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–
localizing Conditional AutoRegressive Expectiles

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BERLIN

ECONOMIC RISK

SFB 649

lCARE - localizing Conditional AutoRegressive Expectiles[*]

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Abstract

We account for time-varying parameters in the conditional expectile based value at risk (EVaR) model. EVaR appears more sensitive to the magnitude of portfolio losses compared to the quantile-based Value at Risk (QVaR), nevertheless, by fitting the models over relatively long ad-hoc fixed time intervals, research ignores the potential time-varying parameter properties. Our work focuses on this issue by exploiting the local parametric approach in quantifying tail risk dynamics. By achieving a balance between parameter variability and modelling bias, one can safely fit a parametric expectile model over a stable interval of homogeneity. Empirical evidence at three stock markets from 2005- 2014 shows that the parameter homogeneity interval lengths account for approximately 1-6 months of daily observations. Our method outperforms models with one-year fixed intervals, as well as quantile based candidates while employing a time invariant portfolio protection (TIPP) strategy for the DAX portfolio. The tail risk measure implied by our model finally provides valuable insights for asset allocation and portfolio insurance.

JEL classification: C32, C51, G17

Keywords: expectiles, tail risk, local parametric approach, risk management

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1   Introduction

Value at risk (VaR) is commonly used to measure the downside risk in finance, especially in portfolio risk management. Given a predetermined probability level, VaR evaluates the potential maximum loss for the targeted portfolio value; statistically it represents the quantile of the portfolio loss distribution, see Jorion (2000). Although it is straightforward to understand the VaR concept, it has been recently criticized. VaR lacks the property of sub-additivity, that is, under the VaR risk measure, the risk of a diversified portfolio is larger than the sum of each individual asset risk, which in turn contradicts the common wisdom of diversification. In light of this, Artzner et al. (1999) proposed the expected shortfall (ES) to measure portfolio risk, i.e., the expected loss below a given threshold (e.g., VaR) given the risk probability level.

Another undesirable aspect of the VaR measure is its insensitivity to the magnitude of the portfolio loss. Kuan et al. (2009) provide an example where, under a given probability level, the potential downside risk changes under different tail loss distributions while the corresponding VaR remains the same. Since VaR merely depends on the probability value and neglects the size of the downside loss, Kuan et al. (2009) proposed a downside risk measure, the expectile-based Value at Risk (EVaR), a more sensitive measure of the magnitude of extreme losses than the conventional quantile-based VaR (QVaR). The expectile at given level is estimated by minimizing the asymmetric weighted least squared errors, exploring the method proposed by Newey and Powell (1987). The expectile level is the relative cost of the expected margin shortfall, explained as the level of prudentiality. EVaR may be interpreted as a flexible QVaR (Kuan et al., 2009), because of the one-to-one mapping between quantiles and expectiles for a given loss distribution, see Efron (1991), Jones (1994) and Yao and Tong (1996).

Models based on the expectile risk measure framework have thus been proposed, see e.g. Taylor (2008) and Kuan et al. (2009) after Engle and Manganelli (2004) successfully initialize the conditional autoregressive framework to model VaR. Kuan et al. (2009) moreover extend the EVaR to conditional EVaR and propose various Conditional AutoRegressive Expectile (CARE) specifications as well as establishing the asymptotic results of Newey and Powell (1987) to allow for stationary and weakly dependent data.

Potential time-varying parameters resulting from the dynamic state of the economic and financial environment are however barely analysed. This is where this research comes into play. We focus on incorporating and reacting to potential structural breaks in order to estimate the tail risk measure.

The proposed local parametric approach (LPA) targets a parametric stable model in an adaptively chosen interval. The essential idea of the LPA is to find the longest interval length guaranteeing a relatively small modelling bias, see e.g. Spokoiny (1998) and Spokoiny (2009). The main advantage of the approach is the achievement of a balance between modelling bias and parameter variability. This approach has been successfully applied in many research areas: Čížek et al. (2009) analyse the GARCH$(1, 1)$ models, Chen et al. (2010) explore it to forecast realised volatilities, Chen and Niu (2014) predict the interest rate term structure, whereas Härdle et al. (2015) utilise it successfully in high frequency time series modelling and forecasting.

In this paper, we locally estimate the expectile risk measure rather than following a traditional approach of assuming constant CARE parameters. Based on one of the conditional expectile model specifications in Kuan et al. (2009) and assuming that the error term follows the asymmetric normal distribution, Gerlach et al. (2012) and Gerlach and Chen (2014), we dynamically estimate the time-varying CARE parameters over potentially varying intervals of homogeneity. The desired interval of homogeneity is found by multiple testing the null hypothesis that the model parameters are constant. The resulting time-varying interval lengths indicate potential structural changes in tail risk assessment.

It is worth mentioning that several articles consider the dynamic window selection of time-varying parameters, Pesaran and Timmermann (2007) and Inoue et al. (2014), or introduce varying-coefficient models for tail risk measure estimation, Honda (2004), Kim (2007) and Cai and Xu (2008). Most of the research however mainly explores non-parametric approaches or considers polynomial splines to estimate the conditional quantile. A state space signal extraction algorithm to iteratively formulate quantile and non-parametrically obtain the quantile and expectile has been applied by De Rossi and Harvey (2009), whereas Xie et al. (2014) develop a nonparametric varying-coefficient approach to model the expectile-based value at risk.

In our research it turns out that the proposed localised conditional autoregressive expectile (lCARE) model successfully captures tail risk dynamics by taking the time-varying parameter characteristics and potential market condition structure changes into account while measuring the risk associated with tail events. Based on empirical results, we find that at the 5% expectile level the typical interval lengths that strike a balance between bias and variability in daily time series include approximately 100 days. At the lower, 1% expectile level, the selected interval lengths range roughly between 40-60 days. The resulting time-varying expectile series allows us to consider the dynamics of other tail risk measures, most prominently quantiles or expected shortfall.

The methodology presented here is successfully applied to a portfolio insurance strategy for the DAX index portfolio. A portfolio insurance strategy is designed to guarantee a minimum value for the asset portfolio over a selected investment horizon, where the downside risk can be reduced and controlled while investors can participate in the potential gains. The proportion of the value invested into the risky asset (here the DAX portfolio), denoted as the multiplier, is directly related to the estimated tail risk measure. A standard approach keeps the multiplier fixed regardless of the market conditions, Estep and Kritzman (1988), Hamidi et al. (2014), whereas we exercise the protection strategy with the dynamic tail risk measure implied by the lCARE model. Comparison to the benchmarks - one-year fixed rolling window CARE estimation and quantile-based (CAViaR) estimation - reveals that the lCARE model presents a striking outperformance in portfolio insurance.

This paper is structured as follows: firstly, the data is presented in section 2 whereas section 3 introduces the lCARE model based on the CARE model setup and the local parametric approach in the tail risk modelling. Section 4 presents the empirical results and finally, section 5 concludes.

2 Data

In risk modelling we consider three stock markets and focus on the dynamics of the representative index time series, namely, DAX30, FTSE100 and S&P500 series. Daily
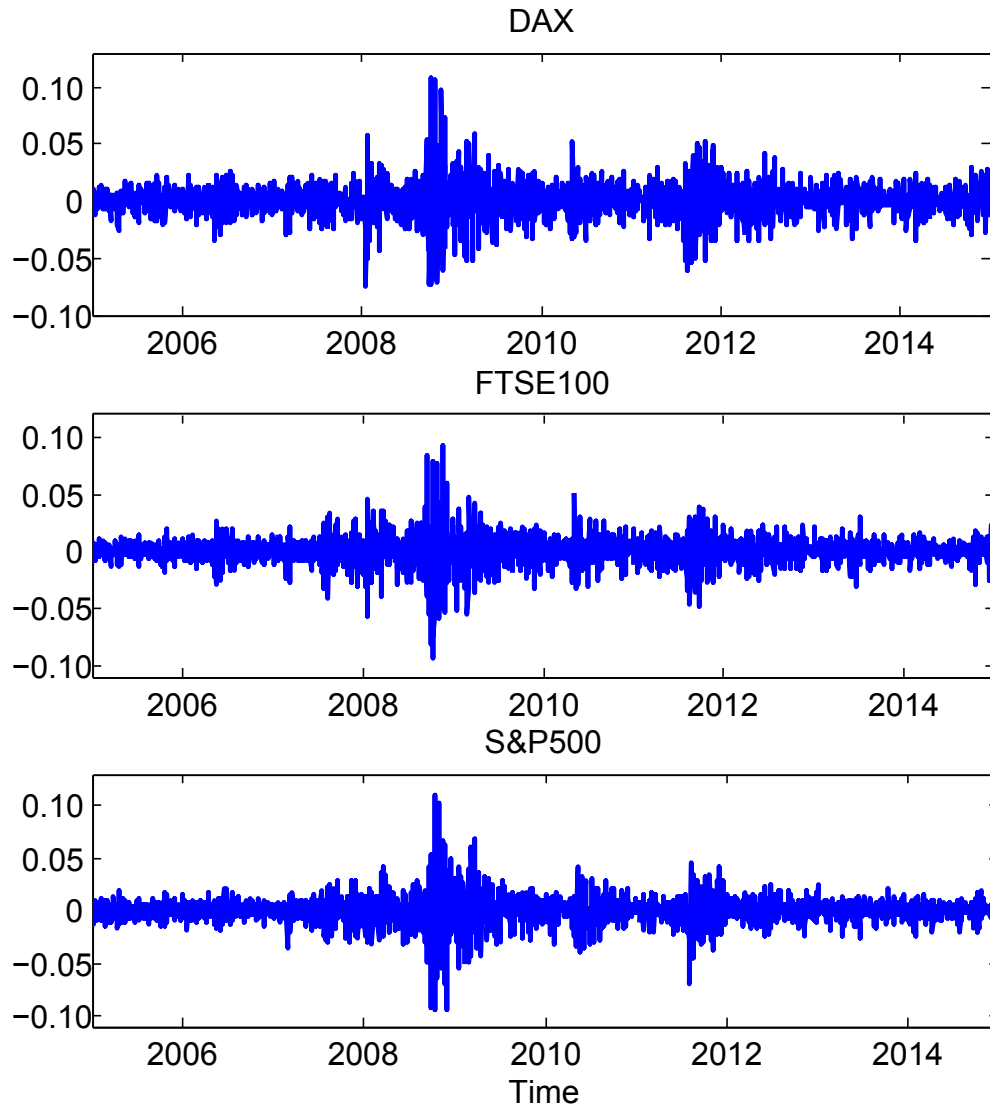
Figure 1: Selected index return time series from 3 January 2005 to 31 December 2014 (2608 trading days).　　　　　　　　　　　　　　　Q LCARE_Index_Returns

