

Fibred Kähler and quasi-projective groups

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Dedicated to Adriano Barlotti on the occasion of his 80th birthday

Abstract. We formulate a new theorem giving several necessary and sufficient conditions in order that a surjection of the fundamental group $\pi_1(X)$ of a compact Kähler manifold onto the fundamental group Π_g of a compact Riemann surface of genus $g \geq 2$ be induced by a holomorphic map. For instance, it suffices that the kernel be finitely generated.

We derive as a corollary a restriction for a group G , fitting into an exact sequence $1 \rightarrow H \rightarrow G \rightarrow \Pi_g \rightarrow 1$, where H is finitely generated, to be the fundamental group of a compact Kähler manifold.

Thanks to the extension by Bauer and Arapura of the Castelnuovo–de Franchis theorem to the quasi-projective case (more generally, to Zariski open sets of compact Kähler manifolds) we first extend the previous result to the non-compact case. We are finally able to give a topological characterization of quasi-projective surfaces which are fibred over a (quasi-projective) curve by a proper holomorphic map of maximal rank, and we extend the previous restriction to the monodromy of any fibration onto a curve.

1 Introduction

The study of fibrations of algebraic (or Kähler) manifolds $f : X \rightarrow C$ over curves C of genus at least 2, called classically irrational pencils, has a long history.

Around 1905 almost simultaneously de Franchis and Castelnuovo–Enriques ([14], [7]) found that the existence of such a fibration is equivalent to the existence of at least two linearly independent holomorphic 1-forms whose wedge product yields a holomorphic 2-form which is identically zero.

Hodge theory was yet to be developed and only much later ([9]) it was shown that combining the Hodge decomposition with the theorem of Castelnuovo–de Franchis one obtains a topological characterization of such fibrations via any subspace in de Rham cohomology obtained as the pull-back of a maximal isotropic subspace in the cohomology of C .

Other topological characterizations in terms of the induced surjection of fundamental groups $f_* : \pi_1(X) \rightarrow \pi_1(C)$, or other statements in this direction, had been

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obtained earlier by several authors (cf. Jost–Yau and Siu, [35], [22], [23], [36], who used the theory of harmonic maps, Beauville, [6], used instead the generic vanishing theorems of Green and Lazarsfeld, [17]).

In [10] I tried to show how the isotropic subspace theorem, which predicts the genus of the image curve C (it equals the dimension of the corresponding maximal isotropic subspace), unlike the other statements, could be used to obtain also simple proofs of statements concerning surjections of fundamental groups, as the one given by Gromov ([19]).

Hodge theory also works for quasi-projective manifolds, and it turns out that in the non-compact case it works much better than the other methods ([4], [2]). These results were then used in [11] to give topological characterizations of varieties isogenous to a product and of isotrivial fibrations of surfaces. Kotschick ([28]) instead used very similar methods to give a topological characterization of Kodaira fibrations, which was independently also obtained by Hillman ([21]). Our first motivation was to extend this result to any fibration, a goal that we have achieved in the following

Theorem 6.4. *Assume that U is a non-complete Zariski open set of an algebraic surface and that the following properties hold:*

- (P1) *We have an exact sequence $1 \rightarrow \Pi_r \rightarrow \pi_1(U) \rightarrow \mathbb{F}_g \rightarrow 1$, where $g \geq 2$.*
- (P2) *The topological Euler–Poincaré characteristic $e(U)$ of U is $2(g-1)(r-1)$.*
- (P3) *For each end \mathcal{E} of U , the corresponding fundamental group $\pi_1^{\mathcal{E}}$ surjects onto a cyclic subgroup of \mathbb{F}_g , and each simple geometric generator γ_i has a non-trivial image in \mathbb{F}_g .*

Then U is a good open set of a fibration, more precisely, there exists a proper holomorphic submersion $f : U \rightarrow C$ inducing the previous exact sequence.

For this purpose, we started to put together the existing results, both in the compact and in the non-compact case, with some small addition: we refer to Theorems 4.3 and 5.4 for full statements. We only indicate here some new results:

Theorem A. *Let X be a compact Kähler manifold, and assume that its fundamental group admits a non-trivial homomorphism ψ to the fundamental group Π_g of a compact Riemann surface of genus $g \geq 2$, with kernel H . Then the following conditions are equivalent:*

- (4) *ψ is induced by an irrational pencil of genus g without multiple fibres.*
- (5) *ψ is surjective and its kernel H is finitely generated.*

Theorem A'. *Let X be a compact Kähler manifold and $Y = X - D$ be a Zariski open set. Assume that the fundamental group of Y admits a homomorphism $\psi : \pi_1(Y) \rightarrow \mathbb{F}_g$ to a free group of rank g , with kernel H . Then the following are equivalent:*

- (1') *ψ is induced by a pencil $f : Y \rightarrow C$ of type g without multiple fibres and by a surjection $\pi_1(C) \rightarrow \mathbb{F}_g$.*
- (2') *ψ is surjective and the kernel H of ψ is finitely generated.*

The new ingredient here is a remarkable property of free groups and of the fundamental groups of compact curves of genus $g \geq 2$. This property is that every non-trivial Normal subgroup of Infinite index is Not Finitely generated. We abbreviate this property by the acronym *NINF*, and we devote Section 3 to establishing this result for the above mentioned groups.

