On the average behaviour of greedy algorithms for the knapsack problem

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Abstract

We study the average behaviour of the well-known greedy algorithms for the one-dimensional knapsack problem with Boolean variables when the number of variables n tends to infinity. It is supposed that the right-hand side b of the constraint depends linearly on n, i.e. $b = \lambda n$. It is shown that if $\lambda > \frac{1}{2} - \frac{t}{3}$ then the primal and the dual greedy algorithms have an asymptotical tolerance t.

1 Introduction

Our main object is the classical knapsack problem with Boolean variables. It consists in finding

$$f^* = \max\{\sum_{j=1}^{n} c_j x_j | \sum_{j=1}^{n} a_j x_j \le b,\ x \in \mathbb{B}^n\}. \quad (1)$$

All coefficients in (1) are positive. The standard interpretation of the problem (1) is the following: we have to fill a knapsack of capacity b with the most profitable subset of items from $\{1, \dots, n\}$, where each item j is characterized by its utility c_j and weight a_j. The Boolean variables x_j equal 1 if the item j is chosen, and 0 otherwise.

Without loss of generality, we can suppose that $a_j < b$ for all j (otherwise the variables x_j for wich this inequality is violated can be excluded), and that $\sum_{j=1}^{n} a_j > b$ (otherwise the problem (1) is trivial, and its optimal solution is $x^* = (1, \dots, 1)$). Besides, we shall suppose that

$$\frac{c_1}{a_1} \ge \frac{c_2}{a_2} \ge \dots \ge \frac{c_n}{a_n}, \quad (2)$$

i.e., the variables x_j are numbered in the non-increasing order of their "utility densities" c_j/a_j. The condition (2) is often called the *regularity condition*.

The problen (1) has numerous applications, and it is one of the main models of combinatorial optimization. From the viewpoint of the general complexity theory, it is NP-hard. This means that exact algorithms with polynomial complexity can only exist in the case $P = NP$. Therefore, the main research efforts are now concentrated around approximate methods for the problem (1), and this tendency is characteristic for the entire combinatorial optimization.

Among these approximate methods, the so-called *greedy* methods play a major role. They can be interpreted as discrete analogs of gradient (or steepest-ascent) methods in continuous optimization. Their undoubted advantage is that for the problem (1) they work in linear time

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(if the regularity condition (2) is fulfilled). The greedy methods do not guarantee optimality; however, theoretical estimations of their worst-case performance can be given. Details can be found in the review paper [1].

The idea of the greedy algorithm for the problem (1) consists in a consecutive selection of items with the largest density c_j/a_j until the knapsack capacity admits it. More formally, the algorithm starts with a feasible solution $x = (0, \ldots, 0)$ and consecutively replaces zeroes by ones in the order of decreasing ratios c_j/a_j (i.e., from the left to the right); every time the feasibility of the corresponding solution is checked. The process terminates after obtaining the last feasible solution. This solution x^G is called the *greedy solution*; the corresponding objective function value is denoted by f^G.

An idea which is in some sense opposite, consists in a consecutive rejecting the least profitable items (again, in the sense of the ratios c_j/a_j) until the remaining ones fit in the knapsack. In accordance with the usual terminology, such algorithms can be called *dual* algorithms. Therefore the greedy algorithm described above will sometimes be termed *primal.* More formally, the dual greedy algorithm starts with an infeasible solution $x = (1, \ldots, 1)$ and consecutively replaces ones by zeroes in the order of increasing ratios c_j/a_j (i.e., from the right to the left). Every time the feasibility of the current solution is checked. The process terminates when the first feasible solution is obtained. This solution x^{DG} is called the *dual greedy* solution; the corresponding objective function value is denoted by f^{DG}.

First of all we note that the primal and the dual greedy solutions can be different (even if all inequalities in the regularity condition (2) are strict). The simplest example is

$$\max\{6x_1 + 4x_2 + x_3 | 2x_1 + 2x_2 + x_3 \leq 3,\ x \in \mathbb{B}^3\}. \tag{3}$$

Here we have $x^G = (1, 0, 1)$, $x^{DG} = (1, 0, 0)$.

Up to now, practically no attention to the analysis of dual greedy algorithms was paid. The reason was probably the following "folklore theorem".

Proposition 1 *The dual greedy algorithm for the problem (1) can be arbitrarily bad.*

It is natural to estimate the performance of the dual greedy algorithm by the ratio $R_{DG} = f^{DG}/f^*$. The assertion means that R_{DG} can take arbitrarily small values. To prove this, we consider the following one-parametric family of instances of (1):

$$\max\{3x_1 + 2\lambda x_2 | x_1 + \lambda x_2 \leq \lambda,\ x \in \mathbb{B}^2\}, \tag{4}$$

where $\lambda > 3/2$. We have $x^* = (0, 1)$ and $f^* = 2\lambda$. At the same time, $x^{DG} = (1, 0)$, $f^{DG} = 3$. Thus, $R_{DG} = 3/2\lambda$ tends to zero when $\lambda \to \infty$.

We discuss briefly some connections between primal and dual greedy solutions. To this end, we shall need several definitions. For any vector $x \in \mathbb{B}^n$, we call its *fragment* any set of its consecutive components equal to 1. A fragment is *maximal* if it cannot be extended to the left and to the right. There is a natural ordering of maximal fragments. Vectors having a unique maximal fragment are called *connected.* Consider now vectors from $\mathbb{B}^n$ satisfying one of the conditions

$$x_1 \geq x_2 \geq \cdots \geq x_n, \tag{5}$$
$$x_1 \leq x_2 \leq \cdots \leq x_n. \tag{6}$$

It is clear that such vectors either do not contain any fragments (i.e., they are null vectors) or they have exactly one maximal fragment, that is, they are connected. We call vectors satisfying (5) *lower connected*, and vectors satisfying (6) *upper connected*. Thus, lower connected vectors have the form $(1,\dots,1,0,\dots,0)$, and upper connected vectors - the form $(0,\dots,0,1,\dots 1)$.

Now we consider again the problem (1). We note that the primal greedy solution is not necessarily connected (cf. the example (3)). The following assertion holds.

Proposition 2 *Let x^G, x^{DG} be respectively the primal and the dual greedy solutions to the problem (1). Then*

1) x^{DG} is lower connected;

2) $x^{DG} \le x^G$ (the inequality is componentwise) where the equality takes place if and only if x^G is lower connected;

3) x^{DG} contains the first maximal fragment of x^G and only this fragment.

From the assertion 2) and the positivity of the objective function coefficients it follows that $f^{DG} \le f^G$. In other words, the dual greedy algorithm cannot be better than the primal one. We formulate the last assertion in a somewhat extended form which will be important later. Consider the linear relaxation of (1) which consists in finding

$$f^{LR} = \max\{\sum_{j=1}^{n} c_j x_j | \sum_{j=1}^{n} a_j x_j \le b, 0 \le x_j \le 1, j = 1,\dots,n\}. \tag{7}$$

Corollary. The following inequalities hold

$$f^{DG} \le f^G \le f^* \le f^{LR}. \tag{8}$$

It is well-known that the optimal solution x^{LR} to the linear relaxation (7) has the form

$$x^{LR} = (x_1,\dots,x_{k-1},x_k,0,\dots,0),$$

where $x_1 = \cdots = x_{k-1} = 1$, $0 \le x_k < 1$, and the index k is determined as

$$k = \max\{s | \sum_{j=1}^{s-1} a_j \le b\}$$

or, equivalently, as

$$k = \min\{s | \sum_{j=1}^{s} a_j > b\}.$$

The number k is called the *critical index* for the problem (1). Its definition implies that

$$\sum_{j=1}^{k-1} a_j \le b, \;\; \sum_{j=1}^{k} a_j > b.$$

In another terminology, this means that k is the first index "skipped" by the primal greedy algorithm, or, equivalently, that $(x_1,\dots,x_{k-1})$ is the first maximal fragment of the vector x^G (or, according to the assertion 3) of Proposition 2, the unique maximal fragment of x^{DG}). The value of x_k can be easily found explicitly:

$$x_k = \frac{b - \sum_{j=1}^{k-1} a_j}{a_k}. \tag{9}$$

Thus we have

$$x^{LR} = (F, x_k, 0, \ldots, 0),$$

where F is the maximal fragment of x^{DG}, $0 \leq x_k < 1$. From this if follows immediately that

$$f^{LR} - f^{DG} = c_k x_k, \tag{10}$$

where k is the critical index, and x_k is determined by (9).

The proofs together with a generalization of these results to more general combinatorial structures (independence systems) can be found in [2], [3].

We showed above that the dual greedy algorithm can be arbitrarily bad. On the other hand, this algorithm was successfully used (as a subroutine) in solving some applied large-scale problems where it gave remarkably good results. This controversy between bad theoretical and good actual performance of an algorithm can be, as usually, resolved by studying not the worst-case but the average behaviour of this algorithm.

We began to implement this general program in [2] (see also the preliminary publication [4]). Roughly speaking, it was shown there that if, for a random set of coefficients, the "feasibility probability" tends to 1 with the growth of the number of variables then the objective function values for the primal and the dual greedy algorithms, with a large probability, differ insignificantly from the optimal value. However, this result had in a certain sense a conditional character. It remained to find bounds for the right-hand side of the problem for which this theorem is non-trivially true (see some short comments concluding [2]). This is the main subject of the present paper.

2 The main result

Our main assumption is rather traditional in the investigation of the average behaviour of algorithms. We suppose that all coefficients c_j, a_j, $j = 1, \ldots, n$ are independent random variables uniformly distributed on [0,1]. The assumption about b will be made a little later.

We denote $z_j = c_j/a_j$. Let z^k be the random variable equal to the $(n-k+1)$-th term of the variational series determined by the n-tuple of random variables $z_1, \ldots, z_n$, i.e. $z^1 \geq z^2 \geq \cdots \geq z^n$. We introduce now the random variables c^k, a^k as follows. If $z^k = c_{q(k)}/a_{q(k)}$, we let

$$c^k = c_{q(k)}, \;\; a^k = a_{q(k)}.$$

We are interested in the behaviour of approximate algorithms for problems with n variables when n grows. Suppose that we are applying to a combinatorial optimization problem some approximate algorithm $\mathcal{A}$. We denote this algorithm by $\mathcal{A}_n$ to stress the dependence on the number of variables. We say that the algorithm $\mathcal{A}_n$ has the *asymptotical tolerance* t if

$$\mathbf{P}(f^* - f^{\mathcal{A}_n} \leq t) \underset{n\to\infty}{\longrightarrow} 1,$$

where f^* is the optimal value, $t > 0$.

