum using (H.40). Fourier transformation yields

$$\langle \hat{\mathcal{O}}_j(p) \bar{\hat{\mathcal{O}}}_j(-p) \rangle^{(7.4d)} = -g^4 \delta^{aa} 2^{2d+2j-7} \frac{g^2 N}{\pi^h} \frac{\Gamma\left(\frac{3d}{2} + 2j - 4\right)}{\Gamma(4-d)} \frac{(\hat{x}_{12})^{2j}}{(-x_{12}^2)^{3d/2-4+2j}} S^{(7.4d)}[j]. \quad (\text{H.82})$$

Leading divergent term

The leading divergent term in the sum is

$$S^{(7.4d)}[j] = \frac{1}{\epsilon^2} \frac{1}{2} \frac{1}{2j+1} \left(2H_j + \frac{1}{j+1} - 1 \right) \frac{1}{(4\pi)^d} + \mathcal{O}(\epsilon^{-1}) \quad (\text{H.83})$$

and therefore

$$\begin{aligned} \langle \hat{\mathcal{O}}_j^A(x_1) \bar{\hat{\mathcal{O}}}_j^{\text{free}}(x_2) \rangle^{7.4d} &= -\frac{g^6 N \delta^{aa}}{\epsilon} \left(2H_j + \frac{1}{j+1} - 1 \right) \frac{2^{2j-7} \Gamma(2j+1)}{\pi^6} \frac{(\hat{x}_{12})^{2j}}{(-x_{12}^2)^{2+2j}} + \mathcal{O}(\epsilon^0) \\ &= \left(-\frac{1}{\epsilon} \frac{g^2 N}{4\pi^2} \left(2H_j + \frac{1}{j+1} - 1 \right) \right) \langle \hat{\mathcal{O}}_j \bar{\hat{\mathcal{O}}}_j \rangle^{(0)}. \end{aligned}$$

Thus the contribution to the anomalous dimension is

$$\gamma_j^{7.4d} = \frac{g^2 N}{4\pi^2} \left(2H_j + \frac{1}{j+1} - 1 \right). \quad (\text{H.84})$$

H.7.6 Diagram 7.4e

We consider the tree-level contraction of two operators $\hat{\mathcal{O}}_j^A$ as given by (H.43)

$$\begin{aligned} \langle \hat{\mathcal{O}}_j \bar{\hat{\mathcal{O}}}_j \rangle^{7.4e} &= \langle \hat{\mathcal{O}}_j^A \bar{\hat{\mathcal{O}}}_j^A \rangle = \frac{1}{4} f^{abc} f^{def} \left(1 + (-1)^j \right)^2 \sum_{k,k'=1}^j a_{jk}^{1/2} a_{jk'}^{1/2} \sum_{m,m'=1}^{k,k'} \binom{k}{m} \binom{k'}{m'} \\ &\quad \langle \hat{\partial}^{m-1} \hat{A}^a \hat{\partial}^{m'-1} \hat{A}^d \rangle \langle \hat{\partial}^{k-m} \phi^{12,b} \hat{\partial}^{j-k} \phi^{12,c} \hat{\partial}^{k'-m'} \bar{\phi}^{12,e} \hat{\partial}^{j-k'} \bar{\phi}^{12,f} \rangle. \end{aligned}$$

In Feynman-gauge we have

$$\langle \hat{A}^a \hat{A}^d \rangle = -z^\mu z^\nu \langle A_\mu^a A_\nu^d \rangle = z^2 \frac{\Gamma(h-1) \delta^{ad}}{4\pi^h (-x_{12}^2)^{h-1}} = 0, \quad (\text{H.85})$$

since $z^2 = 0$ and thus the diagram is zero in Feynman gauge.

H.8 Tree-Level Three-Point Function in the Limit

$$x_2 \rightarrow \infty$$

The 3-point correlator $\langle \mathcal{O} \tilde{\mathcal{O}} \hat{\mathcal{O}}_j \rangle$ simplifies considerably in the limit $x_2 \rightarrow \infty$. We will now explicitly evaluate the correlation function in this limit in order to extract the structure constant $C_{\mathcal{O} \tilde{\mathcal{O}}_j}$. Using the representation (7.14) and $a_{jk} = (-1)^j a_{j,j-k}$ we have

$$\begin{aligned} \langle \mathcal{O}(x_1) \tilde{\mathcal{O}}(x_2) \hat{\mathcal{O}}_j(x_3) \rangle &= \frac{g^6 \delta^{aa} \Gamma(h-1)^3}{8} \frac{1}{(4\pi^h)^3} \frac{1}{(-x_{12}^2)^{h-1}} \\ &\quad \sum_{k=0}^j a_{jk}^\nu \left(\hat{\partial}_3^k \frac{1}{(-x_{13}^2)^{h-1}} \hat{\partial}_3^{j-k} \frac{1}{(-x_{23}^2)^{h-1}} + \hat{\partial}_3^k \frac{1}{(-x_{23}^2)^{h-1}} \hat{\partial}_3^{j-k} \frac{1}{(-x_{13}^2)^{h-1}} \right). \end{aligned} \quad (\text{H.86})$$