This property plays an important role, for instance it shows that, contrary to what is stated by some author, the kernel of the homomorphism between fundamental groups $f_* : \pi_1(X) \rightarrow \pi_1(C)$ needs not be finitely generated, it is finitely generated if and only if there are no multiple fibres.

In the case where there are multiple fibres, one can take a ramified base change which eliminates the multiple fibres, and indeed one could extend Theorems A and A' to include the case where H is not finitely generated, but we believed that the results stated in this article are already sufficiently complicated, so we omitted to treat this extension.

We preferred instead to concentrate on some important consequence, concerning the monodromy of fibrations over curves. For instance, putting together Theorem A with the old isotropic subspace theorem we obtain

Corollary 7.3. *If a finitely presented group Γ admits a surjection $\Gamma \rightarrow \Pi_g$ with finitely generated kernel H , then Γ cannot be the fundamental group of a compact Kähler manifold X if there is a non-zero element $u \in H^1(H, \mathbb{Z})^{\Pi_g}$ such that the cup product with u yields the zero map*

$$H^1(\Pi_g, \mathbb{Z}) \rightarrow H^1(\Pi_g, H^1(H, \mathbb{Z})).$$

We then spell out in detail the meaning of degeneracy of the above cup product: it means that there exists a bad monodromy submodule.

By extending everything to the non-compact case one obtains then a restriction for the monodromy of fibrations over curves which is in the same spirit as Deligne's Semisimplicity Theorem (4.2.6. of [14]).

2 Notation

Π_g denotes the fundamental group of a compact Riemann surface C_g of genus $g \geq 2$, $\Pi_g := \langle a_1, \dots, a_g, b_1, \dots, b_g \mid [a_1, b_1] \dots [a_g, b_g] = 1 \rangle$. By \mathbb{F}_g we denote a free group of rank $g \geq 2$, and X will be a compact Kähler manifold.

3 Non-finitely generated subgroups

This section is devoted to a remarkable property enjoyed, for $g \geq 2$, by the free groups \mathbb{F}_g and by the fundamental groups Π_g of a compact Riemann surface C_g of genus $g \geq 2$.

Definition 3.1. A group G is said to satisfy property *NINF* (to be more precise, but less concise, we should call it *NIINFG*) if every normal non-trivial subgroup K of infinite index is not finitely generated. We shall also say that G is a *NINF*.

The application of the above notion that we shall need is the following

Lemma 3.2. *Let $1 \rightarrow A \rightarrow \Pi \rightarrow B \rightarrow 1$ be an exact sequence of group homomorphisms such that the following conditions are satisfied.*

- (1) *A is finitely generated.*
- (2) *B is infinite.*
- (3) *$\varphi : \Pi \rightarrow B$ factors as $\rho \circ \psi$, where $\psi : \Pi \rightarrow G$ is surjective.*
- (4) *G is a NINF.*

Then $\rho : G \rightarrow B$ is an isomorphism.

Proof. Let $j : A \rightarrow \Pi$ be the inclusion and define $K' := \ker(\psi \circ j)$, $K := \ker(\psi)$. Then $K' = K$ since $K \subset A = \ker(\varphi)$. Set $A' := A/K$, so A' injects into $G = \Pi/K$. Moreover, A' is normal in G with quotient $B = \Pi/A$ which is infinite by assumption. Since A' is finitely generated, as a quotient of A , and G is a NINF it follows that A' is trivial. Whence $A = K$ and $\rho : G \rightarrow B$ is an isomorphism, as desired. \square

Lemma 3.3. *A free group \mathbb{F}_n enjoys property NINF.*

Proof. We may assume that $n \geq 2$. We view \mathbb{F}_n as the fundamental group $\pi_1(Y)$, where Y is a bouquet of n circles. Let Z be the covering space corresponding to a normal subgroup K of infinite index. Z is indeed the Cayley graph for the infinite group $G := \mathbb{F}_n/K$ with respect to the finite set of n generators, g_1, \dots, g_n , corresponding to the surjection $\mathbb{F}_n \rightarrow G$.

If K is not trivial, then Z is not simply connected, and there is a non-trivial minimal closed simplicial path ξ based on the base point $x_0 = 1$: $(x_0 = 1, x_1 = \gamma_1, \dots, x_m = \gamma_1 \dots \gamma_m)$, where the γ_i belong to the given set of generators $\{g_1, \dots, g_n\}$. Let $M \subset G$ be the set $\{x_0 = 1, x_1, \dots, x_m\}$ and let M' be the finite set $MM^{-1} \subset G$. Since G is infinite, there exist infinitely many h_α such that $h_\alpha \notin h_\beta M'$ for $\alpha \neq \beta$. Whence, for $\alpha \neq \beta$ and for all x_i and x_j , we have $h_\alpha x_i \neq h_\beta x_j$. It follows that the cycles $h_\alpha(\xi)$ are homologically independent.