This definition is rather general. Further we shall consider the knapsack problem (1), and $\mathcal{A}_n$ will be the greedy algorithm. We formulate now our main result.

Theorem 3 *If $b = \lambda n$ where*

$$\lambda > \frac{1}{2} - \frac{t}{3},$$

then the primal and the dual greedy algorithms have the asymptotical tolerance t.

The proof of this theorem will be split into several stages.

First of all, we define the following events (the notation was introduced in the Introduction):

$A^n : f^{LR} - f^{DG} \le t$

$B_k^n : x^{LR} = (1, \dots, 1, \alpha, 0, \dots, 0)$, where the component $\alpha \in [0,1)$ has the number k, $k = 1, \dots, n$.

$B_{n+1}^n : x^{LR} = (1, \dots, 1)$

$C_k^n : c^k \le t$, $k = 1, \dots, n$.

Choose a $t' < t$ and denote

$$N = n\Big(1 - \frac{t'}{2}\Big).$$

We formulate now two conditions.

C o n d i t i o n 1. $\sum_{k=N}^{n} \mathbf{P}(\overline{C}_k^n) \underset{n\to\infty}{\longrightarrow} 0.$

C o n d i t i o n 2. $\sum_{k=N}^{n+1} \mathbf{P}(B_k^n) \underset{n\to\infty}{\longrightarrow} 1.$

The general idea of the proof of the main Theorem 3 is that the validity of Conditions 1 and 2 implies the assertion of this theorem. We establish first some auxilary results.

Lemma 4 *The following implication holds*

$$A^n \supset \sum_{k=N}^{n} B_k^n C_k^n + B_{n+1}^n$$

.

P r o o f. We have $B_{n+1}^n A^n = B_{n+1}^n$, and the assertion follows from the relations

$$A^n = B_1^n A^n + \dots + B_n^n A^n + B^{n+1} A^n \supset$$

$$\supset B_1^n C_1^n + \dots + B_n^n C_n^n + B_{n+1}^n \supset \sum_{k=N}^{n} B_k^n C_k^n + B_{n+1}^n.$$

Lemma 5 *The following inequalities hold*

$$\mathbf{P}(A^n) \ge \sum_{k=N}^{n} \mathbf{P}(B_k^n C_k^n) \ge \sum_{k=N}^{n} [\mathbf{P}(B_k^n) - \mathbf{P}(\overline{C}_k^n)].$$

P r o o f. We have

$$1 \ge \mathbf{P}(B_k^n + C_k^n) = \mathbf{P}(B_k^n) + \mathbf{P}(C_k^n) - \mathbf{P}(B_k^n C_k^n).$$

Therefore

$$\mathbf{P}(B_k^n C_k^n) \ge \mathbf{P}(B_k^n) + \mathbf{P}(C_k^n) - 1 = \mathbf{P}(B_k^n) - \mathbf{P}(\overline{C}_k^n),$$

and this implies the assertion.

It can be seen from Lemma 5 that if in the conditions of Theorem 3 there is a $N = N(n)$ for which Conditions 1 and 2 hold, then the main Theorem 3 will be proved. We shall seek N of the form $N = nt_0$, $0 < t_0 < 1$. It will be shown that if we take $t_0 = 1 - \frac{t'}{2}$ where

$$1 - \frac{t}{3} < 1 - \frac{t'}{3} < \lambda,$$

then for $N = nt_0$ the Conditions 1 and 2 will be fulfilled. In the next section the validity of Condition 1 for $N = nt_0$ will be demonstrated.

3 The proof of Condition 1

First of all we find the main probabilistic characteristics of the random variables c^j, a^j introduced at the beginning of the previous section. We use the standard notation (cf. [5], Ch.VI)

$$b(j-1,n-1,1-t)=\binom{n-1}{j-1}(1-t)^{j-1}t^{n-j} \qquad (11)$$

for the probability of $j-1$ successes in $n-1$ Bernoulli trials with the success probability $1-t$. The values (realizations) of the random variables c^j, a^j will be denoted by γ_j, α_j respectively.

Lemma 6 *The density $f_j(\gamma_j)$ of the random variable c^j is*

$$\begin{aligned} f_j(\gamma_j) &= 2n\binom{n-1}{j-1}\gamma_j\int_0^{\frac{1}{2}} x^{j-1}(1-x)^{n-j}dx + \\ &+ \frac{1}{2}n\binom{n-1}{j-1}\gamma_j\int_{\frac{\gamma_j}{2}}^{\frac{1}{2}}(1-y)^{j-1}y^{n-j-2}dy. \end{aligned} \qquad (12)$$

P r o o f. It is easy to see that

$$\mathbf{P}(z_j\le z)=\begin{cases} 0 & z<0 \\ \dfrac{z}{2} & 0\le z\le 1 \\ 1-\dfrac{1}{2z} & z>1. \end{cases} \qquad (13)$$

Fix an index $j\in\{1,\dots,n\}$. We get from (13) that the probability that some $j-1$ of the random variables z_j will take the value $\le \gamma_j/\alpha_j$ and the remaining random variables will be greater than γ_j/α_j is (taking into account that any random variable can occupy the place j)

$$p(\gamma_j/\alpha_j)=\begin{cases} nb(j-1,n-1,\dfrac{\alpha_j}{2\gamma_j}) & \dfrac{\gamma_j}{\alpha_j}\ge 1 \\ nb(j-1,n-1,1-\dfrac{\gamma_j}{2\alpha_j}) & \dfrac{\gamma_j}{\alpha_j}\le 1. \end{cases} \qquad (14)$$

It follows that the probability of the event

$$c^j\in[\gamma_j,\gamma_j+\triangle\gamma_j],\ \ a^j\in[\alpha^j,\alpha^j+\triangle\alpha_j]$$

equals

$$p(\gamma_j/\alpha_j)\triangle\alpha_j\triangle\gamma_j+o(\triangle\alpha_j\triangle\gamma_j).$$

Therefore the joint density $r(\alpha_j,\gamma_j)$ of the random variables c^j, a^j is

$$r(\alpha_j,\gamma_j)=p(\gamma_j/\alpha_j), \qquad (15)$$

and the density of c^j will be

$$f_j(\gamma_j)=\int_0^1 r(\alpha_j,\gamma_j)d\alpha_j. \qquad (16)$$

Using the representation (11), we get from (14) - (16)

$$f_j(\gamma_j) = n\binom{n-1}{j-1}\int_0^{\gamma_j}\left(\frac{\alpha_j}{2\gamma_j}\right)^{j-1}\left(1-\frac{\alpha_j}{2\gamma_j}\right)^{n-j}d\alpha_j +$$

$$+\quad n\binom{n-1}{j-1}\int_{\gamma_j}^{1}\Big(1-\frac{\gamma_j}{2\alpha_j}\Big)^{j-1}\Big(\frac{\gamma_j}{2\alpha_j}\Big)^{n-j}d\alpha_j.$$

Changing the variables $\alpha_j/(2\gamma_j) = x$ in the first integral and $\gamma_j/(2\alpha_j) = y$ in the second one, we get the representation (12).

Lemma 7 *The distribution function $F_j(t)$ of the random variable c^j for $0 \le t \le 1$ is*

$$\begin{aligned} F_j(t) &= n\binom{n-1}{j-1}t^2\int_0^{\frac{1}{2}} x^{j-1}(1-x)^{n-j}dx + \\ &+ \frac{1}{4}n\binom{n-1}{j-1}t^2\int_{\frac{t}{2}}^{\frac{1}{2}}(1-x)^{j-1}x^{n-j-2}dx + \\ &+ n\binom{n-1}{j-1}\int_0^{\frac{t}{2}}(1-x)^{j-1}x^{n-j}dx. \end{aligned} \tag{17}$$

P r o o f. We have

$$\begin{aligned} F_j(t) &= \int_0^t f_j(\gamma_j)d\gamma_j = \\ &= 2n\binom{n-1}{j-1}\int_0^t \gamma_j\int_0^{\frac{1}{2}} x^{j-1}(1-x)^{n-j}dx\, d\gamma_j + \\ &+ \frac{1}{2}n\binom{n-1}{j-1}\int_0^t \gamma_j\int_{\frac{\gamma_j}{2}}^{\frac{1}{2}}(1-y)^{j-1}y^{n-j-2}dy\, d\gamma_j. \end{aligned}$$

The first of these integrals is calculated directly, and the second one can be easily found by partial integration. Letting

$$u = \int_{\frac{\gamma_j}{2}}^{\frac{1}{2}}(1-y)^{j-1}y^{n-j-2}dy, \quad dv = \gamma_j\, d\gamma_j,$$

we get, after elementary transformations, the representation (17).

Lemma 8 *The density $g_j(\alpha_j)$ of the random variable a^j is*

$$\begin{aligned} g_j(\alpha_j) &= 2n\binom{n-1}{n-j}\alpha_j\int_0^{\frac{1}{2}} x^{n-j}(1-x)^{j-1}dx + \\ &+ \frac{1}{2}n\binom{n-1}{n-j}\alpha_j\int_{\frac{\alpha_j}{2}}^{\frac{1}{2}}(1-y)^{n-j}y^{j-3}dy. \end{aligned} \tag{18}$$

P r o o f. The formula (18) could be obtained similarly to (16) (that is, by integration of the joint density $r(\alpha_j, \gamma_j)$ over γ_j). Another reasoning consists in the following. The inequalities

$$\frac{c^1}{a^1} \ge \frac{c^2}{a^2} \ge \cdots \ge \frac{c^n}{a^n}$$

are equivalent to

$$\frac{a^n}{c^n} \ge \frac{a^{n-1}}{c^{n-1}} \ge \cdots \ge \frac{a^1}{c^1}.$$

Therefore the density of a^j is equal to the density of c^{n-j+1}. Substituting in (12) α_j instead of γ_j and replacing j in the exponents by $n-j+1$, we get (18).

Now we can proceed to the proof of Condition 1.

Theorem 9 *Let $N = nt_0$, where $t_0 = 1 - \frac{t'}{2}$, $t' < t$. Then the Condition 1*

$$\sum_{k=N}^{n} \mathbf{P}(\overline{C_k^n}) \underset{n\to\infty}{\longrightarrow} 0$$

is fulfilled.

