

Flexible Stochastic Volatility Structures for High Frequency Financial Data

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Abstract

Stochastic Volatility (SV) models are widely used in financial applications. To decide whether standard parametric restrictions are justified for a given dataset, a statistical test is required. In this paper, we develop such a test based on the linear state space representation. We provide a simulation study and apply the test to the HFDF96 data set. Our results confirm a linear AR(1) structure for the analyzed stock indices S&P500, Dow Jones Industrial Average and for the exchange rate DEM/USD.

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1 Introduction

A good knowledge of path-dependent volatility structures is important for a wide variety of tasks in the analysis of high frequency data in finance (HFDF). Such knowledge enables multi-step forecasts of volatility, which can be used for derivative pricing, evaluation of risk exposure, and prediction intervals for the mean. Potential applications of this knowledge are tests of economic or financial theories concerning the stock, bond and currency markets or studies of the link between short and long term interest rates. Another important set of applications concern interventions on the markets based on portfolio choice, hedging portfolios, values at Risk (VaR), the size and times of block trading. Several statistical software packages include the estimation of volatility structures.

Typically, the conditional volatility exhibits a strong dependence on past values of the observed process. In this context, autoregressive conditional heteroskedasticity (ARCH) models (Engle, 1982; Gouriéroux, 1997) and stochastic volatility (SV) models (Taylor, 1986) have been studied intensively. For example, Duan (1995) used an ARCH model for option pricing under time-varying volatility. Volatility models have consequences for the stationary distribution of the process and, thus, influence the calculation of tail indices and Value at Risk, see e.g. de Haan (1990) and de Vries (1994). A comparison of the prediction performance of standard GARCH and some nonlinear GARCH models is provided by Franses and van Dijk (1996).

Maximum Likelihood estimation of ARCH models is much easier than of SV models. On the other hand, for SV models the continuous-time representation is simpler, and the definition of a risk-premium for volatility is more natural.

Starting with Taylor (1986), SV models are mostly specified as parametric AR(1)-type models. The question arises, whether the parametric structure adequately describes the data. Stylized facts of HFDF show that GARCH volatility models do not sufficiently capture the structure of HFDF. Therefore, the same question of appropriateness of simple parametric models has been posed e.g. by Gouriéroux, Monfort (1992), Härdle, Tsybakov (1997) and Hafner (1998) in the framework of nonparametric ARCH and CHARN models (conditionally heteroskedastic autoregressive nonlinear model). Tests for nonparametric structures have been developed by Leblanc, Lepski (1996) and by Gouriéroux, Monfort, Tenreiro (1995) in the time series context.

In this paper, we adopt nonparametric hypothesis testing to the case of stochastic volatility models. First, we show that the discrete time SV model can be represented in statistical terms as an errors in variables model. Second, we introduce the test statistic, which allows to separate the standard parametric hypothesis (e.g. linear) from the family of alternatives and describe the sensitivity of the test. Third, we investigate the finite sample behavior by a simulation study. Finally, we apply it to the HFDF96 data set: the S&P500 and the Dow Jones stock price indices, as well as the DEM/USD exchange rate. Our findings support the hypothesis of a parametric volatility structure for all analyzed data sets.

2 Test of a Parametric Hypothesis

Let S_t denote the underlying asset price at time t , $t = 1, \dots, n$, and define returns y_t as $y_t = \log S_t/S_{t-1}$. The standard SV model as in Taylor (1986) can be written as

$$y_t = \exp(h_t/2)\xi_t^* \quad (1)$$

$$h_t = \vartheta h_{t-1} + \varepsilon_t, \quad (2)$$

where ξ_t^* and ε_t are i.i.d. random variables with $E[\xi_t^*] = 0$ and $E[\varepsilon_t] = 0$. Let $\xi_t = \log(\xi_t^*)^2 - \omega$ with $\omega = E[\log(\xi_t^*)^2]$, and let $z_t = \log y_t^2$. Then we obtain the following linear state space model for the observables z_1, \dots, z_n ,

$$z_t = \omega + h_t + \xi_t, \quad (3)$$

$$h_t = \vartheta h_{t-1} + \varepsilon_t. \quad (4)$$

We can write (4) as $h_t = m(h_{t-1}) + \varepsilon_t$, where $m(\cdot)$ is an unknown function. The shape of this function determines the type of impact of volatility on financial decision variables. Our goal is to test the hypothesis that the function $m(\cdot)$ is linear, i.e.

$$\mathbf{H}_0 : m(x) = \vartheta x, \vartheta \in [a, b] \subset (-1, 1), \quad (5)$$

where a, b are some known constants. The following assumptions imposed on the noises are supposed to be true.