A fortiori, we have shown $\text{Rank}(H_1(Z, \mathbb{Z})) = \infty$ and K is not finitely generated. \square

Lemma 3.4. *A fundamental group Π_g enjoys property NINF for $g \geq 2$.*

Proof. Let Γ be a non-trivial normal subgroup of $\Pi := \Pi_g$, and let $f : D \rightarrow C$ be the corresponding unramified covering of a compact Riemann surface of genus g . We have that $g \geq 2$, whence we may view D as a quotient of the upper half plane \mathbb{H} by the action of the group Γ acting freely and properly discontinuously.

As in [34], Theorem 4, page 35, we consider fundamental domains, \mathcal{F}_Π resp. \mathcal{F}_Γ , bounded by non-Euclidean segments (possibly also lines or half-lines). While \mathcal{F}_Π has finite area, the area of \mathcal{F}_Γ is the area of \mathcal{F}_Π multiplied by the index of Γ , whence \mathcal{F}_Γ has infinite area.

Assume that Γ is finitely generated: then (cf. [5] Theorem 10.1.2, page 254) there is such a fundamental domain \mathcal{F}_Γ with finitely many sides. Since however its area is infinite, it cannot be a non-Euclidean ideal polygon, and there are intervals in the real line $\mathbb{P}_\mathbb{R}^1$ which need to be added for a compactification of \mathcal{F}_Γ .

Let us recall the following standard definitions (cf. [34], and especially [33], [5]).

Definition 3.5. (1) A subgroup Γ of $\mathrm{PSL}(2, \mathbb{R})$ acting properly discontinuously on \mathbb{H} is called a *Fuchsian group* (more generally, a Fuchsian group is a conjugate in $\mathrm{PSL}(2, \mathbb{C})$ of such a subgroup).

(2) Γ is *properly discontinuous* if and only if it is discrete in $\mathrm{PSL}(2, \mathbb{R})$.

(3) The *limit set* $L(\Gamma)$ is defined as

$$L(\Gamma) := \bigcup_{z \in \mathbb{H}} \overline{\Gamma z} \cap \mathbb{P}_\mathbb{R}^1.$$

(4) Equivalently, (cf. [33], page 108)

$$L(\Gamma) := \{z \in \mathbb{P}_\mathbb{R}^1 \mid \exists \gamma \in \Gamma, \gamma \neq 1, \text{ such that } \gamma(z) = z\}.$$

(5) A Fuchsian subgroup Γ is said to be of the *first kind* if $L(\Gamma) = \mathbb{P}_\mathbb{R}^1$ (else, it is said to be of the *second kind*).

We have (cf. e.g. Lemma 3.12.2, page 108 of [33]) that if \mathbb{H}/Γ is compact, then Γ is of the first kind. This implies a consequence for our group Π . In fact, since Π normalizes Γ , the group Π carries the limit set $L(\Gamma)$ to itself.

Let I be an interval in the real line $\mathbb{P}_\mathbb{R}^1$ which is in the boundary of \mathcal{F}_Γ . Since Π is of the first kind, there is $x \in I$ and $g \in \Pi - \{1\}$ such that $gx = x$. Since, as we observed, $g(L(\Gamma)) = L(\Gamma)$, g carries the interior of the complement to $L(\Gamma)$ in $\mathbb{P}_\mathbb{R}^1$ to itself. Let the interval (a, b) be the connected component of this interior containing x . Since moreover $gx = x$, g carries (a, b) to itself.

Assume that $a \neq b$: then g^2 has three fixed points (a, b, x) , thus g^2 is the identity, contradicting the hyperbolicity of g .

If however $a = b$, this means that the limit set $L(\Gamma)$ consists of a single point a (fixed by each $g \in \Pi$), contradicting the fact that Γ is of the first kind. \square

4 Mappings to curves

If a manifold X is fibred (with connected fibres) onto a curve C , certainly we have a surjection of fundamental groups $\pi_1(X) \rightarrow \pi_1(C)$. In fact, let p_1, \dots, p_r be the critical values of f , and set $C^* := C - \{p_1, \dots, p_r\}$, $X^* := f^{-1}(C^*)$: then we have surjections $\pi_1(X^*) \rightarrow \pi_1(X)$, $\pi_1(C^*) \rightarrow \pi_1(C)$, and an exact homotopy sequence

$$\pi_1(X) \rightarrow \pi_1(X^*) \rightarrow \pi_1(C^*) \rightarrow 1.$$

It suffices to observe that the surjection $\pi_1(X^*) \rightarrow \pi_1(C)$ factors through $\pi_1(X) \rightarrow \pi_1(C)$.

If however there is a non-trivial holomorphic map $F : C \rightarrow C'$ such that F does not factor as $F = F' \circ F''$, where F' is unramified, then it is not difficult to show that there is a surjection of fundamental groups $F_* : \pi_1(C) \rightarrow \pi_1(C')$.

This is the reason why one needs some extra assumptions on a surjection of fundamental groups $\pi_1(X) \rightarrow \pi_1(C')$ in order to decide whether the corresponding map is a fibration (i.e., it has connected fibres).