index returns are obtained from Datastream and our data cover the period from 3 January 2005 to 31 December 2014, in total 2608 trading days. The daily returns evolve similarly across the selected markets and all present relatively large variations during the financial crisis period from 2008-2010, see, e.g., Figure 1. Although the return time series are nearly zero-mean with slightly pronounced skewness values, all present comparatively high kurtosis, see, e.g., Table 1 that collects the summary statistics. Noting that the results and the corresponding Matlab codes can be found in the folder at https://github.com/QuantLet/lCARE-BTU-HUB and http://quantlet.de/d3/ia/.

Index	Mean	Median	Min	Max	Std	Skew.	Kurt.
DAX	0.0003	0.0007	-0.0743	0.1080	0.0137	0.0357	10.1654
FTSE100	0.0001	0.0001	-0.0927	0.0938	0.0120	-0.1498	11.9066
S&P500	0.0002	0.0005	-0.0947	0.1096	0.0127	-0.3364	14.5131

Table 1: Descriptive statistics for the selected index return time series from 3 January 2005 to 31 December 2014 (2608 trading days): mean, median, minimum (Min), maximum (Max), standard deviation (Std), skewness (Skew.) and kurtosis (Kurt.).

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3 Localized Conditional Autoregressive Expectiles

Understanding tail risk plays an essential role in asset pricing, portfolio allocation, investment performance evaluation and external regulation. Tail event dynamics is commonly assessed through the employment of parametric, semi-parametric or nonparametric techniques, see, e.g., Taylor (2008). Our paper contributes to the econometric literature by localizing parametric CARE specifications by Kuan et al. (2009) and, while modelling tail risk, explores the effects of potential market structure changes. In this section we summarise the current research on expectile-based risk management and conduct a detailed empirical study concerning the parameter dynamics. The results motivate the usage of the local parametric approach by Spokoiny (1998) that is presented at the end of the chapter. The localized Conditional Autoregressive Expectiles (lCARE) model provides a statistical and applicable framework to analyse the downside risk in quantitative finance.

3.1 Conditional Autoregressive Expectile Model

Tail risk exposure can successfully be captured by an expectile-based risk measure, in contrast to modelling risk solely using Value at Risk (VaR). Despite being the most commonly used (not coherent) tail risk measure, VaR exhibits insensitivity to the potential magnitude of the loss, see, e.g., Acerbi and Tasche (2002), Taylor (2008). After the conditional autoregressive value at risk (CAViaR) model by Engle and Manganelli (2004) was proposed, Taylor (2008) found that VaR, based on the conditional autoregressive expectile model, is more sensitive to the tail risk distribution. Finally, the conditional autoregressive expectile (CARE) model specifications by Kuan et al. (2009) directly model the

return time series and extend the asymmetric least square estimation method by Newey and Powell (1987) to analyse stationary but weakly dependent time series data.

The CARE model specifications provide insights into the dynamics of financial data and offer valuable economic interpretation. Although quantiles and expectiles belong to M-quantiles, see, e.g., Jones (1994), the implications in risk assessment differ considerably. VaR is a zero-moment whereas expectile is a first-moment tail risk measure, thus in the former case the proportion of asymmetric downside and upside quantile level is determined only by the ratio between downside and upside probabilities. Expectiles measure the proportion of asymmetric downside and upside expectile level while capturing the ratio between the expected marginal shortfall. Equivalently, the potential cost of more extreme losses and the opportunity cost due to the expected marginal overcharge is captured by expectiles. The CARE specifications furthermore accommodate stylised facts of the return time series, such as weak serial dependence, or volatility heteroskedasticity. Accommodating asymmetric effects on the tail expectiles of the positive and negative returns becomes essential in interpreting tail risk dynamics.

Based on the dynamics of an observed return time series $y = \{y_t\}_{t=1}^n$, the CARE framework is introduced as

$$y_t = e_{t,\tau} + \varepsilon_{t,\tau} \tag{1}$$

$$e_{t,\tau} = \alpha_{0,\tau} + \alpha_{1,\tau} y_{t-1} + \alpha_{2,\tau} \left(y_{t-1}^+ \right)^2 + \alpha_{3,\tau} \left(y_{t-1}^- \right)^2 \tag{2}$$

where $e_{t,\tau}$ and $\varepsilon_{t,\tau}$ denote the expectile and the error term at level $\tau \in (0,1)$ and time t, respectively. $y_{t-1}^+ = \max \{y_{t-1}, 0\}$ and $y_{t-1}^- = \min \{y_{t-1}, 0\}$ denote the positive or negative observed one-period lagged returns, respectively.

Generally, the τ-level expectile $e_{t,\tau}$ in Equation (2) can be estimated by minimising the asymmetric least square (ALS) loss function

$$\sum_{t=2}^n |\tau - \mathbf{I}\left(y_t \leq e_{t,\tau}\right)| \left(y_t - e_{t,\tau}\right)^2 \tag{3}$$

with $\mathbf{I}\left(\cdot\right)$ denoting the indicator function.

Within the CARE framework, Gerlach and Chen (2014) and Gerlach et al. (2012) assume

that the error term $\varepsilon_{t,\tau}$ follows the asymmetric normal distribution (AND). We assume that, conditional on the information set $\mathcal{F}_{t-1}$, the data process follows an asymmetric normal distribution $\mathrm{AND}\left(\mu, \sigma_{\varepsilon_\tau}^2, \tau\right)$ with pdf:

$$f\left(y_t - \mu \mid \mathcal{F}_{t-1}\right) = \frac{2}{\sigma_{\varepsilon_\tau}} \left(\sqrt{\frac{\pi}{|\tau - 1|}} + \sqrt{\frac{\pi}{\tau}}\right)^{-1} \exp\left\{-\eta_\tau\left(\frac{y_t - \mu}{\sigma_{\varepsilon_\tau}}\right)\right\} \tag{4}$$

where $\eta_\tau\left(u\right) = |\tau - \mathbf{I}\left\{u \leq 0\right\}| \, u^2$ is the employed check function, μ represents the expectile value to be estimated and $\sigma_{\varepsilon_\tau}^2$ denotes the variance of the error term. Maximising the likelihood equation with respect to μ for the distribution (4) is asymptotically equivalent to minimising the asymmetric least square loss function (3).

Conditional on the information set $\mathcal{F}_{t-1}$ up to observation $(t - 1)$, the expectile $e_{t,\tau}$ includes a lagged return component and it mimics several financial series features, namely, the volatility clustering and potential asymmetric magnitude effects. Note that at level $\tau = 0.5$, the expectile equals to the mean value. With specification (3), the parameter vector finally contains five elements, namely $\theta_\tau = \left(\alpha_{0,\tau}, \alpha_{1,\tau}, \alpha_{2,\tau}, \alpha_{3,\tau}, \sigma_{\varepsilon_\tau}^2\right)^\top$.

In the specification (2), the parameter $\alpha_{1,\tau}$ indirectly measures the persistence level in the conditional expectile tail through the lagged return series. Since the parameters $\alpha_{2,\tau}$ and $\alpha_{3,\tau}$ potentially differ, (2) accounts for the asymmetric effects of the positive and negative lagged squared returns on the conditional tail expectile magnitude. This similarly mimics the leverage effect associated with volatility modelling, where negative (positive) returns are followed by relatively larger (lower) variability. Under the working assumption that the expectile tail dynamics can be well approximated over a given data interval by a model with constant parameters, it suffices to include one-lag process dynamics.

