The Cauchy-Riemann equation with support conditions in domains with Levi-degenerate boundaries

D I S S E R T A T I O N

zur Erlangung des akademischen Grades
doctor rerum naturalium
(Dr. rer. nat.)
im Fach Mathematik

eingereicht an der
Mathematisch-Naturwissenschaftlichen Fakultät II
der Humboldt-Universität zu Berlin

von
Judith Brinkschulte
geboren am 22.10.1975 in Bonn

Präsident der Humboldt-Universität zu Berlin:
Prof. Dr. Jürgen Mlynek

Dekan der Mathematisch-Naturwissenschaftlichen Fakultät II:
Prof. Dr. Elmar Kulke

Gutachter:

1. Prof. Dr. Jürgen Leiterer
2. Prof. Dr. Ingo Lieb
3. Prof. Dr. Bo Berndtsson

eingereicht am:	26. November 2001
Tag der mündlichen Prüfung:	19. April 2002

Abstract

In a first part, we consider a domain Ω with Lipschitz boundary, which is relatively compact in an n-dimensional Kähler manifold and satisfies some "$\log \delta$-pseudoconvexity" condition. We show that the Cauchy-Riemann equation with exact support in Ω admits a solution in bidegrees (p, q), $1 < q < n$. Moreover, the range of the Cauchy-Riemann operator acting on smooth $(p, n-1)$-forms with exact support in Ω is closed. Applications are given to the solvability of the tangential Cauchy-Riemann equations for smooth forms and currents for all intermediate bidegrees on boundaries of weakly pseudoconvex domains in Stein manifolds and to the solvability of the tangential Cauchy-Riemann equations for currents on Levi-flat CR manifolds of arbitrary codimension.
In a second part, we study the Cauchy-Riemann equation with zero Cauchy data along a hypersurface with constant signature. Applications to the solvability of the tangential Cauchy-Riemann equations for smooth forms with compact support and currents on the hypersurface are given. We also prove that the Hartogs phenomenon holds in weakly 2-convex-concave hypersurfaces with constant signature of Stein manifolds.

Keywords:
Cauchy-Riemann equation, pseudoconvex domain, extension of CR functions, Levi-degenerate hypersurfaces

Zusammenfassung

In einem ersten Teil betrachten wir ein relativ kompaktes Gebiet Ω einer n-dimensionalen Kähler-Mannigfaltigkeit, mit Lipschitz-Rand, welches eine gewisse “$\log\delta$”-Pseudokonvexität besitzt. Wir zeigen, daß die Cauchy-Riemann Gleichung mit exaktem Träger in Ω für alle Bigrade (p,q) mit $0<q<n-1$ eine Lösung besitzt. Außerdem ist das Bild des Cauchy-Riemann Operators auf glatten $(p,n-1)$-Formen mit exaktem Träger in Ω abgeschlossen. Wir geben Anwendungen für die Lösbarkeit der tangentialen Cauchy-Riemann Gleichungen für glatte Formen und Ströme auf Rändern von schwach pseudokonvexen Gebieten Steinscher Mannigfaltigkeiten und fr die Lösbarkeit der tangentialen Cauchy-Riemann Gleichungen für Ströme auf Levi-flachen CR Mannigfaltigkeiten beliebiger Kodimension.
In einem zweiten Teil untersuchen wir die Cauchy-Riemann Gleichung mit Randbedingung Null entlang einer Hyperfläche mit konstanter Signatur. Wir geben Anwendungen für die Lösbarkeit der tangentialen Cauchy-Riemann Gleichung fr glatte Formen mit kompaktem Träger und für Ströme auf der Hyperfläche. Wir zeigen auch, daß Hartogs-Phänomen in schwach 2-konvex-konkaven Hyperflächen mit konstanter Signatur Steinscher Mannigfaltigkeiten gilt.

Schlagwörter:
Cauchy-Riemann Gleichung, pseudoconvexes Gebiet, Fortsetzbarkeit von CR Funktionen, Levi-degenerierte Hyperflächen

Contents

Introduction

In this thesis, we study the $\overline{\partial}$-problem with exact support in certain domains with Levi-degenerate boundaries. This is the following problem:

Consider a complex manifold X and a relatively compact domain $\Omega \subset\subset X$. Let $f \in \mathcal{C}^\infty_{p,q}(X) \cap \mathrm{Ker}\overline{\partial}$ be a smooth $\overline{\partial}$-closed (p,q)-form on X such that $\mathrm{supp} f \subset \overline{\Omega}$ (in other words, f vanishes to infinite order at the boundary of Ω). We want to find a smooth $(p, q-1)$ form u on X satisfying

$$(*)_{p,q} \quad \begin{cases} \overline{\partial} u = f \\ \mathrm{supp}\, u \subset \overline{\Omega} \end{cases}$$

We will give some positive answers to the problem $(*)_{p,q}$ for two different types of domains.

The first type will be a domain satisfying a certain pseudoconvexity condition, which we call "$\log \delta$-pseudoconvexity". More precisely, let (X, ω) be an n-dimensional Kähler manifold and $\Omega \subset\subset X$ a domain. Let δ be the boundary distance function of Ω with respect to ω. We assume that Ω has Lipschitz boundary and is $\log \delta$-pseudoconvex, that is $\partial\overline{\partial}(-\log \delta + h) \geq C\omega$ for some $C > 0$ and some bounded function h on Ω.

Let $E \longrightarrow X$ be a holomorphic vector bundle and set

$$\mathcal{C}^k_{p,q}(X, \overline{\Omega}, E) = \{f \in \mathcal{C}^k_{p,q}(X, E) \mid \mathrm{supp} f \subset \overline{\Omega}\}, \quad k \in \mathbb{N} \cup \{+\infty\},$$

$$H^{p,q}(X, \overline{\Omega}, E) = \mathcal{C}^\infty_{p,q}(X, \overline{\Omega}, E) \cap \mathrm{Ker}\overline{\partial} / \overline{\partial}(\mathcal{C}^\infty_{p,q-1}(X, \overline{\Omega}, E)).$$

Our result is then the following:

Theorem 1
$H^{p,q}(X, \overline{\Omega}, E) = 0$ for $0 \leq p \leq n,\ 0 \leq q \leq n-1$ and $H^{p,n}(X, \overline{\Omega}, E)$ is separated.

For example, if X is a Stein manifold, then any $\Omega \subset\subset X$, which is locally Stein, satisfies the $\log \delta$-pseudoconvexity condition (see [Ele75]). The same is true if (X, ω) has positive holomorphic bisectional curvature, that is $T^{1,0}X$ is positive in the sense of Griffiths (see [Tak64], [Ele75], [Suz76]).

The case where $\Omega \subset\subset \mathbb{C}^n$ and $\partial\Omega$ is piecewise smooth was settled in [MS99] using some kernel method. On the other hand, if X is compact and

$H^{p,q}(X,E) = H^{p,q-1}(X,E) = 0$, then solving the $\overline{\partial}$-problem with exact support $(*)_{p,q}$ in Ω is equivalent to solving the $\overline{\partial}$-equation with regularity up to the boundary in $X \setminus \overline{\Omega}$ in bidegree $(p, q-1)$. This equation has been discussed in [HI00] under the same assumption on Ω. If Ω has smooth boundary, then Theorem 1 implies that smooth functions (more generally, smooth forms of certain bidegrees) satisfying the tangential Cauchy-Riemann equations on $\partial\Omega$ extend to holomorphic functions in $\overline{\Omega}$. This has been previously proved in [Ohs99].

The proof of Theorem 1 consists essentially of two steps. In the first step, we use L^2 estimates with weights δ^{-N} for large $N \in \mathbb{N}$. More precisely, combining the standard L^2 estimates in the form of [Dem82] with some duality argument, we obtain the following result:

Let f be a $\overline{\partial}$-closed $(0,q)$-form on Ω with values in E, $1 \leq q \leq n-1$. Then there exists a $(0, q-1)$-form u on Ω satisfying $\overline{\partial} u = f$ in the sense of distributions and

$$\int_\Omega |u|^2_\omega \delta^{-N} dV_\omega \leq \int_\Omega |f|^2_\omega \delta^{-N-2} dV_\omega,$$

provided the integral on the right hand side is finite and N is sufficiently large.

Now, since Ω has Lipschitz boundary, the integral on the right hand side will be finite for every N if f vanishes to infinite order at the boundary of Ω. Thus we obtain a solution u which is square integrable with respect to the weight δ^{-N}. It is then natural to ask whether this solution maybe vanishes to some finite order at the boundary. In fact, the second step consists of showing that the minimal L^2 solution satisfies $u \in \mathcal{C}^{s(N)}_{0,q-1}(X, \overline{\Omega}, E)$ with $s(N) \sim \sqrt{N}$.

Finally, if one starts with f vanishing to infinite order at the boundary, then, applying a Mittag-Leffler procedure, one gets a solution u which also vanishes to infinite order at the boundary. Also, the separation statement in the theorem is proved similarly. One in fact shows that the range of $\overline{\partial}$ consists of all (p,n)-forms orthogonal to holomorphic $(n-p, 0)$-forms with polynomial growth at the boundary.

Theorem 1 and its dual version yield information about the tangential Cauchy-Riemann equations on boundaries of smooth weakly pseudoconvex domains (by weakly pseudoconvex we mean that the Levi form of the boundary is semi-positive). We prove the following theorem, which generalizes well known statements in case the boundary is strongly pseudoconvex.

Theorem 2
Let X be an n-dimensional Stein manifold and $\Omega \subset\subset X$ a weakly pseudoconvex domain with smooth boundary M. Then we have $H^{p,q}(M) = H^{p,q}_{cur}(M) = 0$ for $0 \leq p \leq n$, $1 \leq q \leq n-2$. Moreover $H^{p,0}(M), H^{p,0}_{cur}(M), H^{p,n-1}(M)$ and $H^{p,n-1}_{cur}(M)$ are infinite dimensional and, if $n \geq 3$, separated.

Let us also mention that under the hypothesis of Theorem 2, if $\Omega \subset\subset \mathbb{C}^n$, the tangential Cauchy-Riemann equations for smooth forms have been studied in [Ros82]. The dual version of Theorem 1 can also be applied to show the following:

Theorem 3
Let X be an n-dimensional Stein manifold and $\Omega \subset\subset X$ a smooth weakly pseudoconvex domain. Let M be a Levi-flat hypersurface in X, such that M intersects $\partial\Omega$ transversally and $\Omega \setminus M$ has exactly two connected components. Then $H^{p,q}_{cur}(M \cap \Omega) = 0$ for $0 \leq p \leq n$, $1 \leq q \leq n-1$.

By an induction argument, the above result can also be generalized to Levi-flat CR manifolds of arbitrary codimension $k \geq 1$ by taking nice generic intersections of Levi-flat hypersurfaces.

Next, we discuss the $\overline{\partial}$-problem with exact support in some weakly q-convex domains. We consider the following situation:

Let X be an n-dimensional Stein manifold and $\Omega \subset\subset X$ a smooth strictly pseudoconvex domain. Let M be a real hypersurface of class $\mathcal{C}^\infty$ intersecting $\partial\Omega$ transversally such that $\Omega \setminus M$ has exactly two connected components. We suppose that $M = \{\varrho = 0\}$ where ϱ is a $\mathcal{C}^\infty$ function whose Levi form has exactly p^+ positive, p^0 zero and p^- negative eigenvalues on $T^{1,0}_x M$ for each $x \in M$, $p^- + p^0 + p^+ = n-1$. We put $D = \Omega \cap \{\varrho < 0\}$. Let $E \longrightarrow X$ be a holomorphic vector bundle.

Our result is then as follows.

Theorem 4
$H^{p,q}(X, \overline{D}, E) = 0$ for $0 \leq p \leq n$, $q \leq p^0 + p^+$ and $H^{p,p^0+p^++1}(X, \overline{D}, E)$ is separated.

Under the assumption that M is strictly q-convex, the $\overline{\partial}$-equation with vanishing along M has been studied by Andreotti and Hill in order to obtain a Poincaré lemma for the tangential Cauchy-Riemann operator on hypersurfaces. Also, in the setting of strictly q-convex (or concave) domains, the $\overline{\partial}$-equation with exact support has been studied by Sambou in his thesis, where he proves some Dolbeault isomorphism between the tangential Cauchy-Riemann cohomology groups of smooth forms and currents on hypersurfaces (see [Sam99], [Sam01]). Let us also mention that M. Derridj [Der81] has studied the $\overline{\partial}$-equation with exact support and L^2 regularity in certain weakly q-convex domains in $\mathbb{C}^n$.

The proof of Theorem 4 follows the same scheme as the proof of Theorem 1, but this time it is far more difficult to obtain the L^2 estimates. The crucial point is to construct a metric on D which permits to prove L^2 estimates with inverse powers of the boundary distance as weight functions as before for some appropriate bidegrees.

The dual version of Theorem 4 then leads to the following application.

Theorem 5
$H^{p,q}_{cur}(M \cap \Omega) = 0$, $q \geq n - \min(p^-, p^+) - p^0$.

In particular, the above theorem gives a Poincaré lemma for currents on this particular type of hypersurfaces. For smooth forms, the corresponding Poincaré lemma was obtained by V. Michel, who studied the $\overline{\partial}$-equation with regularity up to the boundary in weakly q-convex domains near a point where the number of negative eigenvalues of the Levi form is constant (see [Mic93]).

Theorem 4 also leads to the following theorem.

Theorem 6
Assume that M is a closed connected real hypersurface of a Stein manifold which has signature (p^-, p^0, p^+) at each point. Then $H^{p,q}_c(M) = 0$, $q \leq \min(p^-, p^+) + p^0 - 1$.

Here $H^{p,q}_c(M)$ denote the tangential Cauchy-Riemann cohomology groups of smooth forms with compact support on M. This in turn has the following interesting corollary.

Theorem 7
Assume moreover that $p^- + p^0 \geq 2$, $p^+ + p^0 \geq 2$ and that M is globally minimal. Then the Hartogs phenomenon for CR funtions holds in M.

Note that the assumption of global minimality is necessary only to assure that the weak analytic continuation principle for CR functions holds in M. It is however satisfied as long as $p^+ \neq 0$ or $p^- \neq 0$. An interesting case is e.g. the one of signature $(1,1,1)$.

The Hartogs phenomenon has already been previously discussed on hypersurfaces whose Levi form has at least q positive and q negative eigenvalues everywhere. Indeed, Henkin [Hen84] proved that for $q = 1$, the Hartogs phenomenon holds in sufficiently small open sets. For $q = 2$, it was proved in [LT91] that the Hartogs phenomenon holds globally if M is closed in a Stein manifold. For $q = 1$, however, a counterexample was given in [HN96], which shows that the Hartogs phenomenon fails to hold globally.

This thesis is organized as follows. In Chapter 1, we provide a (nonexhaustive) introduction to L^2 estimates for the $\overline{\partial}$-operator on complex ma-nifolds. In Chaper 2, we study the regularity of the equation $Lu = f$, where L is an elliptic operator on an open set $\Omega \subset \mathbb{R}^n$, whose principle symbol can be controlled by some power of the boundary distance of Ω, and f vanishes to some finite order at the boundary of Ω. This regularity result will provide the desired vanishing at the boundary of the minimal L^2 solutions mentioned in the sketch of Theorem 1. In Chapter 3, we discuss the weakly pseudoconvex case and prove Theorem 1. Theorem 4 is proved in Chapter 4, and in Chapter 5, we give the applications to CR manifolds.

Acknowledgements. This thesis was prepared while I was staying at the Institut Fourier in Grenoble and at the Humboldt-Universität in Berlin. The travels between the two institutions were financed by the European network "Complex analysis and analytic geometry". I would also like to thank the mathematiciens which I met at both institutions for valuable mathematical as well as personal discussions. In particular, I am gratefully indebted to Christine Laurent and Jürgen Leiterer for having supervised and encouraged this work, while at the same time letting me a huge amount of liberty in choosing my research topics, which I appreciated very much.

Chapter 1

L^2 estimates for the $\overline{\partial}$-operator

In this chapter we briefly describe the most important L^2 estimates for the $\overline{\partial}$-operator on holomorphic hermitian vector bundles over complex manifolds. We first recall the most basic definitions of hermitian differential geometry related to the concepts of connection and curvature of a vector bundle. We then state some purely functional analytic theorems before turning to L^2 theory on Riemannian manifolds. We introduce the concept of a complete metric. Proving the fundamental approximation theorem for complete metrics, we explain why it is particularly convenient to work with complete metrics. We then turn to the $\overline{\partial}$-operator on holomorphic vector bundles, stating the Bochner-Kodaira-Nakano identity and Nakano's inequality. At the end of this chapter, we prove the general existence theorem on weakly pseudoconvex manifolds, allowing also non complete metrics and singular weights. There is nothing original in this chapter. Almost everything is shamelessly copied from Demailly's beautiful book [Dem]. All the left-out details and proofs can be found there.

1.1 Hermitian vector bundles

Let X be an n-dimensional complex manifold and let $(z_1, \ldots, z_n)$ be holomorphic local coordinates on some open set $\Omega \subset X$ (we usually think of Ω as being just an open set in $\mathbb{C}^n$). We write $z_j = x_j + iy_j$, $\overline{z}_j = x_j - iy_j$, and $dz_j = dx_j + idy_j$, $d\overline{z}_j = dx_j - idy_j$.

A (p, q)-form on X is a differential form of total degree $p+q$ with complex coefficients, which can be written as

$$u(z) = \sum_{|I|=p,|J|=q} u_{IJ}(z) dz_I \wedge d\overline{z}_J$$

where $I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_q)$ are multiindices (arranged in increasing order) and $dz_I = dz_{i_1} \wedge \ldots \wedge dz_{i_p}$, $d\overline{z}_J = d\overline{z}_{j_1} \wedge \ldots \wedge d\overline{z}_{j_q}$.

We denote by $\Lambda^{p,q}T^*X$ the vector bundle of complex-valued (p,q)-forms over X and by $\mathcal{C}^\infty_{p,q}(X)$ (resp. $\mathcal{C}^k_{p,q}(X)$) the smooth (resp. $\mathcal{C}^k$) sections of $\Lambda^{p,q}T^*X$.

In this setting, the exterior derivative du of the (p,q)-form u is

$$du = \sum_{|I|=p,|J|=q,1\leq k\leq n} \left(\frac{\partial u_{IJ}}{\partial z_k} dz_k + \frac{\partial u_{IJ}}{\partial \overline{z}_k} d\overline{z}_k\right) \wedge dz_I \wedge d\overline{z}_J$$

We may therefore write $du = \partial u + \overline{\partial} u$ with uniquely defined forms ∂u of type $(p+1,q)$ and $\overline{\partial} u$ of type $(p,q+1)$ such that

$$\partial u = \sum_{|I|=p,|J|=q,1\leq k\leq n} \frac{\partial u_{IJ}}{\partial z_k} dz_k \wedge dz_I \wedge d\overline{z}_J,$$

$$\overline{\partial} u = \sum_{|I|=p,|J|=q,1\leq k\leq n} \frac{\partial u_{IJ}}{\partial \overline{z}_k} d\overline{z}_k \wedge dz_I \wedge d\overline{z}_J$$

The operator $\overline{\partial}$ is usually called the Cauchy-Riemann operator and satisfies $\overline{\partial} \circ \overline{\partial} = 0$.

Let E be a $\mathcal{C}^\infty$ vector bundle of rank r over X. We denote by $\mathcal{C}^\infty_{p,q}(X,E)$ the space of $\mathcal{C}^\infty$ sections of the bundle $\Lambda^{p,q}T^*X \otimes E$.

Now let us consider a holomorphic vector bundle $E \longrightarrow X$. By definition, this means that we have a collection of trivializations $E_{|U_j} \simeq U_j \times \mathbb{C}^r$, $r = \mathrm{rank} E$, such that the transition matrices $g_{jk}(z)$ are holomorphic. We consider the complex of E-valued smooth (p,q)-forms. Again, $\mathcal{C}^\infty_{p,q}(X,E)$ possesses a canonical $\overline{\partial}$-operator. Indeed, if u is a smooth (p,q)-section of E represented by forms $u_j \in \mathcal{C}^\infty_{p,q}(U_j, \mathbb{C}^r)$ over the open sets U_j, we have the transition relation $u_j = g_{jk}u_k$; this relation implies $\overline{\partial} u_j = g_{jk}\overline{\partial} u_k$ (since $\overline{\partial} g_{jk} = 0$), hence the collection $(\overline{\partial} u_j)$ defines a unique global $(p,q+1)$-section $\overline{\partial} u$.

Let us recall that a *Riemannian metric* on a (real) differentiable manifold M is a positive definite symmetric form

$$g = \sum_{1\leq j,k\leq n} g_{jk}(x) dx_j \otimes dx_k$$

on the tangent bundle TM, where $(x_1, \dots, x_n)$ are local coordinates for M. We usually assume that the coefficients $g_{jk}(x)$ are smooth. Then, for any tangent vector $\xi = \sum \xi_j \frac{\partial}{\partial x_j} \in T_x M$, one defines its norm with respect to g by

$$|\xi|_g^2 = \sum_{1 \le j,k \le n} g_{jk}(x) \xi_j \xi_k.$$

If M is moreover assumed to be oriented, one defines a corresponding volume element

$$dV_g = \sqrt{\det(g_{jk}(x))}\, dx_1 \wedge \ldots \wedge dx_n$$

whenever $(x_1, \dots, x_n)$ fit with the given orientation. It is easy to check by the jacobian formula that this definition of dV_g is independent of the choice of coordinates.
On any coordinate open set $\Omega \subset M$, we can use the Gram-Schmidt orthogonalization procedure in order to construct an orthonormal frame $(\zeta_1, \dots, \zeta_n)$ for $TM_{|\Omega}$ for the metric g. The dual basis $(\zeta_1^*, \dots, \zeta_n^*)$ defines an orthonormal frame for the dual metric, furthermore, any p-form can be written in a unique way $u = \sum_{|I|=p} u_I \zeta_I^*$. We define the (pointwise) Riemannian norm of u to be $|u|_g^2 = \sum_I |u_I|^2$. In this way, we get a Riemannian metric on $\Lambda^p T^* M$, which is actually independent of the initial choice of the orthonormal frame (ζ_j).

Now, we consider the complex case. Let X be a complex n-dimensional manifold. A *hermitian metric* on X is a positive definite hermitian form of class $\mathcal{C}^\infty$ on TX; in a coordinate system $(z_1, \dots, z_n)$, such a form can be written $h(z) = \sum_{1 \le j,k \le n} h_{jk}(z) dz_j \otimes d\overline{z}_k$, where (h_{jk}) is a positive hermitian matrix with $\mathcal{C}^\infty$ coefficients. Thanks to the hermitian condition $\overline{h_{jk}} = h_{kj}$, our form h can be written as $h = g - i\omega$, where

$$h(\xi, \eta) = \sum_{1 \le j,k \le n} h_{jk}(z) \xi_j \overline{\eta}_k,$$

$$\begin{aligned} g(\xi,\eta) = \operatorname{Re} h(\xi,\eta) &= \frac{1}{2} \sum_{1 \le j,k \le n} (h_{jk}(z)\xi_j \overline{\eta}_k + \overline{h}_{jk}(z)\overline{\xi}_j \eta_k) \\ &= \frac{1}{2} \sum_{1 \le j,k \le n} h_{jk}(z)(\xi_j \overline{\eta}_k + \eta_j \overline{\xi}_k), \end{aligned}$$

$$\omega(\xi,\eta) = -\operatorname{Im} h(\xi,\eta) = \frac{i}{2} \sum_{1 \le j,k \le n} h_{jk}(z)(\xi_j \overline{\eta}_k - \eta_j \overline{\xi}_k), \qquad \text{i.e.}$$

$$\omega = -\mathrm{Im}\, h = \frac{i}{2} \sum_{1\le j,k\le n} h_{jk}(z)dz_j \wedge d\overline{z}_k.$$

By definition, ω is the *fundamental* $(1,1)$*-form* associated with h. Since ω and h are "isomorphic" objects, we usually do not make any difference and will think of hermitian metrics as being positive $(1,1)$-forms. A *hermitian manifold* is a pair (X,ω) where ω is a $\mathcal{C}^\infty$ positive definite $(1,1)$-form on X. Here a $(1,1)$-form $\omega = i\sum \omega_{jk} dz_j \wedge d\overline{z}_k$ is said to be *positive definite*, denoted by > 0, resp. *positive* (≥ 0), if and only if

$$\xi \longmapsto \sum \omega_{jk}\xi_j\overline{\xi}_k$$

is a positive (resp. semi-positive) hermitian form on $\mathbb{C}^n$.

Let E be a complex vector bundle of rank r over a smooth differentiable manifold M. A *connection* D on E is a linear differential operator of order one

$$D : \mathcal{C}^\infty_q(M,E) \longrightarrow \mathcal{C}^\infty_{q+1}(M,E)$$

such that

$$D(f\wedge u) = df \wedge u + (-1)^{\deg f} f\wedge Du$$

for all forms $f \in \mathcal{C}^\infty_p(M)$, $u \in \mathcal{C}^\infty_q(M,E)$. On an open set $\Omega \subset M$, where E admits a trivialization $\theta : E_{|\Omega} \simeq \Omega\times\mathbb{C}^r$, a connection D can be written

$$Du \simeq_\theta du + \Gamma\wedge u$$

where $\Gamma \in \mathcal{C}^\infty_1(\Omega, \mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^r))$ is an arbitrary matrix of 1-forms and d acts componentwise. It is then easy to check that

$$D^2u \simeq_\theta (d\Gamma + \Gamma\wedge\Gamma)\wedge u \text{ on } \Omega.$$

Since D^2 is a globally defined operator, there is a global 2-form

$$\Theta(D) \in \mathcal{C}^\infty_2(M, \mathrm{Hom}(E,E))$$

such that $D^2u = \Theta(D)\wedge u$ for every form u with values in E. $\Theta(D)$ is called the *curvature* of D.

Assume now that E is endowed with a $\mathcal{C}^\infty$ hermitian metric along the fibers and that the isomorphism $E_{|\Omega} \simeq \Omega\times\mathbb{C}^r$ is given by a $\mathcal{C}^\infty$ frame (e_λ). We then have a canonical sesquilinear pairing

$$\begin{array}{ccc} \mathcal{C}^\infty_p(M,E)\times\mathcal{C}^\infty_q(M,E) & \longrightarrow & \mathcal{C}^\infty_{p+q}(M,E) \\ (u,v) & \longmapsto & \{u,v\} \end{array}$$

given by

$$\{u, v\} = \sum_{\lambda,\mu} u_\lambda \wedge \overline{v}_\mu \langle e_\lambda, e_\mu \rangle, \qquad u = \sum u_\lambda \otimes e_\lambda, \quad v = \sum v_\mu \otimes e_\mu.$$

The connection D is said to be *hermitian* if it satisfies the additional property

$$d\{u, v\} = \{Du, v\} + (-1)^{\deg u} \{u, Dv\}.$$

Assuming that (e_λ) is orthonormal, one easily checks that D is hermitian if and only if $\Gamma^* = -\Gamma$. In this case $\Theta(D)^* = -\Theta(D)$, thus

$$i\Theta(D) \in \mathcal{C}_2^\infty(M, \mathrm{Herm}(E, E)).$$

We now concentrate ourselves on the complex analytic case. If $M = X$ is a complex manifold X, every connection D on a complex $\mathcal{C}^\infty$ vector bundle E can be split in a unique way as a sum of a $(1, 0)$ and of a $(0, 1)$-connection, $D = D' + D''$. In a local trivialization θ given by a $\mathcal{C}^\infty$ frame, one can write

$$D'u \simeq_\theta \partial u + \Gamma' \wedge u,$$

$$D''u \simeq_\theta \overline{\partial} u + \Gamma'' \wedge u,$$

with $\Gamma = \Gamma' + \Gamma''$. The connection is hermitian if and only if $\Gamma' = -(\Gamma'')^*$ in any orthonormal frame. Thus there exists a unique hermitian connection D corresponding to a prescribed $(0, 1)$ part D''.

Assume now that the bundle E itself has a *holomorphic* structure. The unique hermitian connection for which D'' is the $\overline{\partial}$-operator as defined before is called the *Chern connection* of E. In this situation, we will write ∂ instead of D'.
In a local holomorphic frame (e_λ) of $E_{|\Omega}$, the metric is given by the hermitian matrix $H = (h_{\lambda\mu})$, $h_{\lambda\mu} = \langle e_\lambda, e_\mu \rangle$. We have

$$\{u, v\} = \sum_{\lambda,\mu} h_{\lambda\mu} u_\lambda \wedge \overline{v}_\mu = {}^\dagger u \wedge H\overline{v},$$

where ${}^\dagger u$ is the transposed matrix of u. Easy computations yield that

$$\Theta(D) \simeq_\theta \overline{\partial}(\overline{H}^{-1} \partial \overline{H}) \qquad \text{on } \Omega. \tag{1.1}$$

In particular, the Chern curvature tensor $\Theta(E) := \Theta(D)$ is such that

$$i\Theta(E) \in \mathcal{C}_{1,1}^\infty(X, \mathrm{Herm}(E, E)).$$

Moreover, it is important to observe that

$$\Theta(E \otimes F) = \Theta(E) \otimes \mathrm{Id}_F + \mathrm{Id}_E \otimes \Theta(F) \qquad \text{and}$$

$$\Theta(E^*) = -{}^t\Theta(E)$$

where ^{t} denotes transposition.

Let $(z_1, \ldots, z_n)$ be holomorphic local coordinates on X and let $(e_\lambda)_{1\le\lambda\le r}$ be an orthonormal frame of E. Writing

$$i\Theta(E) = \sum_{1\le j,k\le n, 1\le\lambda,\mu\le r} c_{jk\lambda\mu} dz_j \wedge d\overline{z}_k \otimes e_\lambda^* \otimes e_\mu,$$

we can identify the curvature tensor to a hermitian form

$$\widetilde{\Theta}(E)(\xi \otimes v, \xi \otimes v) = \sum_{1\le j,k\le n, 1\le\lambda,\mu\le r} c_{jk\lambda\mu}\xi_j\overline{\xi}_k v_\lambda \overline{v}_\mu$$

on $TX \otimes E$. This naturally leads to the following concepts of positivity:

The vector bundle E is said to be *positive in the sense of Griffiths* if $\widetilde{\Theta}(E)(\xi\otimes v, \xi\otimes v) > 0$ for all non zero decomposable tensors $\xi \otimes v \in TX \otimes E$.

The vector bundle E is said to be *positive in the sense of Nakano* if

$$\widetilde{\Theta}(E)(\tau, \tau) = \sum c_{jk\lambda\mu}\tau_{j\lambda}\overline{\tau}_{k\mu} > 0$$

for all non zero tensors $\tau = \sum \tau_{j\lambda}\partial/\partial z_j \otimes e_\lambda \in TX \otimes E$. We then write $E > 0$.

Example. Assume that E is a line bundle. The hermitian matrix $H = (h_{11})$ associated to a trivialization $\theta : E_{|\Omega} \simeq \Omega \times \mathbb{C}$ is simply a positive function which we find convenient to denote by $e^{-\varphi}$, $\varphi \in \mathcal{C}^\infty(\Omega, \mathbb{R})$. In this case, the curvature form $\Theta(E)$ can be identifiend with the $(1,1)$-form $\partial\overline{\partial}\varphi$, and

$$i\Theta(E) = i\partial\overline{\partial}\varphi$$

is a real $(1,1)$-form. E is positive in either the sense of Griffiths or the sense of Nakano if and only if $i\partial\overline{\partial}\varphi > 0$.

1.2 L^2 theory on complete manifolds

A few preliminaries of functional analysis will be needed here. Let $\mathcal{H}_1, \mathcal{H}_2$ be complex Hilbert spaces. We consider a linear operator T defined on a subspace $\operatorname{Dom} T \subset \mathcal{H}_1$ (called the domain of T) into $\mathcal{H}_2$. The operator T is said to be *densely defined* if $\operatorname{Dom} T$ is dense in $\mathcal{H}_1$, and *closed* if its graph

$$\operatorname{Gr} T = \{(x, Tx) \mid x \in \operatorname{Dom} T\}$$

is closed in $\mathcal{H}_1 \times \mathcal{H}_2$.
Assume now that T is closed and densely defined. The adjoint T^* of T (in Von Neumann's sense) is constructed as follows: $\operatorname{Dom} T^*$ is the set of $y \in \mathcal{H}_2$ such that the linear form

$$\operatorname{Dom} T \ni x \longmapsto \langle Tx, y \rangle_2$$

is bounded in the $\mathcal{H}_1$-norm. Since $\operatorname{Dom} T$ is dense, there exists for every y in $\operatorname{Dom} T^*$ a unique element $T^*y \in \mathcal{H}_1$ such that $\langle Tx, y\rangle_2 = \langle x, T^*y\rangle_1$ for all $x \in \operatorname{Dom} T^*$. It is immediate to verify that $\operatorname{Gr} T^* = (\operatorname{Gr}(-T))^\perp$ in $\mathcal{H}_1 \times \mathcal{H}_2$. It follows that T^* is closed and that every pair $(u, v) \in \mathcal{H}_1 \times \mathcal{H}_2$ can be written

$$(u, v) = (x, -Tx) + (T^*y, y), \quad x \in \operatorname{Dom} T,\ y \in \operatorname{Dom} T^*.$$

Take in particular $u = 0$. Then

$$x + T^*y = 0, \quad v = y - Tx = y + TT^*y, \quad \langle v, y\rangle_2 = \|y\|_2^2 + \|T^*y\|_1^2.$$

If $v \in (\operatorname{Dom} T^*)^\perp$ we get $\langle v, y\rangle_2 = 0$, thus $y = 0$ and $v = 0$. Therefore T^* is densely defined and our discussion implies:

Theorem 1.2.1
If $T : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a closed and densely defined operator, then its adjoint T^* is also closed and densely defined and $(T^*)^* = T$. Furthermore, we have the relations $\operatorname{Ker} T^* = (\operatorname{Im} T)^\perp$ and $(\operatorname{Ker} T)^\perp = \overline{\operatorname{Im} T^*}$.

Consider now two closed and densely defined operators T, S:

$$\mathcal{H}_1 \xrightarrow{T} \mathcal{H}_2 \xrightarrow{S} \mathcal{H}_3$$

such that $S \circ T = 0$. The starting point of all L^2 estimates is the following abstract existence theorem.

Theorem 1.2.2
There are orthogonal decompositions

$$\mathcal{H}_2 = (\operatorname{Ker} S \cap \operatorname{Ker} T^*) \oplus \overline{\operatorname{Im} T} \oplus \overline{\operatorname{Im} S^*},$$

$$\mathrm{Ker}\, S = (\mathrm{Ker}\, S \cap \mathrm{Ker}\, T^*) \oplus \overline{\mathrm{Im}\, T}.$$

In order that $\mathrm{Im}\, T = \mathrm{Ker}\, S$*, it suffices that*

$$\|T^* x\|_1^2 + \|Sx\|_3^2 \geq C\|x\|_2^2, \quad \text{for all } x \in \mathrm{Dom}\, S \cap \mathrm{Dom}\, T^*$$

for some constant $C > 0$*. In that case, for every* $v \in \mathcal{H}_2$ *such that* $Sv = 0$*, there exists* $u \in \mathcal{H}_1$ *such that* $Tu = v$ *and*

$$\|u\|_1^2 \leq \frac{1}{C}\|v\|_2^2.$$

In particular

$$\overline{\mathrm{Im}\, T} = \mathrm{Im}\, T = \mathrm{Ker}\, S, \quad \overline{\mathrm{Im}\, S^*} = \mathrm{Im}\, S^* = \mathrm{Ker}\, T^*.$$

Let (M, g) be a Riemannian manifold and let F_1, F_2 be hermitian $\mathcal{C}^\infty$ vector bundles over M. Then we can define the spaces $L^2(M, F_i)$ of square-integrable sections of F_i with respect to the metrics of M and F_i. If $P : \mathcal{C}^\infty(M, F_1) \longrightarrow \mathcal{C}^\infty(M, F_2)$ is a differential operator with smooth coefficients, then P induces a non bounded operator

$$\widetilde{P} : L^2(M, F_1) \longrightarrow L^2(M, F_2),$$

as follows: if $u \in L^2(M, F_1)$, we compute $\widetilde{P}u$ in the sense of distribution theory and we say that $u \in \mathrm{Dom}\widetilde{P}$ if $\widetilde{P}u \in L^2(M, F_2)$. It follows that $\widetilde{P}$ is densely defined, since $\mathrm{Dom}P$ contains the set $\mathcal{D}(M, F_1)$ of compactly supported sections of $\mathcal{C}^\infty(M, F_1)$, which is dense in $L^2(M, F_1)$. Furthermore $\mathrm{Gr}\widetilde{P}$ is closed: if $u_\nu \longrightarrow u$ in $L^2(M, F_1)$ and $\widetilde{P}u_\nu \longrightarrow v$ in $L^2(M, F_2)$, then $\widetilde{P}u_\nu \longrightarrow \widetilde{P}u$ in the weak topology of distributions, thus we must have $\widetilde{P}u = v$ and $(u, v) \in \mathrm{Gr}\widetilde{P}$. By the preceeding general results, we see that $\widetilde{P}$ has a closed and densely defined Von Neumann adjoint $(\widetilde{P})^*$. We want to stress, however, that $(\widetilde{P})^*$ *does not always* coincide with the extension $\widetilde{P^*}$ of the formal adjoint $P^* : \mathcal{C}^\infty(M, F_2) \longrightarrow \mathcal{C}^\infty(M, F_1)$, computed in the sense of distribution theory. In fact $u \in \mathrm{Dom}(\widetilde{P})^*$, resp. $u \in \mathrm{Dom}\widetilde{P^*}$, if and only if there is an element $v \in L^2(M, F_1)$ such that $\langle u, \widetilde{P}f\rangle = \langle v, f\rangle$ for all $f \in \mathrm{Dom}\widetilde{P}$, resp. for all $f \in \mathcal{D}(M, F_1)$. Therefore we always have $\mathrm{Dom}(\widetilde{P})^* \subset \mathrm{Dom}\widetilde{P^*}$ and the inclusion may be strict because the integration by parts to perform may involve boundary integrals for $(\widetilde{P})^*$. This is why we have to introduce the concept of complete metrics.