Recall (cf. e.g. [12], Lemma 3, page 283) the following

Definition 4.1. Let m_i for $i = 1, \dots, r$ be the greatest common divisor of the multiplicities of the components of the divisor $f^{-1}(p_i)$ (p_1, \dots, p_r are again the critical values of f). Then the *orbifold fundamental group* $\pi_1^{\text{orb}}(f)$ is defined as the quotient of $\pi_1(C - \{p_1, \dots, p_r\})$ by the subgroup normally generated by $\{\gamma_i^{m_i}\}$, γ_i being a simple geometric path around the point p_i .

As a corollary of the results of the previous section we have

Lemma 4.2. *If X admits a surjective holomorphic map f with connected fibres $f : X \rightarrow C$ where C is a Riemann surface of genus $g \geq 0$, then the induced homomorphism $f_* : \pi_1(X) \rightarrow \Pi_g$ is surjective, and its kernel H is finitely generated exactly when $g = 0$ or when $g \geq 1$ and there are no multiple fibres, i.e., $\pi_1^{\text{orb}}(f) \cong \pi_1(C)$.*

Proof. As is well known (cf. e.g. a slightly more general version given in [12], Lemma 3, page 283, whose notation we will follow) we have an exact sequence

$$\pi_1(F) \rightarrow \pi_1(X) \rightarrow \pi_1^{\text{orb}}(f) \rightarrow 1,$$

where F is a smooth fibre of f .

Let $\psi := f_*$. Thus $\ker(\psi)$ contains the normal subgroup K , the image of $\pi_1(F)$, which is finitely generated since F is compact, and the cokernel $\ker(\psi)/K$ is isomorphic to the kernel of $\rho : \pi_1^{\text{orb}}(f) \rightarrow \pi_1(C)$.

Therefore $\ker(\psi)$ is finitely generated if and only if $\ker \rho$ is finitely generated. This is then the case for $g = 0$, so let us assume that $g \geq 1$. If we moreover assume that there are no multiple fibres, then ρ is an isomorphism, and we are again done.

Otherwise, $\pi_1^{\text{orb}}(f)$ is a Fuchsian group and the same proof as in Lemma 3.4 shows that $\pi_1^{\text{orb}}(f)$ is NINF. Then $\ker \rho$ is finitely generated if only if it is trivial, since the alternative that its index be finite is ruled out by the condition $g \geq 1$.

Finally, if $\ker \rho$ is trivial, then ρ is an isomorphism, and by looking at the Abelianization we see that $r = 1$. But then the orbifold fundamental group $\pi_1^{\text{orb}}(f)$ is a free product of a free group of rank $2g - 1$ with a cyclic group of order m_1 and its Abelianization is then not a free Abelian group of rank $2g$, a contradiction. \square

Theorem 4.3. *Let X be a compact Kähler manifold, and assume that its fundamental group admits a non-trivial homomorphism ψ to the fundamental group Π_g of a compact Riemann surface of genus $g \geq 2$, with kernel H . Then the following conditions are equivalent:*

- (1) ψ is induced by an irrational pencil of genus g , i.e., there is a surjective holomorphic map f with connected fibres $f : X \rightarrow C$ such that $\psi = f_*$, and where C is a Riemann surface of genus g .
- (2) ψ is surjective and the image of $\psi^* : H^1(\Pi_g, \mathbb{Q}) \rightarrow H^1(X, \mathbb{Q})$ contains a g -dimensional maximal isotropic subspace (for the bilinear pairing $H^1(X, \mathbb{Q}) \times H^1(X, \mathbb{Q}) \rightarrow H^2(X, \mathbb{Q})$).
- (3) ψ induces an injective map in cohomology $\psi^* : H^1(\Pi_g, \mathbb{Q}) \rightarrow H^1(X, \mathbb{Q})$, and the image of ψ^* contains a g -dimensional maximal isotropic subspace.

Likewise, the following conditions are also equivalent to each other:

- (4) ψ is induced by an irrational pencil of genus g without multiple fibres, i.e., for each fibre F' the equation of divisors $F' = rD$, with $r \geq 1$, implies $r = 1$.
- (5) ψ is surjective and its kernel H is finitely generated.

Proof. We observe first that (2) implies (3), since a surjective homomorphism induces a surjective homomorphism between the Abelianizations, and dualizing one obtains an injective homomorphism in cohomology.

Recall that, by the isotropic subspace theorem of [9], given a maximal isotropic subspace $V \subset H^1(X, \mathbb{Q})$ of dimension g , there is a holomorphic fibration onto a curve C of genus g such that $V \subset f^*(H^1(C, \mathbb{Q}))$.

Now, (1) implies (2) because, if the pull back of a maximal isotropic subspace from the given curve C is not maximal, then we have another fibration $f' : X \rightarrow C'$ to a curve of genus $g' > g$ such that $f^*(H^0(C, \Omega_C^1)) \subset f'^*(H^0(C', \Omega_{C'}^1))$, whence f factors through f' , contradicting the fact that f has connected fibres.

Let us show that (3) implies (1). The isotropic subspace theorem gives us the desired $f : X \rightarrow C$, where C has genus g . Since however C is a classifying space for Π_g , there is a continuous map $F : X \rightarrow C$ such that $\psi = F_*$. Compose both maps with the Jacobian embedding $\alpha : C \rightarrow J$ and observe that by the proof of the isotropic subspace theorem the two subspaces $f^*(H^1(C, \mathbb{Q}))$ and $F^*(H^1(C, \mathbb{Q}))$ coincide.

