BILATERAL GAMMA DISTRIBUTIONS AND PROCESSES IN FINANCIAL MATHEMATICS

UWE KÜCHLER, STEFAN TAPPE

Abstract. We present a class of Lévy processes for modelling financial market fluctuations: Bilateral Gamma processes. Our starting point is to explore the properties of bilateral Gamma distributions, and then we turn to their associated Lévy processes. We treat exponential Lévy stock models with an underlying bilateral Gamma process as well as term structure models driven by bilateral Gamma processes and apply our results to a set of real financial data (DAX 1996-1998).

Key Words: bilateral Gamma distributions, selfdecomposability, unimodality, bilateral Gamma processes, measure transformations, stock models, option pricing, term structure models

1. Introduction

In recent years more realistic stochastic models for price movements in financial markets have been developed, for example by replacing the classical Brownian motion by Lévy processes. Popular examples of such Lévy processes are generalized hyperbolic processes [2] and their subclasses, Variance Gamma processes [13] and CGMY-processes [4]. A survey about Lévy processes used for applications to finance can for instance be found in [19, Chap. 5.3].

We propose another family of Lévy processes which seems to be interesting: Bilateral Gamma processes, which are defined as the difference of two independent Gamma processes. This four-parameter class of processes is more flexible than Variance Gamma processes, but still analytically tractable, in particular these processes have a simple cumulant generating function.

The aim of this article is twofold: First, we investigate the properties of these processes as well as their generating distributions, and show how they are related to other distributions considered in the literature.

As we shall see, they have a series of properties making them interesting for applications: Bilateral Gamma distributions are selfdecomposable, unimodal, stable under convolution and have a simple cumulant generating function. The associated Lévy processes are finite-variation processes making infinitely many jumps at each interval with positive length, and all their increments are bilateral Gamma distributed. In particular, one can easily provide simulations for the trajectories of bilateral Gamma processes.

So, our second goal is to apply bilateral Gamma processes for modelling financial market fluctuations. We treat exponential Lévy stock market models and derive a closed formula for pricing European Call Options. As an illustration, we apply our results to the evolution of the German stock index DAX over the period of three years. Term structure models driven by bilateral Gamma processes are considered as well.

2. Bilateral Gamma distributions

A popular method for building Lévy processes is to take a subordinator S, a Brownian motion W which is independent of S, and to construct the time-changed Brownian motion $X_t := W(S_t)$. For instance, generalized hyperbolic processes and Variance Gamma processes are constructed in this fashion. We do not go this way. Instead, we define $X := Y - Z$ as the difference of two independent subordinators Y, Z. These subordinators should have a simple characteristic function, because then the characteristic function of the resulting Lévy process X will be simple, too. Guided by these ideas, we choose Gamma processes as subordinators.

To begin with, we need the following slight generalization of Gamma distributions. For $\alpha > 0$ and $\lambda \in \mathbb{R} \setminus \{0\}$, we define the $\Gamma(\alpha, \lambda)$-distribution by the density

$$f(x) = \frac{|\lambda|^\alpha}{\Gamma(\alpha)} |x|^{\alpha-1} e^{-|\lambda||x|} \left(\mathbb{1}_{\{\lambda>0\}} \mathbb{1}_{\{x>0\}} + \mathbb{1}_{\{\lambda<0\}} \mathbb{1}_{\{x<0\}} \right), \quad x \in \mathbb{R}.$$

If $\lambda > 0$, then this is just the well-known Gamma distribution, and for $\lambda < 0$ one has a Gamma distribution concentrated on the negative half axis. One verifies that for each $(\alpha, \lambda) \in (0, \infty) \times \mathbb{R} \setminus \{0\}$ the characteristic function of a $\Gamma(\alpha, \lambda)$-distribution is given by

$$\varphi(z) = \left(\frac{\lambda}{\lambda - iz} \right)^\alpha, \quad z \in \mathbb{R} \tag{2.1}$$

where the power α stems from the main branch of the complex logarithm.

A *bilateral Gamma distribution* with parameters $\alpha^+, \lambda^+, \alpha^-, \lambda^- > 0$ is defined as the convolution

$$\Gamma(\alpha^+, \lambda^+; \alpha^-, \lambda^-) := \Gamma(\alpha^+, \lambda^+) * \Gamma(\alpha^-, -\lambda^-).$$

Note that for independent random variables X, Y with $X \sim \Gamma(\alpha^+, \lambda^+)$ and $Y \sim \Gamma(\alpha^-, \lambda^-)$ the difference has a bilateral Gamma distribution $X - Y \sim \Gamma(\alpha^+, \lambda^+; \alpha^-, \lambda^-)$.

By (2.1), the characteristic function of a bilateral Gamma distribution is

$$\varphi(z) = \left(\frac{\lambda^+}{\lambda^+ - iz} \right)^{\alpha^+} \left(\frac{\lambda^-}{\lambda^- + iz} \right)^{\alpha^-}, \quad z \in \mathbb{R}. \tag{2.2}$$

2.1. **Lemma.**

(1) *Suppose* $X \sim \Gamma(\alpha_1^+, \lambda^+; \alpha_1^-, \lambda^-)$ *and* $Y \sim \Gamma(\alpha_2^+, \lambda^+; \alpha_2^-, \lambda^-)$*, and that* X *and* Y *are independent. Then* $X + Y \sim \Gamma(\alpha_1^+ + \alpha_2^+, \lambda^+; \alpha_1^- + \alpha_2^-, \lambda^-)$.

(2) *For* $X \sim \Gamma(\alpha^+, \lambda^+; \alpha^-, \lambda^-)$ *and* $c > 0$ *it holds* $cX \sim \Gamma(\alpha^+, \frac{\lambda^+}{c}; \alpha^-, \frac{\lambda^-}{c})$.

Proof. The asserted properties follow from expression (2.2) of the characteristic function. □

As it is seen from the characteristic function (2.2), bilateral Gamma distributions are stable under convolution, and they are *infinitely divisible*. It follows from [16, Ex. 8.10] that both, the drift and the Gaussian part in the Lévy-Khintchine formula (with truncation function $h = 0$), are equal to zero, and that the Lévy measure is given by

$$F(dx) = \left(\frac{\alpha^+}{x} e^{-\lambda^+ x} \mathbb{1}_{(0,\infty)}(x) + \frac{\alpha^-}{|x|} e^{-\lambda^- |x|} \mathbb{1}_{(-\infty,0)}(x) \right) dx. \tag{2.3}$$

Thus, we can also express the characteristic function $\hat{\mu}$ as

$$\hat{\mu}(z) = \exp\left(\int_{\mathbb{R}} \left(e^{izx} - 1 \right) \frac{k(x)}{x} dx \right), \quad z \in \mathbb{R} \tag{2.4}$$

where $k : \mathbb{R} \to \mathbb{R}$ is the function

$$k(x) = \alpha^+ e^{-\lambda^+ x} \mathbb{1}_{(0,\infty)}(x) - \alpha^- e^{-\lambda^- |x|} \mathbb{1}_{(-\infty,0)}(x), \quad x \in \mathbb{R} \tag{2.5}$$

which is decreasing on each of $(-\infty, 0)$ and $(0, \infty)$. It is an immediate consequence of [16, Cor. 15.11] that bilateral Gamma distributions are *selfdecomposable*. By (2.3), it moreover holds

$$\int_{|x|>1} e^{zx} F(dx) < \infty \quad \text{for all } z \in (-\lambda^-, \lambda^+).$$

Consequently, the *cumulant generating function*

$$\Psi(z) = \ln \mathbb{E}\left[e^{zX}\right] \quad (\text{where } X \sim \Gamma(\alpha^+, \lambda^+; \alpha^-, \lambda^-))$$

exists on $(-\lambda^-, \lambda^+)$, and Ψ and Ψ' are, with regard to (2.2), given by

$$\Psi(z) = \alpha^+ \ln\left(\frac{\lambda^+}{\lambda^+ - z} \right) + \alpha^- \ln\left(\frac{\lambda^-}{\lambda^- + z} \right), \quad z \in (-\lambda^-, \lambda^+), \tag{2.6}$$

$$\Psi'(z) = \frac{\alpha^+}{\lambda^+ - z} - \frac{\alpha^-}{\lambda^- + z}, \quad z \in (-\lambda^-, \lambda^+). \tag{2.7}$$

Hence, the n-th order cumulant $\kappa_n = \frac{\partial^n}{\partial z^n} \Psi(z)|_{z=0}$ is given by

$$\kappa_n = (n-1)! \left(\frac{\alpha^+}{(\lambda^+)^n} + (-1)^n \frac{\alpha^-}{(\lambda^-)^n} \right), \quad n \in \mathbb{N} = \{1, 2, \ldots\}. \tag{2.8}$$

In particular, for a $\Gamma(\alpha^+, \lambda^+; \alpha^-, \lambda^-)$-distributed random variable X, we can specify

- The expectation

$$\mathbb{E}[X] = \kappa_1 = \frac{\alpha^+}{\lambda^+} - \frac{\alpha^-}{\lambda^-}. \tag{2.9}$$

- The variance

$$\mathrm{Var}[X] = \kappa_2 = \frac{\alpha^+}{(\lambda^+)^2} + \frac{\alpha^-}{(\lambda^-)^2}. \tag{2.10}$$

- The Charliers skewness

$$\gamma_1(X) = \frac{\kappa_3}{\kappa_2^{3/2}} = \frac{2\left(\frac{\alpha^+}{(\lambda^+)^3} - \frac{\alpha^-}{(\lambda^-)^3} \right)}{\left(\frac{\alpha^+}{(\lambda^+)^2} + \frac{\alpha^-}{(\lambda^-)^2} \right)^{3/2}}. \tag{2.11}$$

- The excess

$$\gamma_2(X) = \frac{\kappa_4}{\kappa_2^2} = \frac{6\left(\frac{\alpha^+}{(\lambda^+)^4} + \frac{\alpha^-}{(\lambda^-)^4}\right)}{\left(\frac{\alpha^+}{(\lambda^+)^2} + \frac{\alpha^-}{(\lambda^-)^2}\right)^2}. \tag{2.12}$$

It follows that bilateral Gamma distributions are *leptokurtosic.*

3. Related classes of distributions

As apparent from the Lévy measure (2.3), bilateral Gamma distributions are special cases of *generalized tempered stable distributions* [5, Chap. 4.5]. This six-parameter family is defined by its Lévy measure

$$F(dx) = \left(\frac{\alpha^+}{x^{1+\beta^+}} e^{-\lambda^+ x} \mathbb{1}_{(0,\infty)}(x) + \frac{\alpha^-}{|x|^{1+\beta^-}} e^{-\lambda^- |x|} \mathbb{1}_{(-\infty,0)}(x) \right) dx.$$

The *CGMY-distributions*, see [4], are a four-parameter family with Lévy measure

$$F(dx) = \left(\frac{C}{x^{1+Y}} e^{-Mx} \mathbb{1}_{(0,\infty)}(x) + \frac{C}{|x|^{1+Y}} e^{-G|x|} \mathbb{1}_{(-\infty,0)}(x) \right) dx.$$

We observe that some bilateral Gamma distributions are CGMY-distributions, and vice versa.