The resulting quasi log likelihood function for observed data $\mathcal{Y} = \left\{y_1, \ldots, y_n\right\}$ over a fixed interval I is given by

$$\ell_I\left(\mathcal{Y}; \theta_\tau\right) = \sum_{t \in I} \log f\left(y_t - e_{t,\tau} \mid \mathcal{F}_{t-1}\right) \tag{5}$$

The quasi maximum likelihood estimate (QMLE) for the CARE parameter is then obtained through

$$\widetilde{\theta}_{I,\tau} = \arg\max_{\theta_\tau \in \Theta} \ell_I\left(\mathcal{Y}; \theta_\tau\right) \tag{6}$$

over a right-end fixed interval $I = [t_0 - m, t_0]$ of $(m + 1)$ observations at observation t_0.

3.2   Parameter Dynamics

The idea behind the local parametric approach (LPA) is to find the optimal (in-sample) data interval over which one can safely fit a parametric model with time-invariant parameters. This optimal interval, the so-called interval of homogeneity, is selected among pre-specified right-end interval candidates. Over the resulting optimal data interval the proposed lCARE model allows the structure break properties of the expectile dynamics to be captured and therefore it can be used for expectile estimation. In this part we implement a fixed rolling window exercise in order to provide empirical evidence on the time-varying characteristics of the CARE estimates, as well as to select the 'true' parameter constellation used in the LPA simulation. At the end we discuss the estimation quality of the QMLE (6).

Dynamics and Distributional Characteristics

In the analysis of the selected (daily) stock market indices presented in Section 2, we consider different interval lengths (e.g., 20, 60, 125 and 250 observations) and provide the corresponding estimates. Shorter intervals will, in practice, result in larger variation as compared to longer ones, whereas the modelling bias behaves in the opposite direction: it is quite enlarged in the latter case. The distributional features of CARE parameters are moreover studied through two expectile level cases, namely $\tau = 0.05$ and $\tau = 0.01$. The following rolling window estimation exercise provides valuable insights into the expectile (distribution) dynamics.

Parameter estimates are more volatile while fitting the data over shorter intervals with the modelling bias comparatively smaller as compared to schemes using longer intervals, see, e.g., Figures 2 and 3. Both figures namely present the estimated parameters $\tilde{\alpha}_{1,0.05}$ and $\tilde{\alpha}_{1,0.01}$ in the rolling window exercise across the three selected stock market indices from 2 January 2006 to 31 December 2014 at fixed expectile levels $\tau = 0.05$ and $\tau = 0.01$. The upper panel shows the estimation results with 20 observations and the lower panel with 250 observations.

The above mentioned properties are furthermore supported by the density estimates of the parameters involved, i.e., parameters belonging to the three stock market indices. Kernel density plots (using, e.g., a Gaussian kernel with optimal bandwidth) of estimated parameters show that shorter intervals lead to more variability and vice versa. For the sake of brevity, here we refrain from showing the density estimates. It is further verified that with fewer observations, such as including one month data (20 observations), the parameter density is distinguished from the estimates based on longer sample intervals, such as one year of data.
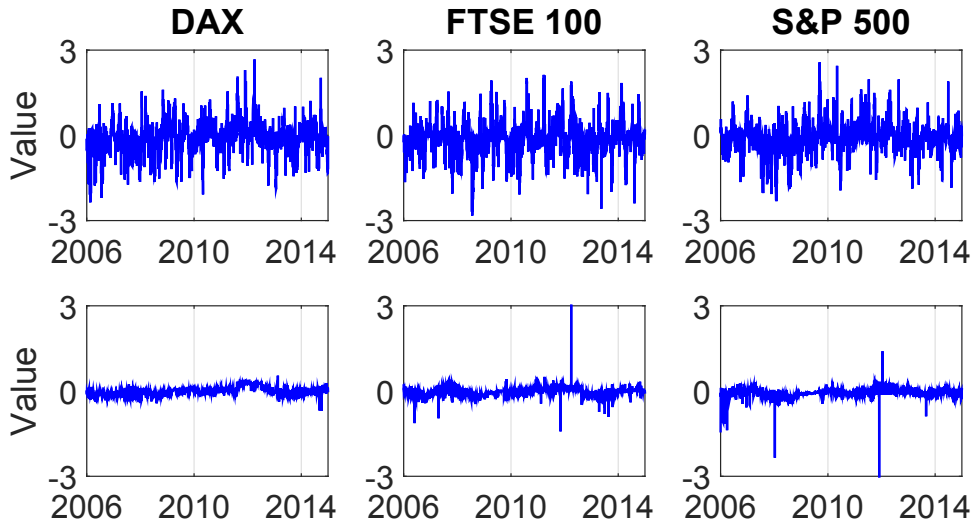
Figure 2: Estimated parameter $\widetilde{\alpha}_{1,0.05}$ across the three selected stock markets from 2 January 2006 to 31 December 2014, with 20 (upper panel) and 250 (lower panel) observations used in the rolling window exercise at fixed expectile level $\tau = 0.05$.
LCARE_Estimation_Rolling_005

Descriptive Statistics

The lCARE testing framework demands a set of simulated critical values that rely on reasonable parameter constellations. A data driven approach to select the 'true' parameters here is based on a sample window covering one year, i.e., 250 observations as the target parameters. Descriptive statistics of the resulting estimated CARE parameters given the ad hoc selected window length of one year, i.e., 250 observations, from 2 January 2006 to 31 December 2014 (2348 trading days) is provided in Table 2. We pool the estimates together for the three market indices, and label the first quartile as 'low', the mean as 'mid' and the third quartile as 'high' at two expectile levels, $\tau = 0.05$ and $\tau = 0.01$. For a

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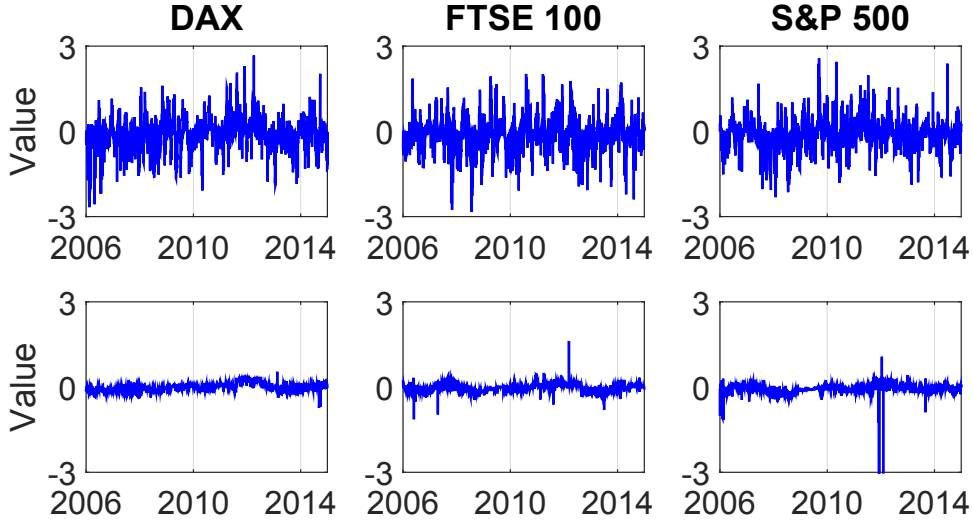
Figure 3: Estimated parameter $\widetilde{\alpha}_{1,0.01}$ across the three selected stock markets from 2 January 2006 to 31 December 2014, with 20 (upper panel) and 250 (lower panel) observations used in the rolling window exercise at fixed expectile level $\tau = 0.01$.

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LCARE_Estimation_Rolling_001
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given expectile level τ, there are three potential 'true' parameter constellations, i.e., the parameters that are most likely to be found in practice.

Estimation Quality

The estimation quality of the quasi-maximum likelihood approach is addressed here. Denote the pseudo *true* parameter vector as θ_τ^* at expectile level τ, the quality of estimating θ_τ^* by quasi-maximum likelihood estimator (QMLE) $\widetilde{\theta}_{I,\tau}$ given in (6) is measured in terms of the Kullback-Leibler divergence

$$\mathsf{E}_{\theta_\tau^*} \left| \ell_I \left(\mathcal{Y}; \widetilde{\theta}_{I,\tau} \right) - \ell_I \left(\mathcal{Y}; \theta_\tau^* \right) \right|^r \leq \mathcal{R}_r \left(\theta_\tau^* \right) \tag{7}$$

with $\mathcal{R}_r \left(\theta_\tau^* \right)$ denoting the risk bound, see, e.g., Mercurio and Spokoiny (2004) and Spokoiny (2009). In practice the modest risk power $r = 0.5$ leads to relatively shorter intervals of homogeneity as compared with the conservative risk case with $r = 1$. According to the pseudo true parameter vector, we simulate thousand time series of the CARE specification and take the largest average value of the (r-th power) difference between the respective log-likelihood values, see equation (7), as the corresponding risk bound. Note

	$\tau = 0.05$			$\tau = 0.01$		
	Low	Mid	High	Low	Mid	High
$\widetilde{\alpha}_{0,\tau}$	-0.01514	-0.00998	0.00000	-0.02892	-0.02323	0.00000
$\widetilde{\alpha}_{1,\tau}$	-0.01034	0.05234	0.12149	-0.00298	0.10132	0.12637
$\widetilde{\alpha}_{2,\tau}$	-0.31360	-0.85700	0.00421	-0.14472	-2.43912	0.00008
$\widetilde{\alpha}_{3,\tau}$	-0.06366	0.56274	0.17589	-0.00037	2.63032	0.03325
$\widetilde{\sigma}^2_{\varepsilon_\tau}$	0.00001	0.00005	0.00007	0.00001	0.00040	0.00004

Table 2: Descriptive statistics of estimated CARE parameters. All estimated CARE parameters based on the window covering one year, i.e., 250 observations, for the three stock market indices from 2 January 2006 to 31 December 2014 (2348 trading days) are pooled together for the two expectile levels $\tau = 0.05$ and $\tau = 0.01$, respectively. We label the first quartile as 'low', the mean as 'mid' and the third quartile as 'high'. 🔍LCARE_Parameter_Dynamics_Quartiles

that the considered interval candidates in this simulation covered

$$\{20, 25, 31, 39, 49, 61, 76, 95, 119, 149, 186, 250\}$$

observations - see the selection details in the following sub-section.