Therefore the two maps $\alpha \circ F$, $\alpha \circ f$ are given by integrals of the same differentiable 1-forms, hence there is an isogeny $p : J \rightarrow J$ such that $\alpha \circ f = p \circ \alpha \circ F$. We get thus, up to changing F in its homotopy equivalence class, a factorization $f = p' \circ F$. Thus, p' is surjective, and actually it has degree 1 or otherwise $f^*(\eta)$, with (η) the positive generator of $H^2(C, \mathbb{Z})$, would be divisible, contradicting the fact that f has connected fibres.

The conclusion is that f and F are homotopy equivalent, thus $\psi = F_* = f_*$.

The implication (4) \Rightarrow (5) is exactly Lemma 4.2, so we are left with showing that (5) implies (4).

Now, (5) implies that the image of ψ^* contains a g -dimensional isotropic subspace. Assume this subspace is not maximal: then there is a fibration $f : X \rightarrow C$ where the genus g' of C is strictly larger than g . Arguing as we did before, we find a factorization of ψ through f'_* . Since ψ is surjective, Lemma 3.2 applies and we get that $g' = g$, and $\psi = f'_*$. Finally, f has no multiple fibres again by Lemma 4.2. \square

5 The logarithmic case

In this section we shall generalize the results of the previous section to the case where we have a Zariski open set Y in a compact Kähler manifold X . One may assume without loss of generality that the complement $X - Y$ is a normal crossings divisor D . We shall consider holomorphic maps $f : Y \rightarrow C$, where C is Zariski open in a compact curve \bar{C} , and the map f is meromorphic on X , whence there is another compactification \bar{X} of X where f extends holomorphically.

When we shall say that f is a *pencil*, we shall mean that f is as above, that the extension \bar{f} of f has connected fibres, and that f is surjective (Arapura calls these maps *admissible maps*). We shall denote by B the complement $\bar{C} - C$, because quite often it will be the branch locus of a fibration of a compact manifold.

However, X will not necessarily be non-compact, the reason for this being that we shall here consider surjective homomorphisms $\pi_1(X) \rightarrow \mathbb{F}_n$ to a non-Abelian free group. Notice moreover that

- any automorphism of Π_g composed with the standard surjection $p : \Pi_g \rightarrow \mathbb{F}_g$ such that $p(a_i) = p(b_i) = x_i$ produces a maximal isotropic subspace of $H^1(C, \mathbb{Z})$.
- There is a surjection $p : \Pi_g \rightarrow \mathbb{F}_n$ iff $g \geq n$ (since $\text{Im}(p^*)$ is an isotropic subspace of dimension n).

The next theorem extends the results of I. Bauer and D. Arapura (Theorems 2.1 and 3.1 of [4] and Corollary 1.8 of [2], cf. also Theorem 2.11 of [11]) using the new ideas introduced in the previous sections.

Observe that also in this context a pencil induces a surjective homomorphism of fundamental groups.

Definition 5.1. Let $f : Y \rightarrow C$ be a pencil as above. We shall say that f is of *type g* if either

C is compact of genus g (then $\pi_1(C) \cong \Pi_g$), or

C is not compact and its first Betti number equals g (then $\pi_1(C) \cong \mathbb{F}_g$).

Definition 5.2. Let $f : Y \rightarrow C$ be a pencil as above. We may assume that f extends to a holomorphic fibration $F : X \rightarrow \bar{C}$. We can separate the complementary divisor $D = X - Y$ into three parts:

D^{hor} : the union of the components dominating C

D^{FVert} : the union of the fibres over $\bar{C} - C$

D^{PVert} : the union of the components mapping to points of C .

We define then the orbifold fundamental group of f by the usual procedure: let p_1, \dots, p_r be the critical values of f , and m_i for $i = 1, \dots, r$, be the respective greatest common divisor of the multiplicities of the components of the divisor $f^{-1}(p_i)$. Then the *orbifold fundamental group* $\pi_1^{\text{orb}}(f)$ is defined as the quotient of $\pi_1(C - \{p_1, \dots, p_r\})$ by the subgroup normally generated by $\{\gamma_i^{m_i}\}$, γ_i being a simple geometric path around the point p_i .

Lemma 5.3. *Let X be a compact Kähler manifold and $Y = X - D$ be a Zariski open set. If Y admits a pencil $f : Y \rightarrow C$, then the induced homomorphism $f_* : \pi_1(X) \rightarrow \pi_1(C)$ is surjective, and its kernel H is finitely generated exactly when $g = 0$ or when $g \geq 1$ and there are no multiple fibres, i.e., $\pi_1^{\text{orb}}(f) \cong \pi_1(C)$.*

Proof. We use once more the exact sequence (the situation being more general, but the proof exactly the same as in [12], Lemma 3, page 283)

$$\pi_1(F) \rightarrow \pi_1(X) \rightarrow \pi_1^{\text{orb}}(f) \rightarrow 1,$$

where F is a smooth fibre of f which is transversal to D^{hor} .