1. The sequences $(\xi_t, t = 1, \dots, n)$ and $(\varepsilon_t, t = 1, \dots, n)$ are stochastically independent.
2. $E\xi_1^2 = \eta^2$, $E\xi_1^3 = \tau$, $E\xi_1^4 = \mu$, and $E|\xi_1|^{4+\delta} < \infty$,
3. $E\varepsilon_1^2 = \sigma^2$, $E\varepsilon_1^3 = \kappa$, $E\varepsilon_1^4 = \nu$, and $E|\varepsilon_1|^{4+\delta} < \infty$,

for some $\delta > 0$. For the theory we can assume that under hypothesis \mathbf{H}_0 the constant ω is known and without loss of generality it is set to zero. Now, consider

$$\bar{\vartheta}_n = \frac{\sum_{t=3}^n z_t z_{t-2}}{\sum_{t=3}^n z_{t-1} z_{t-2}}. \quad (6)$$

Under hypothesis \mathbf{H}_0 , $\bar{\vartheta}_n$ is a \sqrt{n} -consistent estimator for the parameter ϑ . This can be seen by writing

$$\bar{\vartheta}_n = \vartheta + \frac{\sum_{t=3}^n z_{t-2}(\xi_t - \vartheta \xi_{t-1} + \varepsilon_t)}{\sum_{t=3}^n z_{t-1} z_{t-2}}, \quad (7)$$

and noting that z_{t-2} and $\xi_t - \vartheta \xi_{t-1} + \varepsilon_t$ are independent.

Now let us introduce the test statistics T_n and the decision rule Δ_n , corresponding to it. Set

$$\hat{\vartheta}_n = \begin{cases} \bar{\vartheta}_n, & \text{if } a \leq \bar{\vartheta}_n \leq b; \\ a, & \text{if } \bar{\vartheta}_n < a; \\ b, & \text{if } \bar{\vartheta}_n > b; \end{cases}$$

$$\begin{aligned} M_n &= \sigma^2 + \eta^2 (1 + \hat{\vartheta}_n^2); \\ B_n &= B(\hat{\vartheta}_n) \end{aligned}$$

and $B(\vartheta) = (\nu - \sigma^4) + (1 + \vartheta^2)^2 \mu - \eta^4(1 + \vartheta^4) + 2\sigma^2\eta^2(1 + \vartheta^2)$. We define the test statistics and decision rule as follows.

$$T_n = \frac{1}{\sqrt{nB_n}} \sum_{t=1}^n \left\{ (z_t - \hat{\vartheta}_n z_{t-1})^2 - M_n \right\}. \quad (8)$$

Fix some $0 < \alpha < 1$ and set

$$\Delta_n = \begin{cases} 0, & \text{if } T_n \leq \sqrt{2 \ln \frac{1}{\alpha}} \\ 1, & \text{if } T_n > \sqrt{2 \ln \frac{1}{\alpha}}. \end{cases}$$

The hypothesis is accepted if $\Delta_n = 0$ and it is rejected if $\Delta_n = 1$.

Theorem 1 *Let \mathbf{P}_ϑ be the probability measure generated by the observations z_t , $t = 1, \dots, n$ under hypothesis \mathbf{H}_0 . Then*

$$\limsup_{n \rightarrow \infty} \sup_{\vartheta \in [a, b]} \mathbf{P}_\vartheta \{ \Delta_n = 1 \} \leq \alpha.$$

Thus, we have that the first error probability, which is the probability to reject the parametric hypothesis when it is true, is less than a fixed value α .