Let (M, g) be a Riemannian manifold of dimension n, with metric

$$g(x) = \sum g_{jk}(x) dx_j \otimes dx_k.$$

The length of a path $\gamma : [a,b] \longrightarrow M$ is by definition

$$l(\gamma) = \int_a^b |\gamma'(t)|_g dt = \int_a^b (\sum g_{jk}(\gamma(t))\gamma'_j(t)\gamma'_k(t))^{1/2} dt.$$

The geodesic distance of two points $x, y \in M$ is

$$\delta(x,y) = \inf_\gamma l(\gamma) \quad \text{with } \gamma(a) = x,\ \gamma(b) = y,$$

if x, y are in the same connected component of M, $\delta(x,y) = +\infty$ otherwise.

The following standard definitions and properties will be useful in order to deal with the completeness of the metric.

Definitions.

(i) *A riemannian manifold (M,g) is said to be complete if (M,δ) is complete as a metric space.*

(ii) *A continuous function $\psi : M \longrightarrow \mathbb{R}$ is said to be exhaustive if for every $c \in \mathbb{R}$ the sublevel set $M_c = \{x \in M \mid \psi(x) < c\}$ is relatively compact in M.*

(iii) *A sequence $(K_\nu)_{\nu\in\mathbb{N}}$ of compact subsets of M is said to be exhaustive if $M = \cup_\nu K_\nu$ and if K_ν is contained in the interior of $K_{\nu+1}$ for all ν.*

Lemma 1.2.3
The following properties are equivalent:

(i) *(M,g) is complete;*

(ii) *there exists an exhaustive function $\psi \in \mathcal{C}^\infty(M,\mathbb{R})$ such that $|d\psi|_g \leq 1$;*

(iii) *there exists an exhaustive sequence $(K_\nu)_{\nu\in\mathbb{N}}$ of compact subsets of M and functions $\psi_\nu \in \mathcal{C}^\infty(M,\mathbb{R})$ such that*

$$\psi_\nu = 1 \text{ in a neighborhood of } K_\nu, \quad \operatorname{supp}\psi_\nu \subset \overset{\circ}{K}_{\nu+1},$$

$$0 \leq \psi_\nu \leq 1 \text{ and } |d\psi_\nu|_g \leq 2^{-\nu}.$$

Let $E \longrightarrow M$ be a differentiable hermitian vector bundle. Let us consider the Hilbert space $L^2_p(M,E)$ of p-forms u on M with values in E, having measurable coefficients, such that

$$\|u\|^2 = \int_M |u|^2 dV < +\infty.$$

We denote by $\ll\ ,\ \gg$ the global inner product on L^2-forms. Let D be a hermitian connection on E. We denote by δ the formal adjoint of D and put $\triangle = D\delta + \delta D$. Extended in the sense of distribution theory, these operators are thus closed and densely defined operators on $L^2_\bullet(M,E) = \bigoplus_p L^2_p(M,E)$. We also introduce the spaces $\mathcal{D}^p(M,E)$ of compactly supported forms in $\mathcal{C}^\infty_p(M,E)$. The theory relies heavily on the following important result.

Theorem 1.2.4
Assume that (M,g) is complete. Then

(i) $\mathcal{D}_\bullet(M,E)$ is dense in $\mathrm{Dom}D$, $\mathrm{Dom}\delta$ and $\mathrm{Dom}D \cap \mathrm{Dom}\delta$ respectively for the graph norms

$$u \mapsto \|u\| + \|Du\|, \quad u \mapsto \|u\| + \|\delta u\|, \quad u \mapsto \|u\| + \|Du\| + \|\delta u\|.$$

(ii) $D^* = \delta$, $\delta^* = D$ as adjoint operators in Von Neumann's sense.

(iii) One has $\langle\langle u, \triangle u\rangle\rangle = \|Du\|^2 + \|\delta u\|^2$ for every $u \in \mathrm{Dom}\triangle$. In particular

$$\mathrm{Dom}\triangle \subset \mathrm{Dom}D \cap \mathrm{Dom}\delta, \quad \mathrm{Ker}\triangle = \mathrm{Ker}D \cap \mathrm{Ker}\delta,$$

and $\triangle$ is self-adjoint.

(iv) If $D^2 = 0$, there are orthogonal decompositions

$$L^2_\bullet(M,E) = \mathcal{H}^\bullet(M,E) \oplus \overline{\mathrm{Im}D} \oplus \overline{\mathrm{Im}\delta},$$

$$\mathrm{Ker}D = \mathcal{H}^\bullet(M,E) \oplus \overline{\mathrm{Im}D},$$

where $\mathcal{H}^\bullet(M,E) = \{u \in L^2_\bullet(M,E) \mid \triangle u = 0\} \subset \mathcal{C}^\infty_\bullet(M,E)$ is the space of L^2 harmonic forms.

Sketch of the proof. (i) We show that every element $u \in \mathrm{Dom}D$ can be approximated in the graph norm of D by smooth and compactly supported forms. By hypothesis, u and Du belong to $L^2_\bullet(M,E)$. Let (ψ_ν) be a sequence of functions as in Lemma 1.2.3 (iii). Then $\psi_\nu u \longrightarrow u$ in $L^2_\bullet(M,E)$ and $D(\psi_\nu u) = \psi_\nu Du + d\psi_\nu \wedge u$ where

$$|d\psi_\nu \wedge u| \le |d\psi_\nu|\ |u| \le 2^{-\nu}|u|.$$

Therefore $d\psi_\nu \wedge u \longrightarrow 0$ and $D(\psi_\nu u) \longrightarrow Du$. After replacing u by $\psi_\nu u$, we may therefore assume that u has compact support, and by using a finite partition of unity on a neighborhood of supp u, we may also assume that supp u is contained in a coordinate chart of M on which E is trivial. Let Γ be the connection form of D on this chart and (ρ_ε) a family of smoothing kernels. Then $u * \rho_\varepsilon \in \mathcal{D}_\bullet(M, E)$ converges to u in $L^2_\bullet(M, E)$ and

$$D(u * \rho_\varepsilon) - (Du) * \rho_\varepsilon = \Gamma \wedge (u * \rho_\varepsilon) - (\Gamma \wedge u) * \rho_\varepsilon$$

because d commutes with convolutions (as any differential operators with constant coefficients). Moreover $(Du) * \rho_\varepsilon$ converges to Du in $L^2_\bullet(M, E)$ and $\Gamma \wedge (u * \rho_\varepsilon)$, $(\Gamma \wedge u) * \rho_\varepsilon$ both converge to $\Gamma \wedge u$ since $\Gamma \wedge \bullet$ acts continuously on L^2. Thus $D(u * \rho_\varepsilon)$ converges to Du and the density of $\mathcal{D}_\bullet(M, E)$ in Dom$D$ follows. The proof for Domδ and Dom$D \cap$ Domδ is similar, except that the principal part of δ no longer has constant coefficients in general. The convolution technique requires in this case a lemma due to K.O. Friedrichs (see e.g. [Dem]), which we omit here.

The assertion (ii) is equivalent to the fact that

$$\langle\!\langle Du, v\rangle\!\rangle = \langle\!\langle u, \delta v\rangle\!\rangle, \qquad \forall u \in \mathrm{Dom}D,\ \forall v \in \mathrm{Dom}\delta.$$

By (i), we can find $u_\nu, v_\nu \in \mathcal{D}_\bullet(M, E)$ such that

$$u_\nu \to u, \quad v_\nu \to v, \quad Du_\nu \to Du, \quad \text{and } \delta v_\nu \to \delta v \text{ in } L^2_\bullet(M, E),$$

and the required equality is the limit of the equalities
$\langle\!\langle Du_\nu, v_\nu\rangle\!\rangle = \langle\!\langle u_\nu, \delta v_\nu\rangle\!\rangle$.

We skip the proof of (iii) and remark that (iv) is an immediate consequence of (ii), (iii) and Theorem 1.2.2. □

On a complete hermitian manifold (X, ω), there are of course similar results for the operators D', D'', δ', δ'', $\Box'$, $\Box''$ attached to a hermitian vector bundle E.

1.3 General estimates for $\overline{\partial}$

Let (X, ω) be a hermitian manifold, and let E be a hermitian holomorphic vector bundle over X. We denote by $D = \partial + \overline{\partial}$ its Chern connection (or D_E if we want to specify the bundle), and by $\delta = \partial^* + \overline{\partial}^*$ the formal adjoint operator of D. Another important operator is the operator L of type $(1,1)$ defined by

$$Lu = \omega \wedge u$$

and its adjoint Λ:

$$\langle\langle u, \Lambda v\rangle\rangle = \langle\langle Lu, v\rangle\rangle.$$

If A, B are endomorphisms of $\mathcal{C}^\infty_{\bullet,\bullet}(X, E)$, their graded commutator is defined by

$$[A, B] = AB - (-1)^{ab}BA$$

where a, b are the degrees of A and B respectively.

We can now state the fundamental Bochner-Kodaira-Nakano identity, which is the basis of all L^2 vanishing theorems for hermitian holomorphic vector bundles. It expresses the antiholomorphic Laplace operator $\Box'' = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ in terms of its conjugate $\Box' = \partial\partial^* + \partial^*\partial$, plus some extra term involving the curvature of E and the torsion of the metric ω.

Theorem 1.3.1

$$\Box'' = \Box' + [i\Theta(E), \Lambda] + [\partial, \tau^*] - [\overline{\partial}, \overline{\tau}^*],$$

where τ is the operator of type $(1, 0)$ defined by $\tau = [\Lambda, \partial\omega]$ on $\mathcal{C}^\infty_{\bullet,\bullet}(X, E)$.

For a large class of manifolds, called Kähler manifolds, the above identity has a much simpler form, expressing $\Box'' - \Box'$ as an operator of order 0 closely related to the curvature of E.

Definition. ω is a *Kähler metric* if $\partial\omega = 0$.
(X, ω) is said to be a *Kähler manifold* if ω is a Kähler metric.

Corollary 1.3.2
If ω is a Kähler metric, then

$$\Box'' = \Box' + [i\Theta(E), \Lambda].$$

Now assume that ω is a *complete* hermitian metric. Then for every form $u \in \mathrm{Dom}\overline{\partial} \cap \mathrm{Dom}\overline{\partial}^*$ of bidegree (p, q) we have the following a priori inequality

$$\frac{3}{2}(\|\overline{\partial}u\|^2 + \|\overline{\partial}^*u\|^2) \geq \langle\langle[i\Theta(E), \Lambda]u, u\rangle\rangle - \frac{1}{2}(\|\tau u\|^2 + \|\tau^*u\|^2 + \|\overline{\tau}u\|^2 + \|\overline{\tau}^*u\|^2), \tag{1.2}$$

provided the integrals on the right hand side are finite. This inequality is known as Nakano's inequality.

Indeed, for every $u \in \mathcal{D}^{p,q}(X,E)$, since $\langle\langle \Box'' u, u\rangle\rangle = \|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2$ and $\langle\langle \Box' u, u\rangle\rangle = \|\partial u\|^2 + \|\partial^* u\|^2$, we get from Theorem 1.3.1

$$\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2 =$$
$$\|\partial u\|^2 + \|\partial^* u\|^2 + \langle\langle [i\Theta(E), \Lambda] u, u\rangle\rangle + \langle\langle [\partial, \tau^*] u, u\rangle\rangle - \langle\langle [\overline{\partial}, \overline{\tau}^*] u, u\rangle\rangle.$$

Moreover, we have

$$\begin{aligned}
\langle\langle [\partial, \tau^*] u, u\rangle\rangle &= \langle\langle \partial \tau^* u, u\rangle\rangle - \langle\langle \tau^* \partial u, u\rangle\rangle \\
&= \langle\langle \tau^* u, \partial^* u\rangle\rangle - \langle\langle \partial u, \tau u\rangle\rangle \\
&\geq -\frac{1}{2}(\|\tau^* u\|^2 + \|\partial^* u\|^2 + \|\partial u\|^2 + \|\tau u\|^2).
\end{aligned}$$

Analogously, we find

$$\langle\langle [\overline{\partial}, \overline{\tau}^*] u, u\rangle\rangle \geq -\frac{1}{2}(\|\overline{\tau}^* u\|^2 + \|\overline{\partial}^* u\|^2 + \|\overline{\partial} u\|^2 + \|\overline{\tau} u\|^2),$$

thus establishing (1.2) for all $u \in \mathcal{D}^{p,q}(X,E)$. This result is easily extended to every $u \in \mathrm{Dom}\overline{\partial} \cap \mathrm{Dom}\overline{\partial}^*$ by density of $\mathcal{D}^{p,q}(X,E)$ in virtue of Theorem 1.2.4 (i).

In virtue of the general Theorem 1.2.2, Nakano's inequality yields a vanishing theorem for the $\overline{\partial}$-cohomology if, for some bidegree (p,q), the right hand side of (1.2) can be made $\geq C\|u\|^2$ for some $C > 0$.

We would also like to mention that there are far more precise inequalities than (1.2) (see e.g. [Dem86]). However, since we will use exactly this inequality in Chapter 4, we content us with this statement.

In the case where ω is a Kähler metric, the same reasoning as above yields of course

$$\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2 \geq \langle\langle [i\Theta(E), \Lambda] u, u\rangle\rangle \tag{1.3}$$

for every $u \in \mathrm{Dom}\overline{\partial} \cap \mathrm{Dom}\overline{\partial}^*$ of bidegree (p,q) if $[i\Theta(E), \Lambda]$ acting on $\Lambda^{p,q} T^* X \otimes E$ is semi-positive.

1.4 $\overline{\partial}$ on weakly pseudoconvex manifolds

Let (X,ω) be a Kähler manifold and $E \longrightarrow X$ a hermitian holomorphic vector bundle. Then the operator

$$A^{p,q}_{E,\omega} = [i\Theta(E), \Lambda]$$

acting on $\Lambda^{p,q}T^*X \otimes E$ is of fundamental importance, as shown by the following existence theorem, which is the basic result of L^2 theory on Kähler manifolds.

Theorem 1.4.1
Let $E \longrightarrow X$ be a hermitian holomorphic vector bundle over a complete Kähler manifold (X, ω). Suppose $A^{p,q}_{E,\omega}$ is a positive hermitian operator, and let $f \in L^2_{p,q}(X, E)$ satisfy $\overline{\partial} f = 0$ and

$$\int_X \langle (A^{p,q}_{E,\omega})^{-1} f, f\rangle dV_\omega < +\infty,$$

$q \geq 1$. Then there exists $u \in L^2_{p,q-1}(X, E)$ such that $\overline{\partial} u = f$ and

$$\int_X |u|^2 dV_\omega \leq \int_X \langle (A^{p,q}_{E,\omega})^{-1} f, f\rangle dV_\omega.$$

We include a proof of this theorem, since we have not given a proof of Theorem 1.2.2, which is basically the same.

Proof. Consider the Hilbert space orthogonal decomposition

$$L^2_{p,q}(X, E) = \mathrm{Ker}\overline{\partial} \oplus (\mathrm{Ker}\overline{\partial})^\perp,$$

observing that $\mathrm{Ker}\overline{\partial}$ is weakly (hence strongly) closed. Let $v = v_1 + v_2$ be the decomposition of a smooth form $v \in \mathcal{D}^{p,q}(X, E)$ with compact support according to this decomposition (v_1, v_2 do not have compact support in general!). Since $(\mathrm{Ker}\overline{\partial})^\perp = \overline{\mathrm{Im}\overline{\partial}^*} \subset \mathrm{Ker}\overline{\partial}^*$ and $f, v_1 \in \mathrm{Ker}\overline{\partial}$ by hypothesis, we get $\overline{\partial}^* v_2 = 0$ and

$$|\langle f, v\rangle|^2 = |\langle f, v_1\rangle|^2 \leq \int_X \langle (A^{p,q}_{E,\omega})^{-1} f, f\rangle dV_\omega \int_X \langle A^{p,q}_{E,\omega} v_1, v_1\rangle dV_\omega$$

thanks to the Cauchy-Schwarz inequality. The a priori inequality (1.3) applied to $u = v_1$ yields

$$\int_X \langle A^{p,q}_{E,\omega} v_1, v_1\rangle dV_\omega \leq \|\overline{\partial} v_1\|^2 + \|\overline{\partial}^* v_1\|^2 = \|\overline{\partial}^* v_1\|^2 = \|\overline{\partial}^* v\|^2.$$

Combining both inequalities, we find

$$|\langle f, v\rangle|^2 \leq (\int_X \langle (A^{p,q}_{E,\omega})^{-1} f, f\rangle dV_\omega) \|\overline{\partial}^* v\|^2$$

for every smooth (p,q)-form v with compact support. This shows that we have a well defined linear form

$$w = \overline{\partial}^* v \longmapsto \langle v, f\rangle, \quad L^2_{p,q-1}(X,E) \supset \overline{\partial}^*(\mathcal{D}^{p,q}(X,E)) \longmapsto \mathbb{C}$$

on the range of $\overline{\partial}^*$. This linear form is continuous in L^2 norm and has norm $\leq C$ with

$$C = \Big(\int_X \langle (A^{p,q}_{E,\omega})^{-1} f, f\rangle dV_\omega\Big)^{1/2}.$$

By the Hahn-Banach theorem, there is an element $u \in L^2_{p,q-1}(X,E)$ with $\|u\| \leq C$, such that $\langle\langle v, f\rangle\rangle = \langle\langle \overline{\partial}^* v, u\rangle\rangle$ for every v, hence $\overline{\partial} u = f$ in the sense of distributions. The inequality $\|u\| \leq C$ is equivalent to the last estimate in the theorem. □

Remark. One can always find a solution $u \in (\mathrm{Ker}\overline{\partial})^\perp$: otherwise replace u by its orthogonal projection on $(\mathrm{Ker}\overline{\partial})^\perp$. This solution is clearly unique and is precisely the solution of minimal L^2 norm of the equation $\overline{\partial} u = f$. We have $u \in \overline{\mathrm{Im}\overline{\partial}^*}$, thus u satisfies the additional equation

$$\overline{\partial}^* u = 0.$$

Consequently $\Box'' u = \overline{\partial}^* \overline{\partial} u = \overline{\partial}^* f$. If $f \in \mathcal{C}^\infty_{p,q}(X,E)$, the ellipticity of $\Box''$ shows that $u \in \mathcal{C}^\infty_{p,q-1}(X,E)$.

With Theorem 1.4.1 in mind, it is important to compute the term

$$A^{p,q}_{E,\omega} = [i\Theta(E), \Lambda].$$

In particular, we want to know when it is ≥ 0. Unfortunately, this operator can be quite complicated in general. It turns out, however, that $A^{n,q}_{E,\omega}$ is positive under the assumption that E is positive in the sense of Nakano. Moreover, if E is a line bundle and $\lambda_1 \leq \ldots \leq \lambda_n$ are the eigenvalues of $i\Theta(E)$ with respect to ω, we have

$$\langle [i\Theta(E), \Lambda] u, u\rangle \geq (\lambda_1 + \ldots + \lambda_q)|u|^2$$

if u is of bidegree (n,q).

We now introduce a large class of complex manifolds on which the L^2 estimates will be easily tractable.

Definition. A complex manifold is said to be weakly (resp. strongly) pseudoconvex if there exists an exhaustion function $\psi \in \mathcal{C}^\infty(X, \mathbb{R})$ such that $i\partial\overline{\partial}\psi \geq 0$ (resp. > 0) on X, i.e. ψ is plurisubharmonic (resp. strictly plurisubharmonic). A strongly pseudoconvex manifold is also called a Stein manifold.

Lemma 1.4.2
Every weakly pseudoconvex Kähler manifold (X, ω) carries a complete Kähler metric $\widehat{\omega}$.

Proof. Let $\psi \in \mathcal{C}^\infty(X, \mathbb{R})$ be an exhaustion function which is plurisubharmonic on X. After adding a constant to ψ, we can assume $\psi \geq 0$. Then $\widehat{\omega} = \omega + i\partial\overline{\partial}(\psi^2)$ is a Kähler metric and

$$\widehat{\omega} = \omega + 2i\psi\partial\overline{\partial}\psi + 2i\partial\psi \wedge \overline{\partial}\psi \geq \omega + 2i\partial\psi \wedge \overline{\partial}\psi.$$

Since $d\psi = \partial\psi + \overline{\partial}\psi$, we get $|d\psi|_{\widehat{\omega}} = \sqrt{2}|\partial\psi|_{\widehat{\omega}} \leq 1$ and Lemma 1.2.3 shows that $\widehat{\omega}$ is complete. □

If we apply the main L^2 existence theorem (Theorem 1.4.1) to a sequence ω_ε of complete Kähler metrics, we see, by passing to the limit, that the theorem even applies to non necessarily complete metrics if our manifold is pseudoconvex. Precisely, we have the following result:

Theorem 1.4.3
Let (X, ω) be a Kähler manifold (ω is not assumed to be complete). Assume that X is weakly pseudoconvex. Let E be a hermitian holomorphic vector bundle over X and assume that there exists a positive continuous function $\gamma : X \longrightarrow \mathbb{R}$ such that

$$i\Theta(E) \geq \gamma\omega \otimes \mathrm{Id}_E.$$

Then for any (n, q)-form f with L^2_{loc} coefficients, $q \geq 1$, such that $\overline{\partial}f = 0$ and

$$\int_X \gamma^{-1}|f|^2 dV_\omega < +\infty,$$

there exists $u \in L^2_{n,q-1}(X, E)$ such that $\overline{\partial}u = f$ and

$$\int_X |u|^2 dV_\omega \leq \frac{1}{q}\int_X \gamma^{-1}|f|^2 dV_\omega.$$

Proof. Indeed, under the assumption on E, we have

$$\langle A^{n,q}_{E,\omega} u, u\rangle \geq q\gamma |u|^2,$$

hence $\langle (A^{n,q}_{E,\omega})^{-1} u, u\rangle \leq \frac{1}{q}\gamma^{-1}|u|^2$. The assumption that f is only L^2_{loc} instead of $f \in L^2_{n,q}(X, E)$ is not a real problem, since we may restrict ourselves to $X_c = \{x \in X \mid \psi(x) < c\} \subset\subset X$, where ψ is a plurisubharmonic exhaustion function on X. Then X_c is itself weakly pseudoconvex (with exhaustion function $\psi_c = 1/(c-\psi)$), hence X_c can be equipped with a complete Kähler metric $\omega_{c,\varepsilon} = \omega + \varepsilon i\partial\overline{\partial}(\psi_c^2)$ (cf the proof of Lemma 1.4.2). For each (c, ε), Theorem 1.4.1 yields a solution $u_{c,\varepsilon} \in L^2_{\omega_\varepsilon}(X_c, \Lambda^{n,q-1}T^*X \otimes E)$ of the equation $\overline{\partial} u_{c,\varepsilon} = f$ on X_c such that

$$\int_{X_c} |u_{c,\varepsilon}|^2_{\omega_{c,\varepsilon}} dV_{\omega_{c,\varepsilon}} \leq \int_{X_c} \langle (A^{n,q}_{E,\omega_{c,\varepsilon}})^{-1} f, f\rangle dV_{\omega_{c,\varepsilon}}$$

A simple computation shows that the integral on the right hand side is monotonically decreasing with respect to the metric, hence

$$\begin{aligned}\int_{X_c} \langle (A^{n,q}_{E,\omega_{c,\varepsilon}})^{-1} f, f\rangle dV_{\omega_{c,\varepsilon}} &\leq \int_{X_c} \langle (A^{n,q}_{E,\omega})^{-1} f, f\rangle dV_\omega \\ &\leq \int_X \frac{1}{q}\gamma^{-1}|f|^2 dV_\omega.\end{aligned}$$

Therefore the solutions $u_{c,\varepsilon}$ are uniformly bounded in L^2 norm on every compact subset of X. Since the closed unit ball of an Hilbert space is weakly compact (and metrizable if the Hilbert space is separable), we can extract a subsequence

$$u_{c_k,\varepsilon_k} \longrightarrow u \in L^2_{\mathrm{loc}}$$

converging weakly in L^2 on any compact subset $K \subset X$, for some $c_k \to +\infty$ and $\varepsilon_k \to 0$. By the weak continuity of differentiations, we get again in the limit $\overline{\partial} u = f$. Also, for every compact set $K \subset W$, we get

$$\int_K |u|^2_\omega dV_\omega \leq \liminf_{k\to\,+\infty} \int_K |u_{c_k,\varepsilon_k}|^2_{\omega_{c_k,\varepsilon_k}} dV_{\omega_{c_k,\varepsilon_k}}$$

by weak L^2_{loc} convergence. Finally, we let K increase to X and conclude that the desired estimate holds on all of X. □

An important observation is that the above theorem still applies when the hermitian metric on E is a singular metric with positive curvature in the sense of currents.

Theorem 1.4.4
Let (X, ω) be a Kähler manifold. Assume that X is weakly pseudoconvex. Let E be a hermitian holomorphic vector bundle and let $\varphi \in L^1_{\mathrm{loc}}$ be a weight function (no further regularity assumption is made on φ). Suppose that

$$i\Theta(E) + i\partial\overline{\partial}\varphi \otimes \mathrm{Id}_E \geq \gamma\omega \otimes \mathrm{Id}_E$$

for some continuous positive function γ on X. Then for any (n, q)-form f with L^2_{loc} coefficients, $q \geq 1$, satisfying $\overline{\partial} f = 0$ and $\int_X \gamma^{-1}|f|^2 dV_\omega < +\infty$, there exists $u \in L^2_{n,q-1}(X, E)$ such that $\overline{\partial} u = f$ and

$$\int_X |u|^2 e^{-\varphi} dV_\omega \leq \frac{1}{q} \int_X \gamma^{-1}|f|^2 e^{-\varphi} dV_\omega.$$

Proof (Sketch). The general proof is based on regularization techniques for plurisubharmonic function (see e.g. [Dem82]). It is technically involved essentially because the required regularization techniques are difficult in the case of arbitrary manifolds. We will therefore just explain the proof in the simple case when $X = \Omega$ is a weakly pseudoconvex open set in $\mathbb{C}^n$ with a plurisubharmonic exhaustion function ψ. Then the functions $\varphi_\varepsilon = \varphi * \rho_\varepsilon$, where (ρ_ε) is a family of smoothing kernels, is well defined, smooth on $\Omega_c = \{x \in \Omega \mid \psi(x) < c\}$ for ε small enough. Moreover, it satisfies a lower bound of the form

$$i\partial\overline{\partial}\varphi_\varepsilon \otimes \mathrm{Id}_E \geq \gamma_\varepsilon\omega \otimes \mathrm{Id}_E - i\Theta(E)$$

for some continuous function γ_ε converging uniformly to γ on compact subsets of Ω as $\varepsilon \to 0$. We define new hermitian metrics h_ε on the vector bundle E by multiplying the original metric h with the weight $e^{-\varphi_\varepsilon}$, i.e. we set $h_\varepsilon = he^{-\varphi_\varepsilon}$. Then

$$i\Theta_{h_\varepsilon}(E) = i\Theta_h(E) + i\partial\overline{\partial}\varphi_\varepsilon \otimes \mathrm{Id}_E \geq \gamma_\varepsilon\omega \otimes \mathrm{Id}_E.$$

From Theorem 1.4.3 we thus get solutions $u_{c,\varepsilon}$ on X_c such that

$$\int_{X_c} |u_{c,\varepsilon}|^2 e^{-\varphi_\varepsilon} dV_\omega \leq \frac{1}{q} \int_{X_c} \gamma_\varepsilon^{-1}|f|^2 e^{-\varphi_\varepsilon} dV_\omega$$

whenever $\gamma_\varepsilon > 0$ on $\overline{X}_c$. As $\varphi_\varepsilon \geq \varphi$ converges to φ monotonically, we conclude by extracting weak limits and applying Lebesgue's monotone convergence theorem as before. $\square$

Chapter 2

Elliptic operators with polynomial growth at the boundary

In this chapter we prove some regularity results for certain elliptic operators on a bounded domain $\Omega \subset\subset \mathbb{R}^n$. Namely, we will consider an elliptic operator L, whose principal symbol can be controlled by some power of the boundary distance of Ω. We show that if Ω has Lipschitz boundary and if u is a smooth function on Ω satisfying $Lu = f$, where f vanishes to some finite order at the boundary of Ω, then also u vanishes to some finite order at the boundary.

2.1 The Sobolev spaces

The purpose of this section is to fix some terminology and to recall some of the basic properties of Sobolev spaces (see [Fol76] for more details).

Let $\mathcal{D}(\mathbb{R}^n)$ be the space of $\mathcal{C}^\infty$ functions on $\mathbb{R}^n$ with compact support, and $\mathcal{S}$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$, i.e. the space of all $\mathcal{C}^\infty$ functions u such that $\sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta u(x)| < \infty$ for all multiindices α and β. We define the Fourier transform of a function $u \in \mathcal{S}$ by

$$\hat{u}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} u(x) e^{-i\langle x,\xi\rangle} dx,$$

where $\xi = (\xi_1, \ldots, \xi_n)$ and $\langle x, \xi\rangle = x_1\xi_1 + \ldots + x_n\xi_n$.

The Sobolev norms $\| \ \|_s$ of order s on $\mathbb{R}^n$, $s \in \mathbb{R}$, are defined by

$$\|u\|_s^2 \ = \ \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi$$

where $u \in \mathcal{S}$. The Sobolev space $H_s = H_s(\mathbb{R}^n)$ is the completion of $\mathcal{S}$ under the norm $\| \, \|_s$.

For $u \in \mathcal{S}$, a straightforward computation shows that

$$\widehat{D^\alpha u}(\xi) = i^{|\alpha|}\xi^\alpha \hat{u}(\xi).$$

It follows that if k is a positive integer, we have

$$\|u\|_k^2 \sim \sum_{0 \leq |\alpha| \leq k} \|D^\alpha u\|_0^2 \text{ for all } u \in \mathcal{S}.$$

Here we write $a \lesssim b$ (resp. $b \gtrsim a$), if there exists an *absolute* constant $C > 0$ such that $a \leq C \cdot b$ (resp. $b \geq C \cdot a$). $a \sim b$ signifies $a \lesssim b$ and $a \gtrsim b$.

From this remark it follows that $u \in H_k$ admits weak distribution derivatives $D^\alpha u \in L^2(\mathbb{R}^n)$ for $|\alpha| \leq k$. Although Sobolev spaces make the manipulation of distribution derivatives very easy, they would be of limited usefulness if we could not relate derivatives in the L^2 sense to classical pointwise derivatives. Fortunately, the Sobolev lemma provides a simple and beautiful connection between the two.

If u is a function of class $\mathcal{C}^k$ on $\mathbb{R}^n$ whose derivatives up to order k are bounded, we define $|u|_k$ to be the supremum over $x \in \mathbb{R}^n$ and $|\alpha| \leq k$ of $|D^\alpha u(x)|$.

Proposition 2.1.1 *(Sobolev lemma)*
$H_s \subset \mathcal{C}^k$ and $| \, |_k \lesssim \| \, \|_s$ if and only if $s > k + \frac{n}{2}$.

It is also possible to define Sobolev spaces on bounded domains. Namely, let $\Omega \subset \mathbb{R}^n$ be a bounded open set in $\mathbb{R}^n$ and m a nonnegative integer. Consider the space of all those $\mathcal{C}^\infty$ functions $f : \Omega \to \mathbb{C}$ such that

$$\|f\|_{m,\Omega}^2 := \sum_{|\alpha| \leq m} \int_\Omega |D^\alpha f|^2 dx < +\infty.$$

The completion of the above space relative to the norm $\| \, \|_{m,\Omega}$ is called the Sobolev space $H_m(\Omega)$. The completion of the space $\mathcal{D}(\Omega)$ of $\mathcal{C}^\infty$ functions with compact support in Ω relative to $\| \, \|_{m,\Omega}$ is denoted by $\overset{\circ}{H}_m(\Omega)$. Roughly, $\overset{\circ}{H}_m(\Omega)$ is the set of elements of H_m which are supported in $\overline{\Omega}$. Indeed, under mild regularity assumptions on Ω (e.g. Ω with Lipschitz boundary suffices, see [Gri85]), then if f is of class $\mathcal{C}^k$ on $\mathbb{R}^n$ and supported in Ω, then

$f \in \overset{\circ}{H}_k(\Omega)$. On the other hand, if $f \in \overset{\circ}{H}_s(\Omega)$ and $s > k + \frac{n}{2}$, then it follows from the Sobolev lemma that f is of class $\mathcal{C}^k$ on $\mathbb{R}^n$ and supported in $\overline{\Omega}$.

2.2 A regularity theorem for elliptic operators

In this section, we will study the regularity of the equation $Lu = f$, where L is an elliptic operator on a bounded open set in $\mathbb{R}^n$, whose principal symbol can be controlled by some power of the boundary distance.

More precisely, let Ω be an open set in $\mathbb{R}^n$, and let

$$L = \sum_{|\alpha|=m} a_\alpha(x) D^\alpha + \sum_{|\beta|<m} b_\beta(x) D^\beta$$

be a differential operator of order m with smooth coefficients $a_\alpha,\ b_\beta \in \mathcal{C}^\infty(\Omega)$ on Ω. Let $\Delta : \Omega \longrightarrow \mathbb{R}^+$ be a smooth function on Ω.

We say that L is an *elliptic operator of polynomial growth with respect to* Δ *on* Ω if there exist $k, l \in \mathbb{N}$ such that

$$|\sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha| \gtrsim \Delta^k(x)\ |\xi|^m \text{ for every } \xi \in \mathbb{R}^n \tag{2.1}$$

and

$$|D^\gamma a_\alpha(x)| \lesssim \Delta^{-l-|\gamma|}(x),\ |D^\gamma b_\beta(x)| \lesssim \Delta^{-l-|\gamma|}(x) \tag{2.2}$$

for all multiindices $\alpha,\ \beta,\ \gamma$.

We define $\mathcal{C}^r(\mathbb{R}^n, \overline{\Omega}) := \{f \in \mathcal{C}^r(\mathbb{R}^n) \mid \text{supp} f \subset \overline{\Omega}\}$.

Theorem 2.2.1
Let L be a differential operator of order m with smooth coefficients on an open set $\Omega \subset\subset \mathbb{R}^n$, which is of polynomial growth with respect to a smooth function $\Delta \in \mathcal{C}^\infty(\Omega, \mathbb{R}^+)$.
Then we have the following a priori estimate

$$\|u\|^2_{s,\Omega} \lesssim \|\Delta^{-ts} Lu\|^2_{s-m,\Omega} + \|\Delta^{-Ts^2} u\|^2_{0,\Omega} \tag{2.3}$$

for some $t, T \in \mathbb{N}$ *and* $s \gg 1,\ u \in \mathcal{C}^\infty(\Omega)$.
Moreover, let Ω *have Lipschitz boundary and assume that* Δ *has essentially the same features as the boundary distance function* d *of* Ω, *i.e.* $\Delta \sim d$ *and* $|D^\alpha \Delta| \lesssim d^{1-|\alpha|}$ *for every multiindex* α. *Let* $u \in \mathcal{C}^\infty(\Omega)$ *satisfy* $\int_\Omega |u(x)|^2 \Delta^{-N}(x) d\lambda(x) < +\infty$ *and* $Lu \in \mathcal{C}^N(\mathbb{R}^n, \overline{\Omega}) \cap \mathcal{C}^\infty(\Omega)$. *Then* $u \in \mathcal{C}^{s(N)}(\mathbb{R}^n, \overline{\Omega}) \cap \mathcal{C}^\infty(\Omega)$, *where* $s(N) \sim \sqrt{N}$ *for all* $N \gg 1$.

Proof: We will first show that it suffices to prove the a priori estimate (2.3). Let $u \in \mathcal{C}^\infty(\Omega)$ satisfy $\int_\Omega |u(x)|^2 \Delta^{-N}(x) d\lambda(x) < +\infty$ and $Lu \in \mathcal{C}^N(\mathbb{R}^n, \overline{\Omega}) \cap \mathcal{C}^\infty(\Omega)$. We want to show that $u \in \mathcal{C}^{s(N)}(\mathbb{R}^n, \overline{\Omega}) \cap \mathcal{C}^\infty(\Omega)$ with $s(N) \sim \sqrt{N}$ for all $N \gg 1$. As noted in the preceeding section, it suffices by the Sobolev lemma to show that $u \in \overset{\circ}{H}_{s(N)}(\Omega)$.

Since Ω has Lipschitz boundary, it follows from a general result of Grisvard that

$$\mathcal{C}^k(\mathbb{R}^n, \overline{\Omega}) \subset \{f \in \mathcal{C}^k(\Omega) \mid \int_U |f|^2 d^{-2k} d\lambda < +\infty\}$$

(see [Gri85, theorem 1.4.4.4] or Theorem A.2.2). Hence the a priori estimate (2.3) together with the assumptions on u yields $u \in H_{s(N)}(\Omega)$ with $s(N) \sim \sqrt{N}$.

Next, we define the open sets $\Omega_j \subset \Omega$ as follows:

$$\Omega_j = \{z \in \Omega \mid d(z) > \frac{1}{j-1}\} \subset\subset \Omega_{j+1}.$$

For every $j \in \mathbb{N}$, it is then possible to construct $\chi_j \in \mathcal{C}^\infty(\mathbb{R}^n)$ with compact support in Ω_{j+1} such that $\chi_j \equiv 1$ in a neighborhood of $\overline{\Omega}_j$, and moreover, for every multiindex α,

$$\sup_{x \in \mathbb{R}^n} |D^\alpha \chi_j(x)| \leq N_{|\alpha|} j^{2|\alpha|} \tag{2.4}$$

($N_{|\alpha|}$ does not depend on j!). The existence of such functions χ_j is proved in the appendix (cf Lemma A.3.1); note that $\mathrm{dist}(\partial\Omega_j, \partial\Omega_{j+1}) \geq j^{-2}$.