As the upcoming result reveals, bilateral Gamma distributions are not closed under weak convergence.

3.1. **Proposition.** *Let $\lambda^+, \lambda^- > 0$ be arbitrary. Then the following convergence holds:*

$$\Gamma\left(\frac{(\lambda^+)^2 \lambda^- n}{\lambda^+ + \lambda^-}, \lambda^+ \sqrt{n}; \frac{\lambda^+ (\lambda^-)^2 n}{\lambda^+ + \lambda^-}, \lambda^- \sqrt{n} \right) \xrightarrow{w} N(0,1) \quad \textit{for } n \to \infty.$$

Proof. This is a consequence of the Central Limit Theorem, Lemma 2.1 and relations (2.9), (2.10). □

Bilateral Gamma distributions are special cases of *extended generalized Gamma convolutions* in the terminology of [21]. These are all infinitely divisible distributions μ whose characteristic function is of the form

$$\hat{\mu}(z) = \exp\left(izb - \frac{cz^2}{2} - \int_{\mathbb{R}} \left[\ln\left(1 - \frac{iz}{y}\right) + \frac{izy}{1+y^2} \right] dU(y) \right), \quad z \in \mathbb{R}$$

with $b \in \mathbb{R}, c \geq 0$ and a non-decreasing function $U : \mathbb{R} \to \mathbb{R}$ with $U(0) = 0$ satisfying the integrability conditions

$$\int_{-1}^{1} |\ln y| dU(y) < \infty \quad \text{and} \quad \int_{-\infty}^{-1} \frac{1}{y^2} dU(y) + \int_{1}^{\infty} \frac{1}{y^2} dU(y) < \infty.$$

Since extended generalized Gamma convolutions are closed under weak limits, see [21], every limiting case of bilateral Gamma distributions is an extended generalized Gamma convolution.

Let Z be a subordinator (an increasing real-valued Lévy process) and X a Lévy process with values in $\mathbb{R}^d$. Assume that $(X_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$ are independent. According to [16, Thm. 30.1], the process Y defined by

$$Y_t(\omega) = X_{Z_t(\omega)}(\omega), \quad t \geq 0$$

is a Lévy process on $\mathbb{R}^d$. The process $(Y_t)_{t\geq 0}$ is said to be *subordinate* to $(X_t)_{t\geq 0}$. Letting $\lambda = \mathcal{L}(Z_1)$ and $\mu = \mathcal{L}(X_1)$, we define the *mixture* $\mu \circ \lambda := \mathcal{L}(Y_1)$. If μ is a Normal distribution, $\mu \circ \lambda$ is called a *Normal variance-mean mixture* (cf. [3]), and the process Y is called a *time-changed Brownian motion.*

The characteristic function of $\mu \circ \lambda$ is, according to [16, Thm. 30.1],

$$\varphi_{\mu\circ\lambda} = L_\lambda(\log \hat{\mu}(z)), \quad z \in \mathbb{R}^d \tag{3.1}$$

where L_λ denotes the Laplace transform

$$L_\lambda(w) = \int_0^\infty e^{wx}\lambda(dx), \quad w \in \mathbb{C} \text{ with } \operatorname{Re} w \leq 0$$

and where $\log \hat{\mu}$ denotes the unique continuous logarithm of the characteristic function of μ [16, Lemma 7.6].

Generalized hyperbolic distributions $GH(\lambda, \alpha, \beta, \delta, \mu)$ with drift $\mu = 0$ are Normal variance-mean mixtures, because (see, e.g., [6])

$$GH(\lambda, \alpha, \beta, \delta, 0) = N(\beta, 1) \circ GIG(\lambda, \delta, \sqrt{\alpha^2 - \beta^2}), \tag{3.2}$$

where GIG denotes the *generalized inverse Gaussian distribution.* For GIG-distributions it holds the convergence

$$GIG(\lambda, \delta, \gamma) \xrightarrow{w} \Gamma(\lambda, \tfrac{\gamma^2}{2}) \quad \text{as } \delta \downarrow 0, \tag{3.3}$$

see, e.g., [19, Sec. 5.3.5].

The characteristic function of a *Variance Gamma distribution* $VG(\mu, \sigma^2, \nu)$ is (see [13, Sec. 6.1.1]) given by

$$\phi(z) = \left(1 - iz\mu\nu + \frac{\sigma^2\nu}{2}z^2\right)^{-\frac{1}{\nu}}, \quad z \in \mathbb{R}. \tag{3.4}$$

Hence, we verify by using (3.1) that Variance Gamma distributions are Normal variance-mean mixtures, namely it holds

$$VG(\mu, \sigma^2, \nu) = N(\mu, \sigma^2) \circ \Gamma(\tfrac{1}{\nu}, \tfrac{1}{\nu}) = N(\tfrac{\mu}{\sigma^2}, 1) \circ \Gamma(\tfrac{1}{\nu}, \tfrac{1}{\nu\sigma^2}). \tag{3.5}$$

It follows from [13, Sec. 6.1.3] that Variance Gamma distributions are special cases of bilateral Gamma distributions. In Theorem 3.3 we characterize those bilateral Gamma distributions which are Variance Gamma. Before, we need an auxiliary result about the convergence of mixtures.

3.2. **Lemma.** *$\lambda_n \xrightarrow{w} \lambda$ and $\mu_n \xrightarrow{w} \mu$ implies that $\lambda_n \circ \mu_n \xrightarrow{w} \lambda \circ \mu$ as $n \to \infty$.*

Proof. Fix $z \in \mathbb{R}^d$. Since $\log \hat{\mu}_n \to \log \hat{\mu}$ [16, Lemma 7.7], the set

$$K := \{\log \hat{\mu}_n(z) : n \in \mathbb{N}\} \cup \{\log \hat{\mu}(z)\}$$

is compact. It holds $L_{\lambda_n} \to L_\lambda$ uniformly on compact sets (the proof is analogous to that of Lévy's Continuity Theorem). Taking into account (3.1), we thus obtain $\varphi_{\lambda_n \circ \mu_n}(z) \to \varphi_{\lambda \circ \mu}(z)$ as $n \to \infty$. □

Now we formulate and prove the announced theorem.

3.3. **Theorem.** *Let $\alpha^+, \lambda^+, \alpha^-, \lambda^- > 0$ and $\gamma = \Gamma(\alpha^+, \lambda^+; \alpha^-, \lambda^-)$. There is equivalence between:*

(1) *γ is a Variance Gamma distribution.*
(2) *γ is a limiting case of $GH(\lambda, \alpha, \beta, \delta, 0)$, where $\delta \downarrow 0$, and λ, α, β are fixed.*
(3) *γ is a Normal variance-mean mixture.*
(4) *$\alpha^+ = \alpha^-$.*

Proof. Assume $\gamma = VG(\mu, \sigma^2, \nu)$. We set

$$(\lambda, \alpha, \beta) := \left(\frac{1}{\nu}, \sqrt{\frac{2}{\nu\sigma^2} + \left(\frac{\mu}{\sigma^2}\right)^2}, \frac{\mu}{\sigma^2} \right),$$

and obtain by using (3.2), Lemma 3.2, (3.3) and (3.5)

$$GH(\lambda, \alpha, \beta, \delta, 0) = N(\beta, 1) \circ GIG(\lambda, \delta, \sqrt{\alpha^2 - \beta^2}) = N\left(\tfrac{\mu}{\sigma^2}, 1\right) \circ GIG\left(\tfrac{1}{\nu}, \delta, \sqrt{\tfrac{2}{\nu\sigma^2}}\right)$$
$$\xrightarrow{w} N\left(\tfrac{\mu}{\sigma^2}, 1\right) \circ \Gamma\left(\tfrac{1}{\nu}, \tfrac{1}{\nu\sigma^2}\right) = \gamma \quad \text{as } \delta \downarrow 0,$$

showing (1) ⇒ (2). If $GH(\lambda, \alpha, \beta, \delta, 0) = N(\beta, 1) \circ GIG(\lambda, \delta, \alpha^2 - \beta^2) \xrightarrow{w} \gamma$ for $\delta \downarrow 0$, then γ is a Normal variance-mean mixture by Lemma 3.2. The implication (3) ⇒ (4) is valid by [5, Prop. 4.1]. If $\alpha^+ = \alpha^- =: \alpha$, using the characteristic functions (2.2), (3.4) we obtain that $\gamma = VG(\mu, \sigma^2, \nu)$ with parameters

$$(\mu, \sigma^2, \nu) := \left(\frac{\alpha}{\lambda^+} - \frac{\alpha}{\lambda^-}, \frac{2\alpha}{\lambda^+\lambda^-}, \frac{1}{\alpha} \right), \tag{3.6}$$

whence (4) ⇒ (1) follows. □

We emphasize that bilateral Gamma distributions which are not Variance Gamma cannot be obtained as limiting case of generalized hyperbolic distributions. We refer to [6], where all limits of generalized hyperbolic distributions are determined.

4. Properties of the density functions

Bilateral Gamma distributions are absolutely continuous with respect to the Lebesgue measure, because they are the convolution of two Gamma distributions. Since the densities satisfy the symmetry relation

$$f(x; \alpha^+, \lambda^+, \alpha^-, \lambda^-) = f(-x; \alpha^-, \lambda^-, \alpha^+, \lambda^+), \quad x \in \mathbb{R} \setminus \{0\} \tag{4.1}$$

it is sufficient to analyze the density functions on the positive real line. As the convolution of two Gamma densities, they are for $x \in (0, \infty)$ given by

$$f(x) = \frac{(\lambda^+)^{\alpha^+}(\lambda^-)^{\alpha^-}}{(\lambda^+ + \lambda^-)^{\alpha^-}\Gamma(\alpha^+)\Gamma(\alpha^-)} e^{-\lambda^+ x} \int_0^\infty v^{\alpha^- - 1} \left(x + \frac{v}{\lambda^+ + \lambda^-} \right)^{\alpha^+ - 1} e^{-v} dv. \tag{4.2}$$

We can express the density f by means of the *Whittaker function* $W_{\lambda,\mu}(z)$ [8, p. 1014]. According to [8, p. 1015], the Whittaker function has the representation

$$W_{\lambda,\mu}(z) = \frac{z^\lambda e^{-\frac{z}{2}}}{\Gamma(\mu-\lambda+\frac{1}{2})} \int_0^\infty t^{\mu-\lambda-\frac{1}{2}} e^{-t} \left(1+\frac{t}{z}\right)^{\mu+\lambda-\frac{1}{2}} dt \quad \text{for } \mu-\lambda > -\frac{1}{2}. \tag{4.3}$$