Let $\psi := f_*$. Thus $\ker(\psi)$ contains the normal subgroup K , the image of $\pi_1(F)$, which is finitely generated since F is of finite type, and the cokernel $\ker(\psi)/K$ is isomorphic to the kernel of $\rho : \pi_1^{\text{orb}}(f) \rightarrow \pi_1(C)$.

Therefore $\ker(\psi)$ is finitely generated if and only if $\ker \rho$ is finitely generated. Assume that there are no multiple fibres: then ρ is an isomorphism.

Otherwise, $\pi_1^{\text{orb}}(f)$ is a Fuchsian group and the same proof as in Lemma 3.4 shows that $\pi_1^{\text{orb}}(f)$ is NINF. Then $\ker \rho$ is finitely generated if only if it is trivial, its index being infinite for $g \geq 1$. \square

Theorem 5.4. *Let X be a compact Kähler manifold and $Y = X - D$ be a Zariski open set. Assume that the fundamental group of Y admits a homomorphism $\psi : \pi_1(Y) \rightarrow \mathbb{F}_g$ to a free group of rank g , with kernel H . Then the following are equivalent:*

- (1) ψ is induced by a pencil $f : Y \rightarrow C$ of type g and by a surjection $\pi_1(C) \rightarrow \mathbb{F}_g$.
- (2) ψ is surjective and the image of $\psi^* : H^1(\mathbb{F}_g, \mathbb{Q}) \rightarrow H^1(Y, \mathbb{Q})$ is a g -dimensional maximal isotropic subspace (for the bilinear pairing $H^1(Y, \mathbb{Q}) \times H^1(Y, \mathbb{Q}) \rightarrow H^2(Y, \mathbb{Q})$).

Likewise, the following are also equivalent to each other:

- (1') ψ is induced by a pencil of type g without multiple fibres.
- (2') ψ is surjective and the kernel H of ψ is finitely generated.

Finally, the curve C is compact if and only if $\psi^(H^1(\mathbb{F}_g, \mathbb{Q}))$ is also an isotropic subspace in $H^1(X, \mathbb{Q})$ ($\subset H^1(Y, \mathbb{Q})$).*

Proof. (1) implies (2): it suffices to show that $\psi^*(H^1(\mathbb{F}_g, \mathbb{Q}))$ is a maximal isotropic subspace. Assume the contrary: then, there is a strictly larger maximal isotropic subspace V induced (cf. [11], Theorem 2.11) by a pencil $f' : Y \rightarrow C'$. The pencil is induced by integrations of linearly independent forms in $H^0(\Omega^1(\log D))$, whence we get a factorization $Y \rightarrow C' \rightarrow C$: since f has connected fibres $C' \cong C$, contradicting that the logarithmic genus of C' is strictly larger than g .

Assume (2). Then by Arapura's Corollary 1.9 there is a pencil $f : Y \rightarrow C$ and $\tau : H_1(C, \mathbb{Z}) \rightarrow \mathbb{Z}^g$ such that there is a factorization in homology $H_1(\psi) = \tau \circ H_1(f)$. Since ψ is surjective, so is $H_1(\psi)$, whence we may lift τ to a surjection $\pi_1(C) \rightarrow \mathbb{F}^g$ (since $\text{Aut}(\mathbb{F}_g)$ surjects onto $\text{GL}(g, \mathbb{Z})$, cf. [29], Sections 3.5 and 3.6).

The equivalence of (1') and (2') follows as in the proof of Theorem 4.3 in view of Lemma 5.3. The last assertion is already contained in Theorem 2.11, loc. cit. \square

6 Fibred algebraic surfaces and good open sets

In this section we shall consider a smooth compact algebraic surface, and a holomorphic fibration $f : S \rightarrow C$.

Definition 6.1. A *good open set* of a fibration will be any set of the form $U = f^{-1}(C - B)$, where B is any finite set containing the set of critical values of f .

Remark 6.2. In the above situation one has an exact sequence of fundamental groups

$$1 \rightarrow \pi_1(F) \rightarrow \pi_1(U) \rightarrow \pi_1(C - B) \rightarrow 1$$

where F is any fibre of f over a point of $C - B$.

The next theorem will give a topological characterization of good open sets of some fibration. The case where $U = S$ was already treated by Kotschick ([28], Proposition 1) and Hillman ([21]), cf. also Kapovich ([25]), and we only prove an ε^2 more general result:

Theorem 6.3. *Assume that S is a compact Kähler surface and that we have an exact sequence*

$$1 \rightarrow \Pi_r \rightarrow \pi_1(S) \rightarrow \Pi_g \rightarrow 1,$$

where $g \geq 2$. If moreover the topological Euler–Poincaré characteristic of S , $e(S)$, equals $4(g - 1)(r - 1)$, then there exists a holomorphic submersion $f : S \rightarrow C$ inducing the previous exact sequence.

Proof. By Theorem 4.3 we find a fibration $f : S \rightarrow C$, where C has genus g , inducing the given epimorphism of fundamental groups. By the theorem of Zeuthen–Segre the Euler–Poincaré characteristic of S , $e(S)$, equals $4(g - 1)(s - 1) + \mu$ where s is the genus of a smooth fibre of f and where $\mu \geq 0$, equality holding if and only if all the singular fibres are multiples of a smooth elliptic curve. It is clear that $s \geq r$.