The simulated risk bound $\mathcal{R}_r (\theta^*_\tau)$ according to equation (7) across different setups is given in Table 3. We consider the modest ($r = 0.50$) and the conservative ($r = 1.00$) risk case for two expectile levels, $\tau = 0.05$ and $\tau = 0.01$. The risk bounds are obtained by Monte Carlo simulation for each selected parameter vector corresponding to Table 2 where we label the first quartile of estimated parameters as 'low', the mean as 'mid' and the third quartile as 'high'. It turns out that the risk bounds in the conservative risk case ($r = 1$) are relatively larger than the bounds obtained in the modest risk case with $r = 0.5$.

	$\tau = 0.05$			$\tau = 0.01$		
	Low	Mid	High	Low	Mid	High
$r = 0.5$	0.24	0.33	0.25	0.38	0.38	0.15
$r = 1.0$	2.40	4.62	2.75	5.90	5.81	1.15

Table 3: Risk bound $\mathcal{R}_r (\theta^*_\tau)$ given two expectile levels, $\tau = 0.05$ and $\tau = 0.01$. We consider the modest ($r = 0.50$) and the conservative ($r = 1.00$) risk case. The risk bounds are obtained by Monte Carlo simulation for each selected parameter vector from Table 2 where we label the first quartile of estimated parameters as 'low', the mean as 'mid' and the third quartile as 'high'. 🔍LCARE_Risk_Bound_Results

Key empirical results from the presented fixed rolling window exercise can be summarised

as follows: (a) with different estimation sample windows, a tradeoff between the modelling bias and parameter variability exists, (b) the estimated parameter characteristics as well as the estimation quality results demand a method that successfully accommodates time-varying parameters, (c) data intervals covering 60 to 250 observations may be suitable in providing a good balance between the bias and variability, (d) it is reasonable practice to select three data-driven 'true' parameter constellations for each expectile level in daily risk management. Motivated by these findings, we now introduce some more details of lCARE.

3.3 Local Parametric Approach

How to account for the time-varying characteristics of CARE parameters in tail risk modelling? We utilize the aforementioned local parametric approach (LPA), which has been gradually introduced to modelling time series data in econometrics. The essential idea of lCARE is to find the longest interval over which the CARE model can be approximated by constant parameters.

This interval is labelled as the interval of homogeneity. By a sequential testing procedure, we adaptively select the interval of homogeneity among interval candidates. After the corresponding critical values of the sequential test have been simulated by employing a Monte Carlo method, the adaptive parameter estimate at every time point (i.e., trading day) is selected, based on the test outcome. It is worth noting that at each observation, the associated critical values curve is selected based on a data-driven approach.

Interval Selection

There are many possible candidates for these intervals of homogeneity. To alleviate the computational burden, we choose $(K + 1)$ nested intervals of length $n_k = |I_k|$, $k = 0, \ldots, K$, i.e., $I_0 \subset I_1 \subset \cdots \subset I_K$. Interval lengths are assumed to be geometrically increasing with $n_k = \left\lceil n_0 c^k \right\rceil$. Based on the empirical results reported above, it is reasonable to select $(K + 1) = 12$ intervals, starting with 20 observations (one trading month) and for convenience to end with 250 observations (one trading year), i.e., we consider the set

$$\{20, 25, 31, 39, 49, 61, 76, 95, 119, 149, 186, 250\}.$$

where within the initial interval I_0 the local CARE model with a constant parameter fits reasonably well. This shortest interval is therefore assumed to be homogeneous.

Local Change Point Detection Test

Based on the selected nested intervals, we utilize a sequential testing procedure to adaptively find the homogeneous interval at a fixed data point t_0. The initial interval I_0 is assumed to be homogeneous. Consider now $J_k = I_k \setminus I_{k-1}$, and sequentially conduct the test, i.e., over interval index steps $k = 1, \ldots, K$. The hypotheses of the test at step k read as

H_0 : parameter homogeneity of I_k vs H_1 : $\exists$ change point within $J_k = I_k \setminus I_{k-1}$.

The test statistics is

$$T_{k,\tau} = \sup_{s \in J_k} \left\{ \ell_{A_{k,s}} \left(\mathcal{Y}, \widetilde{\theta}_{A_{k,s},\tau} \right) + \ell_{B_{k,s}} \left(\mathcal{Y}, \widetilde{\theta}_{B_{k,s},\tau} \right) - \ell_{I_{k+1}} \left(\mathcal{Y}, \widetilde{\theta}_{I_{k+1},\tau} \right) \right\} \tag{8}$$

where $A_{k,s} = [t_0 - n_{k+1}, s]$ and $B_{k,s} = (s, t_0]$ utilize only part of the observation within the interval I_{k+1}. Since the change point position is unknown, we test every point $s \in J_k$.

The algorithm at step k is visualized in Figure 4. Assuming that the null of homogeneity of interval I_{k-1} has not been rejected, the testing procedure at step k tests for the homogenity of I_k. Since the position of a change point within $J_k = I_k \setminus I_{k-1}$ is unknown, the test statistic is calculated based on all points $s \in J_k$, i.e. $s \in (t_0 - n_{k-1}, t_0 - n_k]$, utilizing data from I_{k+1}. Compute the sum of the log-likelihood values over the sample interval $A_{k,s} = [t_0 - n_{k+1}, s]$ (dotted area) and $B_{k,s} = (s, t_0]$ (solid area) and subtract the log-likelihood value over I_{k+1}. The likelihood ratio test statistics $T_{k,\tau}$ at each predetermined expectile level τ is then determined by (8).

The test statistics (8) at every step k is compared to the corresponding (simulated) critical value $\mathfrak{z}_{k,\tau}$, for a given expectile and significance level at fix point t_0. Then the adaptive estimate is obained by $\widehat{\theta}_\tau = \widetilde{\theta}_{I_{\widehat{k}},\tau}$, with $\widehat{k} = \max_{k \leq K} \{k : T_{\ell,\tau} \leq \mathfrak{z}_{\ell,\tau}, \ell \leq k\}$. Here the index and the length of the interval of homogeneity are denoted by $\widehat{k}$ and $n_{\widehat{k}}$, respectively. If the null is already rejected at the interval I_1, $\widehat{k} = 0$ and similarly, if the null has not been rejected yet up to I_K, $\widehat{k} = K$.
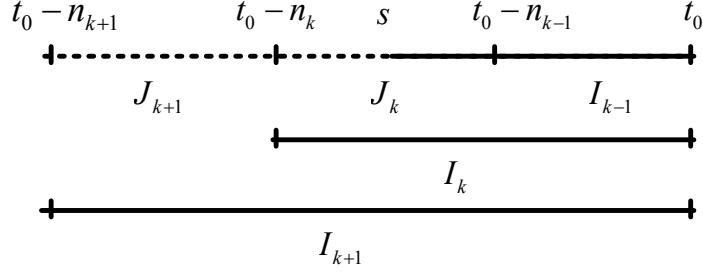
Figure 4: Sequential testing for parameter homogeneity in interval I_k with length n_k ending at fixed time point t_0

Critical Values

The critical value sequence $\mathfrak{z}_{k,\tau}, k = 1, \ldots, K$ essentially controls the threshold values of the likelihood ratio test statistic (8). The true distribution of the test statistics is unknown and thus we resort to simulate the critical values. Critical values are calculated through the propagation condition at each step $k = 1, \ldots, K$

$$\mathsf{E}_{\theta^*_\tau} \left| \ell_{I_k} \left(\mathcal{Y}; \widetilde{\theta}_{I_k,\tau} \right) - \ell_{I_k} \left(\mathcal{Y}; \widehat{\theta}_\tau \right) \right|^r \leq \rho_k \mathcal{R}_r \left(\theta^*_\tau \right) \tag{9}$$

with $\rho_k = \dfrac{\rho k}{K}$. Here ρ is a false alarm level.

