Convex hull approximation of TU integer recourse models: Counterexamples, sufficient conditions, and special cases

Ward Romeijnders · Maarten H. van der Vlerk

November 20, 2012

Abstract We consider a convex approximation for integer recourse models. In particular, we show that the claim of Van der Vlerk (2004) that this approximation yields the convex hull of totally unimodular (TU) integer recourse models is incorrect. We discuss counterexamples, indicate which step of its proof does not hold in general, and identify a class of random variables for which the claim in Van der Vlerk (2004) is not true. At the same time, we derive additional assumptions under which the claim does hold. In particular, if the random variables in the model are independently and uniformly distributed, then these assumptions are satisfied.

Keywords Stochastic Programming · Integer recourse · Convex approximations

1 Introduction

Integer recourse models from the field of stochastic programming deal with discrete decision making under uncertainty (see, e.g., Birge and Louveaux (1997), Shapiro et al. (2009) and the Stochastic Programming Community Home Page). These models have a wide range of possible applications because they combine the modeling power of integer variables with possible uncertainty in the data, making them highly relevant for practice, but at the same time very difficult to solve. Examples of applications range from energy optimization problems (see, e.g., Gollmer et al. (2000) and Zhang et al. (2010)) to scheduling problems (see, e.g., Alonso-Ayuso et al. (2007)). More examples can be found e.g. in the extensive bibliography on stochastic programming Van der Vlerk (1996-2007).

We consider the two-stage integer recourse problem

$$\min_x \{cx + Q(z) : Ax \geq b,\, Tx = z,\, x \in \mathbb{R}^{n_1}_+\}, \tag{1}$$

where Q is a function of the tender variables z,

$$Q(z) := \mathbb{E}_\omega[v(\omega - z)], \quad z \in \mathbb{R}^m, \tag{2}$$

and

$$v(s) := \min_y \{qy : Wy \geq s,\, y \in \mathbb{Z}^{n_2}_+\}, \quad s \in \mathbb{R}^m.$$

The functions Q and v are called the recourse or expected value function and the second-stage value function, respectively. They model the recourse actions y and the corresponding expected recourse costs for satisfying the underlying random goal constraints $Tx \geq \omega$. The right-hand side vector ω is a random vector with known cumulative distribution function (cdf) F_ω.

Throughout this paper we use the following assumptions.

Ward Romeijnders · Maarten. H van der Vlerk
Department of Operations, University of Groningen, PO Box 800, 9700 AV, Groningen, The Netherlands
E-mail: w.romeijnders@rug.nl

(i) W is a complete recourse matrix, i.e., for every $s \in \mathbb{R}^m$ there exists $y \in \mathbb{Z}_+^{n_2}$ such that $Wy \geq s$,
(ii) the recourse structure is sufficiently expensive, i.e., $v(s) > -\infty$ for all $s \in \mathbb{R}^m$, and
(iii) $\mathbb{E}_\omega[|\omega|]$ is finite.

As a result we have that $Q(z)$ is finite for all $z \in \mathbb{R}^m$.

Various algorithms for solving pure and mixed-integer recourse problems are available, see, e.g., Laporte and Louveaux (1993), Schultz et al. (1998), Carøe and Schultz (1999), Takriti and Birge (2000), Ahmed et al. (2004), Sherali and Zhu (2006), Escudero et al. (2009), and Guan et al. (2009), and the survey papers Klein Haneveld and Van der Vlerk (1999), Louveaux and Schultz (2003), and Sen (2005). Typically, these algorithms combine solution techniques developed for either stochastic continuous or deterministic integer programs. In general, these exact methods have difficulties dealing with large problem instances, though substantial progress has been made for special cases (see, e.g., Sen and Higle (2005) and subsequent papers).

The main difficulty in solving integer recourse problems is that the integer recourse function Q is generally non-convex (Rinnooy Kan and Stougie (1988)). A possible approach to deal with this difficulty is to approximate Q by a *convex* function $\hat{Q}$. In this way, we do not obtain the exact solution of the integer recourse problem, but as long as $\hat{Q}$ is a close approximation of Q, we expect to find near-optimal first-stage solutions. The advantage is that efficient algorithms exist for solving convex optimization problems, so that the approximation model can be solved much easier than the original integer recourse model.

Van der Vlerk (2004) obtains such a convex approximation by perturbing the distribution of the right-hand side random vector ω. Indeed, Van der Vlerk (2004) claims that this approximation yields the convex hull of Q if the recourse matrix W is totally unimodular (TU), which would justify to expect to find near-optimal first-stage solutions using this approximation. However, we will show that this claim does not hold in general. We discuss counterexamples of this claim, indicate which step of its proof does not hold in general, and identify a class of random variables for which the claim in Van der Vlerk (2004) is not true. At the same time, we derive additional assumptions under which the claim does hold. In particular, if the random variables in the model are independently and uniformly distributed, then these assumptions are satisfied.

Preliminary results on this topic and more (extensive) examples can be found in the Master's thesis Romeijnders (2011).