From (4.2) and (4.3) we obtain for $x > 0$

$$\begin{aligned} f(x) = {} & \frac{(\lambda^+)^{\alpha^+}(\lambda^-)^{\alpha^-}}{(\lambda^+ + \lambda^-)^{\frac{1}{2}(\alpha^+ + \alpha^-)}\Gamma(\alpha^+)} x^{\frac{1}{2}(\alpha^+ + \alpha^-)-1} e^{-\frac{x}{2}(\lambda^+ - \lambda^-)} \\ & \times W_{\frac{1}{2}(\alpha^+ - \alpha^-), \frac{1}{2}(\alpha^+ + \alpha^- - 1)}(x(\lambda^+ + \lambda^-)). \end{aligned} \tag{4.4}$$

By [8, p. 1014], we can express the Whittaker function $W_{\lambda,\mu}(z)$ by the Whittaker functions $M_{\lambda,\mu}(z)$, namely it holds

$$W_{\lambda,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda)} M_{\lambda,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda)} M_{\lambda,-\mu}(z).$$

For these Whittaker functions the identities [8, p. 1014]

$$\begin{aligned} M_{\lambda,\mu}(z) &= z^{\mu+\frac{1}{2}} e^{-\frac{z}{2}} \Phi(\mu-\lambda+\tfrac{1}{2}, 2\mu+1; z), \\ M_{\lambda,-\mu}(z) &= z^{-\mu+\frac{1}{2}} e^{-\frac{z}{2}} \Phi(-\mu-\lambda+\tfrac{1}{2}, 2\mu+1; z) \end{aligned}$$

are valid, with $\Phi(\alpha,\gamma;z)$ denoting the *confluent hypergeometric function* [8, p. 1013]

$$\Phi(\alpha,\gamma;z) = 1 + \frac{\alpha}{\gamma}\frac{z}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{z^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{z^3}{3!} + \dots \tag{4.5}$$

Because of the series representation (4.5) of $\Phi(\alpha,\gamma;z)$, we can use (4.4) in order to obtain density plots with a computer program.

The symmetry relation (4.1) and the identity [8, p. 1017]

$$W_{0,\mu}(z) = \sqrt{\frac{z}{\pi}} K_\mu\left(\frac{z}{2}\right),$$

where $K_\mu(z)$ denotes the Bessel function of the third kind, imply that in the case $\alpha^+ = \alpha^- =: \alpha$ the density (4.4) is of the form

$$f(x) = \frac{1}{\Gamma(\alpha)} \left(\frac{\lambda^+\lambda^-}{\lambda^+ + \lambda^-}\right)^\alpha |x|^{\alpha-1} e^{-\frac{x}{2}(\lambda^+ + \lambda^-)} \sqrt{\frac{|x|(\lambda^+ + \lambda^-)}{\pi}} K_{\alpha-\frac{1}{2}}\left(\frac{|x|}{2}(\lambda^+ + \lambda^-)\right) \tag{4.6}$$

for $x \in \mathbb{R} \setminus \{0\}$. The density of a $VG(\mu,\sigma^2,\nu)$-distribution is, according to [13, Sec. 6.1.5], given by

$$h(x) = \frac{2\exp(\frac{\mu x}{\sigma^2})}{\nu^{1/\nu}\sqrt{2\pi}\sigma\Gamma(\frac{1}{\nu})} \left(\frac{x^2}{\frac{2\sigma^2}{\nu}+\mu^2}\right)^{\frac{1}{2\nu}-\frac{1}{4}} K_{\frac{1}{\nu}-\frac{1}{2}}\left(\frac{1}{\sigma^2}\sqrt{x^2\left(\frac{2\sigma^2}{\nu}+\mu^2\right)}\right). \tag{4.7}$$

Inserting the parametrization (3.6) into (4.7), we obtain indeed the density (4.6) of a bilateral Gamma distribution with $\alpha^+ = \alpha^- =: \alpha$ (cf. Theorem 3.3).

As we have shown in Section 2, bilateral Gamma distributions are *selfdecomposable*, and hence of class L in the sense of [17] and [18], because the characteristic function is of the form (2.4) with the function k defined in (2.5). We can therefore apply the deep results of these two articles to bilateral Gamma distributions.

The smoothness of the density depends on the parameters α^+ and α^-. In the sequel, let $N := \lceil \alpha^+ + \alpha^- \rceil - 1$, which is an element of $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

4.1. **Theorem.** *It holds* $f \in C^N(\mathbb{R} \setminus \{0\})$ *and* $f \in C^{N-1}(\mathbb{R}) \setminus C^N(\mathbb{R})$.

Proof. This is a direct consequence of [17, Thm. 1.2]. □

It follows from Theorem 4.1 that the N-th order derivative of the density f is not continuous. The only point of discontinuity is zero. Therefore, we study the behaviour of $f^{(N)}$ near zero. For the proof of the upcoming result, Theorem 4.2, we need the following properties of the *Exponential Integral* [1, Chap. 5]

$$E_1(x) := \int_1^\infty \frac{e^{-xt}}{t} dt, \quad x > 0.$$

The Exponential Integral has the series expansion

$$E_1(x) = -\gamma - \ln x - \sum_{n=1}^\infty \frac{(-1)^n}{n \cdot n!} x^n, \tag{4.8}$$

where γ denotes Euler's constant

$$\gamma = \lim_{n \to \infty} \left[\sum_{k=1}^n \frac{1}{k} - \ln(n) \right].$$

The derivative of the Exponential Integral is given by

$$\frac{\partial}{\partial x} E_1(x) = -\frac{e^{-x}}{x}. \tag{4.9}$$

Due to symmetry relation (4.1) it is, concerning the behaviour of $f^{(N)}$ near zero, sufficient to treat the case $x \downarrow 0$.

4.2. **Theorem.** *Let* $N := \lceil \alpha^+ + \alpha^- \rceil - 1$.

(1) $\lim_{x \downarrow 0} f^{(N)}(x)$ *is finite if and only if* $\alpha^+ \in \mathbb{N}$.
(2) *If* $\alpha^+ \notin \mathbb{N}$ *and* $\alpha^+ + \alpha^- \notin \mathbb{N}$, *then* $f^{(N)}(x) \sim \frac{C_1}{x^\alpha}$ *as* $x \downarrow 0$ *for constants* $C_1 \neq 0, \alpha \in (0, 1)$.
(3) *Let* $\alpha^+ \notin \mathbb{N}$ *be such that* $\alpha^+ + \alpha^- \in \mathbb{N}$. *Then* $f^{(N)}(x) \sim M(x)$ *as* $x \to 0$, *where* M *is a slowly varying function as* $x \to 0$ *satisfying* $\lim_{x \to 0} M(x) = \infty$. *Moreover, it holds* $\lim_{x \downarrow 0}(f^{(N)}(x) - f^{(N)}(-x)) = C_2 \in \mathbb{R}$.

The constants in Theorem 4.2 are given by

$$\alpha = N + 1 - \alpha^+ - \alpha^-, \tag{4.10}$$

$$C_1 = \frac{(\lambda^+)^{\alpha^+} (\lambda^-)^{\alpha^-} \sin(\alpha^+ \pi)}{\Gamma(\alpha^+ + \alpha^- - N) \sin((\alpha^+ + \alpha^-)\pi)}, \tag{4.11}$$

$$C_2 = \frac{(\lambda^+)^{\alpha^+} (\lambda^-)^{\alpha^-}}{2} \Big((-1)^{N+1} \cos(\alpha^+ \pi) + \cos(\alpha^- \pi) \Big). \tag{4.12}$$

Proof. For $\alpha^+ \in \mathbb{N}$ we conclude the finiteness of the limit in the first statement from [18, Thm. 3], since for each $\beta \in (0,1)$ (recall that the function k was defined in (2.5))

$$\lim_{u \downarrow 0} u^{\beta-1}(\alpha^+ - k(u)) = \lim_{u \downarrow 0} u^{\beta-1}\alpha^+(1 - e^{-\lambda u}) = 0.$$

In order to prove the rest of the theorem, we evaluate expressions (1.8)-(1.10) in [17], and then we apply [17, Thm. 1.7]. The constant c in [17, eqn. (1.8)] is in the present situation

$$c = \exp\bigg((\alpha^+ + \alpha^-) \int_0^1 \frac{e^{-u} - 1}{u} du + (\alpha^+ + \alpha^-) \int_1^\infty \frac{e^{-u}}{u} du - \int_1^\infty \frac{\alpha^+ e^{-\lambda^+ u} + \alpha^- e^{-\lambda^- u}}{u} du \bigg). \tag{4.13}$$

The first integral appearing in (4.13) is by (4.9) and the series expansion (4.8)

$$\int_0^1 \frac{e^{-u} - 1}{u} du = \lim_{x \downarrow 0} \Big[- E_1(u) - \ln u \Big]_x^1 = -E_1(1) - \gamma.$$

Using (4.9), for each constant $\lambda > 0$ the identity

$$\int_1^\infty \frac{e^{-\lambda u}}{u} du = \lim_{x \to \infty} \Big[- E_1(\lambda u) \Big]_1^x = E_1(\lambda)$$

is valid. Thus, we obtain

$$c = e^{-(\alpha^+ + \alpha^-)\gamma - \alpha^+ E_1(\lambda^+) - \alpha^- E_1(\lambda^-)}. \tag{4.14}$$

The function $K(x)$ in [17, eqn. (1.9)] is in the present situation

$$K(x) = \exp\left(\int_{|x|}^1 \frac{\alpha^+ + \alpha^- - \alpha^+ e^{-\lambda^+ x} - \alpha^- e^{-\lambda^- x}}{u} du \right).$$

Since by (4.9)

$$\int_{|x|}^1 \frac{1}{u} du = -\ln|x| \quad \text{and} \quad \int_{|x|}^1 \frac{e^{-\lambda u}}{u} du = E_1(\lambda|x|) - E_1(\lambda) \quad \text{for } \lambda > 0,$$

we obtain

$$K(x) = e^{\alpha^+ E_1(\lambda^+) + \alpha^- E_1(\lambda^-)} |x|^{-(\alpha^+ + \alpha^-)} e^{-\alpha^+ E_1(\lambda^+ |x|) - \alpha^- E_1(\lambda^- |x|)}. \tag{4.15}$$

Using the series expansion (4.8), we get

$$\lim_{x \to 0} K(x) = (\lambda^+)^{\alpha^+} (\lambda^-)^{\alpha^-} e^{(\alpha^+ + \alpha^-)\gamma + \alpha^+ E_1(\lambda^+) + \alpha^- E_1(\lambda^-)}, \tag{4.16}$$

showing that for the slowly varying function

$$L(x) = \int_{|x|}^1 \frac{K(u)}{u} du$$

in [17, eqn. (1.10)] it holds

$$\lim_{x \to 0} L(x) = \infty. \tag{4.17}$$

Applying [17, Thm. 1.7] and relations (4.14)-(4.17) completes the proof. □

Bilateral Gamma distributions are *strictly unimodal*, which is the contents of the next result.