We can also find functions $\eta_j \in \mathcal{C}^\infty(\mathbb{R}^n)$ satisfying $0 \leq \eta_j \leq 1$, supp $\eta_j \subset \Omega_{j+2} \setminus \overline{\Omega}_{j-1}$, $\eta_j \equiv 1$ in a neighborhood of $\overline{\Omega}_{j+1} \setminus \Omega_j$ and

$$\sup_{x \in \mathbb{R}^n} |D^\alpha \eta_j(x)| \leq M_{|\alpha|} j^{2|\alpha|}$$

for every multiindex α, where $M_{|\alpha|}$ does not depend on j. Thus $|\eta_j|_s^2 \lesssim j^{4s}$.

Let us now estimate $\|u-\chi_j u\|^2_{s,\Omega}$. Using the a priori estimate (2.3) and (2.4), we obtain

$$\begin{aligned}\|u-\chi_j u\|^2_{s,\Omega} &\lesssim \|\Delta^{-ts}L(u-\chi_j u)\|^2_{s-m,\Omega}+\|\Delta^{-Ts^2}(u-\chi_j u)\|^2_{0,\Omega}\\ &\lesssim \|\Delta^{-ts}(Lu-\chi_j Lu)\|^2_{s-m,\Omega}+\|\Delta^{-Ts^2}(u-\chi_j u)\|^2_{0,\Omega}\\ &\quad +j^{cs}\|\eta_j u\|^2_{s-1,\Omega}\end{aligned}$$

for some large $c\in\mathbb{N}$.

We also have

$$\begin{aligned}\|\eta_j u\|^2_{s-1,\Omega} &\sim \int_{\mathbb{R}^n}(1+|\xi|^2)^{s-1}|\widehat{\eta_j u(\xi)}|^2 d\xi\\ &= \int_{\{1+|\xi|^2\geq j^{(4+c)s+1}\}}(1+|\xi|^2)^{s-1}|\widehat{\eta_j u(\xi)}|^2 d\xi\\ &\quad +\int_{\{1+|\xi|^2\leq j^{(4+c)s+1}\}}(1+|\xi|^2)^{s-1}|\widehat{\eta_j u(\xi)}|^2 d\xi\\ &\lesssim j^{-(4+c)s-1}\|\eta_j u\|^2_{s,\Omega}+j^{c's^2}\|\eta_j u\|^2_{0,\Omega}\\ &\lesssim j^{-(4+c)s-1}\|\eta_j u\|^2_{s,\Omega}+j^{-cs-1}\|\Delta^{-c''s^2}u\|^2_{0,\Omega}\\ &\lesssim j^{-cs-1}(\|u\|^2_{s,\Omega}+\|\Delta^{-c''s^2}u\|^2_{0,\Omega})\end{aligned}$$

for some large $c',c''\in\mathbb{N}$; note that $j\lesssim\Delta^{-1}$ on $\Omega\setminus\Omega_j$.

Combining this with the above inequalities, we obtain

$$\begin{aligned}\|u-\chi_j u\|^2_{s,\Omega} &\lesssim \|\Delta^{-ts}(Lu-\chi_j Lu)\|^2_{s-m,\Omega}+\|\Delta^{-Ts^2}(u-\chi_j u)\|^2_{0,\Omega}\\ &\quad +\frac{1}{j}(\|u\|^2_{s,\Omega}+\|\Delta^{-c''s^2}u\|^2_{0,\Omega})\end{aligned}$$

We have already shown that for some $s\sim\sqrt{N}$, $\|u\|^2_{s,\Omega}<+\infty$. By hypothesis on u, we also have $\|\Delta^{-c''s^2}u\|^2_{0,\Omega}<+\infty$ for some $s\sim\sqrt{N}$, thus the last term in the above inequality tends to zero as $j\to+\infty$. Moreover, the assumptions on u imply that also the first two terms tend to zero as $j\to+\infty$ for some $s\sim\sqrt{N}$ (use Grisvard's result, see Theorem A.2.2). We have therefore proved the last assertion of the theorem.

Now, let us finally turn to the proof of the a priori estimate (2.3). We prove this estimate by simply expliciting the dependence on Δ of all the constants involved in the classical proof of the hypoellipticity of uniformly elliptic operators (see [Fol76]).

Let us fix $x_0 \in \Omega$ and let $B_\delta(x_0)$ be the ball of radius $\delta \ll 1$ centered at x_0. Let u be a smooth function with support in $B_\delta(x_0)$.

First, we assume that $b_\beta = 0$ for every multiindex β. Then we have

$$(\widehat{L_{x_0} u})(\xi) = i^m \sum_{|\alpha|=m} a_\alpha(x_0)\ \xi^\alpha\ \hat{u}(\xi)$$

where $L_{x_0} = L(x_0)$ is the differential operator with frozen coefficients at x_0.

This implies

$$\begin{aligned}(1+|\xi|^2)^s|\hat{u}(\xi)|^2 &\le 2^m(1+|\xi|^2)^{s-m}\ (1+|\xi|^{2m})\ |\hat{u}(\xi)|^2 \\ &\lesssim (1+|\xi|^2)^{s-m}\ |\hat{u}(\xi)|^2 + \Delta^{-2k}(x_0)\ (1+|\xi|^2)^{s-m}\ |(\widehat{L_{x_0} u})(\xi)|^2\end{aligned}$$

by (2.1). Integrating both sides and using the inequality $\|u\|_{s-m,\Omega} \le \|u\|_{s-1,\Omega}$, one obtains

$$\|u\|^2_{s,\Omega} \lesssim \Delta^{-2k}(x_0)\ \|L_{x_0} u\|^2_{s-m,\Omega} + \|u\|^2_{s-1,\Omega}.$$

Hence there exists $C_0 > 0$ such that

$$\|u\|^2_{s,\Omega} \le C_0\ \Delta^{-2k}(x_0)\ (\|L_{x_0} u\|^2_{s-m,\Omega} + \|u\|^2_{s-1,\Omega}). \tag{2.5}$$

We now wish to estimate

$$\|L_x u - L_{x_0} u\|^2_{s-m,\Omega} = \|\sum_\alpha (a_\alpha(x) - a_\alpha(x_0))\ D^\alpha u\|^2_{s-m,\Omega}.$$

The estimates (2.2) yield

$$|a_\alpha(x) - a_\alpha(x_0)| \le C_1\ \Delta^{-l-1}(x_0)\ |x_0 - x|$$

for some $C_1 > 0$ and all α, x, x_0.

Set $\delta = (8C_0C_1^2 n^m \Delta^{-2k-2l-2}(x_0))^{-\frac{1}{2}}$ and fix $\phi \in \mathcal{D}(B_{2\delta}(0))$ with $0 \le \phi \le 1$ and $\phi \equiv 1$ on $B_\delta(0)$. Suppose u is a smooth function supported in $B_\delta(x_0)$. Then

$$(a_\alpha(x) - a_\alpha(x_0))\ D^\alpha u(x) = \phi(x_0 - x)\ (a_\alpha(x) - a_\alpha(x_0))\ D^\alpha u(x)$$

and

$$\sup_x |\phi(x_0 - x)(a_\alpha(x) - a_\alpha(x_0))|^2 \le 4C_1^2\Delta^{-2l-2}(x_0)\delta^2 = (2n^m C_0 \Delta^{-2k}(x_0))^{-1}.$$

Hence by (2.2)

$$\begin{aligned}\|(a_\alpha(x) - a_\alpha(x_0))\ D^\alpha u\|^2_{s-m,\Omega} \ &\le\ (2n^m C_0 \Delta^{-2k}(x_0))^{-1}\ \|u\|^2_{s,\Omega} \\ &\quad + C_2 \Delta^{-s_1 s - s_0}(x_0)\ \|u\|^2_{s-1,\Omega}\end{aligned}$$

for some $C_2 > 0,\ s_0, s_1 \in \mathbb{N}$.

Thus, since there are at most n^m multiindices α with $|\alpha| = m$, we have

$$\|L_x u - L_{x_0} u\|^2_{s-m,\Omega} \le (2C_0 \Delta^{-2k}(x_0))^{-1}\ \|u\|^2_{s,\Omega} + n^m C_2 \Delta^{-s_1 s - s_0}(x_0)\ \|u\|^2_{s-1,\Omega}$$

Combining this with (2.5), we then obtain

$$\|u\|^2_{s,\Omega} \le C_0 \Delta^{-2k}(x_0)(\|Lu\|^2_{s-m,\Omega} + \|u\|^2_{s-1,\Omega}) + \frac{1}{2}\|u\|^2_{s,\Omega},$$

hence

$$\|u\|^2_{s,\Omega} \lesssim \Delta^{-2k}(x_0)\ \|Lu\|^2_{s-m,\Omega} + \Delta^{-m_0 s - k_0}(x_0)\|u\|^2_{s-1,\Omega}$$

for some $m_0, k_0 \in \mathbb{N}$.

Next, we consider the case $b_\beta \not\equiv 0$. Replacing m_0, k_0 by larger integers if necessary, we can absorb the additional terms of Lu in the term $\Delta^{-m_0 s - k_0}(x_0)\|u\|^2_{s-1,\Omega}$ and still have the estimate

$$\|u\|^2_{s,\Omega} \lesssim \Delta^{-2k}(x_0)\ \|Lu\|^2_{s-m,\Omega} + \Delta^{-m_0 s - k_0}(x_0)\|u\|^2_{s-1,\Omega}$$

We emphasize that all the constants involved are independent of $x_0 \in \Omega$.

Next, one can cover Ω by balls $B_{\delta_i}(x_i)$ of the above type, $i \in \mathbb{N}$, such that there exists a partition of unity $(\theta_i)_{i\in\mathbb{N}}$ with respect to this covering satisfying $\sum_{|\alpha|\le s} |D^\alpha \theta_i|^2 \le \theta_i |P_s(\delta_i^{-2})|$ where P_s is a polynomial of degree s in one variable (see Lemma A.4.1). One has

$$\|\theta_i u\|^2_{s,\Omega} \lesssim \Delta^{-2k}(x_i)\ \|L\theta_i u\|^2_{s-m,\Omega} + \Delta^{-m_0 s - k_0}(x_i)\|\theta_i u\|^2_{s-1,\Omega}$$

for every smooth function u on Ω.

Replacing m_0, k_0 by larger integers if necessary, we get

$$
\begin{aligned}
\|\theta_i u\|^2_{s,\Omega} \leq C\ \{&\Delta^{-k_0}(x_i)\|\theta_i Lu\|^2_{s-m,\Omega} + \Delta^{-m_0 s-k_0}(x_i)\|\theta_i u\|^2_{s-1,\Omega} \\
&+ \Delta^{-m_0 s-k_0}(x_i)\int_\Omega \theta_i |u|^2 d\lambda\} \\
\leq\ & M\Delta^{-m_0 s-k_0}(x_i)\ \{\sum_{|\alpha|\leq s-m}\int_\Omega \theta_i |D^\alpha(Lu)|^2 d\lambda \\
&+ \|\theta_i u\|^2_{s-1,\Omega} + \int_\Omega \theta_i |u|^2 d\lambda\} \qquad (2.6)
\end{aligned}
$$

for some $C, M > 0$; note that $\delta_i^{-2} \sim \Delta^{-2k-2l-2}(x_i)$.

Moreover,

$$
\begin{aligned}
&M\Delta^{-m_0 s-k_0}(x_i)\ \|\theta_i u\|^2_{s-1,\Omega} \\
&= M\Delta^{-m_0 s-k_0}(x_i)\int_{\mathbb{R}^n}(1+|\xi|^2)^{s-1}|\widehat{\theta_i u}(\xi)|^2 d\lambda \\
&= M\Delta^{-m_0 s-k_0}(x_i)\int_{\{1+|\xi|^2\geq 2M\Delta^{-m_0 s-k_0}(x_i)\}}(1+|\xi|^2)^{s-1}|\widehat{\theta_i u}(\xi)|^2 d\lambda \\
&+ M\Delta^{-m_0 s-k_0}(x_i)\int_{\{1+|\xi|^2\leq 2M\Delta^{-m_0 s-k_0}(x_i)\}}(1+|\xi|^2)^{s-1}|\widehat{\theta_i u}(\xi)|^2 d\lambda \\
&\leq \frac{1}{2}\|\theta_i u\|^2_{s,\Omega} + C'\Delta^{-m_0 s^2+m_0 s-k_0 s+k_0}(x_i)\ \|\theta_i u\|^2_{0,\Omega}
\end{aligned}
$$

for some $C' > 0$. Thus, by (2.6),

$$
\|\theta_i u\|^2_{s,\Omega} \lesssim \sum_{|\alpha|\leq s-m}\int_\Omega \theta_i \Delta^{-2ts}|D^\alpha(Lu)|^2 d\lambda + \int_\Omega \theta_i \Delta^{-2Ts^2}|u|^2 d\lambda
$$

for some $t, T \in \mathbb{N}$ and $s \gg 1$. So

$$
\|u\|^2_{s,\Omega} = \|\sum_i \theta_i u\|^2_{s,\Omega} \leq \sum_i \|\theta_i u\|^2_{s,\Omega} \lesssim \|\Delta^{-ts}Lu\|^2_{s-m,\Omega} + \|\Delta^{-Ts^2}u\|^2_{0,\Omega}
$$

which completes the proof. $\square$

Chapter 3

The pseudoconvex case

In this chapter, we consider a domain Ω, which is relatively compact in an n-dimensional Kähler manifold X and has Lipschitz boundary. We moreover assume that Ω satisfies some pseudoconvexity condition, which we call "$\log\delta$-pseudoconvexity". Roughly speaking, this means that there exists a metric on X such that $-\log(\text{boundary distance})$ admits a strictly plurisubharmonic extension to Ω. We then show that the $\overline{\partial}$-equation with exact support in Ω admits a solution in bidegrees (p,q), $0 \leq p \leq n$, $1 \leq q \leq n-1$. Moreover, the range of $\overline{\partial}$ acting on smooth $(p, n-1)$-forms with support in $\overline{\Omega}$ is closed. This result can be applied to solve the $\overline{\partial}$-equation with regularity up to the boundary in the domain $X \setminus \overline{\Omega}$ as well as the $\overline{\partial}$-equation for currents on Ω, wich are the restriction of a currents defined on X. This in turn gives the vanishing of the $\check{C}$ech-cohomology groups of the sheaf of germs of holomorphic functions on Ω admitting a distribution boundary value.

3.1 Pseudoconvex domains in Kähler manifolds

In order to prove a solvability result for the $\overline{\partial}$-problem with exact support in pseudoconvex domains, we have to make a global assumption on the ambient complex manifold as well as an additional assumption on the domain itself.

We will denote by (X,ω) an n-dimensional Kähler manifold. Let $\Omega \subset\subset X$ be an open set. Let $\delta(z)$ be the distance from $z \in \Omega$ to the boundary of Ω with respect to the metric ω.

Definition.
We say that Ω is $\log\delta$-pseudoconvex, if there exists a smooth bounded func-

tion h on Ω such that

$$i\partial\overline{\partial}(-\log\delta + h) \geq C\,\omega \text{ in } \Omega. \tag{3.1}$$

for some $C > 0$.

In particular, every $\log\delta$-pseudoconvex domain Ω admits a strictly plurisubharmonic exhaustion function, therefore Ω is a Stein manifold.

Example 1.
Let X be a Stein manifold and let $\Omega \subset\subset X$ be a domain which is locally Stein, i.e. for every $x \in \partial\Omega$, there exists a neighborhood U_x of x in X such that $\Omega \cap U_x$ is Stein. It was shown in [Ele75] that there exists a Kähler metric ω on X such that Ω is $\log\delta$-pseudoconvex.
The same remains true if X is only assumed to admit a strictly plurisubharmonic function (see [Ele75]).
In particular, every bounded weakly pseudoconvex domain with smooth boundary in $\mathbb{C}^n$ is $\log\delta$-pseudoconvex.

Example 2.
Let (X, ω) be a Kähler manifold with *positive holomorphic bisectional curvature*, that is $T^{1,0}X$ is positive in the sense of Griffiths. Then every domain $\Omega \subset\subset X$, which is locally Stein, is $\log\delta$-pseudoconvex (see [Tak64] for the case $X = \mathbb{P}^n$, [Ele75], [Suz76]).

In particular, the complex projective space $\mathbb{P}^n$ is a Kähler manifold with positive holomorphic bisectional curvature. Indeed, let ω_{FS} be the natural Kähler metric on $\mathbb{P}^n$, called the Fubini-Study metric, which is defined by

$$p^*\omega_{FS} = \frac{i}{2}\partial\overline{\partial}\log(|\zeta_0|^2 + |\zeta_1|^2 + \cdots + |\zeta_n|^2)$$

where $\zeta_0, \ldots, \zeta_n$ are coordinates of $\mathbb{C}^{n+1}$ and where $p : \mathbb{C}^{n+1} \longrightarrow \mathbb{P}^n$ is the projection. Let $z = (\zeta_1/\zeta_0, \ldots, \zeta_n/\zeta_0)$ be non homogeneous coordinates on $\mathbb{C}^n = \{\zeta_0 \neq 0\} \subset \mathbb{P}^n$. Then, since $\partial\overline{\partial}\log|\zeta_0|^2 = 0$ on $\{\zeta_0 \neq 0\}$, we see that

$$\omega_{FS} = \frac{i}{2}\partial\overline{\partial}\log(1 + |z|^2) = \frac{i}{2}\sum_{1\leq i,j\leq n}\frac{(1+|z|^2)\delta_{ij} - \overline{z}_i z_j}{(1+|z|^2)^2}\,dz_i \wedge d\overline{z}_j,$$

thus

$$h_{ij}(z) = \langle\frac{\partial}{\partial z_i}(z), \frac{\partial}{\partial z_j}(z)\rangle_{\omega_{FS}} = \frac{\delta_{ij}}{1+|z|^2} - \frac{\overline{z}_i z_j}{(1+|z|^2)^2}.$$

To calculate the curvature of $T^{1,0}\mathbb{P}^n$ at a point z_0, we may without loss of generality suppose $z_0 = 0$. A Taylor expansion around $z = 0$ shows that

$$h_{kl}(z) = (1 - |z|^2)\delta_{kl} - \overline{z}_k z_l + O(|z|^3).$$

Formula (1.1) then shows that the curvature coefficients are as follows:

$$c_{ijkl}(0) = -\frac{\partial^2 \overline{h}_{lk}}{\partial \overline{z}_j \partial z_i}(0) = \delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}.$$

Hence

$$c_{iiii}(0) = 2,\ c_{iijj}(0) = 1 \text{ if } i \neq j,\ c_{ijji}(0) = 1 \text{ if } i \neq j,\ c_{ijkl}(0) = 0 \text{ otherwise.}$$

Thus

$$\begin{aligned}\widetilde{\Theta}(T^{1,0}\mathbb{P}^n)(\xi \otimes v, \xi \otimes v) &= 2\sum_{i=1}^{n} |\xi_i|^2|v_i|^2 + \sum_{i\neq j} |\xi_i|^2|v_j|^2 + \sum_{i\neq j} \xi_i\overline{\xi}_j v_j \overline{v}_i \\ &= |\xi|^2|v|^2 + |\langle \xi, v\rangle|^2 \geq |\xi|^2|v|^2 > 0\end{aligned}$$

if $0 \neq \xi = \sum_{i=1}^n \xi_i \frac{\partial}{\partial z_i} \in T^{1,0}\mathbb{P}^n$ and $0 \neq v = \sum_{j=1}^n v_j \frac{\partial}{\partial z_j} \in T^{1,0}\mathbb{P}^n$, which shows that $\mathbb{P}^n$ has positive holomorphic bisectional curvature.

By [SY80] we moreover know that a compact Kähler manifold with positive holomorphic bisectional curvature is biholomorphic to $\mathbb{P}^n$.

In general, δ is not a smooth function in Ω. However, in [Ste70, p.171], the existence of a regularized distance having essentially the same profile as δ is proved:

There exists a function $\Delta \in \mathcal{C}^\infty(\Omega, \mathbb{R})$ satisfying

$$c_1\delta(x) \leq \Delta(x) \leq c_2\delta(x) \qquad \text{and}$$

$$|\frac{\partial^\alpha}{\partial x^\alpha}\Delta(x)| \leq B_\alpha(\delta(x))^{1-|\alpha|}$$

where $x = (x_1, \ldots, x_{2n})$ are local coordinates on X. B_α, c_1 and c_2 are independent of Ω.

3.2 The L^2 estimates

Let (E, h) be a hermitian holomorphic vector bundle on X, and let $N \in \mathbb{Z}$. We denote by $L^2_{p,q}(\Omega, E, N)$ the Hilbert space of (p, q)-forms u with values in E which satisfy

$$\|u\|^2_N \ := \int_\Omega |u|^2_{\omega,h} \Delta^N \mathrm{dV}_\omega < +\infty.$$

Here dV_ω is the canonical volume element associated to the metric ω, and $|\ |_{\omega,h}$ is the norm of (p,q)-forms induced by ω and h.

Proposition 3.2.1
Let Ω be a relatively compact domain in a Kähler manifold (X,ω). We assume that Ω is $\log\delta$-pseudoconvex. Let (E,h) be a hermitian holomorphic vector bundle on X and let $N \gg 1$ and $1 \leq q \leq n$. Suppose $f \in L^2_{n,q}(\Omega, E, N) \cap \mathrm{Ker}\overline{\partial}$. Then there exists $u \in L^2_{n,q-1}(\Omega, E, N)$ such that $\overline{\partial}u = f$ and $\|u\|_N \leq \|f\|_N$.

Proof. This follows immediately from Theorem 1.4.4. Indeed, since Δ has essentially the same features as $\delta\exp(-h)$(cf (3.1)), it suffices to prove the statement with Δ replaced by $\delta\exp(-h)$ in the definition of the spaces $L^2_{p,q}(\Omega, E, N)$. But for N sufficiently large, we clearly have

$$i\Theta(E) + Ni\partial\overline{\partial}(-\log\delta + h) \otimes \mathrm{Id}_E \geq \omega \otimes \mathrm{Id}_E$$

by (3.1), thus Theorem 1.4.4 yields the desired vanishing result; note that $-\log\delta + h = -\log(\delta\exp(-h)))$. □

Proposition 3.2.2
Let Ω be a relatively compact domain in an n-dimensional Kähler manifold (X,ω). We assume that Ω is $\log\delta$-pseudoconvex. Let (E,h) be a hermitian vector bundle on X and let $N \gg 1$. Suppose $f \in L^2_{0,q}(\Omega, E, -N) \cap \mathrm{Ker}\overline{\partial}$, $1 \leq q \leq n-1$. Then there exists $u \in L^2_{0,q-1}(\Omega, E, -N+2)$ such that $\overline{\partial}u = f$ and $\|u\|_{-N+2} \leq \|f\|_{-N}$.

Proof. Suppose $1 \leq q \leq n-1$ and let $f \in L^2_{0,q}(\Omega, E, -N) \cap \mathrm{Ker}\overline{\partial}$, $N \gg 1$. We define the linear operator

$$\begin{array}{rrcl} L_f: & \overline{\partial}L^2_{n,n-q}(\Omega, E^*, N-2) & \longrightarrow & \mathbb{C} \\ & \overline{\partial}\varphi & \longmapsto & \displaystyle\int_\Omega f \wedge \varphi \end{array}$$

Note that the integral on the right hand side is finite, since

$$|\int_\Omega f\wedge\varphi|^2 \leq (\int_\Omega |f|^2_\omega \Delta^{-N} dV_\omega)\cdot(\int_\Omega |\varphi|^2_\omega \Delta^N dV_\omega) \leq \|f\|^2_{-N}\|\varphi\|^2_{N-2}.$$

Let us first show that L_f is well defined.

Indeed, let $\varphi_1,\ \varphi_2 \in L^2_{n,n-q}(\Omega, E^*, N-2)$ such that $\overline{\partial}\varphi_1 = \overline{\partial}\varphi_2$. Then $\overline{\partial}(\varphi_1-\varphi_2)=0$, and by Proposition 3.2.1, since $n-q\geq 1$, there exists $\alpha \in L^2_{n,n-q-1}(\Omega, E^*, N-2)$ such that $\overline{\partial}\alpha = \varphi_1-\varphi_2$. But then

$$\begin{aligned}
\int_\Omega f\wedge(\varphi_1-\varphi_2) &= \int_\Omega f\wedge\overline{\partial}\alpha \\
&= \lim_{\varepsilon\to 0}(-1)^q\int_{\partial\Omega_\varepsilon} f\wedge\alpha \\
&= -\lim_{\varepsilon\to 0}\int_{\Omega\setminus\Omega_\varepsilon} f\wedge\overline{\partial}\alpha \\
&= -\lim_{\varepsilon\to 0}\int_{\Omega\setminus\Omega_\varepsilon} f\wedge(\varphi_1-\varphi_2)
\end{aligned}$$

with $(\Omega_\varepsilon)_{\varepsilon>0}$ an exhaustion of Ω by smoothly bounded domains such that $\Omega_\varepsilon \supset \{z\in\Omega \mid \Delta(z)>\varepsilon\}$. Here we have used Stoke's theorem several times. The third equality is obtained as follows: Fix $\varepsilon<0$ and choose for each large $j>\frac{2}{\varepsilon}$ a $\mathcal{C}^\infty$ function χ_j such that $\chi_j\equiv 1$ on $\Omega_{\frac{2}{j}}$, $\chi_j\equiv 0$ on $\Omega_{\frac{1}{j}}$, $0\leq\chi_j\leq 1$, $|D\chi_j|\leq Cj$, and set $\alpha_j=\chi_j\alpha\in\mathcal{D}^{n,n-q-1}(\Omega)$. Then we have

$$\int_{\Omega\setminus\Omega_\varepsilon} f\wedge\overline{\partial}\alpha_j = \int_{\Omega\setminus\Omega_\varepsilon}\chi_j f\wedge\overline{\partial}\alpha + \int_{\Omega\setminus\Omega_\varepsilon} f\wedge\overline{\partial}\chi_j\wedge\alpha$$

and

$$\begin{aligned}
|\int_{\Omega\setminus\Omega_\varepsilon} f\wedge\overline{\partial}\chi_j\wedge\alpha|^2 &\leq C\int_{\Omega\setminus\Omega_\varepsilon}|f|^2_\omega\Delta^{-N}dV_\omega\cdot\int_{\Omega\setminus\Omega_{\frac{2}{j}}} j^2|\alpha|^2_\omega\Delta^N dV_\omega \\
&\leq C\|f\|^2_{-N}\|\alpha\|^2_{N-2}.
\end{aligned}$$

Hence the dominated convergence theorem gives

$$\begin{aligned}
\int_{\Omega\setminus\Omega_\varepsilon} f\wedge\overline{\partial}\alpha &= \lim_j\int_{\Omega\setminus\Omega_\varepsilon} f\wedge\overline{\partial}\alpha_j = (-1)^q\lim_j\int_{\Omega\setminus\Omega_\varepsilon}\overline{\partial}(f\wedge\alpha_j) \\
&= -(-1)^q\lim_j\int_{\partial\Omega_\varepsilon} f\wedge\alpha_j = -(-1)^q\int_{\partial\Omega_\varepsilon} f\wedge\alpha.
\end{aligned}$$

Moreover,

$$|\int_{\Omega\setminus\Omega_\varepsilon} f\wedge(\varphi_1-\varphi_2)| \leq (\int_{\Omega\setminus\Omega_\varepsilon} |f|^2_\omega \Delta^{-N})^{1/2}\ (\int_{\Omega\setminus\Omega_\varepsilon} |\varphi_1-\varphi_2|^2_\omega \Delta^{N})^{1/2} \longrightarrow_{\varepsilon\to 0} 0$$

(note that $(\int_{\Omega\setminus\Omega_\varepsilon} |\varphi_1-\varphi_2|^2_\omega \Delta^N)^{1/2} \leq \varepsilon(\int_{\Omega\setminus\Omega_\varepsilon} |\varphi_1-\varphi_2|^2_\omega \Delta^{N-2})^{1/2} \leq \varepsilon\|\varphi_1-\varphi_2\|_{N-2} \to 0$ as $\varepsilon \to 0$ since $\varphi_1,\varphi_2 \in L^2_{n,n-q}(\Omega, E^*, N-2)$).

Thus $L_f(\varphi_1) = L_f(\varphi_2)$.

Now let $\varphi \in \mathrm{Dom}(\overline{\partial} : L^2_{n,n-q}(\Omega, E^*, N-2) \to L^2_{n,n-q+1}(\Omega, E^*, N-2))$. Applying Proposition 3.2.1, there exists $\widetilde{\varphi} \in L^2_{n,n-q}(\Omega, E^*, N-2)$ satisfying $\overline{\partial}\widetilde{\varphi} = \overline{\partial}\varphi$ and $\|\widetilde{\varphi}\|_{N-2} \leq \|\overline{\partial}\varphi\|_{N-2}$. This yields

$$\begin{aligned}|L_f(\overline{\partial}\varphi)| = |L_f(\overline{\partial}\widetilde{\varphi})| = |\int_\Omega f\wedge\widetilde{\varphi}| &\leq \|f\|_{-N}\|\widetilde{\varphi}\|_N \\ &\leq \|f\|_{-N}\|\widetilde{\varphi}\|_{N-2} \leq \|f\|_{-N}\|\overline{\partial}\varphi\|_{N-2}.\end{aligned}$$

Thus L_f is a continuous linear operator of norm $\leq \|f\|_{-N}$ and therefore, using the Hahn-Banach theorem, L_f extends to a continuous linear operator with norm $\leq \|f\|_{-N}$ on the Hilbert space $L^2_{n,n-q+1}(\Omega, E^*, N-2)$. By the theorem of Riesz, there exists $u \in L^2_{0,q-1}(\Omega, E, -N+2)$ with $\|u\|_{-N+2} \leq \|f\|_{-N}$ such that for every $\varphi \in L^2_{n,n-q}(\Omega, E^*, N-2)$ we have

$$(-1)^q \int_\Omega u\wedge\overline{\partial}\varphi = L_f(\varphi) = \int_\Omega f\wedge\varphi,$$

i.e. $\overline{\partial}u = f$. □

3.3 The $\overline{\partial}$-problem with exact support

In this section, we will show some vanishing and separation theorems for the $\overline{\partial}$-cohomology groups with values in a vector bundle E supported in $\overline{\Omega}$:

$$\begin{aligned}H^{p,q}(X,\overline{\Omega},E) \quad = \quad & \{f\in\mathcal{C}^\infty_{p,q}(X,E) \mid \mathrm{supp} f\subset\overline{\Omega}\}\cap \mathrm{Ker}\overline{\partial}/ \\ & \overline{\partial}\{f\in\mathcal{C}^\infty_{p,q-1}(X,E) \mid \mathrm{supp} f\subset\overline{\Omega}\}.\end{aligned}$$

This is done by solving the $\overline{\partial}$-equation in the L^2-sense as in the last section and then applying the results of Chapter 2 to the operator $\Box_{-N} = \overline{\partial}\overline{\partial}^*_{-N} + \overline{\partial}^*_{-N}\overline{\partial}$ for $N > 0$. Here $\overline{\partial}^*_{-N}$ is the Von Neumann adjoint of $\overline{\partial} : L^2_{p,q}(\Omega, E, -N+2) \to L^2_{p,q+1}(\Omega, E, -N)$. An easy computation shows that $\overline{\partial}^*_{-N}u = \Delta^{N-2}\overline{\partial}^*_{\omega}(\Delta^{-N}u)$, where $\overline{\partial}^*_{\omega}$ is the Von Neumann adjoint of $\overline{\partial}$ with respect to the metric ω on X.

Theorem 3.3.1
Assume that Ω has Lipschitz boundary. Let $u \in L^2_{p,q}(\Omega, E, -N)$ satisfy $\overline{\partial}u = f$ and $\overline{\partial}^*_{-N}u = 0$ with $f \in \mathcal{C}^N_{p,q}(X, \overline{\Omega}, E) \cap \mathcal{C}^\infty_{p,q}(\Omega, E)$.
Then $u \in \mathcal{C}^{s(N)}_{p,q}(X, \overline{\Omega}, E) \cap \mathcal{C}^\infty_{p,q}(\Omega, E)$ where $s(N)$ is a function proportional to $\sqrt{N}$, $N \gg 1$.

Proof: The above theorem is a consequence of the results of Chapter 2. Indeed, since $\overline{\partial}^*_{-N}u = \Delta^{N-2}\overline{\partial}^*_{\omega}(\Delta^{-N}u)$, where $\overline{\partial}^*_{\omega}$ is the adjoint of $\overline{\partial}$ with respect to the metric ω on X, it is clear that $\Box_{-N}$ is an elliptic operator of polynomial growth with respect to Δ on Ω. Since $\overline{\partial}^*_{-N}u = 0$, and $\overline{\partial}u = f$, we have $\Box_{-N}u = \overline{\partial}^*_{-N}f$. From general results on domains with Lipschitz boundaries (see [Gri85]), we deduce that $\overline{\partial}^*_{-N}f \in \mathcal{C}^{N-k_0}_{p,q-1}(X, \overline{\Omega}, E) \cap \mathcal{C}^\infty_{p,q-1}(\Omega, E)$ for some k_0 not depending on N. The result then follows from Theorem 2.2.1, using a finite partition of unity.

More precisely, fix $z \in \partial\Omega$ and let U be a coordinate neighborhood of z. We assume that $\Lambda^{p,q}T^*X \otimes E$ is trivial over U. On $U \cap \Omega$, $u = (u_1, \ldots, u_r)$ can then be regarded as a mapping $U \cap \Omega \longrightarrow \mathbb{C}^r$, $r = \mathrm{rank}(\Lambda^{p,q}T^*X \otimes E)$. Moreover, $\Box_{-N}u$ is of the form $(Lu_1, \ldots, Lu_r)+$ lower order terms, where the lower order terms involve only derivatives of order at most 1 of u and multiplication by functions whose derivatives can be bounded by some power of Δ; L is an elliptic operator of order 2 on $U \cap \Omega$, which is of polynomial growth with respect to Δ.

Choose a function $\chi \in \mathcal{D}(U)$ which equals one in a neighborhood of z. Multiplying all functions by χ, we may assume that we are in $\mathbb{C}^n$ and may define Sobolev norms for mappings componentwise; hence we get from the a priori estimate (2.3)

$$\begin{aligned}\|\chi u\|^2_{s,U\cap\Omega} &\lesssim \|\Delta^{-ts}\Box_{-N}(\chi u)\|^2_{s-2,U\cap\Omega} + \|\Delta^{-Ts^2}\chi u\|^2_{0,U\cap\Omega} \\ &\lesssim \|\Delta^{-ts}\chi\Box_{-N}u\|^2_{s-2,U\cap\Omega} + \|\Delta^{-ts}\chi u\|^2_{s-1,U\cap\Omega} + \|\Delta^{-Ts^2}\chi u\|^2_{0,U\cap\Omega}\end{aligned}$$

(note that it follows from the proof of Theorem 2.2.1 that the lower order terms have no essential importance). By carefully looking at the proof of Theorem 2.2.1, we see that the term $\|\Delta^{-ts}\chi u\|^2_{s-1,U\cap\Omega}$ can be absorbed by the term $\|\Delta^{-Ts^2}\chi u\|^2_{0,U\cap\Omega}$, replacing t and T by larger integers if necessary. Putting this together with the above inequality, we have

$$\|\chi u\|^2_{s,U\cap\Omega} \lesssim \|\Delta^{-ts}\chi\Box_{-N}u\|^2_{s-2,U\cap\Omega} + \|\Delta^{-Ts^2}\chi u\|^2_{0,U\cap\Omega},$$

i.e. χu verifies the a priori estimate (2.3). Since Ω is relatively compact in X, we may even assume that t and T are independent of $z \in \partial\Omega$. Note that in the proof of Theorem 2.2.1, we have seen that it suffices to show the a priori estimate in order to prove the vanishing to some finite order at the boundary. Hence the theorem is proved. $\square$

We are now ready to prove the main theorem of this section.

Theorem 3.3.2
Let Ω be a relatively compact domain with Lipschitz boundary in an n-dimensional Kähler manifold (X,ω). We assume that Ω is $\log\delta$-pseudoconvex. Let E be a holomorphic vector bundle on X. Then we have

$$H^{p,q}(X,\overline{\Omega},E) = 0 \qquad \text{for } 1 \leq q \leq n-1$$

and

$$H^{p,n}(X,\overline{\Omega},E) \text{ is separated for the usual } \mathcal{C}^\infty - \text{topology}.$$

Moreover,

$$\overline{\partial}(\mathcal{C}^\infty_{p,n-1}(X,\overline{\Omega},E)) =$$
$$\bigcap_{N\in\mathbb{N}}\{f \in \mathcal{C}^\infty_{p,n}(X,\overline{\Omega},E) \mid \int_\Omega f\wedge h = 0 \ \forall h \in L^2_{n-p,0}(\Omega,E^*,N)\cap \mathrm{Ker}\overline{\partial}\}.$$

Proof: Replacing the vector bundle E by $\Lambda^p(T^{1,0}X)^*\otimes E$, it is no loss of generality to assume $p=0$.

We will begin by proving the following claim:
Let $f \in \mathcal{C}^k_{0,q}(X,\overline{\Omega},E)\cap\mathcal{C}^\infty_{0,q}(\Omega,E)\cap\mathrm{Ker}\overline{\partial}$, $1\leq q\leq n-1$, $k \gg 1$. Then there exists $u \in \mathcal{C}^{j(k)}_{0,q-1}(X,\overline{\Omega},E)\cap\mathcal{C}^\infty_{0,q-1}(\Omega,E)$ such that $\overline{\partial}u = f$ with $j(k)\sim\sqrt{k}$.

