Strings in plane wave backgrounds

D I S S E R T A T I O N

zur Erlangung des akademischen Grades
doctor rerum naturalium
(Dr. rer. nat.)
im Fach Physik

eingereicht an der
Mathematisch-Naturwissenschaftlichen Fakultät I
Humboldt-Universität zu Berlin

von
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geboren am 02.11.1974 in Heidelberg

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eingereicht am:	03. April 2003
Tag der mündlichen Prüfung:	13. Juni 2003

Abstract

The interplay between string and gauge theory has led to many new insights in recent years. The most prominent example is the AdS/CFT correspondence, a duality between string theory on Anti-de Sitter (AdS) spaces and conformal gauge theories defined on their boundary. The study of string theory on plane wave backgrounds, which are connected to AdS by the Penrose limit, opens up the possibility of testing this duality beyond the low-energy supergravity approximation. Generalized plane wave geometries are interesting in themselves, as they provide a large class of exact classical space-time backgrounds for string theory.

In this thesis aspects of string theory on plane wave backgrounds are studied, with an emphasis on the connection to gauge theory. String interactions in the plane wave space-time with maximal supersymmetry are investigated in the framework of light-cone string field theory. In the process, many results that had been found for the case of flat Minkowski space-time are generalized to the more complex plane wave background. The leading non-planar corrections to the anomalous dimensions of gauge theory operators dual to string states are recovered within light-cone string field theory.

Keywords: String theory, AdS/CFT correspondence, Penrose limit and pp-wave background, Light-cone string field theory

Zusammenfassung

Das Wechselspiel zwischen String- und Eichtheorien hat in den letzten Jahren zu vielen neuen Einsichten geführt. Das herausragendste Beispiel ist die sogenannte AdS/CFT Korrespondenz, eine Dualität zwischen Stringtheorien auf Anti-de Sitter-Räumen (AdS) und konformen Eichtheorien auf deren Rand. Die Untersuchung von Stringtheorie auf ebenfrontigen Gravitationswellen, die sich im sogenannten Penrose-Limes aus AdS-Raumzeiten gewinnen lassen, erlaubt es, diese Dualität über die niederenergetische Supergravitationsnäherung hinausgehend zu überprüfen. Verallgemeinerte ebenfrontige Gravitationswellen sind auch für sich gesehen interessant, da sie eine grosse Klasse von Raumzeiten bilden, die exakte klassische Lösungen der Stringtheorie sind.

In dieser Arbeit werden Aspekte der Stringtheorie auf ebenfrontigen Gravitationswellen untersucht. Besonderes Interesse gilt dabei der Verbindung dieser Stringtheorien zu Eichtheorien. Wechselwirkungen von Strings in derjenigen Gravitationswellen-Raumzeit mit maximaler Supersymmetrie werden im Rahmen der Lichtkegel-Stringfeldtheorie behandelt. Viele Ergebnisse, die für den Fall der flachen Minkowski-Raumzeit bekannt sind, werden dabei vollständig auf die komplizierteren ebenfrontigen Gravitationswellen verallgemeinert. Die führenden nicht-planaren Korrekturen zu den anomalen Dimensionen von Operatoren in der Eichtheorie, die eine duale Beschreibung von Stringzuständen liefern, werden innerhalb der Lichtkegel-Stringfeldtheorie reproduziert.

Schlagwörter: Stringtheorie, AdS/CFT Korrespondenz, Penrose-Limes und ebenfrontige Gravitationswellen, Lichtkegel-Stringfeldtheorie

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Chapter 1

Introduction

1.1 Motivation

The intimate connection between string and gauge theories has been one of the dominant themes in theoretical high energy physics over the last years. A famous example is the equivalence (duality) of string theory on Anti-de Sitter spaces with conformal field theories, the AdS/CFT correspondence [1, 2, 3], see [4] for a review.

Several arguments support the expectation of a duality between string and gauge theories or, even more generally, gravitational and non-gravitational theories. For example, a qualitative one comes from the fact that QCD, the $SU(3)$ gauge theory of strong interactions, confines chromoelectric flux to flux tubes – the QCD string – at low energies. After all, this is how string theory was originally discovered. A quantitative argument is 't Hooft's analysis of the large N limit of $SU(N)$ gauge theories [5]. 't Hooft showed that for large N and fixed 't Hooft coupling $\lambda = g_{\rm YM}^2 N$, the Feynman diagram expansion can be rearranged according to the genus g of the Riemann surface which the diagram can be drawn on and every amplitude can be written in an expansion of the form $\sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda)$, i.e. $1/N^2$ is the effective genus counting parameter. This is like the perturbation series of a string theory, where the string coupling $g_{\rm s}$ is identified with $1/N$ and λ corresponds to the loop-counting parameter of the string non-linear σ-model. This a very general argument for the large N duality between gauge theories and certain string theories, but it does not give an answer to what kind of string theory one should look for.

Further hints come from the study of black holes. The simplest exam-

ple is the Schwarzschild solution of general relativity depending on a single parameter, the mass M of the black hole. They have a horizon and are black classically, everything crossing the horizon is inevitably pulled into the black hole singularity. However, semi-classical analysis shows that due to quantum processes black holes start to emit Hawking radiation: the emission spectrum is roughly that of a blackbody with temperature $T \sim 1/M$; the deviation of the pure blackbody spectrum is encoded in the so called 'greybody factor'. As radiating systems black holes are expected to obey the laws of thermodynamics. If one defines the black hole entropy, as first proposed by Bekenstein and Hawking by $S = \frac{1}{4}A \sim M^2$, A the area of the black hole horizon, these laws are in fact satisfied. A quantum theory of gravity should e.g. provide the framework for a microscopic derivation of the black hole entropy via a counting of states and predict its greybody factor. As the Bekenstein-Hawking entropy involves the area instead of the volume, as is the case for statistical mechanics and local quantum field theories, one may wonder if one can find a holographic description in terms of local quantum field theories 'living' on the horizon, such that $S_{\rm QFT} \sim A$. More generally, the holographic principle [6, 7] asserts that the number of degrees of freedom of quantum gravity on some manifold scales as the area of its boundary: this suggests that a field theory on the boundary of space-time might capture the physics of gravity in the bulk. For reviews of the holographic principle see [8, 9]; for an introduction on black holes in string theory see e.g. [10].

The AdS/CFT correspondence explicitly realizes the general principles of large N duality and holography. The simplest and best understood example is the equivalence of string theory on $AdS_5 \times S^5$ and the maximally supersymmetric gauge theory in four dimensions, $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills (SYM). The latter arises as the low-energy (i.e. energies much smaller than the string scale $1/\sqrt{\alpha'}$) effective theory on the world-volume of N D3-branes. As these are charged under the R-R four-form potential [11], their presence generates a five-form flux in the (flat) transverse six-dimensional space. This flux contributes to the energy-momentum tensor, so the geometry backreacts and curves. The backreaction is negligible if the effective gravitational coupling is small, which is the case if $g_{\rm s}N \sim g^2_{\rm YM}N \ll 1$. In this regime the gauge theory is weakly coupled. In the regime of strong coupling, the large N limit, the backreaction is no longer small and the geometry will change significantly. To be more precise, for $1 \ll g_{\rm s}N < N$ we can use the dual description of D3-branes in terms of extremal three-branes in type IIB supergravity [11]: in this picture, considering low-energy excitations on the D3-brane, trans-

lates to going to the near-horizon region of the three-brane since energies are red-shifted for an asymptotic observer [1]. The near-horizon region has the geometry of $AdS_5 \times S^5$ with radii $R^4/\alpha'^2 = g_{\rm YM}^2 N$ and the five-form flux on the S^5 equals N, the number of colors in the gauge theory. Strongly coupled $\mathcal{N} = 4$ SYM is identified with supergravity (since the curvature $\alpha'/R^2 \ll 1$) on $AdS_5 \times S^5$. It is believed that this duality is true for all values of parameters and extends to the full string theory; this however is difficult to verify with the present technology, though there are some exceptions, see [4]. For a review of attempts to use AdS/CFT as a starting point to obtain a string description of QCD or at least of pure $\mathcal{N} = 1$ SYM, see [12].

It was realized by Berenstein, Maldacena and Nastase (BMN) [13] that plane (or pp) wave backgrounds provide an interesting example where the string/gauge correspondence can be studied beyond the supergravity approximation. As will be explained in detail in what follows, on the geometric side this involves the Penrose limit [14] applied for example to $AdS_5 \times S^5$; roughly speaking, one focuses on the neighborhood of the geodesic of a massless particle, in the center of AdS_5 and rotating on the S^5. String theory in the resulting plane wave background can be exactly quantized in light-cone gauge [15]. On the other hand, in the gauge theory this limit singles out composite operators carrying a large charge [13]. Though I will not discuss this here, let me mention that one can also consider macroscopic rotating strings vs. large spin operators [16].

1.2 Outline

This thesis is organized as follows: chapter 2 starts with a fairly general introduction to pp-wave backgrounds in ten/eleven-dimensional supergravities. I discuss various basic aspects of these backgrounds, in particular their (super)symmetries, emphasizing the emergence of special maximally supersymmetric solutions that will play a major role in the rest of the thesis. Then I introduce the notion and properties of the Penrose limit of a spacetime and show that this connects maximally supersymmetric pp-waves to the $AdS \times S$ spaces. Having introduced the necessary background material, the correspondence between IIB string theory on the maximally supersymmetric plane wave and a double scaling limit of $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills will be derived from the AdS/CFT correspondence. Several features of this novel BMN correspondence, for example symmetries, the comparison of states and

spectra, and holography, will be discussed in detail both from the (free) string theory and the gauge theory point of view.

Chapter 3 presents extensions of the BMN duality. First an overview over various possible approaches is given to provide a feeling for the general picture that emerges. The ingredients are then used to describe in detail the specific example of the duality between strings on supersymmetric orbifolds of the plane wave background and $\mathcal{N} = 2$ quiver gauge theories. In addition to these generalizations, further issues to be discussed include D-branes on the plane wave and more complicated pp-wave backgrounds leading to interacting world-sheet theories.

We return to string theory on the plane wave background in chapter 4, where string interactions are introduced. These correspond to non-planar corrections in the (interacting) dual gauge theory. I explain why it is natural to describe string interactions in the setup of light-cone string field theory and discuss its principles, in particular additional complications arising in the superstring as compared to its bosonic version. To make the presentation self-contained a review of the free string is included. In the following, the full construction of the cubic interaction vertex as well as the dynamical supercharges is presented; the focus is mostly on the general methods and technical details are relegated to two appendices. The results thus obtained are applied to compute the mass shift of certain string states induced by interactions. In an approximation to be specified, the leading non-planar corrections to the anomalous dimension of the dual gauge theory operators are exactly recovered within string theory.

Finally, I conclude in chapter 5 and discuss some open problems and possible directions for further research.

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This thesis is based on the publications [17, 18, 19].

Chapter 2

Strings on the plane wave from gauge theory

2.1 pp-waves in supergravity

It is known that maximally supersymmetric backgrounds of 11-dimensional supergravity include flat Minkowski space, $AdS_4 \times S^7$ and $AdS_7 \times S^4$ [20]. In addition to these three spaces there is another maximally supersymmetric solution discovered by Kowalski-Glikman [21]. This solution – which will be referred to as the KG space – arises as a special case of the more general pp-wave[1] solutions [22] of the form

$$\begin{aligned} ds^2 &= 2dx^+dx^- + H(x^I, x^+)\big(dx^+\big)^2 + dx^I dx^I \,, \\ F_4 &= dx^+ \wedge \varphi(x^I, x^+) \,, \end{aligned} \tag{2.1}$$

where I labels the transverse nine-dimensional space, F_4 is the four-form field strength of 11d supergravity and H obeys

$$\Delta H = -\varphi^2 \,, \qquad \varphi^2 \equiv \frac{1}{3!}\varphi_{IJK}\varphi^{IJK} \,. \tag{2.2}$$

Δ is the Laplacian in the transverse space $\mathbb{E}^9$ and φ is closed and co-closed in $\mathbb{E}^9$. $\partial/\partial x^-$ is a covariantly constant null vector. For constant φ this solution preserves at least 16 supercharges [22, 23]. An important subclass of solutions are the homogeneous plane wave space-times, where the field

[1] pp-wave geometries are space-times admitting a covariantly constant null vector field.

strength is constant and H is independent of x^+ and quadratic in the x^I

$$H(x^I) = A_{IJ}x^I x^J \,, \tag{2.3}$$

with A a constant, symmetric matrix. In this case the metric describes a Lorentzian symmetric space G/K with $K = \mathbb{R}^9$ and G a (solvable) Lie group depending on A [24, 23]. Solutions of this kind are space-times with a null ($F_4^2 = 0$) homogeneous flux and were referred to as Hpp-waves in [23]. Up to the overall scale and permutations these solutions are parameterized by the eigenvalues of A. Modulo diffeomorphisms, there is precisely one choice for A for which the solution is maximally supersymmetric. This is the KG solution

$$A_{IJ} = \begin{cases} -\frac{1}{9}\delta_{IJ} \,, & I,J = 1,2,3 \\ -\frac{1}{36}\delta_{IJ} \,, & I,J = 4,\ldots,9 \end{cases} \qquad \varphi = dx^1 \wedge dx^2 \wedge dx^3 \,. \tag{2.4}$$

Let me briefly sketch the derivation of some of the statements that I made above. It is possible to verify that the pp-wave geometry in equation (2.1) is a solution of the supergravity equations of motion provided the conditions on φ and H are satisfied. To analyze the number of preserved supersymmetries one has to consider the Killing spinor equation. A solution to the supergravity equations of motion is supersymmetric if it is left invariant under some non-trivial supersymmetry transformation. If the fermions have been put to zero in the solution non-trivial conditions following from the requirement of unbroken supersymmetry only arise in the transformation of the fermions themselves. The gravitino transformation law gives rise to the Killing spinor equation

$$\delta_\varepsilon \psi_M = \mathcal{D}_M \varepsilon = 0 \,, \tag{2.5}$$

where the supercovariant derivative is

$$\mathcal{D}_M \varepsilon = \nabla_M \varepsilon - \frac{1}{288}\left(\Gamma^{PQRS}{}_M + 8\Gamma^{PQR}\delta^S_M\right) F_{PQRS}\varepsilon \,. \tag{2.6}$$

Iterating the first order Killing equation implies the second order supergravity equations of motion. In other supergravities containing additional bosonic and fermionic fields the number of unbroken supersymmetries may be further constrained by algebraic equations arising from the variations of other fermions, such as for example the dilatino in type IIB supergravity. Computing the supercovariant derivative in the background equation (2.1) and

solving the Killing equation leads to [22]

$$\partial_+\varepsilon = \frac{1}{24}\varphi_{IJK}\Gamma^{IJK}\varepsilon\,, \qquad \Gamma_-\varepsilon = 0\,, \tag{2.7}$$

where $\varepsilon = \varepsilon(x^+)$ is only a function of x^+ and φ is assumed to be constant. This is a first order ordinary differential equation, which has a unique solution for each initial value. Hence, for constant field strength, the background in equation (2.1) generically preserves 16 supersymmetries. If one chooses the three-form φ and the matrix A to be of the form given in equation (2.4) spinors satisfying $\Gamma_+\varepsilon = 0$ solve the Killing equation as well [25, 23] and hence the KG solution is maximally supersymmetric. The fact that the Hpp-wave geometry is a Lorentzian symmetric space can be seen as follows [23]: consider the 20-dimensional Lie algebra

$$[e_+, e_I] = e_I^*\,, \qquad [e_+, e_I^*] = A_{IJ}e_J\,, \qquad [e_I^*, e_J] = A_{IJ}e_-\,, \tag{2.8}$$

which is isomorphic to $\mathfrak{h}(9) \rtimes \mathbb{R}$, $\mathfrak{h}(9)$ the Heisenberg algebra generated by $\{e_I, e_I^*, e_-\}$, e_- being the central element, and e_+ an outer automorphism which rotates coordinates $\{e_I\}$ and momenta $\{e_I^*\}$. The Hpp-wave space-time can then be constructed as the coset G/K, where G is the Lie group with the algebra in (2.8) and K is generated by $\{e_I^*\}$ [23]. To verify this one proceeds in the standard way by choosing a representative of the coset and solving the Cartan-Maurer equations. Notice that the inclusion of the form flux respects these symmetries as F_4 is parallel. For a generic Hpp-wave background these are all the isometries, in special cases however, the number of isometries is enlarged due to symmetries of A and F_4. For example, for the KG solution the isometry is enhanced to a semi-direct product

$$\mathfrak{h}(9) \rtimes \big(\mathfrak{so}(3) \oplus \mathfrak{so}(6) \oplus \mathbb{R}\big)\,, \tag{2.9}$$

due to the degeneracy of the eigenvalues of A. Notice that the dimension of the isometry algebra of the KG solution is 38, which equals the dimension of the isometry algebras of the two other non-trivial maximally supersymmetric solutions $AdS_4 \times S^7$ and $AdS_7 \times S^4$ ($\mathfrak{so}(3,2) \oplus \mathfrak{so}(8)$ and $\mathfrak{so}(6,2) \oplus \mathfrak{so}(5)$, respectively). One suspects that this is not merely a coincidence. Recall that flat space and $AdS_4 \times S^7$ ($AdS_7 \times S^4$) play the role of asymptotic and near-horizon limits of the M2-brane (M5-brane) and as such are related to each other. Is there a connection to the KG solution as well? I will say more about this in the next section. The full superalgebra can be obtained by

utilizing the fact that for ε_1, ε_2 Killing spinors, $\bar{\varepsilon}_1\Gamma^M\varepsilon_2$ is a Killing vector and by analyzing the transformations of Killing spinors induced by the action of the Killing vectors. This has been done in [23] to which I refer for details.

The story is similar for type IIB supergravity [26]. The analogue of equation (2.1) is

$$\begin{aligned} ds^2 &= 2dx^+dx^- + H(x^I,x^+)\big(dx^+\big)^2 + dx^I dx^I\,, \\ F_5 &= dx^+\wedge\varphi(x^I,x^+)\,, \end{aligned} \tag{2.10}$$

with the dilaton being constant and all other supergravity fields set to zero. The equations of motion for F_5 require that the four-form φ is self-dual and closed in $\mathbb{E}^8$ and hence also co-closed. Again, H has to satisfy the Poisson equation in transverse space

$$\Delta H = -\frac{1}{2}\varphi^2\,, \qquad \varphi^2 \equiv \frac{1}{4!}\varphi_{IJKL}\varphi^{IJKL}\,. \tag{2.11}$$

For φ constant, this solution preserves as least 16 supersymmetries [26]. In analogy with the 11d case, the subclass of solutions in which H is of the form (2.3), describe Lorentzian symmetric space-times with homogeneous five-form flux. There is again one exceptional, maximally supersymmetric solution [26]

$$A_{IJ} = -\mu^2\delta_{IJ}\,, \qquad \varphi = 4\mu\big(dx^1\wedge dx^2\wedge dx^3\wedge dx^4 + dx^5\wedge dx^6\wedge dx^7\wedge dx^8\big)\,. \tag{2.12}$$

Here μ is a parameter with dimension of mass, which by a rescaling of x^+ and x^- can be set to any non-zero value. It has become common in the literature to refer to this solution as the *plane wave background*. The isometry algebra of the plane wave solution is

$$\mathfrak{h}(8) \rtimes \big(\mathfrak{so}(4)\oplus\mathfrak{so}(4)\oplus\mathbb{R}\big)\,. \tag{2.13}$$

Notice that the metric by itself has an $\mathfrak{so}(8)$ symmetry, which however, is broken by the R-R field strength to $\mathfrak{so}(4)\oplus\mathfrak{so}(4)$. The isometry group also contains a discrete $\mathbb{Z}_2$ exchanging the two transverse $\mathbb{R}^4$'s. The dimension of the isometry algebra is 30 – again the same as of the $\mathfrak{so}(4,2)\oplus\mathfrak{so}(6)$ of $AdS_5\times S^5$. Let me be more explicit about the Killing vectors of the plane wave solution generating the algebra $\mathfrak{h}(8)\rtimes\mathbb{R}$. A convenient parametrization

is [26] [2]

$$\begin{aligned} P^- &= -i\partial_+ \,, \qquad P^+ = -i\partial_- \,, \\ P^I &= -i\cos(\mu x^+)\partial_I - i\mu \sin(\mu x^+) x_I \partial_- \,, \\ J^{+I} &= -i\mu^{-1}\sin(\mu x^+)\partial_I + i\cos(\mu x^+) x_I \partial_- \,. \end{aligned} \tag{2.14}$$

They obey the algebra

$$[P^-, P^I] = i\mu^2 J^{+I} \,, \qquad [P^I, J^{+J}] = i\delta_{IJ} P^+ \,, \qquad [P^-, J^{+I}] = -iP^I \,, \tag{2.15}$$

and transform in the obvious way under the transverse $\mathfrak{so}(4) \oplus \mathfrak{so}(4)$. The generators $\{P^I, J^{+I}, P^+, P^-\}$ are hermitian and related to $\{e_I, e_I^*, e_-, e_+\}$ by trivial rescaling. It is convenient to work with the former to make the flat space limit $\mu \to 0$ manifest. I will present some of the remaining (anti)commutation relations of the plane wave superalgebra in chapter 4 when I need them, see [26] for the full algebra.

One might wonder if there are any further maximally supersymmetric solutions of ten/eleven-dimensional supergravities, however, as was proved in [27] by careful analysis of the constraints arising from the supersymmetry variations, this is not the case. It is instructive to discuss the issue of supersymmetry in Hpp-wave backgrounds in more detail, in particular the dependence of the Killing spinors on the coordinate x^+. For φ constant and hence H independent of x^+, the Killing spinors of the background (2.10) are independent of x^- and can be expressed as [26]

$$\varepsilon = \left(1 + \frac{i}{2} x^I \Gamma_- [\Gamma_I, W]\right) \chi \,, \qquad W \equiv \frac{1}{4!} \varphi_{IJKL} \Gamma^{IJKL} \,, \tag{2.16}$$

where χ has only x^+ dependence determined by

$$\big(\partial_+ + iW\big)\chi = 0 \,. \tag{2.17}$$

Additionally one has the requirement that

$$\big(x^I W^2 + 32\partial^I H\big)\Gamma_I \Gamma_- \chi = 0 \,. \tag{2.18}$$

This equation determines the number of Killing spinors. As $\chi = \Gamma_- \chi_0$ is a solution for any H satisfying equation (2.11), the generic Hpp-wave background has 16 *standard* Killing spinors [28]. By equation (2.16) these are also

[2] Strictly speaking one should write P_+ instead of P^- since indices are raised and lowered with the plane wave metric and g_{++} is non-zero. So $P^- \equiv P_+$ by definition.

independent of the x^I. Generically the standard spinors depend on the coordinate x^+ but they are independent of it if $W\chi = 0$. This equation may or may not have solutions depending on the explicit form of the four-form φ. If H is quadratic in x^I the above equation may admit additional Killing spinors χ that are annihilated by Γ_+. These *supernumerary* spinors are always independent of x^+ [28] but depend on the x^I via equation (2.16). Performing a T-duality along x^+, those Killing spinors which are independent of x^+ will survive at the level of the low-energy effective field theory and the resulting type IIA solution will also be supersymmetric.[3] So in the generic case (only standard Killing spinors, all depending on x^+), performing a T-duality along x^+ results in a non-supersymmetric solution of type IIA supergravity. In special cases like the plane wave background (16 supernumerary spinors), the IIA solution will be supersymmetric. Lifting this solution to 11 dimensions gives rise to a supersymmetric deformed M2-brane with additional four-form flux [28]. One can also revert this logic [29] and analyze the Killing spinors of the 11d Hpp-waves. In this case the supernumerary Killing spinors generically also depend on x^+. Dimensionally reducing the Hpp-wave on x^+ or x^I (provided the latter is a Killing direction) one gets a D0-brane or IIA pp-wave, respectively and the number of preserved supersymmetries is again determined by the coordinate dependence of the Killing spinors in 11 dimensions.

2.2 The Penrose-Güven limit

We have seen in the previous section that ten/eleven-dimensional supergravities admit maximally supersymmetric solutions of the pp-wave type, the plane-wave background and the KG solution, respectively. These are on equal footing with the other more standard maximally supersymmetric backgrounds, that is flat space and the $AdS \times S$ solutions. But whereas the latter are connected being the asymptotic and near-horizon regions respectively of fundamental branes, no such connection was known for the pp-waves. I have already mentioned that the dimensions of the superalgebras of the KG and plane wave solutions agree with those of $AdS \times S$, so one might expect that there exists a connection between the two. In fact it does [30] and the connection is the Penrose-Güven limit as defined originally by Penrose [14] and

[3] In the full string theory including winding states, all supersymmetries must survive as T-duality is an exact symmetry.

extended to supergravity by Güven [31]. I review this limit below.

Consider a Lorentzian space-time and a null geodesic γ in it. According to [14, 31] for a sufficiently well-behaved geodesic one can introduce local coordinates U, V and Y^I such that the metric in the neighborhood of γ takes the form

$$ds^2 = dV\left(dU + \alpha dV + \beta_I dY^I\right) + C_{IJ}dY^I dY^J\,, \tag{2.19}$$

where α, β_I and C_{IJ} are functions of the coordinates. The coordinate U is the affine parameter of γ and for γ to be well-behaved C must be invertible, otherwise the coordinate system breaks down. Supergravities contain additional fields besides the metric, such as the dilaton Φ and p-form potentials A_p. In particular the p-forms have a gauge symmetry and this gauge freedom can be used to eliminate some of the components of A_p. Indeed, one can choose locally [31]

$$A_{UVI_1\cdots I_{p-2}} = 0 = A_{UI_1\cdots I_{p-1}}\,. \tag{2.20}$$

This is the starting point of the Penrose-Güven limit: a null geodesic γ which locally is described by the metric in equation (2.19) plus (possibly) additional background fields which are gauge fixed to have the local form in equation (2.20). The next step consists in introducing a real, positive constant Ω and rescaling the coordinates as

$$U = u\,, \qquad V = \Omega^2 v\,, \qquad Y^I = \Omega y^I\,. \tag{2.21}$$

This diffeomorphism results in a Ω-dependent family of fields $g(\Omega)$, $A_p(\Omega)$ and $\Phi(\Omega)$ and the coordinate choices in equations (2.19) and (2.20) ensure that the following Penrose limit [14], extended by Güven [31] to fields other than the metric, is well-defined:

$$\bar{g} = \lim_{\Omega\to 0}\Omega^{-2}g(\Omega)\,, \qquad \bar{A}_p = \lim_{\Omega\to 0}\Omega^{-p}A_p(\Omega)\,, \qquad \bar{\Phi} = \lim_{\Omega\to 0}\Phi(\Omega)\,. \tag{2.22}$$

Due to the rescaling of coordinates in (2.21) the limiting fields only depend on u and the background takes the form

$$\begin{aligned} ds^2 &= dudv + \bar{C}_{IJ}(u)dy^I dy^J\,,\\ \bar{F}_{p+1} &= du\wedge \bar{A}'_p(u)\,. \end{aligned} \tag{2.23}$$

Here $\bar{F}_{p+1}$ is the $(p+1)$-form field strength of $\bar{A}_p$ and $'$ denotes d/du. This background describes a pp-wave with null flux in *Rosen* coordinates [30]. It

is possible to change to *Brinkmann* coordinates, where the resulting metric takes the form

$$ds^2 = 2dx^+dx^- + A_{IJ}(x^+)x^I x^J \big(dx^+\big)^2 + dx^I dx^I \,, \tag{2.24}$$

considered in the previous section. For more details, see [30]. Before I explicitly show that this mechanism connects the KG and plane wave solutions with the *AdS* ones, it is instructive to discuss some important *hereditary* properties of the Penrose limit [32]. As we have seen, the Penrose limit basically consists of two steps, performing a diffeomorphism and gauge-fixing with a subsequent rescaling of the supergravity fields. It is a general property of supergravity actions that they transform homogeneously under the rescaling of fields in equation (2.22). Hence, if the original background is a solution to the supergravity equations of motion, so is the new Ω-dependent one for any $\Omega > 0$ and by continuity the limiting configuration (2.22) is a valid supergravity background [14, 31]. The Penrose limit inherits further properties of its parent solution, involving for example the curvature tensor; the Penrose limit of a conformally flat space-time is conformally flat, that of an Einstein space is Ricci-flat and another hereditary property is that of being locally symmetric, see for example [32]. One may also wonder about the fate of isometries and supersymmetries; these are hereditary in the sense that the resulting background has *at least* as many isometries and supersymmetries as the parent background [32]. Let me show that this is the case. Consider a Killing vector ξ of the metric g. Performing the rescaling of coordinates and fields in equations (2.21) and (2.22), $\xi \to \xi(\Omega)$ and $\xi(\Omega)$ is a Killing vector for the transformed metric $\Omega^{-2}g(\Omega)$ for non-zero Ω. The question is if a weight Δ_ξ exists such that the limit

$$\bar{\xi} = \lim_{\Omega\to 0} \Omega^{\Delta_\xi}\xi(\Omega) \,, \tag{2.25}$$

is both non-singular and non-zero. In the local coordinates adapted to the null geodesic ξ can be written as

$$\xi = \alpha(U,V,Y^I)\partial_U + \beta(U,V,Y^I)\partial_V + \gamma^I(U,V,Y^I)\partial_{Y^I} \,. \tag{2.26}$$

Performing the rescaling of coordinates one can expand $\xi(\Omega)$ around $\Omega = 0$ as

$$\Omega^2\xi(\Omega) = \bar{\beta}(u)\partial_v + \Omega\big(\bar{\gamma}^I(u)\partial_{y^I} + y^I\partial_{y^I}\bar{\beta}(u)\partial_v\big) + \cdots \tag{2.27}$$

Then for Ω^{k_ξ} being the coefficient of the first non-vanishing term in this expansion

$$\bar{\xi} = \lim_{\Omega \to 0} \Omega^{2-k_\xi} \xi(\Omega) \tag{2.28}$$

is finite and non-zero. Now suppose we have two linearly independent Killing vectors ξ_1 and ξ_2. Then it might happen that their leading order terms in a small-Ω expansion are linearly dependent, for definiteness assume they are equal. Do we loose a Killing vector here? Consider the difference

$$\xi_-(\Omega) = \xi_1(\Omega) - \xi_2(\Omega)\,. \tag{2.29}$$

By construction the leading order term is zero. The next to leading term defines a new Killing vector $\bar{\xi}_-$. If $\bar{\xi}_-$ and $\bar{\xi}_1$ are linearly independent we are done, if not one has to iterate the procedure. One can show [32] that eventually one ends up with two linearly independent Killing vectors of the limiting space-time. Hence the number of Killing vectors never decreases in the Penrose-Güven limit. Notice however that it may very well happen that it *increases*. This is because we have seen that the resulting space-time is of the Hpp-wave form and as we know from the previous section this space-time has always an isometry algebra isomorphic to a $(2D-3)$-dimensional Heisenberg algebra plus outer automorphism (in D dimensions). So some isometries need not have a counterpart in the original space-time and can arise only in the limit $\Omega \to 0$. It is also important to realize that because different Killing vectors ξ may have to be rescaled with different weights Δ_ξ the original isometry algebra may get contracted in the limit. The discussion of the hereditary properties of Killing spinors is similar. Again, no supersymmetries are lost in the limit, though the number of Killing spinors may increase (as we have seen Hpp-waves preserve at least 16 supersymmetries). For a more detailed and rigorous discussion see [32].

The information acquired above is already quite powerful. Consider for example the Penrose limit of AdS. Anti de-Sitter is a conformally flat, locally symmetric, Einstein space. The limiting space-time is Ricci-flat, conformally flat and locally symmetric and hence isometric to flat Minkowski space. We are primarily interested in the maximally supersymmetric $AdS \times S$ backgrounds. Now the result depends on the geodesic: if it lies purely in AdS we get Minkowski space (the sphere is blown up to flat space in the limit as well); if not it follows from the hereditary properties that we have to get the KG solution and the plane wave background as limiting space-times [30, 32]. I will also show this explicitly below for the case of $AdS_5 \times S^5$. For $AdS_4 \times S^7$

and $AdS_7 \times S^4$ the Penrose-Güven limits are isomorphic to each other and result in the KG solution [30].