P r o o f. We have

$$\mathbf{P}(\overline{C_k^n}) = 1 - F_k(t),$$

where $F_k(t)$ is the distribution function of the random variable c^k. It follows from the representation (17) that

$$F_k(t) \geq n\binom{n-1}{n-j}\int_0^{\frac{t}{2}}(1-x)^{k-1}x^{n-k}dx = \sum_{\nu=0}^{k-1}\binom{n}{\nu}\left(1-\frac{t}{2}\right)^{\nu}\left(\frac{t}{2}\right)^{n-\nu}.$$

From this we get

$$\mathbf{P}(\overline{C_k^n}) \leq 1 - \sum_{\nu=0}^{k-1}\binom{n}{\nu}\left(1-\frac{t}{2}\right)^{\nu}\left(\frac{t}{2}\right)^{n-\nu}.$$

Therefore

$$\mathbf{P}(\overline{C_k^n}) \leq \sum_{\nu=k}^{n}\binom{n}{\nu}\left(1-\frac{t}{2}\right)^{\nu}\left(\frac{t}{2}\right)^{n-\nu}.$$

Thus,

$$\sum_{k=N}^{n}\mathbf{P}(\overline{C_k^n}) \leq \sum_{k=N}^{n}\sum_{\nu=k}^{n}\binom{n}{\nu}\left(1-\frac{t}{2}\right)^{\nu}\left(\frac{t}{2}\right)^{n-\nu}.$$

Transforming the double sum in the right-hand side, we get

$$\sum_{k=N}^{n}\mathbf{P}(\overline{C_k^n}) \leq \sum_{j=N}^{n}(j-N+1)\binom{n}{j}\left(1-\frac{t}{2}\right)^{j}\left(\frac{t}{2}\right)^{n-j}. \tag{19}$$

Now we can prove that the right-hand side of (19) tends to zero when $n \to \infty$, and the required assertion follows. The technical details of this proof are given in Appendix 1.

The proof of Condition 2 will be performed by using the Tchebyshev's inequality; this requires some preparations.

4 The joint distribution of a^k and a^l

Our next goal is the computation of the joint distribution of two random variables a^k and a^l. We introduce first the random variables $u^j = a^j/c^j$. We have $u^j = 1/z^j$ where z^j was defined at the beginning of Section 2. Thus, $u^1 \leq u^2 \leq \cdots \leq u^n$.

Similarly to the proof of Lemma 6, it is easy to show that

$$\mathbf{P}(u_j \leq u) = \begin{cases} 0 & u < 0 \\ \dfrac{u}{2} & 0 \leq u \leq 1 \\ 1 - \dfrac{1}{2u} & u > 1. \end{cases} \tag{20}$$

We choose two arbitrary indices k, l from $\{1,\ldots,n\}$ such that $k<l$. Further, choose two arbitrary ordered indices from $\{1,\ldots,n\}$ and denote then by i_k, i_l. The number of such choices is

$$2\binom{n}{2}=n(n-1).$$

Consider the sets

$$\begin{array}{ll} S_1\subset\{1,\ldots,n\}\setminus\{i_k,i_l\}, & |S_1|=k-1\\ S_2\subset\{1,\ldots,n\}\setminus(S_1\cup\{i_k,i_l\}), & |S_2|=l-k-1\\ S_3=\{1,\ldots,n\}\setminus(S_1\cup S_2\cup\{i_k,i_l\}), & |S_3|=n-l. \end{array}$$

The number of possible choices of i_k, i_l, S_1, S_2, S_3 is

$$\begin{aligned} n(n-1)\frac{(n-2)!}{(k-1)!(l-k-1)!(n-l)!} &= \\ =\frac{n!}{(k-1)!(l-k-1)!(n-l)!} &= c(n,k,l). \end{aligned}$$

Choose four number α_k, γ_k, α_l, γ_l between 0 and 1 such that $\alpha_k/\gamma_k<\alpha_l/\gamma_l$. Consider the conditional probability

$$\pi(i_k,i_l,S_1,S_2,S_3,\alpha_k/\gamma_k,\alpha_l/\gamma_l)=$$
$$=\mathbf{P}\Big(u_i\le\frac{\alpha_k}{\gamma_k},i_k\in S_1;\frac{\alpha_k}{\gamma_k}<u_i<\frac{\alpha_l}{\gamma_l},i\in S_2;u_i\ge\frac{\alpha_l}{\gamma_l},i\in S_3|u_{i_k}=\frac{\alpha_k}{\gamma_k},u_{i_l}=\frac{\alpha_l}{\gamma_l}\Big).$$

From the independence of u^i we get

$$\pi(i_k,i_l,S_1,S_2,S_3,\alpha_k/\gamma_k,\alpha_l/\gamma_l)=$$
$$=\mathbf{P}(u_i\le\frac{\alpha_k}{\gamma_k},i_k\in S_1)\mathbf{P}(\frac{\alpha_k}{\gamma_k}<u_i<\frac{\alpha_l}{\gamma_l},i\in S_2)\mathbf{P}(u_i\ge\frac{\alpha_l}{\gamma_l},i\in S_3). \tag{21}$$

For i_k, i_l, S_1, S_2, S_3 defined above, all probabilities $\pi(i_k,i_l,S_1,S_2,S_3,\alpha_k/\gamma_k,\alpha_l/\gamma_l)$ are equal. Therefore the sum of all probabilities $\pi(i_k,i_l,S_1,S_2,S_3,\alpha_k/\gamma_k,\alpha_l/\gamma_l)$ over all i_k, i_l, S_1, S_2, S_3 equals

$$p(\alpha_k/\gamma_k,\alpha_l/\gamma_l)=c(n,k,l)\pi(i_k,i_l,S_1,S_2,S_3,\alpha_k/\gamma_k,\alpha_l/\gamma_l).$$

This implies that the probability of events

$$\begin{array}{ll} a^k\in[\alpha_k,\alpha_k+\triangle\alpha_k], & c^k\in[\gamma_k,\gamma_k+\triangle\gamma_k],\\ a^l\in[\alpha_l,\alpha_l+\triangle\alpha_l], & c^l\in[\gamma_l,\gamma_l+\triangle\gamma_l] \end{array}$$

is

$$p(\alpha_k/\gamma_k,\alpha_l/\gamma_l)\triangle\alpha_k\triangle\gamma_k\triangle\alpha_l\triangle\gamma_l+o(\triangle\alpha_k\triangle\gamma_k\triangle\alpha_l\triangle\gamma_l)$$

if $\alpha_k/\gamma_k<\alpha_l/\gamma_l$, and is zero otherwise. Therefore the joint density $r(\alpha_k,\gamma_k,\alpha_l,\gamma_l)$ of the random variables a^k, c^k, a^l, c^l will be

$$r(\alpha_k,\gamma_k,\alpha_l,\gamma_l)=\begin{cases} p(\alpha_k/\gamma_k,\alpha_l/\gamma_l) & \alpha_k/\gamma_k<\alpha_l/\gamma_l\\ 0 & \alpha_k/\gamma_k>\alpha_l/\gamma_l. \end{cases}$$

Now, using (20), we find the probabilities in the right-hand side of (21). Here three cases are to be considered.

C a s e 1. $\dfrac{\alpha_k}{\gamma_k} \le 1$, $\dfrac{\alpha_l}{\gamma_l} \le 1$, $\dfrac{\alpha_k}{\gamma_k} < \dfrac{\alpha_l}{\gamma_l}$. In this case we have

$$p_1 = c(n,k,l)\left(\frac{\alpha_k}{2\gamma_k}\right)^{k-1}\left(\frac{\alpha_l}{2\gamma_l} - \frac{\alpha_k}{2\gamma_k}\right)^{l-k-1}\left(1 - \frac{\alpha_l}{2\gamma_l}\right)^{n-l}.$$

C a s e 2. $\dfrac{\alpha_k}{\gamma_k} \le 1$, $\dfrac{\alpha_l}{\gamma_l} > 1$. In this case

$$p_2 = c(n,k,l)\left(\frac{\alpha_k}{2\gamma_k}\right)^{k-1}\left(1 - \frac{\gamma_l}{2\alpha_l} - \frac{\alpha_k}{2\gamma_k}\right)^{l-k-1}\left(\frac{\gamma_l}{2\alpha_l}\right)^{n-l}.$$

C a s e 3. $\dfrac{\alpha_k}{\gamma_k} > 1$, $\dfrac{\alpha_l}{\gamma_l} > 1$, $\dfrac{\alpha_l}{\gamma_l} > \dfrac{\alpha_k}{\gamma_k}$. Here we have

$$p_3 = c(n,k,l)\left(1 - \frac{\gamma_k}{2\alpha_k}\right)^{k-1}\left(\frac{\gamma_k}{2\alpha_k} - \frac{\gamma_l}{2\alpha_l}\right)^{l-k-1}\left(1 - \frac{\gamma_l}{2\alpha_l}\right)^{n-l}.$$

Thus,

$$r(\alpha_k,\gamma_k,\alpha_l,\gamma_l) = \begin{cases} p_1\left(\dfrac{\alpha_k}{\gamma_k}, \dfrac{\alpha_l}{\gamma_l}\right) & \text{if } \dfrac{\alpha_k}{\gamma_k} \le 1, \quad \dfrac{\alpha_l}{\gamma_l} \le 1, \quad \dfrac{\alpha_k}{\gamma_k} < \dfrac{\alpha_l}{\gamma_l} \\ p_2\left(\dfrac{\alpha_k}{\gamma_k}, \dfrac{\alpha_l}{\gamma_l}\right) & \text{if } \dfrac{\alpha_k}{\gamma_k} \le 1, \quad \dfrac{\alpha_l}{\gamma_l} > 1 \\ p_3\left(\dfrac{\alpha_k}{\gamma_k}, \dfrac{\alpha_l}{\gamma_l}\right) & \text{if } \dfrac{\alpha_k}{\gamma_k} > 1, \quad \dfrac{\alpha_l}{\gamma_l} > 1, \quad \dfrac{\alpha_k}{\gamma_k} < \dfrac{\alpha_l}{\gamma_l} \\ 0 & \text{if } \dfrac{\alpha_k}{\gamma_k} > \dfrac{\alpha_l}{\gamma_l}. \end{cases}$$