2 The convex approximation of Van der Vlerk (2004)

The convex approximation of Van der Vlerk (2004) can be applied to general complete integer recourse models. However, the earliest version of this approximation was developed for so-called one-sided simple integer recourse (SIR) models (when $W = I_m$). In this simple case the recourse function $Q(z)$ is separable in the components of z and can be written as

$$Q(z) = \mathbb{E}_\omega \Big[\min_y \Big\{ qy : y \geq \omega - z,\, y \in \mathbb{Z}_+^m \Big\} \Big] = \sum_{i=1}^m q_i Q_i(z_i), \quad z \in \mathbb{R}^m,$$

with $Q_i(z_i) := \mathbb{E}_{\omega_i}[\lceil \omega_i - z_i \rceil^+]$, $z_i \in \mathbb{R}$, $i = 1, \dots, m$, and $\lceil s \rceil^+ := \max\{0, \lceil s \rceil\}$, $s \in \mathbb{R}$. The generic one-dimensional recourse function

$$\mathcal{Q}(z) := \mathbb{E}_\omega[\lceil \omega - z \rceil^+], \quad z \in \mathbb{R}, \tag{3}$$

has been studied extensively in the literature (see Louveaux and Van der Vlerk (1993)). Obviously, if the random variable ω is discretely distributed, then $\mathcal{Q}$ is non-convex because of the round-up function involved. Klein Haneveld et al. (1995, 1996) develop efficient algorithms to construct the convex hull of $\mathcal{Q}$ in this case. If ω is continuously distributed, then $\mathcal{Q}$ is generally non-convex as well, but exceptions do exist. Klein Haneveld et al. (2006) give a complete description of the class of probability density functions (pdf) for which $\mathcal{Q}$ is convex.

Theorem 1 (see Corollary 1 in Klein Haneveld et al. (2006)) *Consider the SIR function $\mathcal{Q}$ as defined in (3) and let $\omega \in \mathbb{R}$ be a continuously distributed random variable with pdf f_ω. Then $\mathcal{Q}$ is convex if and only if $f_\omega(x) = G(x+1) - G(x)$, $x \in \mathbb{R}$, for some cdf G with finite mean. We say that f_ω is generated by G.*

A natural approach to construct a convex approximation of $\mathcal{Q}$ is to approximate the original random variable ω by a random variable $\hat{\omega}$ which has a pdf $f_{\hat{\omega}}$ that is generated by some cdf G. Obviously, we want $f_{\hat{\omega}}$ to be a close approximation of f_ω. The so-called α-*approximation* developed by Klein Haneveld et al. (2006) is a good candidate. Here, $f_{\hat{\omega}}$ is generated by a cdf G corresponding to a discrete distribution with support in $\alpha + \mathbb{Z}$ for some $\alpha \in [0, 1)$. Following Van der Vlerk (2004) we define this approximation for m-dimensional distributions.

Definition 1 Let $\omega \in \mathbb{R}^m$ be a random vector with arbitrary continuous or discrete distribution, and choose $\alpha = (\alpha_1, \dots, \alpha_m) \in [0, 1)^m$. Define the α-*approximation* ω_α as the random vector with joint pdf f_{ω_α} that is constant on every hypercube

$$C_\alpha^l := \prod_{i=1}^{m} (\alpha_i + l_i - 1,\ \alpha_i + l_i], \quad l \in \mathbb{Z}^m,$$

such that

$$\mathbb{P}\left\{\omega_\alpha \in C_\alpha^l\right\} = \mathbb{P}\left\{\omega \in C_\alpha^l\right\}, \quad l \in \mathbb{Z}^m.$$

From this definition it follows that for every $\omega \in \mathbb{R}$ and $\alpha \in [0, 1)$, the cdf F_{ω_α} of ω_α is piecewise linear with knots contained in $\alpha + \mathbb{Z}$ and $F_{\omega_\alpha}(x) = F_\omega(x)$ for $x \in \alpha + \mathbb{Z}$.

The α-approximation $\mathcal{Q}_\alpha$ of $\mathcal{Q}$ is defined for every $\alpha \in [0, 1)$ as

$$\mathcal{Q}_\alpha(z) := \mathbb{E}_{\omega_\alpha}[\lceil \omega_\alpha - z \rceil^+], \quad z \in \mathbb{R}. \tag{4}$$

Interestingly, it can be shown (see Klein Haneveld et al. (2006)) that

$$\mathcal{Q}_\alpha(z) = \mathbb{E}_{\phi_\alpha}[(\phi_\alpha - z)^+], \quad z \in \mathbb{R},$$

with $\phi_\alpha := \lceil \omega - \alpha \rceil + \alpha$ a discrete random variable with support in $\alpha + \mathbb{Z}$. That is, the recourse function $\mathcal{Q}_\alpha$ of an *integer* recourse model with *continuous* random variable can be expressed as the recourse function of a *continuous* recourse model with a *discrete* random variable. Continuous simple recourse models can be solved very efficiently by special purpose algorithms (see e.g. Wets (1983)), and thus the approximation model can be solved much more easily than the original model. Similarly, for continuous recourse models in general there are efficient algorithms available, most of them based on the L-shaped algorithm of Van Slyke and Wets (1969). This implies that if we replace an integer recourse model by a continuous recourse approximation (with discrete right-hand side), then the approximation model is computationally much more tractable than the original integer recourse version.