The spaces $AdS_{p+2} \times S^{D-p-2}$ with radii of curvature related by $R_{AdS}/R_S = \rho$ provide an example which illustrates the above behavior of isometries [32]. The original isometry algebra is $\mathfrak{so}(2, p+1) \oplus \mathfrak{so}(D-p-1)$. The $\mathfrak{so}(2, p+1)$ factor is contracted to $\mathfrak{h}(p+1) \rtimes \mathfrak{so}(p+1)$. The $p+1$ creation- and $p+1$ annihilation operators transform as vectors under $\mathfrak{so}(p+1)$. Similarly $\mathfrak{so}(D-p-1)$ contracts to $\mathfrak{h}(D-p-3) \rtimes \mathfrak{so}(D-p-3)$. The central elements of the two Heisenberg algebras coincide; this is due to the fact that two Killing vectors of the parent space-time agree to leading order in small Ω. Thus the two Heisenberg algebras combine into $\mathfrak{h}(D-2)$. The remaining Killing vector $\bar{\xi}_-$ becomes an outer automorphism and the resulting contracted algebra is [32]

$$\mathfrak{h}(D-2) \rtimes \big(\mathfrak{so}(p+1) \oplus \mathfrak{so}(D-p-3) \oplus \mathbb{R}\big)\,. \tag{2.30}$$

If the radii of curvature are equal (as is the case for $p = 3$) the subalgebra $\mathfrak{so}(p+1) \oplus \mathfrak{so}(D-p-3)$ is enlarged to the full $\mathfrak{so}(D-2)$. This has no counterpart in the original background.

Finally, consider the Penrose-Güven limit of $AdS_5 \times S^5$ explicitly. The dilaton is constant and in global coordinates the metric and five-form flux is

$$\begin{aligned} ds^2 &= R^2\big[-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_3^2 + \cos^2\theta d\psi^2 + d\theta^2 + \sin^2\theta d\Omega_3'^2\big]\,, \\ F_5 &= 4R^4\big[\cosh\rho \sinh^3\rho dt \wedge d\rho \wedge d\Omega_3 + \cos\theta \sin^3\theta d\psi \wedge d\theta \wedge d\Omega_3'\big], \end{aligned} \tag{2.31}$$

where $R^4 \equiv 4\pi g_s \alpha'^2 N$ and $\rho \geq 0$, $t \in \mathbb{R}$, $\psi \in [0, 2\pi]$ and $\theta \in [0, \frac{\pi}{2}]$. As alluded to above, in order that the limiting space-time will be non-trivial the null geodesic must not lie purely within AdS_5; so consider a massless particle sitting at the origin of AdS_5 ($\rho = 0$) and rotating around the circle of the S^5 parameterized by ψ and $\theta = 0$ [30, 13]. To focus on the geometry in the neighborhood of this geodesic the coordinates are rescaled such that a tube around the geodesic is blown up. Explicitly, introduce light-cone coordinates $x^\pm$ and perform a rescaling

$$x^+ = \frac{1}{2\mu}(t+\psi)\,, \qquad x^- = -\mu R^2(t-\psi)\,, \qquad \rho = \frac{r}{R}\,, \qquad \theta = \frac{y}{R}\,, \tag{2.32}$$

where μ is an arbitrary (non-zero) mass parameter. Blowing up the neighborhood of the geodesic is equivalent to taking $R \to \infty$ and the metric and

five-form flux become

$$\begin{aligned} ds^2 &= 2dx^+dx^- - \mu^2\vec{x}^2\big(dx^+\big)^2 + d\vec{x}^2\,, \\ F_5 &= 4\mu dx^+ \wedge \big(dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 + dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8\big)\,. \end{aligned} \tag{2.33}$$

This is the plane wave solution of type IIB supergravity [30].

2.3 The BMN correspondence

In the previous section I reviewed the connection of $AdS_5 \times S^5$ and the plane wave background via the Penrose-Güven limit. This has far reaching consequences. First of all it turns out that strings on the plane wave can be exactly quantized in light-cone gauge [15], in spite of the presence of the constant R-R flux. Moreover, as IIB string theory on $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills by the AdS/CFT correspondence [1, 2, 3, 4] the implications of the Penrose-Güven limit on the dual CFT can be studied. One may hope that this simpler setup allows to extend our understanding of the AdS/CFT duality beyond the supergravity approximation by the inclusion of string states on the plane wave. This is indeed the case as was demonstrated by Berenstein, Maldacena and Nastase in [13]. The formulation of the BMN correspondence is the subject of this section.

Following the construction of the type IIB superstring action on $AdS_5 \times S^5$ using superspace coset methods [33], the action on the plane wave background was constructed by Metsaev in [15]. Let me briefly sketch this construction. The action has to obey the following conditions: its bosonic part is the σ-model with the plane wave geometry being the target space; it is globally supersymmetric with respect to the plane wave superalgebra and locally κ-symmetric; it reduces to the standard Green-Schwarz action in the flat space limit. As shown in [15] this conditions uniquely specify the action, which as in flat space can be written as a sum of a 'kinetic' σ-model term and a Wess-Zumino term. The latter is needed to obey the condition of κ-symmetry. To find the explicit form of the superstring action in terms of the coordinate (super)fields a parametrization of the coset representative has to be specified and the Cartan-Maurer equations have to solved. Not surprisingly, the resulting covariant action is non-polynomial [15]. The simplest way to proceed is to study the action in light-cone gauge. As in flat space the light-cone gauge-fixing procedure consists of two steps, first κ-symmetry is fixed by the fermionic light-cone gauge choice $\Gamma^+ S = 0$, then the diffeomorphism

and Weyl-symmetry on the world-sheet is fixed by the bosonic light-cone gauge $\sqrt{-g}g^{ab} = \eta^{ab}$ and $x^+(\sigma,\tau) = \tau$. The resulting action is quadratic in both bosonic and fermionic superstring $2d$ fields, and hence can be quantized explicitly [15]. In fact, from the form of the metric in equation (2.33), it is obvious that the action for the eight transverse directions in light-cone gauge is just that for eight bosons of mass μ. Similarly the fermions acquire masses due to the coupling to the R-R background [34]. Masses of bosons and fermions are equal due to world-sheet supersymmetry: after imposing the light-cone gauge conditions the world-sheet κ-symmetry and space-time supersymmetries transmute into rigid world-sheet supersymmetries. As in flat space 16 of the 32 supersymmetries are linearly realized in light-cone gauge and commute with the Hamiltonian [15]. It was shown in [35] that the linearly realized supersymmetries correspond to the supernumerary Killing spinors of the pp-wave backgrounds. This is in agreement with their independence of x^+ [28] (cf. section 2.1).

After gauge-fixing the light-cone action becomes [15, 34]

$$S_{\text{l.c.}} = \frac{1}{2\pi\alpha'}\int d\tau \int_0^{2\pi\alpha' p^+} d\sigma \left[\frac{1}{2}\dot{x}^2 - \frac{1}{2}x'^2 - \frac{1}{2}\mu^2 x^2 + i\bar{S}\big(\partial\!\!\!/ + \mu\Pi\big)S\right], \tag{2.34}$$

where $\Pi = \Gamma^1\Gamma^2\Gamma^3\Gamma^4$ and S is a Majorana spinor on the world-sheet and a positive chirality $SO(8)$ spinor under rotations in the eight transverse directions. It is not difficult to quantize this action and the resulting light-cone Hamiltonian is [15, 34]

$$H = \mu\sum_{n\in\mathbb{Z}} N_n \sqrt{1 + \frac{n^2}{\big(\mu\alpha' p^+\big)^2}}\,. \tag{2.35}$$

Here n is a label for the Fourier mode and N_n is the occupation number of that mode including bosons and fermions. The ground state energy is cancelled between bosons and fermions. In contrast to flat space, modes with $n = 0$ are also harmonic oscillators due to the mass terms on the world-sheet and string theory on the plane wave has a unique ground state $|v, p^+\rangle$, p^+ the light-cone momentum. The single string Hilbert space is built by acting with the bosonic and fermionic creation oscillators (for all n) on $|v, p^+\rangle$ subject to the level-matching condition for physical states

$$\sum_{n\in\mathbb{Z}} nN_n = 0\,. \tag{2.36}$$

Truncation to the zero-mode sector gives rise to the spectrum of IIB supergravity on the plane wave [34]. I will provide more details on the quantization of strings on the plane wave in section 4.1, where I need them.

To understand the effect of the Penrose-Güven limit on the dual CFT, consider the scaling behavior of the energy $E = i\partial_t$ and angular momentum $J = -i\partial_\psi$ of a state in $AdS_5 \times S^5$. Recall that the AdS/CFT correspondence relates the energy of a string state in $AdS_5 \times S^5$ to the energy of a state in $\mathcal{N} = 4$ SYM living on $\mathbb{R} \times S^3$ [2, 3], which is the (conformal) boundary of $AdS_5 \times S^5$ in global coordinates. By the operator-state map, the energy of a state on $\mathbb{R} \times S^3$, where the S^3 has unit radius, translates to the conformal dimension Δ of an operator on $\mathbb{R}^4$. Likewise, the angular momentum J on the S^5 translates to the R-charge under a $U(1)_R$ subgroup of the full $SU(4)_R \simeq SO(6)_R$ R-symmetry of $\mathcal{N} = 4$ SYM. Then we have the following relations

$$\begin{aligned} H &= -p_+ = i\partial_+ = i\mu(\partial_t + \partial_\psi) = \mu(\Delta - J)\,, \\ p^+ &= p_- = -i\partial_- = \frac{i}{2\mu R^2}(\partial_t - \partial_\psi) = \frac{\Delta + J}{2\mu R^2}\,. \end{aligned} \tag{2.37}$$

Now what happens if we apply the limit $R \to \infty$? Firstly, $R \to \infty$ means $N \to \infty$, the string coupling g_s and hence also $g_{YM}^2 = 4\pi g_s$ should be kept fixed. Then a configuration with fixed, non-zero p^+ requires to scale Δ, $J \sim \sqrt{N}$. In fact, the plane wave superalgebra implies that H and p^+ are non-negative or equivalently $\Delta \geq |J|$; this also follows from the representation theory of the 4d superconformal algebra. So the Penrose-Güven limit induces the following double-scaling or BMN limit in $\mathcal{N} = 4$ $SU(N)$ SYM [13]

$$N \to \infty \quad \text{and} \quad J \to \infty \quad \text{with} \quad \frac{J^2}{N} \quad \text{fixed}\,, \quad g_{YM} \quad \text{fixed}\,. \tag{2.38}$$

As a first check consider how the bosonic part of the plane wave superalgebra $\mathfrak{h}(8) \rtimes (\mathfrak{so}(4) \oplus \mathfrak{so}(4) \oplus \mathbb{R})$ is realized in the gauge theory on $\mathbb{R} \times S^3$. The conformal group $SO(4,2)$ is generated by the seven Killing vectors of $\mathbb{R} \times SO(4)$ and eight additional conformal Killing vectors. By singling out a $U(1)_R$ subgroup with generator J the $SO(6)_R$ symmetry is broken to $SO(4)_R \times U(1)_R$. So we see that the transverse symmetry corresponds to $SO(4)_R$ and the isometry group of the S^3 [13, 36]. In the BMN limit, the eight conformal Killing vectors together with the eight broken generators of R-symmetry give rise to a Heisenberg algebra $\mathfrak{h}(8)$ with central element J and outer automorphism

$E - J$, see for example [37, 38]. In other words the $\mathcal{N} = 4$ superalgebra contracts to the plane wave superalgebra in the Penrose-Güven limit. In the previous section I have argued that this is the case, see also [39] for an explicit demonstration. It is an important question how the unitary irreducible representations – e.g. composite operators in $\mathcal{N} = 4$ SYM – behave under the contraction [40]. In the limit they should form representations of the plane wave superalgebra. In particular, as the conformal dimension diverges in the BMN limit, the space-time dependence of their correlation functions is ill-defined and hence requires special treatment. One way to achieve this was proposed in [40] and requires to combine space-time with an auxiliary R-symmetry space much in the same way that Δ and J combine into the finite quantity $\Delta - J$. The manifestation of the discrete $\mathbb{Z}_2$ exchanging the two transverse $\mathbb{R}^4$'s in the gauge theory is somewhat mysterious.

The BMN limit is different from the 't Hooft limit of $SU(N)$ gauge theories and at first sight puzzling. To see why this is so, recall that the 't Hooft limit takes $N \to \infty$, $g_{\rm YM} \to 0$, such that the 't Hooft coupling $\lambda \equiv g_{\rm YM}^2 N$ is fixed. As shown by 't Hooft [5], away from the strict $N \to \infty$ limit all Feynman diagrams which contribute to a given order in $1/N$ can be drawn on a Riemann surface whose Euler number is precisely the power of N to which these diagrams contribute. So $1/N^2$ is identified with the genus counting parameter and the perturbation series of the gauge theory may then be organized in a double series expansion in the effective coupling λ and the genus counting parameter $1/N^2$. This is the standard lore why large N gauge theories are expected to be dual to some weakly coupled string theory with coupling $1/N$. The AdS/CFT correspondence provides a concrete example where this is realized. The above reasoning breaks down because operators in the field theory are not held fixed in the limit but acquire an infinite charge as $N \to \infty$. Indeed, using equation (2.37) and $(\Delta - J) \ll J$, in the BMN limit

$$\frac{1}{\left(\mu\alpha' p^+\right)^2} = \frac{g_{\rm YM}^2 N}{J^2} \equiv \lambda' \,, \qquad 4\pi g_{\rm s}\left(\mu\alpha' p^+\right)^2 = \frac{J^2}{N} \equiv g_2 \,. \tag{2.39}$$

These relations are quite suggestive. It looks like a new effective coupling λ' and a new effective genus counting parameter g_2^2 might develop as a consequence of the simultaneous infinite scaling of N and J. This is in some sense correct as I will explain in more detail below.

While most of the (unprotected) operators acquire infinite anomalous dimension and decouple in the BMN limit, it is conceivable that some (BMN)

operators with a suitable scaling of charge survive and be dual to string states in the plane wave background (for a general discussion, see [41]). At the planar level this class of operators has been identified in [13]. Recall that $\mathcal{N} = 4$ SYM contains six scalar fields ϕ^r of conformal dimension one transforming in the **6** of $SO(6)_R$. Take J to be the $U(1)_R$ generator rotating the 5-6-plane and define $Z = \frac{1}{\sqrt{2}}\big(\phi^5 + i\phi^6\big)$. Z carries unit J-charge and the remaining four scalars ϕ^i, $i = 1, \ldots, 4$ are invariant under $U(1)_R$. For simplicity, consider only single-trace operators for the moment. The operator corresponding to the string ground state should carry large J charge and have $\Delta - J = 0$. There is a unique single-trace operator satisfying this requirement which subsequently is identified with $|v, p^+\rangle$ [13]

$$\frac{1}{\sqrt{JN^J}}\mathrm{Tr}\big[Z^J\big] \longleftrightarrow |v, p^+\rangle\,, \tag{2.40}$$

where the trace is over color indices. At weak coupling the dimension of this operator is J since each Z field has dimension one. As the operator is a chiral primary [13] it is protected by supersymmetry and $\Delta - J = 0$ for all values of the coupling. The normalization is chosen such that the operator has normalized two-point function when we restrict ourselves to planar diagrams. However, non-planar diagrams do give a non-vanishing contribution in the BMN limit and the two-point function of $\mathrm{Tr}\big[Z^J\big]$ can be computed exactly for all genera [42, 43]. This can be understood by noting that at genus h diagrams are weighted by $1/N^{2h}$ as expected, but at the same time the number of diagrams grows as J^{4h}, see also [36, 44]. So we see the quantity g_2^2 emerging as the effective genus counting parameter for the above operator. This will also be true for more general BMN operators, to be described below. There is an additional complication: at finite g_2 single-trace operators are no longer orthogonal to multi-trace operators and it is therefore no longer justified to restrict attention to single-trace operators only. To simplify matters let me assume $g_2 = 0$ in what follows; then equation (2.40) is a precise identification. I will return to the issue of operator mixing below.

Next consider the supergravity states obtained by acting with the eight bosonic and fermionic zero-mode oscillators $a_0^{I\,\dagger}$ and $S_0^{a\,\dagger}$ on the plane wave vacuum. Each oscillator raises the energy by μ. In the gauge theory these are obtained by the action of the broken symmetries on the trace of Z's [13]. For example we can rotate Z into ϕ^i by a broken $SO(6)_R$ transformation.

Applying this to $\mathrm{Tr}\big[Z^{J+1}\big]$ one obtains [13]

$$\frac{1}{\sqrt{N^{J+1}}}\mathrm{Tr}\big[\phi^i Z^J\big] \longleftrightarrow a_0^{i\,\dagger}|v,p^+\rangle\,, \tag{2.41}$$

where the cyclicity of the trace was used. Acting a second time with such a transformation changes another Z to ϕ^j or, if $i=j$, ϕ^i to $\bar{Z}$. For $i\neq j$

$$\frac{1}{\sqrt{JN^{J+2}}}\sum_{l=0}^{J}\mathrm{Tr}\big[\phi^i Z^l\phi^j Z^{J-l}\big] \longleftrightarrow a_0^{i\,\dagger}a_0^{j\,\dagger}|v,p^+\rangle\,. \tag{2.42}$$

Similarly the action of broken superconformal symmetries give rise to insertions of $D_iZ=\partial_iZ+[A_i,Z]$ and the components of the gaugino with $J=1/2$, $\chi^a_{J=1/2}$, in the trace of Z's [13]. In this way one obtains a precise correspondence between supergravity states on the plane wave and (at the planar level) single-trace chiral primary operators. This is already known from the AdS/CFT correspondence [2, 3]. One of the crucial insights of [13] was to extend this identification to 'massive' string states. These are constructed similarly to the above but now each insertion is accompanied with a phase. For example, the operator

$$\sum_{l=0}^{J}e^{\frac{2\pi inl}{J+1}}\mathrm{Tr}\big[Z^l\phi^i Z^{J-l}\big] \tag{2.43}$$

reduces to the supergravity state considered above for $n=0$, but it vanishes for nonzero n due to the cyclicity of the trace. This is precisely how it should be: a single non-zero-mode acting on the vacuum does not satisfy the level-matching condition (2.36). So the next-simplest possibility is to consider [13]

$$\frac{1}{\sqrt{JN^{J+2}}}\sum_{l=0}^{J}e^{\frac{2\pi inl}{J}}\mathrm{Tr}\big[\phi^i Z^l\phi^j Z^{J-l}\big] \longleftrightarrow a_n^{i\,\dagger}a_{-n}^{j\,\dagger}|v,p^+\rangle\,, \tag{2.44}$$

where $i\neq j$, the cyclicity of the trace was used to put one operator at the first position and $1/J$ contributions have been neglected in the power of the phase factor. The general rule is quite simple, each insertion of an 'impurity' is accompanied with a phase depending on the world-sheet momentum; those operators where the momenta do not sum to zero vanish due to cyclicity of the trace, in this way implementing the level matching condition; the dictionary

between impurity insertions and string oscillators is thus roughly (cf. the discussion below) as follows [13]

$$\begin{aligned} a^{i\,\dagger} &\longleftrightarrow \phi^i\,, \quad i=1,2,3,4\,, \\ a^{i'\,\dagger} &\longleftrightarrow D_{i'-4}Z\,, \quad i'=5,6,7,8\,, \\ S^{a\,\dagger} &\longleftrightarrow \chi^a_{J=\frac{1}{2}}\,. \end{aligned} \tag{2.45}$$

To check this identification it is useful to expand the string theory Hamiltonian (2.35) for large $\mu\alpha' p^+$ or equivalently for small λ' (cf. equation (2.39))

$$\frac{1}{\mu}H \simeq \sum_{n\in\mathbb{Z}} N_n \left(1+\frac{1}{2}\frac{n^2}{\left(\mu\alpha' p^+\right)^2}\right) = \sum_{n\in\mathbb{Z}} N_n \left(1+\frac{1}{2}\frac{\lambda}{J^2}n^2\right)\,. \tag{2.46}$$

We see that for $\mu\alpha' p^+ \gg 1$ all string states have approximately the same energy; this is reproduced by the construction of the BMN operators: in free field theory the inclusion of the phases does not make a difference, it is only in the interacting theory that this gets important because these operators are no longer protected. Notice however, that the BMN operators proposed to be dual to string states are built by sewing together protected operators with varying phases. One might imagine that these operators are nearly BPS in the sense that a delicate cancellation of renormalization and large J effects protects them from leaving the spectrum in the BMN limit. This is exactly what happens [13]. Remarkably it turns out that the anomalous dimensions of these operators are not just finite in the BMN limit, but as has been argued in [13], they are perturbatively computable with λ' playing the role of the effective coupling. Indeed, notice that the first correction in (2.46) involves the 't Hooft coupling λ so it seems one might reproduce this from a *perturbative* (in $g^2_{\rm YM}$ or λ) field theory computation. Consider for example the operator in (2.44). Taking into account interactions the relevant diagrams arise from the quartic vertex

$$\sim g^2_{\rm YM}{\rm Tr}\big([Z,\phi^i][\bar{Z},\phi^i]\big)\,. \tag{2.47}$$

The effect of this vertex can be analyzed as follows. The above interaction can be split into two parts, depending on whether the position of the operator ϕ in the 'string' of Z's is effectively moved to a neighboring position or not. Since at the planar level operators with ϕ's sitting at different positions are orthogonal to each other, contracting all the fields gives a result

which, for the first class, does not depend on the insertion of the phases, whereas for the second class it does. Combining the relevant contributions, utilizing the fact that other interactions involving gauge bosons and scalar loops cancel due to supersymmetry and taking the large N and J limit one precisely reproduces the first non-trivial correction in (2.46) [13]. For a careful treatment see for example [42]. Notice that the computation was done perturbatively in λ, but to take the BMN limit requires to send $\lambda \to \infty$. But the result for small λ equals the one for large λ obtained from the string Hamiltonian and it is tempting to assume that it is correct for all λ at the planar level. Further support to this conjecture comes from [45] which extended the above computation to two loops and presented arguments for higher loops, again matching the expectation coming from the expansion of the square root in (2.46). In [46] superconformal representation theory was used to argue that the full square root is reproduced; alternatively this was seen to be the case in [13] by exponentiating the quartic vertex; let me sketch how this works. SYM on $\mathbb{R} \times S^3$ can be expanded in spherical harmonics on the S^3. In particular the zero-modes of scalar fields on the S^3 have unit energy and the 'string' of oscillators corresponding to the zero-mode of Z carries $\Delta - J = 0$. To raise the energy we insert for example the zero-mode of $\phi \sim b + b^\dagger$ at some position along the string of Z oscillators. In the free theory the position of ϕ is unchanged and operators with ϕ inserted at different positions are orthogonal in the planar approximation. So we can think of the J Z's as defining a lattice with $J+1$ sites and an insertion of ϕ at different positions corresponds to the excitations $b_l^\dagger$ at the l-th site of the lattice. As alluded to above, the interaction in (2.47) can move an operator ϕ to a neighboring position, so when acting on the string of Z oscillators the effective Hamiltonian for ϕ consisting of the free and interacting parts is [13]

$$H \sim \sum_l \left(b_l^\dagger b_l + \frac{\lambda}{4\pi^2} \left[(b_{l+1} + b_{l+1}^\dagger) - (b_l + b_l^\dagger) \right]^2 \right) . \tag{2.48}$$

In the large N and J continuum limit the discretized Hamiltonian reduces to

$$H \sim \int_0^L d\sigma \left[\dot{\phi}^2 + \phi'^2 + \mu^2 \phi^2 \right] , \qquad L = \frac{2\pi}{\sqrt{\lambda}} \frac{J}{\mu} = 2\pi\alpha' p^+ . \tag{2.49}$$

This *is* the bosonic part of the string light-cone Hamiltonian on the plane wave. Consequently the full square root is reproduced from planar gauge

theory in the BMN limit and the 'string' of Z's plus insertion of impurities becomes equivalent to the physical string [13]. So there is evidence that λ' emerges as a new effective coupling in the BMN limit and one might think that the perturbation series of SYM in the BMN limit can be reorganized as a double series expansion in the effective coupling λ' and the effective genus counting parameter g_2^2. If true, the BMN duality has the interesting property that regimes in string theory on the plane wave and SYM in the BMN limit are *simultaneously* perturbatively accessible. This is in contrast to the usual AdS/CFT correspondence, where due to our limited ability to perform calculations for finite λ in SYM – or equivalently in the full string theory on $AdS_5 \times S^5$ – the relation is a strong/weak coupling duality. Note however, while perturbative calculations in λ of BMN operator two- and three-point functions can be reorganized in λ' [13, 42, 43, 45, 46] – and hence an extrapolation to large λ seems viable – this is no longer the case for higher point functions: computing for example the 4-point function of $\mathrm{tr}\big[Z^J\big]$ perturbatively in λ, a naive extrapolation to large λ leads to divergences [47].

The above heuristic discussion is in fact oversimplifying. Consider for example the BMN operators with $\Delta - J = 2$, that is a *defect charge* of two. Instead of inserting two impurities (defects) into the trace of Z's we could also insert one $\bar{Z}$, $D\phi^i$, D^2Z etc., that is fields carrying multiple defect charge. Indeed, all of these are present, even at the planar level [48]. However, they do not give rise to additional string states (there are none) but are *hidden* within the ordinary operators with single charge defects by *operator mixing* [48]. One example where this happens is the $SO(4)$ singlet [47, 48]

$$\mathcal{O}_n^J \sim \sum_{l=0}^{J} \cos\frac{\pi n(2l+3)}{J+3}\mathrm{Tr}\big[\phi^i Z^l \phi^i Z^{J-l}\big] - 4\cos\frac{\pi n}{J+3}\mathrm{Tr}\big[\bar{Z}Z^{J+1}\big]\,. \quad (2.50)$$

Written like this it is in fact an exact one-loop eigenstate of Δ even for finite J [48]. Roughly speaking the above mixing is needed to cancel singularities that occur when the two ϕ impurities collide [47]. For non-zero n the above operator is the primary of a long $\mathcal{N} = 4$ superconformal multiplet and all the other defect charge two operators dual to string states in the BMN limit are contained in this multiplet as descendants [48]. All fields with defect charge two *do* appear in these generalized BMN operators. Analogously, for $n = 0$ the operator in equation (2.50) is the primary of a half BPS multiplet; all operators dual to supergravity states with up to two oscillators are descendants. One might conjecture that this pattern generalizes to higher

defect charge [48].

At finite g_2 mixing of single-trace with multi-trace operators has to be taken into account [42, 43]. For example, to compute the anomalous dimension on the torus single- and double-trace operators have to be redefined (mixed) in order to normalize and diagonalize their two-point functions. For the (redefined) operator in (2.50) one finds at order $\mathcal{O}(g_2^2\lambda')$ [47, 49]

$$\big(\Delta - J\big)_n = 2 + \lambda' \left[n^2 + \frac{g_2^2}{4\pi^2} \left(\frac{1}{12} + \frac{35}{32\pi^2 n^2} \right) \right] . \tag{2.51}$$

In fact, the above result holds for all BMN operators with defect charge two transforming in the various irreducible representations of $SO(4) \times SO(4)$; this is a consequence of superconformal symmetry [48]. For the explicit form of some of the redefined operators at this order see [47, 49]. It is actually simpler to consider directly the dilatation operator, work with the 'bare' operators and diagonalize the resulting anomalous dimension matrix. This approach was followed in [50, 51] and results in a simple derivation of equation (2.51). Further results on higher genus correlators include [52, 53], scalar/vector, vector/vector and multi-trace BMN operators have also been considered in [54, 55, 56, 50]. For an extension of equation (2.51) to order $\mathcal{O}(g_2^4\lambda')$ see [50]. The contribution of higher genus corrections to the anomalous dimension is related to a mass-shift of the dual string states due to interactions. A detailed study of string interactions will be deferred to chapter 4. Let me however mention a route – which will not be pursued in what follows – to study interacting strings on the plane wave, the string-bit formalism [57]. Inspired by the emergence of the free string, discretized into J bits along the string coordinate σ as in (2.48) and from matrix string theory [58, 59, 60], one interprets the J small strings as describing the quantization of the J-th symmetric product of the plane wave target space. This leads to a quantum-mechanical orbifold model. In a spirit reminding of the matrix string, string splitting and joining is then realized by an operator that roughly speaking exchanges two string bits; see [57] for details. This approach was further studied in [61, 62, 63] and led to results in agreement with field theory. Very recently, doubts on the consistency of this model have been voiced in [64]. The reason for this is the so-called fermion doubling problem, which leads to the loss of supersymmetry – inevitably broken by the discretization – even in the continuum limit. Moreover, repeating the above derivation of the string Hamiltonian (2.49) by truncation to the lowest modes corresponding to the operators DZ and the fermions, apparently does

not lead to the correct string Hamiltonian [37].

Finally, let me briefly discuss the issue of holography on the plane wave. As already mentioned, the conformal boundary of $AdS_5 \times S^5$ in global coordinates is $\mathbb{R} \times S^3$ on which the dual SYM theory lives. However, in the Penrose-Güven limit one focuses on the neighborhood of a null geodesic located at the origin of AdS_5 and rotating around a great circle of the S^5. It was shown in [36] that the conformal boundary of the plane wave is a one-dimensional null line. This can be seen by a conformal mapping of the plane wave to the Einstein static universe $\mathbb{R} \times S^9$. Since the Einstein static universe is regular, the boundary consists of the space-time region for which the Weyl factor is divergent. This is the case for a null line, a S^7 inside the S^9 shrinks to zero size and the spatial projection of the null line is a circle on the S^9 [36]. One can picture this as a line winding in time on the Einstein cylinder, see [36]. For a thorough discussion of the causal structure of more general pp-wave geometries, which are not conformally flat and hence the above trick of identifying the boundary by a conformal mapping does not work, see [65, 66]. For a large class of pp-waves satisfying certain conditions, the boundary is again one-dimensional. The conformal boundaries and geodesics of $AdS_5 \times S^5$ and the plane wave and how the former approach the latter in the Penrose limit have been analyzed in [67].

So the boundary of the plane wave is a null line, whereas SYM lives on $\mathbb{R} \times S^3$ before the limit is taken. Here one should recall again that the geodesic is rotating on the S^5, so when projected on the boundary it is time-like and can be identified with t. As the S^3 has disappeared in the process this supports the expectation [36] (see also [68]) that the holographic dual of string theory on the plane wave is a quantum mechanical matrix model obtained by a truncation of SYM on the S^3. It would be nice to gain a precise understanding in which sense such a truncation can be consistently performed. An alternative approach, the construction of a holographic screen consisting of a four-dimensional hypersurface in the plane wave, was followed in [69, 70]. It would be interesting to understand if this has some connection to [71], where supersymmetric D3-branes and $\mathcal{N} = 4$ SYM on a four-dimensional plane wave, arising from a Penrose limit of $\mathbb{R} \times S^3$, was studied. One would also like to go beyond the comparison of masses vs. anomalous dimensions in both theories. Some ideas in this respect have been formulated in [36] (see however, also [72]), a consistent truncation of SYM in the BMN limit would suggest to compare finite time transition amplitudes in this model to string amplitudes on the plane wave.