The resulting critical value curves satisfying equation (9) for the selected six 'true' parameter constellations from Table 2 and associated risk bounds from Table 3 are displayed in Figure 5. The upper (lower) panel represents critical values in the modest (conservative) risk case. The blue and red lines consider the expectile levels $\tau = 0.05$ and $\tau = 0.01$, respectively. Critical values evolve in a decreasing route, with a similar magnitude across all cases except for the middle 'true' parameter constellations in the first few steps. It is therefore reasonable to choose the critical value set in a data-driven fashion: at a fixed time point, the yearly estimate $\widehat{\alpha}_{1,\tau}$ serves as a benchmark to select the appropriate curve. If its value is, for example, higher than the reported upper quartile case in Table 2, then the corresponding critical value curve is selected.
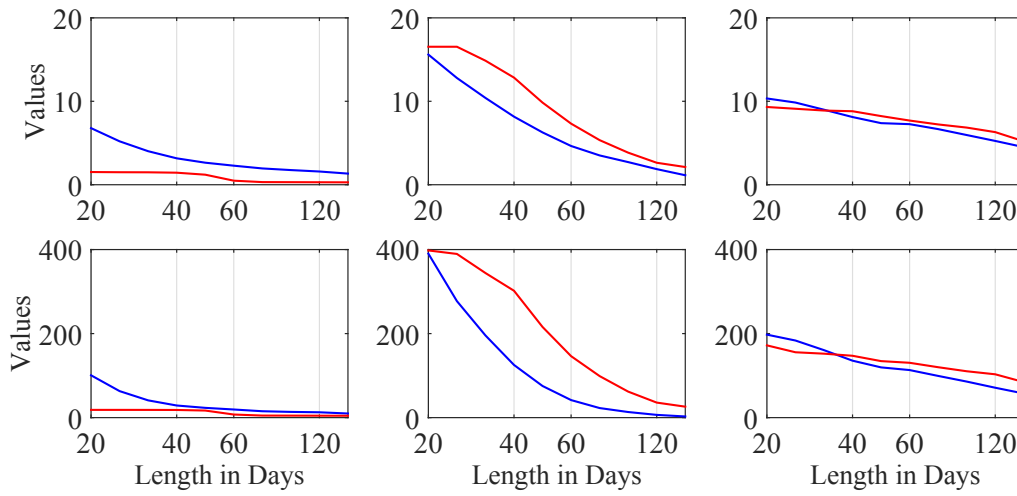
Figure 5: Simulated critical values across different parameter constellations given in Table 2 for the modest (upper panel, $r = 0.5$) and conservative (lower panel, $r = 1$) risk cases. We consider two expectile levels, $\tau = 0.05$ (blue) and $\tau = 0.01$ (red). LCARE_Critical_Values LCARE_Critical_Values_Th1_001 LCARE_Critical_Values_Th1_005 LCARE_Critical_Values_Th2_001 LCARE_Critical_Values_Th2_005 LCARE_Critical_Values_Th3_001 LCARE_Critical_Values_Th3_005

4   Empirical Results

lCARE accommodates and reacts to structural changes. From the fixed rolling window exercise in subsection 3.2 one observes time-varying parameter characteristics while facing the trade-off between parameter variability and the modelling bias. How to account for the effects of potential market changes on the tail risk based on the intervals of homogenity? In this section, we utilize the lCARE model to estimate the tail risk exposure across three stock markets. Using the time series of the adaptively selected interval length, we improve a portfolio insurance strategy employing our tail risk estimate and furthermore enhance its performance in the following application part.

4.1   Intervals of Homogeneity

The interval of homogeneity in tail expectile dynamics is obtained here by the lCARE framework for the time series of DAX, FTSE 100 and S&P 500 returns. Using the sequential local change point detection test, the optimal interval length is considered at two expectile levels, $\tau = 0.05$ and $\tau = 0.01$. We set the significance level $\rho = 0.25$. Interestingly, the homogeneity intervals are relatively longer at the end of 2009 and at the beginning of 2010 especially at $\tau = 0.05$, the period following the financial crisis across all three stock markets, see, e.g., Figures 6 and 7. Figure 6 presents the estimated length of the interval of homogeneity in trading days across the selected three stock market indices from 2 January 2006 to 31 December 2014 at the expectile level $\tau = 0.05$, while Figure 7 denotes the results given $\tau = 0.01$. The upper panel depicts the modest risk case $r = 0.5$, whereas the lower panel denotes the conservative risk case $r = 1$.

Recall that the lCARE model aims to select the longest interval over which the null hypothesis of time homogeneity of CARE parameters is not rejected. In the financial crisis initial period, the homogeneity intervals became shorter, due to the increasing market volatility and obvious market turmoil. During the post-crisis period, characterised by the high volatile regime, the homogeneity intervals became relatively longer.

In a similar way, the intervals of homogeneity are relatively shorter in the modest risk case $r = 0.5$, as compared to the conservative risk case $r = 1$. The average daily selected

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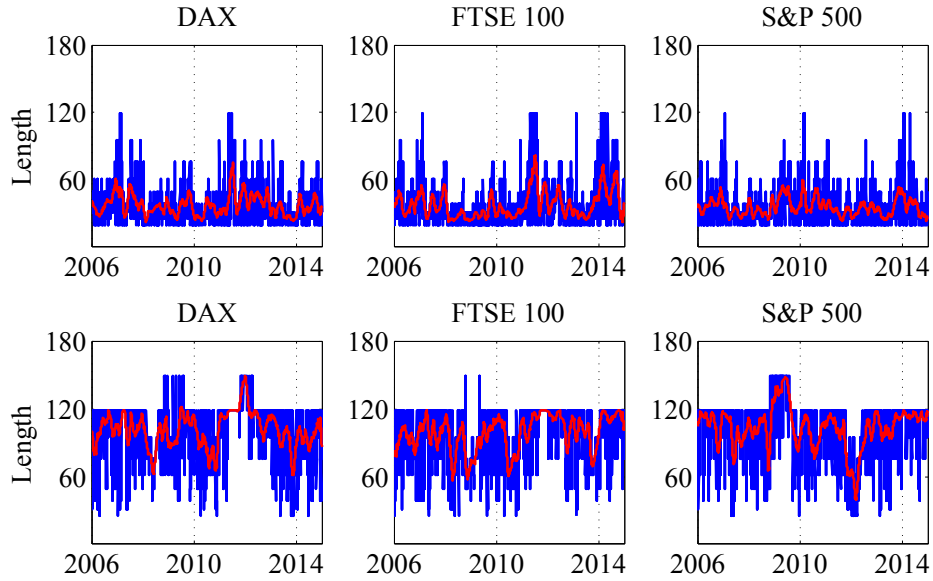
Figure 6: Estimated length of the interval of homogeneity in trading days across the selected three stock markets from 2 January 2006 to 31 December 2014 for the modest (upper panel, $r = 0.5$) and the conservative (lower panel, $r = 1$) risk cases. The expectile level equals $\tau = 0.05$. ◙ LCARE_Adaptive_Estimation_Length_005 ◙LCARE_Adaptive_Estimation_005
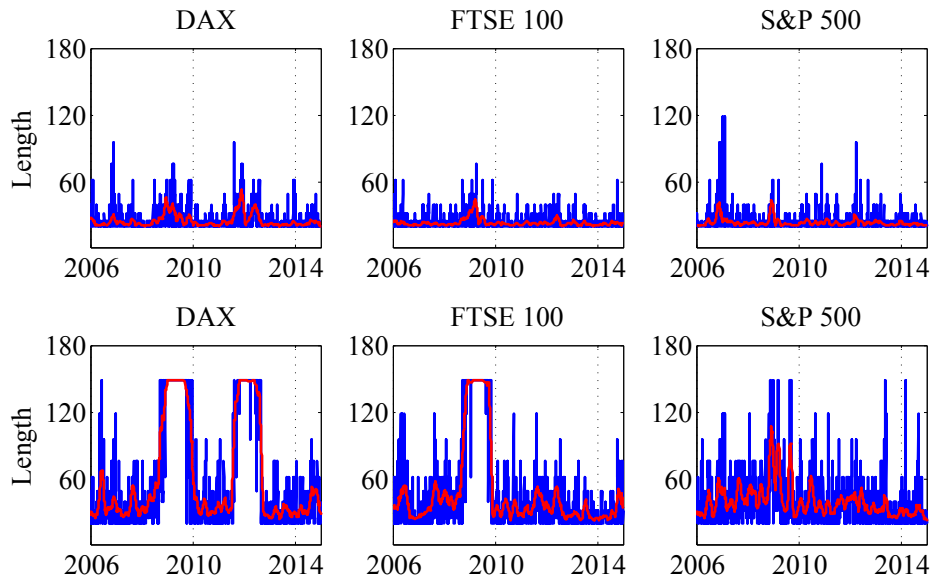
Figure 7: Estimated length of the interval of homogeneity in trading days across the selected three stock markets from 2 January 2006 to 31 December 2014 for the modest (upper panel, $r = 0.5$) and the conservative (lower panel, $r = 1$) risk cases. The expectile level equals $\tau = 0.01$. ◙ LCARE_Adaptive_Estimation_Length_001 ◙LCARE_Adaptive_Estimation_001

optimal interval length supports this, see, e.g., Table 4. The results are presented for both expectile levels, $\tau = 0.05$ and $\tau = 0.01$, at the modest and the conservative risk cases, $r = 0.50$ and $r = 1$, respectively. At expectile level $\tau = 0.01$, the interval of homogeneity is comparatively shorter than the interval at $\tau = 0.05$, due to more severe tail events. This fact is also implied by the associated parameter variability.