Our assumption implies $4(g - 1)(s - r) + \mu = 0$, whence $s = r$ and $\mu = 0$. We are therefore done in the case where $r = 0$ or $r \geq 2$. If finally $r = 1$, we are done unless there is some multiple fibre. But the existence of a multiple fibre is excluded by Lemma 4.2. \square

We come now to the non-compact case.

Theorem 6.4. *Assume that U is a non-complete Zariski open set of an algebraic surface and that the following properties hold:*

- (P1) *we have an exact sequence $1 \rightarrow \Pi_r \rightarrow \pi_1(U) \rightarrow \mathbb{F}_g \rightarrow 1$, where $g \geq 2$.*
- (P2) *The topological Euler–Poincaré characteristic $e(U)$ of U is $2(g - 1)(r - 1)$.*
- (P3) *For each end \mathcal{E} of U , the corresponding fundamental group $\pi_1^{\mathcal{E}}$ surjects onto a cyclic subgroup of \mathbb{F}_g , and each simple geometric generator γ_i has a non-trivial image in \mathbb{F}_g .*

Then U is a good open set of a fibration, more precisely, there exists a proper holomorphic submersion $f : U \rightarrow C$ inducing the previous exact sequence.

Proof. By Theorem 5.4 we have a fibration $f : U \rightarrow C$ inducing the surjection $\pi_1(U) \rightarrow \mathbb{F}_g \rightarrow 1$, and without loss of generality we have an extension $\bar{f} : \bar{S} \rightarrow \bar{C}$, where \bar{S} is a blow-up of S . By condition (P3) there is no component of D which is horizontal, and each component of D pulls back to a fibre of \bar{f} .

Therefore it turns out that there is no point of indeterminacy of f on D , whence $\bar{S} = S$, and again by condition (P3) U is the full inverse image of C under \bar{f} .

It suffices to apply the logarithmic version of the Zeuthen–Segre theorem, similarly to Theorem 2.14 of [11], and we conclude that ($e(U)$ being the same in ordinary and Borel–Moore homology by virtue of Poincaré duality) $e(U) = 2(g - 1)(r - 1) \geq 2(g - 1)(s - 1) + \mu$, where $s \geq r$ is the genus of a smooth fibre of f . Whence, as usual, $0 \geq 2(g - 1)(s - r) + \mu$, thus $s = r$ and $\mu = 0$. We conclude as in Theorem 6.3. □

Note. The next question: when is the fibration f a constant moduli fibration? was already answered, with similar methods, in the previous paper [11], cf. 5.4 and 5.7.

7 Restrictions for the monodromy

Before we present some interesting corollary of the previous theorem, we need to recall some well known results

Lemma 7.1. *Let X be a topological manifold and Γ its fundamental group. Then $H^1(X, \mathbb{Z}) = H^1(\Gamma, \mathbb{Z})$ and $H^2(\Gamma, \mathbb{Z})$ injects into $H^2(X, \mathbb{Z})$.*

Proof. Let \tilde{X} be the universal covering of X , so that $X \cong \tilde{X}/\Gamma$. The proof is a direct consequence of the spectral sequence for group cohomology with terms $H^p(\Gamma, H^p(\tilde{X}, \mathbb{Z}))$, converging to a suitable graded quotient of $H^{p+q}(X, \mathbb{Z})$, in view of the fact that $H^1(\tilde{X}, \mathbb{Z}) = 0$. □

Lemma 7.2. *Let $1 \rightarrow H \rightarrow \Gamma \rightarrow B \rightarrow 1$ be an exact sequence of groups, where B is a finitely generated free group, or the fundamental group Π_g of a compact hyperbolic Riemann surface. Assume that*

()** $H^2(B, \mathbb{Z})$ injects into $H^2(\Gamma, \mathbb{Z})$.

Then $H^1(B, \mathbb{Z}) \subset H^1(\Gamma, \mathbb{Z})$ with quotient $H^1(H, \mathbb{Z})^B$, and the cup product $H^1(B, \mathbb{Z}) \times H^1(\Gamma, \mathbb{Z}) \rightarrow H^2(\Gamma, \mathbb{Z})$ lands in the subgroup F fitting into the exact sequence $0 \rightarrow H^2(B, \mathbb{Z}) \rightarrow F \rightarrow H^1(B, H^1(H, \mathbb{Z})) \rightarrow 0$.

In particular, let V a maximal isotropic subspace of $H^1(B, \mathbb{Z})$: then V remains a maximal isotropic subspace in $H^1(\Gamma, \mathbb{Z})$ only if (resp.: if and only if, in the case where B is free)

(***) the cup product $H^1(B, \mathbb{Z}) \times H^1(H, \mathbb{Z})^B \rightarrow H^1(B, H^1(H, \mathbb{Z}))$ is non-degenerate in the second factor.

Proof. The proof of the first assertions is a direct consequence of the spectral sequence for group cohomology, with terms $H^p(B, H^q(H, \mathbb{Z}))$, converging to a suitable graded quotient of $H^{p+q}(\Gamma, \mathbb{Z})$, in view of the fact that, by assumption (**), the differential $d_2 : H^1(H, \mathbb{Z})^B = H^0(B, H^1(H, \mathbb{Z})) \rightarrow H^2(B, \mathbb{Z})$ is zero.