4.3. **Theorem.** *There exists a point $x_0 \in \mathbb{R}$ such that f is strictly increasing on $(-\infty, x_0)$ and strictly decreasing on (x_0, ∞).*

Proof. If $\alpha^+ + \alpha^- = 1$, then it holds $\alpha^+ e^{-\lambda^+ x} < 1$ for all $x > 0$ and $\alpha^- e^{-\lambda^- x} < 1$ for all $x > 0$. Consequently, neither the distribution function of a bilateral Gamma distribution nor its reflection is of type I_4 in the sense of [17]. Hence, the assertion follows from [17, Thm. 1.4]. □

We emphasize that the *mode* x_0 from Theorem 4.3 can, in general, not be determined explicitly. Let us consider some examples and discuss the smoothness, the behaviour near zero and the location of the mode of the density f.

4.4. **Examples.**

(1) *If $\alpha^+ + \alpha^- \leq 1$, then f is not continuous at zero by Theorem 4.1. According to Theorem 4.2, it holds*
$$\lim_{x \uparrow 0} f(x) = \infty \quad \textit{and} \quad \lim_{x \downarrow 0} f(x) = \infty.$$
We infer that the mode x_0 is equal to zero. Notice that in the special case $\alpha^+ + \alpha^- = 1$ the difference $f(x) - f(-x)$ tends to a finite value as $x \downarrow 0$ by the third statement of Theorem 4.2.

(2) *If $1 < \alpha^+ + \alpha^- \leq 2$, then, by Theorem 4.1, f is continuous on $\mathbb{R}$, but its derivative is not continuous at zero. Let us have a closer look at the behaviour of f' near zero.*
   - *If $\alpha^+, \alpha^- \in (0,1)$ and $\alpha^+ + \alpha^- \in (1,2)$, then it holds, according to Theorem 4.2,*
   $$\lim_{x \uparrow 0} f'(x) = \infty \quad \textit{and} \quad \lim_{x \downarrow 0} f'(x) = -\infty.$$
   In particular, the mode x_0 is equal to zero.
   - *If $\alpha^- < 1 < \alpha^+$, applying Theorem 4.2 yields*
   $$\lim_{x \uparrow 0} f'(x) = \infty \quad \textit{and} \quad \lim_{x \downarrow 0} f'(x) = \infty.$$
   Hence, the mode x_0 is located at the positive half axis. We remark that in the special case $\alpha^+ + \alpha^- = 2$ the difference $f'(x) - f'(-x)$ tends to a finite value as $x \downarrow 0$ by the third statement of Theorem 4.2.
   - *If $\alpha^+, \alpha^- = 1$, we have a two-sided exponential distribution (which is in particular Variance Gamma by Theorem 3.3). We obtain*
   $$\lim_{x \uparrow 0} f'(x) = C^- \quad \textit{and} \quad \lim_{x \downarrow 0} f'(x) = -C^+$$
   with finite constants $C^-, C^+ \in (0, \infty)$. Consequently, the mode x_0 is zero.

The asymptotic behaviour of the Whittaker function for large values of $|z|$ is, according to [8, p. 1016],
$$W_{\lambda,\mu}(z) \sim e^{-\frac{z}{2}} z^\lambda H(z)$$

with H denoting the function

$$H(z) = 1 + \sum_{k=1}^{\infty} \frac{\left[\mu^2 - (\lambda - \frac{1}{2})^2\right]\left[\mu^2 - (\lambda - \frac{3}{2})^2\right] \cdots \left[\mu^2 - (\lambda - k + \frac{1}{2})^2\right]}{k! z^k}.$$

Obviously, it holds $H(z) \sim 1$ for $z \to \infty$. Taking into account (4.1) and (4.4), for $x \to \pm\infty$ the density has the asymptotic behaviour

$$f(x) \sim C_3 x^{\alpha^+ - 1} e^{-\lambda^+ x} \quad \text{as } x \to \infty,$$
$$f(x) \sim C_4 |x|^{\alpha^- - 1} e^{-\lambda^- |x|} \quad \text{as } x \to -\infty,$$

where the constants $C_3, C_4 > 0$ are given by

$$C_3 = \frac{(\lambda^+)^{\alpha^+}(\lambda^-)^{\alpha^-}}{(\lambda^+ + \lambda^-)^{\alpha^-}\Gamma(\alpha^+)}, \quad C_4 = \frac{(\lambda^+)^{\alpha^+}(\lambda^-)^{\alpha^-}}{(\lambda^+ + \lambda^-)^{\alpha^+}\Gamma(\alpha^-)}.$$

As a consequence, we obtain for the logarithmic density function $\ln f$

$$\lim_{x \to \infty} \frac{\ln f(x)}{x} = -\lambda^+ \quad \text{and} \quad \lim_{x \to -\infty} \frac{\ln f(x)}{x} = \lambda^-.$$

In particular, the density of a bilateral Gamma distribution is *semiheavy tailed.*

5. Statistics of bilateral Gamma distributions

The results of the previous sections show that bilateral Gamma distributions have a series of properties making them interesting for applications.

Assume we have a set of data, and suppose its law actually is a bilateral Gamma distribution. Then we need to estimate the parameters. This section is devoted to the statistics of bilateral Gamma distributions.

Let $X_1, \ldots, X_n$ be an i.i.d. sequence of $\Gamma(\Theta)$-distributed random variables, where $\Theta = (\alpha^+, \alpha^-, \lambda^+, \lambda^-)$, and let $x_1, \ldots, x_n$ be a realization. We would like to find an estimation $\hat{\Theta}$ of the parameters. We start with the *method of moments* and estimate the k-th moments $m_k = \mathbb{E}[X_1^k]$ for $k = 1, \ldots, 4$ as

$$\hat{m}_k = \frac{1}{n} \sum_{i=1}^{n} x_i^k. \tag{5.1}$$

By [14, p. 346], the following relations between the moments and the cumulants $\kappa_1, \ldots, \kappa_4$ in (2.8) are valid:

$$\begin{cases} \kappa_1 &= m_1 \\ \kappa_2 &= m_2 - m_1^2 \\ \kappa_3 &= m_3 - 3m_1 m_2 + 2m_1^3 \\ \kappa_4 &= m_4 - 4m_3 m_1 - 3m_2^2 + 12 m_2 m_1^2 - 6m_1^4 \end{cases} . \tag{5.2}$$

Inserting the cumulants (2.8) for $n = 1, \dots, 4$ into (5.2), we obtain

$$\begin{cases} \alpha^+\lambda^- - \alpha^-\lambda^+ - c_1\lambda^+\lambda^- &= 0 \\ \alpha^+(\lambda^-)^2 + \alpha^-(\lambda^+)^2 - c_2(\lambda^+)^2(\lambda^-)^2 &= 0 \\ \alpha^+(\lambda^-)^3 - \alpha^-(\lambda^+)^3 - c_3(\lambda^+)^3(\lambda^-)^3 &= 0 \\ \alpha^+(\lambda^-)^4 + \alpha^-(\lambda^+)^4 - c_4(\lambda^+)^4(\lambda^-)^4 &= 0 \end{cases}, \tag{5.3}$$

where the constants $c_1, \dots, c_4$ are given by

$$\begin{cases} c_1 &= m_1 \\ c_2 &= m_2 - m_1^2 \\ c_3 &= \frac{1}{2}m_3 - \frac{3}{2}m_1m_2 + m_1^3 \\ c_4 &= \frac{1}{6}m_4 - \frac{2}{3}m_3m_1 - \frac{1}{2}m_2^2 + 2m_2m_1^2 - m_1^4 \end{cases}.$$

We can solve the system of equations (5.3) explicitly. In general, it has finitely many, but more than one solution. However, in practice, the restriction $\alpha^+, \alpha^-, \lambda^+, \lambda^- > 0$ ensures uniqueness of the solution. This yields a vector $\hat{\Theta}_0$ as first estimation for the parameters.

The logarithm of the *likelihood function* for $\Theta = (\alpha^+, \alpha^-, \lambda^+, \lambda^-)$ is, by the symmetry relation (4.1) and the representation (4.4) of the density, given by

$$\begin{aligned} \ln L(\Theta) = &-n^+ \ln(\Gamma(\alpha^+)) - n^- \ln(\Gamma(\alpha^-)) \\ &+ n\left(\alpha^+ \ln(\lambda^+) + \alpha^- \ln(\lambda^-) - \frac{\alpha^+ + \alpha^-}{2} \ln(\lambda^+ + \lambda^-)\right) \\ &+ \left(\frac{\alpha^+ + \alpha^-}{2} - 1\right)\left(\sum_{i=1}^n \ln|x_i|\right) - \frac{\lambda^+ - \lambda^-}{2}\left(\sum_{i=1}^n x_i\right) \\ &+ \sum_{i=1}^n \ln\left(W_{\frac{1}{2}\mathrm{sgn}(x_i)(\alpha^+ - \alpha^-), \frac{1}{2}(\alpha^+ + \alpha^- - 1)}(|x_i|(\lambda^+ + \lambda^-))\right), \end{aligned} \tag{5.4}$$

where n^+ denotes the number of positive, and n^- the number of negative observations. We take the vector $\hat{\Theta}_0$, obtained from the method of moments, as starting point for an algorithm, for example the Hooke-Jeeves algorithm [15, Sec. 7.2.1], which maximizes the logarithmic likelihood function (5.4) numerically. This gives us a *maximum likelihood estimation* $\hat{\Theta}$ of the parameters. We shall illustrate the whole procedure in Section 10.

6. Bilateral Gamma processes

As we have shown in Section 2, bilateral Gamma distributions are infinitely divisible. Let us list the properties of the associated Lévy processes, which are denoted by X in the sequel.

From the representation (2.3) of the Lévy measure F we see that $F(\mathbb{R}) = \infty$ and $\int_{-1}^1 |x|F(dx) < \infty$. Since the Gaussian part is zero, X is of type B in the terminology of [16, Def. 11.9]. We obtain the following properties. Bilateral Gamma processes are *finite-variation processes* [16, Thm. 21.9] making infinitely many jumps at each interval with positive length [16, Thm. 21.3], and they are equal to the sum of their jumps [16, Thm. 19.3], i.e.

$$X_t = \sum_{s \leq t} \Delta X_s = x * \mu^X, \quad t \geq 0$$

where μ^X denotes the random measure of jumps of X. Bilateral Gamma processes are *special semimartingales* with canonical decomposition [10, Cor. II.2.38]

$$X_t = x * (\mu^X - \nu)_t + \left(\frac{\alpha^+}{\lambda^+} - \frac{\alpha^-}{\lambda^-} \right) t, \quad t \geq 0$$

where ν is the compensator of μ^X, which is given by $\nu(dt, dx) = dt F(dx)$ with F denoting the Lévy measure given by (2.3).

We immediately see from the characteristic function (2.2) that all increments of X have a bilateral Gamma distribution, more precisely

$$X_t - X_s \sim \Gamma(\alpha^+(t-s), \lambda^+; \alpha^-(t-s), \lambda^-) \quad \text{for } 0 \leq s < t. \tag{6.1}$$

There are many efficient algorithms for generating Gamma random variables, for example Johnk's generator and Best's generator of Gamma variables, chosen in [5, Sec. 6.3]. By virtue of (6.1), it is therefore easy to simulate bilateral Gamma processes.