Now let us describe the sensitivity of the test based on decision rule Δ_n . The standard way of doing this in nonparametric hypothesis testing consists in the following. A set of alternatives is introduced and one needs to show that the probability to accept the parametric hypothesis for the case that the function m belongs to this set is less than some given value, say, β . In other words, the probability to accept the parametric hypothesis if the data are not described by a parametric model (i.e. the second error probability) is less than β . In order to introduce a set of alternatives, let us fix some $c \in (0, 1)$, $d > 0$ and denote by $\mathcal{M} = \mathcal{M}(c, d)$ the set of functions m possessing the following property,

$$|m(x)| \leq c|x| + d, \quad \forall x \in \mathbb{R}^1.$$

Denote

$$d_n(m) = \inf_{\vartheta \in [a, b]} \frac{1}{n} \sum_{t=2}^n \mathbf{E}_m (m(h_{t-1}) - \vartheta h_{t-1})^2. \quad (9)$$

where \mathbf{E}_m is the expectation with respect to the probability measure \mathbf{P}_m generated by the observations (3).

For any $\lambda > 0$ and any $n \geq 1$ denote

$$\mathcal{M}_n(\lambda) = \left\{ m \in \mathcal{M} : d_n(m) \geq \frac{\lambda}{\sqrt{n}} \right\}$$

and consider the set of alternatives

$$\mathbf{H}_n : m \in \mathcal{M}_n(\lambda).$$

Theorem 2 Let random variable ε_1 have a density p and $p(x) > 0$, $\forall x \in \mathbb{R}^1$. For any $0 < \alpha < 1$ and any $0 < \beta < 1$ there exists a constant $\lambda(\alpha, \beta)$ such that

$$\limsup_{n \rightarrow \infty} \sup_{m \in \mathcal{M}_n(\lambda(\alpha, \beta))} \mathbf{P}_m \{ \Delta_n = 0 \} \leq \beta.$$

The values α and β are the chosen first and second error probability, respectively.

Now we turn to the situation in practice where under hypothesis \mathbf{H}_0 the constant ω is not known. Note that we can rewrite model (4) as

$$z_t = \omega(1 - \vartheta) + \vartheta z_{t-1} + \varepsilon_t + \xi_t - \vartheta \xi_{t-1} \quad (10)$$

$$= \gamma + \vartheta z_{t-1} + \eta_t, \quad (11)$$

where $\gamma = \omega(1 - \vartheta)$, and $\eta_t = \varepsilon_t + \xi_t - \vartheta \xi_{t-1}$ with $E(\eta_t) = 0$. The mean of z_t is a \sqrt{n} -consistent estimator for ω ,

$$\hat{\omega} = n^{-1} \sum_{t=1}^n z_t.$$

In the following, we establish an iterative procedure to obtain improved estimates of ϑ . Consider the centered observations $z_t^* = z_t - \hat{\omega}$ and calculate preliminary estimates for ϑ and γ ,

$$\vartheta_n^{(1)} = \frac{\sum_{t=3}^n z_t^* z_{t-2}^*}{\sum_{t=3}^n z_{t-1}^* z_{t-2}^*}, \quad (12)$$

and $\gamma^{(1)} = n^{-1} \sum (z_t - \vartheta_n^{(1)} z_{t-1})$.

This estimator can be modified iteratively. Write (11) as $\tilde{z}_t = \vartheta z_{t-1} + \eta_t$ with $\tilde{z}_t = z_t - \gamma$. At the i th step, calculate $\tilde{z}_t^{(i)} = z_t - \gamma^{(i-1)}$,

$$\vartheta_n^{(i)} = \frac{\sum_{t=3}^n \tilde{z}_t^{(i)} z_{t-2}^{(i)}}{\sum_{t=3}^n z_{t-1} z_{t-2}}, \quad (13)$$

and $\gamma^{(i)} = \hat{\omega}(1 - \vartheta_n^{(i)})$.