Proof of the claim:
Let $f \in \mathcal{C}^k_{0,q}(X,\overline{\Omega},E)\cap\mathcal{C}^\infty_{0,q}(\Omega,E)\cap\mathrm{Ker}\overline{\partial}$, $1\leq q\leq n-1$, $k\gg 1$. General results on Lipschitz domains (see e.g. [Gri85, Theorem 1.4.4.4] or Theorem

A.2.2) show that $f \in L^2_{0,q}(\Omega, E, -2k)$. Proposition 3.2.2 implies that there exists $u \in L^2_{0,q-1}(\Omega, E, -2k+2)$ such that $\overline{\partial} u = f$ in Ω. Moreover, choosing the minimal solution, we may assume $\overline{\partial}^*_{-2k} u = 0$. Applying Theorem 3.3.1, we then have $u \in \mathcal{C}^{j(k)}_{0,q-1}(X, \overline{\Omega}, E) \cap \mathcal{C}^{\infty}_{0,q-1}(\Omega, E)$ with $j(k) \sim \sqrt{k}$.

Let us now prove the theorem.

$H^{0,1}(X, \overline{\Omega}, E) = 0$ follows immediately from the above claim and the hypoellipticity of $\overline{\partial}$ in bidegree $(0,1)$.

Now assume $1 < q \leq n-1$ and let $f \in \mathcal{C}^{\infty}_{0,q}(X, \overline{\Omega}, E) \cap \mathrm{Ker}\overline{\partial}$. By induction, we will construct $u_k \in \mathcal{C}^{k}_{0,q-1}(X, \overline{\Omega}, E) \cap \mathcal{C}^{\infty}_{0,q-1}(\Omega, E)$ such that $\overline{\partial} u_k = f$ and $|u_{k+1} - u_k|_{j(k)-1} < 2^{-k}$. It is then clear that $(u_k)_{k\in\mathbb{N}}$ converges to $u \in \mathcal{C}^{\infty}_{0,q-1}(X, \overline{\Omega}, E)$ such that $\overline{\partial} u = f$.

Suppose that we have constructed $u_1, \ldots, u_k$. By the above claim, since $f \in \mathcal{C}^{\infty}_{0,q}(X, \overline{\Omega}, E)$, there exists $\alpha_{k+1} \in \mathcal{C}^{k+1}_{0,q-1}(X, \overline{\Omega}, E) \cap \mathcal{C}^{\infty}_{0,q-1}(\Omega, E)$ such that $f = \overline{\partial} \alpha_{k+1}$. We have $\alpha_{k+1} - u_k \in \mathcal{C}^{k}_{0,q-1}(X, \overline{\Omega}, E) \cap \mathcal{C}^{\infty}_{0,q-1}(\Omega, E) \cap \mathrm{Ker}\overline{\partial}$. Once again by the above claim, there exists $g \in \mathcal{C}^{j(k)}_{0,q-2}(X, \overline{\Omega}, E) \cap \mathcal{C}^{\infty}_{0,q-2}(\Omega, E)$ satisfying $\alpha_{k+1} - u_k = \overline{\partial} g$.

Since $\mathcal{C}^{\infty}_{0,q-2}(X, \overline{\Omega}, E)$ is dense in $\mathcal{C}^{j(k)}_{0,q-2}(X, \overline{\Omega}, E)$, there exists $g_{k+1} \in \mathcal{C}^{\infty}_{0,q-2}(X, \overline{\Omega}, E)$ such that $|g - g_{k+1}|_{j(k)} < 2^{-k}$.

Define $u_{k+1} = \alpha_{k+1} - \overline{\partial} g_{k+1} \in \mathcal{C}^{k+1}_{0,q-1}(X, \overline{\Omega}, E) \cap \mathcal{C}^{\infty}_{0,q-1}(\Omega, E)$. Then $\overline{\partial} u_{k+1} = f$ and $|u_{k+1} - u_k|_{j(k)-1} = |\overline{\partial} g - \overline{\partial} g_{k+1}|_{j(k)-1} \leq |g - g_{k+1}|_{j(k)} < 2^{-k}$. Thus u_{k+1} has the desired properties.

It remains to show that

$$\overline{\partial}(\mathcal{C}^{\infty}_{0,n-1}(X, \overline{\Omega}, E)) =$$

$$\bigcap_{N\in\mathbb{N}} \{f \in \mathcal{C}^{\infty}_{0,n}(X, \overline{\Omega}, E) \mid \int_{\Omega} f \wedge h = 0 \ \forall h \in L^2_{n,0}(\Omega, E^*, N) \cap \mathrm{Ker}\overline{\partial}\}.$$

This clearly implies that $H^{0,n}(X, \overline{\Omega}, E)$ is separated.

First of all, suppose $f = \overline{\partial} \alpha$ with $\alpha \in \mathcal{C}^{\infty}_{0,n-1}(X, \overline{\Omega}, E)$ and let $h \in$

$L^2_{n,0}(\Omega, E^*, N) \cap \mathrm{Ker}\overline{\partial}$. Then we have

$$\begin{aligned}\int_\Omega f \wedge h &= \int_\Omega \overline{\partial}\alpha \wedge h \\ &= \lim_{\varepsilon \to 0} \int_{\partial\Omega_\varepsilon} \alpha \wedge h \\ &= -\lim_{\varepsilon\to 0} \int_{\Omega\setminus\Omega_\varepsilon} \overline{\partial}\alpha \wedge h \\ &= -\lim_{\varepsilon\to 0} \int_{\Omega\setminus\Omega_\varepsilon} f \wedge h\end{aligned}$$

with $\Omega_\varepsilon \supset \{z \in \Omega \mid \Delta(z) > \varepsilon\}$ and

$$\begin{aligned}|\int_{\Omega\setminus\Omega_\varepsilon} f \wedge h| &\leq (\int_{\Omega\setminus\Omega_\varepsilon} |f|^2_\omega \Delta^{-N-2})^{1/2} \, (\int_{\Omega\setminus\Omega_\varepsilon} |h|^2_\omega \Delta^{N+2})^{1/2} \\ &\leq \varepsilon \|f\|_{-N-2} \|h\|_N \longrightarrow_{\varepsilon\to 0} 0\end{aligned}$$

which shows the inclusion $\subset$ (note that $f \in \mathcal{C}^\infty_{0,n}(X, \overline{\Omega}, E)$ implies $f \in L^2_{0,n}(\Omega, E, -N-1)$ for all $N \in \mathbb{N}$, cf Theorem A.2.2, and see the proof of Theorem 3.2.2 for the justification of some of the equalities).

Now, let us take $f \in \bigcap_{N\in\mathbb{N}}\{f \in \mathcal{C}^\infty_{0,n}(X, \overline{\Omega}, E) \mid \int_\Omega f \wedge h = 0 \ \forall h \in L^2_{n,0}(\Omega, E^*, N) \cap \mathrm{Ker}\overline{\partial}\}$.

We first show that for each $N \in \mathbb{N}$, $N \gg 1$, there exists $\beta_N \in L^2_{0,n-1}(\Omega, E, -N)$ satisfying $\overline{\partial}\beta_N = f$.

To see this, we define the linear operator

$$\begin{aligned}L_f : \qquad \mathrm{Im}(\overline{\partial} : L^2_{n,0}(\Omega, E^*, N) \to L^2_{n,1}(\Omega, E^*, N)) &\longrightarrow \mathbb{C} \\ \overline{\partial}\varphi &\longmapsto \int_\Omega f \wedge \varphi.\end{aligned}$$

First of all, notice that L_f is well-defined because of the moment conditions imposed on f.

By Proposition 3.2.1, $\mathrm{Im}(\overline{\partial} : L^2_{n,0}(\Omega, E^*, N) \to L^2_{n,1}(\Omega, E^*, N))$ is a closed subspace of $L^2_{n,1}(\Omega, E^*, N)$. Applying Banach's open mapping theorem, we know that L_f is a continuous linear operator and therefore extends to a continuous linear operator on the Hilbert space $L^2_{n,1}(\Omega, E^*, N)$ by the Hahn-Banach theorem. By the theorem of Riesz, there exists $\beta_N \in L^2_{0,n-1}(\Omega, E, -N)$

such that for every $\varphi \in L^2_{n,0}(\Omega, E^*, N)$ we have

$$(-1)^r \int_\Omega \beta_N \wedge \overline{\partial}\varphi = L_f(\varphi) = \int_\Omega f \wedge \varphi,$$

i.e. $\overline{\partial}\beta_N = f$.

Now the proof follows the same lines as above, and we construct $(u_k)_{k\in\mathbb{N}} \in \mathcal{C}^k_{0,n-1}(X, \overline{\Omega}, E)$ converging to $u \in \mathcal{C}^\infty_{0,n-1}(X, \overline{\Omega}, E)$ such that $\overline{\partial}u = f$, which concludes the proof. □

Corollary 3.3.3 (see [HI00])
Let $\Omega \subsetneq X$ be a $\mathcal{C}^\infty$-smooth domain in a compact Kähler manifold (X, ω) of complex dimension n. We assume that Ω is $\log\delta$-pseudoconvex. Let E be a holomorphic vector bundle on X. Assume that $H^{p,q}(X, E) = 0$ and put $D = X \setminus \overline{\Omega}$.
Then for every $\overline{\partial}$-closed form $f \in \mathcal{C}^\infty_{p,q}(\overline{D}, E)$, which is smooth up to the boundary, there exists $u \in \mathcal{C}^\infty_{p,q-1}(\overline{D}, E)$ such that $\overline{\partial}u = f$, $1 \leq q \leq n-2$.
For $q = n-1$, the same holds true if there exists $\tilde{f} \in \mathcal{C}^\infty_{p,n-1}(X, E)$ such that $\tilde{f}_{|\overline{D}} = f$, $\overline{\partial}\tilde{f}$ vanishes to infinite order on $\partial\Omega$ and $\int_\Omega \overline{\partial}\tilde{f} \wedge h = 0$ for all $h \in L^2_{n-p,0}(\Omega, E^*, N) \cap \mathrm{Ker}\overline{\partial}$, for all $N \in \mathbb{N}$.

Proof: Choose $\tilde{f} \in \mathcal{C}^\infty_{p,q}(X, E)$ such that $\tilde{f}_{|\overline{D}} = f$. Then $\overline{\partial}\tilde{f}$ vanishes to infinite order on $\partial\Omega$. Applying Theorem 3.3.2, there exists $h \in \mathcal{C}^\infty_{p,q}(X, \overline{\Omega}, E)$ such that $\overline{\partial}h = \overline{\partial}\tilde{f}$. $F := \tilde{f} - h$ is then a $\overline{\partial}$-closed $\mathcal{C}^\infty$ extension of f to X. As $H^{p,q}(X, E) = 0$, we have $F = \overline{\partial}u$ for some $u \in \mathcal{C}^\infty_{p,q-1}(X, E)$. Then $u_{|\overline{D}}$ has the desired properties. □

3.4 The $\overline{\partial}$-equation for extensible currents

The results of the previous section will allow us to solve the $\overline{\partial}$-equation for extensible currents by duality.

Let $\Omega \subset X$ be an open set in an n-dimensional complex manifold X. A current T defined on Ω is said to be *extensible*, if T is the restriction to Ω of a current defined on X.

It was shown in [Mar66] that if Ω satisfies $\overset{\circ}{\overline{\Omega}} = \Omega$ (which is always satisfied in our case), the vector space $\check{\mathcal{D}}'^{p,q}_{\Omega}(X)$ of extensible currents on Ω of bidegree (p,q) is the topological dual of $\mathcal{C}^{\infty}_{n-p,n-q}(X,\overline{\Omega}) \cap \mathcal{D}^{n-p,n-q}(X)$.

Theorem 3.4.1
Let Ω be a relatively compact domain with Lipschitz boundary in a Kähler manifold (X,ω). We assume that Ω is $\log\delta$-pseudoconvex.
Let $T \in \check{\mathcal{D}}'^{p,q}_{\Omega}(X)$ be an extensible current on Ω of bidegree (p,q), $q \geq 1$ such that $\overline{\partial}T = 0$ in Ω. Then there exists $S \in \check{\mathcal{D}}'^{p,q-1}_{\Omega}(X)$ satisfying $\overline{\partial}S = T$ in Ω.

Proof: Since Ω is relatively compact in X, we have $\mathcal{C}^{\infty}_{n-p,n-q}(X,\overline{\Omega}) \cap \mathcal{D}^{n-p,n-q}(X) = \mathcal{C}^{\infty}_{n-p,n-q}(X,\overline{\Omega})$. Let $T \in \check{\mathcal{D}}'^{p,q}_{\Omega}(X)$ be an extensible current on Ω of bidegree (p,q), $q \geq 1$, such that $\overline{\partial}T = 0$ in Ω.

Consider the operator

$$\begin{array}{rccc} L_T : & \overline{\partial}\mathcal{C}^{\infty}_{n-p,n-q}(X,\overline{\Omega}) & \longrightarrow & \mathbb{C} \\ & \overline{\partial}\varphi & \longmapsto & < T,\varphi > \end{array}$$

We first notice that L_T is well-defined. Indeed, let $\varphi \in \mathcal{C}^{\infty}_{n-p,n-q}(X,\overline{\Omega})$ be such that $\overline{\partial}\varphi = 0$.
If $q = n$, the analytic continuation principle for holomorphic functions yields $\varphi = 0$, so $< T,\varphi >= 0$.
If $1 \leq q \leq n-1$, one has $\varphi = \overline{\partial}\alpha$ with $\alpha \in \mathcal{C}^{\infty}_{n-p,n-q-1}(X,\overline{\Omega})$ by Theorem 3.3.2. As $\mathcal{D}^{n-p,n-q-1}(\Omega)$ is dense in $\mathcal{C}^{\infty}_{n-p,n-q-1}(X,\overline{\Omega})$, there exists $(\alpha_j)_{j\in\mathbb{N}} \in \mathcal{D}^{n-p,n-q-1}(\Omega)$ such that $\overline{\partial}\alpha_j \underset{j\to+\infty}{\longrightarrow} \overline{\partial}\alpha$ in $\mathcal{C}^{\infty}_{n-p,n-q}(X,\overline{\Omega})$.
Hence $< T,\varphi >=< T,\overline{\partial}\alpha >= \lim_{j\to+\infty} < T,\overline{\partial}\alpha_j >= 0$, because $\overline{\partial}T = 0$.

By Theorem 3.3.2, $\overline{\partial}\mathcal{C}^{\infty}_{n-p,n-q}(X,\overline{\Omega})$ is a closed subspace of $\mathcal{C}^{\infty}_{n-p,n-q+1}(X,\overline{\Omega})$, thus a Fréchet space. Using Banach's open mapping theorem, L_T is in fact continuous, so by the Hahn-Banach theorem, we can extend L_T to a continuous linear operator $\tilde{L}_T : \mathcal{C}^{\infty}_{n-p,n-q+1}(X,\overline{\Omega}) \longrightarrow \mathbb{C}$, i.e. $\tilde{L}_T$ is an extensible current on Ω satisfying

$$< \overline{\partial}\tilde{L}_T,\varphi >= (-1)^{p+q} < \tilde{L}_T,\overline{\partial}\varphi >= (-1)^{p+q} < T,\varphi >$$

for every $\varphi \in \mathcal{C}^\infty_{n-p,n-q}(X,\overline{\Omega})$. Therefore $T = (-1)^{p+q}\overline{\partial}\tilde{L}_T$. □

For the notion of differential forms admitting distribution boundary values, which is used in the following corollary, we refer the reader to [LT78].

Corollary 3.4.2
Let $\Omega \subset\subset X$ be a $\mathcal{C}^\infty$-smooth relatively compact domain in a Kähler manifold (X,ω). We assume that Ω is $\log\delta$-pseudoconvex. Let f be a smooth $\overline{\partial}$-closed $(0,1)$-form on Ω admitting a distribution boundary value on $\partial\Omega$.
Then there exists a smooth function g on Ω admitting a distribution boundary value on $\partial\Omega$ such that $\overline{\partial}g = f$ on Ω.

Proof: As f admits a distribution boundary value, we may view f as an extensible $\overline{\partial}$-closed current on Ω (see [LT78]). Applying Theorem 3.4.1, there exists an extensible current S of bidegree $(0,0)$ on Ω such that $\overline{\partial}S = T$.

The hypoellipticity of $\overline{\partial}$ in bidegree $(0,1)$ yields that S is in fact a $\mathcal{C}^\infty$-smooth function on Ω. But a $\mathcal{C}^\infty$-smooth function S, extensible as a current, such that $\overline{\partial}S$ admits a distribution boundary value, admits itself a distribution boundary value (see [Sam99, Lemme 4.3]). □

Corollary 3.4.3
Let $\Omega \subset\subset X$ be a $\mathcal{C}^\infty$-smooth domain in a Kähler manifold (X,ω). We assume that Ω is $\log\delta$-pseudoconvex. Then we have

$$H^q(\Omega,\check{\mathcal{O}}_\Omega) = 0$$

for every $q \geq 1$, where $\check{\mathcal{O}}_\Omega$ is the sheaf of germs on $\overline{\Omega}$ of holomorphic functions admitting a distribution boundary value.

Proof: We will show that

$$0 \longrightarrow \check{\mathcal{O}}_\Omega \longrightarrow \check{\mathcal{D}}'^{0,0}_\Omega(X) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \check{\mathcal{D}}'^{0,n}_\Omega(X) \longrightarrow 0$$

is an exact sequence of sheaves on $\overline{\Omega}$. Then the de Rham-Weil theorem yields

$$H^q(\Omega,\check{\mathcal{O}}_\Omega) \cong \frac{\mathrm{Ker}(\check{\mathcal{D}}'^{0,q}_\Omega(X) \xrightarrow{\overline{\partial}} \check{\mathcal{D}}'^{0,q+1}_\Omega(X))}{\mathrm{Im}(\check{\mathcal{D}}'^{0,q-1}_\Omega(X) \xrightarrow{\overline{\partial}} \check{\mathcal{D}}'^{0,q}_\Omega(X))}$$

and by Theorem 3.4.1, the right hand side of the above isomorphism is 0 for $q \geq 1$.

First of all, $\mathrm{Ker}(\check{\mathcal{D}}'^{0,0}_{\Omega}(X) \xrightarrow{\overline{\partial}} \check{\mathcal{D}}'^{0,1}_{\Omega}(X)) = \check{\mathcal{O}}_{\Omega}$ was proved in [LT78].

Now fix $z_0 \in \overline{\Omega}$ and let $B(z_0)$ be a small ball around z_0 such that either $B(z_0) \cap \partial\Omega = \emptyset$ or $B(z_0)$ intersects $\partial\Omega$ transversally. Set $V = \Omega \cap B(z_0)$. Applying Theorem 3.4.1, we conclude that for every $T \in \check{\mathcal{D}}'^{0,q}_{V}(X)$ satisfying $\overline{\partial}T = 0$ in V, there exists $S \in \check{\mathcal{D}}'^{0,q-1}_{V}(X)$ such that $\overline{\partial}S = T$ in V, $q \geq 1$. This proves the exactness of the rest of the above sequence. $\square$

Chapter 4

The weakly q-convex case

In this chapter, we consider the following situation:

Let Ω be a smooth bounded completely strictly pseudoconvex domain in a complex n-dimensional manifold X and M a real hypersurface of class $\mathcal{C}^\infty$ intersecting $\partial\Omega$ transversally, such that $\Omega \setminus M$ has exactly two connected components. We suppose that $M = \{\varrho = 0\}$ where ϱ is a $\mathcal{C}^\infty$ function whose Levi form has exactly p^+ positive, p^0 zero and p^- negative eigenvalues on $T_x^{1,0}M$ for each $x \in M$, $p^- + p^0 + p^+ = n-1$. We put $D = \Omega \cap \{\varrho < 0\}$.

We show that the $\overline{\partial}$-equation with exact support in D admits a solution in bidegrees (p,q), $0 \leq p \leq n$, $1 \leq q \leq p^+ + p^0$. Moreover, the range of $\overline{\partial}$ acting on smooth $(p, p^+ + p^0)$-forms with support in $\overline{D}$ is closed.

4.1 Basic properties of weakly q-convex domains

Let X be an n-dimensional complex manifold. Let ψ be a real-valued $\mathcal{C}^2$-function on X and $x \in X$. Then we define the hermitian form $\mathcal{L}(\psi, x)$ on $T_x^{1,0}X$ -the Levi form of ψ- as follows:
Choose holomorphic coordinates $(z_1, \ldots, z_n)$ in a neighborhood of x and set

$$\mathcal{L}(\psi, x)\xi = \sum_{j,k=1}^{n} \frac{\partial^2 \psi}{\partial \overline{z}_j \partial z_k}(x)\overline{\xi}_j \xi_k$$

if $\xi = \sum_{j=1}^n \xi_j \frac{\partial}{\partial z_j}(x) \in T_x^{1,0}X$.

By $p^+_\psi(x)$ (resp. $p^0_\psi(x)$, resp. $p^-_\psi(x)$) we denote the number of positive (resp. zero, resp. negative) eigenvalues of $\mathcal{L}(\psi, x)$.

We say that ψ is a q-convex function if $\mathcal{L}(\psi, x)$ has at least q positive eigenvalues for each $x \in X$, i.e. $p^+_\psi(x) \geq q$ for all $x \in X$.

Let M be a smooth hypersurface in X. We denote by $T^{1,0}_x M$ the holomorphic tangent space of M at x.

Such a hypersurface can be represented locally in the form

$$M \cap U = \{z \in U \mid \varrho(z) = 0\},$$

where ϱ is a real valued $\mathcal{C}^\infty$ function in an open subset U of X. In this representation, we have

$$T^{1,0}_x M = \{\sum_{j=1}^n \zeta_j \frac{\partial}{\partial z_j} \in T^{1,0}_x X \mid \sum_{j=1}^n \frac{\partial \varrho}{\partial z_j}(x)\zeta_j = 0\},$$

where $(z_1, \ldots, z_n)$ are local holomorphic coordinates in a neighborhood of x.

Now let $\Omega \subset\subset X$ be a domain with $\mathcal{C}^2$ boundary:

$$\Omega \cap U = \{z \in U \mid \varrho(z) < 0\}$$

where U is an open neighborhood of $\partial\Omega$ and ϱ is a function of class $\mathcal{C}^2$ on U satisfying $d\varrho \neq 0$ on $\partial\Omega = \{z \in U \mid \varrho(z) = 0\}$.

Let $z \in \partial\Omega$. By $p^+_{\partial\Omega}(z)$ (resp. $p^0_{\partial\Omega}(z)$, resp. $p^-_{\partial\Omega}(z)$) we denote the number of positive (resp. zero, resp. negative) eigenvalues of $\mathcal{L}(\varrho, z)_{|T^{1,0}_z \partial\Omega}$. This number is independent of the defining function ϱ for Ω. We say that Ω is weakly q-convex if $p^-_{\partial\Omega}(z) \leq n - q - 1$ for all $z \in \partial\Omega$.

It is well known that an open set with $\mathcal{C}^2$ boundary, whose Levi form is semi-positive at each boundary point, admits a strictly plurisubharmonic exhaustion function. The following two lemmas generalize this property to domains whose Levi form also has some negative eigenvalues. They have been proved in [Mic93].

Lemma 4.1.1
Let $\Omega \subset \mathbb{C}^n$ be an open set with $\mathcal{C}^2$ boundary. Let r_Ω be the function defined

by $r_\Omega(z) = -\text{dist}(z, \partial\Omega)$ *if* $z \in \Omega$, $r_\Omega(z) = \text{dist}(z, \partial\Omega)$ *if* $z \notin \Omega$.
Let $\zeta \in \partial\Omega$ *and assume that* $p^-_{\partial\Omega}(z) \leq s$ *for every* $z \in \partial\Omega$ *close to* ζ. *Then there exists an open neighborhood* U *of* ζ *such that* $\mathcal{L}(-\log|r_\Omega|, z)$ *has at most* s *negative eigenvalues for each* $z \in \Omega \cap U$.

Proof: Define $\varphi = -\log|r_\Omega|$. Let V be a neighborhood of ζ such that r_Ω is of class $\mathcal{C}^2$ on V and such that on $V \cap \partial\Omega$, $p^-_{\partial\Omega} \leq s$. Let $U \subset V$ be a sufficiently small open neighborhood of ζ such that the orthogonal projection on $\partial\Omega$, π, is defined on U and satisfies $\pi(U) \subset V \cap \partial\Omega$.

Let $z \in U \cap \Omega$ and define $k = p^-_\varphi(z)$. Then there exists a k-dimensional subspace E of $\mathbb{C}^n$ such that $\mathcal{L}(\varphi, z)_{|E}$ is negative definite.

Let $w = (w_1, \ldots, w_n) \in E \setminus \{0\}$. We consider the function f, defined in an open neighborhood of 0 in $\mathbb{C}^n$ by

$$f(\tau) = \varphi(z + \tau w) = -\log|r_\Omega(z + \tau w)|.$$

f is of class $\mathcal{C}^2$ in a neighborhood of 0 and Taylor's formula implies

$$-f(\tau) = -\varphi(z) + \text{Re}(A\tau + B\tau^2) + c|\tau|^2 + o(|\tau|^2), \quad \tau \to 0 \tag{4.1}$$

with $A = -2\partial\varphi(z)(w) = -2\sum_{j=1}^n \frac{\partial\varphi}{\partial z_j}(z)w_j$, $B = -\sum_{j,k=1}^n \frac{\partial^2\varphi}{\partial z_j \partial z_k}(z) w_j w_k$ and $c = -\mathcal{L}(\varphi, z)w > 0$.

Let $P = \pi(z) \in V \cap \partial\Omega$. Set $a = P - z$ and

$$z(\tau) = z + \tau w + a e^{A\tau + B\tau^2}.$$

If τ is suffiently small, (4.1) yields

$$\text{dist}(z(\tau), \partial\Omega) \geq \text{dist}(z + \tau w, \partial\Omega) - |a e^{A\tau + B\tau^2}| \geq |a|(e^{c|\tau|^2/2} - 1)|e^{A\tau + B\tau^2}|.$$

The function

$$g : \tau \longmapsto -r_\Omega(z(\tau))$$

has therefore a minimum at the point $\tau = 0$. We must therefore have $dg(0) = 0$ and $(-\frac{\partial^2 r_\Omega \circ z}{\partial\tau\partial\bar\tau})_{|\tau=0} > 0$. Hence $z'(0) = w - 2\partial\varphi(z)(w)a \in T^{1,0}_P\partial\Omega$ and $\mathcal{L}(r_\Omega, P)z'(0) < 0$. We observe that $\mathcal{L}(r_\Omega, P)z'(0) < 0$ implies in particular that $z'(0) \neq \{0\}$.

Let L be the endomorphism of $\mathbb{C}^n$ defined by $L(u) = u - 2\partial\varphi(z)(u)a$; in fact, L is the orthogonal projection of $\mathbb{C}^n$ onto $T^{1,0}_P\partial\Omega$. We set $F = L(E)$.

Then the preceeding computations show that $F \subset T_P^{1,0}\partial\Omega$, $\dim F = k$ and $\mathcal{L}(r_\Omega, P)_{|F}$ is negative definite. But by hypothesis, $p^-_{\partial\Omega}(P) \le s$. Hence $k \le s$, which proves the lemma. $\square$

Lemma 4.1.2
Let $\Omega \subset\subset \mathbb{C}^n$ be an open set with C^k boundary, $k \ge 2$. We assume that $p^-_{\partial\Omega}(\zeta) \le s$ for all $\zeta \in \partial\Omega$. Then, for all $A > 0$, there exists a defining function $r \in \mathcal{C}^k(\overline{\Omega}, \mathbb{R})$ for Ω such that $\mathcal{L}(-\log|r|, z)$ has $n-s$ positive eigenvalues, greater than or equal to A, for each $z \in \Omega$.

Proof: We set $\varphi_\Omega = -\log|r_\Omega|$ (see Lemma 4.1.1). Let V be a small open neighborhood of $\partial\Omega$ where r_Ω is of class C^k. By Lemma 4.1.1, $\mathcal{L}(\varphi_\Omega, s)$ has $n-s$ nonnegative eigenvalues for all $z \in V \cap \Omega$ if V is small enough. We set

$$\varphi = \chi \circ \varphi_\Omega + A|z|^2,$$

where χ is a convex increasing $\mathcal{C}^\infty$ function on $\mathbb{R}$ such that

$$\chi(x) = c + \frac{1}{2} \text{ if } x \in]-\infty, c],$$

$$\chi(x) = x \text{ if } x \in [c+1, +\infty[$$

with c such that $\Omega \setminus V \subset \{\varphi_\Omega < c\}$. Since χ is a convex increasing function, we have

$$\mathcal{L}(\varphi, \cdot)\xi = \chi'(\varphi_\Omega)\mathcal{L}(\varphi_\Omega, \cdot)\xi + \chi''(\varphi_\Omega)|\partial\varphi_\Omega(\xi)|^2 + A|z|^2 \ge \chi'(\varphi_\Omega)\mathcal{L}(\varphi_\Omega, \cdot)\xi + A|z|^2.$$

If $z \in \{\varphi_\Omega \le c\}$, $\mathcal{L}(\varphi, z)\xi = A|\xi|^2$, thus all the eigenvalues of $\mathcal{L}(\varphi, z)$ are $\ge A$. If $z \in \Omega \setminus \{\varphi_\Omega \le c\}$, $z \in \Omega \cap V$ and $\mathcal{L}(\varphi, z)$ has $n-s$ positive eigenvalues $\ge A$.

Let r be the function defined by

$$r(z) = e^{-A|z|^2} r_\Omega(z) \text{ if } z \notin \Omega,$$

$$r(z) = -e^{-\varphi(z)} \text{ if } z \in \Omega.$$

By definition, $r = \exp(-A|z|^2)r_\Omega$ on the set $\{|r_\Omega| \le \exp(-c-1)\}$ and r is thus a defining function of class $\mathcal{C}^k$ of Ω. As $\varphi = -\log|r|$ in Ω, r has the desired properties. $\square$

If one replaces $\mathbb{C}^n$ in Lemma 4.1.2 by an arbitrary Stein manifold X, one obtains a similar result, which is, however, more difficult to prove (see

[Mat96, Proposition 7.2] and its proof):

Theorem 4.1.3
Let X be an n-dimensional Stein manifold, ω a complete Kähler metric on X and Ω be a weakly q-convex domain with $\mathcal{C}^2$ boundary. Then there exists a positive function $\delta_{\partial\Omega}$ of class $\mathcal{C}^2$ on Ω, which coincides with the boundary distance function of Ω with respect to ω near $\partial\Omega$, having the following property:
There exists $c > 0$ and a smooth bounded plurisubharmonic function h on Ω such that $\mathcal{L}(-\log \delta_{\partial\Omega} + h, \cdot)$ has at least $(q+1)$ positive eigenvalues which are $\geq c$ with respect to ω.

Let $\Omega \subset\subset X$ be a nonempty domain. We say that Ω is completely strictly pseudoconvex if there exists a function ϱ of class $\mathcal{C}^2$ in a neighborhood $U_{\overline{\Omega}}$ of $\overline{\Omega}$ such that $\Omega = \{z \in U_{\overline{\Omega}} \mid \varrho(z) < 0\}$ and such that $\mathcal{L}(\varrho, x)$ is positive definite for all $x \in U_{\overline{\Omega}}$.

The remainder of this chapter is dedicated to the study of the $\overline{\partial}$-equation with exact support in a certain domain which is a transversal intersection of a completely strictly pseudoconvex domain with smooth boundary and a weakly q-convex domain with smooth boundary. In particular, it follows from Theorem 4.1.3 that such a domain is piecewise smooth and is a q-convex manifold, i.e. it admits a $(q+1)$-convex exhaustion function.

4.2 Construction of a family of metrics

Let Ω be a smooth bounded completely strictly pseudoconvex domain in a complex n-dimensional manifold X and M a real hypersurface of class $\mathcal{C}^\infty$ intersecting $\partial\Omega$ transversally, such that $\Omega \setminus M$ has exactly two connected components. We suppose that $M = \{\varrho = 0\}$ where ϱ is a $\mathcal{C}^\infty$ function whose Levi form has exactly p^+ positive, p^0 zero and p^- negative eigenvalues on $T_x^{1,0}M$ for each $x \in M$, $p^- + p^0 + p^+ = n-1$. We put $D = \Omega \cap \{\varrho < 0\}$.

As Ω is completely strictly pseudoconvex, there exists a neighborhood $U_{\overline{\Omega}}$ of $\overline{\Omega}$ in X and a strictly pseudoconvex smooth function ψ on $U_{\overline{\Omega}}$ such that $\Omega = \{z \in U_{\overline{\Omega}} \mid \psi(z) < 0\}$. We define $\omega_g = i\partial\overline{\partial}\psi$. ω_g is then a hermitian metric on $U_{\overline{\Omega}}$.

We can find a weakly $(0+p+)$-convex domain $\widetilde{\Omega} \subset\subset X$ with smooth boundary such that $M \cap \overline{\Omega} \subset \partial\widetilde{\Omega}$. Then by Theorem 4.1.3, there exists $c>0$ and a smooth defining function δ_M for M, defined on a neighborhood V of $M \cap \overline{\Omega}$ such that for every $x \in V \cap \{\varrho < 0\}$, $\mathcal{L}(-\log \delta_M, x)$ has p^- negative eigenvalues less than or equal to $\frac{-c}{\delta_M(x)}$, p^0 positive eigenvalues greater or equal to c and $p^+ + 1$ positive eigenvalues greater than or equal to $\frac{c}{\delta_M(c)}$ with respect to ω_g. For later convenience, we set $V^- = V \cap \{\varrho < 0\}$.

The proof of the following lemma basically follows from the proof of Proposition 2.3 in [Mic93]. However, since we have made some adjustments and precisions, we include the complete proof.

Lemma 4.2.1
Fix $x_0 \in M \cap \overline{\Omega}$. Then there exists a neighborhood U of x_0 in X and a smooth orthonormal basis $(\zeta_1(x), \dots, \zeta_n(x))$ of $(T_x^{1,0}X)^*$ with respect to ω_g on U such that on $U \cap D$ we have

$$\begin{aligned}
\mathcal{L}(x) &:= -i\partial\overline{\partial} \log \delta_M(x) \\
&= \sum_{\mu,\nu=1}^{p^-} a^-_{\mu\nu}(x)\zeta_\mu(x) \wedge \overline{\zeta}_\nu(x) + \sum_{\mu,\nu=p^-+1}^{p^-+p^0} a^0_{\mu\nu}(x)\zeta_\mu(x) \wedge \overline{\zeta}_\nu(x) \\
&\quad + \sum_{\mu,\nu=p^-+p^0+1}^{n-1} a^+_{\mu\nu}(x)\zeta_\mu(x) \wedge \overline{\zeta}_\nu(x) + a_n(x)\zeta_n(x) \wedge \overline{\zeta}_n(x) \\
&= \mathcal{L}^-(x) \oplus \mathcal{L}^0(x) \oplus \mathcal{L}^+(x) \oplus \mathcal{L}^n(x)
\end{aligned}$$

such that $\mathcal{L}^-(x)$ has p^- eigenvalues smaller than the p^0 eigenvalues of $\mathcal{L}^0(x)$, which in turn are smaller than the p^+ eigenvalues of $\mathcal{L}^+(x)$, and $a_n(x)$ is the biggest eigenvalue of $\mathcal{L}(x)$.

Moreover, if $(L_1(x), \dots, L_n(x))$ is the dual basis of $(\zeta_1(x), \dots, \zeta_n(x))$, we can arrange that

(i) $[L_\alpha, L_\beta](x) \in \mathrm{Span}(L_{p^-+1}(x), \dots, L_{p^-+p^0}(x))$ *for $x \in M$ and $\alpha, \beta \in \{p^-+1, \dots, p^-+p^0\}$*

(ii) $[L_\alpha, L_\beta](x) \in \mathrm{Span}(L_1(x), \dots, L_{n-1}(x))$ *for $\alpha, \beta \in \{1, \dots, n-1\}$ and $x \in M$*

(iii) $[L_\alpha, \overline{L}_\beta](x) \in \mathrm{Span}(L_{p^-+1}(x), \dots, L_{p^-+p^0}(x), \overline{L}_{p^-+1}(x), \dots, \overline{L}_{p^-+p^0}(x))$ *for $x \in M$ and $\alpha, \beta \in \{p^-+1, \dots, p^-+p^0\}$*

(iv) $[L_\alpha, \overline{L}_\beta](x) \in \mathrm{Span}(L_1(x), \ldots, L_{n-1}(x), \overline{L}_1(x), \ldots, \overline{L}_{n-1}(x))$ *for* $\alpha \in \{1, \ldots, n-1\}$, $\beta \in \{p^- + 1, \ldots, p^- + p^0\}$ *and* $x \in M$

Proof: The Levi form of M at the point x is the bilinear map $\mathcal{L}_x$: $\{T_x^{1,0}M \oplus \overline{T_x^{1,0}M}\} \times \{T_x^{1,0}M \oplus \overline{T_x^{1,0}M}\} \longrightarrow \{T_xM \otimes \mathbb{C}\}/\{T_x^{1,0}M \oplus \overline{T_x^{1,0}M}\}$ defined by $\mathcal{L}_x(X_x, Y_x) = \frac{1}{2i}\pi_x[X_x, Y_x]_x$, where π_x is the projection $\{T_xM \otimes \mathbb{C}\} \longrightarrow \{T_xM \otimes \mathbb{C}\}/\{T_x^{1,0}M \oplus \overline{T_x^{1,0}M}\}$. Since, by hypothesis on M, the Levi form of M has exactly p^0 zero eigenvalues everywhere, $N^{1,0}M = \cup_{x \in M} N_x^{1,0}M$, where

$$N_x^{1,0}M = \{L_x \in T_x^{1,0}M \mid \mathcal{L}_x(L_x, Y_x) = 0 \ \forall\ Y_x \in T_x^{1,0}M\}$$

is the Levi null set at x, forms a subbundle of $T_x^{1,0}M$ of rank p^0. Moreover, it is easy to see (use the Jacobi identity, cf [Fre76]) that $N^{1,0}M \oplus \overline{N}^{1,0}M$ is involutive.