7. Measure transformations for bilateral Gamma processes

Equivalent changes of measure are important in order to define arbitrage-free financial models. In this section, we deal with equivalent measure transformations for bilateral Gamma processes.

We assume that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is given as follows. Let $\Omega = \mathbb{D}$, the collection of functions $\omega(t)$ from $\mathbb{R}_+$ into $\mathbb{R}$, right-continuous with left limits. For $\omega \in \Omega$, let $X_t(\omega) = \omega(t)$ and let $\mathcal{F} = \sigma(X_t : t \in \mathbb{R}_+)$ and $(\mathcal{F}_t)_{t \geq 0}$ be the filtration $\mathcal{F}_t = \sigma(X_s : s \in [0,t])$. We consider a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ such that X is a bilateral Gamma process.

7.1. **Proposition.** *Let X be a $\Gamma(\alpha_1^+, \lambda_1^+; \alpha_1^-, \lambda_1^-)$-process under the measure $\mathbb{P}$ and let $\alpha_2^+, \lambda_2^+, \alpha_2^-, \lambda_2^- > 0$. The following two statements are equivalent.*

1. *There is another measure $\mathbb{Q} \overset{\text{loc}}{\sim} \mathbb{P}$ under which X is a bilateral Gamma process with parameters $\alpha_2^+, \lambda_2^+, \alpha_2^-, \lambda_2^-$.*
2. *$\alpha_1^+ = \alpha_2^+$ and $\alpha_1^- = \alpha_2^-$.*

Proof. All conditions of [16, Thm. 33.1] are obviously satisfied, with exception of

$$\int_{\mathbb{R}} \left(1 - \sqrt{\Phi(x)} \right)^2 F_1(dx) < \infty, \tag{7.1}$$

where $\Phi = \frac{dF_2}{dF_1}$ denotes the Radon-Nikodym derivative of the respective Lévy measures, which is by (2.3) given by

$$\Phi(x) = \frac{\alpha_2^+}{\alpha_1^+} e^{-(\lambda_2^+ - \lambda_1^+)x} \mathbf{1}_{(0,\infty)}(x) + \frac{\alpha_2^-}{\alpha_1^-} e^{-(\lambda_2^- - \lambda_1^-)|x|} \mathbf{1}_{(-\infty,0)}(x), \quad x \in \mathbb{R}. \tag{7.2}$$

The integral in (7.1) is equal to

$$\int_{\mathbb{R}} \left(1 - \sqrt{\Phi(x)}\right)^2 F_1(dx) = \int_0^\infty \frac{1}{x} \left(\sqrt{\alpha_2^+} e^{-(\lambda_2^+/2)x} - \sqrt{\alpha_1^+} e^{-(\lambda_1^+/2)x} \right)^2 dx + \int_0^\infty \frac{1}{x} \left(\sqrt{\alpha_2^-} e^{-(\lambda_2^-/2)x} - \sqrt{\alpha_1^-} e^{-(\lambda_1^-/2)x} \right)^2 dx.$$

Hence, condition (7.1) is satisfied if and only if $\alpha_1^+ = \alpha_2^+$ and $\alpha_1^- = \alpha_2^-$. Applying [16, Thm. 33.1] completes the proof. □

Proposition 7.1 implies that we can transform any Variance Gamma process, which is according to Theorem 3.3 a bilateral Gamma process $\Gamma(\alpha, \lambda^+; \alpha, \lambda^-)$, into a symmetric bilateral Gamma process $\Gamma(\alpha, \lambda; \alpha, \lambda)$ with arbitrary parameter $\lambda > 0$.

Now assume the process X is $\Gamma(\alpha^+, \lambda_1^+; \alpha^-, \lambda_1^-)$ under $\mathbb{P}$ and $\Gamma(\alpha^+, \lambda_2^+; \alpha^-, \lambda_2^-)$ under the measure $\mathbb{Q} \overset{\text{loc}}{\sim} \mathbb{P}$. According to Proposition 7.1, such a change of measure exists. For the computation of the *likelihood process*

$$\Lambda_t(\mathbb{Q}, \mathbb{P}) = \frac{d\mathbb{Q}|_{\mathcal{F}_t}}{d\mathbb{P}|_{\mathcal{F}_t}}, \quad t \geq 0$$

we will need the following auxiliary result.

7.2. **Lemma.** *For all $\lambda_1, \lambda_2 > 0$ it holds*

$$\int_0^\infty \frac{e^{-\lambda_2 x} - e^{-\lambda_1 x}}{x} dx = \ln\left(\frac{\lambda_1}{\lambda_2}\right).$$

Proof. Due to relation (4.9) and the series expansion (4.8) of the Exponential Integral E_1 we obtain

$$\begin{aligned} \int_0^\infty \frac{e^{-\lambda_2 x} - e^{-\lambda_1 x}}{x} dx &= \lim_{b \to \infty} [E_1(\lambda_1 b) - E_1(\lambda_2 b)] - \lim_{a \to 0} [E_1(\lambda_1 a) - E_1(\lambda_2 a)] \\ &= \lim_{b \to \infty} E_1(\lambda_1 b) - \lim_{b \to \infty} E_1(\lambda_2 b) \\ &\quad + \ln\left(\frac{\lambda_1}{\lambda_2}\right) + \lim_{a \to 0} \sum_{n=1}^\infty \frac{1}{n \cdot n!} (\lambda_1 a)^n - \lim_{a \to 0} \sum_{n=1}^\infty \frac{1}{n \cdot n!} (\lambda_2 a)^n. \end{aligned}$$

Each of the four limits is zero, so the claimed identity follows. □

For our applications to finance, the *relative entropy* $\mathcal{E}_t(\mathbb{Q}, \mathbb{P}) = \mathbb{E}_{\mathbb{Q}}[\ln \Lambda_t(\mathbb{Q}, \mathbb{P})]$, also known as *Kullback-Leibler distance*, which is often used as measure of proximity of two equivalent probability measures, will be of importance. The upcoming result provides the likelihood process and the relative entropy. In the degenerated cases $\lambda_1^+ = \lambda_2^+$ or $\lambda_1^- = \lambda_2^-$, the associated Gamma distributions in (7.3) are understood to be the Dirac measure $\delta(0)$.

7.3. **Proposition.** *It holds $\Lambda_t(\mathbb{Q},\mathbb{P}) = e^{U_t}$, where U is under $\mathbb{P}$ the Lévy process with generating distribution*

$$U_1 \sim \Gamma\left(\alpha^+, \frac{\lambda_1^+}{\lambda_1^+ - \lambda_2^+}\right) * \Gamma\left(\alpha^-, \frac{\lambda_1^-}{\lambda_1^- - \lambda_2^-}\right) * \delta\left(\alpha^+ \ln\left(\frac{\lambda_2^+}{\lambda_1^+}\right) + \alpha^- \ln\left(\frac{\lambda_2^-}{\lambda_1^-}\right)\right). \tag{7.3}$$

Setting $f(x) = x - 1 - \ln x$, it holds for the relative entropy

$$\mathcal{E}_t(\mathbb{Q},\mathbb{P}) = t\left[\alpha^+ f\left(\frac{\lambda_1^+}{\lambda_2^+}\right) + \alpha^- f\left(\frac{\lambda_1^-}{\lambda_2^-}\right)\right]. \tag{7.4}$$

Proof. According to [16, Thm. 33.2], the likelihood process is of the form $\Lambda_t(\mathbb{Q},\mathbb{P}) = e^{U_t}$, where U is, under the measure $\mathbb{P}$, the Lévy process

$$U_t = \sum_{s\leq t} \ln(\Phi(\Delta X_s)) - t\int_{\mathbb{R}} (\Phi(x) - 1)F_1(dx), \tag{7.5}$$

and where Φ is the Radon-Nikodym derivative given by (7.2) with $\alpha_1^+ = \alpha_2^+ =: \alpha^+$ and $\alpha_1^- = \alpha_2^- =: \alpha^-$. For every $t > 0$ denote by X_t^+ the sum $\sum_{s\leq t}(\Delta X_s)^+$ and by X_t^- the sum $\sum_{s\leq t}(\Delta X_s)^-$. Then $X = X^+ - X^-$. By construction and the definition of $\mathbb{Q}$, the processes X^+ and X^- are independent $\Gamma(\alpha^+, \lambda_1^+)$- and $\Gamma(\alpha^-, \lambda_1^-)$-processes under $\mathbb{P}$ and independent $\Gamma(\alpha^+, \lambda_2^+)$- and $\Gamma(\alpha^-, \lambda_2^-)$-processes under $\mathbb{Q}$, respectively. From (7.2) it follows

$$\sum_{s\leq t} \ln(\Phi(\Delta X_s)) = (\lambda_1^+ - \lambda_2^+)X_t^+ + (\lambda_1^- - \lambda_2^-)X_t^-.$$

The integral in (7.5) is, by using Lemma 7.2, equal to

$$\begin{aligned}\int_{\mathbb{R}} (\Phi(x) - 1)F_1(dx) &= \alpha^+ \int_0^\infty \frac{e^{-\lambda_2^+ x} - e^{-\lambda_1^+ x}}{x} dx + \alpha^- \int_0^\infty \frac{e^{-\lambda_2^- x} - e^{-\lambda_1^- x}}{x} dx \\ &= \alpha^+ \ln\left(\frac{\lambda_1^+}{\lambda_2^+}\right) + \alpha^- \ln\left(\frac{\lambda_1^-}{\lambda_2^-}\right).\end{aligned}$$

Hence, we obtain

$$U_t = (\lambda_1^+ - \lambda_2^+)X_t^+ + (\lambda_1^- - \lambda_2^-)X_t^- + \left[\alpha^+ \ln\left(\frac{\lambda_2^+}{\lambda_1^+}\right) + \alpha^- \ln\left(\frac{\lambda_2^-}{\lambda_1^-}\right)\right] t. \tag{7.6}$$

Equation (7.6) yields (7.4) and, together with Lemma 2.1, the relation (7.3). □

Since the likelihood process is of the form $\Lambda_t(\mathbb{Q},\mathbb{P}) = e^{U_t}$, where the Lévy process U is given by (7.6), one verifies that

$$\begin{aligned}\Lambda_t(\mathbb{Q},\mathbb{P}) &= \exp\Big((\lambda_1^+ - \lambda_2^+)X_t^+ - t\Psi^+(\lambda_1^+ - \lambda_2^+)\Big) \\ &\quad \times \exp\Big((\lambda_1^- - \lambda_2^-)X_t^- - t\Psi^-(\lambda_1^- - \lambda_2^-)\Big),\end{aligned} \tag{7.7}$$

where Ψ^+, Ψ^- denote the respective cumulant generating functions of the Gamma processes X^+, X^- under the measure $\mathbb{P}$.