For n fixed and $i \rightarrow \infty$, $\vartheta_n^{(i)}$ converges to some value $\bar{\vartheta}_n$. As before, determine $\hat{\vartheta}_n$ and replace the test statistic (8) by

$$T_n = \frac{1}{\sqrt{n} B_n} \sum_{t=1}^n \left\{ \left(\tilde{z}_t - \hat{\vartheta}_n z_{t-1} \right)^2 - M_n \right\}. \quad (14)$$

In practice, we also do not know the moments of the errors. However, we will give an advice how to use estimates in Section 4.

3 A Simulation Study

In this section, we provide simulation evidence of the finite sample behavior of the test statistic derived in the previous section. We consider the following two functions m ,

1.

$$m(x) = \vartheta x + \frac{\lambda}{n^{1/4}} x \sin(2\pi x).$$

In this case, $|m(x)| \leq |x|(|\vartheta| + \frac{\lambda}{n^{1/4}})$ and therefore $m \in \mathcal{M}$ if $|\vartheta| + \frac{\lambda}{n^{1/4}} < 1$.

2.

$$m(x) = \vartheta x + \frac{\lambda}{n^{1/4}} \sin(2\pi x).$$

In this case, $|m(x)| \leq |x| |\vartheta| + \frac{\lambda}{n^{1/4}}$ and therefore $m \in \mathcal{M}$ if $|\vartheta| < 1$ and $\frac{\lambda}{n^{1/4}} < \infty$.

In the following, we present simulation results for the second function, i.e. we generated 1000 replications of the series

$$z_t = h_t + \xi_t, \tag{15}$$

$$h_t = \vartheta h_{t-1} + \frac{\lambda}{n^{1/4}} \sin(2\pi h_{t-1}) + \varepsilon_t, \tag{16}$$

where ξ_t and ε_t are independent i.i.d. $N(0,1)$ random variables, $n = 10000, 6561, 4096$, $\vartheta = 0.95$, and λ determines the deviation from linearity. The odd numbers of n were chosen to obtain simple values for the sensitivity coefficient, $\lambda/n^{1/4}$. We have not included a constant into (16), so that we can directly calculate parameter estimates and test statistics without an iterative procedure. For the constants M_n and B_n , we calculated the moments implied by the normality and independence of the errors. Summary statistics of the T_n test statistic are given in Table 1. The estimates of ϑ were always very close to the true value of 0.95, so they are not reported.

The distribution of T_n for $n = 4096$ and $n = 10000$ is depicted in Figure 1 and 2, respectively, for $\lambda = 0$ to $\lambda = 1.5$. Obviously, the distribution moves to the right when λ is increased, which shows the consistency of the test. We also present the power functions for the levels $\alpha = 0.05$ and $\alpha = 0.1$ in Figure 3 (for $n = 4096$) and Figure 4 (for $n = 10000$). We see that the test has large power for $\lambda > 1$ and that the power increases fast.

4 Application to the HFDF96 data set

We extracted two stock price indices, the Dow Jones Industrial Average and the Standard & Poors 500, and the DEM/USD exchange rate from the HFDF96 data set, provided by Olsen & Associates. The data are half-hourly sampled index values. For the stock indices, we skipped the intervals corresponding to non-trading hours at the New York Stock Exchange. For DEM/USD, we skipped those intervals for which the time of the last quote was more than half an hour behind. This left us with 3680 observations for the stock indices and 14234 observations for DEM/USD. The time series are depicted in Figure 5, 6 and 7.

First, we estimated ϑ under the null hypothesis as described in the previous section, and obtained $\hat{\vartheta} = 0.9004$ for DEM/USD, $\hat{\vartheta} = 1.0153$ for DJIA, and $\hat{\vartheta} = 0.9241$ for S&P500. These results confirm previous results of SV models for high frequency financial data, see e.g. Mahieu,

n	10000	6561	4096
λ	0.000	0.000	0.000
mean	-0.048	0.019	0.014
std.dev.	1.502	1.551	1.490
λ	0.500	0.450	0.400
mean	0.383	0.365	0.357
std.dev.	1.553	1.569	1.418
λ	1.000	0.900	0.800
mean	1.801	1.418	1.052
std.dev.	1.521	1.520	1.492
λ	1.500	1.350	1.200
mean	4.251	3.445	2.747
std.dev.	1.545	1.664	1.643
λ	2.000	1.800	1.600
mean	8.486	6.868	5.377
std.dev.	1.903	1.864	1.823
λ	2.500	2.250	2.000
mean	15.406	12.445	9.881
std.dev.	3.306	3.974	3.332

Table 1: Summary statistics of simulated test statistics T_n . The first rows of each row-triple gives value of λ , the second the mean of T_n for 1000 replications, the third the standard deviation.