Klein Haneveld et al. (2006) derive an error bound for the α-approximation $\mathcal{Q}_\alpha$ of the SIR function $\mathcal{Q}$. They show that if ω is continuously distributed with pdf f_ω of bounded variation, then for all $\alpha \in [0, 1)$,

$$\sup_{z \in \mathbb{R}} |\mathcal{Q}(z) - \mathcal{Q}_\alpha(z)| \le \min\left\{1, \frac{|\Delta| f_\omega}{4}\right\},$$

where $|\Delta| f_\omega$ denotes the total variation of f_ω. This error bound shows that $\mathcal{Q}_\alpha$ is a good approximation of $\mathcal{Q}$ when the total variation of f_ω is low. For example, for unimodal densities this is the case if the variance is large.

Van der Vlerk (2004) generalizes the concept of α-approximations to m-dimensional recourse functions Q as defined in (2). As in the simple integer case, we simultaneously relax the integrality constraints and replace the random vector ω by a discrete random vector $\phi_\alpha := \lceil \omega - \alpha \rceil + \alpha$ for some $\alpha \in [0, 1)^m$.

Definition 2 For every $\alpha \in [0,1)^m$ the *α-approximation* Q_α of the recourse function Q is given by

$$Q_\alpha(z) = \mathbb{E}_{\phi_\alpha}\big[\min_y \Big\{qy : Wy \geq \phi_\alpha - z,\ y \in \mathbb{R}^{n_2}_+\Big\}\big], \quad z \in \mathbb{R}^m,$$

where $\phi_\alpha := \lceil \omega - \alpha \rceil + \alpha$.

From now on we assume that ω follows a continuous distribution, as in Van der Vlerk (2004).

3 Rectification of a false claim on the convex hull

In general, the α-approximation Q_α is neither a lower nor an upper bound of Q. However, Van der Vlerk (2004) claims that there exists some $\alpha^* \in [0,1)^m$ such that the α^*-approximation Q_{α^*} does provide a lower bound. This $\alpha^* = (\alpha^*_1, \ldots, \alpha^*_m)$ is defined for each component α^*_i as

$$\alpha^*_i \in \operatorname*{argmin}_{x \in [0,1)} \mathbb{E}_{\omega_i}\Big[\lceil \omega_i - x \rceil + x\Big], \quad i = 1, \ldots, m. \tag{5}$$

In fact, for TU integer recourse models, Van der Vlerk (2004) claims that this α^*-approximation Q_{α^*} yields the convex hull of Q. We repeat this claim here because we discuss it in detail in the remainder of this paper.

Proposition 1 *Consider the integer recourse function Q, defined as*

$$Q(z) = \mathbb{E}_\omega\left[\min_y qy : Wy \geq \omega - z,\ y \in \mathbb{Z}^{n_2}_+\right], \quad z \in \mathbb{R}^m.$$

Under the assumptions (i)-(iii) of Section 1, and in addition that W is totally unimodular, the convex hull of Q is the continuous recourse function Q_{α^*}, defined as

$$Q_{\alpha^*}(z) = \mathbb{E}_{\phi_{\alpha^*}}\left[\min_y qy : Wy \geq \phi_{\alpha^*} - z,\ y \in \mathbb{R}^{n_2}_+\right], \quad z \in \mathbb{R}^m,$$

where α^* is defined in (5), and $\phi_{\alpha^*} := \lceil \omega - \alpha^* \rceil + \alpha^*$ is a discrete random vector with support in $\alpha^* + \mathbb{Z}^m$, and

$$\mathbb{P}\Big\{\phi_{\alpha^*} = \alpha^* + l\Big\} = \mathbb{P}\Big\{\omega \in C^l_{\alpha^*}\Big\}, \quad l \in \mathbb{Z}^m.$$

The proof of Proposition 1 in Van der Vlerk (2004) is based on the following line of reasoning. First observe that Q_{α^*} is a convex polyhedral function with vertices contained in $\alpha^* + \mathbb{Z}^m$. Moreover, $Q_{\alpha^*}(z) = Q(z)$ for all $z \in \alpha^* + \mathbb{Z}^m$. If, in addition, Q_{α^*} is a lower bound of Q, then the polyhedral function Q_{α^*} is equal to the convex hull of Q. Van der Vlerk (2004) argues that this is indeed the case. However, in the next section we give counterexamples where Q_{α^*} is not a lower bound of Q, and we show that the convex hull of Q is not necessarily a polyhedral function. In these examples we analyze the SIR function $\mathcal{Q}$ defined in (3), which is a special case of the TU integer recourse functions considered in Proposition 1.

3.1 Counterexamples for Proposition 1

Example 1 Consider the generic one-dimensional SIR function $\mathcal{Q}$ defined in (3), and let ω be a continuous random variable with pdf f_ω defined as

$$f_\omega(x) = \begin{cases} \frac{3}{2}, & 0 \leq x < \frac{1}{2}, \\ \frac{1}{2}, & \frac{1}{2} \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Using (5) we obtain by straightforward calculation that $\alpha^* = 1/2$. Moreover, we have $\mathbb{P}\{\phi_{\alpha^*} = 1/2\} = 3/4$ and $\mathbb{P}\{\phi_{\alpha^*} = 3/2\} = 1/4$, so that

$$\mathcal{Q}_{\alpha^*}(z) = \tfrac{1}{4}\left(\tfrac{3}{2} - z\right)^+ + \tfrac{3}{4}\left(\tfrac{1}{2} - z\right)^+, \quad z \in \mathbb{R}.$$

We observe that $\mathcal{Q}_{\alpha^*}(1) = 1/8 > \mathcal{Q}(1) = 0$ and conclude that $\mathcal{Q}_{\alpha^*}$ is not a lower bound for $\mathcal{Q}$, and thus $\mathcal{Q}_{\alpha^*}$ is not the convex hull of $\mathcal{Q}$, see Figure 1. That is, Proposition 1 as stated is false.