Chapter 3

Extensions of the BMN duality

3.1 Various approaches

It is an interesting question whether the BMN proposal is applicable to other less trivial backgrounds. Can the string spectrum in less supersymmetric situations again be deduced from a subsector of a dual gauge theory with reduced, possibly even no supersymmetry? This question was addressed in several publications [73, 74, 75, 76, 17] appearing shortly after [13]. Recall that orbifolds of type IIB string theory on $AdS_5 \times S^5$ [77] provide a simple way to reduce the amount of supersymmetry in the AdS/CFT correspondence. For example, the world-volume theory of kN D3-branes located at the $\mathbb{Z}_k$ orbifold singularity of an ALE space is a $\mathcal{N} = 2$ $[U(N)]^k$ quiver gauge theory [78] which is dual to string theory on $AdS_5 \times (S^5/\mathbb{Z}_k)$ [77]. $\mathcal{N} = 1$ field theories can arise from D3-branes on orbifold singularities of the form $\mathbb{C}^3/\Gamma$, with Γ a discrete proper subgroup of $SU(3)$. These are dual to strings on $AdS_5 \times (S^5/\Gamma)$ [77]. One can also consider N D3-branes located at a conifold singularity of a Calabi-Yau three-fold. In this case the world-volume theory is a $\mathcal{N} = 1$ $SU(N) \times SU(N)$ field theory coupled to four bi-fundamental chiral multiplets with a IR fixed point and an exactly marginal superpotential [79]. This theory is dual to string theory on $AdS_5 \times T^{1,1}$, $T^{1,1}$ being the base of the conifold.

What happens if we apply the Penrose-Güven limit to these situations?[1]

[1] I remind the reader that in general there exist distinct classes of geodesics which give rise to different space-times in the limit. The statements I make usually refer to the generic case if not stated otherwise.

Let me sketch the case of $AdS_5 \times T^{1,1}$ which was studied in [73, 74, 75]. Topologically $T^{1,1}$ is a $U(1)$ bundle over $S^2 \times S^2$ and its $SU(2) \times SU(2) \times U(1)$ isometry is identified with a $SU(2) \times SU(2)$ global symmetry and $U(1)_R$ symmetry of the dual superconformal field theory [79]. The surprising result found in [73, 74, 75] is that blowing up the neighborhood of a null geodesic rotating around the $U(1)$ fiber one ends up with the maximally supersymmetric plane wave background again. Consequently a subsector of the gauge theory with enhancement from $\mathcal{N} = 1$ to $\mathcal{N} = 4$ supersymmetry should emerge in the BMN limit. Indeed, one finds that the string Hamiltonian in this case is related to that of the plane wave by a twisting [73, 74, 75]

$$H_{T^{1,1}} = H_{S^5} + J_1 + J_2 \,, \tag{3.1}$$

where J_1 and J_2 are rotation generators of a $\mathbb{R}^2 \times \mathbb{R}^2$ subspace of the plane wave transverse geometry. From the gauge theory perspective $H_{T^{1,1}}$ is identified with $\Delta - \frac{3}{2}R$, where R is the generator of the $U(1)_R$ symmetry and $J_a = Q_a - \frac{1}{2}R$, where Q_a are the Cartan generators of the $SU(2) \times SU(2)$ global symmetry. All these combinations remain fixed in the limit, similarly to $\Delta - J$ in the $\mathcal{N} = 4$ case. In particular the sector in the $\mathcal{N} = 1$ theory with supersymmetry enhancement is specified by [73, 74, 75]

$$H_{S^5} = \Delta - \frac{1}{2}R - Q_1 - Q_2 \,. \tag{3.2}$$

One can explicitly identify these operators in the gauge theory. The matter content consists of chiral multiplets A_i and B_i with R-charge $1/2$ and conformal dimension $3/4$ transforming as $(\mathbf{2}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{2})$ under the global symmetry. Then the unique operator corresponding to the string ground state is $\mathrm{tr}(A_1 B_1)^R$, analogous to $\mathrm{tr} Z^J$ in $\mathcal{N} = 4$. Oscillators in the $\mathbb{R}^2 \times \mathbb{R}^2$ direction are roughly speaking identified with the action of the raising operators of $SU(2) \times SU(2)$ on the ground state and a possible addition of phases. For more details, see [73, 74, 75]. Another example where $\mathcal{N} = 1$ is enhanced to $\mathcal{N} = 4$ arises from the Penrose-Güven limit of the dual pair obtained from N D3 branes on a $\mathbb{C}^3/\mathbb{Z}_3$ orbifold singularity [74]. Further discussion of supersymmetry enhancement in $\mathcal{N} = 1$ theories arising from various orbifolds of S^5 and $T^{1,1}$ can be found in [80].

However, supersymmetry enhancement is not a generic feature, as can be seen from the examples involving $\mathcal{N} = 2$ $[U(N)]^k$ quiver gauge theory [76, 17, 81] (the case $k = 2$ has also been discussed in [73]). The reason for this

is that in the generic case the Penrose-Güven limit of $AdS_5 \times (S^5/\mathbb{Z}_k)$ yields the $\mathbb{Z}_k$ orbifold of the plane wave background and hence breaks half of the supersymmetry. This example will be discussed in more detail in the next section. Penrose-Güven limits of various orbifolds and orientifolds of $AdS \times S$ spaces have also been considered in [82]. I have said above that generically supersymmetry is not enhanced in the Penrose limit of $AdS_5 \times (S^5/\mathbb{Z}_k)$. A precise statement is the following: if the null geodesic is fixed by the group action, the resulting space-time will be an orbifold of the plane wave; if this is not the case one recovers the pure plane wave again [76]. Following the logic above this means that strings on plane waves can also arise in a sector of $\mathcal{N} = 2$ theory with enhancement to $\mathcal{N} = 4$. This observation leads to a further interesting development. Suppose we have N_1 D3-branes placed on a $\mathbb{C}^2/\mathbb{Z}_{N_2}$ singularity. Blowing up the region around a null geodesic not fixed by the group action one can also take N_1, $N_2 \to \infty$ and keep the R charge finite [83, 84]. How does this affect the resulting geometry? Again, introduce light-cone coordinates

$$x^+ = \frac{1}{2\mu}(t+\psi)\,, \qquad x^- = -\mu R^2(t-\psi)\,, \qquad R^4 \equiv 4\pi g_s \alpha'^2 N_1 N_2\,, \tag{3.3}$$

however, this time $\psi \sim \psi + \frac{2\pi}{N_2}$ since the geodesic is not fixed by $\mathbb{Z}_{N_2}$. Taking $N_1 \sim N_2 \to \infty$ yields the standard plane wave geometry with the difference that due to (3.3) the light-like coordinate x^- becomes *compact* with period

$$x^- \sim x^- + 2\pi R^-\,, \qquad R^- \equiv \mu\alpha' \sqrt{4\pi g_s \frac{N_1}{N_2}}\,. \tag{3.4}$$

Consequently the light-cone momentum p^+ is quantized in units of $1/R^-$ and we have a description of discrete light-cone quantization of strings on the plane wave in terms of a quiver gauge theory [83, 84]. An interesting new feature is for example the appearance of momentum and winding states along the compact direction. These are also realized in the gauge theory [83, 84]: the dual gauge theory is a $[U(N_1)]^{N_2}$ quiver gauge theory, in particular it contains N_2 bi-fundamental hypermultiplets [78] or, in $\mathcal{N} = 1$ language, $2N_2$ chiral multiplets in the bi-fundamental. Denote their scalar components by (A_I, B_I). The operator $\mathrm{tr}(A_1 \cdots A_{N_2})$ has precisely the correct quantum numbers to describe a state with one unit of light-cone momentum and zero winding. This looks like a 'string' winding once around the quiver diagram (which is a circle). Similarly an operator with k units of momentum winds

k times around the quiver. Winding states are shown to be dual to operators with insertions of adjoint scalars from the vector multiplet together with a phase. The picture that emerges is quite suggestive: strings carrying momentum are described by operators winding around a large quiver circle, whereas strings with non-zero winding are dual to operators which carry 'momentum' (the phase). Indeed it was argued in [83, 84], using T-duality, that the 'strings' winding the quiver circle *are* so called non-relativistic winding strings in the T-dual description. I refer the reader to [83, 84] for more details. One can also study compactifications of string theory on the plane wave along space-like circles [85]. The plane wave with a manifest space-like isometry is related to the standard one by a coordinate transformation, resulting in a shift of the Hamiltonian by a rotation generator. For a classification of the preserved supersymmetry under toroidal compactifications see [85]. Plane waves with space-like isometries can also arise from non-standard Penrose limits of $AdS_5 \times S^5$ and $AdS_5 \times S^5/\mathbb{Z}_k$ and are dual to triple scaling limits of $\mathcal{N} = 4$ or $\mathcal{N} = 2$ gauge theories [86]. The identification of momentum and winding states along the space-like circle with operators in the dual gauge theory is similar in spirit to [83, 84], see [86] for the details.

A further interesting direction is the generalization of the BMN correspondence to non-conformal backgrounds [75]. In particular one can consider examples known to be dual to RG flows from $\mathcal{N} = 4$ in the UV to $\mathcal{N} = 1$ IR fixed points and take the Penrose-Güven limit 'along the flow' [87, 88]. Non-conformal backgrounds do, however, not lead to solvable string theories, rather they share the generic feature that the Penrose limit leads to time-dependent mass terms for the world-sheet theory in light-cone gauge [75]. Despite of this fact it has been argued in [87] that some features of the RG flow, such as the branching of a given operator in the UV into operators of the IR, can be captured by studying the corresponding problem of a point particle propagating in this time dependent background. This system is exactly solvable [87]. One may also focus on the geometry in the IR [88, 89] and the resulting background will be one of a deformed Hpp-wave containing additional constant three-form fluxes. This leads again to a solvable string theory, see also [90]. By choosing a non-standard geodesic, one use the resulting string theory to study heavy hadrons with mass proportional to a large global charge in the confining dual IR gauge theory [89]. An interesting *solvable* example of a time-dependent plane wave background supported by a non-constant dilaton was considered in [91].

3.2 Strings on orbifolded plane waves from quiver gauge theory

In the previous section I tried to give a flavor of the possible extensions of the BMN duality. In this section the case of the plane wave orbifold [76, 17, 81] will be discussed in more detail. Specifically, I will consider a $\mathbb{Z}_k$ orbifold of one of the two $\mathbb{R}^4$ subspaces transverse to the propagation null vector and show that first-quantized free string theory is described correctly by the large N, fixed gauge coupling limit of $\mathcal{N} = 2$ $[U(N)]^k$ quiver gauge theory. Apart from being an interesting example with less supersymmetry, a further motivation comes from the fact that, as shown in [13, 15, 34], the plane wave background acts as a harmonic potential to the string, and hence the dynamical distinction between untwisted and twisted states is less clear. It is thus of intrinsic interest to see if one can find a precise map between type IIB string oscillation modes and quiver gauge theory operators, both for untwisted and twisted sectors. Indeed, we will see that operators dual to untwisted and twisted sector states are quite similar.

3.2.1 IIB superstring on plane wave orbifold

As explained in the previous chapter, the dynamics of superstrings on the maximally supersymmetric plane wave geometry supported by homogeneous R-R 5-form flux and constant dilaton

$$\begin{aligned} ds^2 &= 2dx^+dx^- - \mu^2(\vec{x}^2 + \vec{y}^2)(dx^+)^2 + d\vec{x}^2 + d\vec{y}^2\,, \\ F_{+1234} &= F_{+5678} = 4\mu\,, \end{aligned} \tag{3.5}$$

$(\vec{x}, \vec{y}) \in \mathbb{R}^4 \times \mathbb{R}^4$, is governed by an exactly solvable light-cone world-sheet theory of *free*, albeit massive fields [15]. The isometry group of the eight-dimensional space transverse to the null propagation direction is $SO(4)_1 \times SO(4)_2$: while the space-time geometry is invariant under $SO(8)$, the 5-form field strength breaks it to $SO(4)_1 \times SO(4)_2$. In the Green-Schwarz action on the plane wave background, the reduction of the isometry is due to the coupling of spinor fields to the background R-R 5-form field strength.

One is interested in reducing the number of supersymmetries preserved by the background. As alluded to above, one can break one half of the 32 supersymmetries by taking a $\mathbb{Z}_k$ orbifold of the $\mathbb{R}^4$ subspace parameterized

by $\vec{y}$. The orbifold action is defined by

$$g: \quad (z^1, z^2) \longrightarrow \omega(z^1, z^2)\,, \qquad \omega = e^{\frac{2\pi i}{k}}\,, \tag{3.6}$$

where

$$z^1 \equiv \frac{1}{\sqrt{2}}(y^6 + iy^7)\,, \qquad z^2 \equiv \frac{1}{\sqrt{2}}(y^8 - iy^9)\,, \tag{3.7}$$

and g acts on space-time fields as

$$g = \exp\left(\frac{2\pi i}{k}(J_{67} - J_{89})\right)\,. \tag{3.8}$$

J_{67} and J_{89} are the rotation generators in the 6-7 and 8-9 planes, respectively. Defined so, the orbifold of the plane wave background is actually derivable from the Penrose limit of $AdS_5 \times S^5/\mathbb{Z}_k$ taken along the great circle of the S^5 that is fixed by the $\mathbb{Z}_k$ action.

In the light-cone gauge, the superstring on the background (3.5) is described by eight world-sheet scalars x^I and eight world-sheet fermions S^a, all of which are free but massive. The masses of scalars and fermions are equal by world-sheet supersymmetry (which descends from the light-cone gauge fixing of the Green-Schwarz action, cf. the remark above equation (2.34)) and equal the R-R 5-form field strength μ. S is a positive chirality Majorana-Weyl spinor of $SO(9,1)$, obeying the light-cone gauge condition $\Gamma^+ S = 0$ and hence transforming as a positive chirality spinor of $SO(8)$ under rotations in the transverse directions. Decompose the world-sheet fields into representations of $SO(4)_1 \times SO(4)_2$

$$x^I = (\vec{x}, \vec{y}) \to (\vec{x}, z^1, z^2)\,, \qquad S^a \to (\chi^\alpha, \xi^{\dot\alpha})\,, \tag{3.9}$$

where α and $\dot\alpha$ are spinor indices of $SO(4)_2$, ranging over 1, 2 and I have suppressed the spinor indices of $SO(4)_1$ under which χ^α and $\xi^{\dot\alpha}$ carry positive and negative chirality, respectively. Then the fields $\vec{x}$ and χ^α transform trivially under g whereas

$$g: \quad z^m \longrightarrow \omega z^m\,, \qquad \xi^{\dot\alpha} \longrightarrow \Omega^{\dot\alpha}{}_{\dot\beta}\xi^{\dot\beta}\,, \tag{3.10}$$

and $\Omega = \mathrm{diag}(\omega, \omega^{-1})$, that is $\xi^{\dot 1}$ and $\xi^{\dot 2}$ transform oppositely under the $\mathbb{Z}_k$ action. It is convenient to combine $\xi^{\dot 1}$, $\bar\xi^{\dot 2}$ into a Dirac spinor ξ, and $\bar\xi^{\dot 1}$ and $\xi^{\dot 2}$ into its conjugate $\bar\xi$ and analogously for χ and $\bar\chi$. As the world-sheet

theory is free, it is straightforward to quantize the string in each twisted sector, the only difference among various sectors being the monodromy of the world-sheet fields sensitive to the orbifolding, that is z^m and ξ. The other world-sheet fields remain periodic. The monodromy conditions in the q-th twisted sector, $q = 0, \ldots, k-1$, are

$$z^m(\sigma + 2\pi\alpha' p^+, \tau) = \omega^q z^m(\sigma, \tau)\,, \qquad \xi(\sigma + 2\pi\alpha' p^+, \tau) = \omega^q \xi(\sigma, \tau)\,, \quad (3.11)$$

and the corresponding oscillator modes depend on $n(q) = n + \frac{q}{k}$ $(n \in \mathbb{Z})$.

Physical states are obtained by applying the bosonic and fermionic creation operators to the light-cone vacuum $|v, p^+\rangle_q$ of each twisted sector. They should satisfy additional constraints ensuring the level-matching condition:

$$\sum_{n\in\mathbb{Z}} n N_n = 0\,, \quad \sum_{n\in\mathbb{Z}} n(q)\left(N_{n(q)} - \bar{N}_{-n(q)}\right) = 0\,, \quad (3.12)$$

and $\mathbb{Z}_k$ invariance. The bosonic creation operators are

$$\vec{a}_n^{\,\dagger}\,, \qquad \text{and} \qquad \alpha_{n(q)}^{\dagger\, m}\,, \quad \bar{\alpha}_{n(-q)}^{\dagger\, m}\,, \qquad (n \in \mathbb{Z})\ . \quad (3.13)$$

Here, $\vec{a}_n$ are the $\vec{x}$ oscillators, whereas $\alpha^m_{n(q)}$ and $\bar{\alpha}^m_{n(-q)}$ are z^m and $\bar{z}^m$ oscillators, respectively. The fermionic creation operators consist, in obvious notation, of

$$\chi_n^\dagger\,, \quad \bar{\chi}_n^\dagger \qquad \text{and} \qquad \xi_{n(q)}^\dagger\,, \quad \bar{\xi}_{n(-q)}^\dagger\,. \quad (3.14)$$

Acting with the fermionic zero-mode oscillators on the light-cone vacua and projecting onto $\mathbb{Z}_k$ invariant states, one fills out $\mathcal{N} = 2$ gravity and tensor supermultiplets of the plane wave background. The action of the bosonic zero-mode oscillators on these gives rise to a whole tower of multiplets [34], much as in the $AdS_5 \times S^5$ case. As an example, we have four invariant states with a single bosonic oscillator

$$\vec{a}_0^{\,\dagger}|v, p^+\rangle_q\,, \quad (3.15)$$

and states with two bosonic oscillators are

$$a_n^{\dagger\mu} a_{-n}^{\dagger\nu}|v, p^+\rangle_q\,, \qquad \alpha_{n(q)}^{\dagger\, l}\bar{\alpha}_{-n(q)}^{\dagger\, m}|v, p^+\rangle_q\,. \quad (3.16)$$

In the $\mathbb{Z}_2$ case there are additional invariant states built from two z^m or two $\bar{z}^m$ oscillators. However, they do not satisfy the level matching condition (3.12). The light-cone Hamiltonian in the q-th twisted sector is

$$H_q = \sum_{n\in\mathbb{Z}} N_n \sqrt{\mu^2 + \frac{n^2}{(\alpha' p^+)^2}} + \sum_{n\in\mathbb{Z}} \left(N_{n(q)} + \bar{N}_{-n(q)}\right) \sqrt{\mu^2 + \frac{n(q)^2}{(\alpha' p^+)^2}}\,. \quad (3.17)$$

The first sum is over those oscillators which are not sensitive to the orbifold and N_n ($N_{n(q)}$ and $\bar{N}_{-n(q)}$) is the total occupation number of bosons and fermions. The ground state energy is cancelled between bosons and fermions. This corresponds to a choice of fermionic zero-mode vacuum that explicitly breaks the $SO(8)$ symmetry, which is respected by the metric but not the field strength background, to $SO(4)_1 \times SO(4)_2$ [34].

3.2.2 Operator analysis in $\mathcal{N} = 2$ quiver gauge theory

It is known [77] that type IIB string theory on $AdS_5 \times (S^5/\mathbb{Z}_k)$ is dual to $\mathcal{N} = 2$ $[U(N)]^k$ quiver gauge theory, the world-volume theory of kN D3-branes placed at the orbifold singularity. In light of the discussion in the previous section, one can anticipate that string theory on the plane wave orbifold is dual to a new perturbative expansion of the quiver gauge theory at large N and *fixed* gauge coupling $g^2_{\rm YM} = 4\pi g_s k$. The factor of k in the relation between the string and the gauge coupling is standard and can be deduced by moving the D3-branes off the tip of the orbifold into the Higgs branch, see also [92]. In the new expansion, one focuses primarily on states with conformal weight Δ and $U(1)_R$ charge J which scale as Δ, $J \sim \sqrt{N}$, whose difference $(\Delta - J)$ remains finite in the large N limit. $U(1)_R$ is the subgroup of the original $SU(4)_R$ symmetry of $\mathcal{N} = 4$ super Yang-Mills theory, which on the gravity side corresponds to the S^1 fixed under the orbifolding; this $U(1)_R$ together with the $SU(2)_1$ subgroup of the remaining $SO(4) \simeq SU(2)_1 \times SU(2)_2$ that commutes with $\mathbb{Z}_k \subset SU(2)_2$ forms the R-symmetry group of $\mathcal{N} = 2$ supersymmetric gauge theory.

The reason for the above scaling behavior is that $(\Delta - J)$ is identified with the light-cone Hamiltonian on the string theory side, whereas[2] $\frac{J}{\sqrt{kN}} \sim p^+$, p^+ being the longitudinal momentum carried by the string. When $(\Delta - J) \ll J$, the light-cone Hamiltonian in (3.17) implies that on the gauge theory side there are operators obeying the following relation between the dimension Δ and the $U(1)_R$ charge J

$$(\Delta - J)_n = \sqrt{1 + \lambda' n^2} \qquad \text{and} \qquad (\Delta - J)_{n(q)} = \sqrt{1 + \lambda' \left(n(q)\right)^2}\,. \quad (3.18)$$

In the gauge theory, before orbifolding we have $N \times N$ matrix valued fields,

[2] Since $\int_{S^5/\mathbb{Z}_k} F_5 = N$, the radius of AdS_5 is proportional to $(kN)^{1/4}$.

that is the gauge field and three complex scalars

$$A_\mu\,, \quad Z = \frac{1}{\sqrt{2}}(\phi^4 + i\phi^5)\,, \quad \varphi^m = (\varphi^1, \varphi^2) \equiv \frac{1}{\sqrt{2}}(\phi^6 + i\phi^7, \phi^8 - i\phi^9)\,, \tag{3.19}$$

and in addition their superpartners, fermions χ and ξ. The fields χ and ξ are spinors of $SO(5,1)$, transforming as $\mathbf{4}$ and $\mathbf{4}'$, respectively. To define the $\mathbb{Z}_k$ orbifolding in the gauge theory, we promote these fields to $kN \times kN$ matrices $\mathcal{A}_\mu$, $\mathcal{Z}$, Φ^m, $\mathcal{X}$ and Ξ and project onto the $\mathbb{Z}_k$ invariant components. The projection is ensured by the conditions

$$S\mathcal{A}_\mu S^{-1} = \mathcal{A}_\mu\,, \qquad S\mathcal{Z}S^{-1} = \mathcal{Z}\,, \qquad S\mathcal{X}S^{-1} = \mathcal{X} \tag{3.20}$$

and

$$S\Phi^m S^{-1} = \omega\Phi^m\,, \qquad S\Xi S^{-1} = \omega\Xi\,. \tag{3.21}$$

where $S = \mathrm{diag}(1, \omega^{-1}, \omega^{-2}, \ldots, \omega^{-k+1})$, each block being proportional to the $N \times N$ unit matrix.

The resulting spectrum is that of a four-dimensional $\mathcal{N} = 2$ quiver gauge theory [78] with $[U(N)]^k$ gauge group, containing hypermultiplets in the bifundamental representations of $U(N)_i \times U(N)_{i+1}$, $i \in \mathbb{Z}$ mod(k). More precisely, $\mathcal{A}_\mu$, $\mathcal{Z}$ and $\mathcal{X}$ fill out k $\mathcal{N} = 2$ vector multiplets with the fermions transforming as doublets under $SU(2)_R$ (as its Cartan generator is proportional to $(J_{67} + J_{89})$). The $\mathcal{Z}$ field has the block-diagonal form

$$\mathcal{Z} = \begin{pmatrix} Z_1 & & & & & \\ & Z_2 & & & & \\ & & Z_3 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & Z_k \end{pmatrix} \tag{3.22}$$

with zeros on the off-diagonal and the diagonal blocks being $N \times N$ matrices of $U(N)_i$'s. The $\mathcal{A}_\mu$ and $\mathcal{X}$ fields take an analogous form. Likewise, the Φ^m and Ξ fields fill out k hypermultiplets, in which the scalars are doublets under $SU(2)_R$, whereas the fermions are neutral. The Φ^m fields take the form

$$\Phi^m = \begin{pmatrix} 0 & \varphi^m_{12} & & & & & \\ & 0 & \varphi^m_{23} & & & & \\ & & 0 & \cdot & & & \\ & & & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \\ \varphi^m_{k1} & & & & & \cdot & \cdot \end{pmatrix} \tag{3.23}$$

and analogously for Ξ.

The light-cone vacua of string theory on the plane wave orbifold ought to be described by $H_q = 0$ and in the quiver gauge theory this translates to operators with $\Delta - J = 0$. One can build k mutually orthogonal, $\mathbb{Z}_k$ invariant single-trace operators $\mathrm{Tr}[S^q \mathcal{Z}^J]$ and associate these operators to the vacuum in the q-th twisted sector

$$\frac{1}{\sqrt{kJN^J}} \mathrm{Tr}[S^q \mathcal{Z}^J] \longleftrightarrow |v, p^+\rangle_q \,, \qquad (q = 0, \ldots, k-1) \,. \tag{3.24}$$

In what sense is this identification unique? After all, in the quiver gauge theory it appears that the operators $\mathrm{Tr}[S^q \mathcal{Z}^J]$ for any q stand on equal footing. However, the orbifold action renders an additional 'quantum' $\mathbb{Z}_k$ symmetry (see for example [93]) that acts on fields in the quiver gauge theory.[3] Specifically, one can take an element g in this quantum $\mathbb{Z}_k$ to act on an arbitrary field T_{ij}, $i, j \in \mathbb{Z} \bmod(k)$, as $g : T_{ij} \longrightarrow T_{i+1,j+1}$. In particular, one notes that $g : \mathrm{Tr}[S^q \mathcal{Z}^J] \longrightarrow \omega^q \mathrm{Tr}[S^q \mathcal{Z}^J]$. So one can indeed distinguish classes of operators on the quiver gauge theory side by their eigenvalues under the quantum $\mathbb{Z}_k$ symmetry.

Next, consider the eight twist invariant operators with $\Delta - J = 1$. They are

$$\frac{1}{k\sqrt{N^{J+1}}} \mathrm{Tr}[S^q \mathcal{Z}^J \mathcal{D}_\mu \mathcal{Z}] \longleftrightarrow a_0^{\dagger\mu} |v, p^+\rangle_q \,, \tag{3.25}$$

$$\frac{1}{k\sqrt{N^{J+1}}} \mathrm{Tr}[S^q \mathcal{Z}^J \mathcal{X}_{J=1/2}] \longleftrightarrow \chi_0^\dagger |v, p^+\rangle_q \,, \tag{3.26}$$

$$\frac{1}{k\sqrt{N^{J+1}}} \mathrm{Tr}[S^q \mathcal{Z}^J \bar{\mathcal{X}}_{J=1/2}] \longleftrightarrow \bar{\chi}_0^\dagger |v, p^+\rangle_q \,. \tag{3.27}$$

These are identified with IIB supergravity modes built out of a single zero-mode oscillator acting on the q-th vacuum. Here, $\mathcal{D}_\mu \mathcal{Z} = \partial_\mu \mathcal{Z} + [\mathcal{A}_\mu, \mathcal{Z}]$. Operators corresponding to higher string states on the plane wave orbifold arise as follows. Oscillators of non-zero level n corresponding to the fields not sensitive to the orbifold are identified with insertions of the operators $\mathcal{D}_\mu \mathcal{Z}$, $\mathcal{X}_{J=1/2}$ and $\bar{\mathcal{X}}_{J=1/2}$ with a position dependent phase factor in the trace $\mathrm{Tr}[S^q \mathcal{Z}^J]$. For instance, for $\Delta - J = 2$, $\mu \neq \nu$,

$$\frac{1}{\sqrt{kJN^{J+2}}} \sum_{l=0}^{J} e^{\frac{2\pi i l n}{J}} \mathrm{Tr}[S^q \mathcal{Z}^l \mathcal{D}_\mu \mathcal{Z} \mathcal{Z}^{J-l} \mathcal{D}_\nu \mathcal{Z}] \longleftrightarrow a_n^{\mu\,\dagger} a_{-n}^{\nu\,\dagger} |v, p^+\rangle_q \,. \tag{3.28}$$

[3] This $\mathbb{Z}_k$ should not to be confused with the space-time $\mathbb{Z}_k$ used for constructing the orbifold. By construction, under the orbifold action all the fields are invariant.

This is exactly the same as in the unorbifolded case – the insertion of the position-dependent phase factor ensures that the level-matching condition is satisfied and that the light-cone energy of the string states is reproduced correctly [13].

The remaining string states involving oscillators with a fractional modding $n(q)$ in the twisted sectors, should be identified with insertions of the operators Φ^m and $\Xi_{J=1/2}$ together with position-dependent phase factors of the form $e^{2\pi i l n(q)/J}$. Similarly, insertions of $\bar{\Phi}^m$ and $\bar{\Xi}_{J=1/2}$ are accompanied with the phase factor $e^{2\pi i l n(-q)/J}$. Again, the prescription implements the level-matching condition and yields the correct energy of the corresponding string states. For $r \neq s$

$$\frac{1}{\sqrt{kJN^{J+2}}} \sum_{l=0}^{J} e^{\frac{2\pi i l n(q)}{J}} \mathrm{Tr}[S^q \mathcal{Z}^l \Phi^r \mathcal{Z}^{J-l} \bar{\Phi}^s] \longleftrightarrow \alpha^{r\,\dagger}_{n(q)} \bar{\alpha}^{s\,\dagger}_{-n(q)} |v, p^+\rangle_q \,. \tag{3.29}$$

For the $\mathbb{Z}_2$ orbifold, the operator corresponding to $\alpha^{r\,\dagger}_{n(q)} \alpha^{s\,\dagger}_{m(q)} |v, p^+\rangle_1$, though being $\mathbb{Z}_2$ invariant, vanishes for all m, n due to the cyclicity of the trace, as it should, cf. the remark below equation (3.16).

Finally, operators with insertions such as $\mathcal{D}^2\mathcal{Z}$, $\bar{\mathcal{Z}}$ or $\mathcal{X}_{J=-1/2}$ are expected to be hidden by operator mixing, much in the same way as discussed in the previous chapter, section 2.3. One can compute the leading order anomalous dimensions of the $\Delta - J = 2$ operators in equation (3.29), perturbatively in $\mathcal{N} = 2$ quiver gauge theory and confirm that the proposal for the twisted sector operators reproduces the correct light-cone string energy spectrum. In fact, in the setup I have outlined above one can proceed with the computations essentially parallel to those of [13], see for example [17] for more details.

3.3 Further directions

So far I mainly considered closed strings in IIB string theory on the plane wave background, their duality to $\mathcal{N} = 4$ SYM in the BMN limit and generalizations thereof. In this section I would like to discuss two further interesting issues: D-branes on the plane wave and string theory on more general pp-wave backgrounds.

3.3.1 D-branes on the plane wave

Since D-branes capture non-perturbative effects in string theory, their understanding in the plane wave background is important. They can be studied by various means: in perturbative string theory they are defined as hypersurfaces on which open strings end and hence can be analyzed by finding consistent boundary conditions for open strings; alternatively they can be described using boundary states, that is coherent states in closed string theory. The boundary state imposes certain gluing conditions on the closed string fields that arise through the presence of the D-brane. Interactions between two static D-branes through the exchange of closed strings at tree level can then be computed by sandwiching the closed string propagator between two boundary states. The same process can be re-interpreted as an open string one-loop diagram, i.e. the open string partition function. This is open-closed duality, which has to be satisfied for a D-brane to be consistent. Yet another way to describe D-branes is by considering their world-volume theory, consisting of a Dirac-Born-Infeld and a Wess-Zumino term. Solutions to the resulting field equations describe the embedding of the D-brane into the target space. Finally, at low energies D-branes arise as solitonic solutions to the supergravity equations of motion.

All of these different approaches have been used to obtain a rather detailed picture of supersymmetric D-branes in the plane wave background via open strings in light-cone gauge [94, 95, 96], covariant open strings [97], boundary states [98, 99, 100, 96] and the open-closed consistency conditions [99, 100], D-brane embeddings [101] and supergravity solutions [102]. I will summarize these results below, overviews over many aspects on D-branes on the plane wave can be found in [103, 104]. For a discussion of open strings in the plane wave with a constant B-field turned on, see [105].

