Exponential Dichotomy and Smooth Invariant Center Manifolds for Semilinear Hyperbolic Systems

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Abstract

A "spectral gap mapping theorem", which characterizes exponential dichotomy, is proven for a general class of semilinear hyperbolic systems of PDEs in a Banach space X of continuous functions. This resolves a key problem on existence and smoothness of invariant manifolds for semilinear hyperbolic systems.

The system is of the following form: For $0 < x < l$ and $t > 0$

$$
\text{(SH)} \left\{ \begin{array}{l} \frac{\partial}{\partial t} \begin{pmatrix} u(t,x) \\ v(t,x) \\ w(t,x) \end{pmatrix} + K(x) \frac{\partial}{\partial x} \begin{pmatrix} u(t,x) \\ v(t,x) \\ w(t,x) \end{pmatrix} + H(x, u(t,x), v(t,x), w(t,x)) = 0, \\ \frac{d}{dt} \left[v(t,l) - Du(t,l) \right] = F(u(t,\cdot), v(t,\cdot)), \\ u(t,0) = E\, v(t,0), \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x),\ w(0,x) = w_0(x), \end{array} \right.
$$

where $u(t,x) \in \mathbb{R}^{n_1}$, $v(t,x) \in \mathbb{R}^{n_2}$ and $w(t,x) \in \mathbb{R}^{n_3}$, $K(x) = \operatorname{diag}\left(k_i(x)\right)_{i=1,\dots,n}$ is a diagonal matrix of functions $k_i \in C^1([0,l], \mathbb{R})$, $k_i(x) > 0$ for $i = 1, \dots, n_1$ and $k_i(x) < 0$ for $i = n_1 + 1, \dots n_1 + n_2$, $k_i \equiv 0$ for $i = n_1 + n_2 + 1, \dots, n_1 + n_2 + n_3 = n$, and D and E are matrices.

It is shown that weak solutions to (SH) form a smooth semiflow in X under natural conditions on H and F. For linearizations of (SH) high frequency estimates of spectra and resolvents in terms of reduced diagonal and blockdiagonal systems are given. Using these estimates and the theory [36, 42] of Kaashoek, Lunel and Latushkin a spectral gap mapping theorem for linearizations of (SH) in the "small" Banach space X is proven: An open spectral gap of the generator is mapped exponentially to an open spectral gap of the semigroup and vice versa. Hence, a phenomenon like in the counterexample [61] of Renardy cannot appear for linearizations of (SH). The results here differ to the work [48] of Lopes, Neves and Ribeiro in essential directions: First, the focus is on the "small" Banach space X (not L^p spaces), which is required for nonlinear problems like (SH). Second, degenerate and equal speed systems are considered needed for applications to laser dynamics. Existence of smooth center manifolds for (SH) is shown by applying the above results and general theory on persistence and smoothness of invariant manifolds, obtained by Bates, Lu and Zeng [7, 8], in the Banach space X.

The results are applied to traveling wave models of semiconductor laser dynamics. For such models mode approximations (ODE systems which approximately describe the dynamics on center manifolds) are derived and justified, and generic bifurcations of modulated waves from rotating waves are shown. Global existence and smooth dependence of nonautonomous traveling wave models with more general solutions, which possess jumps, are considered, and mode approximations are derived for such nonautonomous models. In particular the theory applies to stability and bifurcation analysis for Turing models with correlated random walk [33, 31]. Moreover, the class (SH) includes neutral and retarded functional differential equations.
Keywords:
Semilinear Hyperbolic Systems, Smooth Dependence on Data, Center Manifold Theorem, Linearized Stability, Linear Hyperbolic Systems, Estimates for Spectra and Resolvents, Exponential Dichotomy, Spectral Mapping Theorem, C_0 Semigroups, Laser Dynamics