Therefore the joint density $h_{k\,l}(\alpha_k, \alpha_l)$ of the random variables a^k and a^l equals

$$\begin{aligned} h_{k\,l}(\alpha_k,\gamma_k) &= \int_0^1\int_0^1 r(\alpha_k,\gamma_k,\alpha_l,\gamma_l)d\gamma_k\,d\gamma_l = \\ &= \int_{\Omega_1} p_1\left(\frac{\alpha_k}{\gamma_k}, \frac{\alpha_l}{\gamma_l}\right)d\gamma_k\,d\gamma_l + \int_{\Omega_2} p_2\left(\frac{\alpha_k}{\gamma_k}, \frac{\alpha_l}{\gamma_l}\right)d\gamma_k\,d\gamma_l + \\ &+ \int_{\Omega_3} p_3\left(\frac{\alpha_k}{\gamma_k}, \frac{\alpha_l}{\gamma_l}\right)d\gamma_k\,d\gamma_l, \end{aligned} \tag{22}$$

where

$$\begin{aligned} \Omega_1 &= \Omega_1(\alpha_k,\alpha_l) = \{(\gamma_k,\gamma_l)| \frac{\alpha_k}{\gamma_k} \le 1, \frac{\alpha_l}{\gamma_l} \le 1, \frac{\alpha_k}{\gamma_k} < \frac{\alpha_l}{\gamma_l}, 0 \le \gamma_k \le 1, 0 \le \gamma_l \le 1\} \\ \Omega_2 &= \Omega_2(\alpha_k,\alpha_l) = \{(\gamma_k,\gamma_l)| \frac{\alpha_k}{\gamma_k} \le 1, \frac{\alpha_l}{\gamma_l} > 1, 0 \le \gamma_k \le 1, 0 \le \gamma_l \le 1\} \\ \Omega_3 &= \Omega_3(\alpha_k,\alpha_l) = \{(\gamma_k,\gamma_l)| \frac{\alpha_k}{\gamma_k} > 1, \frac{\alpha_l}{\gamma_l} > 1, \frac{\alpha_k}{\gamma_k} \le \frac{\alpha_l}{\gamma_l}, 0 \le \gamma_k \le 1, 0 \le \gamma_l \le 1\}. \end{aligned}$$

We denote the integrals in the right-hand side of (22) by I_1, I_2, I_3 respectively. Consider the first of these integrals

$$I_1 = c(n,k,l)\int_{\Omega_1}\left(\frac{\alpha_k}{2\gamma_k}\right)^{k-1}\left(\frac{\alpha_l}{2\gamma_l} - \frac{\alpha_k}{2\gamma_k}\right)^{l-k-1}\left(1 - \frac{\alpha_l}{2\gamma_l}\right)^{n-l}d\gamma_k\,d\gamma_l.$$

Changing the variables $\alpha_k/(2\gamma_k) = x,\ \alpha_l/(2\gamma_l) = y$, we get

$$I_1 = \frac{1}{4}c(n,k,l)\alpha_k\alpha_l \int_{\Omega_1'} x^{k-3}(y-x)^{l-k-1}(1-y)^{n-l}y^{-2}dx\,dy.$$

Here the integration domain Ω_1' in the new variables turns out to be dependent on the ratio α_l/α_k. For the case $\alpha_l/\alpha_k < 1$ we have

$$I_1 = \frac{1}{4}c(n,k,l)\alpha_k\alpha_l \int_{\frac{\alpha_k}{2}}^{\frac{1}{2}} (1-y)^{n-l}y^{-2} \int_{\frac{\alpha_k}{2}}^{y} x^{k-3}(y-x)^{l-k-1}dx\,dy, \tag{23}$$

and for the case $\alpha_l/\alpha_k > 1$ -

$$I_1' = \frac{1}{4}c(n,k,l)\alpha_k\alpha_l \int_{\frac{\alpha_l}{2}}^{\frac{1}{2}} (1-y)^{n-l}y^{-2} \int_{\frac{\alpha_k}{2}}^{y} x^{k-3}(y-x)^{l-k-1}dx\,dy. \tag{24}$$

Elementary technical details are omitted. Consider the second integral in (22):

$$I_2 = c(n,k,l)\int_{\Omega_2}\left(\frac{\alpha_k}{2\gamma_k}\right)^{k-1}\left(1-\frac{\gamma_l}{2\alpha_l}-\frac{\alpha_k}{2\gamma_k}\right)^{l-k-1}\left(\frac{\gamma_l}{2\alpha_l}\right)^{n-l}d\gamma_k\,d\gamma_l.$$

Changing the variables $\alpha_k/(2\gamma_k) = x,\ \gamma_l/(2\alpha_l) = y$, we get

$$I_2 = c(n,k,l)\alpha_k\alpha_l \int_0^{\frac{1}{2}} y^{n-l} \int_{\frac{\alpha_k}{2}}^{\frac{1}{2}} x^{k-3}(1-y-x)^{l-k-1}dx\,dy. \tag{25}$$

Here the integration domain in the new variables x, y is a rectangle independent on the ratio α_l/α_k.

In the third integral from (22)

$$I_3 = c(n,k,l)\int_{\Omega_3}\left(1-\frac{\gamma_k}{2\alpha_k}\right)^{k-1}\left(\frac{\gamma_k}{2\alpha_k}-\frac{\gamma_l}{2\alpha_l}\right)^{l-k-1}\left(\frac{\gamma_l}{2\alpha_l}\right)^{n-l}d\gamma_k\,d\gamma_l$$

we change the variables $\gamma_k/(2\alpha_k) = x,\ \gamma_l/(2\alpha_l) = y$ and obtain

$$I_3 = 4c(n,k,l)\alpha_k\alpha_l \int_0^{\frac{1}{2}} (1-x)^{k-1} \int_0^{x} (x-y)^{l-k-1}y^{n-l}dy\,dx. \tag{26}$$

In this case the initial integration domain depends on the ratio α_l/α_k, but the integration domain in the new variables x, y turns out to be the same triangle.

Thus, we can summarize the considerations of this section in the form of the following assertion.

Theorem 10 *nsity $h_{k\,l}(\alpha_k, \alpha_l)$ of the random variables a^k, a^l equals*

$$h_{k\,l}(\alpha_k,\alpha_l) = \begin{cases} I_1 + I_2 + I_3 & \dfrac{\alpha_l}{\alpha_k} < 1 \\ I_1' + I_2 + I_3 & \dfrac{\alpha_l}{\alpha_k} > 1 \end{cases}$$

where I_1, I_1', I_2, I_3 are defined in (23), (24), (25), (26) respectively.

5 The expected value and the variance of $a^1 + \cdots + a^N$

Let again

$$N = n\Big(1 - \frac{t'}{2}\Big),$$

where $t' < t$ (cf. Section 2). We introduce the notation

$$A_N = \sum_{k=1}^{N} a^k.$$

Our next goal is the computation of $\mathbf{E}(A_N)$.

Lemma 11 *The following representation holds*

$$\begin{aligned} \mathbf{E}(A_N) &= \frac{2}{3}n\int_0^{\frac{1}{2}} \sum_{k=0}^{N-1} \binom{n-1}{k} x^{n-k-1}(1-x)^k\,dx + \\ &+ \frac{4}{3}n\int_0^{\frac{1}{2}} y \sum_{k=0}^{N-1} \binom{n-1}{k} (1-y)^{n-k-1} y^k\,dy. \end{aligned} \tag{27}$$

The proof will be given in Appendix 2.

In the sequel we shall use the following lemma which is an analogue of the Law of Great Numbers. Introduce the notation (cf. [5], . VI)

$$B_n = B(m_n + \mu, n-1, 1-x) = \sum_{j=0}^{m_n+\mu} \binom{n-1}{j} (1-x)^j x^{n-j-1}, \tag{28}$$

where μ is an arbitrary fixed number (may be, non-positive). It is understood here that if μ is not an integer then the summation is extended to $\lceil m_n + \mu \rceil$.

Lemma 12 *The following assertions hold:*

1) Let $m_n \geq \lambda n$, where $\lambda > 1 - x$. Then $B_n \to 1$ if $n \to \infty$;

2) Let $m_n < \lambda n$, where $\lambda < 1 - x$. Then $B_n \to 0$ if $n \to \infty$.

P r o o f. The value B_n in (28) is the probability of at most $m_n + \mu$ successes in $n-1$ Bernoulli trials with the success probability $1-x$. Denote by s_{n-1} the random variable equal to the number of successes in $n-1$ Bernoulli trials. Since from 1) we have $1 - x < \lambda$, it is possible to choose an $\varepsilon > 0$ such that

$$1 - x + \varepsilon < \lambda.$$

Then for n sufficiently large

$$B_n \geq \mathbf{P}(s_{n-1} < (n-1)(1-x+\varepsilon)) \geq \mathbf{P}\left(\left|\frac{s_{n-1}}{n-1} - (1-x)\right| < \varepsilon\right).$$

According to the Law at Great Numbers, the last probability tends to 1 when $n \to \infty$. The first assertion is established. The second assertion can be proved similarly.

Further, for $N = n(1 - \frac{t'}{2})$ we have the following assertion

Theorem 13 *The expected value of A_N is*

$$\mathbf{E}(A_N) = n\Big(\frac{1}{2} - \frac{t'}{3}\Big) + o(n). \tag{29}$$

P r o o f. Consider the integral representation (27) for $\mathbf{E}(A_N)$, given in Lemma 11. According to Lemma 12 (for $m_n = N$, $\mu = -1$), the first integrand in (27) for n sufficiently large will tend to 1 for $1 - \frac{t'}{2} > 1 - x$, (i.e., for $x > \frac{t'}{2}$) and will tend to 0 for $x < \frac{t'}{2}$. Now we use Lebesgue's bounded convergence theorem. The integral of the limiting function is

$$\int_{\frac{t'}{2}}^{\frac{1}{2}} 1 dx = \frac{1}{2} - \frac{t'}{2}.$$

Thus, the first summand in the right-hand side of (27) will be

$$\frac{2}{3}n\Big(\frac{1}{2} - \frac{t'}{2}\Big) + o(n) = n\Big(\frac{1}{3} - \frac{t'}{3}\Big) + o(n).$$

The second integrand in (27) by Lemma 12 will be tend to 1 for all $y < 1 - \frac{t'}{2}$, that is, for all y in the integration domain. Therefore the second summand is

$$\frac{4}{3}n\int_0^{\frac{1}{2}} y\, dy + o(n) = \frac{1}{6}n + o(n).$$

Thus, both summands together yield

$$\mathbf{E}(A_N) = n\Big(\frac{1}{2} - \frac{t'}{3}\Big) + o(n),$$

as asserted.

Now we compute the variance $\mathbf{V}(A_N)$ taking again $N = n(1 - \frac{t'}{2})$.