Now fix $x_0 \in M$. We may then choose a subbundle $N = \cup_x N_x$ of rank p^0 of $T^{1,0}X$ on a neighborhood V of x_0 in X such that $N_x = N_x^{1,0}M$ for $x \in M \cap V$. Moreover, we may assume that N is itself a subbundle of $T = \mathrm{Ker}\partial\delta_M \cap T^{1,0}X$. Note that T is a subbundle of $T^{1,0}X$ of rank $(n-1)$ on V such that $T_x = T_x^{1,0}M$ for $x \in M \cap V$.

Let $\lambda_1^x \leq \ldots \leq \lambda_{n-1}^x$ be the eigenvalues of $\mathcal{M}(x) := i\partial\overline{\partial}\delta_M(x)_{|T_x}$. It is well known that the functions $x \mapsto \lambda_\nu^x$ are continuous on V. Using the assumptions on M, we have

$$\lambda_{p^-}^x < 0 = \lambda_{p^-+1}^x = \ldots = \lambda_{p^-+p^0}^x < \lambda_{p^-+p^0+1}^x$$

for every $x \in M \cap V$. For a small $\varepsilon > 0$, we therefore get a neighborhood W of x_0 in X such that for $x \in W$

$$\lambda_{p^-}^x < -\varepsilon,\ \lambda_{p^-+p^0+1}^x > \varepsilon,$$

$$\lambda_i^x \in (-\varepsilon, \varepsilon) \text{ for } i = p^- + 1, \ldots, p^- + p^0$$

Moreover, we can find $R > 0$ such that all the λ_ν^x are of absolute value smaller than R for each $x \in W$.

Intersecting the cercle of radius R centered at 0 with the lines $[-\varepsilon + i\mathbb{R}]$ and $[\varepsilon + i\mathbb{R}]$, we obtain three closed paths $\Gamma^-,\ \Gamma^0$ and Γ^+ such that for $x \in W$, none of the eigenvalues of $\mathcal{M}(x)$ lies on $\Gamma^-,\ \Gamma^0$ or Γ^+.

We may assume that W is small enough such that there exists a smooth orthonormal basis $X_1, \ldots, X_{n-1}$ of T on W such that $X_{p^-+1}, \ldots, X_{p^-+p^0}$ is a smooth basis of N on W.

We denote by $M(x)$ the matrix of $\mathcal{M}(x)$ in the basis $X_1, \ldots, X_{n-1}$ and by $(e_1, \ldots, e_{n-1})$ the standard basis of $\mathbb{C}^{n-1}$. We then have $\mathrm{Ker} M(x) = \mathrm{Span}(e_{p^-+1}, \ldots, e_{p^-+p^0})$ for every $x \in M \cap W$.

For $x \in W$, we may set

$$\begin{aligned} \Pi^-(x) &= \frac{1}{2i\pi} \int_{\Gamma^-} (M(x) - z\mathrm{Id})^{-1} dz, \\ \Pi^0(x) &= \frac{1}{2i\pi} \int_{\Gamma^0} (M(x) - z\mathrm{Id})^{-1} dz, \\ \Pi^+(x) &= \frac{1}{2i\pi} \int_{\Gamma^+} (M(x) - z\mathrm{Id})^{-1} dz \end{aligned}$$

Then Π^-, Π^0 and Π^+ are $\mathcal{C}^\infty$ mappings in a neighborhood of x_0 (e.g. Π^- is the composition of the $\mathcal{C}^\infty$ map $x \mapsto M(x)$ and the holomorphic mapping from the space of hermitian $(n-1) \times (n-1)$ matrices to itself given by $A \mapsto \frac{1}{2i\pi} \int_{\Gamma^-} (A - z\mathrm{Id})^{-1} dz$). It is easy to see that $\Pi^-(x)$, $\Pi^0(x)$ and $\Pi^+(x)$ are the orthogonal projections of $\mathbb{C}^{n-1}$ onto

$$\begin{aligned} E^-(x) &= \sum_{\nu=1}^{p^-} \mathrm{Ker}(M(x) - \lambda_\nu^x \mathrm{Id}), \\ E^0(x) &= \sum_{\nu=p^-+1}^{p^-+p^0} \mathrm{Ker}(M(x) - \lambda_\nu^x \mathrm{Id}) \qquad \text{and} \\ E^+(x) &= \sum_{\nu=p^-+p^0+1}^{n-1} \mathrm{Ker}(M(x) - \lambda_\nu^x \mathrm{Id}) \end{aligned}$$

For every $x \in M \cap W$ we have $E^0(x) = \mathrm{Span}(e_{p^-+1}, \ldots, e_{p^-+p^0})$. Therefore, if W is small enough, the vectors

$$\tilde{e}_{p^-+1}(x) := \Pi^0(x)(e_{p^-+1}), \ldots, \tilde{e}_{p^-+p^0} := \Pi^0(x)(e_{p^-+p^0})$$

form a basis for $E^0(x)$. After a permutation of some indices, we can also achieve that

$$\tilde{e}_1(x) := \Pi^-(x)(e_1), \ldots, \tilde{e}_{p^-}(x) := \Pi^-(x)(e_{p^-})$$

span $E^-(x)$ and that

$$\tilde{e}_{p^-+p^0+1}(x) := \Pi^+(x)(e_{p^-+p^0+1}), \ldots, \tilde{e}_{n-1}(x) := \Pi^+(x)(e_{n-1})$$

span $E^+(x)$. Due to the Gram-Schmidt orthonormalization procedure and the fact that eigenvectors associated to different eigenvalues are orthogonal, we may assume that $(\tilde{e}_1(x), \ldots, \tilde{e}_{n-1}(x))$ is an orthonormal basis for the standard scalar product on $\mathbb{C}^{n-1}$.

We define $l_{ij}(x)$ by $\tilde{e}_i(x) = \sum_{j=1}^{n-1} l_{ij}(x)e_j$ and set $L_i(x) = \sum_{j=1}^{n-1} l_{ij}(x)X_j(x)$. Then $(L_1(x), \ldots, L_{n-1}(x))$ is an orthonormal basis of T on W. Moreover, we have $N^{1,0}_x M = \mathrm{span}(L_{p^-+1}(x), \ldots, L_{p^-+p^0}(x))$ for $x \in W \cap M$.

Now we apply the same procedure as above to the hermitian form $\delta_M^2(x)\mathcal{L}(x) = -i\delta_M(x)\partial\overline{\partial}\delta_M + i\partial\delta_M \wedge \overline{\partial}\delta_M$ on $T^{1,0}X$. We observe that this hermitian form has $(n-1)$ eigenvalues which vanish on M as well as 1 eigenvalue which is positive on M. After possibly shrinking W, we then obtain a unitary vector $L_n \in T^{1,0}X$ on W, depending smoothly on x, which is an eigenvector of $\mathcal{L}(x)$ and which is orthogonal to $L_1(x), \ldots, L_{n-1}(x)$ with respect to ω_g.

Let $(\zeta_1(x), \ldots, \zeta_n(x)) \in (T^{1,0}X)^*$ be the dual basis of $(L_1(x), \ldots, L_n(x))$ on W. This basis then gives the desired decomposition of $\mathcal{L}(x)$ on W. The assertion (ii) follows because $T^{1,0}M$ is stable under $[\ ,\ ]$. Moreover, since $N^{1,0}M \oplus \overline{N^{1,0}M}$ is involutive, we get (i) and (iii). Finally, (iv) follows by definition of $N^{1,0}M$. $\square$

Let $\delta_{D,g}$ be the boundary distance function of D with respect to ω_g. $\delta_{D,g}$ will not be smooth since D is only a Lipschitz domain. However, like in Chapter 3, Theorem A.1.2 (see also [Ste70]) provides us with a regularized distance having essentially the same profile as $\delta_{D,g}$:

There exists a function $\Delta \in \mathcal{C}^\infty(D, \mathbb{R})$ satisfying

$$c_1\delta_{D,g}(x) \leq \Delta(x) \leq c_2\delta_{D,g}(x) \qquad \text{and}$$

$$\left|\frac{\partial^\alpha}{\partial x^\alpha}\Delta(x)\right| \leq B_\alpha(\delta_{D,g}(x))^{1-|\alpha|},$$

where $x = (x_1, \ldots, x_{2n})$ are local coordinates on X. B_α, c_1 and c_2 are independent of D.

We also need to define a regularized maximum function. For each $\beta > 0$, let χ_β be a fixed non negative real $\mathcal{C}^\infty$-function on $\mathbb{R}$ such that, for all $x \in \mathbb{R}$, $\chi_\beta(x) = \chi_\beta(-x)$, $|x| \le \chi_\beta(x) \le |x| + \beta$, $|\chi'_\beta(x)| \le 1$, $\chi''_\beta(x) \ge 0$ and $\chi_\beta(x) = |x|$ if $|x| \ge \frac{\beta}{2}$. We moreover assume that $\chi'_\beta(x) > 0$ if $x > 0$ and $\chi'_\beta(x) < 0$ if $x < 0$. We set $\max_\beta(t,s) = \frac{t+s}{2} + \chi_\beta(\frac{t-s}{2})$ for $t, s \in \mathbb{R}$.

We omit the proof of the following simple lemma:

Lemma 4.2.2
Let φ, ψ be two real-valued $\mathcal{C}^2$-functions on some real $\mathcal{C}^2$ manifold X. Then, for all $\beta > 0$, and $x \in X$, the following assertions hold:

(i) $\max(\varphi(x), \psi(x)) \le \max_\beta(\varphi(x), \psi(x)) \le \max(\varphi(x), \psi(x)) + \beta$

(ii) $\max_\beta(\varphi(x), \psi(x)) = \max(\varphi(x), \psi(x))$ *if* $|\varphi(x) - \psi(x)| \ge \beta$

(iii) There is a number $\lambda_x(\varphi, \psi)$ with $0 \le \lambda_x(\varphi,\psi) \le 1$, namely

$$\lambda_x(\varphi, \psi) = \frac{1}{2} + \frac{1}{2}\chi'_\beta(\frac{\varphi(x) - \psi(x)}{2}),$$

such that

$$\begin{aligned} \mathcal{L}(\max_\beta(\varphi, \psi), x) &= \lambda_x(\varphi,\psi)\mathcal{L}(\varphi, x) + (1 - \lambda_x(\varphi,\psi))\mathcal{L}(\psi, x) \\ &\quad + \frac{1}{4}\chi''_\beta(\frac{\varphi - \psi}{2})\partial(\varphi - \psi) \wedge \overline{\partial}(\varphi - \psi)(x) \end{aligned}$$

Finally, we write $a \lesssim b$ (resp. $b \gtrsim a$) if there exists an *absolute* constant $C > 0$ such that $a \le C \cdot b$ (resp. $b \ge C \cdot a$). We write $a \sim b$ if $a \lesssim b$ and $a \gtrsim b$.

For some $\beta > 0$, we define $\varphi = \max_\beta(-\log \delta_M, -\log(-\psi)) \in \mathcal{C}^\infty(D)$. Then φ is an exhaustion function for D and (i) of Lemma 4.2.2 implies

$$\max(-\log \delta_M, -\log(-\psi)) \le \varphi \le \max(-\log \delta_M, -\log(-\psi)) + \beta,$$

thus

$$e^{-\beta} \min(\delta_M, -\psi) \le e^{-\varphi} \le \min(\delta_M, -\psi).$$

Hence $e^{-\varphi} \sim \Delta$.

We set $D_j = \{z \in D \mid e^{-\varphi(x)} > \frac{1}{j}\}$.

The following technical lemma is the key point of this chapter. It permits to obtain L^2-vanishing theorems on D with powers of the boundary distance as weight functions.

Lemma 4.2.3
There exists a hermitian metric ω_M on D and a family $(\omega_j)_{j\in\mathbb{N}}$ of complete hermitian metrics on D with the following properties:

(i) $\omega_j = \omega_M$ *on a neighborhood of* $\overline{D}_j$, $\omega_j \geq \omega_M$ *on* D.

(ii) Let $\gamma_1 \leq \ldots \leq \gamma_n$ *be the eigenvalues of* $i\partial\overline{\partial}\varphi$ *with respect to* ω_M. *There exists* $\sigma > 0$ *such that* $\gamma_1 + \ldots + \gamma_r > \sigma$ *for* $r \geq n - p^+ - p^0$.

(iii) There are constants $a, b > 0$ *such that* $a\ \omega_g \leq \omega_M \leq b\ \delta_M^{-2}\omega_g$ *for all* $j \in \mathbb{N}$.

(iv) There is a constant $C > 0$ *such that* $|\partial\omega_M|_{\omega_M} \leq C$.

(v) Let $\omega_M = i\sum_{\mu\nu}\omega_M^{\mu\nu} dz_\mu \wedge d\overline{z}_\nu$ *on* $U \cap D$, *where* U *is a neighborhood of* $x \in M$ *and* $(z_1, \ldots, z_n)$ *are local holomorphic coordinates on* U. *Then, for every multiindex* α, *there exists a constant* C_α *such that* $\sup_{\mu\nu} |D^\alpha\omega_M^{\mu\nu}(z)| \leq C_\alpha\ \delta_M^{-2-|\alpha|}(z)$ *for every* $z \in U \cap D$.

Proof: Let $A_g \in \mathcal{C}^\infty(\mathrm{End}T\Omega)$ be the hermitian endomorphism associated to the hermitian form $-i\partial\overline{\partial}\log\delta_M$ with respect to ω_g and let $\gamma_1^g \leq \ldots \leq \gamma_n^g$ be the eigenvalues of A_g.

We have $-i\partial\overline{\partial}\log\delta_M = \frac{i}{-\delta_M}\partial\overline{\partial}\delta_M + i\partial\log\delta_M \wedge \overline{\partial}\log\delta_M$. Thus there is a constant $c > 0$ such that

$$\begin{aligned}
&\gamma_1^g(x) \leq -\frac{c}{\delta_M}, \ldots,\ \gamma_{p^-}^g(x) \leq -\frac{c}{\delta_M},\\
&\gamma_{p^-+1}^g(x) \geq c, \ldots,\ \gamma_{p^-+p^0}^g(x) \geq c,\\
&\gamma_{p^-+p^0+1}^g(x) \geq \frac{c}{\delta_M}, \ldots,\ \gamma_{n-1}^g(x) \geq \frac{c}{\delta_M},\ \text{and}\\
&\gamma_n^g(x) \geq c\ |\partial\log\delta_M|_g^2(x)
\end{aligned}$$

for every $x \in V^-$, after possibly shrinking V.

Moreover, we claim that there exists a constant $c' > 0$ such that

$$\gamma_{p^-+1}^g(x) \leq c', \ldots,\ \gamma_{p^-+p^0}^g(x) \leq c'$$

This can be seen as follows:
Fix $x_0 \in M$. As in the proof of Lemma 4.2.1, there exists a neighborhood U of x_0 in X and a smooth extension T of $T^{1,0}M$ on U, such that for every $x \in U$

$$\mathcal{M}(x) := i\partial\overline{\partial}\delta_{M|T_x} = \mathcal{M}^-(x) \oplus \mathcal{M}^0(x) \oplus \mathcal{M}^+(x)$$

in a smooth orthonormal basis with respect to ω_g on U, such that the eigenvalues of $\mathcal{M}^-(x)$ are the p^- smallest eigenvalues of $\mathcal{M}(x)$ and those of $\mathcal{M}^+(x)$ are the p^+ biggest. Since M has exactly p^0 zero eigenvalues everywhere, this implies that for $x \in M \cap U$, $\mathcal{M}^0(x) \equiv 0$. Therefore the eigenvalues of $\mathcal{M}^0(x)$ are of absolute value smaller than $c'\delta_M(x)$ for some c', which proves the claim.

Choose a strictly positive function $\theta \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ such that

$$\theta(t) = \begin{cases} -nt & \text{for } t \leq -c \\ c & \text{for } 0 \leq t \leq c' \\ t & \text{for } t \geq c' + 1 \end{cases}$$

We use the following notation:
Let $\phi \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$. If A is a hermitian $n \times n$ matrix with eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n$ and corresponding eigenvectors $v_1, \ldots, v_n$, we define $\phi[A]$ as the hermitian matrix with eigenvalues $\phi(\lambda_j)$ and eigenvectors v_j, $1 \leq j \leq n$.

We let ω_M be the hermitian metric defined by the hermitian endomorphism $A(x) = \theta[A_g(x)]$. ω_M is then a smooth metric (cf [Dem]). By construction, the eigenvalues of $A(x)$ are $\sigma_\nu(x) = \theta(\gamma_\nu^g(x))$ and we have

$$\begin{aligned} &\sigma_1(x) = n\ |\gamma_1^g(x)|, \ldots,\ \sigma_{p^-}(x) = n\ |\gamma_{p^-}^g(x)|, \\ &\sigma_{p^-+1}(x) = c, \ldots,\ \sigma_{p^-+p^0}(x) = c, \\ &\sigma_{p^-+p^0+1}(x) = \gamma_{p^-+p^0+1}^g(x), \ldots,\ \sigma_n(x) = \gamma_n^g(x) \end{aligned}$$

for every $x \in V^-$, after possibly shrinking V.

The eigenvalues of $-i\partial\overline{\partial}\log\delta_M$ with respect to ω_M are $\alpha_\nu(x) = \frac{\gamma_\nu^g(x)}{\sigma_\nu(x)}$. Thus we have for every $x \in V^-$ $\alpha_1(x) = -\frac{1}{n}$ and $\alpha_{n-p^+-p^0}(x) \geq 1$, hence

$$\alpha_1 + \ldots + \alpha_r \geq 1 - \frac{1}{n}(n - p^+ - p^0 - 1) \geq \frac{1}{n} \text{ for } r \geq n - p^+ - p^0 \qquad (4.2)$$

Let us now estimate $|\partial\omega_M|_{\omega_M}$.

Fix $x_0 \in M \cap \overline{\Omega}$. Using Lemma 4.2.1, there exists a neighborhood U of x_0 in X such that we have on $U \cap \Omega$

$$\begin{aligned} -i\partial\overline{\partial}\log\delta_M(x) &= \sum_{\mu,\nu=1}^{p^-} a^-_{\mu\nu}(x)\zeta_\mu(x)\wedge\overline{\zeta}_\nu(x) + \sum_{\mu,\nu=p^-+1}^{p^-+p^0} a^0_{\mu\nu}(x)\zeta_\mu(x)\wedge\overline{\zeta}_\nu(x) \\ &\quad + \sum_{\mu,\nu=p^-+p^0+1}^{n-1} a^+_{\mu\nu}(x)\zeta_\mu(x)\wedge\overline{\zeta}_\nu(x) + a_n(x)\zeta_n(x)\wedge\overline{\zeta}_n(x) \end{aligned}$$

where $(\zeta_1(x),\dots,\zeta_n(x))$ is an orthonormal basis of $T^{1,0}_x X$ with respect to ω_g on U.

By construction of ω_M, we have

$$\begin{aligned} \omega_M &= \sum_{\mu,\nu=1}^{p^-} b^-_{\mu\nu}(x)\ \zeta_\mu(x)\wedge\overline{\zeta}_\nu(x) + c\sum_{\nu=p^-+1}^{p^-+p^0} \zeta_\nu(x)\wedge\overline{\zeta}_\nu(x) \\ &\quad + \sum_{\mu,\nu=p^-+p^0+1}^{n-1} b^+_{\mu\nu}(x)\ \zeta_\mu(x)\wedge\overline{\zeta}_\nu(x) + a_n(x)\ \zeta_n(x)\wedge\overline{\zeta}_n(x) \end{aligned}$$

where $(\overset{\pm}{b}_{\mu\nu})_{\mu,\nu} = \theta[(\overset{\pm}{a}_{\mu\nu})_{\mu,\nu}]$. In order to get more condensed formulae, we extend $\overset{\pm}{b}_{\mu\nu}$ to all pairs $(\mu,\nu)\in\{1,\dots,n\}\times\{1,\dots,n\}$ by setting it equal to zero whenever it is not defined for such a pair.

Let $(L_1(x),\dots,L_n(x))$ be the dual basis of $(\zeta_1(x),\dots,\zeta_n(x))$. The well known Cartan formula for d implies that

$$\partial\zeta_\mu(L_\alpha,L_\beta) = L_\alpha(\zeta_\mu(L_\beta)) - L_\beta(\zeta_\mu(L_\alpha)) - \zeta_\mu([L_\alpha,L_\beta]) = -\zeta_\mu([L_\alpha,L_\beta]),$$
$$\partial\overline{\zeta}_\nu(L_\alpha,\overline{L}_\beta) = L_\alpha(\overline{\zeta}_\nu(\overline{L}_\beta)) - \overline{L}_\beta(\overline{\zeta}_\nu(L_\alpha)) - \overline{\zeta}_\nu([L_\alpha,\overline{L}_\beta]) = -\overline{\zeta}_\nu([L_\alpha,\overline{L}_\beta]).$$

Thus

$$\partial\zeta_\mu = \sum_{\alpha,\beta} c^\mu_{\alpha\beta}\zeta_\alpha\wedge\zeta_\beta,$$
$$\partial\overline{\zeta}_\nu = -\sum_{\alpha,\beta} d^\nu_{\alpha\beta}\zeta_\alpha\wedge\overline{\zeta}_\beta,$$

where the $c^\nu_{\alpha\beta}$ and $d^\nu_{\alpha\beta}$ are determined by the conditions

$$\begin{aligned} [L_\alpha,L_\beta](x) &= -\sum_{\nu=1}^{n} c^\nu_{\alpha\beta}(x)L_\nu(x) \bmod(\overline{L}_1(x),\dots,\overline{L}_n(x)) \\ [L_\alpha,\overline{L}_\beta](x) &= \sum_{\nu=1}^{n} d^\nu_{\alpha\beta}(x)\overline{L}_\nu(x) \bmod(L_1(x),\dots,L_n(x)) \end{aligned}$$

(i)-(iv) of Lemma 4.2.1 therefore yield

$$c^{\mu}_{\alpha\beta} \sim \delta_M,\ d^{\mu}_{\alpha\beta} \sim \delta_M \tag{4.3}$$

for (α, β, μ) such that $\mu \notin \{p^-+1, \dots, p^-+p^0\}$, $\alpha, \beta \in \{p^-+1, \dots, p^-+p^0\}$ and

$$c^{n}_{\alpha\beta} \sim \delta_M,\ d^{n}_{\alpha\beta} \sim \delta_M \tag{4.4}$$

for (α, β) such that $\alpha \in \{1, \dots, n-1\}$, $\beta \in \{p^- + 1, \dots, p^- + p^0\}$.

Moreover, by definition of $c^{\nu}_{\alpha\beta}$ and $d^{\mu}_{\alpha\beta}$, we have

$$\partial\omega_M = \sum_{\alpha=1}^{n} \sum_{\mu,\nu=1}^{p^-} \sum_{\epsilon\in\{-,+\}} L_\alpha(b^{\epsilon}_{\mu\nu})(x)\ \zeta_\alpha \wedge \zeta_\mu \wedge \overline{\zeta}_\nu \tag{4.5}$$

$$+ \sum_{\alpha,\beta=1}^{n} \sum_{\mu,\nu=1}^{p^-} \sum_{\epsilon\in\{-,+\}} b^{\epsilon}_{\mu\nu}(x) c^{\mu}_{\alpha\beta}(x)\ \zeta_\alpha \wedge \zeta_\beta \wedge \overline{\zeta}_\nu \tag{4.6}$$

$$+ \sum_{\alpha,\beta=1}^{n} \sum_{\mu,\nu=1}^{p^-} \sum_{\epsilon\in\{-,+\}} b^{\epsilon}_{\mu\nu}(x) d^{\nu}_{\alpha\beta}(x)\ \zeta_\mu \wedge \zeta_\alpha \wedge \overline{\zeta}_\beta \tag{4.7}$$

$$+c \sum_{\alpha,\beta=1}^{n} \sum_{\nu=p^-+1}^{p^-+p^0} c^{\nu}_{\alpha\beta}(x)\ \zeta_\alpha \wedge \zeta_\beta \wedge \overline{\zeta}_\nu \tag{4.8}$$

$$+c \sum_{\alpha,\beta=1}^{n} \sum_{\nu=p^-+1}^{p^-+p^0} d^{\nu}_{\alpha\beta}(x)\ \zeta_\nu \wedge \zeta_\alpha \wedge \overline{\zeta}_\beta \tag{4.9}$$

$$+ \sum_{\alpha=1}^{n} L_\alpha(a_n)(x)\ \zeta_\alpha \wedge \zeta_n \wedge \overline{\zeta}_n \tag{4.10}$$

$$+ \sum_{\alpha,\beta=1}^{n} a_n(x) c^{n}_{\alpha\beta}(x)\ \zeta_\alpha \wedge \zeta_\beta \wedge \overline{\zeta}_n \tag{4.11}$$

$$+ \sum_{\alpha,\beta=1}^{n} a_n(x) d^{n}_{\alpha\beta}(x)\ \zeta_n \wedge \zeta_\alpha \wedge \overline{\zeta}_\beta \tag{4.12}$$

As A_g is the hermitian endomorphism associated to $-i\partial\overline{\partial}\log\delta_M = \frac{i}{-\delta_M}\partial\overline{\partial}\delta_M + i\partial\log\delta_M \wedge \overline{\partial}\log\delta_M$, it is easy to see that we have $b^{\pm}_{\mu\nu} = \frac{1}{\delta_M}\tilde{b}^{\pm}_{\mu\nu}$, where $\tilde{b}^{\pm}_{\mu\nu}$ is defined and positive definite on U. Moreover, we see that $a_n = \frac{1}{\delta_M^2}\tilde{a}_n$, where $\tilde{a}_n$ is also defined and positive on U. From this we conclude that

$$|\zeta_\nu|^2_{\omega_M} \sim \delta_M \text{ for } \nu \in \{1, \dots, p^-, p^- + p^0 + 1, \dots, n-1\}, \tag{4.13}$$

$$|\zeta_\nu|^2_{\omega_M} \sim 1 \text{ for } \nu \in \{p^- + 1, \ldots, p^- + p^0\} \text{ and } |\zeta_n|^2_{\omega_M} \sim \delta_M^2. \tag{4.14}$$

By construction of ω_M, we clearly have $\omega_M \gtrsim \partial \log \delta_M \wedge \overline{\partial} \log \delta_M$,so

$$|\partial \log \delta_M|^2_{\omega_M} \lesssim 1. \tag{4.15}$$

We have

$$\begin{aligned}\sum_{\alpha=1}^n L_\alpha(b^\epsilon_{\mu\nu})(x)\ \zeta_\alpha(x) &= (\partial b^\epsilon_{\mu\nu})(x) = (\partial(\frac{1}{\delta_M}\tilde{b}^\epsilon_{\mu\nu}))(x) \\ &= -b^\epsilon_{\mu\nu}(x)\partial \log \delta_M(x) + \frac{1}{\delta_M}\sum_{\alpha=1}^n L_\alpha(\tilde{b}^\epsilon_{\mu\nu})(x)\ \zeta_\alpha(x), \\ \sum_{\alpha=1}^n L_\alpha(a_n)(x)\zeta_\alpha(x) &= -2a_n(x)\partial \log \delta_M(x) + \frac{1}{\delta_M^2}\sum_{\alpha=1}^n L_\alpha(\tilde{a}_n)(x)\zeta_\alpha(x),\end{aligned}$$

therefore (4.5) and (4.10) are bounded with respect to ω_M by (4.15), (4.13) and (4.14).

(4.8) and (4.9) are bounded with respect to ω_M by (4.13) and (4.14). Finally, (4.3), (4.4), (4.13) and (4.14) imply that (4.6), (4.7), (4.11) and (4.12) are bounded with respect to ω_M.

It is also clear that (iii) and (v) of Lemma 4.2.3 are satisfied.

Let us now prove (ii). We assume $p^- \geq 1$ (the weakly pseudoconvex case $p^- = 0$ was settled in Chapter 3). We then have $r \geq 2$.

From Lemma 4.2.2, we get

$$i\partial\overline{\partial}\varphi \geq -\lambda i\partial\overline{\partial} \log \delta_M - (1-\lambda) i\partial\overline{\partial} \log \psi$$

where $\lambda = \frac{1}{2} + \frac{1}{2}\chi'_\beta(\frac{\log(-\psi) - \log \delta_M}{2})$. On the set where $\lambda \geq \frac{1}{2}$, the assertion (ii) is clear by (4.2). On the other hand, on $\{\lambda \leq \frac{1}{2}\}$, we have $-\psi \leq \delta_M$ (see the definition of χ_β), and thus by construction of ω_M we get $\omega_M \lesssim \frac{1}{\delta_M}\omega_g \leq \frac{1}{-\psi}\omega_g \lesssim -i\partial\overline{\partial}\log(-\psi)$ on $\mathrm{Ker}\partial\delta_M \cap T^{1,0}X$, which is a subbundle of rank $(n-1)$ of $T^{1,0}X$. If $0 < \beta_1 \leq \ldots \leq \beta_n$ are the eigenvalues of $-i\partial\overline{\partial}\log(-\psi)$ with respect to ω_M, we thus have $\beta_2 \geq 2\sigma$ on $\{\lambda \leq \frac{1}{2}\}$ for some $\sigma > 0$. Since $\alpha_1 + \ldots + \alpha_r > 0$ for $r \geq n - p^+ - p^0 \geq 2$, we then have $\gamma_1 + \ldots + \gamma_r \geq \frac{1}{2}(\beta_1 + \ldots + \beta_r) \geq \sigma$ on $\{\lambda \leq \frac{1}{2}\}$. This establishes (ii).

We define $\omega_j = \omega_M + i\theta_j \partial\varphi \wedge \overline{\partial}\varphi$ where $\theta_j \in \mathcal{C}^\infty(D)$ vanishes on a neighborhood of $\overline{D}_j$ and equals one on $D \setminus D_{j+1}$. Then $|\partial\varphi|_{\omega_j}$ is bounded (j is fixed!), thus, by Lemma 1.2.3, ω_j is complete and has all the desired properties. $\square$

4.3 The L^2 estimates

From now on, D will be equipped with the metric ω_M given by Lemma 4.2.3. Properties (ii) and (iv) will be used to obtain L^2-solutions of some $\overline{\partial}$-equation. Property (v) will yield regularity results for these solutions.

Let (E, h) be a hermitian vector bundle on X, and let $N \in \mathbb{Z}$. We denote by $L^2_{p,q}(D, E, N)$ the Hilbert space of (p, q)-forms u on D with values in E which satisfy

$$\|u\|^2_N \ := \int_D |u|^2_{\omega_M,h} \Delta^N \mathrm{dV}_{\omega_M} < +\infty.$$

Here dV_{ω_M} is the canonical volume element associated to the metric ω_M, and $|\ |_{\omega_M,h}$ is the norm of (p, q)-forms induced by ω_M and h.

Proposition 4.3.1
Let $N \gg 1$. Suppose $f \in L^2_{n,r}(D, E, N) \cap \mathrm{Ker}\overline{\partial}$, $r \geq n - p^+ - p^0$. Then there exists $u \in L^2_{n,r-1}(D, E, N)$ such that $\overline{\partial}u = f$ and $\|u\|_N \leq \|f\|_N$.

Proof: We have already seen that $\Delta \sim e^{-\varphi}$. Also $\Delta^N \sim e^{-N\varphi}$ for $N \in \mathbb{N}$. Thus it suffices to prove the statement with Δ^N replaced by $e^{-N\varphi}$ in the definition of the spaces $L^2_{p,q}(D, E, N)$.

For $j \in \mathbb{N}$, let us denote by $L^2_{p,q}(D, E, N, j)$ the Hilbert space of (p, q)-forms u on D with values in E which satisfy

$$\|u\|^2_{N,j} \ := \int_{D_j} |u|^2_{\omega_j,h} e^{-N\chi_j(\varphi)} \mathrm{dV}_{\omega_j} < +\infty.$$

where $\chi_j \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ with $\chi_j(t) = t$ if $t \leq \log j$, $\chi_j(t) \geq t$ for all $t \in \mathbb{R}$.

Let $\overline{\partial}^*_{N,j}$ be the Hilbert adjoint of $\overline{\partial}$ with respect to the canonical scalar product $\langle\!\langle\ ,\ \rangle\!\rangle_{N,j}$ of (p, q)-forms with values in E induced by $\|\ \|_{N,j}$.

Nakano's inequality (1.2) yields

$$\frac{3}{2}\,(\|\overline{\partial}u\|^2_{N,j}+\|\overline{\partial}^*_{N,j}u\|^2_{N,j}) \geq \langle\!\langle [i\Theta(E_{N,j}),\Lambda_j]u,u\rangle\!\rangle_{N,j}$$
$$-\frac{1}{2}(\|\tau_j u\|^2_{N,j}+\|\tau_j^* u\|^2_{N,j}+\|\overline{\tau}_j u\|^2_{N,j}+\|\overline{\tau}_j^* u\|^2_{N,j}) \tag{4.16}$$

where $\Theta(E_{N,j})$ is the curvature of the bundle $E_{N,j} = (E, e^{-N\chi_j(\varphi)}h)$, Λ_j is the adjoint of multiplication by ω_j and $\tau_j = [\Lambda_j, \partial\omega_j]$. ω_j is the metric given by Lemma 4.2.3.

As $i\Theta(E_{N,j}) = iN\partial\overline{\partial}\chi_j(\varphi) \otimes \mathrm{Id}_E + i\Theta(E)$, a standard calculation (cf [Dem86]) yields

$$\begin{aligned}[][i\Theta(E_{N,j}),\Lambda_j] &= N[i\partial\overline{\partial}\chi_j(\varphi)\otimes \mathrm{Id}_E,\Lambda_j] + [i\Theta(E),\Lambda_j]\\ &\geq N\chi_j'(\varphi)(\gamma_1^j+\ldots+\gamma_r^j)\otimes \mathrm{Id}_E + [ic(E),\Lambda_j]\end{aligned}$$

when this curvature tensor acts on (n,r)-forms. Here γ_l^j are the eigenvalues of $i\partial\overline{\partial}\varphi$ with respect to ω_j.

For $r \geq n-p^+-p^0$, we have $\gamma_1+\ldots+\gamma_r \geq \sigma$ on D_j. Since $|\partial\omega_M|_{\omega_M}$ is bounded on D by (iv) of Lemma 4.2.3 and $\omega_j = \omega_M$ on D_j, the pointwise norms $|\tau_j u|_{\omega_j}$, $|\overline{\tau}_j u|_{\omega_j}$, $|\tau_j^* u|_{\omega_j}$ and $|\overline{\tau}_j^* u|_{\omega_j}$ are uniformly bounded with respect to j by some constant times $|u|_{\omega_M}$ on D_j. Thus, choosing N big enough and χ_j sufficiently rapidly increasing on $\{t > \log j\}$, the right hand side of (4.16) can be made $\geq \frac{3}{2}\|u\|^2_{N,j}$.

Let $f \in L^2_{n,r}(D,E,N)\cap\mathrm{Ker}\overline{\partial}$, $r \geq n-p^+-p^0$. Since f is of bidegree (n,r) and $\chi_j(\varphi) \geq \varphi$, a standard calculation (see [Dem82]) yields $\|f\|_{N,j} \leq \|f\|_N$. By standard L^2-theory (cf Chapter 1 or [Dem82], [Dem], [Ohs87]), we then get $u_j \in L^2_{n,r-1}(D,E,N,j)$ satisfying $\overline{\partial}u_j = f$ and $\|u_j\|_{N,j} \leq \|f\|_{N,j} \leq \|f\|_N$. Therefore the solutions u_j are uniformly bounded in L^2 norm on every compact subset of D. Since the unit ball of a Hilbert space is weakly compact, we can extract a subsequence $u_{\ell_j} \to u \in L^2_{\mathrm{loc}}$ converging weakly in L^2 on any compact subset $K \subset D$, for some $\ell_j \to +\infty$. By the weak continuity of differentiation, we get again in the limit $\overline{\partial}u = f$. Also, since $\chi_j(\varphi) = \varphi$ on D_j, we have

$$\int_{D_j} |u|^2_{\omega_M} e^{-N\varphi} dV_{\omega_M} \leq \liminf_{j\to\,+\infty} \int_{D_j} |u_{\ell_j}|^2_{\omega_{\ell_j},h} e^{-N\chi_j(\varphi)} dV_{\omega_{\ell_j}} \leq \|f\|^2_N,$$

hence $\|u\|^2_N \leq \|f\|^2_N$. □

Proposition 4.3.2
Let $N \gg 1$. Suppose $f \in L^2_{0,r}(D, E, -N) \cap \mathrm{Ker}\overline{\partial}$, $r \leq p^+ + p^0$. Then there exists $u \in L^2_{0,r-1}(D, E, -N+2)$ such that $\overline{\partial}u = f$ and $\|u\|_{-N+2} \leq \|f\|_{-N}$. Moreover, $\mathrm{Im}(\overline{\partial} : L^2_{0,p^++p^0}(D, E, -N+2) \to L^2_{0,p^++p^0+1}(D, E, -N))$ is closed in $L^2_{0,p^++p^0+1}(D, E, -N)$.

Proof. The line of the proof follows exactly the proof of Proposition 3.2.2. Suppose $r \leq p^+ + p^0$ and let $f \in L^2_{0,r}(D, E, -N) \cap \mathrm{Ker}\overline{\partial}$, $N \gg 1$. We define the linear operator

$$\begin{array}{rrcl} L_f : & \overline{\partial}L^2_{n,n-r}(D, E^*, N-2) & \longrightarrow & \mathbb{C} \\ & \overline{\partial}g & \longmapsto & \displaystyle\int_D f \wedge g \end{array}$$

Note that the integral on the right hand side is finite, since

$$|\int_D f \wedge g|^2 \leq (\int_D |f|^2_{\omega_M} \Delta^{-N} dV_{\omega_M}) \cdot (\int_D |g|^2_{\omega_M} \Delta^N dV_{\omega_M}) \leq \|f\|^2_{-N} \|g\|^2_{N-2}.$$

Let us first show that L_f is well defined.