Zusammenfassung

Es wird gezeigt, dass ein Satz über die Abbildung spektraler Lücken, welcher exponentielle Dichotomie charakterisiert, für eine allgemeine Klasse von semilinearen hyperbolischen Systemen von partiellen Differentialgleichungen in einem Banach-Raum X von stetigen Funktionen gilt. Dies beantwortet ein Schlüsselproblem für die Existenz und Glattheit invarianter Mannigfaltigkeiten semilinearer hyperbolischer Systeme. Das System besitzt die folgende Gestalt: Für $x \in]0, l[$ und $t > 0$ gelte

$$
(\mathrm{SH}) \begin{cases}
\frac{\partial}{\partial t} \begin{pmatrix} u(t,x) \\ v(t,x) \\ w(t,x) \end{pmatrix} + K(x) \frac{\partial}{\partial x} \begin{pmatrix} u(t,x) \\ v(t,x) \\ w(t,x) \end{pmatrix} + H(x, u(t,x), v(t,x), w(t,x)) = 0, \\
\frac{d}{dt} \left[v(t,l) - Du(t,l) \right] = F(u(t,\cdot), v(t,\cdot)), \\
u(t,0) = E\, v(t,0), \\
u(0,x) = u_0(x),\ v(0,x) = v_0(x),\ w(0,x) = w_0(x),
\end{cases}
$$

wobei $u(t,x) \in \mathbb{R}^{n_1}$, $v(t,x) \in \mathbb{R}^{n_2}$ und $w(t,x) \in \mathbb{R}^{n_3}$, $K(x) = \operatorname{diag}\,(k_i(x))_{i=1,\dots,n}$ ist eine Diagonalmatrix von Funktionen $k_i \in C^1([0,l], \mathbb{R})$, $k_i(x) > 0$ für $i = 1, \dots, n_1$ und $k_i(x) < 0$ für $i = n_1+1, \dots n_1+n_2$, $k_i \equiv 0$ für $i = n_1+n_2+1, \dots, n_1+n_2+n_3 = n$, D und E sind Matrizen. Unter natürlichen Annahmen an H und F wird gezeigt, dass schwache Lösungen von (SH) einen glatten Halbfluß im Raum X bilden. Für Linearisierungen von (SH) werden Abschätzungen für Spektren sowie Resolventen unter Verwendung von reduzierten diagonal und blockdiagonal Systemen hergestellt. Darauf aufbauend wird unter Verwendung der Theorie [36, 42] von Kaashoek, Lunel und Latushkin der Abbildungssatz für spektrale Lücken im "kleinen" Raum X bewiesen: Eine offene spektrale Lücke des Generators wird exponentiell auf eine offene spektrale Lücke der Halbruppe abgebildet und umgekehrt. Es folgt, dass ein Phänomen wie im Gegenbeispiel von Renardy [61] nicht auftreten kann. Die Ergebnisse unterscheiden sich von der Arbeit [48] von Lopes, Neves und Ribeiro in wesentlichen Punkten: Erstens liegt das Hauptaugenmerk beim "kleinen" Banach-Raum X (nicht bei L^p Räumen), welcher für nichtlineare Probleme wie (SH) benötigt wird. Zweitens werden Systeme betrachtet, die sowohl degeneriert als auch gleiche Geschwindigkeiten besitzen dürfen, was für Anwendungen in der Laserdynamik benötigt wird. Die Existenz von glatten Zentrumsmannigfaltigkeiten für (SH) wird gezeigt, indem die genannten Ergebnisse sowie die allgemeine Theorie [7, 8] von Bates, Lu und Zeng über die Persistenz und Glattheit invarianter Mannigfaltigkeiten im Rahmen des Banach Raumes X angewandt werden. Die Ergebnisse werden auf traveling wave Modelle für die Dynamik von Halbleiter Lasern angewandt. Für diese werden Moden Approximationen (Systeme von gewöhnlichen Differentialgleichungen, welche die Dynamik auf gewissen Zentrumsmannigfaltigkeiten approximativ beschreiben) hergeleitet und gerechtfertigt, die generische Bifurkation von modulierten Wellen aus rotierenden Wellen wird gezeigt. Globale Existenz und glatte Abhängigkeit von nichtautonomen traveling wave Modellen mit allgemeineren schwachen Lösungen, welche Sprünge beinhalten können, werden betrachtet, außerdem werden Moden Approximationen für solche nichtautonomen Modelle rigoros hergeleitet. Insbesondere arbeitet die Theorie für die Stabilitäts- und Bifurkationsanalyse von Turing Modellen mit korellierter Zufallsbewegung [33, 31]. Ferner beinhaltet die Klasse (SH) neutrale und retardierte funktionale Differentialgleichungen.