Let me start with the open string analysis. The covariant action for strings in the plane wave [15] is invariant under local κ-symmetry. For open strings additional boundary terms arise under κ-variations and for supersymmetry preserving configurations these have to be cancelled by imposing suitable boundary conditions. In [97] this analysis was performed for longitudinal Dp-branes $(+,-,m,n)$, i.e. branes whose world-volume is along x^+, x^- and m and n denote the number of coordinates along the two transverse $\mathbb{R}^4$'s. Branes with $p = 3,\ 5,\ 7$ and $|m-n| = 2$ are half-supersymmetric[4] if they are

[4]This means that half of the kinematical as well as half of the dynamical supercharges are preserved. Kinematical (non-linearly realized) supercharges square to P^+, whereas

located 'at the origin', whereas 'outside the origin' only one quarter of the supercharges, namely half of the kinematical ones, are preserved [97]; these results agree with the analysis of open strings in light-cone gauge performed previously in [94], as well as the supergravity analysis [102] and D-brane embeddings [101]. Moreover, the D1-brane $(+,-,0,0)$ at any position only preserves half of the dynamical supercharges [97]. As the plane wave is a homogeneous space it is rather counterintuitive that the number of preserved supersymmetries may depend on the position of the brane. In fact, a more precise statement is that these branes are flat in Brinkmann coordinates. As the P^I are time dependent in these coordinates and do not simply generate translations along the x^I (cf. equation (2.14)), a half-supersymmetric brane related to a flat brane at the origin by a translation is curved [102]. Hence flat branes at different transverse positions do not fall in the same equivalence class with respect to translations generated by the P^I, see also [95].

In light-cone gauge boundary states can only describe instantonic D$(p+1)$-branes [106]. These are formally related to the longitudinal branes discussed above by a double Wick rotation and will be denoted by (m,n). Boundary states in the plane wave preserving half of both kinematical and dynamical supercharges were first constructed in [98] closely following the flat space description of [106]. Assume as in flat space that the D-brane preserves half of the dynamical supersymmetries, i.e.

$$\left(Q + i\eta M\widetilde{Q}\right)_{\dot{a}} \| \, (m,n), \mathbf{y}_t, \eta\rangle\rangle = 0\,, \tag{3.30}$$

where $\eta = \pm 1$ distinguishes a brane from an anti-brane, $\mathbf{y}_t$ is the transverse position and

$$M_{\dot{a}\dot{b}} = \left(\prod_{I\in N} \gamma^I\right)_{\dot{a}\dot{b}}\,. \tag{3.31}$$

Here γ^I are the gamma-matrices of $SO(8)$ and the product is over the Neumann directions. Together with *standard* Neumann and Dirichlet boundary conditions on the transverse bosons this implies that half of the kinematical supersymmetries are preserved (see e.g. [103])

$$\left(Q + i\eta M\widetilde{Q}\right)_{a} \| \, (m,n), \mathbf{y}_t, \eta\rangle\rangle = 0\,. \tag{3.32}$$

dynamical (linearly realized) supercharges square to the Hamiltonian plus additional generators.

Here M_{ab} is analogous to $M_{\dot{a}\dot{b}}$. The structure of the boundary state and consistency of the corresponding brane is crucially dependent on the choice of M. It is useful to distinguish the cases $\Pi M \Pi M = \mp 1$, the resulting branes will be sometimes called D$_-$- and D$_+$-branes, respectively. Boundary states for D$_-$ were constructed in [98, 99]. The condition on M is equivalent to $|m-n| = 2$ and thus leads to an analogous splitting of transverse coordinates as found from the open string analysis [94]. The allowed values for p are $p = 1, 3, 5$ and moreover, the condition (3.30) is only satisfied if $\mathbf{y}_t = 0$, otherwise only half of the kinematical supercharges are preserved. A detailed analysis and proof of the open-closed consistency conditions was given in [99]. In flat space the cylinder diagram can be expressed in terms of certain standard ϑ-functions and open-closed duality arises as a consequence of the properties of ϑ-functions under modular transformations. In the plane wave the cylinder diagram involves deformed ϑ-functions, where the deformation depends on the mass parameter [99]. It has been proven in [99] that these deformed ϑ-functions satisfy certain transformation properties that assure that the open-closed consistency conditions are precisely satisfied for the half-supersymmetric branes. On the other hand, branes away from the origin, i.e. those preserving only half of the kinematical supercharges, appeared to violate open-closed duality and hence be inconsistent. It is also worthwhile to note, that the kinematical conditions (3.32) are not preserved as a function of time x^+ [99]. Indeed, the open string kinematical supercharge does not commute with the Hamiltonian and hence is spectrum generating as is the case for closed strings. The open string ground state is an unmatched boson [94] and it follows that the open string partition function does not vanish [99].

Boundary states for D$_+$ and the analysis of open-closed duality was considered in [100]; independently this class was studied in detail in [95] from the open string side. As mentioned above, these branes also arose in the supergravity analysis [101, 102] and from the covariant open string [97]. In this case the condition on M is equivalent to $|m-n| = 0, 4$, however the coupling of $(0,4)$ and $(4,0)$-branes to the background R-R flux induces a flux on the world-volume [101] and correspondingly the boundary conditions for bosons have to be modified. From the analysis of [100] it seems that the only consistent boundary state with standard bosonic boundary conditions is the $(0,0)$ at any position, i.e. the D-instanton. Again, this is in agreement with the open string analysis of [101, 102, 97] where the corresponding object, the D1-brane, is found to preserve half of the dynamical supersymmetries at any

position. In this case the kinematical conditions (3.32) are preserved as a function of time x^+ [99], corresponding to a vanishing mass term for the open string zero-modes. Hence in this case the ground states form a degenerate supermultiplet and the open string partition function vanishes [100].

However, it appears that this is not yet the full story [95, 96]. The reason for this is that the world-sheet theory being free, it possesses an countably infinite set of world-sheet symmetries. These simply correspond to transformations shifting the fields by a parameter satisfying the free field equations. For the open string such a shift changes the action by a boundary term, so it is a symmetry if it satisfies appropriate boundary conditions. As shown in [95] the dynamical supercharges broken by D_-branes located outside the origin and the kinematical supercharges broken by the D1-brane can be combined with world-sheet transformations that generate a *non-vanishing* boundary term in such a way that the combined transformation *is* a symmetry of the open string. Together with open string symmetries originating from closed string symmetries compatible with the boundary conditions they generate a superalgebra similar to that of the other half-supersymmetric branes [96]. An analysis of the boundary states for D_-branes located outside the origin showed that these do preserve a combination of eight dynamical and kinematical closed string supercharges in addition to the eight standard kinematical ones. It would be very interesting to see whether these D_-branes turn out to be consistent with open-closed duality.

The BMN correspondence can be extended to open strings [107, 108, 101, 109]. It was shown in [101] that the D_-branes located at the origin, descend from supersymmetric AdS embeddings in $AdS_5 \times S^5$ through the Penrose limit; these originate from the near-horizon limit of supersymmetric intersections of the Dp-branes with a stack of D3-branes. For example, in the near-horizon limit, a suitable D3-D5 system leads to a D5 wrapping a $AdS_4 \times S^2$ submanifold in $AdS_5 \times S^5$. AdS/CFT is then supposed to act twice and the holographic dual is SYM coupled to a three-dimensional defect. The defect theory lives on the boundary of AdS_4 and as such is a CFT. The physics of closed strings and 5-5 open strings is described by the bulk theory, whereas the boundary theory captures 3-3, 3-5 and 5-3 strings [110, 111, 112]. In particular, the 3-5 and 5-3 strings give rise to hypermultiplets in the fundamental of the gauge group. Applying the Penrose limit results in the D5 $(+,-,3,1)$ brane at the origin. The dual description is through the BMN limit of SYM coupled to the three-dimensional defect. The closed string vacuum is dual to the trace of Z's and intuitively one expects the open string

vacuum also to be dual to a large number of Z's, but instead of the trace with 'quarks' at the end of the 'string'. This is indeed the case, the 'quarks' are scalars in the hypermultiplet originating from 3-5 and 5-3 strings and $\bar{q}Z^J q$ represents the open string vacuum [108]. Open string excitations are then dual to insertions of defect fields and, for non-zero-modes, in analogy with the insertion of phases for the closed string, cosines and sines for Neumann and Dirichlet boundary conditions, respectively [108]. The D7 $(+,-,4,2)$ was discussed in [107], this is more involved as orientifold planes have to be added to have a consistent theory, but the basic idea remains the same. A further interesting example is the giant graviton, i.e. a D3-brane wrapped on a S^3 in the S^5, which in the Penrose limit gives rise to the $(+,-,0,2)$ brane. Here the open string fluctuations arise from subdeterminant operators in SYM with large R-charge, see [109] for details.

3.3.2 Strings on pp-waves and interacting field theories

So far we have seen that we can get solvable string theories in light-cone gauge turning on null, *constant* R-R field-strengths in a plane wave geometry. As first discussed in [113], a large class of *interacting* string models with world-sheet supersymmetry, can be engineered in more general pp-wave geometries with *non-constant* fluxes and possibly transverse spaces with special holonomy; for example

$$\begin{aligned} ds^2 &= -2dx^+dx^- + H(x^i)\big(dx^+\big)^2 + ds_8^2\,, \\ F_5 &= dx^+ \wedge \varphi(x^i)\,, \end{aligned} \tag{3.33}$$

and all other background fields set to zero. It is convenient to split the candidate Killing spinor ε into two parts of opposite $SO(8)$ chiralities, $\varepsilon = \varepsilon_+ + \varepsilon_-$. Analyzing the gravitino variation, one finds that ε_+ is independent of all the coordinates; at lowest order in φ this is the supernumerary spinor we encountered before and gives rise to linearly realized supersymmetry on the world-sheet in light-cone gauge. On the other hand, it is useful to split ε_- into two parts as well: one, independent of x^+ (and x^-) is determined through ε_+ by the Killing equation, see [113] for the explicit solution. This completes the supernumerary Killing spinor for non-constant φ, however, as it is annihilated by Γ^+ it does not survive as part of the linearly realized supersymmetry in light-cone gauge. Depending on φ one might also have a number of kinematical supersymmetries; these correspond to the part of ε_-

depending only on x^+ and solving the Killing equation with $\varepsilon_+ = 0$; they imply that an even number of fermions (and hence also bosons) are free on the world-sheet and decouple from the remaining interacting fields. Generically there will be no kinematical supersymmetries. If the transverse space is curved, space-time supersymmetry requires it to have special holonomy. For example, for solutions with at least $N = (2,2)$ world-sheet supersymmetry the most general possibility is a Calabi-Yau four-fold. The Killing spinor equation determines the bosonic potential H in terms of φ and imposes additional constraints on the allowed four-forms. For $N = (2,2)$ the solution is parameterized in terms of a holomorphic function W and a real, harmonic Killing potential U. Moreover, the Lie-derivative of W along the holomorphic Killing vector $V_\mu = i\nabla_\mu U$ has to vanish [113]. Explicitly, the general solution leading to $N = (2,2)$ world-sheet theories in light-cone gauge is [113]

$$ds^2 = -2dx^+dx^- - 32\big(|dW|^2 + |V|^2\big)\big(dx^+\big)^2 + 2g_{\mu\bar\nu}dz^\mu d\bar z^{\bar\nu}\,, \tag{3.34}$$

$$\varphi_{\mu\nu} \equiv \frac{1}{3!}\varphi_{\mu\overline{\rho\sigma\tau}}\varepsilon^{\overline{\rho\sigma\tau}}{}_\nu = \nabla_\mu\nabla_\nu W\,, \qquad \varphi_{\overline{\mu\nu}} = \varphi^*_{\mu\nu}\,, \tag{3.35}$$

$$\varphi_{\mu\bar\nu} \equiv \frac{1}{2}\varphi_{\mu\bar\nu\rho}{}^\rho = \nabla_\mu\nabla_{\bar\nu}U\,. \tag{3.36}$$

Holomorphicity of W follows because the $(1,3)$ forms in the **10** of $SU(4)$ are co-closed, whereas U is harmonic due to tracelessness of the $(2,2)$ forms in the **15**. To get interesting interacting world-sheet theories the transverse space needs to be non-compact [113]. As the geometry is that of a pp-wave, one can still choose the light-cone gauge; the form of the resulting world-sheet theory is dictated by supersymmetry [114]. Notice that, pp-wave string theories do not lead to the most general 2d supersymmetric field theories: the target space is always eight-dimensional of special holonomy and the Killing potential U has to be harmonic due to the self-duality of F_5. Turning on an additional null R-R three-form leads to a second Killing vector (commuting with the first one), and again the corresponding potential is harmonic as a consequence of the variation of the dilatino [115]. In the case of $N = (1,1)$ the transverse space has Spin(7) holonomy, one gets a real harmonic superpotential [113] and, if the R-R three-form is non-zero, one harmonic Killing potential [115].

This general class of pp-wave solutions of type IIB supergravity is interesting for several reasons. They are *exact* string solutions, i.e. they do not receive α' corrections. In particular this is true for the plane wave background, see [116] for a proof based on the pure spinor approach for a covariant

description of strings in R-R backgrounds. As shown in [117], for the pp-wave space-times it is more advantageous to use the $U(4)$ formalism, where strings are governed by exact *interacting* $N = 2$ superconformal world-sheet theories. This proves the exactness of this general class of solutions, see also [118] for an extension to a larger class of R-R backgrounds, some of which cannot be studied in light-cone gauge. For an alternative argument, based on space-time properties, essentially the existence of a covariantly constant null vector, see [119]. Another interesting feature is the possibility to choose the superpotential such that the world-sheet theory becomes integrable [113]; in that case one may hope to use known properties of integrable models to learn about strings propagating in these backgrounds, see also [119, 120] for further discussions and examples.

D-branes in these backgrounds have been analyzed in [121], for example for $N = (2, 2)$ branes are supersymmetric if they wrap complex manifolds and the superpotential (and Killing potential) are constant on the world-volume; one can also have supersymmetric D5-branes wrapped on special Lagrangian submanifolds and appropriate conditions on the potentials. These results were derived in [121] in two ways, in the same spirit I described in the previous chapter: by analyzing supersymmetry preserving boundary conditions in the world-sheet theory and by finding supersymmetric embeddings in target space. Interestingly, for the special case of the plane wave, the branes found in [121] are 'oblique', that is they are oriented in directions that couple the two transverse $\mathbb{R}^4$'s; these however, preserve less supersymmetry than the branes considered in the previous section.

Chapter 4

String interactions in the plane wave background

In the previous two chapters I have among other things discussed and explained how free strings on the plane wave background and its orbifold arise in a double-scaling limit of $\mathcal{N} = 4$ SYM and $\mathcal{N} = 2$ quiver gauge theory, respectively. A computation of the anomalous dimensions of BMN single-trace operators in interacting *planar* $\mathcal{N} = 4$ SYM [13, 45, 46] reproduces the mass spectrum of *free* string theory [15, 34]. It is obviously an interesting question how string interactions and the non-planar sector of (interacting) gauge theory will fit into this picture. Before dwelling on the details let me first make a few general remarks. The proposed duality between free string theory and planar, interacting $\mathcal{N} = 4$ SYM in the BMN limit

$$\frac{1}{\mu} H \cong \Delta - J \tag{4.1}$$

should encompass interactions and non-planar effects, respectively. This follows from the fact that the global symmetries of both sides of the duality are not expected to be broken by quantum effects and hence the relation (4.1) should hold to all orders in the string coupling as a consequence of the AdS/CFT correspondence [44]. As the two operators act on different Hilbert spaces, this identity should be interpreted with some care. One information encoded in (4.1) is the identification of eigenvalues of the two operators. This is a basis-independent statement, on both sides of the duality we can choose any suitable basis, compute the matrix elements of the operator and obtain the eigenvalues by diagonalization. Subsequently the

corresponding eigenstates can be identified (up to degeneracy ambiguities). Recall once more the relations

$$\frac{1}{(\mu\alpha' p^+)^2} = \lambda' , \qquad 4\pi g_{\rm s}(\mu\alpha' p^+)^2 = g_2 . \tag{4.2}$$

As already stated, considering planar ($g_2 = 0$) gauge theory for small λ' is equivalent to free ($g_{\rm s} = 0$) string theory for large $\mu\alpha' p^+$. Now what happens if we take g_2 to be non-zero in the *free* gauge theory? We see from (4.2) that this means to take $\mu\alpha' p^+ \to \infty$, $g_{\rm s} \to 0$, such that $g_{\rm s}(\mu\alpha' p^+)^2$ is finite. As single- and multi-string states are orthogonal to each other, whereas single-trace BMN operators start to mix with multi-trace ones at finite g_2 in the free gauge theory [42, 43, 47, 44, 49], the identification of states with operators is modified for finite g_2. The fact that the required transformation is not unique [47, 44, 49, 62, 122] can be intuitively understood from string theory, because string states become highly degenerate for $\mu\alpha' p^+ = \infty$. Taking into account string interactions is equivalent to considering non-planar, *interacting* gauge theory. Then the freedom of mixing is getting more constrained because the dual operators now have to be eigenstates of the interacting dilatation operator. The ambiguity is still present for protected operators or ones where the interaction does not lift degeneracies present in the free theory.

As we are only able to obtain the free string spectrum in light-cone gauge, we should ask how interactions can be studied in this picture. In flat space, the usual strategy is the vertex operator approach and the difficulties associated with the fact that x^- is quadratic in the transverse coordinates are circumvented by using the ten-dimensional Lorentz invariance to set $p^+ = 0$ in general scattering amplitudes. However, in the plane wave background transverse momentum is not a good quantum number due to the harmonic oscillator potential confining the string to the origin of transverse space. Moreover ten-dimensional Lorentz invariance is broken by the non-zero R-R flux, in particular there is no J^{+-} generator. This obstruction significantly hinders the vertex operator approach to string interactions. There is only one other known way of studying string interactions in light-cone gauge, namely light-cone string field theory pioneered by Mandelstam [123, 124] for the bosonic string, see also [125, 126, 127, 128], and extended to the superstring in [129, 130, 131, 132]. The construction of light-cone string field theory in the plane wave geometry [133, 134, 18, 19] and the derivation of the leading non-planar correction to the anomalous dimension of BMN operators with

two defects (cf. equation (2.51)) from string theory [135, 136] is the main subject of this chapter and will be discussed in detail in the following sections. For a qualitative discussion of closed and open string interactions from the gauge theory point of view see [36].

Further studies of string interactions and their comparison with gauge theory in the BMN limit include [137, 138], where an alternative construction of the string field theory vertex is pursued, however a complete solution is still lacking. I will discuss this issue in more detail in section 4.3. In [139, 140] cubic interactions of IIB supergravity scalars arising from the dilaton-axion sector and the chiral primary sector – corresponding to mixtures of the metric and the five-form – were analyzed, the role of the bosonic prefactor in string field theory on the plane wave was studied in [141, 142]. For an investigation of the S-matrix for strings in the plane wave see [143]. In [144, 145] interactions of supergravity and string states were computed to leading and subleading order in $\mu\alpha' p^+$ and agreement with the planar three-point functions of BMN operators was established. For an extension to non-planar corrections and higher string interactions see [146, 147]. Here the comparison was based on the earlier proposal of [43] that the coefficient of the three-point function of BMN operators is proportional to the matrix element of the cubic interaction in the plane wave. With the work of [44] (see also [148]) this proposal has been replaced by the more rigorous expression in equation (4.1), see however [149, 150] for a revival and [72] for a derivation of a vertex-correlator duality slightly different from [43]. By identifying *single* string states with mixtures of single and multi-trace BMN operators – defined such that the redefined single/multi-trace operators are orthogonal in the non-planar, *free* gauge theory – general matrix elements of the two sides in (4.1) have been compared in [44, 62, 63, 122].

The algebraic structure of the cubic interaction vertex, in particular its expansion in powers of $\mu\alpha' p^+$ was first examined in [134, 151] and subsequently studied in [152, 18]. For comments on a non-trivial dependence of the string coupling on $\mu\alpha' p^+$ see [153]. Most notably, closed expressions for all the quantities appearing in the interaction vertex as functions of $\mu\alpha' p^+$ were provided in [154].

This chapter is organized as follows. To make the presentation self-contained and to introduce necessary notation I briefly review the free string on the plane wave in section 4.1. In section 4.2 I discuss the general features of light-cone string field theory. The construction of the kinematical and dynamical parts of the vertex and the (dynamical) supercharges in the number

basis is described in detail in sections 4.3 and 4.4. The functional expressions for the dynamical generators are given in section 4.5. The results are applied in section 4.6 to recover in light-cone string field theory the leading non-planar correction to the anomalous dimension . Several technical details that are not included in this chapter are given in appendices A and B.

4.1 Review of free string theory on the plane wave

In this section I briefly review some basic properties of free string theory on the plane wave background [15] and introduce some notation. After fixing fermionic κ-symmetry and world-sheet diffeomorphism and Weyl-symmetry in light-cone gauge, the r-th free string propagating on the plane wave is described by $x_r^I(\sigma_r)$ and $\vartheta_r^a(\sigma_r)$[1] in position space or by $p_r^I(\sigma_r)$ and $\lambda_r^a(\sigma_r)$ in momentum space, where $I = 1, \ldots, 8$ is a transverse SO(8) vector index, $a = 1, \ldots, 8$ is a SO(8) spinor index. I will often suppress these indices in what follows. The bosonic part of the light-cone action is [15]

$$S_{\mathrm{B}(r)} = \frac{e(\alpha_r)}{4\pi\alpha'} \int d\tau \int_0^{2\pi|\alpha_r|} d\sigma_r \left[\dot{x}_r^2 - x_r'^2 - \mu^2 x_r^2\right], \tag{4.3}$$

where

$$\dot{x}_r \equiv \partial_\tau x_r\,, \qquad x_r' \equiv \partial_{\sigma_r} x_r\,, \qquad \alpha_r \equiv \alpha' p_r^+\,, \qquad e(\alpha_r) \equiv \frac{\alpha_r}{|\alpha_r|}\,. \tag{4.4}$$

In a collision process p_r^+ will be negative for an incoming string and positive for an outgoing one. The mode expansions of the fields $x_r^I(\sigma_r, \tau)$ and $p_r^I(\sigma_r, \tau)$ at $\tau = 0$ are

$$\begin{aligned} x_r^I(\sigma_r) &= x_{0(r)}^I + \sqrt{2}\sum_{n=1}^{\infty}\left(x_{n(r)}^I \cos\frac{n\sigma_r}{|\alpha_r|} + x_{-n(r)}^I \sin\frac{n\sigma_r}{|\alpha_r|}\right), \\ p_r^I(\sigma_r) &= \frac{1}{2\pi|\alpha_r|}\left[p_{0(r)}^I + \sqrt{2}\sum_{n=1}^{\infty}\left(p_{n(r)}^I \cos\frac{n\sigma_r}{|\alpha_r|} + p_{-n(r)}^I \sin\frac{n\sigma_r}{|\alpha_r|}\right)\right]. \end{aligned} \tag{4.5}$$

[1] ϑ_r are the non-vanishing components of the $SO(9,1)$ spinor S satisfying the light-cone gauge $\Gamma^+ S = 0$.

The Fourier modes can be re-expressed in terms of creation and annihilation operators as

$$x^I_{n(r)} = i\sqrt{\frac{\alpha'}{2\omega_{n(r)}}}\left(a^I_{n(r)} - a^{I\,\dagger}_{n(r)}\right), \qquad p^I_{n(r)} = \sqrt{\frac{\omega_{n(r)}}{2\alpha'}}\left(a^I_{n(r)} + a^{I\,\dagger}_{n(r)}\right), \quad (4.6)$$

where

$$\omega_{n(r)} = \sqrt{n^2 + \left(\mu\alpha_r\right)^2}\,. \quad (4.7)$$

Canonical quantization of the bosonic coordinates

$$[x^I_r(\sigma_r), p^J_s(\sigma_s)] = i\delta^{IJ}\delta_{rs}\delta(\sigma_r - \sigma_s) \quad (4.8)$$

yields the usual commutation relations

$$[a^I_{n(r)}, a^{J\,\dagger}_{m(s)}] = \delta^{IJ}\delta_{nm}\delta_{rs}\,. \quad (4.9)$$

The fermionic part of the light-cone action in the plane wave is [15]

$$S_{\mathrm{F}(r)} = \frac{1}{8\pi}\int d\tau \int_0^{2\pi|\alpha_r|} d\sigma_r[i(\bar\vartheta_r\dot\vartheta_r + \vartheta_r\dot{\bar\vartheta}_r) - \vartheta_r\vartheta'_r + \bar\vartheta_r\bar\vartheta'_r - 2\mu\bar\vartheta_r\Pi\vartheta_r]\,, \quad (4.10)$$

where ϑ^a_r is a complex, positive chirality SO(8) spinor and

$$\Pi_{ab} \equiv (\gamma^1\gamma^2\gamma^3\gamma^4)_{ab} \quad (4.11)$$

is symmetric, traceless and squares to one.[2] The matrix Π breaks the transverse $SO(8)$ symmetry of the metric to $SO(4)\times SO(4)$ and induces a projection of $SO(8)$ spinors to subspaces of positive (negative) chirality under both $SO(4)$'s. The mode expansion of ϑ^a_r and its conjugate momentum $i\lambda^a_r \equiv i\frac{1}{4\pi}\bar\vartheta^a_r$ at $\tau = 0$ is

$$\begin{aligned}
\vartheta^a_r(\sigma_r) &= \vartheta^a_{0(r)} + \sqrt{2}\sum_{n=1}^{\infty}\left(\vartheta^a_{n(r)}\cos\frac{n\sigma_r}{|\alpha_r|} + \vartheta^a_{-n(r)}\sin\frac{n\sigma_r}{|\alpha_r|}\right), \\
\lambda^a_r(\sigma_r) &= \frac{1}{2\pi|\alpha_r|}\left[\lambda^a_{0(r)} + \sqrt{2}\sum_{n=1}^{\infty}\left(\lambda^a_{n(r)}\cos\frac{n\sigma_r}{|\alpha_r|} + \lambda^a_{-n(r)}\sin\frac{n\sigma_r}{|\alpha_r|}\right)\right].
\end{aligned} \quad (4.12)$$

[2] In comparison with chapter 2, here γ^I are the gamma-matrices of $SO(8)$. Throughout this chapter I use the gamma matrix conventions of [131].

The Fourier modes satisfy

$$\lambda^a_{n(r)} = \frac{|\alpha_r|}{2}\bar{\vartheta}^a_{n(r)}\,, \tag{4.13}$$

and, due to the canonical anti-commutation relations for the fermionic coordinates

$$\{\vartheta^a_r(\sigma_r), \lambda^b_s(\sigma_s)\} = \delta^{ab}\delta_{rs}\delta(\sigma_r - \sigma_s)\,, \tag{4.14}$$

they obey the following anti-commutation rules

$$\{\vartheta^a_{n(r)}, \lambda^b_{m(s)}\} = \delta^{ab}\delta_{nm}\delta_{rs}\,. \tag{4.15}$$

It is convenient to define a new set of fermionic operators [133]

$$\vartheta_{n(r)} = \frac{c_{n(r)}}{\sqrt{|\alpha_r|}}\left[(1+\rho_{n(r)}\Pi)b_{n(r)} + e(\alpha_r)e(n)(1-\rho_{n(r)}\Pi)b^\dagger_{-n(r)}\right]\,, \tag{4.16}$$

which explicitly break the $SO(8)$ symmetry to $SO(4)\times SO(4)$. Here

$$\rho_{n(r)} = \rho_{-n(r)} = \frac{\omega_{n(r)} - |n|}{\mu\alpha_r}\,, \qquad c_{n(r)} = c_{-n(r)} = \frac{1}{\sqrt{1+\rho^2_{n(r)}}}\,. \tag{4.17}$$

These modes satisfy

$$\{b^a_{n(r)}, b^{b\,\dagger}_{m(s)}\} = \delta^{ab}\delta_{nm}\delta_{rs}\,. \tag{4.18}$$

The free string light-cone Hamiltonian is

$$H_{2(r)} = \frac{1}{\alpha_r}\sum_{n\in\mathbb{Z}}\omega_{n(r)}\big(a^\dagger_{n(r)}a_{n(r)} + b^\dagger_{n(r)}b_{n(r)}\big)\,. \tag{4.19}$$

In the above the zero-point energies cancel between bosons and fermions. Since the Hamiltonian only depends on two dimensionful quantities μ and α_r, α' and p^+_r should not be thought of as separate parameters.

The single string Hilbert space is built out of creation operators acting on the vacuum $|v\rangle_r$ defined by

$$a_{n(r)}|v\rangle_r = 0\,, \qquad b_{n(r)}|v\rangle_r = 0\,, \qquad n\in\mathbb{Z}\,. \tag{4.20}$$

Physical states have to satisfy the level-matching constraint

$$\sum_{n\in\mathbb{Z}} n\big(a^\dagger_{n(r)}a_{n(r)} + b^\dagger_{n(r)}b_{n(r)}\big) = 0\,, \tag{4.21}$$

which expresses the fact that there is no physical significance to the choice of origin for σ_r.

The isometries of the plane wave background are generated by H, P^+, P^I, J^{+I}, J^{ij} and $J^{i'j'}$. The latter two are angular momentum generators of the transverse $SO(4) \times SO(4)$ symmetry of the plane wave. The 32 supersymmetries are generated by Q^+, $\bar{Q}^+$ and Q^-, $\bar{Q}^-$. The former correspond to inhomogeneous shift symmetries on the world-sheet ('non-linearly realized' supersymmetries), whereas the latter generate the linearly realized world-sheet supersymmetries. In sigma models the isometries of the target space-time result in conserved currents on the world-sheet. These have been obtained in [15] by the standard Noether method. I will need the following expressions (at $\tau = 0$)

$$P^I_{(r)} = \int_0^{2\pi|\alpha_r|} d\sigma_r \, p^I_r \,, \qquad J^{I+}_{(r)} = \frac{e(\alpha_r)}{2\pi\alpha'} \int_0^{2\pi|\alpha_r|} d\sigma_r \, x^I_r \,. \tag{4.22}$$

Conservation of (angular) momentum at the time of interaction ($\tau = 0$) will then be achieved by *local* conservation of $\sum p^I_r(\sigma_r)$ and $\sum e(\alpha_r) x^I_r(\sigma_r)$, see equation (4.47) below. The supercharges are

$$Q^+_{(r)} = \sqrt{\frac{2}{\alpha'}} \int_0^{2\pi|\alpha_r|} d\sigma_r \, \sqrt{2}\lambda_r \,, \tag{4.23}$$

$$Q^-_{(r)} = \sqrt{\frac{2}{\alpha'}} \int_0^{2\pi|\alpha_r|} d\sigma_r \, \left[2\pi\alpha' e(\alpha_r) p_r \gamma \lambda_r - i x'_r \gamma \bar{\lambda}_r - i\mu x_r \gamma \Pi \lambda_r\right] \,, \tag{4.24}$$

and $\bar{Q}^\pm_{(r)} = e(\alpha_r)\big[Q^\pm_{(r)}\big]^\dagger$. Conservation of the non-linearly realized supercharges by the interaction is established by local conservation of $\sum \lambda_r(\sigma_r)$ and $\sum e(\alpha_r)\vartheta_r(\sigma_r)$, cf. equation (4.62). Expanding Q^- in modes one finds

$$\begin{aligned} Q^-_{(r)} = \frac{e(\alpha_r)}{\sqrt{|\alpha_r|}} \gamma \Big(& \sqrt{\mu} \left[a_{0(r)}(1 + e(\alpha_r)\Pi) + a^\dagger_{0(r)}(1 - e(\alpha_r)\Pi) \right] \lambda_{0(r)} \\ & + \sum_{n\neq 0} \sqrt{|n|} \left[a_{n(r)} P^{-1}_{n(r)} b^\dagger_{n(r)} + e(\alpha_r) e(n) a^\dagger_{n(r)} P_{n(r)} b_{-n(r)} \right] \Big) \,, \end{aligned} \tag{4.25}$$

where

$$P_{n(r)} = \frac{1}{\sqrt{1 - \rho^2_{n(r)}}} (1 - \rho_{n(r)}\Pi) \,. \tag{4.26}$$

4.2 Principles of light-cone string field theory

The basic object in string field theory is an operator Ψ that, roughly speaking, creates or annihilates strings and is acting on a Hilbert space $\mathcal{H}$.[3] In light-cone string field theory Ψ is a functional of the light-cone time x^+, the string length $|\alpha|$ and the momentum densities $p^I(\sigma)$ and $\lambda^a(\sigma)$ specifying the configuration of the created/annihilated string. Observables of the free theory are expressed in terms of Ψ, for example for the free light-cone Hamiltonian

$$H_2 = \frac{1}{2}\int d|\alpha|\mathcal{D}^8 p(\sigma)\mathcal{D}^8\lambda(\sigma)\Psi^\dagger\left(\frac{\alpha'^2}{4}p^2 - \frac{\mu^2\alpha^2}{4}\frac{\delta^2}{\delta p^2} + \mu|\alpha|\frac{\alpha'}{2}\lambda\Pi\frac{\delta}{\delta\lambda}\right)\Psi\,. \tag{4.27}$$

To add interactions to the theory we have to ask the following question: what are the guiding principles in the construction of the interaction? For the bosonic string the answer is very intuitive and geometric [123, 124], the interaction should couple the string world-sheets in a continuous way. For example, the interaction vertex for the scattering of three strings depicted in Figure 4.1 is constructed with a Delta-functional enforcing world-sheet continuity. The functional approach [123, 124, 127, 128] can be extended to

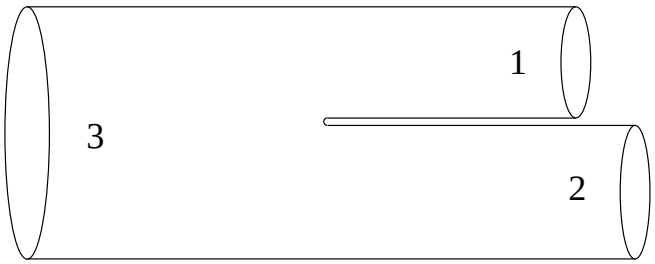

Figure 4.1: The world-sheet of the three string interaction vertex.

the superstring [129, 130, 131, 132]. Here the situation is slightly more complicated, but the basic principle governing the construction of interactions is very simple: the superalgebra has to be realized in the full interacting theory.