The second assertion holds with “if and only if” in the case where B is a free group, since then $H^2(B, \mathbb{Z}) = 0$, $F = H^1(B, H^1(H, \mathbb{Z}))$, and the question is whether $H^1(B, \mathbb{Z})$ is a maximal isotropic subspace in $H^1(\Gamma, \mathbb{Z})$.

In the other case, observe that $H^2(B, \mathbb{Z}) = \mathbb{Z}$, and that we can find two maximal isotropic subspaces V, V' such that $H^1(B, \mathbb{Z}) = V \oplus V'$: moreover then the cup product yields an isomorphism of V' with V^\vee .

If we get an element $w' \in H^1(H, \mathbb{Z})^B$ annihilating $H^1(B, \mathbb{Z})$, this means that there is a lift $w \in H^1(\Gamma, \mathbb{Z}) - H^1(B, \mathbb{Z})$ such that $w \cup H^1(B, \mathbb{Z}) \subset H^2(B, \mathbb{Z})$. In particular, there is $u \in V^\vee$ such that $(w - u) \cup V = 0$.

We easily conclude then that the span of V and of $w - u$ is isotropic. \square

Corollary 7.3. *If the finitely presented group Γ admits a surjection $\Gamma \rightarrow \Pi_g$ with finitely generated kernel H , then Γ cannot be the fundamental group of a compact Kähler manifold X if there is a non-zero element $u \in H^1(H, \mathbb{Z})^{\Pi_g}$ such that the cup product with u yields the zero map*

$$H^1(\Pi_g, \mathbb{Z}) \rightarrow H^1(\Pi_g, H^1(H, \mathbb{Z})).$$

We now want to write down explicitly, for Π equal either to Π_g or to a free group \mathbb{F}_g , the condition that there is a non-zero element $u \in H^1(H, \mathbb{Z})^\Pi$ such that the cup product with u yields the zero map

$$H^1(\Pi, \mathbb{Z}) \rightarrow H^1(\Pi, H^1(H, \mathbb{Z})).$$

Observe first that $H^1(\Pi, \mathbb{Z}) = \text{Hom}_{\mathbb{Z}}(\Pi, \mathbb{Z}) \cong \mathbb{Z}^b$, where $b = g$ in the free case, otherwise $b = 2g$.

The condition that $\varphi u = 0$ in $H^1(\Pi, H^1(H, \mathbb{Z}))$ for each $\varphi \in \text{Hom}_{\mathbb{Z}}(\Pi, \mathbb{Z})$ means that there is an element $v_\varphi \in H^1(H, \mathbb{Z})$ such that

$$\varphi(\gamma)u = \gamma v_\varphi - v_\varphi \quad \text{for all } \gamma \in \Pi.$$

Taking a basis $\varphi_1, \dots, \varphi_b$, we get v_1, \dots, v_b such that

$$\gamma v_j = v_j + \varphi_j(\gamma)u \quad \text{for all } \gamma \in \Pi. \tag{1}$$

Recall moreover that u is invariant, whence

$$\gamma u = u \quad \text{for all } \gamma \in \Pi. \quad (2)$$

Conditions (1), (2) and the \mathbb{Z} -linear independence of the characters φ_j imply the \mathbb{Z} -linear independence of u, v_1, \dots, v_b since the \mathbb{Z} -module $H^1(H, \mathbb{Z})$ is torsion free.

Definition 7.4. A *bad monodromy module* is a free \mathbb{Z} -module of rank $b + 1$, with basis u, v_1, \dots, v_b , and with an action of Π given by (1) and (2).

Example 7.5. Let H be a finitely generated group, and let $1 \rightarrow H \rightarrow \Gamma \rightarrow \Pi_g \rightarrow 1$ be an exact sequence such that the induced action of Π_g on H by conjugation induces on the \mathbb{Z} -dual of the Abelianization of H a Π_g -module structure which contains a bad monodromy module. Then Γ cannot be the fundamental group of a compact Kähler manifold.

Remark 7.6. One can use the same type of restriction in the case where $U \neq X$ is the inverse image of the non-critical values of a fibration $f : X \rightarrow \bar{C}$, and obtain in this way a restriction for the monodromy in the case where $\bar{C} - B$ has first Betti number at least 2.

Remark 7.7. To see finally the relation of the above condition with the theory of Lefschetz pencils (in particular with the splitting in invariant and vanishing cycles), let us observe that our cup product is non-degenerate if we have a monodromy invariant splitting $H^1(H, \mathbb{Z}) = H^1(H, \mathbb{Z})^\Pi \oplus W$. Because then we may write $v_\varphi = u_\varphi + w_\varphi$ and we obtain $\varphi(\gamma)u = \gamma v_\varphi - v_\varphi = \gamma w_\varphi - w_\varphi \in W$ for each $\gamma \in \Pi$, whence $u = 0$. This splitting is proven by Deligne's Semisimplicity Theorem (4.2.6. of [14], cf. also thm. 3.1., page 37 of Chapter II of [20]).

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