	$\tau = 0.05$			$\tau = 0.01$		
	DAX	FTSE 100	S&P 500	DAX	FTSE 100	S&P 500
$r = 0.5$	38	38	36	25	23	23
$r = 1.0$	101	98	103	63	48	40

Table 4: Mean value of the adaptively selected intervals. Note: the average number of trading days of the adaptive interval length is provided for the DAX, FTSE 100 and S&P 500 market indices at two expectile levels, $\tau = 0.05$ and $\tau = 0.01$, and the modest ($r = 0.50$) and the conservative ($r = 1.00$) risk case.

4.2   Dynamic Tail Risk Exposure

Based on the lCARE model, one can directly estimate dynamic tail risk exposure measures through the adaptively selected intervals. The tail risk at smaller expectile level is lower than risk at higher levels, see, e.g., Figure 8. Here the estimated expectile risk exposure for the three stock market indices from 2 January 2006 to 31 December 2014 is displayed for levels $\tau = 0.05$ and $\tau = 0.01$, respectively. The left panel represents the conservative risk case $r = 1$ results, whereas the right panel considers the modest risk case $r = 0.5$. The former leads on average to lower variability, as compared to the modest risk which results in shorter homogeneity intervals.

Estimated expectiles allow us to calculate other tail risk measures, most prominently expected shortfall that represents the expected value of portfolio loss above a certain threshold, e.g., Value at Risk (VaR). The quantile estimation can be improved by employing an expectile-based expected shortfall (ES) framework. In its derivation one notes a one-to-one mapping between quantiles and expectiles with the expectile level τ_α being
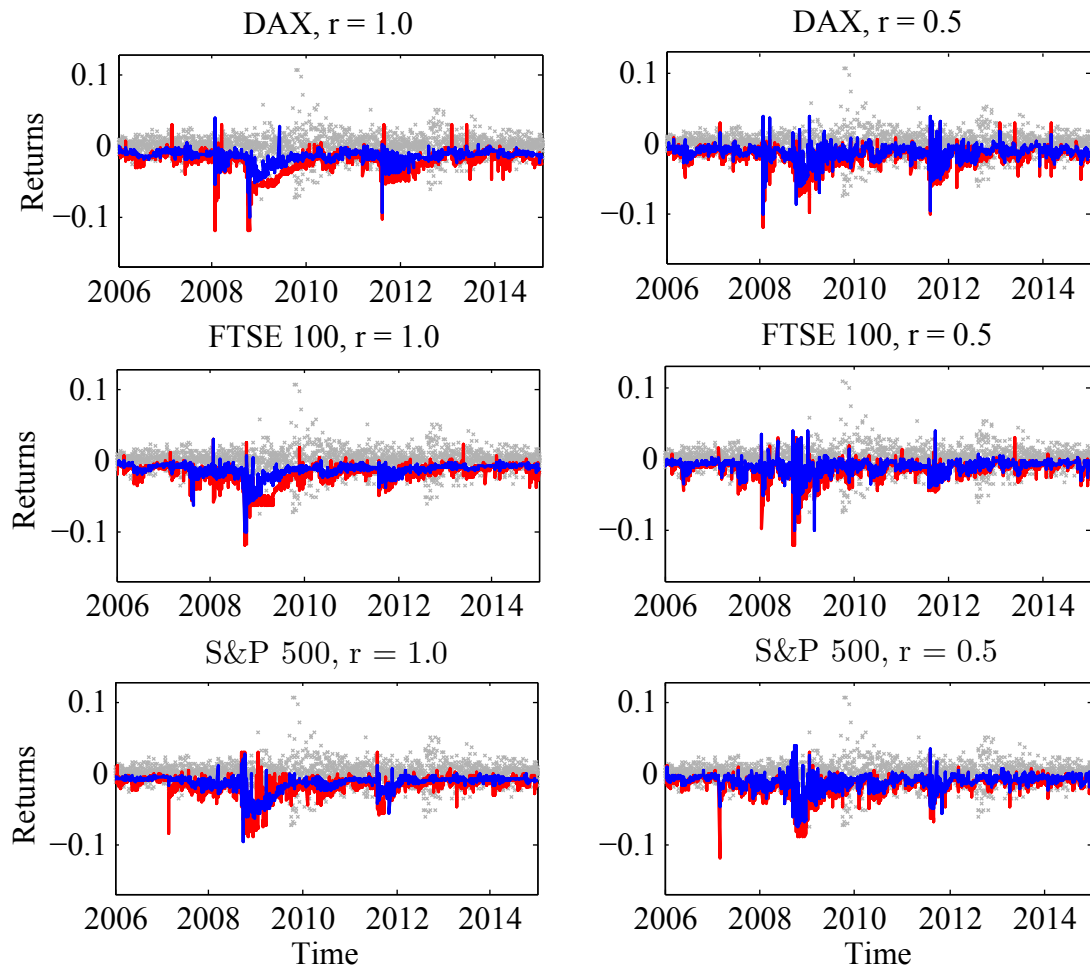
Figure 8: Estimated expectile risk exposure at level $\tau = 0.05$ (blue) and $\tau = 0.01$ (red) for return time series of DAX, FTSE 100, and S&P 500 indices from 2 January 2006 to 31 December 2014. The left panel shows results of the conservative risk case $r = 1$ and the right panel depicts the results of the modest risk case $r = 0.5$.

selected such that $e_{t,\tau_\alpha} = q_\alpha$, i.e., α-quantile

$$\tau_\alpha = \frac{\alpha \cdot q_\alpha - \int_{-\infty}^{q_\alpha} y dF(y)}{\mathsf{E}[Y] - 2\int_{-\infty}^{q_\alpha} y dF(y) - (1 - 2\alpha) q_\alpha} \tag{10}$$

where $F(\cdot)$ denotes the cumulative density function (cdf) of a random variable Y. The corresponding expected shortfall can be expressed as

$$ES_{e_{t,\tau_\alpha}} = \left|1 + \tau_\alpha (1 - 2\tau_\alpha)^{-1} \alpha^{-1}\right| e_{t,\tau_\alpha} \tag{11}$$

with e_{t,τ_α} denoting the expectile at level τ_α. In order to apply (11), one needs to fix a certain cdf $F(\cdot)$ in (10). The asymmetric normal distribution is chosen here considering the consistency with the aforementioned model specification.

Consider the tail risk exposure of DAX index series at expectile level $\tau = 0.05$ and conservative risk case $r = 1.0$. During market distress periods, e.g., the 2008 financial crisis or the 2012 European sovereign debt crisis, the estimated expected shortfall (11) exhibits a high variation, as depicted in Figure 9. Note that the asymmetric normal distribution from subsection 3.1 has been employed in ES calculation. Similarly to current research developments, the estimated expected shortfall using the proposed lCARE model exceeds (by magnitude) the estimated expectile $e_{t,\tau}$ value.
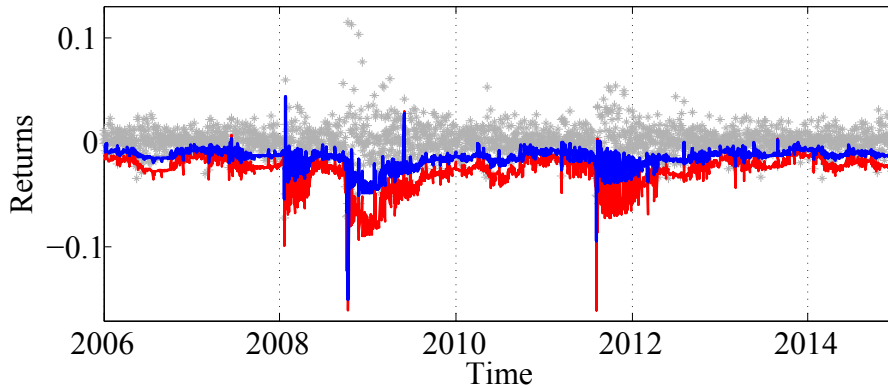
Figure 9: Adaptively estimated expectile (blue) and expected shortfall (red) for DAX index returns from 2 January 2006 to 31 December 2014. We choose $r = 1$ and $\tau = 0.05$.

4.3 Application: Portfolio Insurance

In practice, dynamic tail risk measures are useful tools in many areas, for example in portfolio insurance - a portfolio protection strategy tailored especially for mutual fund management and portfolio optimization. Particularly, a given proportion of an initial asset portfolio value is preserved at the end of the predetermined time horizon. In this aspect the downside risk is limited under bearish market conditions. Simultaneously, the optimal profit return emerges in bullish market situations and thus fund managers can utilize the time invariant portfolio protection (TIPP) strategy, Estep and Kritzman (1988), Hamidi et al. (2014). It turns out that this represents an extension of the constant proportion portfolio insurance (CPPI) strategy by Black and Jones (1987), Black and Perold (1992).