Lemma 14 *The following representation holds*

$$\begin{aligned}
\mathbf{V}(A_N) &= \frac{1}{2}n\int_0^{\frac{1}{2}} \sum_{k=0}^{N-1} \binom{n-1}{k} x^{n-k-1}(1-x)^k dx + \\
&+ 2n\int_0^{\frac{1}{2}} \sum_{k=0}^{N-1} \binom{n-1}{k} (1-y)^{n-k-1} y^k dy + \\
&+ \frac{8}{9}n(n-1)\int_0^1 \sum_{l=2}^{N} \binom{n-2}{l-2} \Big(1 - \frac{\alpha}{2}\Big)^{n-l} \Big(\frac{\alpha}{2}\Big)^{l+1} d\alpha + \\
&+ \frac{2}{3}n(n-1)\int_0^{\frac{1}{2}} \sum_{l=2}^{N} \binom{n-2}{l-2} y^{n-l}(1-y)^{l-2} dy - \\
&- \frac{8}{9}n(n-1)\int_0^1 \sum_{l=2}^{N} \binom{n-2}{l-2} y^{n-l+1}(1-y)^{l-2} dy - \\
&- [\mathbf{E}(A_N)]^2.
\end{aligned} \tag{30}$$

The proof which is quite technical is given in Appendix 2.

Theorem 15 *The following assertion holds*

$$\mathbf{V}(A_N) = o(n^2). \tag{31}$$

The proof is similar to the proof of Theorem 13; it is based on Lemma 12 and Lebesgue's theorem. Consider the representation (30). The first two summands in (30) are processed in the same way as in the proof of Theorem 13. The first summand is

$$\frac{1}{2}n\int_{\frac{t'}{2}}^{\frac{1}{2}} dx + o(n) = \frac{1}{2}n\Big(\frac{1}{2} - \frac{t'}{2}\Big) + o(n),$$

and the second one

$$2n\int_0^{\frac{1}{2}} y^2 dy + o(n) = \frac{1}{4}n + o(n).$$

Thus, the first two summand together yield

$$\frac{1}{2}n\Big(1 - \frac{t'}{2}\Big) + o(n),$$

which is of order $O(n)$.

We proceed similarly. Rewrite the integrand in the third summand of (30) in the form

$$\Big(\frac{\alpha}{2}\Big)^3 \sum_{k=0}^{N-2} \binom{n-2}{k} \Big(1 - \frac{\alpha}{2}\Big)^{n-k-2} \Big(\frac{\alpha}{2}\Big)^k.$$

It will tend to 1 for $1 - \frac{t'}{2} > \frac{\alpha}{2}$, that is for $\alpha < 2 - t'$. Thus, the third summand will be

$$\frac{8}{9}n(n-1)\int_0^1 \Big(\frac{\alpha}{2}\Big)^3 d\alpha + o(n^2) = \frac{1}{36}n(n-1) + o(n^2).$$

The fourth integrand by Lemma 12 will tend to 1 for $1 - \frac{t'}{2} > 1 - y$, i.e. for $y > t'/2$. Therefore the fourth integral equals

$$\frac{2}{3}n(n-1)\int_{\frac{t'}{2}}^{\frac{1}{2}} dy = \frac{1}{3}n(n-1)(1-t') + o(n^2).$$

Rewriting the fifth integrand in the form

$$y\sum_{k=0}^{N-2} \binom{n-2}{k} (1-y)^k y^{n-2-k},$$

we see that it tends to 1 for $y > t'/2$. Therefore the fifth integral is

$$-\frac{8}{9}n(n-1)\int_{\frac{t'}{2}}^{\frac{1}{2}} y dy + o(n^2) = \frac{1}{9}n(n-1)((t')^2 - 1) + o(n^2).$$

The sum of the third, the fourth anf the fifth integrals is then

$$n(n-1)\Big(\frac{1}{36} + \frac{1}{3}(1-t') + \frac{1}{9}((t')^2 - 1)\Big) + o(n^2) = n(n-1)\Big(\frac{1}{2} - \frac{t'}{3}\Big)^2 + o(n^2).$$

Further, from Theorem 10 we have

$$[\mathbf{E}(A_N)]^2 = n^2\Big(\frac{1}{2} - \frac{t'}{3}\Big)^2 + o(n^2).$$

Collecting all summands in (30) yields

$$\mathbf{V}(A_N) = O(n) + n(n-1)\Big(\frac{1}{2} - \frac{t'}{3}\Big)^2 - n^2\Big(\frac{1}{2} - \frac{t'}{3}\Big)^2 + o(n^2) = o(n^2).$$

The theorem is proved.

6 The proof of Condition 2

Let, as earlier, $N = n(1 - \frac{t'}{2})$, $t' < t$. We introduce the random variables

$$Y_n = \frac{a^1 + \ldots + a^N}{n}.$$

Theorem 16 *For any* $\varepsilon > 0$

$$\mathbf{P}(|Y_n - \mathbf{E}(Y_n)| \le \varepsilon) \to 1 \quad for \quad n \to \infty.$$

P r o o f. According to Tchebysheff's inequality,

$$\mathbf{P}(|Y_n - \mathbf{E}(Y_n)| > \varepsilon) \le \frac{\mathbf{V}(Y_n)}{\varepsilon^2}.$$

Furthermore we have

$$\mathbf{V}(Y_n) = \frac{1}{n^2}\mathbf{V}(A_N) = o(1),$$

since, by Theorem 15, $\mathbf{V}(A_N) = o(n^2)$. Therefore

$$\mathbf{P}(|Y_n - \mathbf{E}(Y_n)| > \varepsilon) \to 0 \quad \text{if} \quad n \to \infty,$$

and the complementary probability tends to 1. The theorem is proved.

We proceed now to the proof of Condition 2

$$\sum_{k=N}^{n+1} \mathbf{P}(B_k^n) \underset{n\to\infty}{\longrightarrow} 1.$$

We suppose that $b = b(n)$.

Theorem 17 *If* $b(n) = \lambda n$, *where*

$$\lambda > \frac{1}{2} - \frac{t'}{3} > \frac{1}{2} - \frac{t}{3},$$

then the Condition 2 is satisfied.

P r o o f. We have

$$\sum_{k=N}^{n+1} \mathbf{P}(B_k^n) \ge \sum_{k=N+1}^{n+1} \mathbf{P}(B_k^n) = \mathbf{P}(A_N \le b(n)). \tag{32}$$

We proved earlier (Theorem 13) that

$$\mathbf{E}(A_N) = n\left(\frac{1}{2} - \frac{t'}{3}\right) + o(n).$$

It follows from the conditions of our theorem that

$$\frac{\mathbf{E}(A_N)}{n} + \varepsilon < \lambda$$

or

$$\mathbf{E}(Y_n) + \varepsilon < \lambda.$$

Therefore

$$\begin{aligned}
\mathbf{P}(A_N \le \lambda n) &= \mathbf{P}(a^1 + \cdots + a^N \le \lambda n) = \mathbf{P}\Big(\frac{a^1 + \cdots + a^N}{n} \le \lambda\Big) \ge \\
&\ge \mathbf{P}\Big(\frac{a^1 + \cdots + a^N}{n} - \mathbf{E}(Y_n) \le \varepsilon\Big) \ge \mathbf{P}\Big(-\varepsilon \le \frac{a^1 + \cdots + a^N}{n} - \mathbf{E}(Y_n) \le \varepsilon\Big) = \\
&= \mathbf{P}\Big(\Big|\frac{a^1 + \cdots + a^N}{n} - \mathbf{E}(Y_n)\Big| \le \varepsilon\Big) = \mathbf{P}(|Y_n - \mathbf{E}(Y_n)| \le \varepsilon) \to 1.
\end{aligned}$$

The last equality follows from Theorem 16. Thus, we see from (32) that the Condition 2 is satisfied.

Now the validity of our main theorem is implied by Theorems 9 and 17.

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9 Appendix 1

Completion of the proof of Theorem 9

We stopped at the inequality (19):

$$\sum_{k=N}^{n} \mathbf{P}(\overline{C}_k^n) \le \sum_{j=N}^{n} (j-N+1)\binom{n}{j}\Big(1-\frac{t}{2}\Big)^j\Big(\frac{t}{2}\Big)^j. \tag{33}$$

We estimate the sum in the right-hand side of this inequality. Denote

$$b\Big(j,n,1-\frac{t}{2}\Big) = b_j = \binom{n}{j}\Big(1-\frac{t}{2}\Big)^j\Big(\frac{t}{2}\Big)^{n-j}.$$

The well-known calculation (cf. [5], Ch. VI, n.3) shows that

$$\frac{b_j}{b_{j-1}} = \frac{(n-j+1)(1-\frac{t}{2})}{j\cdot\frac{t}{2}} = 1 + \frac{(n+1)(1-\frac{t}{2})-j}{j\cdot\frac{t}{2}}.$$

It follows that the sequence $\{b_j\}$ decreases for $j > (n+1)(1-\frac{t}{2})$. Therefore for our choice of N the first summand in the right-hand side of (33) will be maximal for n sufficiently large. Indeed, for $N = n(1-\frac{t'}{2})$, $t' < t$ we have

$$n\Big(1-\frac{t'}{2}\Big) > n\Big(1-\frac{t}{2}\Big),$$

which implies

$$n > \Big\lceil \frac{2-t}{t-t'} \Big\rceil.$$

Carrying out in (33) the maximal summand, we get

$$\sum_{k=N}^{n} \mathbf{P}(\overline{C}_k^n) \le \binom{n}{N}\Big(1-\frac{t}{2}\Big)^N\Big(\frac{t}{2}\Big)^{n-N}\Big(1+\sum_{j=N+1}^{n}(j-N+1)\frac{b_j}{b_N}\Big) =$$

$$= \binom{n}{N}\Big(1-\frac{t}{2}\Big)^N\Big(\frac{t}{2}\Big)^{n-N}\Big(1+2\frac{b_{N+1}}{b_N}+3\frac{b_{N+2}}{b_N}+\cdots+(n-N+1)\frac{b_n}{b_N}\Big).$$

We have

$$\begin{aligned} \frac{b_{N+1}}{b_N} &= \frac{(n-N)(1-\frac{t}{2})}{(N+1)\frac{t}{2}} = \alpha, \\ \frac{b_{N+2}}{b_N} &= \frac{(n-N)(n-N-1)(1-\frac{t}{2})^2}{(N+1)(N+2)(\frac{t}{2})^2}. \end{aligned} \tag{34}$$

Replacing $n-N-1$ in the nominator by $n-N$ and $N+2$ in the denominator by $N+1$, we increase the fraction. Thus,

$$\frac{b_{N+2}}{b_N} \le \frac{(n-N)^2(1-\frac{t}{2})^2}{(N+1)^2(\frac{t}{2})^2} = \alpha^2.$$