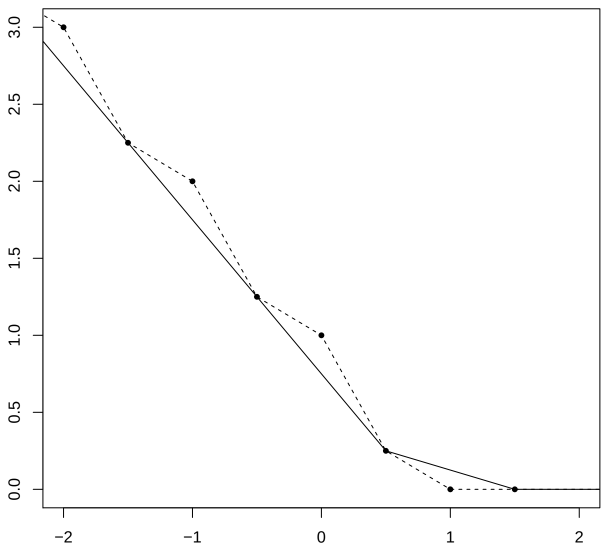

Fig. 1 The recourse function $\mathcal{Q}$ (dashed) and its α^*-approximation $\mathcal{Q}_{\alpha^*}$ (solid) from Example 1, showing that $\mathcal{Q}_{\alpha^*}$ is not a lower bound and hence not the convex hull of $\mathcal{Q}$.

Example 2 Again consider the SIR function $\mathcal{Q}$ defined in (3), and let ω be a random variable following a triangular distribution on $[0, 1]$ with mode $1/2$. Thus, the pdf of ω is given by

$$f_\omega(x) = \begin{cases} 4x, & 0 \le x \le \frac{1}{2}, \\ 4(1-x), & \frac{1}{2} \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

By straightforward calculation it follows that the convex hull of $\mathcal{Q}$, denoted $\mathcal{Q}^{**}$, is given by

$$\mathcal{Q}^{**}(z) = \begin{cases} \frac{7}{8} - z, & z \le \frac{3}{4}, \\ 2(1-z)^2, & \frac{3}{4} \le z \le 1, \\ 0, & 1 \le z. \end{cases}$$

Both functions $\mathcal{Q}$ and $\mathcal{Q}^{**}$ are depicted in Figure 2. We see that Q^{**} is convex quadratic on the interval (3/4,1), implying that it cannot be obtained as an α-approximation which is polyhedral for every $\alpha \in [0, 1)$.

3.2 Error in the proof of Proposition 1

The counterexamples in the previous subsection clearly show that Proposition 1 does not hold in general and needs additional assumptions. We will point out which step of the proof, repeated here

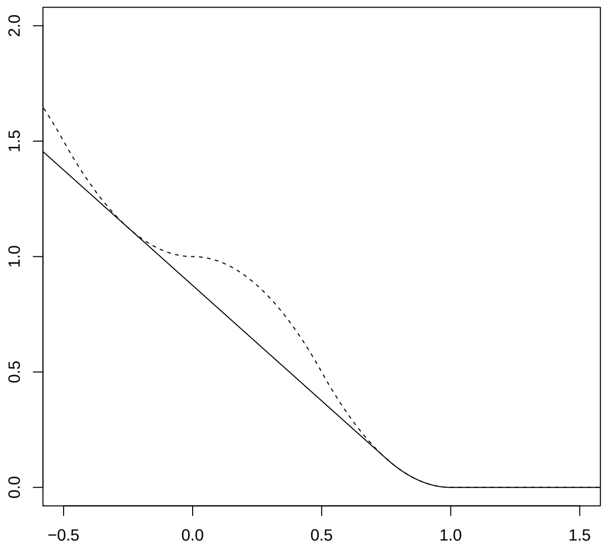

Fig. 2 The SIR function $\mathcal{Q}$ (dashed) and its convex hull $\mathcal{Q}^{**}$ (solid) from Example 2, showing that the convex hull of Q is not necessarily a polyhedral function.