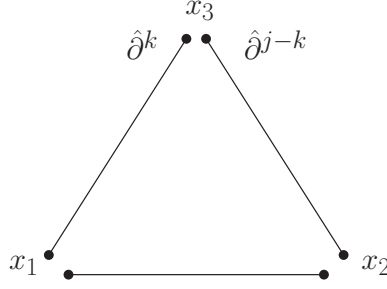


Figure H.1: 3-point function at tree-level

Multiplying this with the factors of x_{13}^2 and x_{23}^2 as given in (7.4) and taking the limit $x_2 \rightarrow \infty$ it is easy to see that the right-hand side can only be produced by the term in which all derivatives act on x_{13}^2 , i.e.

$$\lim_{x_2 \rightarrow \infty} (-x_{12}^2)^{h-1} (-x_{23}^2)^{h-1} \langle \mathcal{O} \tilde{\mathcal{O}} \hat{\mathcal{O}}_j \rangle = \frac{g^6 \delta^{aa}}{8} \frac{\Gamma(h-1)^3}{(4\pi^h)^3} (a_{jj}^\nu + a_{j0}^\nu) \hat{\partial}_3^j \frac{1}{(-x_{13}^2)^{h-1}}. \quad (\text{H.87})$$

Taking j even and using $a_{jk}^\nu = (-1)^j a_{j,j-k}^\nu$ as well as

$$a_{jj}^\nu = \sum_{m=0}^j c_{m_j}^\nu = C_j^\nu(1) = \binom{2\nu - 1 + j}{j}, \quad \nu = h - 3/2, \quad (\text{H.88})$$

performing the derivatives and setting $\nu = 1/2$ we find

$$\begin{aligned} \lim_{x_2 \rightarrow \infty} (-x_{12}^2)^{h-1} (-x_{23}^2)^{h-1} \langle \mathcal{O} \tilde{\mathcal{O}} \hat{\mathcal{O}}_j \rangle & \quad (\text{H.89}) \\ &= g^6 \delta^{aa} 2^{j-2} \frac{\Gamma(h-1)^3}{(4\pi^h)^3} \frac{\Gamma(j+h-1)}{\Gamma(h-1)} \binom{2\nu - 1 + j}{j} \frac{(\hat{x}_{13})^j}{(-x_{13}^2)^{j+h-1}}. \end{aligned}$$

Comparing with (7.4)

$$\lim_{x_2 \rightarrow \infty} (-x_{12}^2)^{h-1} (-x_{23}^2)^{h-1} \langle \mathcal{O} \tilde{\mathcal{O}} \hat{\mathcal{O}}_j \rangle = C_{\mathcal{O} \tilde{\mathcal{O}}_j} \frac{(\hat{x}_{13})^j}{(-x_{13}^2)^{j+h-1}}, \quad (\text{H.90})$$

we thus get

$$C_{\mathcal{O} \tilde{\mathcal{O}}_j} = g^6 \delta^{aa} 2^{j-2} \frac{\Gamma(h-1)^2}{(4\pi^h)^3} \Gamma(h+j-1) \binom{2\nu - 1 + j}{j}. \quad (\text{H.91})$$

This expression is valid for a scalar theory in general integer dimension d , e.g. (ϕ^3) -theory in six dimensions, up to the factors of $\delta^{aa}/8$ that is due to the trace over the gauge group. For $d = 4$ we find

$$C_{\mathcal{O} \tilde{\mathcal{O}}_j} = g^6 \delta^{aa} \frac{2^{j-8}}{\pi^6} \Gamma(j+1) \quad (\text{H.92})$$

in agreement with (7.56).

H.9 Anomalous Dimension from Three-Point Functions

As a check of the relative factors and signs between the diagrams we read off the anomalous dimension from the three-point function. After the cancellations between diagrams 7.7b, 7.7a, 7.7h, 7.7e resp. the cancellations between diagrams 7.7a, 7.7c, 7.7i, 7.7g as shown in section 7.6.3 resp. 7.6.4 the only divergent contributions that remain are