Further we can proceed inductively. We have

$$\sum_{k=N}^{n} \mathbf{P}(\overline{C}_k^n) \le \binom{n}{N}\left(1-\frac{t}{2}\right)^N \left(\frac{t}{2}\right)^{n-N} (1+2\alpha+3\alpha^2+\cdots+(n-N+1)\alpha^{n-N}),$$

where $\alpha < 1$. Passing to the limit by n, we have to take into account that α, as defined in (34), depends on n. Substituting into (34) $N = n(1-\frac{t'}{2})$, we have

$$\alpha = \frac{(n-n(1-\frac{t'}{2}))(1-\frac{t}{2})}{(n(1-\frac{t'}{2})+1)\frac{t}{2}} = \frac{\frac{t'}{2}(1-\frac{t}{2})}{\frac{t}{2}(1-\frac{t'}{2})+\frac{1}{n}\frac{t}{2}}.$$

Discarding in the denominator the summand $\frac{1}{n}\frac{t}{2}$, we increase the fraction. Thus

$$\alpha < \beta = \frac{\frac{t'}{2}(1-\frac{t}{2})}{\frac{t}{2}(1-\frac{t'}{2})}.$$

We see that $\beta < 1$ (because $t' < t$) and that $\alpha = \alpha(n) \to \beta$ when $n \to \infty$. This yields

$$\sum_{k=N}^{n} \mathbf{P}(\overline{C}_k^n) < \binom{n}{N}\left(1-\frac{t}{2}\right)^N \left(\frac{t}{2}\right)^{n-N} (1+2\beta+3\beta^2+\cdots).$$

Obviously

$$1+2\beta+3\beta^2+\cdots = (\beta+\beta^2+\beta^3+\cdots)' = \left(\frac{\beta}{1-\beta}\right)' = \frac{1}{(1-\beta)^2}.$$

Therefore, passing in the last inequality to the limit, we get

$$\lim_{n\to\infty} \sum_{k=N}^{n} \mathbf{P}(\overline{C}_k^n) \le \lim_{n\to\infty} \binom{n}{N}\left(1-\frac{t}{2}\right)^N \left(\frac{t}{2}\right)^{n-N} \frac{1}{(1-\beta)^2}.$$

Now we see that the right-hand side tends to zero when $n \to \infty$, since the last multiplicand does not depend on n. The theorem is proved.

10 Appendix 2

Proof of Lemma 11

We find first the expected value of a^k. The density $g_k(\alpha_k)$ of the random variable a^k was found in Lemma 8 (cf. (18)). We have

$$\begin{aligned}
\mathbf{E}(a^k) &= \int_0^1 \alpha_k g_k(\alpha_k) d\alpha_k = \\
&= 2n\binom{n-1}{n-k}\int_0^1 \alpha_k^2 \int_0^{\frac{1}{2}} x^{n-k}(1-x)^{k-1} dx\, d\alpha_k + \\
&+ \frac{1}{2}n\binom{n-1}{n-k}\int_0^1 \alpha_k^2 \int_{\frac{\alpha_k}{2}}^{\frac{1}{2}} (1-y)^{n-k} y^{k-3} dy\, d\alpha_k.
\end{aligned} \tag{35}$$

The first integral in the right-hand side of (35) is

$$\frac{2}{3}n\binom{n-1}{n-k}\int_0^{\frac{1}{2}} x^{n-k}(1-x)^{k-1}dx, \tag{36}$$

and the second one will be found by partial integration. Let

$$\int_{\frac{\alpha_k}{2}}^{\frac{1}{2}}(1-y)^{n-k}y^{k-3}dy = u, \quad \alpha_k^2\, d\alpha_k = dv.$$

We have

$$v = \frac{1}{3}\alpha_k^3, \quad du = -\frac{1}{2}\left(1-\frac{\alpha_k}{2}\right)^{n-k}\left(\frac{\alpha_k}{2}\right)^{k-3}d\alpha_k.$$

The second integral in the right-hand side of (35) will be then

$$\frac{1}{2}n\binom{n-1}{n-k}\left[\frac{1}{3}\int_{\frac{\alpha_k}{2}}^{\frac{1}{2}}(1-y)^{n-k}y^{k-3}dy\cdot\alpha_k^3\Big|_0^1 + \frac{1}{6}\int_0^1 \alpha_k^3\left(1-\frac{\alpha_k}{2}\right)^{n-k}\left(\frac{\alpha_k}{2}\right)^{k-3}d\alpha_k\right].$$

The first term in brackets is zero, and the second one, after obvious simplifications (we let $\alpha_k/2 = y$) can be rewritten as

$$\frac{4}{3}n\binom{n-1}{n-k}\int_0^{\frac{1}{2}}(1-y)^{n-k}y^k dy. \tag{37}$$

The expected value of a^k is the sum of (36) and (37), i.e.

$$\mathbf{E}(a^k) = \frac{2}{3}n\binom{n-1}{n-k}\int_0^{\frac{1}{2}} x^{n-k}(1-x)^{k-1}dx + \frac{4}{3}n\binom{n-1}{n-k}\int_0^{\frac{1}{2}}(1-y)^{n-k}y^k dy.$$

For the expected value of $\sum_{k=1}^N a^k$ we have, after elementary transformations,

$$\begin{aligned}\mathbf{E}(\sum_{k=1}^N a^k) &= \frac{2}{3}n\int_0^{\frac{1}{2}}\sum_{k=1}^N\binom{n-1}{n-k}x^{n-k}(1-x)^{k-1}dx + \frac{4}{3}n\int_0^{\frac{1}{2}}\sum_{k=1}^N\binom{n-1}{n-k}(1-y)^{n-k}y^k dy = \\ &= \frac{2}{3}n\int_0^{\frac{1}{2}}\sum_{k=0}^{N-1}\binom{n-1}{k}x^{n-k-1}(1-x)^k dx + \frac{4}{3}n\int_0^{\frac{1}{2}} y\sum_{k=0}^{N-1}\binom{n-1}{k}(1-y)^{n-k-1}y^k dy.\end{aligned}$$

Thus, the representation (27) is proved.

11 Appendix 3

The proof of Lemma 14

We have from the definition

$$\begin{aligned}\mathbf{V}(A_N) &= \mathbf{E}(A_N^2) - (\mathbf{E}(A_N))^2 = \\ &= \sum_{k=1}^N \mathbf{E}((a^k)^2) + 2\sum_{k<l\leq N}\mathbf{E}(a^k a^l) - (\mathbf{E}(A_N))^2.\end{aligned} \tag{38}$$

We calculate the first sum in the right-hand side of (38). The density $g_k(\alpha_k)$ of the random variable a^k is given by Lemma 8 (cf. (18)). We have

$$\begin{aligned}
\mathbf{E}((a^k)^2) &= \int_0^1 \alpha_k^2 g_k(\alpha_k) d\alpha_k = \\
&= 2n\binom{n-1}{n-k}\int_0^1 \alpha_k^3 \int_0^{\frac{1}{2}} x^{n-k}(1-x)^{k-1}dx\, d\alpha_k + \\
&+ \frac{1}{2}n\binom{n-1}{n-k}\int_0^1 \alpha_k^3 \int_{\frac{\alpha_k}{2}}^{\frac{1}{2}} (1-y)^{n-k}y^{k-3}dy\, d\alpha_k. \qquad (39)
\end{aligned}$$

The subsequent calculations are identical to those in the proof of Lemma 11 (cf. Appendix 2). We get

$$\mathbf{E}((a^k)^2) = \frac{1}{2}n\binom{n-1}{n-k}\int_0^{\frac{1}{2}} x^{n-k}(1-x)^{k-1}dx + 2n\binom{n-1}{n-k}\int_0^{\frac{1}{2}} y^{k+1}(1-y)^{n-k}dy. \qquad (40)$$

Summing these expressions, we have

$$\begin{aligned}
\sum_{k=1}^{N}\mathbf{E}((a^k)^2) &= \frac{1}{2}n\int_0^{\frac{1}{2}}\sum_{k=0}^{N-1}\binom{n-1}{k}x^{n-k-1}(1-x)^k dx + \\
&+ 2n\int_0^{\frac{1}{2}} y^2 \sum_{k=0}^{N-1}\binom{n-1}{k}(1-y)^{n-k-1}y^k dy.
\end{aligned}$$

Thus, we have found the first two terms in the representation (30).
Now we proceed to the calculation of the second sum in the right-hand side of (38). Recall, that the joint density $h_{kl}(\alpha_k,\alpha_l)$ of the random variables a^k and a^l equals (Theorem 10)

$$h_{kl}(\alpha_k,\alpha_l) = \begin{cases} I_1 + I_2 + I_3 & \text{for } \alpha_l < \alpha_k \\ I_1' + I_2 + I_3 & \text{for } \alpha_l > \alpha_k \end{cases} \qquad (41)$$

where

$$\begin{aligned}
I_1 &= \frac{1}{4}c(n,k,l)\alpha_k\alpha_l \int_{\frac{\alpha_k}{2}}^{\frac{1}{2}} (1-y)^{n-l}y^{-2}\int_{\frac{\alpha_k}{2}}^{y} x^{k-3}(y-x)^{l-k-1}dy\, dx, \qquad (42)\\
I_1' &= \frac{1}{4}c(n,k,l)\alpha_k\alpha_l \int_{\frac{\alpha_l}{2}}^{\frac{1}{2}} (1-y)^{n-l}y^{-2}\int_{\frac{\alpha_k}{2}}^{y} x^{k-3}(y-x)^{l-k-1}dy\, dx, \qquad (43)\\
I_2 &= c(n,k,l)\alpha_k\alpha_l \int_0^{\frac{1}{2}} y^{n-l}\int_{\frac{\alpha_k}{2}}^{\frac{1}{2}} x^{k-3}(1-y-x)^{l-k-1}dy\, dx, \qquad (44)\\
I_3 &= 4c(n,k,l)\alpha_k\alpha_l \int_0^{\frac{1}{2}} (1-x)^{k-1}\int_0^{x} (x-y)^{l-k-1}y^{n-l}dy\, dx. \qquad (45)
\end{aligned}$$