The CPPI method is applied for a dynamic portfolio allocation along the whole management period. The fund managers firstly predetermine a floor, which is the lowest acceptable portfolio value at the end of the investment horizon, and then invest the exposure, the multiple amount of the excess of the portfolio value above the floor by a multiplier, into the risky asset and the remaining part into riskless asset or cash. The TIPP strategy is an extension of the CPPI method, i.e., its floor is time-varying, relating the floor to a proportion of the highest previous portfolio value, which seems more conservative but more actively responds to the prevailing market conditions.

The proportion of the total portfolio invested in risky assets is determined by the so-called asset multiplier. The multiplier is the leverage value of the risky exposure. It is a challenging task to obtain a reasonable multiplier figure. A traditional approach assumes that the multiplier is a constant, i.e., it is insensitive to the current market conditions. Our lCARE model certainly adapts to the risk exposure, say at different states of the economy (bearish or bullish market), and we account for a time-varying nature of the asset multiplier in portfolio allocation. It is expected that during favourable conditions, more assets can be allocated into risky investments and vice versa. The trading idea of the TIPP strategy is explained and thereafter the relationship between the multiplier and the return of the risky asset is derived. The methodologies (constant vs adaptive multiplier selection) are then applied to the DAX index return series and evaluated afterwards.

Time Invariant Portfolio Protection Strategy (TIPP)

Denote the initial asset portfolio value as V_t at time $t \in (0, T]$. An investor aims to preserve a predetermined protection value F_t^s, the so-called floor, at each day

$$V_t \geq s \times \max \left\{ F \cdot e^{-rf_t \cdot (T-t)}, \sup_{p \leq t} V_p \right\} = F_t^s \tag{12}$$

with an exogenous parameter $s \in (0, 1)$ and the cushion value, $C_t = V_t - F_t^s \geq 0$. rf_t is the risky free rate and we set the initial value $F = 100$ and the proportion value $s = 0.9$. The allocation decision states that $G_t = m \cdot C_t$ is invested into the risky asset with return r_t (here the DAX portfolio). Here m denotes a non-negative multiplier that controls the portfolio performance. The remaining amount $V_t - G_t$ is invested into a riskless asset.

The portfolio value V_t and consequently the cushion value $C_t = V_t - F_t^s$ evolve as:

$$V_{t+1} = V_t + G_t r_{t+1} + (V_t - G_t) \, rf_{t+1} \tag{13}$$

$$C_{t+1} = C_t \left\{ 1 + m \cdot r_{t+1} + (1 - m) \, rf_{t+1} \right\} \tag{14}$$

Since the cushion value $C_t \geq 0$, for all $t \leq T$, an upper bound of the multiple m can be derived from equation (14) when rf_t is negligibly small and the risky asset return is negative

$$m \leq \left(-r_{t+1}^- \right)^{-1}, \ \forall t \leq T \tag{15}$$

with $r_{t+1}^- = \min(0, r_{t+1})$.

Formula (15) reflects a relationship between m and the tail structure of the distribution of r_t. When the downside return loss is, for example, 10%, $m \leq 10$, and for a downside of 20%, $m \leq 5$. When the market is bullish, the investor is prone to invest more into the risky asset and vice versa in the bearish situations.

It is worth to note that in the above TIPP strategy, the cushion value is always expected to be near or above zero. This property only holds in the continuous time and assumes that the investor could timely modify their portfolio allocation before a large downside return happens. In practice, fund managers have to account for the risk that the cushion value may be negative since there may happen a unpredictable large downside market crash whereupon the managers fail to reschedule their portfolio allocations in the discontinuous

rebalancing. This risk is known as the gap risk.

How to deal with gap risk and correspondingly calculate the multiplier? There are two common approaches: the first is through the quantile hedging method, see e.g. Föllmer and Leukert (1999), exploiting Value at Risk to imply the multiplier; another method is based on expected shortfall, see e.g. Hamidi et al. (2014), Ameur and Prigent (2014).

In the quantile hedging framework, for a given level α, the protection portfolio condition is given by

$$\mathrm{P}\left(C_t \geq 0, \ \forall t \leq T\right) \geq 1 - \alpha.$$

Similar to the derivation of (15), the multiplier can now be expressed as the $1 - \alpha$ quantile of the distribution of r_t

$$\mathrm{P}\left\{m_t \leq \left(-r_{t+1}^-\right)^{-1}, \ \forall t \leq T\right\} \geq 1 - \alpha$$

where the upbound of m with quantile can be obtained by the above equation.

Note that the quantile technique does not take the magnitude of tail risk into account. The expected shortfall is a coherent risk measure and is more suitable to reflect the tail risk. When the investor is prone to more conservative asset allocation, ES is proposed to estimate the multiplier, see Hamidi et al. (2014).

Performance Comparison

Here we employ the lCARE method to estimate the ES in order to deal with the gap risk. The corresponding multiplier selection is expressed by the lCARE-based ES as

$$m_{t,\tau} = \left|ES_{e_{t,\tau}}\right|^{-1} \tag{16}$$

with $e_{t,\tau}$ denoting the associated expectile value. In the ES calculation, the data process follows an asymmetric normal distribution. The conditional multiplier is the inverse of the expected shortfall. In practice, a threshold range for $m_{t,\tau} \in \{1, 2, \ldots 12\}$ is used. Figure 10 presents the dynamics of the implied multiplier for the DAX index corresponding to ES in Figure 9 based on the lCARE model with $r = 1$ and $\tau = 0.05$ from 2 January 2006 to 31 December 2014.
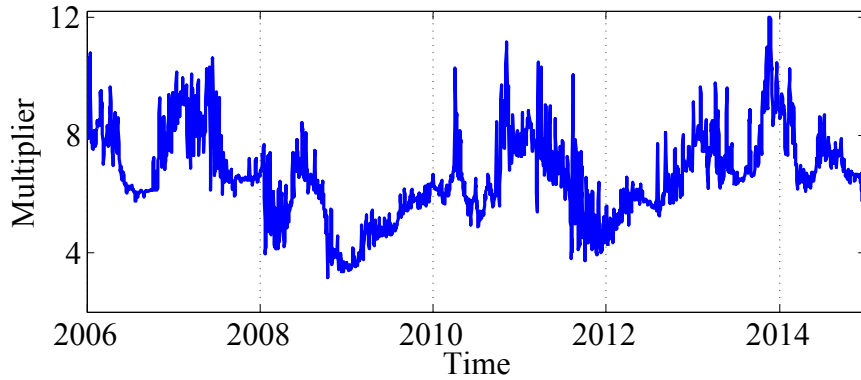
Figure 10: Time-varying multiplier $m_{t,\tau}$ for DAX index returns corresponding to the expected shortfall in Figure 9 based on lCARE ($r = 1$ and $\tau = 0.05$) from 2 January 2006 to 31 December 2014
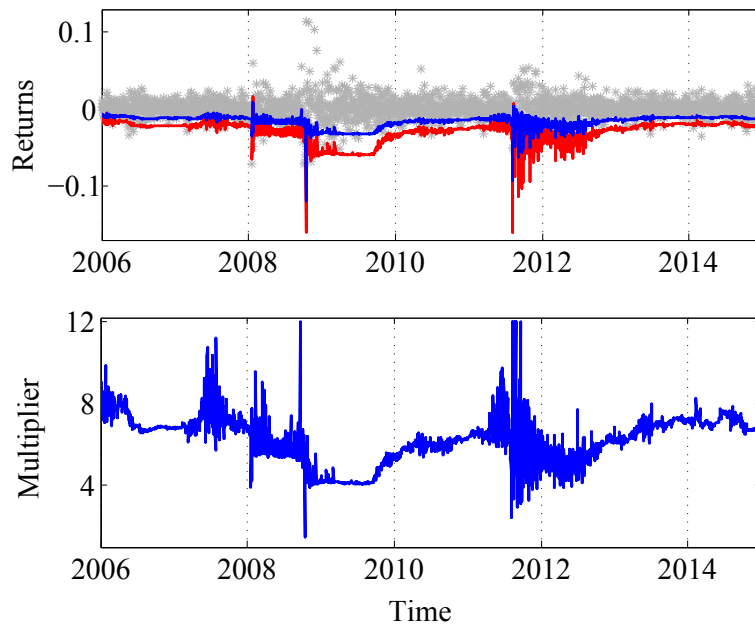
Figure 11: Estimated expectile (blue) and expected shortfall (red) by one-year fixed rolling window (upper panel), and the corresponding time-varying multiplier (lower panel) for DAX index returns from 2 January 2006 to 31 December 2014

In order to better understand adaptive estimation window methods, the one-year rolling window estimation strategy is also selected as one of the benchmark models. In Figure 11, the upper panel presents the estimated expectile and ES based on a one-year fixed rolling window estimation while the lower panel shows the corresponding multipliers. The constant multiplier cases (from 1 to 12) are included for benchmark comparisons.

ES can also be implied by the CAViaR framework, one of the popular conditional autoregressive modeling approaches for Value at Risk. Given that there is a one-to-one mapping between expectiles and quantiles, the expected shortfall can be formulated by the quantile at the corresponding quantile level when the expectile and quantile values are equal, see (10). Here we also list the CAViaR based ES and the corresponding multiplier to implement the insurance strategy as one of the benchmarks. We firstly choose the corresponding quantile level, then illustrate the CAViaR specification from Engle and Manganelli (2004), before presenting the results.