Schlagwörter:
Semilineare Hyperbolische Systeme, Glatte Abhängigkeit von den Daten, Zentrumsmannigfaltigkeiten, Linearisierte Stabilität, Lineare Hyperbolische Systeme, Abschätzungen von Spektrum und Resolvente, Exponentielle Dichotomie, Spektraler Abbildunssatz, C_0 Halbgruppen, Laserdynamik

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Chapter 1

Introduction

This work has been motivated by the investigation of so called traveling wave models, which have been used successfully in recent years to investigate the longitudinal dynamics of distributed feedback multisection semiconductor lasers, see for example [3, 21, 47, 76, 60, 5, 59, 50, 58, 57, 55, 66, 39, 67, 68, 9, 54]. Such lasers exhibit a very rich and complicated dynamics including bifurcations, selfpulsations, hysteresis, excitability, frequency synchronization etc., and so do the models also. One feature is their S^1 symmetry which implies the existence of rotating waves, also called on-states or relative equilibria of the laser. The properties of such stationary states, their stability, domain of attraction and bifurcations, are important from the applications viewpoint. Other objects of interest are high frequency self pulsations branching from the rotating waves via a S^1 equivariant Hopf bifurcation. Potential applications are high frequency signal generation and clock recovery in optical networks.

A lot of such dynamical behavior is described numerically, see e.g. [5, 9, 66, 79, 54] and figure 1.1, but only a few of these results are mathematically rigorously founded [50, 62, 65, 68, 69]. The reason is that for applying, for example, abstract dynamical systems theory, one needs a smooth Fréchet differentiable semiflow, existence and persistence of smooth invariant manifolds, that the linearized semigroup exhibits a spectrum (of the generator) determined exponential dichotomy or a spectral gap mapping property, etc. All these properties are in general well understood for ordinary differential equations, semilinear parabolic equations [28] and functional differential equations [27], but not for semilinear hyperbolic systems, even in one space dimension. Some of these have been verified within the hierarchy of traveling wave models, a hyperbolic system of partial differential equations (two coupled traveling wave equations describing the forward and backward propagating complex amplitudes of the light) coupled to a spatially extended ordinary differential equation (carrier rate equation), only in some exceptional cases. These exceptional cases require that the partial differential equations must be linear and are just nonlinearly coupled to ordinary differential equations, which one obtains from the general model by a Galerkin projection of the carrier rate equation (averaged densities) and neglecting of nonlinear terms in the PDE (due to nonlinear gain compression). Averaging of carrier densities neglects an important physical effect called longitudinal spacial hole burning [4, 16, 18, 19, 64].

The general traveling wave model, used in numerical simulations (e.g. by LDSL tool [54]), is a quite complicated (degenerated) semilinear hyperbolic system including discontinuous coefficients. The discontinuities are due to the heterostructure of the semiconductor laser device which is composed of several different laser sections with significant different electrical and optical features. The question arises if it is possible to state the above mathematical properties needed for a rigorous description of the dynamics in a suitable function space setting for general semilinear hyperbolic systems in one space dimension that appear in many applications including the traveling wave model, see the examples section.

In this work I show that general semilinear hyperbolic systems of partial differential equations in one space dimension can be viewed as smooth dynamical systems in suitable function spaces of continuous functions. I prove a spectral gap mapping theorem for these spaces which characterizes growth and exponential dichotomy in terms of the spectrum of the generator for the linearized system. This solves a key problem in the proof of the main theorems on linearized stability and existence of smooth exponentially attracting local center manifold for a general class of semilinear hyperbolic systems in 1d.