for convenience, is invalid in general, so that we can derive these additional assumptions. First, as in Van der Vlerk (2004), we rewrite the second-stage value function under the assumptions (i) and (ii) and that the recourse matrix W is TU. We have for every $s \in \mathbb{R}^m$,

$$\begin{aligned} v(s) &:= \min_y \{qy : Wy \geq s,\ y \in \mathbb{Z}_+^{n_2}\} \\ &= \min_y \{qy : Wy \geq \lceil s \rceil,\ y \in \mathbb{R}_+^{n_2}\} \qquad (6) \\ &= \max_\lambda \{\lambda \lceil s \rceil : \lambda W \leq q,\ \lambda \in \mathbb{R}_+^m\}, \qquad (7) \end{aligned}$$

where (6) holds because W is TU, and (7) because of strong LP duality. Since the recourse is complete and sufficiently expensive it follows that the dual feasible region $\Lambda := \{\lambda \in \mathbb{R}_+^m : \lambda W \leq q\}$ is nonempty and bounded. Moreover, since Λ is a polytope, it has finitely many extreme points which we denote by λ^k, $k = 1, \dots, K$. It follows immediately that we can rewrite v as

$$v(s) = \max_{k=1,\dots,K} \lambda^k \lceil s \rceil, \quad s \in \mathbb{R}^m.$$

Consequently,

$$Q(z) = \mathbb{E}_\omega \left[\max_{k=1,\dots,K} \lambda^k \lceil \omega - z \rceil \right], \quad z \in \mathbb{R}^m,$$

and analogously

$$Q_{\alpha^*}(z) = \mathbb{E}_{\phi_{\alpha^*}} \left[\max_{k=1,\dots,K} \lambda^k (\phi_{\alpha^*} - z) \right], \quad z \in \mathbb{R}^m.$$

Van der Vlerk (2004) proceeds as follows. By conditioning on the events $\{\omega \in C^l_{\alpha^*}\}$ we have for all $z \in \mathbb{R}^m$,

$$Q(z) = \mathbb{E}_\omega \left[\max_{k=1,\dots,K} \lambda^k \lceil \omega - z \rceil \right] = \sum_{l \in \mathbb{Z}^m} \mathbb{P}\Big\{ \omega \in C^l_{\alpha^*} \Big\} \mathbb{E}_\omega \left[\max_{k=1,\dots,K} \lambda^k \lceil \omega - z \rceil \,\Big|\, \omega \in C^l_{\alpha^*} \right],$$

and by interchanging expectation and maximization,

$$Q(z) \geq \sum_{l \in \mathbb{Z}^m} \mathbb{P}\Big\{\omega \in C^l_{\alpha^*}\Big\} \max_{k=1,\dots,K} \lambda^k \, \mathbb{E}_\omega \left[\lceil \omega - z \rceil \,\Big|\, \omega \in C^l_{\alpha^*}\right].$$

Van der Vlerk (2004) argues that due to the choice of α^*,

$$\lambda^k \, \mathbb{E}_\omega \left[\lceil \omega - z \rceil \,\Big|\, \omega \in C^l_{\alpha^*}\right] \geq \lambda^k \, \mathbb{E}_{\omega_{\alpha^*}} \left[\lceil \omega_{\alpha^*} - z \rceil \,\Big|\, \omega_{\alpha^*} \in C^l_{\alpha^*}\right], \quad \text{for } (z,l) \in \mathbb{R}^m \times \mathbb{Z}^m, \tag{8}$$

for every $k = 1, \dots, K$. As a consequence,

$$\begin{aligned} Q(z) &\geq \sum_{l \in \mathbb{Z}^m} \mathbb{P}\Big\{\omega_{\alpha^*} \in C^l_{\alpha^*}\Big\} \max_{k=1,\dots,K} \lambda^k \, \mathbb{E}_{\omega_{\alpha^*}} \left[\lceil \omega_{\alpha^*} - z \rceil \,\Big|\, \omega_{\alpha^*} \in C^l_{\alpha^*}\right] &(9)\\ &= \sum_{l \in \mathbb{Z}^m} \mathbb{P}\Big\{\omega_{\alpha^*} \in C^l_{\alpha^*}\Big\} \max_{k=1,\dots,K} \lambda^k (\alpha^* + l - z) &(10)\\ &= \sum_{l \in \mathbb{Z}^m} \mathbb{P}\Big\{\phi_{\alpha^*} = \alpha^* + l\Big\} \max_{k=1,\dots,K} \lambda^k (\alpha^* + l - z)\\ &= \mathbb{E}_{\phi_{\alpha^*}} \left[\max_{k=1,\dots,K} \lambda^k (\phi_{\alpha^*} - z)\right]\\ &= Q_{\alpha^*}(z), \end{aligned}$$

where (10) holds because $f_{\omega_{\alpha^*}}$ is constant on $C^l_{\alpha^*}$ for every $l \in \mathbb{Z}^m$.

This proof does not hold in general because (8) is incorrect and thus (9) does not hold. It is true that $\lambda^k \, \mathbb{E}_\omega[\lceil \omega - z \rceil] \geq \lambda^k \, \mathbb{E}_{\omega_{\alpha^*}}[\lceil \omega_{\alpha^*} - z \rceil]$ for every $z \in \mathbb{R}^m$, due to the choice of α^*. However, restricted to individual subsets $C^l_{\alpha^*}$ as required in (8), the inequality does not hold as detailed below.

In the next section we derive a sufficient condition for the inequalities in (8) to be true, which can be used to identify classes of random vectors for which Proposition 1 does hold.

4 Additional assumptions for Proposition 1

Before we derive sufficient conditions for (8) we show that in the one-dimensional setting the inequalities in (8) are *equivalent* to ω_{α^*} being stochastically dominated by ω.