Indeed, let g_1, $g_2 \in L^2_{n,n-r}(D, E^*, N-2)$ such that $\overline{\partial}g_1 = \overline{\partial}g_2$. Then $\overline{\partial}(g_1 - g_2) = 0$ and by Proposition 4.3.1, since $n - r \geq n - p^+ - p^0$, there exists $\alpha \in L^2_{n,n-r-1}(D, E^*, N-2)$ such that $\overline{\partial}\alpha = g_1 - g_2$. But then

$$\begin{aligned} \int_D f \wedge (g_1 - g_2) &= \int_D f \wedge \overline{\partial}\alpha \\ &= \lim_{\varepsilon \to 0} (-1)^r \int_{\partial D_\varepsilon} f \wedge \alpha \\ &= -\lim_{\varepsilon \to 0} \int_{D \setminus D_\varepsilon} f \wedge \overline{\partial}\alpha \\ &= -\lim_{\varepsilon \to 0} \int_{D \setminus D_\varepsilon} f \wedge (g_1 - g_2) \end{aligned}$$

with $(D_\varepsilon)_\varepsilon$ an exhaustion of D by smooth open sets such that $D_\varepsilon \supset \{z \in D \mid \Delta(z) > \varepsilon\}$. Here we have used Stoke's theorem several times. The third equality is obtained as in the proof of Theorem 3.2.2.

Moreover,

$$|\int_{D \setminus D_\varepsilon} f \wedge (g_1 - g_2)| \leq (\int_{D \setminus D_\varepsilon} |f|^2_{\omega_M} \Delta^{-N})^{1/2} \; (\int_{D \setminus D_\varepsilon} |g_1 - g_2|^2_{\omega_M} \Delta^N)^{1/2}$$

$$\longrightarrow_{\varepsilon\to 0} 0.$$

(note that $(\int_{D\setminus D_\varepsilon} |g_1 - g_2|^2_{\omega_M} \Delta^N)^{1/2} \lesssim \varepsilon \int_{D\setminus D_\varepsilon} |g_1 - g_2|^2_{\omega_M} \Delta^{N-2})^{1/2} \leq \varepsilon \|g_1 - g_2\|_{-N-2} \to 0$ as $\varepsilon \to 0$ since $g_1, g_2 \in L^2_{n,n-r}(D, E^*, N-2)$).

Thus $L_f(g_1) = L_f(g_2)$.

Now let
$g \in \mathrm{Dom}(\overline{\partial} : L^2_{n,n-r}(D, E^*, N-2) \to L^2_{n,n-r+1}(D, E^*, N-2))$. Applying Proposition 4.3.1, there exists $\widetilde{g} \in L^2_{n,n-r}(D, E^*, N-2)$ satisfying $\overline{\partial}\widetilde{g} = \overline{\partial} g$ and $\|\widetilde{g}\|_{N-2} \leq \|\overline{\partial} g\|_{N-2}$. This yields

$$|L_f(\overline{\partial} g)| = |L_f(\overline{\partial}\widetilde{g})| = |\int_D f \wedge \widetilde{g}| \leq \|f\|_{-N} \|\widetilde{g}\|_N$$

$$\leq \|f\|_{-N} \|\widetilde{g}\|_{N-2} \leq \|f\|_{-N} \|\overline{\partial} g\|_{N-2}.$$

Thus L_f is a continuous linear operator of norm $\leq \|f\|_{-N}$ and therefore, using the Hahn-Banach theorem, L_f extends to a continuous linear operator with norm $\leq \|f\|_{-N}$ on the Hilbert space $L^2_{n,n-r+1}(D, E^*, N-2)$. By the theorem of Riesz, there exists $u \in L^2_{0,r-1}(D, E, -N+2)$ with $\|u\|_{-N+2} \leq \|f\|_{-N}$ such that for every $g \in L^2_{n,n-r}(D, E^*, N-2)$ we have

$$(-1)^r \int_D u \wedge \overline{\partial} g = L_f(g) = \int_D f \wedge g,$$

i.e. $\overline{\partial} u = f$.

To prove the last assertion, we show that

$$\mathrm{Im}(\overline{\partial} : L^2_{0,p^+ + p^0}(D, E, -N+2) \longrightarrow L^2_{0,p^+ + p^0 + 1}(D, E, -N)) =$$

$$\{g \in L^2_{0,p^0+p^+ +1}(D, E, -N) \mid \int_D g \wedge h = 0 \ \forall h \in L^2_{n,n-p^0-p^+ -1}(D, E^*, N-2)\}.$$

Suppose $f \in \mathrm{Im}(\overline{\partial} : L^2_{0,p^+ + p^0}(D, E, -N+2) \longrightarrow L^2_{0,p^+ + p^0+1}(D, E, -N))$. Then there exists $\alpha \in L^2_{0,p^+ + p^0}(D, E, -N+2)$ such that $\overline{\partial}\alpha = f$. Thus we get for every $h \in L^2_{n,n-p^0-p^+ -1}(D, E^*, N-2)$

$$\begin{aligned}
\int_D f \wedge h &= \int_D \overline{\partial}\alpha \wedge h \\
&= \lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon} \alpha \wedge h \\
&= -\lim_{\varepsilon \to 0} \int_{D \setminus D_\varepsilon} \overline{\partial}\alpha \wedge h \\
&= -\lim_{\varepsilon \to 0} \int_{D \setminus D_\varepsilon} f \wedge h
\end{aligned}$$

with $(D_\varepsilon)_\varepsilon$ an exhaustion of D by smooth open sets such that $D_\varepsilon \supset \{z \in D \mid \Delta(z) > \varepsilon\}$ and

$$\begin{aligned}
|\int_{D \setminus D_\varepsilon} f \wedge h| &\leq (\int_{D \setminus D_\varepsilon} |f|^2_{\omega_M} \Delta^{-N})^{1/2} (\int_{D \setminus D_\varepsilon} |h|^2_{\omega_M} \Delta^{N})^{1/2} \\
&\leq \varepsilon \|f\|_{-N} \|h\|_{N-2} \longrightarrow_{\varepsilon \to 0} 0,
\end{aligned}$$

which shows the inclusion $\subset$ (see the proof of Theorem 3.2.2 for the justification of some of the equalities).

Conversely, we show that for every $f \in \{g \in L^2_{0,p^0+p^++1}(D, E, -N) \mid \int_D g \wedge h = 0 \ \forall h \in L^2_{n,n-p^0-p^+-1}(D, E^*, N-2)\}$, there exists $u \in L^2_{0,p^++p^0}(D, E, -N+2)$ satisfying $\overline{\partial} u = f$. Again, we define the linear operator

$$\begin{aligned}
L_f : \qquad \overline{\partial} L^2_{n,n-p^+-p^0-1}(D, E^*, N-2) &\longrightarrow \mathbb{C} \\
\overline{\partial} g &\longmapsto \int_D f \wedge g
\end{aligned}$$

Here we write $\overline{\partial} L^2_{n,n-p^+-p^0-1}(D, E^*, N-2)$ for $\mathrm{Im}(\overline{\partial} : L^2_{n,n-p^+-p^0-1}(D, E^*, N-2) \to L^2_{n,n-p^+-p^0}(D, E^*, N-2))$. L_f is well defined because of the moment conditions imposed on f. We then show the existence of the desired u as in the first part of the proof. $\square$

Let $U \subset X$ be an open set and E a holomorphic vector bundle on X. For $k \in \mathbb{N} \cup \{+\infty\}$, we define

$$\mathcal{C}^k_{p,q}(X, \overline{U}, E) = \{f \in \mathcal{C}^k_{p,q}(X, E) \mid \mathrm{supp}\ f \subset \overline{U}\}$$

As in Chapter 3, we get a regularity theorem for $\Box_{-N}$.
Here $\Box_{-N} = \overline{\partial}\,\overline{\partial}^*_{-N} + \overline{\partial}^*_{-N}\overline{\partial}$ where $\overline{\partial}^*_{-N}$ is the Von Neumann adjoint of $\overline{\partial}$: $L^2_{p,q}(D,E,-N+2) \to L^2_{p,q+1}(D,E,-N)$.

Theorem 4.3.3
If $u \in L^2_{p,q}(D,E,-N)$ satisfies $\overline{\partial}u = f$ and $\overline{\partial}^*_{-N}u = 0$ with $f \in \mathcal{C}^N_{p,q}(X,\overline{D},E) \cap \mathcal{C}^\infty_{p,q}(D,E)$, then $u \in \mathcal{C}^{s(N)}_{p,q}(X,\overline{D},E) \cap \mathcal{C}^\infty_{p,q}(D,E)$ where $s(N) \sim \sqrt{N}$ for all $N \gg 1$.

Proof: We will show that $\Box_{-N}$ is an elliptic operator of polynomial growth with respect to Δ on D. Then all the assertions follow in the same way from Theorem 2.2.1 as Theorem 3.3.1 if we keep in mind property (iii) of Lemma 4.2.3.

In order to avoid too many sums over too many indices, we will assume that E is the trivial bundle and restrict our attention to $(0,1)$-forms. The general case is handled analogously.

An easy computation yields that $\overline{\partial}^*_{-N}u = \Delta^{N-2}\overline{\partial}^*_{\omega_M}(\Delta^{-N}u)$ where $\overline{\partial}^*_{\omega_M}$ is the Von Neumann adjoint of $\overline{\partial}$ for the metric ω_M. Hence

$$\Box_{-N}u = \Delta^{-2}\Box_{\omega_M}u + \text{ lower order terms}$$

where $\Box_{\omega_M} = \overline{\partial}\,\overline{\partial}^*_{\omega_M} + \overline{\partial}^*_{\omega_M}\overline{\partial}$ and the lower order terms are sums and products of terms like Δ^k, $\overline{\partial}(\Delta^k u)$ and $\overline{\partial}^*_{\omega_M}(\Delta^k u)$ for some integers $k \in \mathbb{Z}$. It therefore suffices to calculate $\overline{\partial}^*_{\omega_M}$ and $\Box_{\omega_M}$.

Let $z_0 \in \partial D$ and let $(z_1,\ldots,z_n)$ be local holomorphic coordinates of X in a neighborhood U of z_0.

We have $\omega_M = i\sum_{j,k=1}^n \omega_M^{\mu\nu} dz_\mu \wedge d\overline{z}_\nu$ on $U \cap D$, where the coeffients $\omega_M^{\mu\nu}$ satisfy (v) of Lemma 4.2.3.

Let $L_1,\ldots,L_n$ be an orthonormal basis of $T^{1,0}X_{|U\cap D}$ with respect to ω_M, i.e. $L_k = \sum_{j\le k} l_{jk}\frac{\partial}{\partial z_j}$ where the l_{jk} have to be determined by the condition $\sum_{l\le k}\sum_{i\le j} l_{lk}\overline{l}_{ij}\omega_M^{li} = \delta_{jk}$. It is therefore clear that all derivatives of l_{jk} can be bounded by some power of δ_M.

Let $\epsilon_i, \dots, \epsilon_j \in (T^{1,0}X)^*_{|U\cap D}$ be the dual basis of $L_1, \dots, L_n$.

For $u = \sum_{j=1}^n u_j \bar{\epsilon}_j$ we then have

$$\overline{\partial} u = \sum_{j,k} \overline{L}_k(u_j)\bar{\epsilon}_k \wedge \bar{\epsilon}_j - \sum_{j,k,l} c^l_{kj} u_l \bar{\epsilon}_k \wedge \bar{\epsilon}_j$$

where c^l_{jk} can be determined by the condition $[\overline{L}_j, \overline{L}_k] = \sum_l c^l_{jk}\overline{L}_l$, because we have $\overline{\partial}\epsilon_l(\overline{L}_k, \overline{L}_j) = -\bar{\epsilon}_l([\overline{L}_k, \overline{L}_j])$ by the Cartan formula for $\overline{\partial}$. Therefore also all derivatives of the c^l_{jk} can be bounded by some power of δ_M.

Now let $v = \sum_{j,k} v_{kj}\bar{\epsilon}_k \wedge \bar{\epsilon}_j$ be a smooth $(0,2)$-form with compact support in $U \cap D$. Then we have

$$\begin{aligned}
\langle\langle \overline{\partial} u, v\rangle\rangle_{\omega_M} &= 2^n \int_{D_j} (\sum_{k,j} \overline{L}_k(u_j)\overline{v}_{kj} - \sum_{k,j,l} c^l_{kj} u_l \overline{v}_{kj}) \det(\omega_M^{\alpha\beta}) d\lambda \\
&= 2^n \int_{D_j} \sum_{k,j,l} (\bar{l}_{lk} \frac{\partial u_j}{\partial \bar{z}_l} \overline{v}_{kj} - c^l_{kj} u_l \overline{v}_{kj}) \det(\omega_M^{\alpha\beta}) d\lambda \\
&= -2^n \int_{D_j} \sum_{k,j,l} \{ u_j \frac{\partial}{\partial \bar{z}_l} (\bar{l}_{lk}\overline{v}_{kj} \det(\omega_M^{\alpha\beta})) + c^l_{kj} u_l \overline{v}_{kj} \det(\omega_M^{\alpha\beta}) \} d\lambda
\end{aligned}$$

Thus

$$\overline{\partial}^*_{\omega_M} v = -\sum_{k,j,l} (L_k(v_{kj}) + v_{kj}\frac{\partial l_{lk}}{\partial z_l} + v_{kj} l_{lk} \frac{\partial}{\partial z_l}(\det(\omega_{\alpha\beta}))\det(\omega_M^{\alpha\beta})^{-1} + v_{kj}\bar{c}^l_{kj})\bar{\epsilon}_j$$

Hence the coefficients of $\overline{\partial}^*_{\omega_M}$ satisfy the condition (2.2) and

$$\begin{aligned}
\Box_{\omega_M} u &= \sum_{j,k} L_k \overline{L}_k(u_j)\bar{\epsilon}_j + \text{lower order terms} \\
&= \sum_{i,j,k,l} l_{lk}\bar{l}_{ik} \frac{\partial^2 u_j}{\partial z_l \partial \bar{z}_j} \bar{\epsilon}_j + \text{lower order terms}
\end{aligned}$$

where the lower order terms involve only derivatives of order ≤ 1 of u and multiplication by functions whose derivatives can be bounded by some power of δ_M. $\square$

4.4 The $\overline{\partial}$-equation with exact support

Let Ω be a smooth bounded completely strictly pseudoconvex domain in a complex n-dimensional manifold X and M a real hypersurface of class $\mathcal{C}^\infty$ intersecting $\partial\Omega$ transversally, such that $\Omega \setminus M$ has exactly two connected components. We suppose that $M = \{\varrho = 0\}$ where ϱ is a $\mathcal{C}^\infty$ function whose Levi form has exactly p^+ positive, p^0 zero and p^- negative eigenvalues on $T_x^{1,0}M$ for each $x \in M$, $p^- + p^0 + p^+ = n-1$. We put $D = \Omega \cap \{\varrho < 0\}$.

In this section, we will show some vanishing and separation theorems for the $\overline{\partial}$-cohomology groups with values in a holomorphic vector bundle E supported in $\overline{D}$:

$$H^{p,q}(X, \overline{D}, E) = \mathcal{C}^\infty_{p,q}(X, \overline{D}, E) \cap \mathrm{Ker}\overline{\partial}/\overline{\partial}(\mathcal{C}^\infty_{p,q-1}(X, \overline{D}, E))$$

Theorem 4.4.1
Let E be a holomorphic vector bundle on X. Then we have

$$H^{p,q}(X, \overline{D}, E) = 0 \qquad \textit{for } 1 \leq q \leq p^0 + p^+$$

and

$$H^{p,p^0+p^++1}(X, \overline{D}, E) \textit{ is separated for the usual } \mathcal{C}^\infty\textit{-topology.}$$

Proof: The proof is exactly the same as the proof of Theorem 3.3.2. Replacing the vector bundle E by $\Lambda^p(T^{1,0}X)^* \otimes E$, it is no loss of generality to assume $p = 0$.

We will begin by proving the following claim:
Let $f \in \mathcal{C}^k_{0,q}(X, \overline{D}, E) \cap \mathcal{C}^\infty_{0,q}(D, E) \cap \mathrm{Ker}\overline{\partial}$, $1 \leq q \leq p^0 + p^+$, $k \gg 1$. Then there exists $u \in \mathcal{C}^{s(k)}_{0,q-1}(X, \overline{D}, E) \cap \mathcal{C}^\infty_{0,q-1}(D, E)$ such that $\overline{\partial}u = f$ with $s(k) \sim \sqrt{k}$.

Proof of the claim: Let $f \in \mathcal{C}^k_{0,q}(X, \overline{D}, E) \cap \mathrm{Ker}\overline{\partial}$, $1 \leq q \leq p^0 + p^+$, $k \gg 1$. General results on Lipschitz domains (see e.g. [Gri85, Theorem 1.4.4.4] or Theorem A.2.2) show that $f \in L^2_{0,q}(D, E, -2k)$ if we keep in mind property (iii) of Lemma 4.2.3. Proposition 4.3.2 implies that there exists $u \in L^2_{0,q-1}(D, E, -2k+2)$ such that $\overline{\partial}u = f$ in D and $\|u\|_{-2k+2} \leq \|f\|_{-2k}$. Moreover, choosing the minimal solution, we may assume $\overline{\partial}^*_{-2k}u = 0$. From Theorem 4.3.3 we get that $u \in \mathcal{C}^{s(k)+1}_{0,q-1}(X, \overline{D}, E)$ with $s(k) \sim \sqrt{k}$.

Let us now prove the theorem.

$H^{0,1}(X, \overline{D}, E) = 0$ follows immediately from the above claim and the hypoellipticity of $\overline{\partial}$ in bidegree $(0,1)$ if $1 \leq p^0 + p^+$.

Now assume $1 < q \leq p^0 + p^+$ and let $f \in \mathcal{C}^\infty_{0,q}(X, \overline{D}, E) \cap \mathrm{Ker}\overline{\partial}$. By induction, we will construct $u_k \in \mathcal{C}^k_{0,q-1}(X, \overline{D}, E) \cap \mathcal{C}^\infty_{0,q-1}(D, E)$ such that $\overline{\partial} u_k = f$ and $|u_{k+1} - u_k|_{s(k)-1} < 2^{-k}$. It is then clear that $(u_k)_{k\in\mathbb{N}}$ converges to $u \in \mathcal{C}^\infty_{0,q-1}(X, \overline{D}, E)$ such that $\overline{\partial} u = f$.

Suppose that we have constructed $u_1, \ldots, u_k$. By the above claim, there exists $\alpha_{k+1} \in \mathcal{C}^{k+1}_{0,q-1}(X, \overline{D}, E) \cap \mathcal{C}^\infty_{0,q-1}(D, E)$ such that $f = \overline{\partial}\alpha_{k+1}$. We have $\alpha_{k+1} - u_k \in \mathcal{C}^k_{0,q-1}(X, \overline{D}, E) \cap \mathcal{C}^\infty_{0,q-1}(D, E) \cap \mathrm{Ker}\overline{\partial}$. Once again by the above claim, there exists $g \in \mathcal{C}^{s(k)}_{0,q-2}(X, \overline{D}, E) \cap \mathcal{C}^\infty_{0,q-2}(D, E)$ satisfying $\alpha_{k+1} - u_k = \overline{\partial} g$.

Since $\mathcal{C}^\infty_{0,q-2}(X, \overline{D}, E)$ is dense in $\mathcal{C}^{s(k)}_{0,q-2}(X, \overline{D}, E)$, there exists $g_{k+1} \in \mathcal{C}^\infty_{0,q-2}(X, \overline{D}, E)$ such that $|g - g_{k+1}|_{s(k)} < 2^{-k}$.

Define $u_{k+1} = \alpha_{k+1} - \overline{\partial} g_{k+1} \in \mathcal{C}^{k+1}_{0,q-1}(X, \overline{D}, E) \cap \mathcal{C}^\infty_{0,q-1}(D, E)$. Then $\overline{\partial} u_{k+1} = f$ and $|u_{k+1} - u_k|_{s(k)-1} = |\overline{\partial} g - \overline{\partial} g_{k+1}|_{s(k)-1} \leq |g - g_{k+1}|_{s(k)} < 2^{-k}$. Thus u_{k+1} has the desired properties.

The last assertion is proved similarly, using the "moreover"statement in Proposition 4.3.2 and the fact that the $\mathcal{C}^\infty$ topology is stronger that the L^2 topologies. □

As in Chapter 3, the results of this section will allow us to solve the $\overline{\partial}$-equation for extensible currents by duality.

We recall the notations. A current T defined on D is said to be *extensible*, if T is the restriction to D of a current defined on X.

It was shown in [Mar66] that, since D satisfies $\overset{\circ}{\overline{D}}$, the vector space $\check{\mathcal{D}}'^{p,q}_D(X)$ of extensible currents on D of bidegree (p, q) is the topological dual of $\mathcal{C}^\infty_{n-p,n-q}(X, \overline{D})$.

Theorem 4.4.2
Let $T \in \check{\mathcal{D}}'^{p,q}_D(X)$ be an extensible current on D of bidegree (p, q), $q \geq n - p^0 - p^+$ such that $\overline{\partial} T = 0$ in D. Then there exists $S \in \check{\mathcal{D}}'^{p,q-1}_D(X)$ satisfying $\overline{\partial} S = T$ in D.

Proof: Let $T \in \check{\mathcal{D}}'^{p,q}_D(X)$ be an extensible current on D of bidegree

$(p,q),\ q \geq n-p^0-p^+$, such that $\overline{\partial} T = 0$ in D.

Consider the operator

$$L_T : \qquad \begin{array}{ccc} \overline{\partial}\mathcal{C}^\infty_{n-p,n-q}(X,\overline{D}) & \longrightarrow & \mathbb{C} \\ \overline{\partial}\varphi & \longmapsto & < T, \varphi > \end{array}$$

We first notice that L_T is well-defined. Indeed, let $\varphi \in \mathcal{C}^\infty_{n-p,n-q}(X,\overline{D})$ be such that $\overline{\partial}\varphi = 0$.
If $q = n$, the analytic continuation principle for holomorphic functions yields $\varphi = 0$, so $< T, \varphi >= 0$.
If $n-1 \geq q \geq n-p^0-p^+$, one has $\varphi = \overline{\partial}\alpha$ with $\alpha \in \mathcal{C}^\infty_{n-p,n-q-1}(X,\overline{D})$ by Theorem 4.4.1. As $\mathcal{D}^{n-p,n-q-1}(D)$ is dense in $\mathcal{C}^\infty_{n-p,n-q-1}(X,\overline{D})$, there exists $(\alpha_j)_{j\in\mathbb{N}} \in \mathcal{D}^{n-p,n-q-1}(D)$ such that $\overline{\partial}\alpha_j \underset{j\to+\infty}{\longrightarrow} \overline{\partial}\alpha$ in $\mathcal{C}^\infty_{n-p,n-q}(X,\overline{D})$.
Hence $< T, \varphi >=< T, \overline{\partial}\alpha >= \lim_{j\to+\infty} < T, \overline{\partial}\alpha_j >= 0$, because $\overline{\partial} T = 0$.

By Theorem 4.4.1, $\overline{\partial}\mathcal{C}^\infty_{n-p,n-q}(X,\overline{D})$ is a closed subspace of $\mathcal{C}^\infty_{n-p,n-q+1}(X,\overline{D})$, thus a Fréchet space. Using Banach's open mapping theorem, L_T is in fact continuous, so by the Hahn-Banach theorem, we can extend L_T to a continuous linear operator $\tilde{L}_T : \mathcal{C}^\infty_{n-p,n-q+1}(X,\overline{D}) \longrightarrow \mathbb{C}$, i.e. $\tilde{L}_T$ is an extensible current on D satisfying

$$< \overline{\partial}\tilde{L}_T, \varphi >= (-1)^{p+q} < \tilde{L}_T, \overline{\partial}\varphi >= (-1)^{p+q} < T, \varphi >$$

for every $\varphi \in \mathcal{C}^\infty_{n-p,n-q}(X,\overline{D})$. Therefore $T = (-1)^{p+q}\overline{\partial}\tilde{L}_T$. □

Remark. Analogous results have been obtained in [Sam99] for completely strictly q-convex domains with smooth boundary. These are domains of the form $\Omega = \{z \in U_{\overline{\Omega}} \mid \psi(z) < 0\}$ where ψ is a smooth function defined on an open neighborhood $U_{\overline{\Omega}}$ of $\overline{\Omega}$ whose Levi form has at least $q+1$ positive eigenvalues everywhere. Sambou shows that for such a domain the $\overline{\partial}$-equation is solvable for extensible currents of bidegree (p,r), $r \geq n-q$. In [Sam01], also the strictly q-concave case is discussed.

Chapter 5

Applications to CR manifolds

In this chapter, we apply the results of Chapter 3 and Chapter 4 to the study of the tangential Cauchy-Riemann complexes on CR manifolds. We first define the tangential Cauchy-Riemann complexes for smooth forms and currents on generic CR submanifolds. Then we show that the tangential Cauchy-Riemann cohomology groups for both smooth forms and currents vanish for all intermediate bidegrees on boundaries of weakly pseudoconvex domains in Stein manifolds. We also prove that the tangential Cauchy-Riemann equations for currents can be solved on Levi flat CR submanifolds of arbitrary codimension. Finally, we give some results on the solvability of the tangential Cauchy-Riemann equations for currents and for smooth forms with compact support on hypersurfaces with constant signature. We also prove a new version of the Hartogs phenomenon in weakly 2-convex-concave hypersurfaces in Stein manifolds.

5.1 The tangential Cauchy-Riemann complexes

Let X be a complex manifold of complex dimension n. Let M be a $\mathcal{C}^\infty$-smooth real submanifold of real codimension k in X. Such a manifold M can be represented locally by

$$M = \{z \in \Omega \mid \rho_1(z) = \ldots = \rho_k(z) = 0\}, \tag{5.1}$$

where the ρ_ν's, $1 \le \nu \le k$, are real $\mathcal{C}^\infty$ functions on an open set Ω of X. M is called a *generic CR manifold of real codimension k* if and only if $\overline{\partial}\rho_1 \wedge \ldots \wedge \overline{\partial}\rho_k \neq 0$ on M. In particular, every smooth real hypersurface in X is a generic CR manifold of real codimension 1.

In this situation, the holomorphic tangent spaces to M form a subbundle $T^{1,0}M$ of $T^{1,0}X_{|M}$. In the local representation (5.1) we have

$$T_z^{1,0}M = \{\zeta \in \mathbb{C}^n \mid \sum_{j=1}^{n} \frac{\partial \rho_\nu}{\partial z_j}(z)\zeta_j = 0,\ \nu = 1,\dots,k\}$$

and $\dim_{\mathbb{C}} T_z^{1,0}M = n-k$ for $z \in M \cap \Omega$, where $(z_1,\dots,z_n)$ are local holomorphic coordinates of X in a neighborhood of z.

For $p \in M$, let

$$\pi_p:\ \{T_pM \otimes \mathbb{C}\} \longrightarrow \{T_pM \otimes \mathbb{C}\}/\{T_p^{1,0}M \oplus \overline{T_p^{1,0}M}\}$$

be the natural projection map. The *Levi form* at a point $p \in M$ is the map

$$\mathcal{L}_p : T_p^{1,0}M \longrightarrow \{T_pM \otimes \mathbb{C}\}/\{T_p^{1,0}M \oplus \overline{T_p^{1,0}M}\}$$

defined by $\mathcal{L}_p(X_p) = \frac{1}{2i}\pi_p\{[X,\overline{X}]_p\}$ for $X_p \in T_p^{1,0}M$, where X is any vector field in $T^{1,0}M$ that equals X_p at p. The Levi form of M takes values in a k-dimensional complex vector space.

We say that M is *Levi flat* if and only if $\mathcal{L}_p \equiv 0$ for every $p \in M$.

Now consider the case where M is a real hypersurface in X. Then the Levi form of M takes values in a 1-dimensional complex vector space, and there is a different way of defining it by means of the Levi form of a local defining function: Let ϱ be a smooth local defining function for M in a neighborhood of p with $d\varrho(p) \neq 0$. The Levi form of M at p can then be identified with the hermitian form $\mathcal{L}(\varrho,p)_{|T_p^{1,0}M}$. If $\tilde{\varrho}$ is another defining function for M with $d\tilde{\varrho} \neq 0$ on M, then $\mathcal{L}(\tilde{\varrho},p)_{|T_p^{1,0}M}$ is a nonzero multiple of $\mathcal{L}(\varrho,p)_{|T_p^{1,0}M}$. We say that M *has signature* (p^-,p^0,p^+) *at* $p \in M$ if there exists a smooth local defining function for M in a neighborhood of p with $d\varrho(p) \neq 0$ such that $\mathcal{L}(\varrho,p)_{|T_p^{1,0}M}$ has p^- negative, p^0 zero and p^+ positive eigenvalues at p.

In order to define the tangential Cauchy-Riemann complexes on a generic CR manifold M of real codimension k, we consider the sheaf $\mathcal{J}_M$ of germs of $\mathcal{C}^\infty$ functions on X which vanish on M.

On X, we have the Dolbeault complexes for sheaves of germs of smooth forms:

$$\mathcal{E}^{p,*} : 0 \to \mathcal{E}^{p,0} \xrightarrow{\bar\partial} \mathcal{E}^{p,1} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{E}^{p,n} \to 0,$$

where $\mathcal{E}^{p,j}$ is the sheaf of germs of complex valued $\mathcal{C}^\infty$ forms of bidegree (p,j) on X, $0 \le p,j \le n$. We denote by $\mathcal{I}_M^{*,*}$ the sheaf of $\mathcal{E}_X^{*,*}$-modules which is locally generated by $\mathcal{J}_M$ and $\overline{\partial}\mathcal{J}_M$. We set $\mathcal{I}_M^{p,q} = \mathcal{I}_M^{*,*} \cap \mathcal{E}_X^{p,q}$. Since $\overline{\partial}\mathcal{I}_M^{p,j} \subset \mathcal{I}_M^{p,j+1}$, we have subcomplexes

$$\mathcal{I}_M^{p,*} : 0 \to \mathcal{I}_M^{p,0} \xrightarrow{\overline{\partial}} \mathcal{I}_M^{p,1} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{I}_M^{p,n} \to 0,$$

for each $0 \leq p \leq n$, of the complex $\mathcal{E}^{p,*}$ and hence quotient complexes $[\mathcal{E}^{p,*}]$, defined by the exact sequence of fine sheaf complexes

$$0 \longrightarrow \mathcal{I}_M^{p,*} \longrightarrow \mathcal{E}^{p,*} \longrightarrow [\mathcal{E}^{p,*}] \longrightarrow 0.$$

The induced differentials are denoted by $\overline{\partial}_M$. M being generic, we have $[\mathcal{E}^{p,q}] = 0$ for $q \geq n-k+1$. We write the quotient complex as

$$[\mathcal{E}^{p,*}] : 0 \to [\mathcal{E}^{p,0}] \xrightarrow{\overline{\partial}_M} [\mathcal{E}^{p,1}] \xrightarrow{\overline{\partial}_M} \cdots \xrightarrow{\overline{\partial}_M} [\mathcal{E}^{p,n-k}] \to 0.$$

It is called the *tangential Cauchy-Riemann complex of $\mathcal{C}^\infty$-smooth forms.* If Ω is an open subset of X, the cohomology groups of $[\mathcal{E}^{p,*}]$ on $M \cap \Omega$ are denoted by $H^{p,q}(M \cap \Omega)$.

Let $\mathcal{F}_M$ denote the ideal sheaf of germs of smooth complex valued differential forms on X that are flat on M, i.e. whose coefficients as well as all its derivatives vanish on M. We set $\mathcal{F}_M^{p,j} = \mathcal{F}_M \cap \mathcal{E}^{p,j}$. Note that $\overline{\partial}\mathcal{F}_M^{p,j} \subset \mathcal{F}_M^{p,j+1}$, therefore $\mathcal{F}_M^{p,*}$ is a subcomplex of $\mathcal{E}^{p,*}$, and the short exact sequence of fine sheaf complexes

$$0 \longrightarrow \mathcal{F}_M^{p,*} \longrightarrow \mathcal{E}^{p,*} \longrightarrow W_M^{p,*} \longrightarrow 0$$

defines the complex

$$W_M^{p,*} : 0 \to W_M^{p,0} \xrightarrow{\overline{\partial}} W_M^{p,1} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} W_M^{p,n} \to 0$$

of Whitney germs on M.

From the formal Cauchy-Kowalewski Theorem for generic CR submanifolds of a complex manifold (cf. [AFN81]), we get the following

Theorem 5.1.1
If M is a generic CR submanifold of real codimension k in X, then for all (p,q), $0 \leq p \leq n$, $0 \leq q \leq n-k$, and every open subset Ω of X with $M \cap \Omega \neq \emptyset$, the maps

$$H^q(M \cap \Omega, W_M^{p,*}) \longrightarrow H^{p,q}(M \cap \Omega),$$

induced by the natural map $W_M^{p,*} \longrightarrow [\mathcal{E}^{p,*}]$, are isomorphisms.

In order to define the current $\overline{\partial}_M$-cohomology groups on $M \cap \Omega$, we first consider the spaces $[\mathcal{D}^{p,j}](M \cap \Omega)$ of sections of $[\mathcal{E}^{p,j}]$ having compact support in $M \cap \Omega$ with their usual inductive limit topology.

We define $[\mathcal{D}'^{p,j}](M \cap \Omega)$ as the topological dual of $[\mathcal{D}^{n-p,n-k-j}](M \cap \Omega)$. In this way we obtain, for each $0 \leq p \leq n$, a complex of sheaves

$$[\mathcal{D}'^{p,*}] : 0 \longrightarrow [\mathcal{D}'^{p,0}] \xrightarrow{\overline{\partial}_M} [\mathcal{D}'^{p,1}] \xrightarrow{\overline{\partial}_M} \cdots \xrightarrow{\overline{\partial}_M} [\mathcal{D}'^{p,n-1}] \longrightarrow 0$$

whose cohomology groups on $M \cap \Omega$ will be denoted by $H^{p,q}_{cur}(M \cap \Omega)$.

Let $\mathcal{D}'$ be the sheaf of currents on X, we denote by $\mathcal{D}'_M$ the subsheaf of $\mathcal{D}'$ of currents with support contained in M. Dualizing the formal Cauchy-Kowalewski theorem (cf. [HN95], [NV87]), we get the following

Theorem 5.1.2
If M is a generic CR submanifold of real codimension k in X, then for all (p,q), $0 \leq p \leq n$, $0 \leq q \leq n-k$, and every open subset Ω of X with $M \cap \Omega \neq \emptyset$, there are natural isomorphisms

$$H^{p,q}_{cur}(M \cap \Omega) \longrightarrow H^{p,q+k}(\mathcal{D}'_M(\Omega)).$$

5.2 Boundaries of weakly pseudoconvex domains

Theorem 5.2.1
Let Ω be a relatively compact domain in an n-dimensional Kähler manifold (X,ω) with smooth boundary M. We assume that Ω is $\log\delta$-pseudoconvex. Let $f \in [\mathcal{E}^{p,q}] \cap \mathrm{Ker}\overline{\partial}_M$ satisfy the tangential Cauchy-Riemann equations on M, $0 \leq p \leq n$, $q \leq n-2$.
Then there exists $F \in \mathcal{C}^\infty_{p,q}(\overline{\Omega})$ such that $F_{|M} = f$ and $\overline{\partial} F = 0$ in Ω.

Proof: There exists $\tilde{f} \in \mathcal{C}^\infty_{p,n-1}(\overline{\Omega})$ such that $\tilde{f}_{|M} = f$ and $\overline{\partial}\tilde{f}$ vanishes to infinite order on M. Applying Theorem 3.3.2, one can find a solution u to the equation $\overline{\partial}u = \overline{\partial}\tilde{f}$ in such a way that u is of class $\mathcal{C}^\infty$ on $\overline{\Omega}$ and vanishes on M. $F = \tilde{f} - u$ is then the desired extension of f to $\overline{\Omega}$. $\square$

Theorem 5.2.2
Let X be an n-dimensional Stein manifold and $\Omega \subset\subset X$ a domain with smooth boundary M. We assume that Ω is weakly pseudoconvex. Then $H^{p,q}(M) = H^{p,q}_{cur}(M) = 0$ for $0 \leq p \leq n$, $1 \leq q \leq n-2$. Moreover, $H^{p,0}(M), H^{p,0}_{cur}(M), H^{p,n-1}(M)$ and $H^{p,n-1}_{cur}(M)$ are infinite dimensional and, if $n \geq 3$, separated.

Proof. It was proved in [HN92] that $H^{p,0}(M)$ and $H^{p,n-1}(M)$ are infinite dimensional. $H^{p,0}_{cur}(M)$ is infinite dimensional since $H^{p,0}(M)$ is infinite dimensional. Moreover, it was proved in [HN92] that there exists a point $x_0 \in M$ such that M is strictly pseudoconvex in a neighborhood of x_0. It follows from the failure of the Poincaré Lemma for $\overline{\partial}_M$ (see [AFN81] and its refined version [HN01]) that there exists a smooth $\overline{\partial}_M$-closed $(p, n-1)$ form defined on a neighborhood U of x_0 in M such that the equation $\overline{\partial}_M S = f$ admits no solution in the distribution sense on any neighborhood of x_0 in M. Since all $(p, n-1)$ forms on M are $\overline{\partial}_M$-closed, we may assume that f is defined on all of M. Thus we have $H^{p,n-1}_{cur}(M) \neq 0$. But then the Laufer alternative proved in [BHN01] permits to conclude that $H^{p,n-1}_{cur}(M)$ is infinite dimensional.

Now let $f \in [\mathcal{E}^{p,q}]$ satisfy the tangential Cauchy-Riemann equations, $1 \leq q \leq n-2$. It follows from Theorem 5.2.1 that there exists $F \in \mathcal{C}^\infty_{p,q}(\overline{\Omega})$ satisfying $F_{|M} = f$. Using Kohn's result on the solvability of the $\overline{\partial}$-equation with regularity up to the boundary in weakly pseudoconvex domains [Koh73], [Koh77], there exists $U \in \mathcal{C}^\infty_{p,q-1}(\overline{\Omega})$ satisfying $\overline{\partial} U = F$ in Ω. Then $u = U_{|M}$ satisfies $\overline{\partial}_M u = f$. Hence $H^{p,q}(M) = 0$ for $1 \leq q \leq n-2$.