[3] $\mathcal{H}$ is the direct sum of m-string Hilbert spaces $\mathcal{H}_m$, the latter being the direct product of the single-string Hilbert space $\mathcal{H}_1$.

It is easy to understand why this complicates matters, as the supercharges that square to the Hamiltonian have to receive corrections as well when adding interactions. This is the essential difference to the bosonic string and modifies the form of the vertex [129, 130]. In a way the picture remains quite geometric, but in addition to a Delta-functional enforcing continuity in superspace, one has to insert local operators at the interaction point [129, 130]. These operators represent functional generalizations of derivative couplings.

To be more precise, consider the plane wave geometry and the behavior of the various generators of its superalgebra [26] when interactions are taken into account. In fact, one can distinguish two essentially different sets of generators. The first set consists of the kinematical generators

$$P^+ \, , \, P^I \, , \, J^{+I} \, , \, J^{ij} \, , \, J^{i'j'} \, , \, Q^+ \, , \, \bar{Q}^+ \, , \tag{4.28}$$

which are not corrected by interactions, in other words the symmetries they generate are not affected by adding higher order terms to the action. Hence these generators remain quadratic in the string field Ψ in the interacting field theory and act diagonally on $\mathcal{H}$. On the other hand, as alluded to above, the dynamical generators

$$H \, , \, Q^- \, , \, \bar{Q}^- \, , \tag{4.29}$$

do receive corrections in the presence of interactions and couple different numbers of strings. The requirement that the superalgebra is satisfied in the interacting theory, now gives rise to two kinds of constraints: *kinematical* constraints arising from the (anti)commutation relations of kinematical with dynamical generators and *dynamical* constraints arising from the (anti)commutation relations of dynamical generators alone. As I will explain below, the former will lead to the continuity conditions in superspace, whereas the latter require the insertion of the interaction point operators. In practice these constraints will be solved in perturbation theory, for example H, the full Hamiltonian of the interacting theory, has an expansion in the string coupling

$$H = H_2 + g_{\rm s} H_3 + \cdots \, , \tag{4.30}$$

and H_3 leads to the three-string interaction depicted in Figure 4.1. To illustrate the procedure, consider the commutator

$$[H, P^I] = -i\mu^2 J^{+I} \, , \tag{4.31}$$

which is of course different from the one in flat space. In the plane wave geometry transverse momentum is not a good quantum number due to the

confining harmonic oscillator potential. However, expansion in g_s implies the same kinematical constraint as in flat space

$$[H_3, P^I] = 0\,, \tag{4.32}$$

and, therefore, the interaction is translationally invariant. In fact, the relation (4.32) is also valid for all higher order interactions and as it is identical to the one in flat space many of the techniques developed in [130, 131] may be used in the plane wave case as well. In momentum space the conservation of transverse momentum by the interaction will be implemented by a Delta-functional (cf. (4.22))

$$\Delta^8 \left[\sum_{r=1}^{3} p_r(\sigma) \right] , \tag{4.33}$$

for a precise definition of this functional see Appendix A, equation (A.1). Here the coordinate σ of the three-string world-sheet is related to the coordinates σ_r of the r-th string as

$$\begin{aligned} \sigma_1 &= \sigma && -\pi\alpha_1 \le \sigma \le \pi\alpha_1\,, \\ \sigma_2 &= \begin{cases} \sigma - \pi\alpha_1 & \pi\alpha_1 \le \sigma \le \pi(\alpha_1+\alpha_2)\,, \\ \sigma + \pi\alpha_1 & -\pi(\alpha_1+\alpha_2) \le \sigma \le -\pi\alpha_1\,, \end{cases} \\ \sigma_3 &= -\sigma && -\pi(\alpha_1+\alpha_2) \le \sigma \le \pi(\alpha_1+\alpha_2) \end{aligned} \tag{4.34}$$

and $\alpha_1+\alpha_2+\alpha_3 = 0$, $\alpha_3 < 0$, i.e. the process where the incoming string splits into two strings. The joining of two strings into one is the adjoint of this process, see also section 4.6. In general, when I write an expression like $p_r(\sigma)$ it is understood that the function has support only for σ within the range that coincides with that of the r-th string. So, for example $p_r(\sigma)$ actually denotes $p_r(\sigma) = p_r(\sigma_r)\Theta_r(\sigma)$, where

$$\Theta_1(\sigma) = \theta(\pi\alpha_1 - |\sigma|)\,, \qquad \Theta_2(\sigma) = \theta(|\sigma| - \pi\alpha_1)\,, \qquad \Theta_3(\sigma) = 1\,. \tag{4.35}$$

Analogously from

$$[H, Q^+] = -\mu\Pi Q^+ \quad \Longrightarrow \quad [H_3, Q^+] = 0\,, \tag{4.36}$$

and, since light-cone momentum is a good quantum number, $[H, P^+] = 0$, one concludes that the cubic interaction contains (cf. (4.23))

$$\Delta^8 \left[\sum_{r=1}^{3} \lambda_r(\sigma) \right] \delta \left(\sum_{r=1}^{3} \alpha_r \right) . \tag{4.37}$$

Most interesting is the supersymmetry algebra

$$\{Q^-_{\dot a}, \bar Q^-_{\dot b}\} = 2\delta_{\dot a\dot b} H - i\mu\big(\gamma_{ij}\Pi\big)_{\dot a\dot b} J^{ij} + i\mu\big(\gamma_{i'j'}\Pi\big)_{\dot a\dot b} J^{i'j'}\,, \tag{4.38}$$

which also differs from the one in flat space. Expanding the supercharges $Q^-_{\dot a} = Q^-_{2\,\dot a} + g_{\rm s} Q^-_{3\,\dot a} + \cdots$, and analogously for $\bar Q^-$, the dynamical constraint following from (4.38) at $\mathcal{O}(g_{\rm s})$

$$\{Q^-_{2\,\dot a}, \bar Q^-_{3\,\dot b}\} + \{Q^-_{3\,\dot a}, \bar Q^-_{2\,\dot b}\} = 2\delta_{\dot a\dot b} H_3\,, \tag{4.39}$$

is again the same as in flat space. This constraint will be solved by inserting a prefactor $h_3(\alpha_r, p_r(\sigma), \lambda_r(\sigma))$ into the ansatz for H_3 and analogously for Q^-_3 and $\bar Q^-_3$. As I have already mentioned, the prefactors are operators inserted at the interaction point as required by locality, see also section 4.5. In summary, the structure of the superalgebra implies that the cubic interaction can formally be written in the form

$$H_3 = \int d\mu_3\, h_3(\alpha_r, p_r(\sigma), \lambda_r(\sigma))\Psi(1)\Psi(2)\Psi(3)\,, \tag{4.40}$$

where $\Psi(r)$ is the string field for the r-th string, h_3 is the prefactor determined by the dynamical constraints and the measure is

$$d\mu_3 \equiv \prod_{r=1}^{3} d\alpha_r D^8\lambda_r(\sigma) D^8 p_r(\sigma)\delta\big(\sum_s \alpha_s\big)\Delta^8\big[\sum_s \lambda_s(\sigma)\big]\Delta^8\big[\sum_s p_s(\sigma)\big]\,. \tag{4.41}$$

The expressions for Q^-_3 and $\bar Q^-_3$ are similar with different prefactors but the same measure $d\mu_3$.

To give a precise meaning to the above functional expressions and in particular, to solve the dynamical constraints, it is essential to do computations in the number basis [125, 126]. For simplicity consider the bosonic part, also the dependence on (and integration over) α_r will be suppressed in what follows. The bosonic part of the string field Ψ can be expanded in the number basis as

$$\Psi = \sum_{\{m_k\}} \phi_{m_k} \prod_{k\in\mathbb{Z}} \psi_{m_k}(p_k)\,, \tag{4.42}$$

where ϕ_{m_k} is an operator that creates/annihilates a number basis state $|m_k\rangle$ and $\psi_{m_k}(p_k)$ is the m_k-th oscillator wave function in momentum space. Substituting this into (4.40) yields the cubic coupling $C(m_{k(1)}, m_{k(2)}, m_{k(3)})$ of

three fields $\phi_{m_{k(r)}}$. It is convenient to express H_3 not as an operator mapping $\mathcal{H}_1 \to \mathcal{H}_2$ (or the adjoint process) but as a state in the 3-string Hilbert space via

$$C(m_{k(1)}, m_{k(2)}, m_{k(3)}) = \langle m_{k(1)}|\langle m_{k(2)}|\langle m_{k(3)}|H_3\rangle\,. \tag{4.43}$$

Analogously the operators Q_3^- and $\bar{Q}_3^-$ will be identified with states $|Q_3^-\rangle$ and $|\bar{Q}_3^-\rangle$ in $\mathcal{H}_3$. Then we can write

$$|H_3\rangle \equiv \hat{h}_3|V\rangle\,, \tag{4.44}$$

where $\hat{h}_3$ is the prefactor (operator) and the *kinematical part of the vertex* $|V\rangle$, common to all the dynamical generators, is

$$|V\rangle \equiv |E_a\rangle|E_b\rangle\delta\left(\sum_{r=1}^{3}\alpha_r\right)\,, \qquad |E_a\rangle \equiv \prod_{r=1}^{3}\int \mathcal{D}p_r\,\Delta^8\big[\sum_{s=1}^{3}p_s(\sigma)\big]|p_r\rangle\,, \tag{4.45}$$

and a similar expression for the fermionic contribution $|E_b\rangle$. Here $|p\rangle$ is the momentum eigenstate

$$\begin{aligned}|p\rangle &= \prod_{k\in\mathbb{Z}}|p_k\rangle = \sum_{\{m_k\}}\prod_{k\in\mathbb{Z}}\psi_{m_k}(p_k)|m_k\rangle \\ &= \prod_{k\in\mathbb{Z}}\Big(\frac{\omega_k\pi}{\alpha'}\Big)^{-1/4}\exp\left(-\frac{\alpha'}{2\omega_k}p_k^2 + \sqrt{\frac{2\alpha'}{\omega_k}}a_k^\dagger p_k - \frac{1}{2}a_k^\dagger a_k^\dagger\right)|0\rangle\,,\end{aligned} \tag{4.46}$$

and $|0\rangle$ is annihilated by a_n. Using (4.6) one can check that this is indeed a momentum eigenstate. It is not too difficult to derive the analogous expression for the fermionic contribution, but I will not need it in what follows.

4.3 The kinematical part of the vertex

In the previous section I have explained the general ideas underlying light-cone string field theory and presented formal expressions for the cubic corrections to the dynamical generators of the plane wave superalgebra. In particular we have seen that the solution to the kinematical constraints can be constructed as a functional integral, which is common to all the dynamical generators, cf. (4.45). To obtain the full solution we still need to determine the explicit form of the prefactors and for this it is necessary to explicitly compute the functional integral in the number basis.

The bosonic contribution $|E_a\rangle$ to the exponential part of the three-string interaction vertex has to satisfy the kinematic constraints [130, 131]

$$\sum_{r=1}^{3} p_r(\sigma)|E_a\rangle = 0\,, \qquad \sum_{r=1}^{3} e(\alpha_r)x_r(\sigma)|E_a\rangle = 0\,. \tag{4.47}$$

These are the same as in flat space and arise from the commutation relations of H with P^I and J^{+I}. They guarantee momentum conservation and continuity of the string world-sheet in the interaction. The solution of the constraints in (4.47) can be constructed as the functional integral (cf. (4.45))

$$\begin{aligned} |E_a\rangle &= \prod_{r=1}^{3} \int \mathcal{D}p_r\, \Delta^8\big[\sum_{s=1}^{3} p_s(\sigma)\big]\,|p_r\rangle \\ &= \prod_{r=1}^{3}\prod_{n\in\mathbb{Z}} \int dp_{n(r)}\delta^8\big[\sum_{s=1}^{3}\big(X^{(s)}p_s\big)_n\big]\,|p_{n(r)}\rangle\,. \end{aligned} \tag{4.48}$$

In the second equality the precise definition of the Delta-functional in terms of an infinite product of delta-functions for the individual Fourier modes of its argument was used, see appendix A, equations (A.1)–(A.7) for details and the explicit expressions of the $X^{(r)}$. As the resulting integrals are Gaussian (cf. (4.46)) the evaluation is straightforward and the result is [133]

$$|E_a\rangle \sim \exp\left(\frac{1}{2}\sum_{r,s=1}^{3}\sum_{m,n\in\mathbb{Z}} a^\dagger_{m(r)}\bar{N}^{rs}_{mn}a^\dagger_{n(s)}\right)|0\rangle_{123}\,, \tag{4.49}$$

where $|0\rangle_{123} = |0\rangle_1\otimes|0\rangle_2\otimes|0\rangle_3$ is annihilated by $a_{n(r)}$, $n\in\mathbb{Z}$. Apart from the zero-mode part, the determinant factor coming from the functional integral will be cancelled by the fermionic determinant. In (4.49) the non-vanishing elements of the so called bosonic Neumann matrices $\bar{N}^{rs}_{mn}$ for $m,\ n > 0$ are [133]

$$\bar{N}^{rs}_{mn} = \delta^{rs}\delta_{mn} - 2\sqrt{\frac{\omega_{m(r)}\omega_{n(s)}}{mn}}\left(A^{(r)\,T}\Gamma^{-1}A^{(s)}\right)_{mn}\,, \tag{4.50}$$

$$\bar{N}^{rs}_{m0} = -\sqrt{2\mu\alpha_s\omega_{m(r)}}\varepsilon^{st}\alpha_t\bar{N}^r_m\,, \qquad s\in\{1,2\}\,, \tag{4.51}$$

$$\bar{N}^{rs}_{00} = (1-4\mu\alpha K)\left(\delta^{rs} + \frac{\sqrt{\alpha_r\alpha_s}}{\alpha_3}\right)\,, \qquad r,s\in\{1,2\}\,, \tag{4.52}$$

$$\bar{N}^{r3}_{00} = -\sqrt{-\frac{\alpha_r}{\alpha_3}}\,, \qquad r\in\{1,2\}\,. \tag{4.53}$$

Here

$$\alpha \equiv \alpha_1\alpha_2\alpha_3\,, \qquad \Gamma \equiv \sum_{r=1}^{3} A^{(r)}U_{(r)}A^{(r)\,T}\,, \tag{4.54}$$

where

$$U_{(r)} \equiv C^{-1}\big(C_{(r)} - \mu\alpha_r\big)\,, \qquad C_{mn} \equiv m\delta_{mn}\,, \quad \big(C_{(r)}\big)_{mn} \equiv \omega_{m(r)}\delta_{mn}\,. \tag{4.55}$$

The matrices $A^{(r)}$ are related to the $X^{(r)}$ in a simple way, see equation (A.8). The terms in $\bar{N}^{rs}_{00}$ and $\bar{N}^{r3}_{00}$ that are not proportional to μ give the pure supergravity contribution to the Neumann matrices. The part of $\bar{N}^{rs}_{00}$ that is proportional to μ is induced by positive string modes of p_3. I also defined

$$\bar{N}^r \equiv -C^{-1/2}A^{(r)\,T}\Gamma^{-1}B\,, \qquad K \equiv -\frac{1}{4}B^T\Gamma^{-1}B\,. \tag{4.56}$$

An explicit expression for the vector B is given in (A.9). The quantities Γ, $\bar{N}^r$ and K manifestly reduce to their flat space counterparts, defined in [130, 131], as $\mu \to 0$. The only non-vanishing matrix elements with negative indices are $\bar{N}^{rs}_{-m,-n}$. They are related to $\bar{N}^{rs}_{mn}$ via [133]

$$\bar{N}^{rs}_{-m,-n} = -\left(U_{(r)}\bar{N}^{rs}U_{(s)}\right)_{mn}\,, \qquad m,\, n > 0\,. \tag{4.57}$$

As such the above expressions are already quite useful, though still formal in the sense that I did not present their explicit expressions as functions of μ, α_r yet. As the inverse of the infinite-dimensional matrix Γ appears in the expressions for the Neumann matrices this is a formidable problem. In flat space the results were known [123, 124] due to the identity[4]

$$\bar{N}^{rs}_{mn} = -\alpha\frac{(mn)^{3/2}}{\alpha_r n + \alpha_s m}\bar{N}^r_m\bar{N}^s_n\,, \tag{4.58}$$

and the explicit expressions

$$\bar{N}^r_m = \frac{1}{\alpha_r}f_m\left(-\frac{\alpha_{r+1}}{\alpha_r}\right)e^{m\tau_0/\alpha_r}\,, \qquad K = -\frac{\tau_o}{2\alpha}\,, \tag{4.59}$$

where $\alpha_4 \equiv \alpha_1$ is understood and

$$f_m(\gamma) = \frac{\Gamma(m\gamma)}{m!\Gamma\big(m(\gamma-1)+1\big)}\,, \qquad \tau_0 = \sum_{r=1}^{3}\alpha_r\ln|\alpha_r|\,. \tag{4.60}$$

[4]Notice that in comparison with [130] we have $\bar{N}^{rs}_{\text{here}} = C^{1/2}\bar{N}^{rs}_{\text{there}}C^{1/2}$.

The generalization of equation (4.58) to the plane wave background is [152, 18]

$$\begin{aligned}\bar{N}^{rs}_{mn} = -(1-4\mu\alpha K)^{-1}\frac{\alpha}{\alpha_r\omega_{n(s)}+\alpha_s\omega_{m(r)}}\times \\ \times\left[U^{-1}_{(r)}C^{1/2}_{(r)}C\bar{N}^r\right]_m\left[U^{-1}_{(s)}C^{1/2}_{(s)}C\bar{N}^s\right]_n ,\end{aligned} \tag{4.61}$$

and reduces to equation (4.58) as $\mu \to 0$. This factorization theorem can also be used to verify directly [18] that $|E_a\rangle$ satisfies the kinematic constraints in equation (4.47), see also appendix A.3. It will also prove essential throughout the next section. The remaining problem of deriving explicit expressions for K and $\bar{N}^r$ as in equation (4.59) has been solved in [154], however as I will not need these results in the remainder of this chapter I shall not give them here and refer the reader to [154].

Analogously to the bosonic case, the fermionic exponential part of the interaction vertex has to satisfy [130, 131]

$$\sum_{r=1}^{3}\lambda_r(\sigma)|E_b\rangle = 0\,, \qquad \sum_{r=1}^{3}e(\alpha_r)\vartheta_r(\sigma)|E_b\rangle = 0\,. \tag{4.62}$$

These constraints arise from the commutation relations of H with Q^+ and $\bar{Q}^+$, *cf.* equation (4.36). As in the bosonic case its solution could be obtained by constructing the fermionic analogue of the wavefunction (4.46) and then performing the resulting integrals over the non-zero-modes. The pure zero-mode contribution has to be treated separately. Instead of using the functional integral the exponential can be obtained (up to the normalization) by making a suitable ansatz and imposing the constraints (4.62) [130, 131]. The solution is [18] (cf. appendix A.3 for the details; the notation is defined below)

$$|E_b\rangle \sim \exp\left[\sum_{r,s=1}^{3}\sum_{m,n=1}^{\infty}b^\dagger_{-m(r)}Q^{rs}_{mn}b^\dagger_{n(s)} - \sqrt{2}\Lambda\sum_{r=1}^{3}\sum_{m=1}^{\infty}Q^r_m b^\dagger_{-m(r)}\right]|E^0_b\rangle\,, \tag{4.63}$$

where

$$\Lambda \equiv \alpha_1\lambda_{0(2)} - \alpha_2\lambda_{0(1)} \tag{4.64}$$

and $|E^0_b\rangle$ is the pure zero-mode part of the fermionic vertex

$$|E^0_b\rangle = \prod_{a=1}^{8}\left[\sum_{r=1}^{3}\lambda^a_{0(r)}\right]|0\rangle_{123} \tag{4.65}$$

and satisfies $\sum_{r=1}^{3}\lambda_{0(r)}|E_b^0\rangle = 0$ and $\sum_{r=1}^{3}\alpha_r\vartheta_{0(r)}|E_b^0\rangle = 0$. Notice that $|0\rangle_r$ is not the plane wave vacuum defined to be annihilated by the $b_{0(r)}$. Rather, it satisfies $\vartheta_{0(r)}|0\rangle_r = 0$ and $H_{2(r)}|0\rangle_r = 4\mu e(\alpha_r)|0\rangle_r$. In the limit $\mu \to 0$ it coincides with the flat space state that generates the massless multiplet by acting with $\lambda^a_{0(r)}$ on it. The fermionic Neumann matrices can be expressed in terms of the bosonic ones as [18]

$$Q^{rs}_{mn} = e(\alpha_r)\sqrt{\left|\frac{\alpha_s}{\alpha_r}\right|}\left[P^{-1}_{(r)}U_{(r)}C^{1/2}\bar{N}^{rs}C^{-1/2}U_{(s)}P^{-1}_{(s)}\right]_{mn}, \tag{4.66}$$

$$Q^{r}_{n} = \frac{e(\alpha_r)}{\sqrt{|\alpha_r|}}(1-4\mu\alpha K)^{-1}(1-2\mu\alpha K(1+\Pi))\left[P_{(r)}C^{1/2}_{(r)}C^{1/2}\bar{N}^{r}\right]_{n}. \tag{4.67}$$

Let me comment on the choice of zero-mode vertex in equation (4.65). In fact, it was proposed in [137] to use a different zero-mode vertex built on the plane wave vacuum $|v\rangle_r$ annihilated by all the $b_{0(r)}$. This also modifies the non-zero-mode part of $|E_b\rangle$, a complete solution to the kinematic constraints was given in [138, 18]. The motivation for this proposal originally was twofold. First, it was shown in [54] that the torus anomalous dimension of BMN operators with mixed scalar/vector impurities is the same as that for scalar/scalar impurities. This was in disagreement with the proposal of [43] that the coefficient of the three-point function of BMN operators is proportional to the matrix element of the cubic interaction in the plane wave, which would predict vanishing anomalous dimension for these class of operators at the torus level. This statement is not obvious, it follows from the structure of the cubic vertex, see also section 4.6. One possible resolution of this discrepancy was to think about a modification of the string vertex. Another possibility is of course to replace the proposal of [43] which was not derived from first principles. In fact, I will show in section 4.6 that using the identification in equation (4.1), the anomalous dimension of BMN operators transforming as $(\mathbf{4},\mathbf{4})$ under $SO(4)\times SO(4)$ *is* reproduced in string theory using the vertex with fermionic zero-mode part as in (4.65). A second motivation was based on the fact that the plane wave has a discrete $\mathbb{Z}_2$ symmetry that exchanges the two transverse $\mathbb{R}^4$'s. This discrete symmetry should be preserved by the interaction. It was shown in [137] that the $\mathbb{Z}_2$ parity of $|v\rangle$ is opposite to the one of $|0\rangle$, and, therefore, we have to assign positive parity to $|0\rangle$ in order to preserve the full transverse symmetry. This seems strange, as $|v\rangle$ has negative parity although it is the ground state of the theory. How can we decide if this assignment is correct? The spectrum of type IIB string theory

on the plane wave was analyzed in detail in [34], in particular the precise correspondence between the lowest lying string states and the fluctuation modes of supergravity on the plane wave was established. It turns out that the state $|0\rangle$ corresponds to the complex scalar arising from the dilaton-axion system, whereas the state $|v\rangle$ corresponds to a complex scalar being a mixture of the trace of the graviton and the R-R potential on one of the $\mathbb{R}^4$'s, that is the chiral primary sector. As dilaton and axion are scalars under $SO(8)$ and the discrete $\mathbb{Z}_2$ is just a particular $SO(8)$ transformation, we see that the assignment of positive parity to $|0\rangle$ is correct. Moreover, analysis of the interaction Hamiltonian for the chiral primary sector shows that invariance of the Hamiltonian under the $\mathbb{Z}_2$ requires the chiral primaries to have negative parity [140]. This does not mean that it is impossible to construct a vertex on $|v\rangle$ but it appears that it should be physically equivalent to the one presented here, i.e. related to it by a unitary transformation. Solving the dynamical constraints for the proposal of [137] turns out to be difficult and no complete solution has been presented so far. So this remains an interesting open problem.

4.4 The complete $\mathcal{O}(g_{\rm s})$ superstring vertex

In the previous section I reviewed the exponential part of the vertex, which solves the kinematic constraints. The remaining dynamic constraints are much more restrictive and are solved by introducing prefactors [130, 131], polynomial in creation operators, in front of $|V\rangle$ (cf. (4.44)). Within the functional formalism, the prefactors can be re-interpreted as insertions of local operators at the interaction point [129, 130]. In this section I present expressions for the dynamical generators in the number basis and prove that they satisfy the superalgebra at order $\mathcal{O}(g_{\rm s})$ [19]. The functional form of the leading order corrections to the dynamical generators [133, 18, 19] will be discussed in section 4.5.

Define the linear combinations of the free supercharges ($\eta = e^{i\pi/4}$)

$$\sqrt{2}\eta\, Q \equiv Q^- + i\bar{Q}^- \,, \qquad \sqrt{2}\bar{\eta}\,\widetilde{Q} = Q^- - i\bar{Q}^- \tag{4.68}$$

which satisfy

$$\begin{aligned} &\{Q_{\dot{a}}, Q_{\dot{b}}\} = \{\widetilde{Q}_{\dot{a}}, \widetilde{Q}_{\dot{b}}\} = 2\delta_{\dot{a}\dot{b}} H \,, \\ &\{Q_{\dot{a}}, \widetilde{Q}_{\dot{b}}\} = -\mu\big(\gamma_{ij}\Pi\big)_{\dot{a}\dot{b}} J^{ij} + \mu\big(\gamma_{i'j'}\Pi\big)_{\dot{a}\dot{b}} J^{i'j'} \,. \end{aligned} \tag{4.69}$$

Since J^{ij} and $J^{i'j'}$ are not corrected by the interaction, it follows that at order $\mathcal{O}(g_{\rm s})$ the dynamical generators have to satisfy

$$\sum_{r=1}^{3} Q_{\dot{a}(r)}|Q_{3\,\dot{b}}\rangle + \sum_{r=1}^{3} Q_{\dot{b}(r)}|Q_{3\,\dot{a}}\rangle = 2\delta_{\dot{a}\dot{b}}|H_3\rangle\,, \tag{4.70}$$

$$\sum_{r=1}^{3} \widetilde{Q}_{\dot{a}(r)}|\widetilde{Q}_{3\,\dot{b}}\rangle + \sum_{r=1}^{3} \widetilde{Q}_{\dot{b}(r)}|\widetilde{Q}_{3\,\dot{a}}\rangle = 2\delta_{\dot{a}\dot{b}}|H_3\rangle\,, \tag{4.71}$$

$$\sum_{r=1}^{3} Q_{\dot{a}(r)}|\widetilde{Q}_{3\,\dot{b}}\rangle + \sum_{r=1}^{3} \widetilde{Q}_{\dot{b}(r)}|Q_{3\,\dot{a}}\rangle = 0\,. \tag{4.72}$$

In order to derive equations that determine the full expressions for the dynamical generators one has to compute (anti)commutators of the free supercharges $Q_{\dot{a}(r)}$ and $\widetilde{Q}_{\dot{a}(r)}$ with the prefactors appearing in $|Q_{3\,\dot{a}}\rangle$ and $|\widetilde{Q}_{3\,\dot{a}}\rangle$. Moreover, the action of the supercharges on $|V\rangle$ has to be known. Here the factorization theorem (4.61) for the bosonic Neumann matrices and the relation between the bosonic and fermionic Neumann matrices given in equations (4.66) and (4.67) prove to be essential.