Lemma 1 *Let ω be a random variable with cdf F_ω. Then*

$$\mathbb{E}_\omega \Big[\lceil \omega - z \rceil \,\Big|\, \omega \in C^l_{\alpha^*}\Big] \geq \mathbb{E}_{\omega_{\alpha^*}} \Big[\lceil \omega_{\alpha^*} - z \rceil \,\Big|\, \omega_{\alpha^*} \in C^l_{\alpha^*}\Big] \quad \textit{for all } (z,l) \in \mathbb{R} \times \mathbb{Z},$$

if and only if ω_{α^*} is (weakly) first-order stochastically dominated *by ω. That is, if and only if*

$$F_\omega(x) \leq F_{\omega_{\alpha^*}}(x), \quad \textit{for all } x \in \mathbb{R}.$$

Proof Using that $\lceil x - (z+k) \rceil = \lceil x - z \rceil - k, k \in \mathbb{Z}$, it suffices to prove the claim for $l \in \mathbb{Z}$ and $z \in C^l_{\alpha^*}$ only, so that $\lceil x - z \rceil \in \{0, 1\}$ for $x \in C^l_{\alpha^*}$.

Choose arbitrary $l \in \mathbb{Z}$ and $z \in C^l_{\alpha^*}$. Disregarding the trivial case, assume that $\mathbb{P}\{\omega \in C^l_{\alpha^*}\} > 0$. Then,

$$\mathbb{E}_\omega \Big[\lceil \omega - z \rceil \,\Big|\, \omega \in C^l_{\alpha^*}\Big] = \mathbb{P}\Big\{\omega > z \Big| \omega \in C^l_{\alpha^*}\Big\} = \frac{\mathbb{P}\{z < \omega \leq \alpha^* + l\}}{\mathbb{P}\{\omega \in C^l_{\alpha^*}\}} = \frac{F_\omega(\alpha^* + l) - F_\omega(z)}{\mathbb{P}\{\omega \in C^l_{\alpha^*}\}}.$$

Similarly, we have that

$$\mathbb{E}_{\omega_{\alpha^*}} \Big[\lceil \omega_{\alpha^*} - z \rceil \,\Big|\, \omega_{\alpha^*} \in C^l_{\alpha^*}\Big] = \frac{F_{\omega_{\alpha^*}}(\alpha^* + l) - F_{\omega_{\alpha^*}}(z)}{\mathbb{P}\{\omega_{\alpha^*} \in C^l_{\alpha^*}\}}.$$

Observing that $\mathbb{P}\{\omega \in C^l_{\alpha^*}\} = \mathbb{P}\{\omega_{\alpha^*} \in C^l_{\alpha^*}\}$ and $F_\omega(\alpha^* + l) = F_{\omega_{\alpha^*}}(\alpha^* + l)$ by definition of ω_{α^*} completes the proof.

Corollary 1 *Consider the TU integer recourse function*

$$Q(z) = \mathbb{E}_\omega[\min_y\{qy : Wy \geq \omega - z,\, y \in \mathbb{Z}_+^{n_2}\}], \quad z \in \mathbb{R}^m,$$

and its α^*-approximation

$$Q_{\alpha^*}(z) = \mathbb{E}_{\phi_{\alpha^*}}[\min_y\{qy : Wy \geq \phi_{\alpha^*} - z,\, y \in \mathbb{R}_+^{n_2}\}], \quad z \in \mathbb{R}^m.$$

If ω_i stochastically dominates $\omega_{\alpha_i^*}$ for every $i = 1, \dots, m$, then Q_{α^*} is the convex hull of Q.

Proof If ω_i stochastically dominates $\omega_{\alpha_i^*}$ for every $i = 1, \dots, m$, then it follows by Lemma 1 that for every $i = 1, \dots, m$,

$$\mathbb{E}_{\omega_i}\left[\lceil \omega_i - z_i \rceil \,\middle|\, \omega_i \in C_{\alpha_i^*}^{l_i}\right] \geq \mathbb{E}_{\omega_{\alpha_i^*}}\left[\lceil \omega_{\alpha_i^*} - z_i \rceil \,\middle|\, \omega_{\alpha_i^*} \in C_{\alpha_i^*}^{l_i}\right] \quad \text{for all } (z_i, l_i) \in \mathbb{R} \times \mathbb{Z}.$$

Since $\lambda^k \geq 0$ for every $k = 1, \dots, K$, it follows immediately that (8) holds, so that the proof of Van der Vlerk (2004) is valid, and thus Proposition 1 holds.

In Van der Vlerk (2004) an example involving independent uniform distributions is presented. Next we show that this special case indeed satisfies the additional assumptions of Corollary 1.

Corollary 2 *Consider the setting of Corollary 1. If $\omega \in \mathbb{R}^m$ is independently and uniformly distributed, then Q_{α^*} is the convex hull of Q.*

Proof We will show that each component ω_i stochastically dominates $\omega_{\alpha_i^*}$ so that the assumptions of Corollary 1 are satisfied. For ease of notation we drop the index i and we let ω denote a random variable that is uniformly distributed on the interval (a, b) with $a < b$. For the moment we assume that $\alpha^* = \langle b \rangle := b - \lfloor b \rfloor$.