Moreover, we know from abstract duality arguments (see [Ser55], also [LTL99]) that $H^{p,q}_{cur}(M)$ is separated if and only if $H^{n-p,n-q}(M)$ is separated. Furthermore, if any one of these equivalent conditions is satisfied, we have $H^{p,q}_{cur}(M) \simeq (H^{n-p,n-q-1}(M))'$.

Therefore we have $H^{p,q}_{cur}(M) = 0$ for $2 \leq q \leq n-2$ and $H^{p,n-1}_{cur}(M)$ is separated for $n \geq 3$; to complete the proof of the theorem, it remains to show that $H^{p,1}_{cur}(M) = 0$ if $n \geq 3$, $0 \leq p \leq n$.

To prove this, we note that we have a direct splitting

$$H^{p,q}_{cur}(M) \simeq H^{p,q}(\check{\mathcal{D}}'_\Omega(X)) \oplus H^{p,q}(\check{\mathcal{D}}'_{X\setminus\overline{\Omega}}(X)),$$

$q \geq 1$. Here $H^{p,q}(\check{\mathcal{D}}'_{\Omega}(X))$ (resp. $H^{p,q}(\check{\mathcal{D}}'_{X\setminus\overline{\Omega}}(X))$) denote the $\overline{\partial}$-cohomology groups for currents on Ω (resp. on $X \setminus \overline{\Omega}$) which are extendable to X (see Chapter 3, Section 4 for the definitions). Indeed, it is a well known fact that we have the following long exact sequence (cf. [HN95], [NV87])

$$\ldots \to H^{p,q}(X) \to H^{p,q}(\check{\mathcal{D}}'_{X\setminus M}(X)) \to H^{p,q+1}(\mathcal{D}'_{M}(X)) \to H^{p,q+1}(X) \to \ldots,$$

where $H^{p,q}(\check{\mathcal{D}}'_{X\setminus M}(X))$ are the $\overline{\partial}$-cohomology groups of currents on $X \setminus M$, which are extendable across M. Since X is Stein, it follows that $H^{p,q}(X) = 0$ for $q \geq 1$. Together with Theorem 5.1.2, this yields

$$H^{p,q}_{cur}(M) \simeq H^{p,q}(\check{\mathcal{D}}'_{X\setminus M}(X)) \simeq H^{p,q}(\check{\mathcal{D}}'_{\Omega}(X)) \oplus H^{p,q}(\check{\mathcal{D}}'_{X\setminus\overline{\Omega}}(X)),$$

$q \geq 1$. Theorem 3.4.1 implies $H^{p,q}(\check{\mathcal{D}}'_{\Omega}(X)) = 0$ for $q \geq 1$. $H^{p,1}_{cur}(M) = 0$ for $n \geq 3$ is now an immediate consequence of the following lemma. □

Lemma 5.2.3
Let X be an n-dimensional Stein manifold and $\Omega \subset\subset X$ a domain with smooth boundary M. We assume that Ω is weakly pseudoconvex. Then $H^{p,q}(\check{\mathcal{D}}'_{X\setminus\overline{\Omega}}(X)) = 0$ for $1 \leq q \leq n-2$.

Proof. We first prove the following claim:

Let Ω_1, Ω_2 be two weakly pseudoconvex domains with smooth boundary such that $\Omega_1 \subset\subset \Omega_2 \subset\subset X$. Then we have $H^{p,q}(X, \overline{\Omega}_2 \setminus \Omega_1) = 0$ for $2 \leq q \leq n-1$ and $H^{p,n}(X, \overline{\Omega}_2 \setminus \Omega_1)$ is separated, $0 \leq p \leq n$.

Indeed, let $f \in \mathcal{C}^{\infty}_{p,q}(X, \overline{\Omega}_2 \setminus \Omega_1) \cap \operatorname{Ker}\overline{\partial}$, $2 \leq q \leq n-1$. Then, since Ω_2 satisfies the assumptions of Theorem 3.3.2, there exists $u \in \mathcal{C}^{\infty}_{p,q-1}(X, \overline{\Omega}_2)$ satisfying $\overline{\partial}u = f$ in X. This implies that $\overline{\partial}u = 0$ in Ω_1. Hence, since $q-1 \geq 1$, there exists $h \in \mathcal{C}^{\infty}_{p,q-1}(\overline{\Omega}_1)$ satisfying $\overline{\partial}h = u$ in Ω_1 (see [Koh73] and [Koh77]). Let $\tilde{h}$ be a smooth extension of h to X with compact support in Ω_2 and set $g = u - \overline{\partial}\tilde{h}$. Then g satisfies $\overline{\partial}g = f$ and $\operatorname{supp} g \subset \overline{\Omega}_2 \setminus \Omega_1$. The separation statement is proved similarly, using the separation statement of Theorem 3.3.2.

With the same proof as the proof of Theorem 3.4.1, it follows from the above claim that $H^{p,q}(\check{\mathcal{D}}'_{\Omega_2\setminus\overline{\Omega}_1}(X)) = 0$ for $0 \leq p \leq n$, $1 \leq q \leq n-2$.

Now let $\psi \in \mathcal{C}^\infty(X)$ be a strictly plurisubharmonic exhaustion function such that $\psi < 0$ on Ω and set $\Omega_j = \{z \in X \mid \psi(z) < j\}$, where we may suppose that $\partial\Omega_j$ is of class $\mathcal{C}^\infty$.

Let $T \in \check{\mathcal{D}}'^{p,q}_{X\setminus\overline{\Omega}}(X)$ be $\overline{\partial}$-closed in $X \setminus \overline{\Omega}$, $1 \le q \le n-2$. As shown before, there exists $S_j \in \check{\mathcal{D}}'^{p,q-1}_{X\setminus\overline{\Omega}}(X)$ satisfying $\overline{\partial} S_j = T$ in $\Omega_j \setminus \overline{\Omega}$. We then have $\overline{\partial}(S_{j+1} - S_j) = 0$ in $\Omega_j \setminus \overline{\Omega}$.

First assume $q \ge 2$. Then there exists $H \in \check{\mathcal{D}}'^{p,q-2}_{X\setminus\overline{\Omega}}(X)$ such that $\overline{\partial} H = S_{j+1} - S_j$ in $\Omega_j \setminus \overline{\Omega}$. Setting $\widetilde{S}_{j+1} = S_{j+1} - \overline{\partial} H$, we have $\overline{\partial}\widetilde{S}_{j+1} = T$ in $\Omega_{j+1} \setminus \overline{\Omega}$ and $\widetilde{S}_{j+1} = S_j$ in $\Omega_j \setminus \overline{\Omega}$. We can thus find a sequence $(G_j)_{j\in\mathbb{N}}$, $G_j \in \check{\mathcal{D}}'^{p,q-1}_{X\setminus\overline{\Omega}}(X)$, satisfying $\overline{\partial} G_j = T$ in $\Omega_j \setminus \overline{\Omega}$ and $G_{j+1} = G_j$ in $\Omega_j \setminus \overline{\Omega}$. Then $(G_j)_j$ converges to $G \in \check{\mathcal{D}}'^{p,q-1}_{X\setminus\overline{\Omega}}(X)$ such that $\overline{\partial} G = T$ in $X \setminus \overline{\Omega}$.

Now assume $q = 1$. Then $S_{j+1} - S_j$ is a holomorphic p-form on $\Omega_j \setminus \overline{\Omega}$. From the Hartogs phenomenon on Stein manifolds, $S_{j+1} - S_j$ extends to a holomorphic p-form on Ω_j (see [HL88]). Moreover, we may approximate holomorphic p-forms on Ω_j uniformly on $\overline{\Omega}_{j-1}$ by holomorphic p-forms on Ω_{j+2} (cf [HL88]), hence there exists a holomorphic p-form H on Ω_{j+2} satisfying $|\langle H - (S_{j+1} - S_j), \varphi\rangle| \le 2^{-j}|\varphi|$ for every $\varphi \in \mathcal{C}^\infty_{n-p,n}(X, \overline{\Omega}_{j-1} \setminus \Omega)$. Let $\chi \in \mathcal{C}^\infty(X \setminus \overline{\Omega})$ satisfy $\chi \equiv 1$ on $\overline{\Omega}_{j+1}$, $\operatorname{supp}\chi \subset \Omega_{j+2}$. Setting $\widetilde{S}_{j+1} = S_{j+1} - \chi H$, we have $\widetilde{S}_{j+1} \in \check{\mathcal{D}}'^{p,q-1}_{X\setminus\overline{\Omega}}(X)$, $\overline{\partial}\widetilde{S}_{j+1} = T$ on $\Omega_{j+1} \setminus \overline{\Omega}$ and $|\langle \widetilde{S}_{j+1} - S_j, \varphi\rangle| \le 2^{-j}|\varphi|$ for every $\varphi \in \mathcal{C}^\infty_{n-p,n}(X, \overline{\Omega}_{j-1} \setminus \Omega)$. Thus there exists a sequence $(G_j)_{j\in\mathbb{N}}$, $G_j \in \check{\mathcal{D}}'^{p,q-1}_{X\setminus\overline{\Omega}}(X)$, such that $\overline{\partial} G_j = T$ in $\Omega_j \setminus \overline{\Omega}$ and $|\langle G_{j+1} - G_j, \varphi\rangle| \le 2^{-j}|\varphi|$ for every $\varphi \in \mathcal{C}^\infty_{n-p,n}(X, \overline{\Omega}_{j-1} \setminus \Omega)$. It follows that $(G_j)_{j\in\mathbb{N}}$ is a Cauchy sequence for the weak topology. In fact, let $\varphi \in \mathcal{C}^\infty_{n-p,n}(X, X \setminus \Omega) \cap \mathcal{D}^{n-p,n}(X)$. Then there exists $N \in \mathbb{N}$ such that $\operatorname{supp}\varphi \subset \overline{\Omega}_N \setminus \Omega$ and for all $j > N$, $p > 0$

$$|\langle G_{j+p} - G_j, \varphi\rangle| \le (\frac{1}{2^j} + \ldots + \frac{1}{2^{j+p}})\,|\varphi|,$$

hence $\langle G_{j+p} - G_j, \varphi\rangle \longrightarrow_{j\to+\infty} 0$. Thus $(G_j)_{j\in\mathbb{N}}$ converges weakly to G. We claim that G is an extensible current on $X \setminus \overline{\Omega}$. G is obviously linear. Indeed, let $\varphi, \psi \in \mathcal{C}^\infty_{n-p,n}(X, X \setminus \Omega) \cap \mathcal{D}^{n-p,n}(X)$. Then there exists $N \in \mathbb{N}$ such that

$\operatorname{supp}\varphi \subset \overline{\Omega}_N \setminus \Omega$, $\operatorname{supp}\psi \subset \overline{\Omega}_N \setminus \Omega$ and $\operatorname{supp}(\varphi+\psi) \subset \overline{\Omega}_N \setminus \Omega$. Hence

$$\begin{aligned} \langle G, \varphi+\psi \rangle &= \lim_{j>N} \langle G_j, \varphi+\psi \rangle \\ &= \lim_{j>N} \langle G_j, \varphi \rangle + \lim_{j>N} \langle G_j, \psi \rangle \\ &= \langle G, \varphi \rangle + \langle G, \psi \rangle. \end{aligned}$$

Let $(\varphi_\nu)_{\nu\in\mathbb{N}}$ be a sequence of elements of $\mathcal{C}^\infty_{n-p,n}(X, X\setminus\Omega) \cap \mathcal{D}^{n-p,n}(X)$ converging to 0 in $\mathcal{C}^\infty_{n-p,n}(X, X\setminus\Omega) \cap \mathcal{D}^{n-p,n}(X)$. Then there exists $N \in \mathbb{N}$ such that $\operatorname{supp}\varphi_\nu \subset \overline{\Omega}_N \setminus \Omega$ for all $\nu \in \mathbb{N}$. Hence

$$\begin{aligned} |\langle G, \varphi_\nu \rangle| &= |\lim_{j>N} \langle G_j, \varphi_\nu \rangle| \\ &\leq \sum_{k=N+1}^{j-1} \frac{1}{2^k} |\varphi_\nu| + |\langle G_{N+1}, \varphi_\nu \rangle|. \end{aligned}$$

Since G_{N+1} is an extensible current, we have $\langle G_{N+1}, \varphi_\nu \rangle \longrightarrow_{\nu\to+\infty} 0$ and by hypothesis $\varphi_\nu \longrightarrow_{\nu\to+\infty} 0$, i.e. $|\varphi_\nu| \to 0$. Hence $\langle G, \varphi_\nu \rangle \longrightarrow_{\nu\to+\infty} 0$ and G is therefore an extensible current on $X \setminus \overline{\Omega}$ satisfying $\overline{\partial} G = T$ in $X \setminus \overline{\Omega}$. $\square$

5.3 Applications to Levi flat CR manifolds

Here we want to solve the tangential Cauchy-Riemann equation for currents on certain domains in Levi flat submanifolds of a Stein manifold. The submanifolds will be of any codimension where the problem makes sense. More precisely, we consider the following set-up.

Let $M \subset X$ be a smooth generic CR manifold of real codimension k in an n-dimensional Stein manifold X. We moreover assume that M is globally defined by

$$M = \{z \in X \mid \rho_1(z) = \ldots = \rho_k(z) = 0\},$$

where the ρ_ν's, $1 \leq \nu \leq k$ are real $\mathcal{C}^\infty$ functions in X satisfying $\partial\rho_1 \wedge \ldots \wedge \partial\rho_k \neq 0$ on M. Our most important assumption is that M should be Levi flat, i.e.

$$\mathcal{L}(\rho_\nu, z)\xi = 0$$

for $\nu = 1, \ldots, k$, $z \in M$ and every $\xi \in \mathbb{C}^n$ satisfying $\sum_{j=1}^n \frac{\partial\rho_\mu}{\partial z_j}(z)\xi_j = 0$ for $\mu = 1, \ldots, k$.

For each $\nu = 1, \dots, k$, we set $\varphi_\nu = \rho_\nu + \psi \sum_{j=1}^k \rho_j^2$ and $\varphi_0 = -\sum_{j=1}^k \rho_j + \psi \sum_{j=1}^k \rho_j^2$, where ψ is a positive stricly plurisubharmonic function of class $\mathcal{C}^\infty$ on X. Then for every ordered collection of k integers $0 \leq i_1 < \dots < i_k \leq k$ we have $\partial\rho_{i_1} \wedge \dots \wedge \partial\rho_{i_k} \neq 0$ on M. For an adequate choice of ψ, we can arrange that if we set $\Omega_\nu = \{z \in X \mid \varphi_\nu(z) < 0\}$, $\nu = 0, \dots, k$, then each Ω_ν is weakly pseudoconvex and

$$M = \bigcap_{\nu=0}^{k} \overline{\Omega}_\nu, \;\; X \setminus M = \bigcup_{\nu=0}^{k} \Omega_\nu, \;\; X = \bigcup_{\nu=0}^{k} \overline{\Omega}_\nu \text{ and } \bigcap_{\nu=0}^{k} \Omega_\nu = \emptyset.$$

Let Ω be a piecewise $\mathcal{C}^\infty$ bounded weakly pseudoconvex domain such that Ω intersects each Ω_ν transversally.

Theorem 5.3.1
Let M and Ω be as above and $0 \leq p \leq n$, $2 \leq q \leq n-k$.
Then $H^{p,q}_{cur}(M \cap \Omega) = 0$.
Moreover, let Ω' be any open set which is relatively compact in Ω. Then the restriction mapping

$$H^{p,1}_{cur}(M \cap \Omega) \longrightarrow H^{p,1}_{cur}(M \cap \Omega')$$

is the zero mapping. In other words, let $T \in [\mathcal{D}'^{p,1}](M \cap \Omega)$ such that $\overline{\partial}_M T = 0$ in $M \cap \Omega$. Then there exists $S \in [\mathcal{D}'^{p,0}](M \cap \Omega')$ satisfying $\overline{\partial}_M S = T$ in $M \cap \Omega'$.

Proof: It is a well known fact that we have the following exact sequence (cf. [HN95], [NV87])

$$\dots \to H^{p,q}_{cur}(\Omega) \to H^{p,q}(\check{\mathcal{D}}'_{\Omega\setminus M}(\Omega)) \to H^{p,q+1}(\mathcal{D}'_M(\Omega)) \to H^{p,q+1}_{cur}(\Omega) \to \dots$$

Here $H^{p,q}_{cur}(\Omega)$ denote the $\overline{\partial}$-cohomology groups for currents on Ω and $H^{p,q}(\check{\mathcal{D}}'_{\Omega\setminus M}(\Omega)) = \check{\mathcal{D}}'^{p,q}_{\Omega\setminus M}(\Omega) \cap \mathrm{Ker}\overline{\partial}/\overline{\partial}\check{\mathcal{D}}'^{p,q-1}_{\Omega\setminus M}(\Omega)$ denote the $\overline{\partial}$-cohomology groups for currents on $\Omega \setminus M$ which are extendable to Ω (see Chapter 3, Section 4 for the relevant definitions).
We denote by $H^{p,q}(\mathcal{D}'_M(\Omega))$ the $\overline{\partial}$-cohomology groups of currents on Ω with support in M.

Since Ω is weakly pseudoconvex and X is Stein, it follows that Ω is a Stein manifold (see [Ele75]), thus we have $H^{p,q}_{cur}(\Omega) = 0$ for all $q \geq 1$.

Moreover, by Theorem 5.1.2 there are natural isomorphisms

$$H^{p,q}_{cur}(M \cap \Omega) \longrightarrow H^{p,q+k}(\mathcal{D}'_M(\Omega)).$$

We then get

$$H^{p,q}_{cur}(M \cap \Omega) \simeq H^{p,q+k-1}(\check{\mathcal{D}}'_{\Omega\setminus M}(\Omega))$$

for $q \geq 1$. Without loss of generality, we may assume that Ω' is also weakly pseudoconvex. Then the above isomorphisms also holds with Ω replaced by Ω'. The theorem is then an immediate consequence of the following lemma. For the case $q = 1$, note that all diagrams induced by the restriction mapping are commutative. □

Lemma 5.3.2
For $0 \leq p \leq n$ and $q \geq k+1$ we have $H^{p,q}(\check{\mathcal{D}}'_{\Omega\setminus M}(\Omega)) = 0$.
Moreover, let Ω' be any open set which is relatively compact in Ω. Then the restriction mapping

$$H^{p,k}(\check{\mathcal{D}}'_{\Omega\setminus M}(\Omega)) \longrightarrow H^{p,k}(\check{\mathcal{D}}'_{\Omega'\setminus M}(\Omega'))$$

is the zero mapping.

Proof: The proof follows an induction argument of [NV87]. By induction on ℓ, we show the following claim:

Let $\Omega, D_0, \ldots, D_\ell$ be piecewise smooth domains in X which are locally Stein and which intersect pairwise transversally. Set $D = \Omega \cap \bigcup_{j=0}^{\ell} D_j$ and let Ω' be any relatively compact open set of Ω.
If $T \in \check{\mathcal{D}}'^{p,q}_D(\Omega)$ satisfies $\overline{\partial} T = 0$ in $D \cap \Omega'$, $q \geq \ell+1$, then there exists $S \in \check{\mathcal{D}}'^{p,q-1}_D(\Omega)$ satisfying $\overline{\partial} S = T$ in $D \cap \Omega'$.

First assume $\ell = 0$ and let $T \in \check{\mathcal{D}}'^{p,q}_{D_0\cap\Omega'}(\Omega)$ satisfy $\overline{\partial} T = 0$ in $D_0 \cap \Omega'$, $q \geq 1$. Without loss of generality, we may assume that Ω' is a piecewise smooth domain which is locally Stein and which intersects D_0 transversally. Then $\Omega' \cap D_0$ has Lipschitz boundary and is relatively compact in Ω. Moreover, since X is a Stein manifold and since $\Omega' \cap D_0$ is locally Stein, it follows from [Ele75] that $\Omega' \cap D_0$ is $\log\delta$-pseudoconvex for some Kähler metric on X. We apply Theorem 3.4.1 and conclude that there exists $S \in \check{\mathcal{D}}'^{p,q-1}_{D_0\cap\Omega}(\Omega)$ satisfying $\overline{\partial} S = T$ in $\Omega' \cap D_0$. This proves the claim for $\ell = 0$.

Now assume the claim is true for $\ell - 1$ and let us prove it for $\ell \geq 1$. Set $U_1 = \Omega \cap \bigcup_{j=0}^{\ell-1} D_j$ and $U_2 = \Omega \cap D_\ell$. Let $T \in \check{\mathcal{D}}'^{p,q}_D(\Omega)$ satisfy $\overline{\partial} T = 0$ in $D \cap \Omega'$, $q \geq \ell+1$. Then, by the induction hypothesis, there exist $S_1, S_2 \in \check{\mathcal{D}}'^{p,q-1}_D(\Omega)$ such that $\overline{\partial} S_1 = T$ in $U_1 \cap \Omega'$ and $\overline{\partial} S_2 = T$ in $U_2 \cap \Omega'$. Then we have $\overline{\partial}(S_1 - S_2) = 0$ in $U_1 \cap U_2 \cap \Omega'$. Again, since $q - 1 \geq 1$, we may apply

Theorem 3.4.1 to the domain $U_1 \cap U_2 \cap \Omega'$; note that $U_1 \cap U_2 \cap \Omega'$ is relatively compact in Ω and locally Stein with Lipschitz boundary. We conclude that there exists $H \in \check{\mathcal{D}}'^{p,q-2}_D(\Omega)$ satisfying $\overline{\partial} H = S_1 - S_2$ in $U_1 \cap U_2 \cap \Omega'$. We define the current S of bidegree $(p, q-1)$ on D by $S = S_1$ in U_1 and $S = S_2 + \overline{\partial} H$ in U_2. Then S is well defined and $\overline{\partial} S = T$ in $\Omega' \cap \bigcup_{j=0}^{\ell} D_j$. Moreover, S is extendable to a current on Ω. This proves the claim.

Let us now prove that for every relatively compact subset Ω' of Ω and every $T \in \check{\mathcal{D}}'^{p,q}_{\Omega\setminus M}(\Omega)$ satisfying $\overline{\partial} T = 0$ in $\Omega \setminus M$, $q \geq k$, there exists $S \in \check{\mathcal{D}}'^{p,q-1}_{\Omega\setminus M}(\Omega)$ such that $\overline{\partial} S = T$ in $\Omega' \setminus M$.

We recall that we have $\Omega \setminus M = \bigcup_{\nu=0}^{k} (\Omega_\nu \cap \Omega)$ and $\bigcap_{\nu=0}^{k} \Omega_\nu = \emptyset$.

Let $T \in \check{\mathcal{D}}'^{p,q}_{\Omega\setminus M}(\Omega)$ satisfy $\overline{\partial} T = 0$ in $\Omega \setminus M$, $q \geq k$. From the above claim, there exist $S_1, S_2 \in \check{\mathcal{D}}'^{p,q-1}_{\Omega\setminus M}(\Omega)$ such that $\overline{\partial} S_1 = T$ in $\bigcup_{\nu=0}^{k-1} (\Omega_\nu \cap \Omega')$ and $\overline{\partial} S_2 = T$ in $\Omega_k \cap \Omega'$. This settles the case $k = 1$, since in this situation, $\Omega_0 \cap \Omega'$ and $\Omega_1 \cap \Omega'$ are disjoint sets.

Now we assume $k \geq 2$. Then $\overline{\partial}(S_1 - S_2) = 0$ in $\bigcup_{\nu=0}^{k-1} \Omega_\nu \cap \Omega_k \cap \Omega' = \bigcup_{\nu=0}^{k-2} (\Omega_\nu \cap \Omega_k \cap \Omega') \cup (\Omega_{k-1} \cap \Omega_k \cap \Omega')$. We set $W_1 = \bigcup_{\nu=0}^{k-2} (\Omega_\nu \cap \Omega_k \cap \Omega')$ and $W_2 = \Omega_{k-1} \cap \Omega_k \cap \Omega'$. Then, since $\bigcap_{\nu=0}^{k} \Omega_\nu = \emptyset$, W_1 and W_2 are disjoint sets. Thus, in order to solve the $\overline{\partial}$-equation for extensible currents on $W_1 \cup W_2$, it suffices to solve the $\overline{\partial}$-equation for extensible currents separately on W_1 and W_2. Since $q - 1 \geq k - 1$, it then follows from the above claim that there exists $G \in \check{\mathcal{D}}'^{p,q-2}_{\Omega\setminus M}(\Omega)$ satisfying $\overline{\partial} G = S_1 - S_2$ in $(W_1 \cup W_2) \cap \Omega'$. It follows that the current S defined by $S = S_1$ in $\bigcup_{\nu=0}^{k-1} (\Omega_\nu \cap \Omega')$, $S = S_2 + \overline{\partial} G$ in $\Omega_k \cap \Omega'$ is well defined, extendable to Ω and satisfies $\overline{\partial} S = T$ in $\Omega' \setminus M$.

We have thus proved the last assertion of the lemma.

Now suppose $q \geq k+1$ and let $T \in \check{\mathcal{D}}'^{p,q}_{\Omega\setminus M}(\Omega)$ satisfy $\overline{\partial} T = 0$ in $\Omega \setminus M$. Let $(\Omega'_i)_{i\in\mathbb{N}}$ be an exhaustion of Ω by smooth pseudoconvex domains. We just proved that for every $i \in \mathbb{N}$, there exists $S_i \in \check{\mathcal{D}}'^{p,q-1}_{\Omega\setminus M}(\Omega)$ satisfying $\overline{\partial} S_i = T$ in $\Omega'_i \setminus M$. Then $\overline{\partial}(S_{i+1} - S_i) = 0$ in $\Omega'_i \setminus M$. Thus, since $q - 1 \geq k$, there exists $H_i \in \check{\mathcal{D}}'^{p,q-2}_{\Omega\setminus M}(\Omega)$ satisfying $S_{i+1} - S_i = \overline{\partial} H_i$ in $\Omega'_i \setminus M$. We set $\widetilde{S}_{i+1} = S_{i+1} - \overline{\partial} H_i$. Then $\widetilde{S}_{i+1} \in \check{\mathcal{D}}'^{p,q-1}_{\Omega\setminus M}(\Omega)$, $\overline{\partial} \widetilde{S}_{i+1} = T$ in $\Omega'_{i+1} \setminus M$ and $\widetilde{S}_{i+1} = S_i$ in $\Omega'_i \setminus M$. Thus it is possible to construct a sequence $(S_j)_{j\in\mathbb{N}}$, $S_j \in \check{\mathcal{D}}'^{p,q-1}_{\Omega\setminus M}(\Omega)$ satisfying $\overline{\partial} S_j = T$ in $\Omega'_j \setminus M$ and $S_{j+1} = S_j$ in $\Omega'_j \setminus M$. Then

$(S_j)_{j\in\mathbb{N}}$ converges to $S \in \check{\mathcal{D}}'^{p,q-1}_{\Omega\setminus M}(\Omega)$ satisfying $\overline{\partial}S = T$ in $\Omega \setminus M$. □

5.4 Applications to hypersurfaces with constant signature

If M is a CR manifold, then we denote by $H^{p,q}_c(M)$ the $\overline{\partial}_M$-cohomology groups for smooth forms with compact support in M. We have the following result:

Theorem 5.4.1
Let X be a Stein manifold of complex dimension $n \geq 2$ and M a smooth, closed, connected hypersurface in X. Suppose that M has signature (p^-, p^0, p^+) at each point. Then $H^{p,q}_c(M) = 0$ for $0 \leq p \leq n$, $0 \leq q \leq \min(p^-, p^+)+p^0-1$.

Proof: Let $f \in [\mathcal{E}^{p,q}] \cap \mathrm{Ker}\overline{\partial}_M$ such that $\operatorname{supp} f \subset K$, where K is a compact subset of M. Since X is Stein, there exists a smooth bounded completely strictly pseudoconvex domain Ω such that $K \subset \Omega$, $\Omega \setminus M$ has exactly two connected components D^+ and D^-, and M intersects $\partial\Omega$ transversally.

Next, we can find $\tilde{f} \in \mathcal{C}^\infty_{p,q}(X)$ such that $\tilde{f}_{|M} = f$, $\operatorname{supp}\tilde{f} \subset\subset \Omega$ and $\overline{\partial}\tilde{f}$ vanishes to infinite order on M.

Applying Theorem 4.4.1, we conclude that $H^{p,q+1}(X, \overline{D}^{\pm}) = 0$ for $q+1 \leq p^0 + \min(p^-, p^+)$. Therefore there exists a solution $u \in \mathcal{C}^\infty_{p,q}(X)$ to the equation $\overline{\partial}u = \overline{\partial}\tilde{f}$ in such a way that u vanishes on $M \cup (X \setminus \overline{\Omega})$. $F = \tilde{f} - u$ is then $\overline{\partial}$-closed in X and we have $F_{|M} = f$, $\operatorname{supp} F \subset \overline{\Omega}$.

If $q = 0$, the analytic continuation principle yields $F \equiv 0$, thus $f \equiv 0$, proving $H^{p,0}_c(M) = 0$.

Now let $q \geq 1$. Ω being completely strictly pseudoconvex, there exists an open set $\tilde{\Omega} \supset \Omega$ which is also completely strictly pseudoconvex. Then the $\overline{\partial}$-cohomology groups with compact support in $\tilde{\Omega}$, $H^{p,q}_c(\tilde{\Omega})$, vanish for $q \geq 1$ (cf [HL88]). Thus we can find $U \in \mathcal{C}^\infty_{p,q-1}(X)$, $\operatorname{supp} U \subset\subset \tilde{\Omega}$ satisfying $\overline{\partial}U = F$. We then have $\overline{\partial}_M(U_{|M}) = f$, which proves the theorem. □

It is well known that if X is a Stein manifold of complex dimension $n \geq 2$ and K a compact subset of X with $X \setminus K$ connected, then every holomorphic function on $X \setminus K$ extends holomorphically to X. In fact it is sufficient that X satisfies $H_c^{0,1}(X) = 0$, which holds for example under the assumption that X is completely 1-convex in the sense of definition 5.1 of [HL88]. This extension property of holomorphic functions is called the Hartogs phenomenon.

The Hartogs phenomenon has also been studied in so-called q-convex-concave hypersurfaces. These are hypersurfaces, whose Levi form has at least q positive and q negative eigenvalues at each point.

Indeed, it is known that the Hartogs phenomenon holds if M is a 2-convex-concave hypersurface in a Stein manifold or if M is 1-convex-concave and K sufficiently small (see [Hen84] and [LT91]).

On the other hand, the following example given in [HN96] shows that the Hartogs phenomenon fails to hold globally for 1-convex-concave hypersurfaces:
Set $M = \{z \in \mathbb{C}^3 \mid |z_1|^2 + |z_2|^2 - |z_3|^2 = 1\}$ and $K = \{z \in M \mid z_3 = 0\}$. Then M is 1-convex-concave and the CR function $f(z) = \frac{1}{z_3}$ defined on $M \setminus K$ has no CR extension to M.

Here we will prove the following result on the Hartogs phenomenon in hypersurfaces:

Theorem 5.4.2
Let X be a Stein manifold and M a smooth, closed, connected hypersurface in X. Suppose that the signature of M is the same at each point and that M is weakly 2-convex-concave. Let K be a compact subset of M such that $M \setminus K$ is connected and globally minimal. Then every smooth CR function on $M \setminus K$ extends to a smooth CR function on M.

M being weakly 2-convex-concave signifies that the Levi form of M has at least 2 nonnegative and 2 nonpositive eigenvalues at each point. In particular, this class of hypersurfaces contains all Levi flat hypersurfaces of real dimension at least 5. Another interesting case is the one of signature $(1, 1, 1)$.

$M \setminus K$ is globally minimal if any two points $p, q \in M \setminus K$ can be joined by a piecewise smooth curve $\gamma = \gamma_1 \cup \ldots \cup \gamma_r$, $\gamma_i : [0, 1] \longrightarrow M \setminus K$, such that $\gamma_i'(t) \in T_{\gamma_i(t)}M \cap JT_{\gamma_i(t)}M$ for all $t \in (0, 1)$; here J denotes the complex structure on X. This assumption of global minimality is needed in order to assure that the weak analytic continuation principle holds for CR functions. However, this assumption is always satisfied as long as the Levi form is not identically zero (or if M is of finite bracket type).

Proof of Theorem 5.4.2: If M is weakly 2-convex-concave, we have $\min(p^-, p^+) + p^0 - 1 \geq 1$, thus Theorem 5.4.1 implies that $H_c^{0,1}(M) = 0$. Keeping in mind that the weak analytic continuation principle holds for CR functions on minimal CR manifolds, we have thus proved Theorem 5.4.2. Indeed, let K be a compact subset of M such that $M \setminus K$ is connected, and let $f \in \mathcal{C}^\infty(M \setminus K)$ satisfy $\overline{\partial}_M f = 0$. Choose a smooth function χ with compact support in M such that $\chi \equiv 1$ in a neighborhood of K. Set $f_o = (1-\chi)f$, defined as 0 in K. Then $f_o \in \mathcal{C}^\infty(M)$. Define

$$g = \begin{cases} -f\overline{\partial}_M \chi & \text{in } M \setminus K \\ 0 & \text{in } K \end{cases}$$

g is then a $\overline{\partial}_M$-closed $(0,1)$-form with compact support in M. As $H_c^{0,1}(M) = 0$, there exists a smooth function u with compact support in M satisfying $\overline{\partial}_M u = g$. Define $F = f_o - u$. Clearly $\overline{\partial}_M F = 0$ in M. Moreover, there exists an open set in $M \setminus K$ where $u = 0$ and $f_o = f$, thus $F = f$. Since the weak analytic continuation principle for CR functions holds on globally minimal CR manifolds, we therefore get $F = f$ in $M \setminus K$.

Similar to the results of the previous section, we can also prove a result on the solvability of the $\overline{\partial}_M$-equation for currents on hypersurfaces with constant signature.

Theorem 5.4.3
Let X be a complex manifold of dimension n and M a smooth, closed, connected hypersurface in X. Suppose that M has signature (p^-, p^0, p^+) at each point. Let $\Omega \subset\subset X$ be a smooth bounded completely strictly pseudoconvex domain in X such that $\Omega \setminus M$ has exactly two connected components and M intersects $\partial\Omega$ transversally. Then $H_{cur}^{p,q}(M \cap \Omega) = 0$ for $0 \leq p \leq n$, $q \geq n - \min(p^-, p^+) - p^0 + 1$.
Moreover, let Ω' be any open set which is relatively compact in Ω. Then for $q = n - \min(p^-, p^+) - p^0$, the restriction mapping

$$H_{cur}^{p,q}(M \cap \Omega) \longrightarrow H_{cur}^{p,q}(M \cap \Omega')$$

is the zero mapping.

Proof: We denote by D^+ and D^- the connected components of $\Omega \setminus M$.

Let $H^{p,q}(\check{\mathcal{D}}'_{D^+}(\Omega))$ (resp. $H^{p,q}(\check{\mathcal{D}}'_{D^-}(\Omega))$) the $\overline{\partial}$-cohomology groups of currents on D^+ (resp. D^-) which are extendable to Ω. Moreover, we consider the $\overline{\partial}$-cohomology groups $H^{p,q}(M\cap\Omega, \mathcal{D}'_M)$ of currents on Ω with support on $M\cap\Omega$.

We then have the following long exact sequence (cf [HN95], [NV87])

$$\ldots \to H^{p,q}_{cur}(\Omega) \to H^{p,q}(\check{\mathcal{D}}'_{D^+}(\Omega)) \oplus H^{p,q}(\check{\mathcal{D}}'_{D^-}(\Omega)) \to H^{p,q+1}(M\cap\Omega, \mathcal{D}'_{M\cap\Omega}) \to H^{p,q+1}_{cur}(\Omega) \to \ldots$$

Since Ω is completely strictly pseudoconvex, we have $H^{p,q}_{cur}(\Omega) = 0$ for all $q \geq 1$ (see [HL88]). Moreover, it follows from Theorem 5.1.2 that we have a natural isomorphism $H^{p,q}_{cur}(M\cap\Omega) \to H^{p,q+1}(M\cap\Omega, \mathcal{D}'_{M\cap\Omega})$. Hence $H^{p,q}_{cur}(M\cap\Omega) \simeq H^{p,q}(\check{\mathcal{D}}'_{D^+}(\Omega)) \oplus H^{p,q}(\check{\mathcal{D}}'_{D^-}(\Omega))$. The theorem is now a consequence of the following lemma (for the case $q = n - \min(p^-, p^+) - p^0$, note that all diagrams induced by the restriction mapping are commutative). □

Lemma 5.4.4
For $0 \leq p \leq n$ and $q \geq n - \min(p^-, p^+) - p^0 + 1$ we have $H^{p,q}(\check{\mathcal{D}}'_{D^+}(\Omega)) = H^{p,q}(\check{\mathcal{D}}'_{D^-}(\Omega)) = 0$.
Moreover, let Ω' be any relatively compact domain in Ω. Then for $q = n - \min(p^-, p^+) - p^0$, the restriction mappings

$$H^{p,q}(\check{\mathcal{D}}'_{D^+}(\Omega)) \longrightarrow H^{p,q}(\check{\mathcal{D}}'_{D^+\cap\Omega'}(\Omega')),$$

$$H^{p,q}(\check{\mathcal{D}}'_{D^-}(\Omega)) \longrightarrow H^{p,q}(\check{\mathcal{D}}'_{D^-\cap\Omega'}(\Omega'))$$

are the zero mappings.

Proof. Let $(\Omega_j)_{j\in\mathbb{N}}$ be an exhaustion of Ω by smooth bounded strictly pseudoconvex domains such that M intersects $\partial\Omega_j$ transversally.