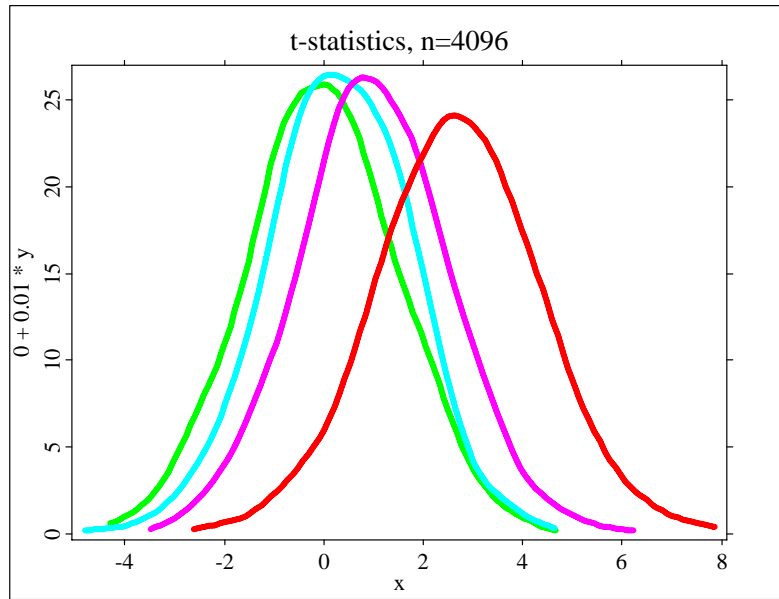


Figure 1: The distribution of T_n for $n = 4096$. From left to right: $\lambda = 0, 0.5, 1, 1.5$.

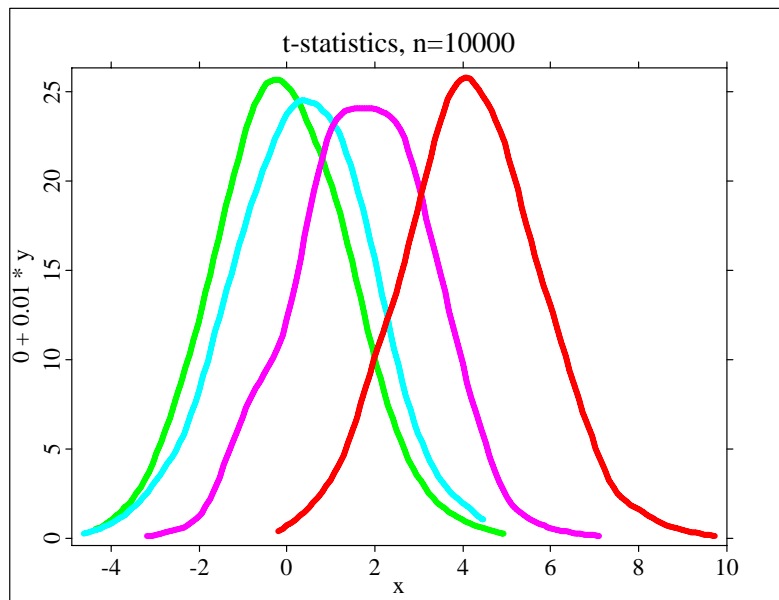


Figure 2: The distribution of T_n for $n = 10000$. From left to right: $\lambda = 0, 0.5, 1, 1.5$.