Keeping $\alpha^+, \alpha^-, \lambda_1^+, \lambda_1^-$ all positive and fixed, then by putting $\vartheta^+ = \lambda_1^+ - \lambda_2^+$, $\vartheta^- = \lambda_1^- - \lambda_2^-$, $\vartheta = (\vartheta^+, \vartheta^-)^\top \in (-\infty, \lambda_1^+) \times (-\infty, \lambda_1^-) =: \Theta$, $\mathbb{Q} = \mathbb{Q}_\vartheta$ we obtain a two-parameter *exponential family* $(\mathbb{Q}_\vartheta, \vartheta \in \Theta)$ of Lévy processes in the sense of [12, Chap. 3], with the canonical process $B_t = (X_t^+, X_t^-)$.

In particular, it follows that for every $t > 0$ the vector B_t is a sufficient statistics for $\vartheta = (\vartheta^+, \vartheta^-)^\top$ based on the observation of $(X_s, s \leq t)$. Considering the subfamily obtained by $\vartheta^+ = \vartheta^-$, we obtain a one-parametric exponential family of Lévy processes with $X_t^+ + X_t^- = \sum_{s \leq t} |\Delta X_s|$ as sufficient statistics and canonical process.

8. Inspecting a typical path

Proposition 7.1 of the previous section suggests that the parameters α^+, α^- should be determinable by inspecting a typical path of a bilateral Gamma process. This is indeed the case. We start with Gamma processes. Let X be a $\Gamma(\alpha, \lambda)$-process. Choose a finite time horizon $T > 0$ and set

$$S_n := \frac{1}{nT} \# \left\{ t \leq T : \Delta X_t \geq e^{-n} \right\}, \quad n \in \mathbb{N}.$$

8.1**. Theorem.** *It holds* $\mathbb{P}\left(\lim_{n \to \infty} S_n = \alpha\right) = 1$.

Proof. Due to [16, Thm. 19.2], the random measure μ^X of the jumps of X is a Poisson random measure with intensity measure

$$\nu(dt, dx) = dt \frac{\alpha e^{-\lambda x}}{x} \mathbf{1}_{(0,\infty)} dx.$$

Thus, the sequence

$$Y_n := \frac{1}{T} \mu^X \left([0, T] \times [e^{-n}, e^{1-n}) \right), \quad n \in \mathbb{N}$$

defines a sequence of independent random variables with

$$\begin{aligned} \mathbb{E}[Y_n] &= \alpha \int_{e^{-n}}^{e^{1-n}} \frac{e^{-\lambda x}}{x} dx = \alpha \int_{n-1}^{n} \exp\left(-\lambda e^{-v}\right) dv \uparrow \alpha \quad \text{as } n \to \infty, \\ \operatorname{Var}[Y_n] &= \frac{\alpha}{T} \int_{e^{-n}}^{e^{1-n}} \frac{e^{-\lambda x}}{x} dx = \frac{\alpha}{T} \int_{n-1}^{n} \exp\left(-\lambda e^{-v}\right) dv \uparrow \frac{\alpha}{T} \quad \text{as } n \to \infty, \end{aligned}$$

because $\exp\left(-\lambda e^{-v}\right) \uparrow 1$ for $v \to \infty$. Hence, we have

$$\sum_{n=1}^{\infty} \frac{\operatorname{Var}[Y_n]}{n^2} < \infty.$$

We may therefore apply Kolmogorov's strong law of large numbers [20, Thm. IV.3.2], and deduce that

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} Y_k + \lim_{n \to \infty} \frac{1}{nT} \mu^X([0, T] \times [1, \infty)) = \alpha,$$

finishing the proof. $\square$

Now, let X be a bilateral Gamma process, say $X_1 \sim \Gamma(\alpha^+, \lambda^+; \alpha^-, \lambda^-)$. We set

$$S_n^+ := \frac{1}{nT} \# \left\{ t \leq T : \Delta X_t \geq e^{-n} \right\}, \quad n \in \mathbb{N},$$
$$S_n^- := \frac{1}{nT} \# \left\{ t \leq T : \Delta X_t \leq -e^{-n} \right\}, \quad n \in \mathbb{N}.$$

8.2. **Corollary.** *It holds* $\mathbb{P}(\lim_{n\to\infty} S_n^+ = \alpha^+$ *and* $\lim_{n\to\infty} S_n^- = \alpha^-) = 1$.

Proof. We define the processes X^+ and X^- as $X_t^+ = \sum_{s\leq t}(\Delta X_s)^+$ and $X_t^- = \sum_{s\leq t}(\Delta X_s)^-$. By construction we have $X = X^+ - X^-$ and the processes X^+ and X^- are independent $\Gamma(\alpha^+, \lambda^+)$- and $\Gamma(\alpha^-, \lambda^-)$-processes. Applying Theorem 8.1 yields the desired result. □

9. Stock models

We move on to present some applications to finance of the theory developed above. Assume that the evolution of an asset price is described by an exponential Lévy model $S_t = S_0 e^{rt+X_t}$, where $S_0 > 0$ is the (deterministic) initial value of the stock, r the interest rate and where the Lévy process X is a bilateral Gamma process $\Gamma(\alpha^+, \lambda^+; \alpha^-, \lambda^-)$ under the measure $\mathbb{P}$, which plays the role of the real-world measure.

In order to avoid arbitrage, it arises the question whether there exists an *equivalent martingale measure*, i.e. a measure $\mathbb{Q} \overset{\text{loc}}{\sim} \mathbb{P}$ such that $Y_t := e^{-rt}S_t$ is a local martingale.

9.1. **Lemma.** *Assume* $\lambda^+ > 1$. *Then* Y *is a local* $\mathbb{P}$*-martingale if and only if*

$$\left(\frac{\lambda^+}{\lambda^+ - 1} \right)^{\alpha^+} = \left(\frac{\lambda^- + 1}{\lambda^-} \right)^{\alpha^-}. \tag{9.1}$$

Proof. Since the Gaussian part of the bilateral Gamma process X is zero, Itô's formula [10, Thm. I.4.57], applied on $Y_t = S_0 e^{X_t}$, yields for the discounted stock prices

$$Y_t = Y_0 + \int_0^t Y_{s-} dX_s + S_0 \sum_{0<s\leq t} \left(e^{X_s} + e^{X_{s-}} + e^{X_{s-}} \Delta X_s \right).$$

Recall from Section 6 that $X = x * \mu^X$ and that the compensator ν of μ^X is given by $\nu(dt, dx) = dtF(dx)$, where F denotes the Lévy measure from (2.3). So we obtain

$$\begin{aligned} Y_t &= Y_0 + \int_0^t \int_{\mathbb{R}} x Y_{s-} \mu^X(ds, dx) + \int_0^t \int_{\mathbb{R}} Y_{s-}(e^x - 1 - x)\mu^X(ds, dx) \\ &= Y_0 + \int_0^t \int_{\mathbb{R}} Y_{s-}(e^x - 1)(\mu^X - \nu)(ds, dx) + \int_0^t Y_{s-} \int_{\mathbb{R}} (e^x - 1)F(dx)ds. \end{aligned} \tag{9.2}$$

Applying Lemma 7.2, the integral in the drift term is equal to

$$\begin{aligned} &\int_{\mathbb{R}} (e^x - 1)F(dx) \\ &= \alpha^+ \int_0^\infty \frac{e^{-(\lambda^+ - 1)x} - e^{-\lambda^+ x}}{x} dx - \alpha^- \int_0^\infty \frac{e^{-\lambda^- x} - e^{-(\lambda^- + 1)x}}{x} dx \\ &= \alpha^+ \ln \left(\frac{\lambda^+}{\lambda^+ - 1} \right) - \alpha^- \ln \left(\frac{\lambda^- + 1}{\lambda^-} \right). \end{aligned} \tag{9.3}$$

The discounted price process Y is a local martingale if and only if the drift in (9.2) vanishes, and by (9.3) this is the case if and only if (9.1) is satisfied. □

As usual in financial modelling with jump processes, the market is free of arbitrage, but not complete, that is there exist several martingale measures. The next result shows that we can find a continuum of martingale measures by staying within the class of bilateral Gamma processes. We define $\phi : (1,\infty) \to \mathbb{R}$ as

$$\phi(\lambda) := \phi(\lambda;\alpha^+,\alpha^-) := \left(\left(\frac{\lambda}{\lambda-1} \right)^{\alpha^+/\alpha^-} - 1 \right)^{-1}, \quad \lambda \in (1,\infty).$$

9.2. **Proposition.** *For each $\lambda \in (1,\infty)$ there exists a martingale measure $\mathbb{Q}_\lambda \overset{\text{loc}}{\sim} \mathbb{P}$ such that under $\mathbb{Q}_\lambda$ we have*

$$X_1 \sim \Gamma(\alpha^+,\lambda;\alpha^-,\phi(\lambda)). \tag{9.4}$$

Proof. Recall that X is $\Gamma(\alpha^+,\lambda^+;\alpha^-,\lambda^-)$ under $\mathbb{P}$. According to Proposition 7.1, for each $\lambda \in (1,\infty)$ there exists a probability measure $\mathbb{Q}_\lambda \overset{\text{loc}}{\sim} \mathbb{P}$ such that under $\mathbb{Q}_\lambda$ relation (9.4) is fulfilled, and moreover this measure $\mathbb{Q}_\lambda$ is a martingale measure, because equation (9.1) from Lemma 9.1 is satisfied. □

So, we have a continuum of martingale measures, and the question is, which one we should choose. There are several suggestions in the literature, see, e.g., [5].

One approach is to minimize the relative entropy, which amounts to finding $\lambda \in (1,\infty)$ which minimizes $\mathcal{E}(\mathbb{Q}_\lambda,\mathbb{P})$, and then taking $\mathbb{Q}_\lambda$. The relative entropy is determined in equation (7.4) of Proposition 7.3. Taking the first derivative with respect to λ, and setting it equal to zero, we have to find the $\lambda \in (1,\infty)$ such that

$$\alpha^-\alpha\lambda^{\alpha-1}\left(\frac{1}{\lambda^\alpha(\lambda-1)-(\lambda-1)^{\alpha+1}} - \frac{\lambda^-}{(\lambda-1)^{\alpha+1}} \right) + \frac{\alpha^+}{\lambda}\left(1-\frac{\lambda^+}{\lambda}\right) = 0, \tag{9.5}$$

where $\alpha := \alpha^+/\alpha^-$. This can be done numerically.

Another point of view is that the martingale measure is given by the market. We would like to *calibrate* the Lévy process X from the family of bilateral Gamma processes to option prices. According to Proposition 9.2 we can, by adjusting $\lambda \in (1,\infty)$, preserve the martingale property, which leaves us one parameter to calibrate.