If $b - a$ is integer, then $f_\omega(x) = f_{\omega_{\alpha^*}}(x)$ for all $x \in \mathbb{R}$ so that

$$F_\omega(x) = F_{\omega_{\alpha^*}}(x), \quad x \in \mathbb{R},$$

and thus ω (weakly) dominates ω_{α^*}.

If $b - a$ is not integer, then choose $l = \lceil a - \alpha^* \rceil \in \mathbb{Z}$ such that $a \in C_{\alpha^*}^l$, that is, $\alpha^* + l - 1 < a < \alpha^* + l$. It is easy to observe that $f_\omega(x) = f_{\omega_{\alpha^*}}(x)$ for all $x \in \mathbb{R} \backslash C_{\alpha^*}^l$, see Figure 3. However, for $x \in C_{\alpha^*}^l$,

$$f_\omega(x) = \begin{cases} 0, & x \leq a, \\ \frac{1}{b-a}, & x > a, \end{cases}$$

and

$$f_{\omega_{\alpha^*}}(x) = \frac{\alpha^* + l - a}{b - a}.$$

Hence, $F_\omega(x) = F_{\omega_{\alpha^*}}(x)$ for all $x \in \mathbb{R} \backslash C_{\alpha^*}^l$, but for $x \in C_{\alpha^*}^l$,

$$F_\omega(x) = \begin{cases} 0, & x \leq a, \\ \frac{x-a}{b-a}, & x \geq a, \end{cases}$$

whereas

$$F_{\omega_{\alpha^*}}(x) = \frac{x - \alpha^* - l + 1}{b - a}.$$

Since $a > \alpha^* + l - 1$, we conclude that

$$F_\omega(x) \leq F_{\omega_{\alpha^*}}(x), \quad x \in \mathbb{R},.$$

that is, ω stochastically dominates ω_{α^*}.

It remains to be shown that $\alpha^* = \langle b \rangle$. If $b-a$ is integer, then $\mathbb{E}_\omega[\lceil \omega - x \rceil + x]$ is constant so that $\alpha^* = \langle b \rangle$ is a valid choice by (5). Moreover, Van der Vlerk (2004) shows that a necessary condition for α^* is that

$$-1 \in \left[-\sum_{k=-\infty}^{\infty} f_\omega^-(\alpha^* + k), -\sum_{k=-\infty}^{\infty} f_\omega^+(\alpha^* + k) \right],$$

where f_ω^- and f_ω^+ represent the left-continuous and right-continuous version of f_ω, respectively. If $b-a$ is not integer, then it can be shown by straightforward computation that this condition is only satisfied for $\alpha^* = \langle b \rangle$.

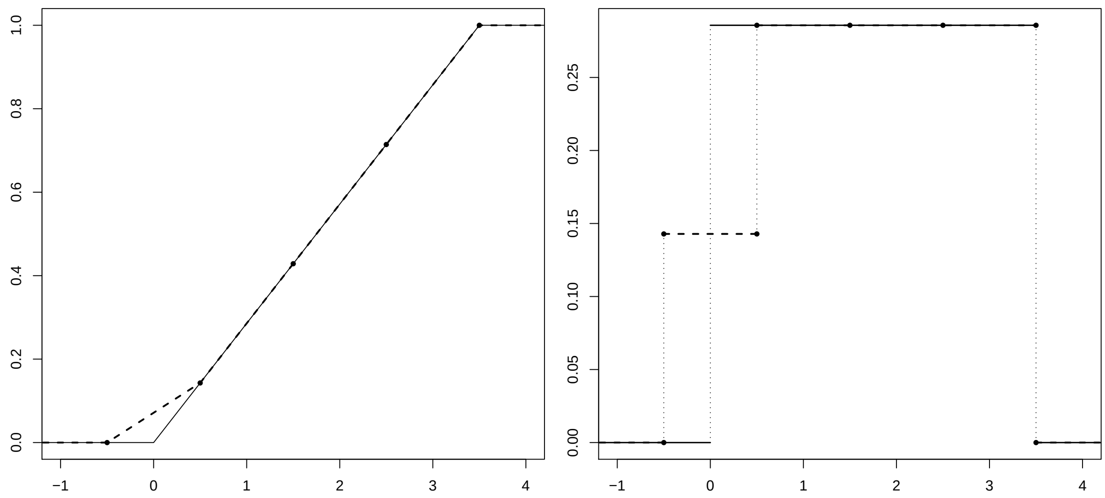

Fig. 3 The cdf (left) and pdf (right) of a uniform distribution on [0,3.5] (solid) and its α^*-approximation (dashed).

5 Density functions with a strictly decreasing right tail

Although the stochastic dominance conditions of Corollary 1 are easy to verify, they will be satisfied only in exceptional cases. Indeed, since $F_{\omega_{\alpha^*}} = F_\omega$ on $\alpha^* + \mathbb{Z}$ and is linear on every set $C^l_{\alpha^*}, l \in \mathbb{Z}$, the conditions are violated if the cdf F_ω is strictly concave on some $C^l_{\alpha^*}, l \in \mathbb{Z}$. For example, this is the case for many unimodal distributions, since they have a pdf f_ω which is strictly decreasing on (ν, ∞), where ν is the mode, so that the cdf F_ω is strictly concave on (ν, ∞), see Figure 4.