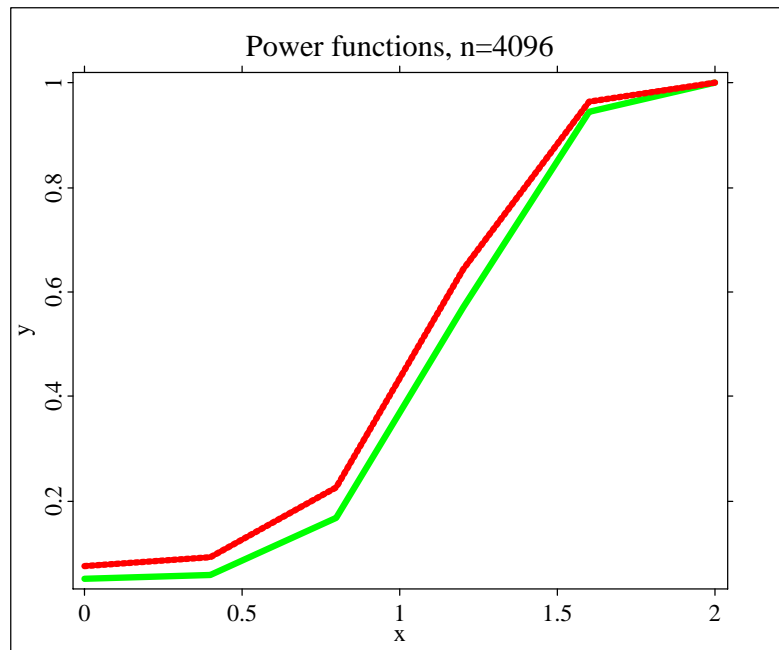


Figure 3: Power functions of T_n for $n = 4096$. The abscissa represents the parameter λ . Under the null hypothesis, $\lambda = 0$. The solid curve is the power function for $\alpha = 0.05$, the dashed curve is the power function for $\alpha = 0.1$.

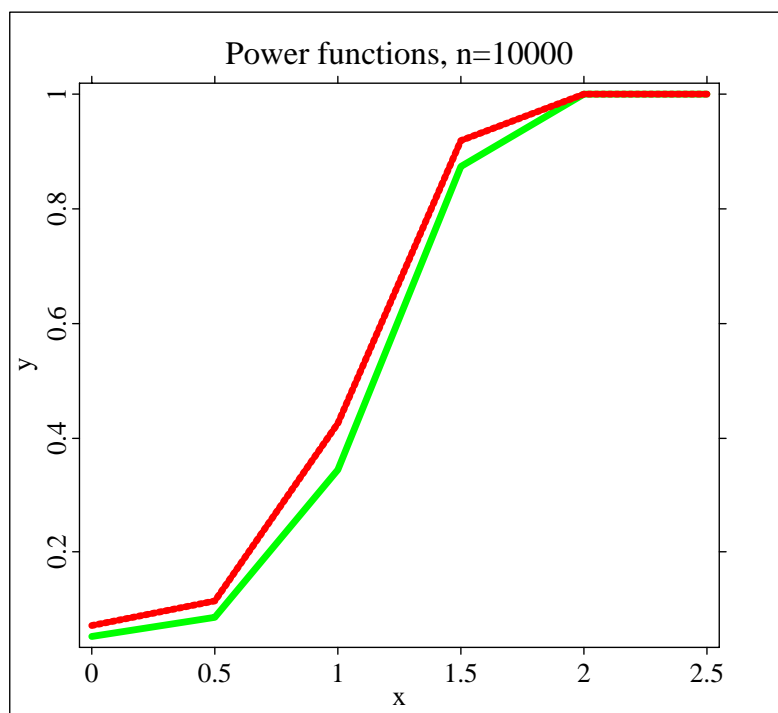


Figure 4: Power functions of T_n for $n = 10000$. The abscissa represents the parameter λ . Under the null hypothesis, $\lambda = 0$. The solid curve is the power function for $\alpha = 0.05$, the dashed curve is the power function for $\alpha = 0.1$.