The latter allows to reduce the local dynamics on a lower, often finite dimensional (nonunique) attracting manifold. Thus one can theoretically justify reduced models and bifurcations on such center manifolds by investigating only the spectrum of the generator (the equations) of the linearization. I apply the results to the traveling wave model by calculating several center manifold reduced or mode approximation equations. My Theorems are formulated for a large class of semilinear hyperbolic systems and apply to many other models (including neutral and retarded functional differential equations, see section 3). I want to mention the hyperbolic Turing model: It follows that the stability analysis, performed by T. Hillen [31, 30] and W. Horstemke [33] in a purely linear context only, in fact implies stability and the occurrence of bifurcations on center manifolds near the homogeneous steady state of the nonlinear system.

Section 2 gives an overview of hyperbolic systems and frequently used symbols.

In section 7 I introduce the general form of autonomous semilinear hyperbolic systems. The systems can be degenerate and the generating functions of the nonlinear Nemytskij operators appearing in the PDE only need to depend measurably on the space and smoothly with respect to the unknown variables without growth conditions on the nonlinearities (hence the results will be local). In particular spatial dependent coefficients in the nonlinear operator are allowed to be discontinuous as it appears in the traveling wave model when written in compact form. It turns out that smoothness assumptions of the nonlinearity with respect to the space variable are not needed and in fact do not simplify the setting: Even if the generating function is arbitrary smooth in all variables including space (or even if it is a most simple constant coefficient linear operator) the Nemytskij operator will not be compatible with boundary conditions of the system and hence map into a larger function space that does not satisfy boundary conditions.

I prove that the solution map is Fréchet differentiable in the chosen space of continuous functions (with derivative generated by the solutions of the formally linearized system). Hence the equations generate a smooth semiflow. This smooth well posedness goes back to my very first work in Project D8 of the DFG Research Center MATHEON when I started to analyse the (general) traveling wave model with the aim to obtain mode reduced equations in the nonautonomous case when the laser is subject to some external optical forcing or injection (see section 11). During that time only mode reductions were known in the autonomous case. One of my first results regarding the nonautonomous traveling wave equations are contained in section 10 where I show that weak solutions depend smoothly on the data in the L^∞ sense. The weak solutions considered there may possess jumps, and discontinuous forcings appearing in the boundary conditions are allowed. Considering such general solutions in L^∞ space has several drawbacks. I only mention that the solution map will not be measurable (in the sense of Bochner) with values in the Banach space L^∞. Hence, because my work focuses on the dynamics (invariant manifolds) I have decided in all other chapters[1] to consider continuous solutions, which satisfy the boundary conditions pointwise. In C space including boundary conditions the solutions form a C_0 (in time) and smooth (with respect to state space) dynamical system or smooth semiflow.

In sections 7 and 10 the following technical difficulty appears: Nemyt-

[1]for the failure of Bochner measurability in L^∞ see remark 7.4

skij operators, mapping "large" function spaces (for example the L^p spaces with $1 \leq p < \infty$) into itself, are continuously differentiable if and only if they are affine, even if the generating functions are arbitrarily smooth (see e.g. [40]). Hence one cannot expect that the weak solutions create a smooth dynamical system (smooth semiflow or process) on such "large" spaces. On the other hand, there are at least three reasons preventing a setting in "too small" function spaces: First, the elements of "small" function spaces have to satisfy certain (homogeneous) boundary conditions, but the Nemytskij operators usually do not respect boundary conditions and therefore map into a larger space. Second, we deal with hyperbolic PDEs, which do not possess a smoothing property, in general. And third, if the coefficients are discontinuous, then the Nemytskij operators don't take values in "small" function spaces.