The coefficient $c(n,k,l)$ equals

$$c(n,k,l) = \frac{n!}{(k-1)!(l-k-1)!(n-l)!}.$$

Using (39), we see that

$$\begin{aligned}
\mathbf{E}(a^k a^l) &= \int_0^1\int_0^{\alpha_k} \alpha_k\alpha_l I_1 d\alpha_l d\alpha_k + \int_0^1\int_{\alpha_k}^1 \alpha_k\alpha_l I_1' d\alpha_l d\alpha_k + \\
&+ \int_0^1\int_0^1 \alpha_k\alpha_l I_2 d\alpha_l d\alpha_k + \int_0^1\int_0^1 \alpha_k\alpha_l I_3 d\alpha_l d\alpha_k. \qquad (46)
\end{aligned}$$

Denote the integrals in the right-hand side of (45) by J_1, J_1', J_2 and J_3 respectively. We write down these integrals explicitly, performing the elementary integration and replacing, for notational convenience, α_k by α and α_l by β. This yields

$$\begin{aligned}
J_1 &= \frac{1}{4}c(n,k,l)\int_0^1\int_0^\alpha \alpha^2\beta^2\int_{\frac{\alpha}{2}}^{\frac{1}{2}}(1-y)^{n-l}y^{-2}\int_{\frac{\alpha}{2}}^{y}x^{k-3}(y-x)^{l-k-1}dy\,dx\,d\beta\,d\alpha = \\
&= \frac{1}{12}c(n,k,l)\int_0^1 \alpha^5\int_{\frac{\alpha}{2}}^{\frac{1}{2}}(1-y)^{n-l}y^{-2}\int_{\frac{\alpha}{2}}^{y}x^{k-3}(y-x)^{l-k-1}dy\,dx\,d\alpha, \\
J_1' &= \frac{1}{4}c(n,k,l)\int_0^1\int_\alpha^1 \alpha^2\beta^2\int_{\frac{\beta}{2}}^{\frac{1}{2}}(1-y)^{n-l}y^{-2}\int_{\frac{\alpha}{2}}^{y}x^{k-3}(y-x)^{l-k-1}dy\,dx\,d\beta\,d\alpha, \\
J_2 &= c(n,k,l)\int_0^1\int_0^1 \alpha^2\beta^2\int_0^{\frac{1}{2}}y^{n-l}\int_{\frac{\alpha}{2}}^{\frac{1}{2}}x^{k-3}(1-y-x)^{l-k-1}dy\,dx\,d\beta\,d\alpha = \\
&= \frac{1}{3}c(n,k,l)\int_0^1 \alpha^2\int_0^{\frac{1}{2}}y^{n-l}\int_{\frac{\alpha}{2}}^{\frac{1}{2}}x^{k-3}(1-y-x)^{l-k-1}dy\,dx\,d\alpha, \\
J_3 &= 4c(n,k,l)\int_0^1\int_0^1 \alpha^2\beta^2\int_0^{\frac{1}{2}}(1-x)^{k-l}\int_0^{x}(x-y)^{l-k-1}y^{n-l}dy\,dx\,d\beta\,d\alpha = \\
&= \frac{4}{9}c(n,k,l)\int_0^{\frac{1}{2}}(1-x)^{k-1}\int_0^{x}(x-y)^{l-k-1}y^{n-l}dy\,dx,
\end{aligned}$$

Taking these representations into account, we can rewrite the second sum in the right-hand side of (38) as

$$\begin{aligned}
&2\sum_{k<l\le N}\mathbf{E}(a^k a^l) = \\
&= 2\Bigg[\frac{1}{12}\int_0^1 \alpha^5\sum_{k<l\le N}c(n,k,l)\int_{\frac{\alpha}{2}}^{\frac{1}{2}}(1-y)^{n-l}y^{-2}\int_{\frac{\alpha}{2}}^{y}x^{k-3}(y-x)^{l-k-1}dy\,dx\,d\alpha + \\
&+ \frac{1}{4}\int_0^1\int_\alpha^1 \alpha^2\beta^2\sum_{k<l\le N}c(n,k,l)\int_{\frac{\beta}{2}}^{\frac{1}{2}}(1-y)^{n-l}y^{-2}\int_{\frac{\alpha}{2}}^{y}x^{k-3}(y-x)^{l-k-1}dy\,dx\,d\alpha\,d\beta + \\
&+ \frac{1}{3}\int_0^1 \alpha^2\int_0^{\frac{1}{2}}\sum_{k<l\le N}c(n,k,l)y^{n-l}\int_{\frac{\alpha}{2}}^{\frac{1}{2}}x^{k-3}(1-y-x)^{l-k-1}dy\,dx\,d\alpha + \\
&+ \frac{4}{9}\sum_{k<l\le N}c(n,k,l)\int_0^{\frac{1}{2}}(1-x)^{k-1}\int_0^{x}(x-y)^{l-k-1}y^{n-l}dy\,dx\Bigg]. \qquad (47)
\end{aligned}$$

Denote the four summands in square brackets in (47) by S_1, S_2, S_3, S_4 respectively. First of all, we have to transform the double sums in (47). An elementary calculation shows that

$$\frac{c(n,k,l)}{\binom{l-2}{k-1}} = n(n-1)\binom{n-2}{l-2}.$$

We have

$$\begin{aligned}
S_1 &= \frac{1}{12}\int_0^1 \alpha^5\sum_{k<l\le N}c(n,k,l)\int_{\frac{\alpha}{2}}^{\frac{1}{2}}(1-y)^{n-l}y^{-2}\int_{\frac{\alpha}{2}}^{y}x^{k-3}(y-x)^{l-k-1}dx\,dy\,d\alpha = \\
&= C\int_0^1 \alpha^5\int_{\frac{\alpha}{2}}^{\frac{1}{2}}\sum_{l=2}^{N}\binom{n-2}{l-2}(1-y)^{n-l}y^{-2}\int_{\frac{\alpha}{2}}^{y}\sum_{k=1}^{l-1}\binom{l-2}{k-1}x^{k-1}(y-x)^{l-k-1}x^{-2}dx\,dy\,d\alpha =
\end{aligned}$$

$$= \quad C\int_0^1 \alpha^5 \int_{\frac{\alpha}{2}}^{\frac{1}{2}} \sum_{l=2}^{N} \binom{n-2}{l-2} (1-y)^{n-l} y^{-2} \int_{\frac{\alpha}{2}}^{y} \sum_{k=0}^{l-2} \binom{l-2}{k} x^k (y-x)^{l-k-2} x^{-2} dx\, dy\, d\alpha,$$

where $C = \frac{1}{12}n(n-1)$. Note that the sum in the inner integral is

$$\sum_{k=0}^{l-2} \binom{l-2}{k} x^k (y-x)^{l-k-2} = (x + (y-x))^{l-2} = y^{l-2}.$$

(the same observation will be used in the sequel). Calculating the remaining integral

$$\int_{\frac{\alpha}{2}}^{y} x^{-2} dx = \frac{2}{\alpha} - \frac{1}{y},$$

we get (taking into account the value of C defined above)

$$\begin{aligned} S_1 &= \frac{1}{6}n(n-1)\int_0^1 \alpha^4 \int_{\frac{\alpha}{2}}^{\frac{1}{2}} \sum_{l=2}^{N} \binom{n-2}{l-2} (1-y)^{n-l} y^{l-4} dy\, d\alpha - \\ &- \frac{1}{12}n(n-1)\int_0^1 \alpha^5 \int_{\frac{\alpha}{2}}^{\frac{1}{2}} \sum_{l=2}^{N} \binom{n-2}{l-2} (1-y)^{n-l} y^{l-5} dy\, d\alpha. \end{aligned}$$

Denote the integrals (without their coefficients) in the last representation of S_1 by S_{11} and S_{12}. Consider S_{11} and compute it by parts letting

$$u = \int_{\frac{\alpha}{2}}^{\frac{1}{2}} \sum_{l=2}^{N} \binom{n-2}{l-2} (1-y)^{n-l} y^{l-4} dy, \quad dv = \alpha^4 d\alpha.$$

Then

$$du = -\frac{1}{2} \sum_{l=2}^{N} \binom{n-2}{l-2} \left(1 - \frac{\alpha}{2}\right)^{n-l} \left(\frac{\alpha}{2}\right)^{l-4} d\alpha, \quad v = \frac{1}{5}\alpha^5.$$

We have

$$\begin{aligned} S_{11} &= \frac{1}{5}\alpha^5 \int_{\frac{\alpha}{2}}^{\frac{1}{2}} \sum_{l=2}^{N} \binom{n-2}{l-2} (1-y)^{n-l} y^{l-4} dy \Big|_0^1 + \\ &+ \frac{1}{10}\int_0^1 \alpha^5 \sum_{l=2}^{N} \binom{n-2}{l-2} \left(1 - \frac{\alpha}{2}\right)^{n-l} \left(\frac{\alpha}{2}\right)^{l-4} d\alpha = \\ &= \frac{16}{5}\int_0^1 \sum_{l=2}^{N} \binom{n-2}{l-2} \left(1 - \frac{\alpha}{2}\right)^{n-l} \left(\frac{\alpha}{2}\right)^{l+1} d\alpha. \end{aligned}$$

Now we apply the partial integration to S_{12}. This yields, in a similar manner

$$S_{12} = \frac{16}{3}\int_0^1 \sum_{l=2}^{N} \binom{n-2}{l-2} \left(1 - \frac{\alpha}{2}\right)^{n-l} \left(\frac{\alpha}{2}\right)^{l+1} d\alpha.$$

(the intermediary calculations are omitted). Thus,

$$\begin{aligned} S_1 &= \frac{1}{6}n(n-1)S_{11} - \frac{1}{12}n(n-1)S_{12} = \\ &= \frac{4}{45}n(n-1)\int_0^1 \sum_{l=2}^{N} \binom{n-2}{l-2} \left(1 - \frac{\alpha}{2}\right)^{n-l} \left(\frac{\alpha}{2}\right)^{l+1} d\alpha. \end{aligned} \tag{48}$$