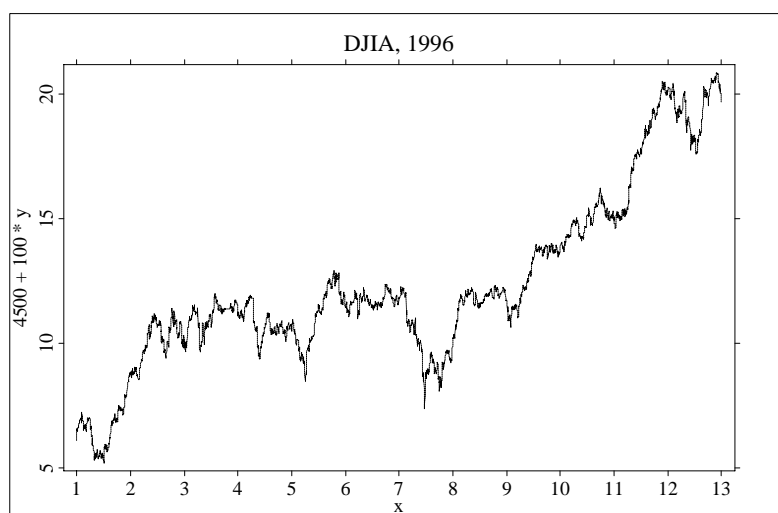


Figure 5: The Dow Jones Index.

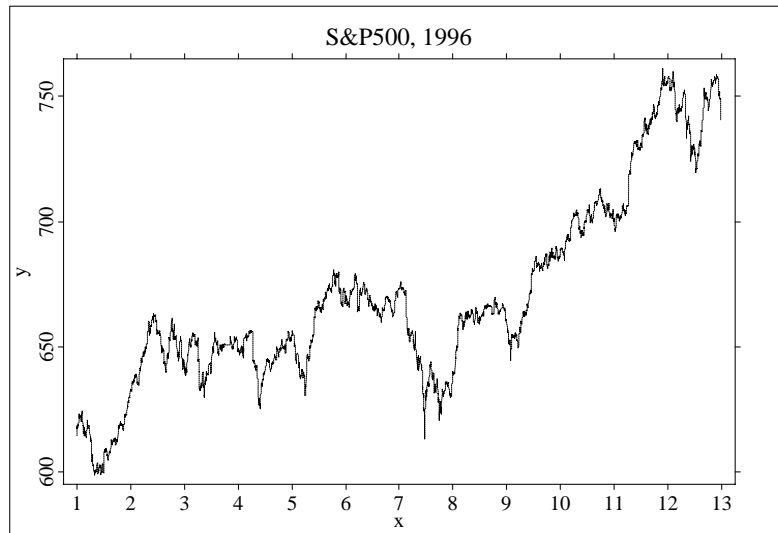


Figure 6: The S&P500 Index.

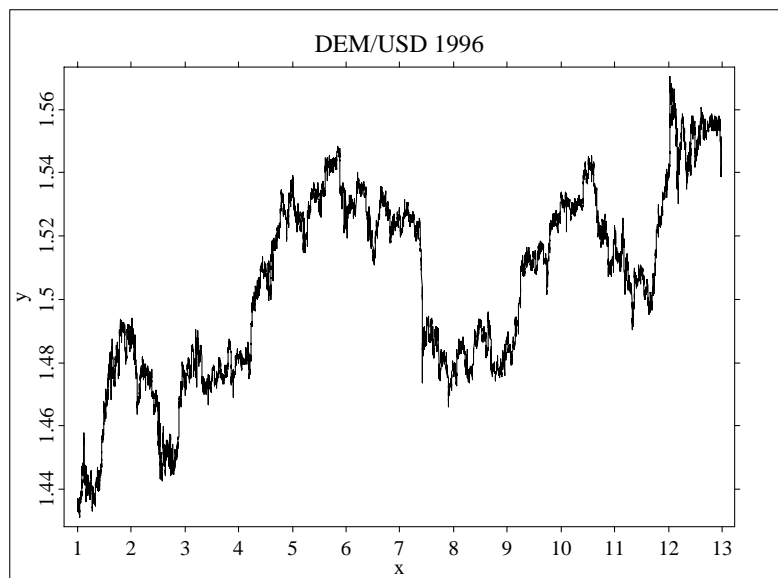


Figure 7: The DEM/USD exchange rate.