4.4.1 The bosonic constituents of the prefactors

An important constraint on the prefactors (that I will collectively denote by $\mathcal{P}$) is that they must respect the local conservation laws ensured by $|E_a\rangle$ and $|E_b\rangle$. For the bosonic part this means that it must commute with [130, 131]

$$\big[\sum_{r=1}^{3} p_r(\sigma), \mathcal{P}\big] = 0 = \big[\sum_{r=1}^{3} e(\alpha_r)x_r(\sigma), \mathcal{P}\big]\,. \tag{4.73}$$

Consider first an expression of the form

$$K_0 + K_+ = \sum_{r=1}^{3}\sum_{m=0}^{\infty} F_{m(r)}a^{\dagger}_{m(r)}\,. \tag{4.74}$$

The Fourier transform of (4.73) leads to the equations [134]

$$\sum_{r=1}^{3}\big[X^{(r)}C_{(r)}^{1/2}F_{(r)}\big]_m = 0 = \sum_{r=1}^{3}\alpha_r\big[X^{(r)}C_{(r)}^{-1/2}F_{(r)}\big]_m\,. \tag{4.75}$$

Here the components $m = 0$ and $m > 0$ decouple from each other. It is convenient to write the solution for $m = 0$ in a form which makes the flat space limit manifest [18]

$$K_0 = (1 - 4\mu\alpha K)^{1/2} \left(\mathbb{P} - i\mu \frac{\alpha}{\alpha'} \mathbb{R} \right) . \tag{4.76}$$

Here

$$\mathbb{P} \equiv \alpha_1 p_{0(2)} - \alpha_2 p_{0(1)} , \qquad \alpha_3 \mathbb{R} \equiv x_{0(1)} - x_{0(2)} , \qquad [\mathbb{R}, \mathbb{P}] = i , \tag{4.77}$$

that is (no sum on r)

$$F_{0(r)} = -(1 - 4\mu\alpha K)^{1/2} \sqrt{\frac{2}{\alpha'}} \varepsilon^{rs} \sqrt{\mu\alpha_r} \alpha_s , \qquad F_{0(3)} = 0 . \tag{4.78}$$

The overall normalization of K_0 is of course not determined by (4.75). The inclusion of the overall factor $(1-4\mu\alpha K)^{1/2}$ will be convenient in what follows. For $m > 0$ we have

$$\sum_{r=1}^{3} \left[A^{(r)} C^{-1/2} C_{(r)}^{1/2} F_{(r)} \right]_m = \frac{1}{\sqrt{\alpha'}} \mu\alpha B_m = \sum_{r=1}^{3} \mu\alpha_r \left[A^{(r)} C^{-1/2} C_{(r)}^{-1/2} F_{(r)} \right] . \tag{4.79}$$

These equations can be solved using the identities (A.12) and (A.19) given in appendix A. One finds [134, 18]

$$F_{m(r)} = -\frac{\alpha}{\sqrt{\alpha'}\alpha_r} (1 - 4\mu\alpha K)^{-1/2} \left[U_{(r)}^{-1} C_{(r)}^{1/2} C \bar{N}^r \right]_m . \tag{4.80}$$

In the limit $\mu \to 0$

$$\lim_{\mu \to 0} \left(K_0 + K_+ \right) = \mathbb{P} - \frac{\alpha}{\sqrt{\alpha'}} \sum_{r=1}^{3} \sum_{m=1}^{\infty} \frac{1}{\alpha_r} \left[C \bar{N}^r \right]_m \sqrt{m} a_{m(r)}^\dagger \tag{4.81}$$

coincides with the flat space result of [131]. Now take into account the negatively moded creation oscillators, i.e. consider

$$K_- = \sum_{r=1}^{3} \sum_{m=1}^{\infty} F_{-m(r)} a_{-m(r)}^\dagger . \tag{4.82}$$

This leads to the equations

$$\sum_{r=1}^{3} \frac{1}{\alpha_r} \big[A^{(r)} C^{1/2} C_{(r)}^{1/2} F_{(r)}\big]_{-m} = 0 = \sum_{r=1}^{3} \big[A^{(r)} C^{1/2} C_{(r)}^{-1/2} F_{(r)}\big]_{-m} \,. \tag{4.83}$$

Comparing the second equation with the difference of the two equations in (4.79) it follows

$$F_{-m(r)} \sim U_{m(r)} F_{m(r)} \,. \tag{4.84}$$

However, if one substitutes this into the first equation one actually sees that the sum is divergent [130, 131, 134]. This phenomenon already appears in flat space and it is known [130] that the function of σ responsible for the divergence is $\delta(\sigma - \pi\alpha_1) - \delta(\sigma + \pi\alpha_1)$. However, since $\pm\pi\alpha_1$ are actually identified this divergence is merely an artifact of our parametrization. I will argue in section 4.4.3 that the appropriate relative normalization is [18]

$$F_{-m(r)} = i U_{m(r)} F_{m(r)} \,. \tag{4.85}$$

4.4.2 The fermionic constituents of the prefactors

The fermionic constituents of the prefactors have to satisfy the conditions

$$\big\{ \sum_{r=1}^{3} \lambda_r(\sigma), \mathcal{P} \big\} = 0 = \big\{ \sum_{r=1}^{3} e(\alpha_r) \vartheta_r(\sigma), \mathcal{P} \big\} \,. \tag{4.86}$$

Consider

$$Y = \sum_{r=1}^{2} G_{0(r)} \lambda_{0(r)} + \sum_{r=1}^{3} \sum_{m=1}^{\infty} G_{m(r)} b^{\dagger}_{m(r)} \,. \tag{4.87}$$

For the zero-modes we can set the coefficient of, say, $\lambda_{0(3)}$ to zero due to the property of the fermionic supergravity vertex that $\sum_{r=1}^{3} \lambda_{0(r)} |E_b^0\rangle = 0$. The Fourier transform of (4.86) leads to the equations

$$\sum_{r=1}^{3} \frac{1}{\sqrt{|\alpha_r|}} \big[A^{(r)} C C_{(r)}^{-1/2} P_{(r)} G_{(r)}\big]_m = 0 \,, \tag{4.88}$$

$$\sum_{r=1}^{3} e(\alpha_r) \sqrt{|\alpha_r|} \big[C^{1/2} A^{(r)} C_{(r)}^{-1/2} P_{(r)}^{-1} G_{(r)}\big]_m = \sum_{r=1}^{3} \alpha_r X^{(r)}_{m0} G_{0(r)} \,. \tag{4.89}$$

The components $m = 0$ and $m > 0$ decouple from each other. For $m = 0$ the solution is

$$Y = (1 - 4\mu\alpha K)^{-1/2}(1 - 2\mu\alpha K(1 + \Pi))\sqrt{\frac{2}{\alpha'}}\Lambda + \cdots \tag{4.90}$$

As in the previous subsection the normalization is not determined and is chosen for further convenience. For $m > 0$ we can rewrite the second equation as

$$\sum_{r=1}^{3} e(\alpha_r)\sqrt{|\alpha_r|}\left[A^{(r)}C_{(r)}^{-1/2}P_{(r)}^{-1}G_{(r)}\right]_m = \frac{\alpha}{\sqrt{\alpha'}}B_m\,. \tag{4.91}$$

Then the solution can be expressed in terms of $F_{(r)}$ as [18]

$$G_{(r)} = \sqrt{|\alpha_r|}P_{(r)}^{-1}U_{(r)}C^{-1/2}F_{(r)}\,. \tag{4.92}$$

As $\mu \to 0$ we have

$$\lim_{\mu\to 0} Y = \sqrt{\frac{2}{\alpha'}}\Lambda + \sum_{r=1}^{3}\sum_{m=1}^{\infty}\frac{F_{m(r)}}{\sqrt{m}}\sqrt{|\alpha_r|}b^{\dagger}_{m(r)}\,. \tag{4.93}$$

Taking into account that $\sqrt{|\alpha_r|}b^{\dagger}_{m(r)} \longleftrightarrow Q^I_{-m(r)}$ in the notation of [131] this is exactly the flat space expression. We will see below that as in flat space [130, 131], it turns out that the prefactors do not involve negatively moded fermionic creation oscillators.

4.4.3 The dynamical generators at order $\mathcal{O}(g_{\rm s})$

Below I present the results [19] necessary to verify the dynamical constraints in equations (4.70) and (4.71), given the ansatz (4.98)-(4.100) for the cubic vertex and dynamical supercharges. Computational details are relegated to appendix B. We need

$$\sqrt{2}\eta\sum_{r=1}^{3}[Q_{(r)}, \widetilde{K}^I]\,|V\rangle = \sqrt{2}\bar{\eta}\sum_{r=1}^{3}[\widetilde{Q}_{(r)}, K^I]\,|V\rangle = \mu\gamma^I(1 + \Pi)Y|V\rangle\,, \tag{4.94}$$

where

$$K^I \equiv K^I_0 + K^I_+ + K^I_-\,, \qquad \widetilde{K}^I \equiv K^I_0 + K^I_+ - K^I_- \tag{4.95}$$

and

$$\sqrt{2}\eta\sum_{r=1}^{3}\{Q_{(r)},Y\}\widetilde{K}^I|V\rangle = i\gamma^J K^J\widetilde{K}^I|V\rangle - i\mu\frac{\alpha}{\alpha'}(1-4\mu\alpha K)\gamma^I(1-\Pi)|V\rangle\,,$$
$$\sqrt{2}\bar{\eta}\sum_{r=1}^{3}\{\widetilde{Q}_{(r)},Y\}K^I|V\rangle = -i\gamma^J\widetilde{K}^J K^I|V\rangle + i\mu\frac{\alpha}{\alpha'}(1-4\mu\alpha K)\gamma^I(1-\Pi)|V\rangle\,. \tag{4.96}$$

Notice that the above identities are only valid when both sides of the equation act on $|V\rangle$. The action of the supercharges on $|V\rangle$ is

$$\begin{aligned}\sqrt{2}\eta\sum_{r=1}^{3}Q_{(r)}|V\rangle &= -\frac{\alpha'}{\alpha}K^I\gamma^I Y|V\rangle\,,\\ \sqrt{2}\bar{\eta}\sum_{r=1}^{3}\widetilde{Q}_{(r)}|V\rangle &= -\frac{\alpha'}{\alpha}\widetilde{K}^I\gamma^I Y|V\rangle\,.\end{aligned} \tag{4.97}$$

The latter two equations actually lead to the insight that one has to consider the combinations K^I and $\widetilde{K}^I$, as they are solely determined by the kinematical part of the vertex and the quadratic pieces of the dynamical supercharges. In this way it is then possible to fix the relative normalization as has been done in equation (4.85) [18]. The results summarized in equations (4.94)-(4.97) motivate the following ansatz for the explicit form of the dynamical supercharges and the three-string interaction vertex [19, 133]

$$|H_3\rangle = \left(\widetilde{K}^I K^J - \mu\frac{\alpha}{\alpha'}\delta^{IJ}\right)v_{IJ}(Y)|V\rangle\,, \tag{4.98}$$
$$|Q_{3\,\dot{a}}\rangle = \widetilde{K}^I s^I_{\dot{a}}(Y)|V\rangle\,, \tag{4.99}$$
$$|\widetilde{Q}_{3\,\dot{a}}\rangle = K^I\tilde{s}^I_{\dot{a}}(Y)|V\rangle\,. \tag{4.100}$$

Substituting the above ansatz into (4.70) and (4.71) and using (4.94)-(4.97), one gets the following equations for v^{IJ}, $s^I_{\dot{a}}$ and $\tilde{s}^I_{\dot{a}}$ [5]

$$\delta_{\dot{a}\dot{b}}v^{IJ} = \frac{i}{\sqrt{2}}\frac{\alpha'}{\alpha}\gamma^J_{a(\dot{a}}D^a s^I_{\dot{b})}\,,\qquad \delta_{\dot{a}\dot{b}}v^{IJ} = -\frac{i}{\sqrt{2}}\frac{\alpha'}{\alpha}\gamma^I_{a(\dot{a}}\bar{D}^a\tilde{s}^J_{\dot{b})}\,, \tag{4.101}$$

[5] Here $(\dot{a}\dot{b})$ denotes symmetrization in $\dot{a}$, $\dot{b}$.

which originate from terms proportional to $\widetilde{K}_I K_J$ and $K_I \widetilde{K}_J$ and are identical to the flat space equations of [131]. Two additional equations, arising from terms proportional to $\mu\delta_{IJ}$, are

$$\begin{aligned} -\delta_{\dot{a}\dot{b}} v^{II} &= \frac{i}{\sqrt{2}} \frac{\alpha'}{\alpha} \gamma^I_{a(\dot{a}} \Big(D^a + i \big[\Pi \bar{D}\big]^a \Big) s^I_{\dot{b})} \,, \\ -\delta_{\dot{a}\dot{b}} v^{II} &= -\frac{i}{\sqrt{2}} \frac{\alpha'}{\alpha} \gamma^I_{a(\dot{a}} \Big(\bar{D}^a - i \big[\Pi D\big]^a \Big) \tilde{s}^I_{\dot{b})} \,. \end{aligned} \tag{4.102}$$

As in flat space [131] one defines

$$D^a \equiv \eta Y^a + \bar{\eta} \frac{\alpha}{\alpha'} \frac{\partial}{\partial Y_a} \,, \qquad \bar{D}^a \equiv \bar{\eta} Y^a + \eta \frac{\alpha}{\alpha'} \frac{\partial}{\partial Y_a} \,. \tag{4.103}$$

Recall first the solution of the flat space equations (4.101) [131]. One introduces the following functions of Y^a

$$w^{IJ} = \delta^{IJ} + \left(\frac{\alpha'}{\alpha}\right)^2 \frac{1}{4!} t^{IJ}_{abcd} Y^a Y^b Y^c Y^d + \left(\frac{\alpha'}{\alpha}\right)^4 \frac{1}{8!} \delta^{IJ} \varepsilon_{abcdefgh} Y^a \cdots Y^h \,, \tag{4.104}$$

$$i y^{IJ} = \frac{\alpha'}{\alpha} \frac{1}{2!} \gamma^{IJ}_{ab} Y^a Y^b + \left(\frac{\alpha'}{\alpha}\right)^3 \frac{1}{2 \cdot 6!} \gamma^{IJ}_{ab} \varepsilon^{ab}{}_{cdefgh} Y^c \cdots Y^h \,, \tag{4.105}$$

$$s^I_{1\,\dot{a}} = 2\gamma^I_{a\dot{a}} Y^a + \left(\frac{\alpha'}{\alpha}\right)^2 \frac{2}{6!} u^I_{abc\dot{a}} \varepsilon^{abc}{}_{defgh} Y^d \cdots Y^h \,, \tag{4.106}$$

$$s^I_{2\,\dot{a}} = -\frac{\alpha'}{\alpha} \frac{2}{3!} u^I_{abc\dot{a}} Y^a Y^b Y^c + \left(\frac{\alpha'}{\alpha}\right)^3 \frac{2}{7!} \gamma^I_{a\dot{a}} \varepsilon^a{}_{bcdefgh} Y^b \cdots Y^h \,. \tag{4.107}$$

Here

$$t^{IJ}_{abcd} \equiv \gamma^{IK}_{[ab} \gamma^{JK}_{cd]} \,, \qquad u^I_{abc\dot{a}} \equiv -\gamma^{IJ}_{[ab} \gamma^J_{c]\dot{a}} \,. \tag{4.108}$$

t^{IJ}_{abcd} is traceless and symmetric in I, J, hence w^{IJ} is a symmetric tensor of $SO(8)$, whereas y^{IJ} is antisymmetric. Apart from the coefficients, in flat space the structure of the individual terms is completely fixed by the $SO(8)$ symmetry. The solution of equations (4.101) is [131]

$$v^{IJ} \equiv w^{IJ} + y^{IJ} \,, \quad s^I_{\dot{a}} \equiv -\frac{2}{\alpha'} \frac{i}{\sqrt{2}} \big(\eta s^I_{1\,\dot{a}} + \bar{\eta} s^I_{2\,\dot{a}} \big) \,, \quad \tilde{s}^I_{\dot{a}} \equiv \frac{2}{\alpha'} \frac{i}{\sqrt{2}} \big(\bar{\eta} s^I_{1\,\dot{a}} + \eta s^I_{2\,\dot{a}} \big) \,. \tag{4.109}$$

Next consider the additional equations (4.102). Using the flat space solution, these can be rewritten as

$$0 = \gamma^I_{a(\dot{a}} \big[\Pi \bar{D}\big]^a s^I_{\dot{b})} \qquad 0 = \gamma^I_{a(\dot{a}} \big[\Pi D\big]^a \tilde{s}^I_{\dot{b})} \,. \tag{4.110}$$

The proof that these equations are also satisfied by (4.109) is given in appendix B.

The proof [19] of equation (4.72) is more involved and provides an important consistency check of the ansatz (4.98)-(4.100). It leads to the equations (cf. appendix B.3)

$$\delta^{IJ} m_{\dot{a}\dot{b}} - \frac{1}{\sqrt{2}} \frac{\alpha'}{\alpha} \gamma^{(I}_{a\dot{a}} D^a \tilde{s}^{J)}_{\dot{b}} = 0 \,, \tag{4.111}$$

$$\delta^{IJ} m_{\dot{a}\dot{b}} - \frac{1}{\sqrt{2}} \frac{\alpha'}{\alpha} \gamma^{(I}_{a\dot{b}} \bar{D}^a s^{J)}_{\dot{a}} = 0 \,, \tag{4.112}$$

$$\sqrt{2}\big(\gamma^I_{a\dot{a}} \eta \tilde{s}^I_{\dot{b}} - \gamma^I_{a\dot{b}} \bar{\eta} s^I_{\dot{a}}\big) - 4i m_{\dot{a}\dot{b}} Y_a = 0 \,, \tag{4.113}$$

$$\big(\gamma^I_{a\dot{a}} \bar{D}_b \tilde{s}^I_{\dot{b}} + \gamma^I_{a\dot{b}} D_b s^I_{\dot{a}}\big)(1-\Pi)^{ab} = 0 \,. \tag{4.114}$$

Here

$$\begin{aligned} m_{\dot{a}\dot{b}} = \delta_{\dot{a}\dot{b}} &+ \frac{i}{4} \frac{\alpha'}{2\alpha} \gamma^{IJ}_{\dot{a}\dot{b}} \gamma^{IJ}_{ab} Y^a Y^b - \frac{1}{4 \cdot 4!} \left(\frac{\alpha'}{2\alpha}\right)^2 \gamma^{IJKL}_{\dot{a}\dot{b}} t^{IJKL}_{abcd} Y^a Y^b Y^c Y^d \\ &- \frac{i}{6!} \left(\frac{\alpha'}{2\alpha}\right)^3 \gamma^{IJ}_{\dot{a}\dot{b}} \gamma^{IJ}_{ab} \varepsilon^{ab}{}_{cdefgh} Y^c \cdots Y^h - \frac{2}{7!} \left(\frac{\alpha'}{2\alpha}\right)^4 \delta_{\dot{a}\dot{b}} \varepsilon_{abcdefgh} Y^a \cdots Y^h \end{aligned} \tag{4.115}$$

and

$$t^{IJKL}_{abcd} \equiv \gamma^{[IJ}_{[ab} \gamma^{KL]}_{cd]} \,. \tag{4.116}$$

The first three equations are identical to those in flat space and have been proven in [131]. The additional equation (4.114) is proved in appendix B.2.

The dynamical constraints do not fix the overall normalization of the dynamical generators which can depend on μ and the α_r's. In flat space, the fact that the J^{-I} generator of the Lorentz algebra is also dynamical imposes further constraints on the other dynamical generators and apart from trivial rescaling uniquely fixes their normalization [132]. As the J^{-I} generator is not part of the plane wave superalgebra this procedure cannot be applied to our setup. A comparison with a supergravity calculation fixes the normalization for small μ to be $\sim (\alpha' \mu^2)/(\alpha_3^4)$ [139], whereas a comparison with the dual

field theory implies that for large μ it is $\sim \alpha'/\alpha^2$ [62, 122, 19]. It was conjectured in [19] that the normalization valid for all μ is

$$16\pi\alpha'\mu^2\alpha_3^{-4}(1-4\mu\alpha K)^2\,, \tag{4.117}$$

which has the correct small- and large-μ behavior [152]. On the other hand, the non-trivial normalization of Y (cf. equation (4.90)) and the fact that the terms $\widetilde{K}^I K^J$ and $\mu\delta^{IJ}$ in equation (4.98) involve different powers of $1-4\mu\alpha K$ is fixed by requiring the closure of the superalgebra at $\mathcal{O}(g_{\rm s})$. In order to obtain the supergravity expressions for the dynamical generators from equations (4.98)-(4.100), one should set K to zero, as it originates from massive string modes, cf. the remark below equation (4.55). Together with $[\mathbb{R}^I,\mathbb{P}^J]=i\delta^{IJ}$, one can check that the supergravity vertex obtained in this way agrees with the supergravity vertex presented in section 4 of [133].

I would like to stress that the part in the cubic interaction vertex proportional to $\mu\delta^{IJ}$ was originally not present in [133, 134] and plays an essential role in the above proof of the superalgebra in the interacting string field theory. In [133], functional expressions for the constituents of the prefactors were used to argue that the vertex of [133] satisfied the superalgebra. It is known [130, 131] that these functional expressions do not in general agree with those in the number basis, and as explained in the next section, it would appear that this subtlety is the origin of the original absence of the $\mu\delta^{IJ}$ term in [133]. In summary the oscillator expressions (4.98)-(4.100) have been shown explicitly to satisfy the superalgebra at $\mathcal{O}(g_{\rm s})$ [19].

4.5 Functional expressions

The functional expressions for the cubic corrections to the dynamical generators can be provided by defining the operator analogues for the constituents of the prefactor. These operators depend on $p_r(\sigma)$, $x_r'(\sigma)$ and $\lambda_r(\sigma)$ and since $p_r(\sigma)$ and $\lambda_r(\sigma)$ correspond to functional derivatives with respect to $x_r(\sigma)$ and $\vartheta_r(\sigma)$ the only physically sensible value of σ to choose is the interaction point $\sigma=\pm\pi\alpha_1$. As operators at this point are singular the prefactor must be carefully defined in the limit $\sigma\to|\pi\alpha_1|$ [130]. Rewriting the operators in the number basis one obtains expressions containing both creation and annihilation operators of the various oscillators. Eliminating the annihilation operators by (anti)commuting them through the exponential factors of the

vertex one recovers the number basis expressions for the constituents of the prefactors [130, 131, 18].

As in flat space [130, 131] consider the following operators

$$P(\sigma) \equiv -2\pi\sqrt{-\alpha}(\pi\alpha_1 - \sigma)^{1/2}\big(p_1(\sigma) + p_1(-\sigma)\big)\,, \tag{4.118}$$

$$\partial X(\sigma) \equiv 4\pi\frac{\sqrt{-\alpha}}{\alpha'}(\pi\alpha_1 - \sigma)^{1/2}\big(x_1'(\sigma) + x_1'(-\sigma)\big)\,, \tag{4.119}$$

$$Y(\sigma) \equiv -2\pi\frac{\sqrt{-2\alpha}}{\sqrt{\alpha'}}(\pi\alpha_1 - \sigma)^{1/2}\big(\lambda_1(\sigma) + \lambda_1(-\sigma)\big)\,. \tag{4.120}$$

One also defines $P|V\rangle \equiv \lim_{\sigma\to\pi\alpha_1} P(\sigma)|V\rangle$ and analogously for ∂X. Acting on the exponential part of the vertex and taking the limit $\sigma \to \pi\alpha_1$ we have [18]

$$\lim_{\sigma\to\pi\alpha_1} K^I(\sigma)|V\rangle \equiv \left(P^I + \frac{1}{4\pi}\partial X^I\right)|V\rangle = K^I|V\rangle\,, \tag{4.121}$$

$$\lim_{\sigma\to\pi\alpha_1} \widetilde{K}^I(\sigma)|V\rangle \equiv \left(P^I - \frac{1}{4\pi}\partial X^I\right)|V\rangle = \widetilde{K}^I|V\rangle\,, \tag{4.122}$$

$$\lim_{\sigma\to\pi\alpha_1} Y(\sigma)|V\rangle = Y|V\rangle\,. \tag{4.123}$$

Here I prove only the last equation, for more details see [18]. Substituting the mode expansion for $\lambda_1(\sigma)$ into (4.120) one gets

$$\lim_{\sigma\to\pi\alpha_1} Y(\sigma)|V\rangle = -\sqrt{\frac{2}{\alpha'}}\sqrt{\frac{-2\alpha}{\alpha_1}}\lim_{\varepsilon\to 0}\varepsilon^{1/2}\sum_{n=1}^{\infty}(-1)^n\cos(n\varepsilon/\alpha_1)\times$$
$$\times\left[\sqrt{2}\Lambda Q_n^1 + \sum_{r=1}^{3}\sum_{m=1}^{\infty}Q_{nm}^{1r}b_{m(r)}^{\dagger}\right]|V\rangle\,. \tag{4.124}$$

Now the singular behavior of the sum as $\varepsilon \to 0$ can be traced to the way it diverges as $n \to \infty$. Therefore to take the limit $\varepsilon \to 0$ we can approximate the summand for large n and using the factorization theorem (4.61) one finds [18]

$$\lim_{\sigma\to\pi\alpha_1} Y(\sigma)|V\rangle = f(\mu)(1 - 4\mu\alpha K)^{-1/2}Y|V\rangle\,, \tag{4.125}$$

where

$$f(\mu) \equiv -2\frac{\sqrt{-\alpha}}{\alpha_1}\lim_{e\to 0}\varepsilon^{1/2}\sum_{n=1}^{\infty}(-1)^n n\cos(n\varepsilon/\alpha_1)\bar{N}_n^1\,. \tag{4.126}$$

The identity

$$f(\mu) = (1 - 4\mu\alpha K)^{1/2} \tag{4.127}$$

was conjectured to hold on general grounds (the closure of the superalgebra) in [19] and shown to be true in [154]. This concludes the proof of equation (4.120).

So up to the overall normalization one can write the functional equivalent of equations (4.98), (4.99) and (4.100) as

$$H_3 = \lim_{\sigma\to\pi\alpha_1} \int d\mu_3 \Big(\widetilde{K}^I(\sigma)K^J(\sigma) - \mu\frac{\alpha}{\alpha'}\delta^{IJ}\Big) v_{IJ}(Y(\sigma))\Psi(1)\Psi(2)\Psi(3)\,, \tag{4.128}$$

$$Q_{3\,\dot{a}} = \lim_{\sigma\to\pi\alpha_1} \int d\mu_3 \widetilde{K}^I(\sigma) s^I_{\dot{a}}(Y(\sigma))\Psi(1)\Psi(2)\Psi(3)\,, \tag{4.129}$$

$$\widetilde{Q}_{3\,\dot{a}} = \lim_{\sigma\to\pi\alpha_1} \int d\mu_3 K^I(\sigma) \tilde{s}^I_{\dot{a}}(Y(\sigma))\Psi(1)\Psi(2)\Psi(3)\,, \tag{4.130}$$

where $d\mu_3$ is the functional expression leading to the kinematical part of the vertex, cf. equation (4.41).

Finally, I would like to point out the following subtlety. One can check for example that

$$\sqrt{2}\bar{\eta}\sum_{r=1}^{3}[\widetilde{Q}_{(r)}, \lim_{\sigma\to\pi\alpha_1} K^I(\sigma)]\,|V\rangle = \mu\gamma^I\Pi Y|V\rangle\,. \tag{4.131}$$

However, this is not equal to the commutator of $\sum_r \widetilde{Q}_{(r)}$ with K^I. Using equation (4.97) and

$$[\lim_{\sigma\to\pi\alpha_1} K^I(\sigma), \widetilde{K}^J]|V\rangle = -\frac{\mu\alpha}{\alpha'}(1 - 4\mu\alpha K)^{-1/2}\delta^{IJ}|V\rangle\,, \tag{4.132}$$

leads to [19]

$$\sqrt{2}\bar{\eta}\sum_{r=1}^{3}[\widetilde{Q}_{(r)}, K^I]|V\rangle = \mu\gamma^I(1+\Pi)Y|V\rangle\,, \tag{4.133}$$

which is equivalent to equation (4.94) of section 4.4. It is this appearance of the matrix $1 + \Pi$ as opposed to just Π, that is responsible for the term proportional to $\mu\delta^{IJ}$ in the cubic interaction vertex.

4.6 Anomalous dimension from string theory

In this section I discuss how the result for the anomalous dimension in equation (2.51) can be recovered in string theory. This has been done for the symmetric-traceless **9** and antisymmetric $\mathbf{6} = \mathbf{3} + \bar{\mathbf{3}}$ of either one of the $SO(4)$'s in [135] and for the trace **1** in [136]. Here I review this work and also extend the analysis to the $(\mathbf{4},\mathbf{4})_{\pm}$[6] of $SO(4) \times SO(4)$. These correspond to BMN operators with mixed scalar/vector impurities and superconformal symmetry of the gauge theory implies that they have the same anomalous dimension as the other representations [48]. As there has been some doubt if the vertex presented in this chapter correctly reproduces this fact, this also constitutes a check of our results.

To compute the mass shift of the single string state due to interactions

$$|n\rangle \equiv \alpha^{I\,\dagger}_{n(3)}\alpha^{J\,\dagger}_{-n(3)}|v\rangle_3\,, \tag{4.134}$$

non-degenerate perturbation theory was used in [135, 136]. In principle one should use degenerate perturbation theory as the single string state can mix with multi-string states having the same energy. The same caveat holds for the computation in gauge theory and we will ignore this complication here. At lowest order the eigenvalue correction comes from two contributions; one-loop diagram and contact term

$$\delta E^{(2)}_n\langle n|n\rangle = g_2^2\sum_{1,2}\left[\frac{1}{2}\frac{|\langle n|H_3|1,2\rangle|^2}{E^{(0)}_n - E^{(0)}_{1,2}} + \frac{1}{8}\sum_{\dot a}|\langle n|Q_{3\,\dot a}|1,2\rangle|^2\right]. \tag{4.135}$$

Factors different from g_2 in the normalization (cf. (4.117)) are absorbed in the definition of H_3 and Q_3, the extra factor of $1/2$ in the first term is due to the reflection symmetry of the one-loop diagram. The sum over 1, 2 is over physical double-string states, that is those obeying the level-matching condition and for the case at hand Q_3^2 is the only relevant contribution to the quartic coupling. As the generators are hermitian we take the absolute value squared of the matrix elements. In fact, time-reversal in the plane wave background consists of the transformation

$$x^+ \to -x^+\,, \qquad x^- \to -x^-\,, \qquad \mu \to -\mu\,, \tag{4.136}$$

[6] We define the states in $(\mathbf{4},\mathbf{4})_{\pm}$ as $\frac{1}{2}\big(\alpha^{i\,\dagger}_{n(3)}\alpha^{j'\,\dagger}_{-n(3)} \pm \alpha^{i\,\dagger}_{-n(3)}\alpha^{j'\,\dagger}_{n(3)}\big)|v\rangle_3$. The change of basis $\alpha_n = \frac{1}{\sqrt{2}}\big(a_{|n|} + ie(n)a_{-|n|}\big)$ for $n \neq 0$ is convenient and an analogous transformation will be made for the fermions.

in particular the reversal of μ is needed due to the presence of the R-R flux. In the previous sections I have always assumed that μ is non-negative and $\alpha_3 < 0$, α_1, $\alpha_2 > 0$. This is, say, the process where a single string *splits* into two strings. One can show that for the process in which two strings *join* to form a single string, i.e. α_1, $\alpha_2 < 0$ and $\alpha_3 > 0$, one should make the additional replacements

$$\mu \to -\mu\,, \qquad \Pi \to -\Pi \tag{4.137}$$

in equations (4.98)-(4.100) and (4.117). This is in agreement with equation (4.136). Notice that the transformation of Π is needed to leave the fermionic mass term invariant, cf. (4.10). From the formal expressions for the Neumann matrices it is not manifest that the cubic corrections to the dynamical generators are hermitian as they have to be. However, from the explicit expressions for the Neumann matrices [154] one can see that all the quantities are in fact invariant under the time-reversal. The string states obey the delta-function normalization $\langle n|n'\rangle = \mathcal{N}|\alpha_3|\delta(\alpha_3-\alpha_4)$, where $\mathcal{N} = \frac{1}{2}(1+\delta^{ij})$ for the **9**, $\mathcal{N} = \frac{1}{4}$ for the **1** and $\mathcal{N} = \frac{1}{2}$ otherwise. The sum over double-string states includes a double integral over light-cone momenta, one integral is trivial due to the string-length conservation of the cubic interaction and the factor of $|\alpha_3|\delta(\alpha_3-\alpha_4)$ can be cancelled on both sides of equation (4.135). The remaining sum is then the usual completeness relation for harmonic oscillators projected on physical states and we have ($\beta \equiv \alpha_1/\alpha_3$)

$$\mathcal{N}\delta E_n^{(2)} = -g_2^2 \int_{-1}^{0} \frac{d\beta}{\beta(\beta+1)} \sum_{\text{modes}} \left[\frac{1}{2} \frac{|\langle n|H_3|1,2\rangle|^2}{E_n^{(0)} - E_{1,2}^{(0)}} + \frac{1}{8} \sum_{\dot{a}} |\langle n|Q_{3\,\dot{a}}|1,2\rangle|^2 \right] . \tag{4.138}$$

The measure arises due to the fact that string states are delta-function normalized.