$$\begin{aligned} \langle \mathcal{O}\tilde{\mathcal{O}}\hat{\mathcal{O}}_j \rangle^{(1)} &= \frac{1}{2} \langle \mathcal{O}\tilde{\mathcal{O}}\hat{\mathcal{O}}_j \rangle^{7.7b} + \frac{1}{2} \langle \mathcal{O}\tilde{\mathcal{O}}\hat{\mathcal{O}}_j \rangle^{7.7c} + \langle \mathcal{O}\tilde{\mathcal{O}}\hat{\mathcal{O}}_j \rangle^{7.7k} + \langle \mathcal{O}\tilde{\mathcal{O}}\hat{\mathcal{O}}_j \rangle^{7.7l} + \langle \mathcal{O}\tilde{\mathcal{O}}\hat{\mathcal{O}}_j \rangle^{7.7d+7.7g} \\ &= \langle \mathcal{O}\tilde{\mathcal{O}}\hat{\mathcal{O}}_j \rangle^{7.7b} + 2 \langle \mathcal{O}\tilde{\mathcal{O}}\hat{\mathcal{O}}_j \rangle^{7.7k} + \langle \mathcal{O}\tilde{\mathcal{O}}\hat{\mathcal{O}}_j \rangle^{7.7d+7.7g} + \mathcal{O}(\epsilon^0), \end{aligned}$$

where we used the symmetry of the diagrams under $p \rightarrow -p$ for j even in the last step. Using

$$c_{jj}^{1/2} = 2^{1-j} \frac{\Gamma(2j)}{\Gamma(j)\Gamma(j+1)}, \quad b_j(4-d/2, 1) = (-1)^j i \frac{\Gamma(j)}{\Gamma(j+1)} \frac{1}{(4\pi)^2} + \mathcal{O}(\epsilon),$$

we find that the divergent part of (7.76) is

$$\langle \mathcal{O}\tilde{\mathcal{O}}\hat{\mathcal{O}}_j \rangle^{7.7b} = i^j g^8 N \delta^{aa} \frac{1}{\epsilon} \frac{\Gamma(2j)}{\Gamma(j+1)\Gamma(j+1)} \frac{\hat{p}^j}{(-p^2)^{5-d}} \frac{1}{(4\pi)^d} + \mathcal{O}(\epsilon^0).$$

The divergent part of (7.75) is

$$\langle \mathcal{O}\tilde{\mathcal{O}}\hat{\mathcal{O}}_j \rangle^{7.7d+7.7g} = -i^j g^8 N \delta^{aa} \frac{1}{\epsilon} \frac{\Gamma(2j)}{\Gamma(j+1)\Gamma(j+1)} \frac{1}{(j+1)} \frac{\hat{p}^j}{(-p^2)^{5-d}} \frac{1}{(4\pi)^d} + \mathcal{O}(\epsilon^0).$$

Furthermore, the divergent part of diagram 7.7k is

$$2 \langle \mathcal{O}\tilde{\mathcal{O}}\hat{\mathcal{O}}_j \rangle^{7.7k} = i^j g^8 N \delta^{aa} \frac{1}{\epsilon} \frac{\Gamma(2j)}{\Gamma(j+1)\Gamma(j+1)} (H_j + H_{j+1} - 1) \frac{\hat{p}^j}{(-p^2)^{5-d}} \frac{1}{(4\pi)^d} + \mathcal{O}(\epsilon^0).$$

Summing up these contributions we find

$$\begin{aligned} \langle \mathcal{O}\tilde{\mathcal{O}}\hat{\mathcal{O}}_j \rangle^{(1)} &= i^j g^8 N \delta^{aa} \frac{\Gamma(2j)}{\Gamma(j+1)\Gamma(j+1)} \left(\frac{H_j}{\epsilon} \right) \frac{\hat{p}^j}{(-p^2)^{5-d}} \frac{1}{(4\pi)^d} + \mathcal{O}(\epsilon^0) \\ &= \left(-\frac{g^2 N}{(4\pi)^h} \frac{4H_j}{\epsilon} \right) \langle \mathcal{O}(p) \mathcal{O}(-p) \bar{\mathcal{O}}_j(0) \rangle^{(0)} + \mathcal{O}(\epsilon^0) \\ &= \left(-\frac{Z_j^{(1)}}{\epsilon} \right) \langle \mathcal{O}(p) \bar{\mathcal{O}}(-p) \bar{\mathcal{O}}_j(0) \rangle^{(0)} + \mathcal{O}(\epsilon^0), \end{aligned}$$

where we used the tree-level three-point function (7.54) and can thus read off the anomalous dimension

$$\gamma_j^{(1)} = \left(\frac{g^2 N}{4\pi^2} \right) 2H_j, \quad (\text{H.93})$$

which is exactly what we expect and thus serves as a check for the relative factors of the divergent diagrams.

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Software

- This thesis was typeset in LaTeX.
- For the calculations Mathematica 8.0 by *Wolfram Research* was used including the package [321].
- Graphics were compiled using JaxoDraw and Mathematica 8.0.

Selbständigkeitserklärung

Hiermit erkläre ich, die vorliegende Arbeit selbständig ohne fremde Hilfe verfasst und nur die angegebene Literatur und Hilfsmittel verwendet zu haben.

Konstantin Wiegandt
Berlin, den 3. Mai 2012