Lemma 2 *Let ω be a random variable with pdf f_ω and cdf F_ω. If f_ω is strictly decreasing on $C^l_{\alpha^*}$ for some $l \in \mathbb{Z}$, then $F_\omega(x) > F_{\omega_{\alpha^*}}(x)$ for all $x \in \operatorname{int} C^l_{\alpha^*}$, so that ω_{α^*} is not first-order stochastically dominated by ω.*

As we will show next, Lemma 2 not only invalidates the proof of Proposition 1, but it also implies that the claim itself is incorrect for a large class of distributions. Consider once more the SIR function $\mathcal{Q}$ defined in (3).

Louveaux and Van der Vlerk (1993) show that

$$\mathcal{Q}(z) = \mathbb{E}_\omega[\lceil \omega - z \rceil^+] = \sum_{k=0}^{\infty} \{1 - F(z+k)\}, \quad z \in \mathbb{R}. \tag{11}$$

This implies that for sufficiently large values of z, the right tail of the distribution of ω determines the value of $\mathcal{Q}(z)$. For this reason we consider density functions with a strictly decreasing right tail.

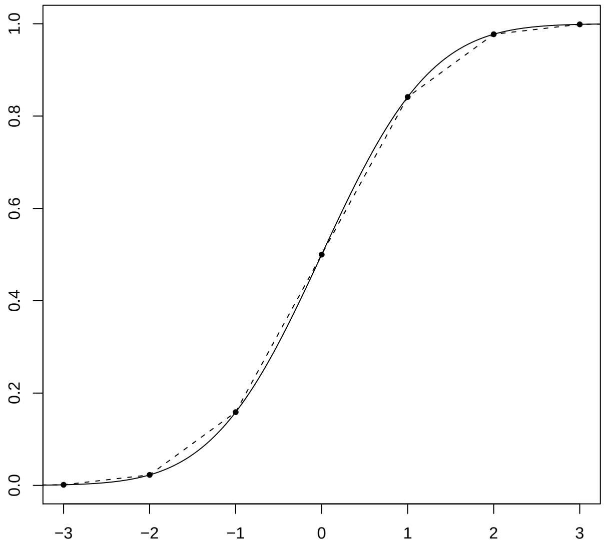

Fig. 4 The cdf (solid) of a standard normal distribution and the cdf of its α-approximation (dashed) with $\alpha = 0$, showing that the stochastic dominance constraints are satisfied in exceptional cases only.

Definition 3 A pdf f_ω has a *strictly decreasing right tail*, if there exists $M \in \mathbb{R}$ such that for every $x, y \in \mathbb{R}$ with $x > y > M$, $f_\omega(x) < f_\omega(y)$.

Remark 1 Note that density functions with a strictly decreasing right tail necessarily have unbounded support. However, the result below also hold for density functions with bounded support, provided that there exists $M \in \mathbb{R}$ such that f_ω is strictly decreasing on $[M, M+2]$ and non-increasing on $(M+2, \infty)$.

Lemma 3 *Consider the SIR function $\mathcal{Q}$ as defined in (3), and let ω be a random variable whose pdf f_ω has a strictly decreasing right tail. Then the α^*-approximation $\mathcal{Q}_{\alpha^*}$ as defined in (4), with α^* as defined in (5), is not a lower bound for $\mathcal{Q}$ and thus not the convex hull of $\mathcal{Q}$.*

Proof Let $\bar{l} \in \mathbb{Z}$ be given such that f_ω is strictly decreasing on $C^l_{\alpha^*}$ for all $l \geq \bar{l}$. Choose $z \in \operatorname{int} C^{\bar{l}}_{\alpha^*}$. Then by Lemma 2 it follows that for every $k \in \mathbb{Z}_+$,

$$F_\omega(z+k) > F_{\omega_{\alpha^*}}(z+k),$$

so that (11) implies

$$\mathcal{Q}_{\alpha^*}(z) - \mathcal{Q}(z) = \sum_{k=0}^{\infty} \Big\{ F_\omega(z+k) - F_{\omega_{\alpha^*}}(z+k) \Big\} > 0.$$

6 Discussion and future research directions

We have shown that the α^*-approximation Q_{α^*} equals the convex hull of the TU integer recourse function Q only in exceptional cases (e.g. if ω is independently uniformly distributed). If so, provided that the first-stage constraints are non-binding and that the matrix T is of full row rank, the first-stage solutions obtained using this approximation will be optimal. In all other cases, either the α^*-approximation does not necessarily yield the convex hull of Q and/or the obtained solutions may not be optimal.

However, this does not imply that Q_{α^*} is not a good approximation of Q. Indeed, Q_{α^*} always coincides with Q on $\alpha^* + \mathbb{Z}$, and for SIR models – a special case – a uniform error bound is available showing that the error is small if the total variation of the pdf f_ω is low. Obtaining such an error bound for the general TU case is an interesting direction for future research. Alternatively, other ways of obtaining the convex hull of Q may be investigated.

Acknowledgements This research is supported by The Netherlands Organisation for Scientific Research (NWO). We are grateful to Wim Klein Haneveld for many beneficial discussions.

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