For each $\lambda \in (1,\infty)$, an arbitrage free pricing rule for a *European Call Option* at time $t \geq 0$ is, provided that $S_t = s$, given by

$$C_\lambda(s,K;t,T) = \mathbb{E}_{\mathbb{Q}_\lambda}[(S_T-K)^+|S_t=s], \tag{9.6}$$

where K denotes the strike price and $T > t$ the time of maturity. We can express the expectation in (9.6) as

$$\mathbb{E}_{\mathbb{Q}_\lambda}[(S_T-K)^+|S_t=s] = \Pi(s,K,\alpha^+(T-t),\alpha^-(T-t),\lambda,\phi(\lambda)), \tag{9.7}$$

where Π is defined as

$$\Pi(s,K,\alpha^+,\alpha^-,\lambda^+,\lambda^-) := \int_{\ln\left(\frac{K}{s}\right)}^\infty (se^x-K)f(x;\alpha^+,\alpha^-,\lambda^+,\lambda^-)dx, \tag{9.8}$$

with $x \mapsto f(x; \alpha^+, \alpha^-, \lambda^+, \lambda^-)$ denoting the density of a bilateral Gamma distribution having these parameters. In order to compute the option prices, we have to evaluate the integral in (9.8). In the sequel, $F(\alpha, \beta; \gamma; z)$ denotes the *hypergeometric series* [8, p. 995]

$$F(\alpha, \beta; \gamma; z) = 1 + \frac{\alpha \cdot \beta}{\gamma \cdot 1} z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1) \cdot 1 \cdot 2} z^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2) \cdot 1 \cdot 2 \cdot 3} z^3 + \dots$$

9.3. **Proposition.** *Assume $\lambda^+ > 1$. For the integral in (9.8) the following identity is valid:*

(9.9)

$$\Pi(s, K, \alpha^+, \alpha^-, \lambda^+, \lambda^-) = \int_{\ln(\frac{K}{s})}^0 (se^x - K) f(x; \alpha^+, \alpha^-, \lambda^+, \lambda^-) dx + \frac{(\lambda^+)^{\alpha^+} (\lambda^-)^{\alpha^-} \Gamma(\alpha^+ + \alpha^-)}{\Gamma(\alpha^+)\Gamma(\alpha^- + 1)} \times \left(\frac{sF(\alpha^+ + \alpha^-, \alpha^-; \alpha^- + 1; -\frac{\lambda^- + 1}{\lambda^+ - 1})}{(\lambda^+ - 1)^{\alpha^+ + \alpha^-}} - \frac{KF(\alpha^+ + \alpha^-, \alpha^-; \alpha^- + 1; -\frac{\lambda^-}{\lambda^+})}{(\lambda^+)^{\alpha^+ + \alpha^-}} \right).$$

Proof. Note that the density of a bilateral Gamma distribution is given by (4.4). The assertion follows by applying identity 3 from [8, p. 816]. □

Proposition 9.3 provides a closed pricing formula for exp-Lévy models with underlying bilateral Gamma process, as the *Black-Scholes formula* for Black-Scholes models. In formula (9.9), it remains to evaluate the integral over the compact interval $[\ln(\frac{K}{s}), 0]$. This can be done numerically. In the special case $K = s$ we get an exact pricing formula.

9.4. **Corollary.** *Assume $\lambda^+ > 1$. In the case $K = s$ it holds for (9.8):*

$$\Pi(s, K, \alpha^+, \alpha^-, \lambda^+, \lambda^-) = \frac{K(\lambda^+)^{\alpha^+} (\lambda^-)^{\alpha^-} \Gamma(\alpha^+ + \alpha^-)}{\Gamma(\alpha^+)\Gamma(\alpha^- + 1)} \times \left(\frac{F(\alpha^+ + \alpha^-, \alpha^-; \alpha^- + 1; -\frac{\lambda^- + 1}{\lambda^+ - 1})}{(\lambda^+ - 1)^{\alpha^+ + \alpha^-}} - \frac{F(\alpha^+ + \alpha^-, \alpha^-; \alpha^- + 1; -\frac{\lambda^-}{\lambda^+})}{(\lambda^+)^{\alpha^+ + \alpha^-}} \right). \tag{9.10}$$

Proof. This is an immediate consequence of Proposition 9.3. □

We will use this result in the upcoming section in order to calibrate our model to an option price observed at the market.

10. An illustration: DAX 1996-1998

We turn to an illustration of the preceding theory. Figure 1 shows 751 observations $S_0, S_1, \dots, S_{750}$ of the German stock index DAX, over the period of three years. We assume that this price evolution actually is the trajectory of an exponential bilateral Gamma model, i.e. $S_t = S_0 e^{X_t}$ with $S_0 = 2307.7$ and X being a $\Gamma(\Theta)$-process, where

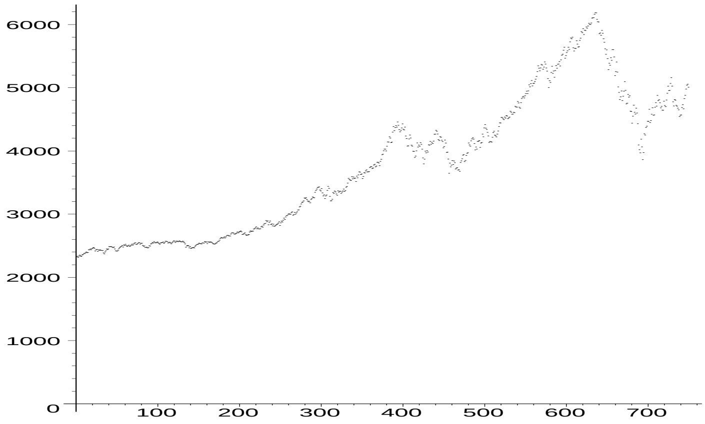

FIGURE 1. DAX, 1996-1998.

$\Theta = (\alpha^+, \alpha^-, \lambda^+, \lambda^-)$. For simplicity we assume that the interest rate r is zero. Then the increments $\Delta X_i = X_i - X_{i-1}$ for $i = 1, \dots, 750$ are a realization of an i.i.d. sequence of $\Gamma(\Theta)$-distributed random variables.

In order to estimate Θ, we carry out the statistical program described in Section 5. For the given observations $\Delta X_1, \dots, \Delta X_{750}$, the *method of moments* (5.1) yields the estimation

$$\begin{aligned}\hat{m}_1 &= 0.001032666257,\\ \hat{m}_2 &= 0.0002100280033,\\ \hat{m}_3 &= -0.0000008191504362,\\ \hat{m}_4 &= 0.0000002735163873.\end{aligned}$$

Taking into account the condition $\alpha^+, \alpha^-, \lambda^+, \lambda^- > 0$, the system of equations (5.3) has a unique solution, given by

$$\hat{\Theta}_0 = (1.28, 0.78, 119.75, 80.82).$$

Proceeding with an algorithm, which maximizes the logarithmic likelihood function (5.4) numerically, with $\hat{\Theta}_0$ as starting point, we obtain the *maximum likelihood estimation*

$$\hat{\Theta} = (1.55, 0.94, 133.96, 88.92).$$

We have estimated the parameters of the bilateral Gamma process X under the measure $\mathbb{P}$, which plays the role of the real-world measure. The next task is to find an appropriate martingale measure $\mathbb{Q}_\lambda \overset{\text{loc}}{\sim} \mathbb{P}$.

Assume that at some point of time $t \geq 0$ the stock has value $S_t = 5000$ DM, and that there is a European Call Option at the market with the same strike price $K = 5000$ DM and exercise time in 100 days, i.e. $T = t + 100$. Our goal is to *calibrate* our model to the price of this option. Since the stock value and the strike price coincide, we can use the exact pricing formula (9.10) from Corollary 9.4. The resulting Figure 2 shows the Call Option prices $C_\lambda(5000, 5000; t, t + 100)$ for $\lambda \in (1, \infty)$. Observe that we get

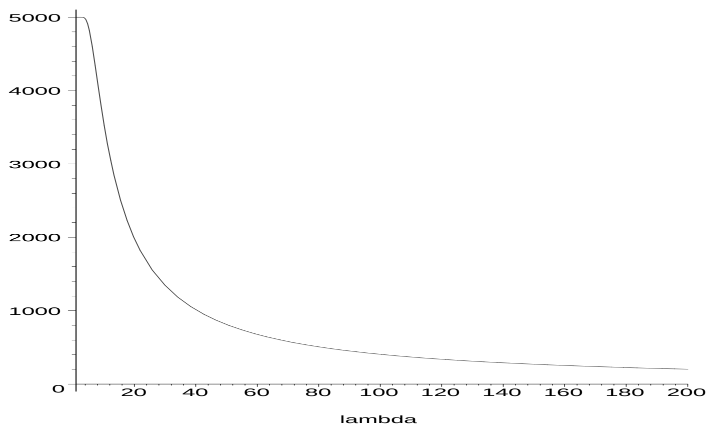

FIGURE 2. Call Option prices $C_\lambda(5000, 5000; t, t + 100)$ for $\lambda \in (1, \infty)$.

the whole interval $[0, 5000]$ of reasonable Call Option prices. This is a typical feature of exp-Lévy models, cf. [7].

Consequently, we can calibrate our model to any observed price $C \in [0, 5000]$ of the Call Option by choosing the $\lambda \in (1, \infty)$ such that $C = C_\lambda(5000, 5000; t, t+100)$.

As described in Section 9, another way to find a martingale measure is to minimize the relative entropy, i.e. finding $\lambda \in (1, \infty)$ which minimizes $\mathcal{E}(\mathbb{Q}_\lambda, \mathbb{P})$. For this purpose, we have to find $\lambda \in (1, \infty)$ such that (9.5) is satisfied. We solve this equation numerically and find the unique solution given by $\lambda = 139.47$. Using the corresponding martingale measure $\mathbb{Q}_\lambda \overset{\text{loc}}{\sim} \mathbb{P}$, we obtain the Call Option price $C_\lambda(5000, 5000; t, t+100) = 290.75$, cf. Figure 2. Under $\mathbb{Q}_\lambda$, the process X is, according to Proposition 9.2, a bilateral Gamma process $\Gamma(1.55, 139.47; 0.94, 83.51)$.

It remains to analyze the goodness of fit of the bilateral Gamma distribution, and to compare it to other families of distributions. Figure 3 shows the empirical and the fitted bilateral Gamma density.

We have provided maximum likelihood estimations for generalized hyperbolic (GH), Normal inverse Gaussian (NIG), hyperbolic (HYP), bilateral Gamma, Variance Gamma (VG) and Normal distributions. In the following table we see the Kolmogorov-distances, the L^1-distances and the L^2-distances between the empirical and the estimated distribution functions. The number in brackets denotes the number of parameters of the respective distribution family.