Schotman (1997). The AR parameter is usually found to be close to one, implying a high persistence of shocks in volatility. Our result for DJIA even implies a nonstationary volatility, and therefore a nonstationary return process. However, since we do not present standard errors we refrain from drawing inference from this result. Also in the ARCH literature it is well known that parameter estimates of standard GARCH models very often are close to the integrated GARCH (IGARCH) model. Even though these models cannot be directly compared, the results appear to be analogous.

Let us turn to the test statistic T_n in (8). In our real data situation, the constants B_n and M_n are unknown, but one can try to estimate them. In the present paper we use the following estimates \hat{M}_n and \hat{B}_n for M_n and B_n , respectively.

$$\begin{aligned}\hat{M}_n &= \frac{1}{n-1} \sum_{t=2}^n (\tilde{z}_t - \bar{\vartheta}_n z_{t-1})^2 \\ \hat{B}_n &= \frac{1}{n-1} \sum_{t=2}^n (\tilde{z}_t - \bar{\vartheta}_n z_{t-1})^4\end{aligned}$$

It is clear that now there is no sense to use the test statistic T_n which is obtained by replacing M_n and B_n in (14) by \hat{M}_n and \hat{B}_n , because, obviously, $T_n = 0$. To avoid this problem, we divide the sample path (z_1, \dots, z_n) into k groups (z_1, \dots, z_{n_1}) , $(z_{n_1+1}, \dots, z_{n_2})$, \dots , $(z_{n_{k-1}+1}, \dots, z_n)$ and study the behavior of k test statistics $T_n^{(j)}$, $j = 1, \dots, k$, which are defined as follows.

$$T_n^{(j)} = \frac{1}{\sqrt{n \hat{B}_n}} \sum_{t=n_{j-1}+1}^{n_j} \left\{ (\tilde{z}_t - \bar{\vartheta}_n z_{t-1})^2 - \hat{M}_n \right\}. \quad (17)$$

In particular, one can take $k = 2$. However, utilization of a larger number of subsamples appears to be reasonable because in this case we have an additional information on how many times the hypothesis is accepted or rejected. On the other hand, k should not be too large, since then the number of observations in the subsamples may become too small. Thus, we obtain k test statistics, and k decisions to accept or reject the null hypothesis at level α . Also, we can estimate ϑ for each subsample. It should be noted that most of these estimates were very close to the estimates reported above for the entire sample.

Table 2 gives the number of rejections for selected k . Ideally, under the null hypothesis we would expect to reject αk times. Especially for the stock indices this holds closely for $k < 100$. Note that for $k = 100$ there are only 36 observations in each subsample for the stock indices. For DEM/USD, we reject slightly more often than one would expect under linearity. However, recall the still moderate sample sizes we obtain in the subsamples and the slow rate of the test.

Overall, the hypothesis of a linear AR(1) structure in log volatility is confirmed by our results. This is astonishing at least for the stock indices, since in the ARCH literature very often asymmetries and nonlinearities were found for stock volatility. But recall that our sample period 1996 does not cover any major crashes of the markets, so volatility exhibits a rather smooth behavior. It would be interesting to apply the test to other time periods.

k	DEM/USD		DJIA		S&P500	
	$\alpha=0.05$	$\alpha=0.1$	$\alpha=0.05$	$\alpha=0.1$	$\alpha=0.05$	$\alpha=0.1$
2	0	0	0	0	0	0
3	0	0	0	0	1	1
4	0	0	1	1	0	0
5	1	1	1	1	0	0
6	0	0	1	1	2	2
10	1	1	1	1	0	2
12	1	1	1	1	1	2
20	2	3	1	1	1	3
50	5	7	2	3	4	6
100	12	15	10	13	9	9

Table 2: Number of rejections for k subsamples, each of size n/k , for two levels, $\alpha=0.05$ and $\alpha=0.1$.

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