Under the asymmetric normal distribution assumption, given expectile level $\tau = 0.05$, Equation (10) implies the corresponding quantile level $\alpha = 0.065$. While Engle and Manganelli (2004) state four CAViaR model specifications, the following asymmetric slope pattern is selected, which appears similar to the focused model specification equation (2),

$$y_t = q_{t,\alpha} + \varepsilon_{t,\alpha} \quad Quant_\alpha(\varepsilon_{t,\alpha}|\mathcal{F}_{t-1}) = 0 \tag{17}$$

$$q_{t,\alpha} = \beta_0 + \beta_1 q_{t-1,\alpha} + \beta_2 y_{t-1}^+ + \beta_3 y_{t-1}^- \tag{18}$$

where $q_{t,\alpha}$ denotes quantile (VaR) at $\alpha \in (0,1)$, and $Quant_\alpha(\varepsilon_{t,\alpha}|\mathcal{F}_{t-1})$ is the α-quantile of $\varepsilon_{t,\alpha}$ conditional on the information set $\mathcal{F}_{t-1}$. In addition, we choose $\alpha = 0.065$ such that $e_{\tau_\alpha} = q_\alpha$ when $\tau_\alpha = 0.05$ under the AND assumption.

The estimated quantile (expectile) and ES based on a one-year rolling window estimation associated to the abovementioned CAViaR model (18), in which ES is implied from equation (11) with the quantile (expectile) value, is presented in the upper panel in Figure 12, while the lower panel shows the corresponding multipliers.

The initial value and the target value of a hypothetical portfolio at the end of the one year investment horizon are both set to 100 ($F = 100$ in equation (12)). Associated to
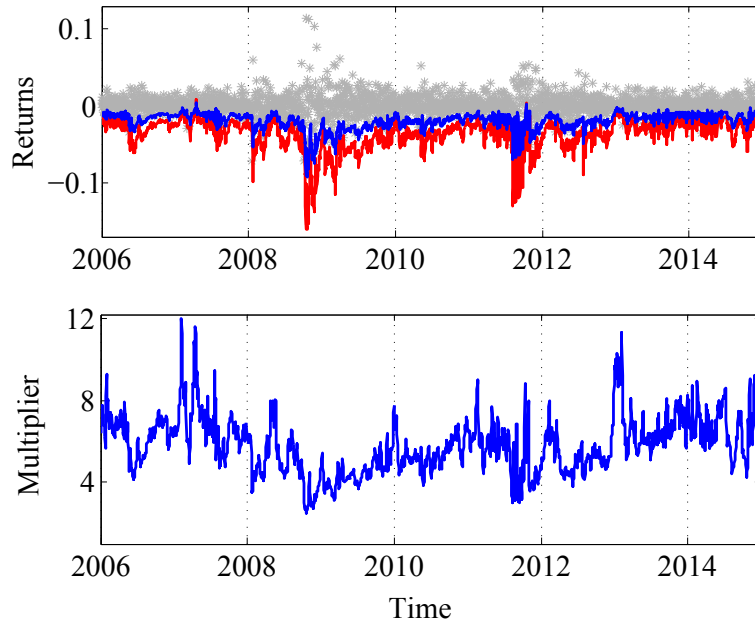
Figure 12: Estimated VaR (blue) ($\alpha = 0.065$) and expected shortfall (red) by CAViaR - based one-year rolling (upper panel), and the corresponding multiplier (lower panel) for DAX index returns from 2 January 2006 to 31 December 2014

the cushioned portfolio strategy, the daily asset allocation decision at time t is to invest the multiple amount of the difference between the portfolio value and the discounted floor up to t into the stock portfolio (DAX), the rest into a riskless asset. Figure 13 presents the performance of the portfolio value based on the cushioned portfolio strategy with unconditional constant multiplier and the conditional time-varying multiplier for the one-year investment horizon. The black line represents the DAX index, the blue line represents the cushioned portfolio with lCARE based conditional dynamic multiplier, the green line represents the portfolio value using a one-year fixed rolling window estimated multiplier, and the brown line presents the value under CAViaR based one-year rolling estimated multiplier. The comparatively best performed portfolio among the constant multipliers considers $m = 5$, denoted by the red line.

The cushioned portfolio with the dynamic multiplier closely tracks the observed DAX index and, by construction, simultaneously guarantees the target portfolio value floor at the end of the investment horizon at every trading day, see Figure 13. It is worth noting that the lCARE strategy does very well in comparison to the cushioned portfolio with a constant multiplier, the one-year rolling window estimation based on expectile or
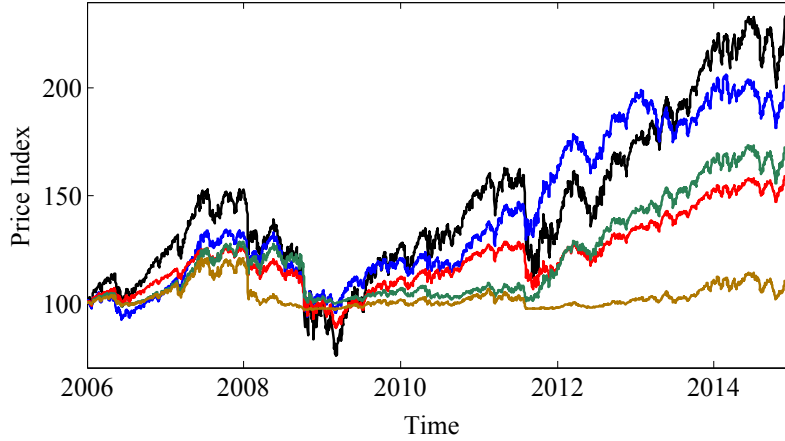
Figure 13: Performance of the portfolio value: (a) DAX index (black), (b) $m = 5$ (red), (c) one-year rolling approach (green), (d) CAViaR based one-year rolling approach ($\alpha = 0.065$) (brown), (e) $m_{t,\tau}$ - lCARE (blue) from 2 January 2006 to 31 December 2014.

quantile.

Table 5 furthermore presents the return moment performance of the portfolio insurance strategy. We list the statistical results of empirical data, the TIPP strategy with lCARE - based multiplier, one-year fixed rolling window CARE - implied multiplier, one-year rolling window CAViaR implied multiplier, and constant multipliers. The average return of lCARE based strategy, 7.36% is larger than the counterpart based on a fixed rolling window, 5.70%. It is also observed that the CAViaR based strategy performs less favourable. Although, the lCARE strategy is slightly lower than the empirical DAX return of 8.79%, it turns out that it performs favourable relative to other benchmark strategies.

5 Conclusions

The localized conditional autoregressive expectiles (lCARE) model accounts for time-varying parameter characteristics and potential structure changes in tail risk exposure modelling. The parameter dynamics implied by a fixed rolling window exercise of three stock market indices, DAX, FTSE 100 and S&P 500, indicates that there is a trade-off between the modelling bias and parameter variability. A local parametric approach (LPA) assumes that locally one can successfully fit a parametric model. Based on a sequential

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	Return(%)	Volatility(%)	VaR 99%	Skewness	Kurtosis	Sharpe
Data	8.79	22.54	-4.24	0.24	10.33	0.02
lCARE	7.36	13.60	-2.31	0.52	9.16	0.03
Rolling one-year	5.70	10.18	-1.59	0.17	10.05	0.04
CAViaR rolling	0.01	7.35	-1.43	-0.90	13.04	0.00
Multiplier 1	3.51	2.25	-0.41	0.20	10.05	0.10
Multiplier 2	3.97	4.50	-0.84	0.19	10.00	0.06
Multiplier 3	4.41	6.74	-1.27	0.17	9.90	0.04
Multiplier 4	4.78	9.00	-1.71	0.15	9.88	0.03
Multiplier 5	4.86	11.17	-2.10	0.11	9.91	0.03
Multiplier 6	3.36	5.36	-0.99	-0.33	6.48	0.04
Multiplier 7	2.65	6.04	-1.08	-0.51	6.49	0.03
Multiplier 8	2.13	6.55	-1.17	-0.59	7.90	0.02
Multiplier 9	1.70	6.96	-1.25	-0.74	10.38	0.02
Multiplier 10	1.46	7.33	-1.38	-0.93	12.90	0.01
Multiplier 12	0.82	7.56	-1.47	-1.25	16.65	0.01

Table 5: Descriptive statistics of the portfolio returns based on the TIPP strategy. We employ several models: the lCARE, one-year rolling window, CAViaR rolling window and constant muliplier approach for the DAX index from 2 January 2006 to 31 December 2014. The investment strategy is based on a one-year investment horizon.

testing procedure, one determines the interval of homogeneity over which a parametric model can be approximated by a constant parameter vector.

The lCARE model adaptively estimates the tail risk exposure by relying on the (in-sample) 'optimal' interval of homogeneity. Setting the expectile levels $\tau = 0.05$ and $\tau = 0.01$, the dynamic expectile tail risk measures for the selected three stock markets are successfully obtained by lCARE. Furthermore, ES has been introduced, evaluated and employed in the asset allocation example: the portfolio protection strategy is improved by the lCARE modelling framework.

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