	Kolmogorov-distance	L^1-distance	L^2-distance
GH (5)	0.0134	0.0003	0.0012
NIG (4)	0.0161	0.0004	0.0013
HYP (4)	0.0137	0.0004	0.0013
Bilateral (4)	0.0160	0.0003	0.0013
VG (3)	0.0497	0.0011	0.0044
Normal (2)	0.0685	0.0021	0.0091

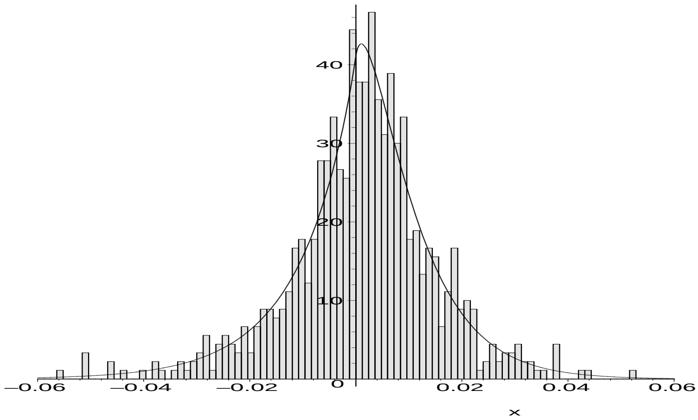

FIGURE 3. Empirical density and fitted bilateral Gamma density.

We remark that the fit provided by bilateral Gamma distributions is of the same quality as that of the four-parameter subclasses of generalized hyperbolic distributions.

We perform the *Kolmogorov test* by using the following table which shows the quantiles $\lambda_{1-\alpha}$ of order $1-\alpha$ of the Kolmogorov distribution divided by the square root of the number n of observations. Recall that in our example we have $n = 750$.

α	0.20	0.10	0.05	0.02	0.01
$\lambda_{1-\alpha}/\sqrt{n}$	0.039	0.045	0.050	0.055	0.059

The table shows that the hypothesis of a Normal distribution can clearly be denied. Variance Gamma distribution can be denied with probability of error 5 percent, whereas the remaining families of distributions cannot be rejected.

11. TERM STRUCTURE MODELS

Let $f(t,T)$ be a Heath-Jarrow-Morton term structure model ([9])

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dX_t,$$

driven by a one-dimensional Lévy process X. We assume that the cumulant generating function Ψ exists on some non-void closed interval I having zero as inner point. By equation (2.6), this condition is satisfied for bilateral Gamma processes.

We assume that the volatility σ is deterministic and that, in order to avoid arbitrage, the drift α satisfies the HJM drift condition

$$\alpha(t,T) = -\sigma(t,T)\Psi'(\Sigma(t,T)), \quad \text{where } \Sigma(t,T) = -\int_t^T \sigma(t,s)ds.$$

Since Ψ is only defined on I, we impose the additional condition

$$\Sigma(t,T) \in I \quad \text{for all } 0 \le t \le T. \tag{11.1}$$

It was shown in [11] that the short rate process $r_t = f(t,t)$ is a *Markov process* if and only if the volatility factorizes, i.e. $\sigma(t,T) = \tau(t)\zeta(T)$. Moreover, provided differentiability of τ as well as $\tau(t) \neq 0$, $t \geq 0$ and $\zeta(T) \neq 0$, $T \geq 0$, there exists an affine *one-dimensional realization*. Since $\sigma(\cdot, T)$ satisfies for each fixed $T \geq 0$ the ordinary differential equation

$$\frac{\partial}{\partial t}\sigma(t,T) = \frac{\tau'(t)}{\tau(t)}\sigma(t,T), \quad t \in [0,T]$$

we verify by using Itô's formula [10, Thm. I.4.57] for fixed $T \geq 0$ that such a realization

$$f(t,T) = a(t,T) + b(t,T)Z_t, \quad 0 \leq t \leq T \tag{11.2}$$

is given by

$$a(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds, \quad b(t,T) = \sigma(t,T) \tag{11.3}$$

and the one-dimensional state process Z, which is the unique solution of the stochastic differential equation

$$\begin{cases} dZ_t &= -\frac{\tau'(t)}{\tau(t)} Z_t dt + dX_t \\ Z_0 &= 0 \end{cases}.$$

We can transform this realization into an affine *short rate realization*. By (11.2), it holds for the short rate $r_t = a(t,t) + b(t,t)Z_t$, $t \geq 0$, implying

$$Z_t = \frac{r_t - a(t,t)}{b(t,t)}, \quad t \geq 0.$$

Inserting this equation into (11.2), we get

$$f(t,T) = a(t,T) + \frac{b(t,T)}{b(t,t)}(r_t - a(t,t)), \quad 0 \leq t \leq T.$$

Incorporating (11.3), we arrive at

$$f(t,T) = f(0,T) - \int_0^t \left[\Psi'(\Sigma(s,T)) - \Psi'(\Sigma(s,t))\right]\sigma(s,T)ds + \frac{\zeta(T)}{\zeta(t)}\left(r_t - f(0,t)\right). \tag{11.4}$$

As an example, let $f(t,T)$ be a term structure model having a *Vasiĉek* volatility structure, i.e.

$$\sigma(t,T) = -\hat{\sigma}e^{-a(T-t)}, \quad 0 \leq t \leq T \tag{11.5}$$

with real constants $\hat{\sigma} > 0$ and $a \neq 0$. We assume that $a > 0$ and $\frac{\hat{\sigma}}{a} < \lambda^+$. Since

$$\Sigma(t,T) = \frac{\hat{\sigma}}{a}\left(1 - e^{-a(T-t)}\right), \quad 0 \leq t \leq T \tag{11.6}$$

the condition (11.1) is then satisfied. By the results above, the short rate r is a Markov process and there exists a short rate realization. Equation (11.4) simplifies to

$$\begin{aligned} f(t,T) = f(0,T) &+ \Psi(\Sigma(0,T)) - \Psi(\Sigma(t,T)) - e^{-a(T-t)}\Psi(\Sigma(0,t)) \\ &+ e^{-a(T-t)}\left(r_t - f(0,t)\right). \end{aligned} \tag{11.7}$$

We can compute the bond prices $P(t,T)$ by using the following result.

11.1. **Proposition.** *It holds for the bond prices*

$$P(t,T) = e^{\phi_1(t,T) - \phi_2(t,T) r_t}, \quad 0 \leq t \leq T$$

where the functions ϕ_1, ϕ_2 *are given by*

$$\begin{aligned} \phi_1(t,T) = &- \int_t^T f(0,s) ds - \int_t^T \Psi\left(\frac{\hat{\sigma}}{a} \left(1 - e^{-as}\right) \right) ds \\ &+ \int_t^T \Psi\left(\frac{\hat{\sigma}}{a} \left(1 - e^{-a(s-t)}\right) \right) ds \\ &+ \frac{1}{a}\left(1 - e^{-a(T-t)}\right) \left[f(0,t) + \Psi\left(\frac{\hat{\sigma}}{a}\left(1 - e^{-a(T-t)}\right)\right)\right], \end{aligned} \tag{11.8}$$

$$\phi_2(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right). \tag{11.9}$$

Proof. The claimed formula for the bond prices follows from the identity

$$P(t,T) = e^{-\int_t^T f(t,s) ds}$$

and equations (11.6), (11.7). □

The problem is that ϕ_1 in (11.8) is difficult to compute for a general driving Lévy process X, because we have to integrate over an expression involving the cumulant generating function Ψ. However, for bilateral Gamma processes we can derive (11.8) in closed form. For this aim, we consider the *dilogarithm function* [1, page 1004], defined as

$$\operatorname{dilog}(x) := - \int_1^x \frac{\ln t}{t-1} dt, \quad x \in \mathbb{R}_+$$

which will appear in our closed form representation. The dilogarithm function has the series expansion

$$\operatorname{dilog}(x) = \sum_{k=1}^{\infty} (-1)^k \frac{(x-1)^k}{k^2}, \quad 0 \leq x \leq 2$$

and moreover the identity

$$\operatorname{dilog}(x) + \operatorname{dilog}\left(\frac{1}{x}\right) = -\frac{1}{2}(\ln x)^2, \quad 0 \leq x \leq 1$$

is valid, see [1, page 1004]. For a computer program, the dilogarithm function is thus as easy to evaluate as the natural logarithm. The following auxiliary result will be useful for the computation of the bond prices $P(t,T)$.

11.2. **Lemma.** *Let* $a, b, c, d, \lambda \in \mathbb{R}$ *be such that* $a \leq b$ *and* $c > 0, \lambda \neq 0$. *Assume furthermore that* $c + de^{\lambda x} > 0$ *for all* $x \in [a,b]$. *Then we have*

$$\int_a^b \ln\left(c + de^{\lambda x}\right) dx = (b-a)\ln(c) - \frac{1}{\lambda}\operatorname{dilog}\left(1 + \frac{d}{c}e^{\lambda b}\right) + \frac{1}{\lambda}\operatorname{dilog}\left(1 + \frac{d}{c}e^{\lambda a}\right).$$

Proof. With $\varphi(x) := 1 + \frac{d}{c} e^{\lambda x}$ we obtain by making a substitution

$$\begin{aligned}\int_a^b \ln\left(c + de^{\lambda x}\right) dx &= (b-a)\ln(c) + \int_a^b \ln\left(1 + \frac{d}{c}e^{\lambda x}\right) dx \\ &= (b-a)\ln(c) + \frac{1}{\lambda}\int_{\varphi(a)}^{\varphi(b)} \frac{\ln t}{t-1} dt \\ &= (b-a)\ln(c) - \frac{1}{\lambda}\operatorname{dilog}\left(1 + \frac{d}{c}e^{\lambda b}\right) + \frac{1}{\lambda}\operatorname{dilog}\left(1 + \frac{d}{c}e^{\lambda a}\right).\end{aligned}$$

□

Now assume the driving process X is a bilateral Gamma process $\Gamma(\alpha^+, \lambda^+; \alpha^-, \lambda^-)$. We obtain a formula for the bond prices $P(t,T)$ in terms of the natural logarithm and the dilogarithm function.

11.3. **Proposition.** *The function ϕ_1 in (11.8) has the representation*

$$\begin{aligned}\phi_1(t,T) = &- \int_t^T f(0,s)ds \\ &+ \frac{\alpha^+}{a}[D_1(\lambda^+,T) - D_1(\lambda^+,t) - D_1(\lambda^+,T-t) + D_1(\lambda^+,0)] \\ &+ \frac{\alpha^-}{a}[D_0(\lambda^-,T) - D_0(\lambda^-,t) - D_0(\lambda^-,T-t) + D_0(\lambda^-,0)] \\ &+ \frac{1}{a}\left(1 - e^{-a(T-t)}\right)[f(0,t) + \alpha^+ L_1(\lambda^+) + \alpha^- L_0(\lambda^-)],\end{aligned}$$

where

$$\begin{aligned}D_\beta(\lambda,t) &= \operatorname{dilog}\left(1 + \frac{\hat{\sigma}e^{-at}}{\lambda^+ a + (-1)^\beta \hat{\sigma}}\right), \quad \beta \in \{0,1\}, \\ L_\beta(\lambda) &= \ln\left(\frac{\lambda}{\lambda + (-1)^\beta(1 - e^{-at})}\right), \quad \beta \in \{0,1\}.\end{aligned}$$

Proof. The assertion follows by inserting the cumulant generating function (2.6) of the bilateral Gamma process X into (11.8) and using Lemma 11.2. □

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