Let $T \in \check{\mathcal{D}}'^{p,q}_{D^+\cap\Omega}(\Omega)$ satisfy $\overline{\partial}T = 0$ in $D^+\cap\Omega$, $q \geq n - \min(p^-, p^+) - p^0$. It follows from Theorem 4.4.2 that there exists $S_j \in \check{\mathcal{D}}'^{p,q-1}_{D^+}(\Omega)$ satisfying $\overline{\partial}S_j = T$ in $D^+\cap\Omega_j$. The same holds true of course with D^+ replaced by D^-. This proves the assertion of the lemma for $q = n - \min(p^-, p^+) - p^0$.

Now let $q \geq n - \min(p^-, p^+) - p^0 + 1$. We have $\overline{\partial}(S_{j+1} - S_j) = 0$ in $D^+\cap\Omega_j$. Hence, again by Theorem 4.4.2, there exists $H \in \check{\mathcal{D}}'^{p,q-2}_{D^+}(\Omega)$ satisfying $\overline{\partial}H = S_{j+1} - S_j$ in $D^+\cap\Omega_j$. Setting $\widetilde{S}_{j+1} = S_{j+1} - \overline{\partial}H$, we have $\overline{\partial}\widetilde{S}_{j+1} = T$ in $D^+\cap\Omega_{j+1}$ and $\widetilde{S}_{j+1} = S_j$ in $D^+\cap\Omega_j$. Thus we can find a sequence $(G_j)_{j\in\mathbb{N}}$, $G_j \in \check{\mathcal{D}}'^{p,q-1}_{D^+}(\Omega)$ satisfying $\overline{\partial} G_j = T$ in $D^+ \cap \Omega_j$ and $G_{j+1} = G_j$ in $D^+ \cap \Omega_j$. Hence (G_j) converges to $G \in \check{\mathcal{D}}'^{p,q-1}_{D^+}(\Omega)$ satisfying $\overline{\partial} G = T$ in $D^+ \cap \Omega$. Since the same holds true also for D^-, we have proved the lemma. □

We remark that Theorem 5.4.3 gives a Poincaré lemma for currents on a certain type of hypersurfaces. Combining this with the results of [NV87], [AH72], [Mic93] and [HN01], we obtain the following corollary:

Corollary 5.4.5
Let X be a smooth hypersurface in $\mathbb{C}^n$ and suppose that M has signature (p^-, p^0, p^+) at each point in a neighborhood of $x_0 \in M$. Then the Poincaré lemma holds for smooth forms and for currents of bidegree (p, q) at the point x_0 if $1 \le q \ne p^-, p^+$, i.e. each smooth form (resp. current) of bidegree (p, q), $1 \le q \ne p^-, p^+$, which is $\overline{\partial}$-closed on some open neighborhood of x_0 is $\overline{\partial}$-exact on some open neighborhood of x_0.
The Poincaré lemma fails to hold at x_0 for smooth forms and currents of bidegree (p, p^-) and (p, p^+).

Proof. Let M be defined by $\{\varrho = 0\}$ in a neighborhood of x_0, where ϱ is a $\mathcal{C}^\infty$ function whose Levi form has exactly p^+ positive, p^0 zero and p^- negative eigenvalues on $T^{1,0}_x M$ for each $x \in M$. Let Ω be a small ball around x_0 such that M intersects $\partial\Omega$ transversally and $\Omega \setminus M$ has exactly two connected components. Set $\Omega^+ = \Omega \cap \{\varrho < 0\}$ and $\Omega^- = \Omega \cap \{\varrho > 0\}$.

It follows from [NV87] that there exists a neighborhood Ω' of x_0 in $\mathbb{C}^n$ such that the restriction mappings

$$H^{p,q^+}(\check{\mathcal{D}}'_{\Omega^+}(\Omega)) \longrightarrow H^{p,q^+}(\check{\mathcal{D}}'_{\Omega^+\cap\Omega'}(\Omega')),$$

$$H^{p,q^-}(\check{\mathcal{D}}'_{\Omega^-}(\Omega)) \longrightarrow H^{p,q^-}(\check{\mathcal{D}}'_{\Omega^-\cap\Omega'}(\Omega'))$$

are the zero mappings for $1 \le q^+ < p^+$ and $1 \le q^- < p^-$. In virtue of Theorem 4.4.2, the same holds true for $q^+ \ge n - p^- - p^0$, $q^- \ge n - p^+ - p^0$. Since we may assume that Ω and Ω' are Stein, this proves the Poincaré lemma for currents of bidegree (p, q), $1 \le q \ne p^-, p^+$; remember that we have a direct splitting

$$H^{p,q}_{cur}(M \cap \Omega) \simeq H^{p,q}(\check{\mathcal{D}}'_{\Omega^+}(\Omega)) \oplus H^{p,q^-}(\check{\mathcal{D}}'_{\Omega^-}(\Omega))$$

(cf the proof of Theorem 5.4.3).

Moreover, it follows from [AH72] and [Mic93] that after possibly shrinking Ω, the restriction mappings

$$H^{p,q}(\overline{\Omega^+}) \longrightarrow H^{p,q}(\overline{\Omega^+ \cap \Omega'}),$$

$$H^{p,q}(\overline{\Omega^-}) \longrightarrow H^{p,q}(\overline{\Omega \cap \Omega'})$$

are the zero mappings for $1 \leq q \neq p^-, p^+$; here $\overline{}$ denotes the closure in Ω (resp. in Ω') and the cohomology groups are the cohomology groups of the $\overline{\partial}$-operator acting on smooth forms with regularity up to the boundary.

In virtue of Theorem 5.1.1, we also have direct splittings

$$H^{p,q}(\overline{\Omega^+}) \oplus H^{p,q}(\overline{\Omega^-}) \simeq H^{p,q}(M \cap \Omega)$$

induced by $(f^+, f^-) \mapsto f^+_{|M} - f^-_{|M}$ (cf [AH72]). This proves the Poincaré lemma for smooth forms on M in bidegree (p, q), $1 \leq q \neq p^-, p^+$.

The failure of the Poincaré lemma in bidegree (p, p^-) and (p, p^+) was proved in [HN01]. $\square$

5.5 Examples

1. Let $p_1, \ldots, p_n$ be positive integers. Then

$$\Omega = \{z \in \mathbb{C}^n \mid \sum_{j=1}^{n} |z_j|^{2p_j} < 1\}$$

   is a smooth bounded weakly pseudoconvex domain in $\mathbb{C}^n$.

2. Let M be a smooth hypersurface in $\mathbb{C}^n$ with signature $(p^-, 0, p^+)$ at each point. Then $\widetilde{M} = M \times \mathbb{C}^{p^0}$ has signature (p^-, p^0, p^+) at each point.

3. Any real-analytic hypersurface in $\mathbb{C}^n$ has constant signature outside a proper real-analytic subvariety (in particular, on a dense open subset).

4. The tube in $\mathbb{C}^n$ defined by

$$\varrho(z) = x_1^2 + \ldots + x_q^2 - x_{q+1}^2 - \ldots - x_n^2 = 0,$$

   $x_j = \operatorname{Re} z_j$, has signature $(n-q-1, 1, q-1)$ at every nonsingular point, i.e. at every point where it is a real submanifold of $\mathbb{C}^n$.

Indeed, it is clear that $\mathcal{L}(\varrho, z)_{T_z^{1,0}M}$ has at least $q-1$ positive and $n-q-1$ negative eigenvalues for every $z \in M \setminus \{0\}$. Moreover, if we define $\xi \in \mathbb{C}^n$ by

$$\xi_j = x_j, \ 1 \le j \le n,$$

then $\xi \in T_z^{1,0}M$ for every $z \in M \setminus \{0\}$ and $\mathcal{L}(\varrho, z)(\xi, \eta) = 0$ for every $\eta \in T_z^{1,0}M$.

In particular, the tube in $\mathbb{C}^n$ over the light cone in $\mathbb{R}^n$, i.e. the variety M defined by

$$\varrho(z) = x_1^2 + \ldots + x_{n-1}^2 - x_n^2 = 0$$

has signature $(0, 1, n-2)$ at every nonsingular point.

5. The following example is taken from [HN00]. Consider the unit sphere S^5 in $\mathbb{C}^3$, where we look at $\mathbb{C}^3$ as one of the standard holomorphic coordinate patches in $\mathbb{P}^3$. Let M denote the smooth subma-nifold of the Grassmannian $G(2, 4)$ of all $\mathbb{P}^1$'s in $\mathbb{P}^3$, consisting of those $\mathbb{P}^1$'s which are tangent to S^5 at some point. Then M is a compact 7-dimensional hypersurface in $G(2, 4)$ with signature $(1, 1, 1)$ at each point.

   Indeed, we can represent $S^5 \subset \mathbb{P}^3$ in homogeneous coordinates by

$$S^5 = \{z_0\overline{z}_3 + z_3\overline{z}_0 + z_1\overline{z}_1 + z_2\overline{z}_2 = 0\}.$$

   This is a homogeneous manifold for the action of $\mathbb{SU}(1, 3)$, i.e. $\mathbb{SU}(1, 3)$ is a group of CR-automorphisms acting transitively on M. Here we identify $\mathbb{SU}(1, 3)$ with the group of 4×4 complex matrices A, with determinant 1, which satisfy $A^*KA = K$ for

$$K = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

   and the action on $\mathbb{P}^3$ is the quotient of the standard action on $\mathbb{C}^4$. With this identification, the line $\ell = \{z_2 = 0, z_3 = 0\}$ is a point of M. We can choose complex coordinates w_1, w_2, w_3, w_4 near ℓ in the Grassmannian of the projective lines of $\mathbb{P}^3$, (w_1, w_2, w_3, w_4) representing the projective line corresponding to the plane V of $\mathbb{C}^4$ generated by the vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ w_1 \\ w_2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ w_3 \\ w_4 \end{pmatrix}.$$

Then $V \in M$ if and only if $V \cap (KV)^{\perp} \neq \{0\}$, i.e.

$$\det(\langle v_i, Kv_j \rangle) = 0.$$

Hence the local equation for M in these coordinates is given by

$$\det \begin{pmatrix} w_2 + \overline{w}_2 + w_1\overline{w}_1 & \overline{w}_1 w_3 + w_4 \\ \overline{w}_4 + w_1\overline{w}_3 & 1 + w_3\overline{w}_3 \end{pmatrix} = 0,$$

i.e.

$$w_2 + \overline{w}_2 + w_1\overline{w}_1 - w_4\overline{w}_4 - \overline{w}_1 w_3 \overline{w}_4 - w_1 \overline{w}_3 w_4 + (w_2 + \overline{w}_2) w_3 \overline{w}_3 = 0.$$

By the homogeneity, it suffices to compute the Levi form at $w_1 = w_2 = w_3 = w_4 = 0$, where w_1, w_3, w_4 can be taken as tangential holomorphic coordinates: it is proportional to the hermitian matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and hence M has signature $(1, 1, 1)$ at every point.

Appendix A

Some results of real analysis

A.1 A regularized distance function

In this section, we recall some results from [Ste70, Chapter VI].

In what follows, F will denote an arbitrary non-empty closed set in $\mathbb{R}^n$, Ω its complement. By a *cube* we mean a closed cube in $\mathbb{R}^n$, with sides parallel to the axes, and two such cubes will be said to be *disjoint* if their interiors are disjoint. For such a cube Q, $\operatorname{diam}(Q)$ denotes its diameter, and $\operatorname{dist}(Q, F)$ its distance from F. Let now Q be any cube with center x. For any ε, $0 < \varepsilon < \frac{1}{4}$, which is arbitrary but will be kept fixed in what follows, denote by Q^* the cube which has the same center as Q but is expanded by the factor $1 + \varepsilon$; that is, $Q^* = (1 + \varepsilon)[Q - x] + x$.

Theorem A.1.1
Let F be given. Then there exists a collection of cubes $\mathcal{F}$, $\mathcal{F} = \{Q_1, Q_2, \dots\}$ such that

(i) $\bigcup_k Q_k = \Omega = ({}^cF)$,

(ii) The Q_k are mutually disjoint,

(iii) $\operatorname{diam}(Q_k) \leq \operatorname{dist}(Q_k, F) \leq 4 \operatorname{diam}(Q_k)$,

(iv) Each point of Ω is contained in a small neighborhood intersecting at most $N = (12)^n$ of the cubes Q_k^*.

Proof. Consider the lattice of points in $\mathbb{R}^n$ whose coordinates are integral. This lattice determines a mesh $\mathcal{M}_0$, which is a collection of cubes; namely all cubes of unit length, whose vertices are points of the above lattice. The mesh $\mathcal{M}_0$ leads to a two-way infinite chain of such meshes $\{\mathcal{M}_k\}_{-\infty}^{+\infty}$, with $\mathcal{M}_k = 2^{-k}\mathcal{M}^0$. Thus each cube in the mesh $\mathcal{M}_k$ gives rise to 2^n cubes in the mesh $\mathcal{M}_{k+1}$ by bisecting the sides. The cubes in the mesh $\mathcal{M}_k$ each have sides of length 2^{-k} and are thus of diameter $\sqrt{n}2^{-k}$.

We also consider the layers Ω_k defined by

$$\Omega_k = \{x \mid 2\sqrt{n}\,2^{-k} < \operatorname{dist}(x, F) < 2\sqrt{n}\,2^{-k+1}\}.$$

Obviously $\Omega = \bigcup_{k=-\infty}^{+\infty} \Omega_k$.

We now make an initial choice of cubes and consider the resulting collection $\mathcal{F}_0$:

$$\mathcal{F}_0 = \bigcup_k \{Q_k \in \mathcal{M}_k \mid Q \cap \Omega_k \neq \emptyset\}.$$

We then have $\bigcup_{Q\in\mathcal{F}_0} Q = \Omega$ and $\operatorname{diam}(Q) \leq \operatorname{dist}(Q, F) \leq 4\operatorname{diam}(Q)$ for all $Q \in \mathcal{F}_0$. Hence the collection $\mathcal{F}_0$ has the required properties (i) and (iii). However, the cubes in it are not necessarily disjoint. We now refine our choice leading to $\mathcal{F}_0$, eliminating those cubes which are really unnecessary.

Start with any cube $Q \in \mathcal{F}_0$ and consider the *maximal* cube in $\mathcal{F}_0$ which contains it. Observe that any cube $Q' \in \mathcal{F}_0$ which contains $Q \in \mathcal{F}_0$ satisfies $\operatorname{diam}(Q') \leq 4\operatorname{diam}(Q)$. Moreover, any two cubes Q' and Q'' which contain Q have obviously a non-trivial intersection. Thus each cube $Q \in \mathcal{F}_0$ has a *unique* maximal cube in $\mathcal{F}_0$ which contains it. By the same token these maximal cubes are also disjoint. We let $\mathcal{F}$ denote the collection of maximal cubes of $\mathcal{F}$. Then $\mathcal{F}$ satisfies (i), (ii) and (iii).

It remains to show that $\mathcal{F}$ also satisfies (iv).

Let us say that two distinct cubes of $\mathcal{F}$, Q_1 and Q_2, *touch* if their boundaries have a common point. Suppose Q_1 and Q_2 touch. Then $\operatorname{diam}(Q_1) \leq \operatorname{dist}(Q_1, F) \leq \operatorname{dist}(Q_2, F) + \operatorname{diam}(Q_2) \leq 5\operatorname{diam}(Q_2)$. However $\operatorname{diam}(Q_2) = 2^k\operatorname{diam}(Q_1)$ for some $k \in \mathbb{Z}$, thus $\operatorname{diam}(Q_1) \leq 4\operatorname{diam}(Q_2)$. Together with the symmetrical implication, this proves

$$\frac{1}{4}\operatorname{diam}(Q_2) \leq \operatorname{diam}(Q_1) \leq 4\operatorname{diam}(Q_2),$$

provided $Q_1, Q_2 \in \mathcal{F}$ touch.

Now let $Q \in \mathcal{F}$. We claim that there are at most $N = (12)^n$ cubes in $\mathcal{F}$ which touch Q. Indeed, if the cube Q belongs to the mesh $\mathcal{M}_k$, then there are 3^n cubes which belong to the mesh $\mathcal{M}_k$ and touch Q. Next, each cube in the mesh $\mathcal{M}_k$ can contain at most 4^n cubes of $\mathcal{F}$ of diameter $\geq \frac{1}{4}\mathrm{diam}(Q)$. Since we have already seen that if a cube of $\mathcal{F}$ touches Q it must be of diameter $\geq \frac{1}{4}\mathrm{diam}(Q)$, this shows that there are at most $(12)^n$ cubes in $\mathcal{F}$ which touch Q.

Now let $x \in \Omega$. We choose a cube $Q \in \mathcal{F}$ such that $x \in Q$. Consider the union of Q_k with all the cubes in $\mathcal{F}$ which touch Q_k. Since the diameters of these cubes are all $\geq \frac{1}{4}\mathrm{diam}(Q_k)$, it is clear that this union contains Q_k^* (we have choosen $\varepsilon < \frac{1}{4}$). Therefore Q intersects Q_k^* only if Q touches Q_k. As we have already seen, there are at most N cubes of $\mathcal{F}$ which touch Q. Thus there are at most N cubes Q_k^* which intersect Q. This proves (iv).$\square$

Again, let F be an arbitrary closed set in $\mathbb{R}^n$, and let $\delta(x)$ denote the distance of x from F. While this function is smooth on F (it vanishes there), it is in general not more differentiable on $\Omega = {}^cF$ than the obvious Lipschitz-condition-inequality $|\delta(x) - \delta(y)| \leq |x - y|$ would indicate. For applications, it is desirable to replace $\delta(x)$ by a regularized distance which is smooth for $x \in \Omega$. In addition, this regularized distance is to have essentially the same profile as $\delta(x)$. Its existence is guaranteed by the following theorem.

Theorem A.1.2
There exists a function $\Delta \in \mathcal{C}^\infty(\Omega)$ such that

(a) $c_1\delta(x) \leq \Delta(x) \leq c_2\delta(x)$,

(b) $|D^\alpha \Delta(x)| \leq B_\alpha (\delta(x))^{1-|\alpha|}$ *for every multiindex* α.

B_α, c_1 *and* c_2 *are independent of* F.

Proof. We keep the notations of Theorem A.1.1. Let Q_0 denote the cube of unit length centered at the origin. Fix a smooth function φ satisfying $0 \leq \varphi \leq 1$, $\varphi(x) = 1$ if $x \in Q_0$ and $\varphi(x) = 0$ if $x \notin Q_0^* = (1+\varepsilon)Q_0$. Let φ_k denote the function φ adjusted to the cube Q_k, that is

$$\varphi_k(x) = \varphi(\frac{x - x^k}{\ell_k}),$$

where x^k is the center of Q_k and ℓ_k is the common length of its sides. Notice that therefore $\varphi_k(x) = 1$ if $x \in Q_k$ and $\varphi_k(x) = 0$ if $x \notin Q_k^*$. We also observe that

$$|D^\alpha \varphi_k(x)| \leq A_\alpha (\mathrm{diam}(Q_k))^{-|\alpha|}. \tag{A.1}$$

We set $\Delta(x) = \sum_k \mathrm{diam}(Q_k)\varphi_k(x)$.

Observe that if $x \in Q_k$, then $\delta(x) = \mathrm{dist}(x, F) \leq \mathrm{dist}(Q_k, F) + \mathrm{diam}(Q_k) \leq 5\,\mathrm{diam}(Q_k)$ by (iii) of Theorem A.1.1. However, if $x \in Q_k$, then $\varphi_k(x) = 1$, so $\Delta(x) \geq \mathrm{diam}(Q_k) \geq \frac{1}{5}\delta(x)$.

Also, if $x \in Q_k^*$, then $\delta(x) \geq \mathrm{dist}(Q_k, F) - \frac{1}{4}\mathrm{diam}(Q_k) \geq \frac{3}{4}\mathrm{diam}(Q_k)$ by (iii) of Theorem A.1.1. On the other hand, any given x lies in at most N of the Q_k^* by (iv) of Theorem A.1.1, thus $\Delta(x) \leq \sum_{x \in Q_k^*} \mathrm{diam}(Q_k) \leq \frac{4}{3}N\delta(x)$. We have therefore proved the conclusion (a) with $c_1 = \frac{1}{5}$ and $c_2 = \frac{4}{3}N$.

To prove conclusion (b), we argue similarly but invoke inequality (A.1) and the observation that if $x \in Q_k^*$, then $\delta(x) \leq \mathrm{dist}(Q_k, F) + \mathrm{diam}(Q_k) + \frac{1}{4}\mathrm{diam}(Q_k) \leq 6\,\mathrm{diam}(Q_k)$. This gives the desired result with $B_\alpha = A_\alpha N 6^{|\alpha|-1}$. $\square$

A.2 Imbeddings of Sobolev spaces on Lipschitz domains

Definition A.2.1
Let $\Omega \subset \mathbb{R}^n$ be an open set. We say that its boundary Γ is Lipschitz if for every $x \in \Gamma$ there exists a neighborhood V of x in $\mathbb{R}^n$ and local coordinates $(y_1, \ldots, y_n)$ such that

(a) V is a cube in the new coordinates:
$V = \{(y_1, \ldots, y_n) \mid -1 < y_j < 1,\ 1 \leq j \leq n\}$

(b) there exists a Lipschitz function φ, defined in
$V' = \{(y_1, \ldots, y_{n-1}) \mid -1 < y_j < 1,\ 1 \leq j \leq n-1\}$
such that
$\Omega \cap V = \{y = (y', y_n) \in V \mid y_n < \varphi(y')\}$,
$\Gamma \cap V = \{y = (y', y_n) \in V \mid y_n = \varphi(y')\}$.

In other words, in a neighborhood of x, Ω is below the graph of a Lipschitz function φ, and Γ is the graph of φ.

The following theorem is taken from [Gri85].

Theorem A.2.2
Let Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary Γ and let $k \in \mathbb{N}$. Then for all $u \in \overset{\circ}{H}_k(\Omega)$ we have $\delta^{-k}u \in L^2(\Omega)$, where $\delta(x)$ is the distance from a point x to Γ. Moreover, we have an estimate $\|\delta^{-k}u\|_{0,\Omega} \lesssim \|u\|_{k,\Omega}$.

Proof. Let us first consider the case when $\Omega = \mathbb{R}^+$ is the nonnegative real axis. Then, for $u \in \mathcal{D}(\mathbb{R}^+)$ we have

$$u(x) = \int_0^x \frac{(x-y)^{k-1}}{(k-1)!} u^{(k)}(y)dy$$

and consequently

$$\frac{|u(x)|}{x^k} \leq \frac{1}{(k-1)!} \frac{1}{x} \int_0^x |u^{(k)}|dy.$$

Hardy's inequality implies that

$$\|x^{-k}u\|_0 \leq \frac{2}{(k-1)!} \|u^{(k)}\|_0.$$

By density of $\mathcal{D}(\mathbb{R}^+)$, this implies the desired result for $\overset{\circ}{H}_k(\mathbb{R}^+)$.

We conclude by extending this result to a general Ω. Let us use the same notation as in Definition A.2.1 and consider a function u whose support is contained in V. One can always reduce the general case to this particular case, using a partition of unity. Now for $y' \in V'$ let us set $u_{y'}(t) = u(y', \varphi(y') - t)$. For almost all $y' \in V'$, we have $u_{y'} \in \overset{\circ}{H}_k(\mathbb{R}^+)$ and consequently $t^{-k}u_{y'} \in L^2(\mathbb{R}^+)$ with $\|t^{-k}u_{y'}\|^2_{0,\mathbb{R}^+} \leq K\|u_{y'}\|^2_{k,\mathbb{R}^+}$, where K does not depend on y'.

Integrating this inequality in y' leads to

$$\|(\varphi(y') - y_n)^{-k}u\|^2_{0,\Omega} \leq K\|u\|^2_{k,\Omega}.$$

Since φ is a Lipschitz function, the weight $\varphi(y') - y_n$ is equivalent to $\delta(y)$, the distance from y to Γ, throughout V. □

A.3 A cut-off function

The following lemma is taken from [Duf79].

Lemma A.3.1
Let F_1, F_2 be two closed subsets of $\mathbb{R}^n$ with $d(F_1, F_2) \geq \varepsilon$. Then there exists $\chi \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that $\chi \equiv 1$ in a neighborhood of F_1, $\chi \equiv 0$ in a neighborhood of F_2 and for every multiindex α, χ satisfies

$$\sup_{x \in \mathbb{R}^n} |D^\alpha \chi(x)| \leq \frac{N_{|\alpha|}}{\varepsilon^{|\alpha|}}$$

(where $N_{|\alpha|}$ does not depend on F_1, F_2).

Proof: Let $\psi \in \mathcal{C}^\infty(\mathbb{R}^n)$ have support in the unit ball of $\mathbb{R}^n$ and satisfy $\int_{\mathbb{R}^n} \psi d\lambda = 1$. We set $\psi_\varepsilon(x) = (\frac{3}{\varepsilon})^n \psi(\frac{3x}{\varepsilon})$.

Let φ be the function defined on $\mathbb{R}^n$ by $\varphi(x) = 1$ if $d(F_1, F_2) < \frac{\varepsilon}{2}$ and $\varphi(x) = 0$ otherwise.

We set $\chi = \varphi * \psi_\varepsilon$. Then it is immediate that $\chi \equiv 1$ in a neighborhood of F_1 and $\chi \equiv 0$ in a neighborhood of F_2. Moreover, we have

$$|D^\alpha \chi(x)| \leq \int_{\mathbb{R}^n} |D^\alpha \psi_\varepsilon(x)| dy \leq (\frac{3}{\varepsilon})^{|\alpha|} \sup_{y \in \mathbb{R}^n} |D^\alpha \psi(y)|$$

and it suffices to take

$$N_{|\alpha|} = 3^{|\alpha|} \max_{|\beta|=|\alpha|} (\sup_{x \in \mathbb{R}^n} |D^\beta \psi(x)|).$$

$\square$

A.4 A partition of unity

Lemma A.4.1
Let Ω be an open set of $\mathbb{R}^n$ and denote by $\delta(x)$ the distance of $x \in \Omega$ to the complement of Ω. Let ε be an arbitrary small positive number $\leq \frac{1}{2}$ and $\ell \in \mathbb{N}$. Then there exists a locally finite open covering of Ω by balls $B(x^i, r_i)$,

with center x^i and radius $r_i = \varepsilon\delta(x^i)^\ell$, and a partition of unity $(\theta_i)_{i\in\mathbb{N}}$ with respect to this covering satisfying

$$\sum_{|\alpha|\leq s} |D^\alpha \theta_i|^2 \leq \theta_i P_s(r_i^{-2}),$$

where P_s is a polynomial of degree s in one variable.

Proof. We may choose a locally finite open covering of Ω by balls $B(x^i, r_i)$, $r_i = \varepsilon\delta(x^i)^\ell$, such that also the balls $B(x^i, \frac{1}{2}r_i)$ cover Ω.

Let $g : \mathbb{R} \longrightarrow [0,1]$ be a smooth function satisfying

$$g(t) = \begin{cases} 1, & |t| < \frac{1}{2} \\ \exp(-\frac{1}{1-t^2}), & \frac{3}{4} < |t| < 1 \\ 0, & |t| \geq 1 \end{cases}$$

We set $\varphi_i = g(\frac{|x-x^i|}{r_i})$. We obviously have $\varphi_i(x) = 1$ if $x \in B(x^i, \frac{1}{2}r_i)$ and $\operatorname{supp}\varphi_i \subset B(x^i, r_i)$. Moreover, a straightforward computation yields

$$\sum_{|\alpha|\leq s} |D^\alpha \varphi_i|^2 \leq \varphi_i |P_s(r_i^{-2})|,$$

where P_s is a polynomial of degree s in one variable.

We set

$$\theta_i = \frac{\varphi_i}{\sum_k \varphi_k}.$$

The family $(\theta_i)_{i\in\mathbb{N}}$ then defines a partition of unity with respect to the covering $(B(x^i, r_i)_{i\in\mathbb{N}}$.

Moreover, since $(B(x^i, \frac{1}{2}r_i))_i$ covers Ω, we clearly have $\sum_k \varphi_k \geq 1$ in Ω. Without loss of generality, we may also assume that $\varepsilon\delta(x^i)^{\ell-1} \leq \frac{1}{2}$ for all $i \in \mathbb{N}$. Then, if $B(x^k, r_k) \cap B(x^i, r_i) \neq 0$, we must have $\delta(x^k) \geq \frac{1}{4}\delta(x^i)$, i.e. $r_k^{-1} \leq 4^\ell r_i^{-1}$. But this implies

$$\sum_{|\alpha|\leq s} |D^\alpha \theta_i|^2 \leq \theta_i |P_s(r_i^{-2})|$$

for some polynomial P_s of degree s. □

Bibliography

[AFN81] A. Andreotti, G. Fredricks, and M. Nacinovich. On the absence of Poincaré lemma in tangential Cauchy-Riemann complexes. *Ann. Sc. Norm. Sup. Pisa*, 8:365–404, 1981.

[AH72] A. Andreotti and C. D. Hill. E.E. Levi convexity and the Hans Lewy problem I,II. *Ann. Sci. Norm. Sup. Pisa*, 26:325–363, 747–806, 1972.

[BHN01] J. Brinkschulte, C.D. Hill, and M. Nacinovich. Obstructions to generic embeddings. *Prépublication de l'Institut Fourier*, 538, 2001.

[Dem] J.-P. Demailly. Complex analytic and algebraic geometry. available at http://www-fourier.ujf-grenoble.fr/∼demailly/books.html.

[Dem82] J.-P. Demailly. Estimations L^2 pour l'opérateur $\overline{\partial}$ d'un fibré holomorphe semi-positif au-dessus d'une variété Kählérienne complete. *Ann. Sci. Ec. Norm. Super.*, 15:457–511, 1982.

[Dem86] J.-P. Demailly. Sur l'identité de Bochner-Kodaira-Nakano en géométrie hermitienne. In *Lecture Notes in Math.*, volume 1198, pages 88–97. Springer-Verlag, Berlin, 1986.

[Der81] M. Derridj. Inégalités de Carleman et extension locale des fonctions holomorphes. *Ann. Sc. Norm. Sup. Pisa*, 4:645–669, 1981.

[Duf79] A. Dufresnoy. Sur l'opérateur d'' et les fonctions différentiables au sens de Whitney. *Ann. Inst. Fourier*, 29:229–238, 1979.

[Ele75] G. Elencwajg. Pseudoconvexité locale dans les variétés Kählériennes. *Ann. Inst. Fourier*, 25:295–314, 1975.

[Fol76] G.B. Folland. *Introduction to partial differential operators.* Princeton University Press and University of Tokyo Press, Princeton, New Jersey, 1976.

[Fre76] M. Freeman. The Levi form and local complex foliations. *Proc. Am. Math. Soc.*, 57:369–370, 1976.

[Gri85] P. Grisvard. *Elliptic problems in non-smooth domains.* Number 24 in Monographs and Studies in Mathematics. Pitman (Advanced Publishing Program), Boston, Mass. -London, 1985.

[Hen84] G.M. Henkin. The Hartogs-Bochner effect on CR manifolds. *Soviet. Math. Dokl.*, 29:78–82, 1984.

[HI00] G.M. Henkin and A. Iordan. Regularity of $\overline{\partial}$ on pseudoconcave compacts and applications. *Asian J. Math.*, 4:885–884, 2000.

[HL88] G.M. Henkin and J. Leiterer. *Andreotti-Grauert theory by integral formulas.* Number 74 in Progress in Math. Akademie-Verlag Berlin and Birkhäuser-Verlag Boston, 1988.

[HN92] C.D. Hill and M. Nacinovich. A necessary condition for global Stein immersion of compact CR manifolds. *Riv. Mat. Univ. Parma*, 5:175–182, 1992.

[HN95] C.D. Hill and M. Nacinovich. Duality and distribution cohomology of CR manifolds. *Ann. Sc. Norm. Sup. Pisa*, 22:315–339, 1995.

[HN96] C.D. Hill and M. Nacinovich. Pseudoconcave CR manifolds. In V. Ancona et al., editors, *Complex analysis and geometry*, pages 275–297. Marcel Dekker, New York, 1996.

[HN00] C.D. Hill and M. Nacinovich. A weak pseudoconcavity condition for abstract almost CR manifolds. *Invent. math.*, 142:251–283, 2000.

[HN01] C.D. Hill and M. Nacinovich. On the failure of the Poincaré lemma for the $\overline{\partial}_M$-complex. *Quaderni sez. Geometria Dip. Matematica Pisa*, 2001.

[Koh73] J.J. Kohn. Global regularity for $\overline{\partial}$ on weakly pseudo-convex manifolds. *Trans. Amer. Math. Soc.*, 181:272–292, 1973.

[Koh77] J.J. Kohn. Methods of partial differential equations in complex analysis. In *Proc. Symp. Pure Math. XXX*, volume 2, pages 215–237, 1977.

[LT78] S. Lojaciewicz and G. Tomassini. Valeurs au bord des formes holomorphes. In *Several complex variables. Proc. of int. conf.*, pages 222–246, Cortona/Italy, 1978.

[LT91] C. Laurent-Thiébaut. Résolution du $\overline{\partial}_b$ à support compact et phénomène de Hartogs-Bochner dans les variétés CR. *Proc. Sympos. Pure Math.*, 52:239–249, 1991.

[LTL99] C. Laurent-Thiébaut and J. Leiterer. Some applications of Serre duality in CR maniolds. *Nagoya Math. J.*, 154:141–156, 1999.

[Mar66] A. Martineau. Distributions et valeurs au bord des fonctions holomorphes. *RCP Strasbourg*, 25, 1966.

[Mat96] K. Matsumoto. Boundary distance functions and q-convexity of pseudoconvex domains of general order in Kähler manifolds. *J. Math. Soc. Japan*, 48:85–107, 1996.

[Mic93] V. Michel. Sur la regularité $\mathcal{C}^\infty$ du $\overline{\partial}$ au bord d'un domaine de ^{n} dont la forme de Levi a exactement s valeurs propres strictement négatives. *Math. Ann.*, 295:135–161, 1993.

[MS99] J. Michel and M.-C. Shaw. $\overline{\partial}$ and $\overline{\partial}_b$ problems on nonsmooth domains. In G. Komatsu et al., editors, *Analysis and geometry in several complex variables*, Trends in Math., pages 239–252, Boston, MA, 1999. Birkhäuser.

[NV87] M. Nacinovich and G. Valli. Tangential Cauchy-Riemann complexes on distributions. *Ann. Mat. Pura Appl.*, 146:123–160, 1987.

[Ohs87] T. Ohsawa. Isomorphism theorems for cohomology groups on weakly 1-complete manifolds. *Publ. Res. Inst. Math. Sci.*, 18:192–232, 1987.

[Ohs99] T. Ohsawa. Pseudoconvex domains in $\mathbb{P}^n$: A question on the 1-convex boundary points. In G. Komatsu et al., editors, *Analysis and geometry in several complex variables*, Trends in Math., pages 239–252. Birkhäuser, 1999.

[Ros82] J.-P. Rosay. Equation de Lewy-Résolubilité globale de l'équation $\overline{\partial}_b u = f$ sur la frontière de domaines faiblement pseudo-convexes de ^{2} (ou ^{n}). *Duke Math. J.*, 49:121–128, 1982.

[Sam99] S. Sambou. Résolution du $\overline{\partial}$ pour les courants prolongeables. *Prépublication de l'Institut Fourier*, 486, 1999. to appear in Math. Nach.

[Sam01] S. Sambou. Résolution du $\overline{\partial}$ pour les courants prolongeables définis dans un anneau. *Prépublication de l'Institut Fourier*, 537, 2001.

[Ser55] J.-P. Serre. Un théorème de dualité. *Comm. Math. Helvetici*, 29:9–26, 1955.

[Ste70] E. Stein. *Singular integrals and differentiability properties of functions*. Princeton University Press, 1970.

[Suz76] O. Suzuki. Pseudoconvex domains on a Kähler manifold with positive holomorphic bisectional curvature. *Publ. RIMS*, 12:191–214, 1976.

[SY80] Y.-T. Siu and S.-T. Yau. Compact Kähler manifolds of positive bisectional curvature. *Invent. Math.*, 59:189–204, 1980.

[Tak64] T. Takeuchi. Domaines pseudoconvexes infinis et la métrique riemannienne dans un espace projectif. *J. Math. Soc. Japan*, 16:159–181, 1964.

Lebenslauf

Name:	Judith Brinkschulte
22.10.1975	geboren in Bonn
1982-1986	Besuch der Grundschule in Kamen
1986-1992	Besuch des Mallinckrodtgymnasiums in Dortmund
1992-1993	Besuch der Marjory Stonemann Douglas High School in Parkland, Florida (USA)
1993-1995	Besuch des Mallinckrodtgymnasiums in Dortmund
1995	Abitur in Dortmund
WS1995-SS1997	Studium der Mathematik und Physik an der Universität Leipzig
1997	Vordiplom in Mathematik und Physik an der Universität Leipzig
WS1997-SS1999	Studium der Mathematik an der Université Joseph Fourier in Grenoble, Frankreich
WS1997-SS1998	Auslandsstipendium des DAAD
seit SS1998	Stipendiatin der Studienstiftung des deutschen Volkes
1999	Diplôme d'études approfondies de mathématiques (DEA) an der Université Joseph Fourier in Grenoble, Frankreich (Note: sehr gut)
seit WS1999	Promotionsstudium im Fach Mathematik an der Humboldt-Universität zu Berlin und an der Université Joseph Fourier in Grenoble; Doktorandenstipendium der französischen Regierung; wissenschaftliche Hilfskraft an der Université Joseph Fourier in Grenoble

Selbständigkeitserklärung

Hiermit erkläre ich, die vorliegende Arbeit selbständig ohne fremde Hilfe verfaßt zu haben und nur die angegebene Literatur und Hilfsmittel verwendet zu haben.

Judith Brinkschulte

26. November 2001