Consider the second summand S_2 from (46) and apply to it the same transformation. This yields

$$\begin{aligned} S_2 &= \frac{1}{4}\int_0^1 \alpha^2 \int_\alpha^1 \beta^2 \sum_{k<l\le N} c(n,k,l) \int_{\frac{\beta}{2}}^{\frac{1}{2}} (1-y)^{n-l} y^{-2} \int_{\frac{\alpha}{2}}^{y} x^{k-3}(y-x)^{l-k-1} dy\, dx\, d\beta\, d\alpha = \\ &= \frac{1}{4} n(n-1) \int_0^1 \alpha^2 \int_\alpha^1 \beta^2 \int_{\frac{\beta}{2}}^{\frac{1}{2}} \sum_{l=2}^{N} \binom{n-2}{l-2} (1-y)^{n-l} y^{-2} y^{l-2} \Big(\frac{2}{\alpha} - \frac{1}{y}\Big) dy\, d\beta\, d\alpha = \\ &= \frac{1}{2} n(n-1) \int_0^1 \alpha \int_\alpha^1 \beta^2 \int_{\frac{\beta}{2}}^{\frac{1}{2}} \sum_{l=2}^{N} \binom{n-2}{l-2} (1-y)^{n-l} y^{l-4} dy\, d\beta\, d\alpha - \\ &- \frac{1}{4} n(n-1) \int_0^1 \alpha^2 \int_\alpha^1 \beta^2 \int_{\frac{\beta}{2}}^{\frac{1}{2}} \sum_{l=2}^{N} \binom{n-2}{l-2} (1-y)^{n-l} y^{l-5} dy\, d\beta\, d\alpha. \end{aligned}$$

Denote the summands (without their coefficients) in this representation of S_2 by S_{21} and S_{22} respectively. Consider S_{21} and denote its "inner part" (without the integration over α) by S_{21}^0. Compute S_{21}^0 by partial integration letting

$$u = \int_{\frac{\beta}{2}}^{\frac{1}{2}} \sum_{l=2}^{N} \binom{n-2}{l-2} (1-y)^{n-l} y^{l-4} dy, \quad dv = \beta^2 d\beta.$$

Then

$$du = -\frac{1}{2} \sum_{l=2}^{N} \binom{n-2}{l-2} \Big(1-\frac{\beta}{2}\Big)^{n-l} \Big(\frac{\beta}{2}\Big)^{l-4} d\beta, \quad v = \frac{1}{3}\beta^3,$$

and S_{21}^0 will be

$$\begin{aligned} S_{21}^0 &= \frac{1}{3}\beta^3 \int_{\frac{\beta}{2}}^{\frac{1}{2}} \sum_{l=2}^{N} \binom{n-2}{l-2} (1-y)^{n-l} y^{l-4} dy \Big|_\alpha^1 + \frac{1}{6}\int_\alpha^1 \beta^3 \sum_{l=2}^{N} \binom{n-2}{l-2} \Big(1-\frac{\beta}{2}\Big)^{n-l} \Big(\frac{\beta}{2}\Big)^{l-4} d\beta = \\ &= -\frac{1}{3}\alpha^3 \int_{\frac{\alpha}{2}}^{\frac{1}{2}} \sum_{l=2}^{N} \binom{n-2}{l-2} (1-y)^{n-l} y^{l-4} dy + \frac{4}{3}\int_\alpha^1 \sum_{l=2}^{N} \binom{n-2}{l-2} \Big(1-\frac{\beta}{2}\Big)^{n-l} \Big(\frac{\beta}{2}\Big)^{l-1} d\beta. \end{aligned}$$

Now we have

$$\begin{aligned} S_{21} &= \frac{1}{2} n(n-1) \int_0^1 \alpha S_{21}^0 d\alpha = \\ &= -\frac{1}{6} n(n-1) \int_0^1 \alpha^4 \int_{\frac{\alpha}{2}}^{\frac{1}{2}} \sum_{l=2}^{N} \binom{n-2}{l-2} (1-y)^{n-l} y^{l-4} dy\, d\alpha + \\ &+ \frac{2}{3} n(n-1) \int_0^1 \alpha \int_\alpha^1 \sum_{l=2}^{N} \binom{n-2}{l-2} \Big(1-\frac{\beta}{2}\Big)^{n-l} \Big(\frac{\beta}{2}\Big)^{l-1} d\beta\, d\alpha. \end{aligned}$$

Here we apply again the partial integration procedure to both integrals in this representation of S_{21}. We omit the intermediary calculations. The final result is

$$S_{21} = \frac{4}{5} n(n-1) \int_0^1 \sum_{l=2}^{N} \binom{n-2}{l-2} \Big(1-\frac{\alpha}{2}\Big)^{n-l} \Big(\frac{\alpha}{2}\Big)^{l+1} d\alpha. \tag{49}$$

We consider now S_{22} and denote by S_{22}^0 its "inner part" (without the integration over α). The integral

$$S_{22}^0 = \int_\alpha^1 \beta^2 \int_{\frac{\beta}{2}}^{\frac{1}{2}} \sum_{l=2}^{n} \binom{n-2}{l-2} (1-y)^{n-l} y^{l-5} dy\, d\beta$$

will be again found by partial integration. It yields

$$\begin{aligned} S_{22}^0 &= -\frac{1}{3}\alpha^3 \int_{\frac{\alpha}{2}}^{\frac{1}{2}} \sum_{l=2}^{n} \binom{n-2}{l-2} (1-y)^{n-l} y^{l-5} dy + \\ &+ \frac{4}{3}\int_{\alpha}^{1} \sum_{l=2}^{n} \binom{n-2}{l-2} \left(1-\frac{\beta}{2}\right)^{n-l} \left(\frac{\beta}{2}\right)^{l-2} d\beta. \end{aligned}$$

Now we have

$$\begin{aligned} S_{22} &= \int_0^1 \alpha^2 S_{22}^0 d\alpha = \\ &= -\frac{1}{3}\int_0^1 \alpha^5 \int_{\frac{\alpha}{2}}^{\frac{1}{2}} \sum_{l=2}^{n} \binom{n-2}{l-2} (1-y)^{n-l} y^{l-5} dy\, d\alpha + \\ &+ \frac{4}{3}\int_0^1 \alpha^2 \int_{\alpha}^{1} \sum_{l=2}^{n} \binom{n-2}{l-2} \left(1-\frac{\beta}{2}\right)^{n-l} \left(\frac{\beta}{2}\right)^{l-2} d\beta\, d\alpha. \end{aligned}$$

Each of these integrals is found again by partial integration (details are omitted). We get

$$S_{22} = -\frac{4}{9} n(n-1) \int_0^1 \sum_{l=2}^{n} \binom{n-2}{l-2} \left(1-\frac{\alpha}{2}\right)^{n-l} \left(\frac{\alpha}{2}\right)^{l+1} d\alpha. \tag{50}$$

The second summand S_2 in (46) is the sum of (48) and (49)

$$S_2 = \frac{16}{45} n(n-1) \int_0^1 \sum_{l=2}^{n} \binom{n-2}{l-2} \left(1-\frac{\alpha}{2}\right)^{n-l} \left(\frac{\alpha}{2}\right)^{l+1} d\alpha. \tag{51}$$

Consider the third integral in (46). After elementary transformations similar to those we made for S_1, we get

$$\begin{aligned} S_3 &= \frac{1}{3}\int_0^1 \alpha^2 \int_0^{\frac{1}{2}} \sum_{k<l\le N} c(n,k,l) y^{n-l} \int_{\frac{\alpha}{2}}^{\frac{1}{2}} x^{k-3} (1-y-x)^{l-k-1} dy\, dx\, d\alpha = \\ &= \frac{1}{3} n(n-1) \int_0^1 \alpha^2 \int_0^{\frac{1}{2}} \sum_{l=2}^{N} \binom{n-2}{l-2} y^{n-l} (1-y)^{l-2} \int_{\frac{\alpha}{2}}^{\frac{1}{2}} x^{-2} dx\, dy\, d\alpha. \end{aligned}$$

The inner integral equals $(2/\alpha) - 2$. Thus,

$$\begin{aligned} S_3 &= \frac{2}{3} n(n-1) \int_0^1 \alpha \int_0^{\frac{1}{2}} \sum_{l=2}^{N} \binom{n-2}{l-2} y^{n-l} (1-y)^{l-2} dy\, d\alpha - \\ &- \frac{2}{3} n(n-1) \int_0^1 \alpha^2 \int_0^{\frac{1}{2}} \sum_{l=2}^{N} \binom{n-2}{l-2} y^{n-l} (1-y)^{l-2} dy\, d\alpha = \\ &= \frac{1}{9} n(n-1) \int_0^{\frac{1}{2}} \sum_{l=2}^{N} \binom{n-2}{l-2} y^{n-l} (1-y)^{l-2} dy. \end{aligned} \tag{52}$$

Consider the last summand S_4 in (46). Applying a similar transformation, we get

$$S_4 = \frac{4}{9} \sum_{k<l\le N} c(n,k,l) \int_0^{\frac{1}{2}} (1-x)^{k-1} \int_0^{x} (x-y)^{l-k-1} y^{n-l} dy\, dx =$$

$$
\begin{aligned}
&= \frac{4}{9}\sum_{k<l\le N} c(n,k,l)\int_0^{\frac12} y^{n-l}\int_y^{\frac12}(1-x)^{k-1}(x-y)^{l-k-1}dy\,dx = \\
&= \frac{4}{9}n(n-1)\sum_{l=2}^{N}\binom{n-2}{l-2}\int_0^{\frac12} y^{n-l}(1-y)^{l-2}\Big(\frac12 - y\Big)dy = \\
&= \frac{2}{9}n(n-1)\int_0^{\frac12}\sum_{l=2}^{N}\binom{n-2}{l-2}(1-y)^{l-2}y^{n-l}dy - \\
&- \frac{4}{9}n(n-1)\int_0^{\frac12} y\sum_{l=2}^{N}\binom{n-2}{l-2}(1-y)^{l-2}y^{n-l}dy. \qquad (53)
\end{aligned}
$$

Now, using the representations for S_1, S_2, S_3, S_4 given in (48), (51), (52), (53) respectively, we can get the final expression for the second sum in the right-hand side of (38). After obvious simplifications this yields

$$
\begin{aligned}
2\sum_{k<l\le N}\mathbf{E}(a^k a^l) &= 2(S_1+S_2+S_3+S_4) = \\
&= \frac{8}{9}n(n-1)\int_0^1\sum_{l=2}^{N}\binom{n-2}{l-2}\Big(1-\frac{\alpha}{2}\Big)^{n-l}\Big(\frac{\alpha}{2}\Big)^{l+1}d\alpha + \\
&+ \frac{2}{3}n(n-1)\int_0^{\frac12}\sum_{l=2}^{N}\binom{n-2}{l-2}y^{n-l}(1-y)^{l-2}dy - \\
&- \frac{8}{9}n(n-1)\int_0^{\frac12} y\sum_{l=2}^{N}\binom{n-2}{l-2}y^{n-l}(1-y)^{l-2}dy.
\end{aligned}
$$

Thus, we got the third, the fourth and the fifth summands in the representation (30). The proof of Lemma 14 is now complete.