It is important to note that in gauge theory the dilatation operator was diagonalized within the subspace of two-impurity BMN operators in perturbation theory in the 't Hooft coupling λ and then extrapolated to λ, $J \to \infty$. But it is not obvious that the large J limit of the perturbation series in λ has to agree order by order with the perturbation series in λ', see for example [151]. Indeed there is evidence from string theory that this is not the case: for large μ the denominator of the first term in equation (4.138) is of order $\mathcal{O}(\mu^{-1})$ in the impurity conserving channel, whereas it is of order $\mathcal{O}(\mu)$ in the impurity non-conserving one. However, as already noticed in [134], matrix

elements where the number of impurities changes by two are of order $\mathcal{O}(1)$ and, therefore potentially can contribute to the mass-shift at leading order, that is $\mathcal{O}(\mu g_2^2 \lambda')$. Notice that impurity non-conserving matrix elements being of order one, means actually $\mathcal{O}(\mu g_2 \sqrt{\lambda'})$ and as the overall factor of μ is simply for dimensional reasons and should not be counted when translating to gauge theory (cf. equation (4.1)) implies contributions $\sim g_2\sqrt{\lambda'}$ to matrix elements of the dilatation operator. It was observed in [135] that the contribution of the impurity non-conserving channel to (4.138) is linearly divergent. This is due to the fact that the large μ limit does not commute with the infinite sums over mode numbers; for finite μ the divergence is regularized. So a linear divergence reflects a contribution $\sim \mu g_2^2 \lambda'(-\mu\alpha_3) = \mu g_2^2 \sqrt{\lambda'}$ and hence of order $g_2^2\sqrt{\lambda'}$ to the anomalous dimension. This constitutes a non-perturbative, 'stringy' effect. It remains a very interesting challenge to investigate the contribution of the impurity non-conserving channel in detail. In principle, it is possible that besides a divergent contribution there is also a finite one; this would then scale as $\mu g_2^2 \lambda'$. However, to reproduce the result (2.51) for the anomalous dimensions of two-impurity BMN operators in string theory one is led to a truncation of equation (4.138) to the impurity conserving channel [135]. This analysis will be performed below.

4.6.1 Contribution of one-loop diagrams

The matrix element $\langle n|H_3|1,2\rangle$ in the impurity conserving channel is non-zero only if the double-string state contains either two bosonic or two fermionic oscillators. The relevant projection operator is

$$\sum_{K,L} \alpha^{\dagger K}_{0(1)} \alpha^{\dagger L}_{0(2)} |v\rangle\langle v| \alpha^{L}_{0(2)} \alpha^{K}_{0(1)} + \frac{1}{2} \sum_{k\in\mathbb{Z}} \sum_{r,K,L} \alpha^{\dagger K}_{k(r)} \alpha^{\dagger L}_{-k(r)} |v\rangle\langle v| \alpha^{L}_{-k(r)} \alpha^{K}_{k(r)}$$
$$+ \sum_{a,b} \beta^{\dagger a}_{0(1)} \beta^{\dagger b}_{0(2)} |v\rangle\langle v| \beta^{b}_{0(2)} \beta^{a}_{0(1)} + \frac{1}{2} \sum_{k\in\mathbb{Z}} \sum_{r,a,b} \beta^{\dagger a}_{k(r)} \beta^{\dagger b}_{-k(r)} |v\rangle\langle v| \beta^{b}_{-k(r)} \beta^{a}_{k(r)} \,.$$

For the first case the fermionic contribution to the matrix elements is simple to determine. Using a γ-matrix representation in which $\Pi = \mathrm{diag}(\mathbf{1}_4, -\mathbf{1}_4)$, the plane wave vacua ${}_r\langle v|$ are related to ${}_r\langle 0|$ (up to an irrelevant phase) via

$${}_r\langle v| = {}_r\langle 0| \left(\frac{\alpha_r}{2}\right)^2 \prod_{a=5}^{8} \vartheta^a_{0(r)} \,, \qquad {}_3\langle v| = -{}_3\langle 0| \left(\frac{\alpha_3}{2}\right)^2 \prod_{a=1}^{4} \vartheta^a_{0(r)} \,. \tag{4.139}$$

Directions $1,\ldots,4$ and $5,\ldots,8$ correspond to positive and negative chirality under $SO(4)\times SO(4)$, respectively. Eight of the zero-modes in equation (4.139), namely $\vartheta^a_{0(3)}$, $a=1,\ldots,4$ and, say, $\vartheta^a_{0(2)}$, $a=5,\ldots,8$ are saturated by $|E^0_b\rangle$, so to give a non-zero contribution the remaining four zero-modes must be contracted with the $\mathcal{O}(Y^4)$ term in $v_{MN}(Y)$. Hence, the fermionic contribution is

$$\left(\frac{\alpha'}{\alpha}\right)^2 \frac{1}{4!} t^{MN}_{abcd123}\langle v|Y^{abcd}|E^0_b\rangle = -\left(\frac{\alpha_3}{2}\right)^4 (1-4\mu\alpha K)^{-2} t^{MN}_{5678}\,. \tag{4.140}$$

One can show that $t^{MN}_{5678} = (\delta^{mn}, -\delta^{m'n'})$ in the γ-matrix basis used here. The bosonic part of the matrix element is not difficult to evaluate and I will not go into details. Using the large μ expansions for the bosonic Neumann matrices [152, 154] (see also appendix A) one finds, for example for $(I,J)=(i,j)$,

$$\begin{aligned}
\langle n|H_3|\alpha^{\dagger\,k}_{0(r)}\alpha^{\dagger\,l}_{0(s)}|v\rangle_{12} &\sim \mu\lambda'\frac{\sin^2 n\pi\beta}{2\pi^2}\left(\delta^{rs}+\frac{\sqrt{\alpha_r\alpha_s}}{\alpha_3}\right)S^{ijkl}\,,\\
\langle n|H_3|\alpha^{\dagger\,K}_{k(r)}\alpha^{\dagger\,L}_{-k(r)}|v\rangle_{12} &\sim \mu\lambda'\beta(\beta+1)\frac{\alpha_3}{\alpha_r}\frac{\sin^2 n\pi\beta}{2\pi^2}S^{ijkl}\,,
\end{aligned} \tag{4.141}$$

and the analogous expression for $(I,J)=(i',j')$ with an (inessential) overall minus sign. Here

$$S^{ijkl}\equiv T^{ijkl}+\frac{1}{4}\delta^{ij}T^{kl}\,,\quad T^{ijkl}=\delta^{ik}\delta^{jl}+\delta^{jk}\delta^{il}-\frac{1}{2}\delta^{ij}\delta^{kl}\,,\quad T^{kl}=-2\delta^{kl} \tag{4.142}$$

can be split into a symmetric-traceless and a trace part. There is no contribution to the **6** nor to $(\mathbf{4},\mathbf{4})_\pm$. The sum over k and the integral over β can be done and the complete contribution of the impurity conserving channel with bosonic excitations at one-loop is

$$\frac{\mu g_2^2\lambda'}{4\pi^2}\frac{15}{16\pi^2 n^2}\begin{cases}\frac{1}{4}\sum_{k,l}T^{ijkl}T^{ijkl}=\frac{1}{2}(1+\frac{1}{2}\delta^{ij})\\ \frac{1}{64}\sum_{k,l}T^{kl}T^{kl}=\frac{1}{4}\end{cases}\,. \tag{4.143}$$

The factors of $\frac{1}{2}(1+\frac{1}{2}\delta^{ij})$ and $\frac{1}{4}$ equal the normalization $\mathcal{N}$ of the string states. Thus the contribution to the **9** and **1** is in both cases [135, 136]

$$\frac{\mu g_2^2\lambda'}{4\pi^2}\frac{15}{16\pi^2 n^2}\,. \tag{4.144}$$

The second case with two fermionic oscillators in the double-string was not analyzed in [135, 136]. For example, one has to evaluate the tensor t^{MN}_{abcd} for spinor indices belonging to different chiralities of $SO(4) \times SO(4)$. Then t^{MN}_{abcd} is non-zero only if M and N are not in the same $SO(4)$. The resulting contribution is the same as in equation (4.144) for the representation $(\mathbf{4},\mathbf{4})_+$.

4.6.2 Contribution of contact terms

To have a non-zero contribution from Q_3^2 the intermediate states need to have an odd number of bosonic oscillators and an odd number of fermionic oscillators. Thus the simplest contribution comes from the impurity conserving channel. In this case the projector is

$$\sum_{K,a} \alpha^{\dagger\, K}_{0(1)} \beta^{\dagger\, a}_{0(2)} |v\rangle\langle v| \beta^{a}_{0(2)} \alpha^{K}_{0(1)} + (1 \leftrightarrow 2) + \sum_{k\in\mathbb{Z}} \sum_{r,K,a} \alpha^{\dagger\, K}_{k(r)} \beta^{\dagger\, a}_{-k(r)} |v\rangle\langle v| \beta^{a}_{-k(r)} \alpha^{K}_{k(r)} \,.$$

At leading order in μ one finds that for the bosonic part of the matrix element the zero-modes contribute only to the antisymmetric representations, whereas the non-zero-modes contribute to all representations. For the fermionic part of the matrix element a simple counting of zero-modes shows that only terms of order $\mathcal{O}(Y^3)$ and $\mathcal{O}(Y^5)$ in $v_{MN}(Y)$ can contribute. One also needs to evaluate the tensor $u^I_{abc\dot{a}}$ and the large μ expansion of the fermionic Neumann matrices, which due to the relation to the bosonic Neumann matrices [18] can be inferred from the latter. The final result is

$$\frac{1}{2} \frac{\mu g_2^2 \lambda'}{4\pi^2} \left(\frac{1}{12} + \frac{35}{32 n^2 \pi^2} \right) , \tag{4.145}$$

for the antisymmetric $\mathbf{6}$ and $(\mathbf{4},\mathbf{4})_-$ and

$$\frac{\mu g_2^2 \lambda'}{4\pi^2} \left(\frac{1}{12} + \frac{5}{32 n^2 \pi^2} \right) \begin{cases} \frac{1}{2}\left(1 + \frac{1}{2}\delta^{IJ}\right) \\ \frac{1}{4} \end{cases} , \tag{4.146}$$

for the $\mathbf{1}$, $\mathbf{9}$ and $(\mathbf{4},\mathbf{4})_+$. Summing the contributions of one-loop and contact diagrams we see that *all* (bosonic) two-impurity irreducible representations of $SO(4)\times SO(4)$ get the same contribution to the mass-shift from the impurity-conserving channels

$$\delta E_n^{(2)} = \frac{\mu g_2^2 \lambda'}{4\pi^2} \left(\frac{1}{12} + \frac{35}{32 n^2 \pi^2} \right) . \tag{4.147}$$

This is in exact agreement with the gauge theory result of [47, 49], cf. (2.51).

Chapter 5

Summary and outlook

The realization of BMN that the Penrose limit of $AdS_5 \times S^5$ and the knowledge of the full string spectrum on the plane wave, allowed for the first time to study AdS/CFT – albeit in a special limit – beyond the supergravity approximation, has ignited a lot of activity. Apart from presenting my own contributions, the goal of this thesis was to give an overview over various developments that have taken place.

In chapter 2 I gave an introduction to the BMN correspondence. Several aspects of this duality were discussed in some detail both from the string theory as well as the gauge theory point of view.

Extensions of the BMN duality to less trivial backgrounds have been the topic of chapter 3. Having first considered several illustrative examples, we studied supersymmetric $\mathbb{Z}_k$ orbifolds of the plane wave space-time and showed that free string theory in the orbifolded plane wave is dual to a subsector of planar $\mathcal{N} = 2$ $[U(N)]^k$ quiver gauge theory. In particular, we gave an explicit identification of gauge theory operators and string states both in the untwisted and twisted sectors. As interesting examples of further aspects of string theory on pp-wave space-times, I discussed D-branes on the plane wave and string theory on pp-waves with non-constant R-R fluxes and curved transverse spaces.

To investigate the BMN correspondence beyond the free string/planar gauge theory level, string interactions and the non-planar gauge theory sector have to be taken into account. In chapter 4 string interactions in the plane wave background were studied in the framework of light-cone string field theory. At first order in the string coupling, interactions in this setup are encoded in a cubic vertex. We analyzed in detail the construction of this

vertex as well as the dynamical supercharges and presented their complete expressions both in the oscillator as well as the continuum basis. We proved that these satisfy the plane wave superalgebra to first order in the string coupling. In the process, several results that had been known in flat space light-cone string field theory, e.g. a factorization theorem for the bosonic Neumann matrices, were generalized to the plane wave space-time. We used the vertex and supercharges to compute the leading order mass shift of certain string states in a truncation to the impurity-conserving channel. The result exactly agreed with the leading non-planar correction to the anomalous dimension of the dual operators in $\mathcal{N} = 4$ SYM.

There are a number of interesting open problems we have encountered: for example, it would be nice to extend the computation of the mass shift for the simplest string states in section 4.6 beyond the contribution of the impurity-conserving channel. As I have explained, in the large μ limit this presumably translates to non-perturbative effects in the dual gauge theory. Indeed, a non-vanishing contribution of order $g_2^2\sqrt{\lambda'}$ to the anomalous dimension would only constitute the leading term in a power series in fractional powers of λ'; verifying the presence of such a contribution could eventually lead to better understanding the nature of the BMN limit in $\mathcal{N} = 4$ SYM. One should be aware, however, that even a computation of the leading order 'stringy' effect along the lines of section 4.6 seems unfeasible, as infinitely many intermediate states have to be taken into account. So the way out seems to be to perform a full-fledged one-loop/contact term computation. Again, this is difficult, as one has to compute the inverse of infinite-dimensional matrices (involving e.g. the product of two Neumann matrices) exactly, before taking the large μ limit. Nevertheless, some progress might be achieved along the lines of [154] using the techniques developed there.

It would also be interesting to go beyond the comparison of masses vs. anomalous dimensions in both theories, that is to establish a vertex-correlator duality. Some promising results in this direction based on a 'phenomenological' string theory vertex have been reported in [149, 150]. This vertex is built on the plane wave vacuum, see the discussion in section 4.3. A derivation of its 'phenomenological' part is so far missing and certainly desirable. It would also be important to derive the vertex-correlator duality used in [149, 150] and originally proposed in [43] from first principles.[1] Possibly the ideas of [36] to relate transition amplitudes in the matrix model truncation of SYM on S^3

[1] A vertex-correlator duality slightly different from [43] was recently derived in [72].

to string theory amplitudes are of importance in that respect. The precise nature of such a truncation deserves further study as well.

It would be natural to extend the research on light-cone string field theory to include open strings, i.e. D-branes on the plane wave. In particular, as explained in section 3.3.1, D$_-$-branes outside the origin preserve dynamical supercharges which involve certain world-sheet symmetries [95]. One way to understand the consistency of these branes in the presence of interactions is to construct the corresponding cubic open string interaction vertex.[2] As a more direct approach, one could try to prove open/closed duality using the boundary states proposed in [96]. It would also be nice to shed light on the covariant origin of these non-standard supercharges. Of course, open/closed string interactions would be interesting as well, given the expected duality to the BMN limit of $\mathcal{N} = 4$ SYM coupled to defect conformal field theories.

As we have seen, the light-cone GS action is well-suited to obtain the spectrum of string theories in simple backgrounds with R-R flux. Although the construction of the cubic interaction vertex is technically quite involved, it is a viable possibility to study simple tree- and – at least in the approximation described in section 4.6 – one-loop interactions. However, as discussed in [116], even for studying higher point tree-amplitudes in flat space this approach is not useful, as the vertex explicitly depends on the interaction point. Moreover, it is difficult to describe physical states with vanishing p^+ in the light-cone formalism. These caveats become even more problematic for backgrounds without the full Lorentz isometry, such as the plane wave. Thus it is a worthwhile prospect to use the $U(4)$ formalism as advocated in [117, 118] to overcome some of these drawbacks. In this approach strings on the plane wave are described by an exact *interacting* $N = 2$ superconformal field theory and standard CFT techniques may be used for computations. One can also naturally describe strings in the more general pp-wave geometries of section 3.3.2 in this setup, which makes this approach potentially even more interesting. A comparison of scattering amplitudes computed in world-sheet perturbation theory with results obtained in the light-cone gauge is of interest and a good starting point for further studies of string theory in backgrounds with R-R flux.

[2] For the D7-brane at the origin this has recently been done in [155].

Appendix A

The kinematical part of the vertex

A.1 The Delta-functional

The precise definition of the Delta-functional is

$$\Delta^8[\sum_{r=1}^{3} p_r(\sigma)] \equiv \prod_{m\geq 0} \delta^8 \left(\int_{-\pi|\alpha_3|}^{\pi|\alpha_3|} d\sigma \, e^{im\sigma/|\alpha_3|} \sum_{r=1}^{3} p_r(\sigma) \right) . \tag{A.1}$$

The pure zero-mode contribution decouples from the Delta-functional, so

$$\Delta^8[\sum_{r=1}^{3} p_r(\sigma)] = \delta^8(\sum_{r=1}^{3} p_{0(r)}) \prod_{m=1}^{\infty} \delta^8 \left(\int_{-\pi|\alpha_3|}^{\pi|\alpha_3|} d\sigma e^{im\sigma/|\alpha_3|} \sum_{r=1}^{3} p_r(\sigma) \right) . \tag{A.2}$$

We need the following integrals for $m > 0$, $n \geq 0$ ($\beta \equiv \alpha_1/\alpha_3$)

$$\begin{aligned} \frac{1}{\pi\alpha_1} \int_{-\pi\alpha_1}^{\pi\alpha_1} d\sigma \cos\frac{m\sigma}{\alpha_3} \cos\frac{n\sigma}{\alpha_1} &= (-1)^n \frac{2m\beta}{\pi} \frac{\sin m\pi\beta}{m^2\beta^2 - n^2} \equiv X^{(1)}_{mn} , \\ \frac{1}{\pi\alpha_1} \int_{-\pi\alpha_1}^{\pi\alpha_1} d\sigma \sin\frac{m\sigma}{\alpha_3} \sin\frac{n\sigma}{\alpha_1} &= \frac{n}{m\beta} X^{(1)}_{mn} , \end{aligned} \tag{A.3}$$

and

$$\begin{aligned} \frac{2}{\pi\alpha_2} \int_{\pi\alpha_1}^{-\pi\alpha_3} d\sigma \cos\frac{m\sigma}{\alpha_3} \cos\frac{n}{\alpha_2}(\sigma - \pi\alpha_1) &= \frac{2m(\beta+1)}{\pi} \frac{\sin m\pi\beta}{m^2(\beta+1)^2 - n^2} \equiv X^{(2)}_{mn}, \\ \frac{2}{\pi\alpha_2} \int_{\pi\alpha_1}^{-\pi\alpha_3} d\sigma \sin\frac{m\sigma}{\alpha_3} \sin\frac{n}{\alpha_2}(\sigma - \pi\alpha_1) &= -\frac{n}{m(\beta+1)} X^{(2)}_{mn} . \end{aligned} \tag{A.4}$$

Then the delta-functions over the non-zero-modes contribute

$$\prod_{m=1}^{\infty}\delta^8\left(\frac{1}{\sqrt{2}}\sum_{r=1}^{3}\left[\sum_{n=1}^{\infty}X^{(r)}_{mn}\big(p_{n(r)}-i\frac{\alpha_3}{\alpha_r}\frac{n}{m}p_{-n(r)}\big)+\frac{1}{\sqrt{2}}X^{(r)}_{m0}p_{0(r)}\right]\right) \tag{A.5}$$

and I have defined $X^{(3)}_{mn}=\delta_{mn}$. We see that negative and non-negative modes decouple from each other. We can extend the range of m,n to $\mathbb{Z}$ by introducing

$$X^{(r)}_{mn}\equiv\begin{cases}X^{(r)}_{mn}\,, & m\,,n>0\\ \frac{\alpha_3}{\alpha_r}\frac{n}{m}X^{(r)}_{-m,-n}\,, & m\,,n<0\\ \frac{1}{\sqrt{2}}X^{(r)}_{m0}\,, & m>0\,,r\in\{1,2\}\\ 1\,, & m=0=n\\ 0\,, & \text{otherwise}\end{cases} \tag{A.6}$$

Then the Delta-functional takes the form

$$\Delta[\sum_{r=1}^{3}p_r(\sigma)]\sim\prod_{m\in\mathbb{Z}}\delta\left(\sum_{r=1}^{3}\sum_{n\in\mathbb{Z}}X^{(r)}_{mn}p_{n(r)}\right)\,. \tag{A.7}$$

Here I ignored factors of $\sqrt{2}$ which can be absorbed in the measure. It is convenient to introduce the matrices for m, $n>0$

$$\begin{aligned}C_{mn}&=m\delta_{mn}\,,\\ A^{(1)}_{mn}&=(-1)^n\frac{2\sqrt{mn}\beta}{\pi}\frac{\sin m\pi\beta}{m^2\beta^2-n^2}=\big(C^{-1/2}X^{(1)}C^{1/2}\big)_{mn}\,,\\ A^{(2)}_{mn}&=\frac{2\sqrt{mn}(\beta+1)}{\pi}\frac{\sin m\pi\beta}{m^2(\beta+1)^2-n^2}=\big(C^{-1/2}X^{(2)}C^{1/2}\big)_{mn}\,,\\ A^{(3)}_{mn}&=\delta_{mn}\end{aligned} \tag{A.8}$$

and the vector ($m>0$)

$$B_m=-\frac{2}{\pi}\frac{\alpha_3}{\alpha_1\alpha_2}m^{-3/2}\sin m\pi\beta \tag{A.9}$$

related to $X^{(r)}_{m0}$ via

$$X^{(r)}_{m0}=-\varepsilon^{rs}\alpha_s\big(C^{1/2}B\big)_m\,. \tag{A.10}$$

These satisfy the following very useful identities [130]

$$-\frac{\alpha_3}{\alpha_r}\,CA^{(r)T}C^{-1}A^{(s)}=\delta^{rs}\mathbf{1}\,,\quad -\frac{\alpha_r}{\alpha_3}\,C^{-1}A^{(r)T}CA^{(s)}=\delta^{rs}\mathbf{1}\,,\quad A^{(r)T}CB=0 \tag{A.11}$$

valid for $r,s\in\{1,2\}$ and

$$\sum_{r=1}^{3}\frac{1}{\alpha_r}A^{(r)}CA^{(r)\,T}=0\,,\qquad \sum_{r=1}^{3}\alpha_r A^{(r)}C^{-1}A^{(r)\,T}=\frac{\alpha}{2}BB^T\,. \tag{A.12}$$

In terms of the big matrices $X^{(r)}_{mn}$, $m,n\in\mathbb{Z}$ the relations (A.11) and (A.12) can be written in the compact form

$$\big(X^{(r)T}X^{(s)}\big)_{mn}=-\frac{\alpha_3}{\alpha_r}\delta^{rs}\delta_{mn}\,,\quad r,s\in\{1,2\}\,,\quad \sum_{r=1}^{3}\alpha_r\big(X^{(r)}X^{(r)T}\big)_{mn}=0\,. \tag{A.13}$$

A.2 Structure of the bosonic Neumann matrices

Evaluating the Gaussian integrals in equation (4.48) one finds the following expressions for the bosonic Neumann matrices [133]

$$\bar{N}^{rs}_{mn}=\delta^{rs}\delta_{mn}-2\big(C^{1/2}_{(r)}X^{(r)T}\Gamma_a^{-1}X^{(s)}C^{1/2}_{(s)}\big)_{mn}\,,\qquad \Gamma_a=\sum_{r=1}^{3}X^{(r)}C_{(r)}X^{(r)T}\,. \tag{A.14}$$

From the structure of the $X^{(r)}$ given in equation (A.6) it follows that Γ_a is block diagonal and using the identities (A.12) one can write the blocks as [133]

$$\big[\Gamma_a\big]_{mn}=\begin{cases}\big(C^{1/2}\Gamma C^{1/2}\big)_{mn}\,, & m\,,n>0\,,\\ -2\mu\alpha_3\,, & m=0=n\,,\\ \big(C^{1/2}\Gamma_- C^{1/2}\big)_{-m,-n}\,, & m\,,n>0\,,\end{cases} \tag{A.15}$$

where

$$\Gamma_-\equiv\sum_{r=1}^{3}A^{(r)}_-U^{-1}_{(r)}A^{(r)\,T}_-\,,\qquad A^{(r)}_-=\frac{\alpha_3}{\alpha_r}C^{-1}A^{(r)}C\,. \tag{A.16}$$

The matrix Γ (which reduces to the flat space Γ of [130, 131] for $\mu \to 0$) exists and is invertible, whereas Γ_- is ill-defined since the above sum is divergent. Nevertheless it is possible to derive a well-defined identity for Γ_-^{-1} [133]

$$\Gamma_-^{-1} = U_{(3)}\big(1 - \Gamma^{-1}U_{(3)}\big)\,. \tag{A.17}$$

Since Γ_-^{-1} is related to Γ^{-1} it is possible to relate the Neumann matrices with positive and negative indices. This results in equation (4.57). To derive the factorization theorem (4.61) [152, 18] introduce

$$\Upsilon \equiv \sum_{r=1}^{3} A^{(r)}U_{(r)}^{-1}A^{(r)\,T} = \Gamma + \mu\alpha BB^T\,, \tag{A.18}$$

where I have used equation (A.12). Its inverse is related to Γ^{-1} by

$$\Upsilon^{-1} = \Gamma^{-1} - \frac{\mu\alpha}{1-4\mu\alpha K}\left(\Gamma^{-1}B\right)\left(\Gamma^{-1}B\right)^T\,. \tag{A.19}$$

For $r,\, s \in \{1,2\}$ one can derive the following relations

$$A^{(r)\,T}C^{-1}U_{(3)}\Gamma^{-1} = A^{(r)\,T}C^{-1} + \frac{\alpha_r}{\alpha_3}C^{-1}U_{(r)}A^{(r)\,T}\Gamma^{-1}\,, \tag{A.20}$$

$$\Upsilon^{-1}U_{(3)}^{-1}C^{-1}A^{(r)} = C^{-1}A^{(r)} + \frac{\alpha_r}{\alpha_3}\Upsilon^{-1}A^{(r)}U_{(r)}^{-1}C^{-1}\,, \tag{A.21}$$

$$\begin{aligned} 2C^{-1} &= \Gamma^{-1}U_{(3)}C^{-1} + C^{-1}U_{(3)}\Gamma^{-1} + \Upsilon^{-1}U_{(3)}^{-1}C^{-1} + C^{-1}U_{(3)}^{-1}\Upsilon^{-1} \\ &\quad - \alpha_1\alpha_2\Upsilon^{-1}B\left(\Gamma^{-1}B\right)^T\,. \end{aligned} \tag{A.22}$$

Using equations (A.19) and (A.11) results in the factorization theorem (4.61).

A.3 The kinematical constraints at $\mathcal{O}(g_{\mathrm{s}})$

A.3.1 The bosonic part

The bosonic constraints the exponential part of the vertex has to satisfy are

$$\sum_{r=1}^{3}\sum_{n\in\mathbb{Z}} X_{mn}^{(r)}p_{n(r)}|V\rangle = 0\,, \qquad \sum_{r=1}^{3}\sum_{n\in\mathbb{Z}} \alpha_r X_{mn}^{(r)}x_{n(r)}|V\rangle = 0\,. \tag{A.23}$$

For $m = 0$ this leads to

$$\sum_{r=1}^{3} p_{0(r)}|V\rangle = 0\,, \qquad \sum_{r=1}^{3} \alpha_r x_{0(r)}|V\rangle = 0\,. \tag{A.24}$$

Substituting (4.6) and commuting the annihilation operators through the exponential this requires

$$\sum_{r,s=1}^{3} \sqrt{|\alpha_r|}\big[\big(\bar{N}^{rs}_{00} + \delta^{rs}\big)a^{\dagger}_{0(s)} + \sum_{n=1}^{\infty} \bar{N}^{rs}_{0n} a^{\dagger}_{n(s)}\big]|V\rangle = 0\,, \tag{A.25}$$

$$\sum_{r,s=1}^{3} e(\alpha_r)\sqrt{|\alpha_r|}\big[\big(\bar{N}^{rs}_{00} - \delta^{rs}\big)a^{\dagger}_{0(s)} + \sum_{n=1}^{\infty} \bar{N}^{rs}_{0n} a^{\dagger}_{n(s)}\big]|V\rangle = 0\,. \tag{A.26}$$

Using the expressions given for $\bar{N}^{rs}_{0n}$ and $\bar{N}^{rs}_{00}$ in (4.51), (4.52) and (4.53) one can check that the above equations are satisfied trivially, i.e. without further use of additional non-trivial identities. For $m > 0$ we find the following constraints

$$B + \sum_{r=1}^{3} A^{(r)} C^{1/2} U_{(r)} \bar{N}^{r} = 0\,, \tag{A.27}$$

$$A^{(s)} C^{-1/2}_{(s)} U^{-1}_{(s)} + \sum_{r=1}^{3} A^{(r)} C^{-1/2}_{(r)} U_{(r)} C^{1/2} \bar{N}^{rs} C^{-1/2} = 0\,, \tag{A.28}$$

$$-\alpha_s A^{(s)} C^{-1/2}_{(s)} + \sum_{r=1}^{3} \alpha_r A^{(r)} C^{-1/2}_{(r)} C^{-1/2} \bar{N}^{rs} C^{1/2} = \alpha B\big[C^{1/2}_{(s)} C^{1/2} \bar{N}^{s}\big]^{T}\,. \tag{A.29}$$

Equation (A.27) is satisfied by the definition for $\bar{N}^{r}$. Equations (A.28) and (A.29) are proved by substituting the expression for $\bar{N}^{rs}$ given in (4.50). For $m < 0$ there is one additional constraint

$$A^{(s)} C^{-1/2}_{(s)} U^{-1}_{(s)} - \alpha_s \sum_{r=1}^{3} \frac{1}{\alpha_r} A^{(r)} C^{1/2}_{(r)} U_{(r)} C^{1/2} \bar{N}^{rs} C^{-1/2} C^{-1}_{(s)} = 0 \tag{A.30}$$

which is verified by subtracting it from equation (A.27) and using (4.61). Here I used the identity

$$\sum_{r=1}^{3} \alpha_r A^{(r)} C^{-1/2} \bar{N}^{r} = 2\alpha K B\,. \tag{A.31}$$

A.3.2 The fermionic part

The fermionic constraints the exponential part of the vertex has to satisfy are

$$\sum_{r=1}^{3}\sum_{n\in\mathbb{Z}} X^{(r)}_{mn}\lambda_{n(r)}|V\rangle = 0\,, \qquad \sum_{r=1}^{3}\sum_{n\in\mathbb{Z}} \alpha_r X^{(r)}_{mn}\vartheta_{n(r)}|V\rangle = 0\,. \tag{A.32}$$

For $m = 0$ this leads to

$$\sum_{r=1}^{3} \lambda_{0(r)}|V\rangle = 0\,, \qquad \sum_{r=1}^{3} \alpha_r\vartheta_{0(r)}|V\rangle = 0\,. \tag{A.33}$$

These equations are satisfied by construction of the zero-mode part of $|V\rangle$. For $m > 0$ we get

$$B + \sum_{r=1}^{3} e(\alpha_r)\sqrt{|\alpha_r|}A^{(r)}C^{-1/2}_{(r)}P_{(r)}Q^r = 0\,, \tag{A.34}$$

$$\sqrt{|\alpha_s|}A^{(s)}C^{-1/2}_{(s)}P^{-1}_{(s)} + \sum_{r=1}^{3} e(\alpha_r)\sqrt{|\alpha_r|}A^{(r)}C^{-1/2}_{(r)}P_{(r)}Q^{rs} = 0\,, \tag{A.35}$$

$$-\sqrt{|\alpha_s|}A^{(s)}C^{-1/2}_{(s)}P_{(s)} + \frac{1}{\alpha_s}\sum_{r=1}^{3} |\alpha_r|^{3/2}A^{(r)}C^{-1/2}_{(r)}P^{-1}_{(r)}C^{-1}Q^{rs}C = \alpha B Q^{s\,T}\,, \tag{A.36}$$

whereas for $m < 0$ the constraints are

$$\sum_{r=1}^{3} \frac{1}{\sqrt{|\alpha_r|}}A^{(r)}CC^{-1/2}_{(r)}P^{-1}_{(r)}Q^r = 0\,, \tag{A.37}$$

$$A^{(s)}CC^{-1/2}_{(s)}P_{(s)} - e(\alpha_s)\sqrt{|\alpha_s|}\sum_{r=1}^{3} \frac{1}{\sqrt{|\alpha_r|}}A^{(r)}CC^{-1/2}_{(r)}P^{-1}_{(r)}Q^{rs} = 0\,. \tag{A.38}$$

Now equations (A.34) and (A.37) uniquely determine

$$Q^r = \frac{e(\alpha_r)}{\sqrt{|\alpha_r|}}(1 - 4\mu\alpha K)^{-1}(1 - 2\mu\alpha K(1 + \Pi))P_{(r)}C^{1/2}_{(r)}C^{1/2}\bar{N}^r\,. \tag{A.39}$$

Furthermore comparing equations (A.35) and (A.28) we see that

$$Q^{rs} = e(\alpha_r)\sqrt{\left|\frac{\alpha_s}{\alpha_r}\right|}P^{-1}_{(r)}U_{(r)}C^{1/2}\bar{N}^{rs}C^{-1/2}U_{(s)}P^{-1}_{(s)} \tag{A.40}$$

solves (A.35). Using

$$P_{(r)}^{-2}U_{(r)}\bar{N}^{rs}U_{(s)}P_{(s)}^{-2} = \bar{N}^{rs} + \mu\alpha(1-4\mu\alpha K)^{-1}C_{(r)}^{1/2}\bar{N}^{r}\left[C_{(s)}^{1/2}\bar{N}^{s}\right]^{T}(1-\Pi) \tag{A.41}$$

establishes (A.36) by virtue of (A.29). Finally, equation (A.38) is satisfied due to the identity

$$A^{(s)}C_{(s)}^{-1/2} - \alpha_s\sum_{r=1}^{3}\frac{1}{\alpha_r}A^{(r)}C_{(r)}^{-1/2}C^{3/2}\bar{N}^{rs}C^{-3/2} = 0 \tag{A.42}$$

which can be proved using the expression for $\bar{N}^{rs}$ given in (4.50). This concludes the determination of the fermionic contribution to the kinematical part of the vertex.

A.4 Neumann matrices at leading order

It is convenient to perform the change of basis for $n \neq 0$

$$\alpha_n = \frac{1}{\sqrt{2}}\left(a_{|n|} + ie(n)a_{-|n|}\right), \qquad \beta_n = \frac{1}{\sqrt{2}}\left(b_{|n|} + ie(n)b_{-|n|}\right). \tag{A.43}$$

In this basis the bosonic Neumann matrices are

$$\widetilde{N}_{mn}^{rs} = \begin{cases} \frac{1}{2}\bar{N}_{|m||n|}^{rs}\left(1 + U_{m(r)}U_{n(s)}\right) & , m\,, n \neq 0 \\ \frac{1}{\sqrt{2}}\bar{N}_{|m|0}^{rs} & , m \neq 0 \\ \bar{N}_{00}^{rs}\,, \end{cases} \tag{A.44}$$

and the components of the bosonic prefactor in an obvious notation

$$\begin{aligned} K_{n(r)} &= \begin{cases} F_{0(r)} & , n = 0 \\ \frac{1}{\sqrt{2}}F_{|n|(r)}\left(1 - U_{n(r)}\right) & , n \neq 0 \end{cases}, \\ \widetilde{K}_{n(r)} &= \begin{cases} F_{0(r)} & , n = 0 \\ \frac{1}{\sqrt{2}}F_{|n|(r)}\left(1 + U_{n(r)}\right) & , n \neq 0 \end{cases}. \end{aligned} \tag{A.45}$$

Using the results for the large μ expansions given in [152, 154] the leading order expressions can be summarized as ($r,\, s \in \{1,2\}$)

$$1 - 4\mu\alpha K \sim \frac{1}{4\pi\beta(\beta+1)\mu\alpha_3}\,, \tag{A.46}$$

$$\widetilde{N}^{r3}_{mn} \sim \frac{(-1)^{(|m|+1)r}}{\pi}\sqrt{-\frac{\alpha_r}{\alpha_3}}\,\frac{\sin n\pi\beta}{n\frac{\alpha_r}{\alpha_3}+m}\,, \tag{A.47}$$

$$\widetilde{N}^{rs}_{mn} \sim (-1)^{(|m|+1)r+(|n|+1)s}\frac{1}{4\pi\mu\sqrt{\alpha_r\alpha_s}}\,, \tag{A.48}$$

$$\widetilde{N}^{33}_{mn} \sim \frac{1}{\pi\mu\alpha_3}\sin n\pi\beta\sin m\pi\beta\,, \tag{A.49}$$

$$K_{n(r)} \sim \widetilde{K}_{n(r)} \sim -(-1)^{(|n|+1)r}\sqrt{\frac{2\mu\alpha_r}{\alpha'}}e^{rs}\alpha_s\,, \tag{A.50}$$

$$K_{n(3)} \sim -\widetilde{K}_{n(3)} \sim \sqrt{\frac{-2\mu\alpha_3}{\alpha'}}\,\frac{\sin n\pi\beta}{\pi}\,. \tag{A.51}$$

For the fermions we write

$$|E_b\rangle = \exp\left[\frac{1}{2}\sum_{r,s=1}^{3}\sum_{m,n\neq 0}\beta^\dagger_{m(r)}\widetilde{Q}^{rs}_{mn}\beta^\dagger_{n(s)} + \Lambda\sum_{r=1}^{3}\sum_{m\neq 0}\widetilde{Q}^{r}_{m}\beta^\dagger_{m(r)}\right]|E^0_b\rangle\,, \tag{A.52}$$

$$Y = \sqrt{\frac{2}{\alpha'}}\Lambda\left[(1-4\mu\alpha K)\frac{1+\Pi}{2} + \frac{1-\Pi}{2}\right] + \sum_{r+1}^{3}\sum_{m\neq 0}\widetilde{G}_{m(r)}\beta^\dagger_{m(r)}\,, \tag{A.53}$$

where

$$\begin{aligned}\widetilde{Q}^{rs}_{mn} &= \frac{i}{2}e(m)Q^{rs}_{|m||n|}\left(1-\left|\frac{n}{m}\right|\frac{\alpha_r}{\alpha_s}\right)\,,\\ \widetilde{Q}^{r}_{m} &= -ie(m)Q^{r}_{|m|}\,,\quad \widetilde{G}_{m(r)} = \frac{1}{\sqrt{2}}G_{|m|(r)}\,.\end{aligned} \tag{A.54}$$

Using the relations between bosonic and fermionic quantities [18] we have

$$\widetilde{Q}^{rs}_{mn} = i\frac{e(m)e(\alpha_r)}{|n|}\sqrt{\left|\frac{\alpha_s}{\alpha_r}\right|}\frac{\sqrt{(\omega_{m(r)}-\mu\alpha_r)(\omega_{n(s)}-\mu\alpha_s)}}{1+U_{m(r)}U_{n(s)}}\widetilde{N}^{rs}_{mn}\times$$
$$\times\left(1-\left|\frac{n}{m}\right|\frac{\alpha_r}{\alpha_s}\right)\left[\frac{1+\Pi}{2}+U_{|m|(r)}U_{|n|(s)}\frac{1-\Pi}{2}\right], \tag{A.55}$$

$$\widetilde{Q}^{r}_{m} = -i\frac{e(m)e(\alpha_r)}{\sqrt{|\alpha_r|}}\sqrt{(\omega_{m(r)}-\mu\alpha_r)\omega_{m(r)}}\bar{N}^{r}_{|m|}\times$$
$$\times\left[\frac{1+\Pi}{2}+(1-4\mu\alpha K)^{-1}U^{-1}_{|m|(r)}\frac{1-\Pi}{2}\right], \tag{A.56}$$

$$\widetilde{G}_{m(r)} = -\frac{e(\alpha_r)}{\sqrt{|\alpha_r|}}\frac{\alpha}{\sqrt{2\alpha'}}\sqrt{(\omega_{m(r)}+\mu\alpha_r)\omega_{m(r)}}\bar{N}^{r}_{|m|}\times$$
$$\times\left[\frac{1+\Pi}{2}+U_{|m|(r)}\frac{1-\Pi}{2}\right]. \tag{A.57}$$

With the help of ($r\in\{1,2\}$)

$$\bar{N}^{r}_{n}\sim\frac{(-1)^{(n+1)r}}{4\pi\mu^2\alpha}\frac{\alpha_3}{\alpha_r}, \qquad \bar{N}^{3}_{n}\sim\frac{1}{\pi\mu\alpha_1\alpha_2}\frac{\sin n\pi\beta}{n} \tag{A.58}$$

one can extract the leading order expressions of the fermionic quantities.

Appendix B

The dynamical constraints

B.1 More detailed computations

Here I give the details leading to equations (4.94), (4.96) and (4.97). The following identities prove very useful ($\alpha_3\Theta \equiv \vartheta_{0(1)} - \vartheta_{0(2)}$)

$$\mathbb{R}|V\rangle = i\sqrt{\alpha'}\left[2K\sqrt{\alpha'}\left(\mathbb{P} - i\frac{\mu\alpha}{\alpha'}\mathbb{R}\right) + \sum_{r,n>0} C^{1/2}_{n(r)}\bar{N}^r_n a^\dagger_{n(r)}\right]|V\rangle\,, \tag{B.1}$$

$$\Theta|V\rangle = -\sqrt{2}\sum_{r,n} Q^r_n b^\dagger_{-n(r)}|V\rangle\,. \tag{B.2}$$

Using the mode expansions of $Q^-_{(r)}$, $\bar{Q}^-_{(r)}$, $K_0 + K_+$, K_- and Y one finds

$$\sum_{r=1}^{3}\{Q^-_{(r)}, Y\} = -\gamma\sum_{r=1}^{3}\frac{1}{\sqrt{|\alpha_r|}}\sum_{n=1}^{\infty}\left[P_{(r)}C^{1/2}G_{(r)}\right]_n a^\dagger_{-n(r)}\,, \tag{B.3}$$

$$\begin{aligned}\sum_{r=1}^{3}\{\bar{Q}^-_{(r)}, Y\} &= (1-4\mu\alpha K)^{-1/2}(1-2\mu\alpha K(1-\Pi))\left(\mathbb{P}\gamma - i\frac{\mu\alpha}{\alpha'}\mathbb{R}\gamma\Pi\right)\\ &\quad + \gamma\sum_{r=1}^{3}\frac{1}{\sqrt{|\alpha_r|}}\sum_{n=1}^{\infty}\left[P^{-1}_{(r)}C^{1/2}G_{(r)}\right]_n a^\dagger_{n(r)}\,,\end{aligned} \tag{B.4}$$

$$\sum_{r=1}^{3}[Q^-_{(r)}, K_0 + K_+] = \mu\gamma(1+\Pi)(1-4\mu\alpha K)^{1/2}\sqrt{\frac{2}{\alpha'}}\Lambda$$
$$+\gamma\sum_{r=1}^{3}\frac{e(\alpha_r)}{\sqrt{|\alpha_r|}}\sum_{n=1}^{\infty}\left[P^{-1}_{(r)}C^{1/2}F_{(r)}\right]_n b^\dagger_{n(r)}\,, \tag{B.5}$$

$$\sum_{r=1}^{3}[Q^-_{(r)}, K_-] = i\gamma\sum_{r=1}^{3}\frac{e(\alpha_r)}{\sqrt{|\alpha_r|}}\sum_{n=1}^{\infty}\left[P^{-1}_{(r)}C^{1/2}U_{(r)}F_{(r)}\right]_n b^\dagger_{-n(r)}\,, \tag{B.6}$$

$$\sum_{r=1}^{3}[\bar{Q}^-_{(r)}, K_0 + K_+] = -\frac{\mu\alpha}{\sqrt{2\alpha'}}\gamma(1-\Pi)(1-4\mu\alpha K)^{1/2}\Theta$$
$$+\gamma\sum_{r=1}^{3}\frac{e(\alpha_r)}{\sqrt{|\alpha_r|}}\sum_{n=1}^{\infty}\left[P_{(r)}C^{1/2}F_{(r)}\right]_n b^\dagger_{-n(r)}\,, \tag{B.7}$$

$$\sum_{r=1}^{3}[\bar{Q}^-_{(r)}, K_-] = -i\gamma\sum_{r=1}^{3}\frac{e(\alpha_r)}{\sqrt{|\alpha_r|}}\sum_{n=1}^{\infty}\left[P_{(r)}C^{1/2}U_{(r)}F_{(r)}\right]_n b^\dagger_{n(r)}\,. \tag{B.8}$$

Using (4.92), (B.1) and (B.2) leads to equations (4.94) and (4.96). The action of the supercharges on $|V\rangle$ given in equation (4.97) can be proven similarly. One needs

$$\bar{N}^{rs}_{nm} + e(\alpha_s)\left(\frac{m}{n}\left|\frac{\alpha_r}{\alpha_s}\right|\right)^{3/2}P_{n(r)}P_{m(s)}Q^{rs}_{nm} = -\frac{\alpha}{\alpha_s}(1-4\mu\alpha K)^{-1}\times$$
$$\times\left[C^{1/2}_{(r)}\bar{N}^r\right]_n\left[U^{-1}_{(s)}C^{1/2}_{(s)}C\bar{N}^s\right]_m\,,$$
$$\bar{N}^{rs}_{-n,-m} + e(\alpha_r)\left(\frac{m}{n}\left|\frac{\alpha_r}{\alpha_s}\right|\right)^{1/2}P_{n(r)}P_{m(s)}Q^{rs}_{nm} = 0\,, \tag{B.9}$$

$$\bar{N}^{rs}_{nm} - e(\alpha_r)\left(\frac{m}{n}\left|\frac{\alpha_r}{\alpha_s}\right|\right)^{1/2}P^{-1}_{n(r)}P^{-1}_{m(s)}Q^{rs}_{nm} = -\mu\alpha(1-4\mu\alpha K)^{-1}\times$$
$$\times(1-\Pi)\left[C^{1/2}_{(r)}\bar{N}^r\right]_n\left[C^{1/2}_{(s)}\bar{N}^s\right]_m,$$
$$\bar{N}^{rs}_{-n,-m} - e(\alpha_s)\left(\frac{m}{n}\left|\frac{\alpha_r}{\alpha_s}\right|\right)^{3/2}P^{-1}_{n(r)}P^{-1}_{m(s)}Q^{rs}_{nm} = \frac{\alpha}{\alpha_s}(1-4\mu\alpha K)^{-1}\times$$
$$\times\left[P^{-2}_{(r)}C^{1/2}_{(r)}\bar{N}^r\right]_n\left[C^{1/2}_{(s)}C\bar{N}^s\right]_m \tag{B.10}$$

which follow from (4.61) and (4.66).

B.2 Proof of the dynamical constraints

In this appendix I prove that

$$\gamma^I_{a(\dot a}\big[\Pi\bar D\big]^a s^I_{\dot b)} = 0\,, \tag{B.11}$$

$$\gamma^I_{a(\dot a}\big[\Pi D\big]^a \widetilde{s}^I_{\dot b)} = 0\,, \tag{B.12}$$

$$\Big(\gamma^I_{a\dot a}\bar D_b\widetilde{s}^I_{\dot b} + \gamma^I_{a\dot b}D_b s^I_{\dot a}\Big)(1-\Pi)^{ab} = 0\,. \tag{B.13}$$

Equations (B.11) and (B.12) are equivalent to

$$\Big(\gamma^I_{a(\dot a}Y_b s^I_{1\,\dot b)} + \frac{\alpha}{\alpha'}\gamma^I_{a(\dot a}\frac{\partial}{\partial Y^b}s^I_{2\,\dot b)}\Big)\Pi^{ab} = 0\,, \tag{B.14}$$

$$\Big(\gamma^I_{a(\dot a}Y_b s^I_{2\,\dot b)} - \frac{\alpha}{\alpha'}\gamma^I_{a(\dot a}\frac{\partial}{\partial Y^b}s^I_{1\,\dot b)}\Big)\Pi^{ab} = 0\,, \tag{B.15}$$

The first equation has terms of order $\mathcal{O}(Y^2)$ and $\mathcal{O}(Y^6)$, whereas the second one has terms of order $\mathcal{O}(Y^0)$, $\mathcal{O}(Y^4)$ and $\mathcal{O}(Y^8)$. There are two contributions to the order $\mathcal{O}(Y^2)$ in equation (B.14) , both vanish separately. The first one is

$$\gamma^I_{a(\dot a}Y_b s^I_{1\,\dot b)}\Pi^{ab} = 2\gamma^I_{a(\dot a}\gamma^I_{c\dot b)}Y^bY^c\Pi^{ab} = -2\delta_{\dot a\dot b}\Pi_{ab}Y^aY^b = 0\,, \tag{B.16}$$

whereas the second one is

$$\begin{aligned}&\frac{\alpha}{\alpha'}\gamma^I_{a(\dot a}\frac{\partial}{\partial Y^b}s^I_{2\,\dot b)}\Pi^{ab} = -\gamma^I_{a(\dot a}u^I_{bcd\dot b)}Y^cY^d\Pi^{ab} = \\ &\frac{1}{16}\left(\gamma^{IJ}\gamma^{KL}\right)_{(\dot a\dot b)}\gamma^{IJ}_{a[b}\gamma^{KL}_{cd]}\Pi^{ab}Y^cY^d = \frac{1}{24}\left(\gamma^{IJ}\gamma^{KL}\right)_{(\dot a\dot b)}\left(\gamma^{IJ}\Pi\gamma^{KL}\right)_{cd}Y^cY^d = 0\,.\end{aligned} \tag{B.17}$$

Here I have used equations (B.29) and (B.32). From the Fourier identities [131]

$$\begin{aligned}s_{1\,\dot a}(\phi) &= \left(\frac{\alpha}{\alpha'}\right)^4\int d^8Y\; s^I_{2\,\dot a}(Y)e^{\frac{\alpha'}{\alpha}\phi Y}\,,\\ s_{2\,\dot a}(\phi) &= \left(\frac{\alpha}{\alpha'}\right)^4\int d^8Y\; s^I_{1\,\dot a}(Y)e^{\frac{\alpha'}{\alpha}\phi Y}\,,\end{aligned} \tag{B.18}$$

it follows that the terms of order $\mathcal{O}(Y^6)$ vanish as well. This proves equation (B.14). The $\mathcal{O}(Y^0)$ term in equation (B.15) is

$$\gamma^I_{a(\dot a}\gamma^I_{b\dot b)}\Pi^{ab} = \delta_{\dot a\dot b}\mathrm{tr}(\Pi) = 0\,, \tag{B.19}$$

and the order $\mathcal{O}(Y^8)$ term vanishes by (B.18). The terms of order $\mathcal{O}(Y^4)$ in equation (B.15) are

$$\begin{aligned}
&\Pi^a{}_b \gamma^I_{a(\dot a} u^I_{cde\dot b)} \left(Y^b Y^c Y^d Y^e + \frac{1}{24} \varepsilon^{cdeb}{}_{ghij} Y^g Y^h Y^i Y^j \right) \\
&= -\frac{1}{16} \Pi^a{}_b \left(\gamma^{IJ} \gamma^{KL} \right)_{(\dot a \dot b)} \gamma^{IJ}_{a[c} \gamma^{KL}_{de]} \left(Y^b Y^c Y^d Y^e + \frac{1}{24} \varepsilon^{cdeb}{}_{ghij} Y^g Y^h Y^i Y^j \right) \\
&= -\frac{1}{16} \Pi^a{}_b \left(\gamma^{IJKL}_{\dot a \dot b} - 2 \delta_{\dot a \dot b} \delta^{IK} \delta^{JL} \right) \gamma^{IJ}_{a[c} \gamma^{KL}_{de]} \left(Y^b Y^c Y^d Y^e \right. \\
&\left. + \frac{1}{24} \varepsilon^{cdeb}{}_{ghij} Y^g Y^h Y^i Y^j \right) \\
&= -\frac{1}{16} \Pi^a{}_b \gamma^{IJKL}_{\dot a \dot b} t^{IJKL}_{acde} \left(Y^b Y^c Y^d Y^e + \frac{1}{24} \varepsilon^{cdeb}{}_{ghij} Y^g Y^h Y^i Y^j \right) = 0 \,. \qquad \text{(B.20)}
\end{aligned}$$

In the last step I used that Π is symmetric and traceless and

$$t^{IJKL}_{abcd} = -\frac{1}{24} \varepsilon_{abcd}{}^{efgh} t^{IJKL}_{efgh} \,. \qquad \text{(B.21)}$$

This proves equation (B.15). Finally, equation (B.13) is equivalent to

$$\left(\gamma^I_{a(\dot a} Y_b s^I_{1\,\dot b)} - \frac{\alpha}{\alpha'} \gamma^I_{a(\dot a} \frac{\partial}{\partial Y^b} s^I_{2\,\dot b)} \right) (1 - \Pi)^{ab} = 0 \,, \qquad \text{(B.22)}$$

$$\left(\gamma^I_{a[\dot a} Y_b s^I_{2\,\dot b]} + \frac{\alpha}{\alpha'} \gamma^I_{a[\dot a} \frac{\partial}{\partial Y^b} s^I_{1\,\dot b]} \right) (1 - \Pi)^{ab} = 0 \,. \qquad \text{(B.23)}$$

The first equation is symmetric in $\dot a$, $\dot b$ and contains terms of order $\mathcal{O}(Y^2)$ and $\mathcal{O}(Y^6)$. These vanish for the same reason as those in equation (B.14). The second equation is antisymmetric in $\dot a$, $\dot b$ and contains terms of order $\mathcal{O}(Y^0)$, $\mathcal{O}(Y^4)$ and $\mathcal{O}(Y^8)$. The $\mathcal{O}(Y^0)$ contribution to equation (B.23) is

$$\gamma^I_{a[\dot a} \gamma^I_{b\dot b]} (1 - \Pi)^{ab} = \frac{1}{4} \gamma^{IJ}_{\dot a \dot b} \gamma^I_{ab} (1 - \Pi)^{ab} = 0 \,. \qquad \text{(B.24)}$$

From equation (B.18) it follows that the term of order $\mathcal{O}(Y^8)$ vanishes as well. Finally, there are two contributions to the terms of order $\mathcal{O}(Y^4)$, both

of them vanish separately. The first one is

$$\begin{aligned}
&\frac{\alpha}{\alpha'}\gamma^I_{a[\dot a}Y_b s^I_{2\,\dot b]}(1-\Pi)^{ab} = -\frac{1}{3}\gamma^I_{a[\dot a}u^I_{cde\dot b]}(1-\Pi)^a{}_b Y^b Y^c Y^d Y^e = \\
&\frac{1}{12}\left(\gamma^{IJ}_{\dot a\dot b}\delta_{a[c}\gamma^{IJ}_{de]} + \frac{1}{4}\left(\gamma^{IJ}\gamma^{KL}\right)_{[\dot a\dot b]}\gamma^{IJ}_{a[c}\gamma^{KL}_{de]}\right)(1-\Pi)^a{}_b Y^b Y^c Y^d Y^e = \\
&\frac{1}{12}\gamma^{IJ}_{\dot a\dot b}\gamma^{IK}_{a[c}\gamma^{KJ}_{de]}(1-\Pi)^a{}_b Y^b Y^c Y^d Y^e = \frac{1}{6}\gamma^{IJ}_{\dot a\dot b}\gamma^{IJ}_{bc}(1-\Pi)_{de}Y^b Y^c Y^d Y^e = 0\,.
\end{aligned} \tag{B.25}$$

In the last step I have used equation (B.30). The second contribution of order $\mathcal{O}(Y^4)$ then vanishes by equation (B.18). This concludes the proof of equation (B.23).
Apart from symmetry and tracelessness of Π I have used the following identities

$$\gamma^{IJ}_{ab} = -\gamma^{IJ}_{ba}\,, \tag{B.26}$$

$$\gamma^I_{a\dot a}\gamma^I_{b\dot b} = \delta_{ab}\delta_{\dot a\dot b} + \frac{1}{4}\gamma^{IJ}_{ab}\gamma^{IJ}_{\dot a\dot b}\,, \tag{B.27}$$

$$\begin{aligned}
\left(\gamma^{IJ}\gamma^{KL}\right)_{ab} &= \gamma^{IJKL}_{ab} + \delta^{IL}\gamma^{JK}_{ab} + \delta^{JK}\gamma^{IL}_{ab} \\
&\quad - \delta^{IK}\gamma^{JL}_{ab} - \delta^{JL}\gamma^{IK}_{ab} + \left(\delta^{JK}\delta^{IL} - \delta^{JL}\delta^{IK}\right)\delta_{ab}\,,
\end{aligned} \tag{B.28}$$

$$\gamma^I_{a\dot a}u^I_{bcd\dot b} = -\frac{1}{4}\gamma^{IJ}_{\dot a\dot b}\delta_{a[b}\gamma^{IJ}_{cd]} - \frac{1}{16}\left(\gamma^{IJ}\gamma^{KL}\right)_{\dot a\dot b}\gamma^{IJ}_{a[b}\gamma^{KL}_{cd]}\,, \tag{B.29}$$

$$\gamma^{IK}_{a[b}\gamma^{JK}_{cd]} = t^{IJ}_{abcd} - 2\delta_{a[b}\gamma^{IJ}_{cd]}\,, \tag{B.30}$$

$$\gamma^{IJ}_{ab}\gamma^{IJ}_{cd} = 8\big(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}\big)\,, \tag{B.31}$$

$$\gamma^{IJKL}_{\dot a\dot b}\left(\gamma^{KL}\Pi\gamma^{IJ}\right)_{[ab]} = 0\,. \tag{B.32}$$

B.3 $\{Q, \widetilde{Q}\}$ at order $\mathcal{O}(g_{\rm s})$

Here I demonstrate that equation (4.72) leads to the constraints (4.111)-(4.114) given in section 4.4. To this end, I adopt a trick introduced in [131]. Namely, associate the world-sheet coordinate dependence with the oscillators as

$$\begin{pmatrix} a_{n(r)} \\ a_{-n(r)} \end{pmatrix} \longrightarrow e^{-i\omega_{n(r)}\tau/\alpha_r}\begin{pmatrix} \cos\frac{n\sigma_r}{\alpha_r} & -\sin\frac{n\sigma_r}{\alpha_r} \\ \sin\frac{n\sigma_r}{\alpha_r} & \cos\frac{n\sigma_r}{\alpha_r} \end{pmatrix}\begin{pmatrix} a_{n(r)} \\ a_{-n(r)} \end{pmatrix}\,, \tag{B.33}$$

and analogously for the fermionic oscillators. Then integrate the constraint equation (4.72) over the σ_r. In dealing with the resulting expressions one can integrate by parts since the integrand is periodic. In addition to the identities in equations (4.96),[1]and (4.97) we have to calculate the commutator of $\sum_r Q_{(r)}$ with K^I and its tilded counterpart. One gets

$$\begin{aligned}\sqrt{2}\eta\sum_{r=1}^{3}[Q_{(r)},K^I]\,|V\rangle &= -2i\gamma^I\left[\dot{Y}+Y'+\frac{i}{2}\mu(1-\Pi)\,(Y-2Y_0)\right]|V\rangle\,,\\ \sqrt{2}\bar{\eta}\sum_{r=1}^{3}[\widetilde{Q}_{(r)},\widetilde{K}^I]\,|V\rangle &= -2i\gamma^I\left[\dot{Y}-Y'+\frac{i}{2}\mu(1-\Pi)\,(Y-2Y_0)\right]|V\rangle\,.\end{aligned} \tag{B.34}$$

Here Y_0 is the zero-mode part of Y, I suppressed the τ, σ_r dependence and

$$\dot{Y}\equiv\partial_\tau Y\,,\qquad Y'\equiv\sum_{r=1}^{3}\partial_{\sigma_r}Y\,. \tag{B.35}$$

The fact that the above equations have a term which only depends on the zero-mode Y_0 is important. Combining the various contributions to equation (4.72), removing the σ_r derivatives from Y by partial integration and using the further identity [131][2]

$$\Big(\gamma^I_{a\dot{a}}\eta\tilde{s}^I_{\dot{b}}+\gamma^I_{a\dot{b}}\bar{\eta}s^I_{\dot{a}}\Big)Y'^a=-\frac{2^{3/2}\alpha}{\alpha'}m'_{\dot{a}\dot{b}} \tag{B.36}$$

and

$$\sum_{r=1}^{3}\partial_{\sigma_r}|V\rangle=-\frac{i}{4}\frac{\alpha'}{\alpha}\Big(\big(K^2-\widetilde{K}^2\big)+4\big(Y\dot{Y}+i\mu(1-\Pi)YY_0\big)\Big)|V\rangle\,, \tag{B.37}$$

[1]In fact, here we need the analogue of equation (4.96) with $K^I\leftrightarrow\widetilde{K}^I$.

[2]There are some minor typos in appendices D and E of [131] which I correct for the interested reader to facilitate the comparison with the results given here. The right-hand-side of equation (D.25) in [131] should be multiplied by $\frac{1}{2\alpha}$; on the r.h.s. of equations (D.26) and (D.27) it should be $\frac{1}{2^{3/2}}$ instead of $\frac{1}{\sqrt{2}}$; the r.h.s. of equation (E.8) should be multiplied by $\frac{1}{\alpha}$. These corrections modify equation (E.9) of [131] slightly, which is now in agreement with our equations (4.111)-(4.113). These typos do not affect the proof presented in appendix E of [131].

we find that equation (4.72) is equivalent to

$$\Big(\Big[\sqrt{2}\big(\gamma^I_{a\dot{a}}\eta\tilde{s}^I_{\dot{b}} - \gamma^I_{a\dot{b}}\bar{\eta}s^I_{\dot{a}}\big) - 4im_{\dot{a}\dot{b}}Y_a\Big]\Big(\dot{Y}^a - \dot{Y}^a_0\Big) - \frac{\mu}{\sqrt{2}}\Big(\gamma^I_{a\dot{a}}\bar{D}_{\dot{b}}\tilde{s}^I_{\dot{b}} + \gamma^I_{a\dot{b}}D_{\dot{b}}s^I_{\dot{a}}\Big)$$
$$(1-\Pi)^{ab} - iK^IK^J\Big[\delta^{IJ}m_{\dot{a}\dot{b}} - \frac{\alpha'}{\sqrt{2}\alpha}\gamma^J_{a\dot{a}}D^a\tilde{s}^I_{\dot{b}}\Big] + i\widetilde{K}^I\widetilde{K}^J\Big[\delta^{IJ}m_{\dot{a}\dot{b}}$$
$$- \frac{\alpha'}{\sqrt{2}\alpha}\gamma^J_{a\dot{b}}\bar{D}^a s^I_{\dot{a}}\Big]\Big)|V\rangle = 0\,. \tag{B.38}$$

This results in equations (4.111)-(4.114).

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Acknowledgments

First and foremost I would like to thank Stefan Theisen for all he has taught me and for his constant support and encouragement. His guidance was always of great value to me and it was a pleasure to work with him. I also wish to thank Albrecht Klemm for his support and advice. I enjoyed the fruitful collaborations with Gleb Arutyunov, Nakwoo Kim, Soo-Jong Rey and Bogdan Stefanski. I greatly benefited from the excellent working conditions at the AEI and its nice atmosphere, thanks in particular to my office mates Niklas Beisert, Thomas Klose, Thomas Quella and Markus Pössel. I would like to acknowledge the financial support from GIF, the German-Israeli foundation for Scientific Research, and from the European Community's Human Potential Programme under contracts HPRN-CT-2000-00131 and HPMT-CT-2001-00296. Last but not least I am grateful to my family and in particular to Silke for their love.

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Selbständigkeitserklärung

Hiermit erkläre ich, die vorliegende Arbeit selbständig ohne fremde Hilfe verfaßt zu haben und nur die angegebene Literatur und Hilfsmittel verwendet zu haben.

Ari Pankiewicz
03. April 2003