In sections 4 and 6 I discuss properties and estimates for spectra and resolvents for linearized hyperbolic systems. One interesting phenomenon appearing with linearized hyperbolic partial differential equations is that the spectral mapping property must not hold. In fact there is a remarkable counterexample found by M. Renardy [61], a lower order derivative perturbation of a two dimensional wave equation with periodic boundary condition, where it happens that growth and spectral bound are different. Hence one sees that for hyperbolic PDEs just the knowledge on the location of the spectrum extracted from the equations does not give the expected information on exponential rates, stability or dichotomy of the linearized system, in general. One has to be extremely careful when one wants to understand stability and bifurcations of hyperbolic PDEs by just looking at the location of the spectrum of the equations (the generator) of the linearization.

For general linearized 1d hyperbolic systems I prove spectral gap mapping Theorems 5.4, 6.7 and 6.16 in the "small" Banach space of continuous function. This implies that growth and spectral bound coincide and proves the presence of an exponential dichotomy under a common spectral gap condition (on the generator), which is a crucial part in the proof on local existence of smooth center manifolds for the semilinear problem. I prove the spectral gap mapping Theorem 5.4 in section 5.2 by using my resolvent estimates, which are obtained in sections 4.2 for nondegenerate and in section 6.2 for more general degenerate hyperbolic systems (allowed to contain identical speeds), see Lemmas 4.16 and 6.14, and checking the conditions of the theory of Kaashoek, Lunel and Latushkin [36, 42], which is based on the Laplace inversion formula for the resolvent and explained in section 5.1.

My results differ to the work [48] of Lopes, Neves and Ribeiro in essential directions: First, I focus on a smaller Banach space X (not only L^p space) which is required to prove stability or the existence of smooth center manifolds for nonlinear hyperbolic systems. Second, degenerate and equal speed systems are considered (by using the more general concept of "blockdiagonal" reduction) needed for applications to laser dynamics.

By considering applications to laser dynamics the following problem appeared: A condition for the vanishing of couplings plays an important role for the resolvent estimates under the presence of equal speed and cannot be removed by sticking to the notion of reduced diagonal system (this condition was also important in the theory [48]). This condition is violated for the traveling wave model (although when written in complex form it seems that the model has different speed this is not true because we must consider it as a real and not complex system of equations, the nonlinearities are of course only real differentiable, then the realified and linearized system has to be complexified, see section 12). Therefore, in section 6.1 I have relaxed this assumption. The idea here is that in the presence of identical speed coupling becomes important and hence one has to modify the notion of reduced linear hyperbolic system which will not be diagonal anymore.

Hence, in "small" C space the solution maps of hyperbolic systems are not only Fréchet differentiable, but a spectral gap mapping theorem holds for the linearized system so that the space can be spectrally decomposed into invariant subspaces with exponential rates given by the location of the spectrum. This allows to apply general results on invariant manifold theory, see the important work of P. W. Bates, K. Lu and C. Zeng [8, 7], which I have summarized in section 8.1 (compare also with the articles "Center Manifold Theory in Infinite Dimensions" by A. Vanderbauwhede and G. Iooss in [75] and "Invariant Manifolds for Semilinear Partial Differential Equations" by P. W. Bates and C. K. R. T. Jones in [6]). It follows that center manifolds persist when one detects a spectral gap near the imaginary axis for the equations (generator) of the linearized system in a neighbourhood of a stationary state which can be easily done in practical applications (see section 4, 6.2). Therefore, as usual for ordinary differential equations, semilinear parabolic equations and functional differential equations, by just locating the spectrum one can perform a linearized stability analysis, have the existence of center manifolds, calculate reduced equations on such manifolds. Here I could not use the results of Lopes, Neves and Ribeiro for my applications to nonlinear problems mainly because they excluded the case $p = \infty$, their results were in the context of large L^p spaces, $1 \leq p < \infty$, probably not having nonlinear problems in mind[2]. The problem appearing again is that L^p-space

[2]in the work of [50, 65, 68, 69] it was possible to work in a large L^p (with $p = 2$) space

for $1 \leq p < \infty$ is too large and does not have the Algebra property (multiplication property with compatible norms). Hence (nonlinear) Nemytskij operators are not Fréchet differentiable as a map of L^p, $p < \infty$, into itself. In the limit $p = \infty$ the Nemytskij operators become smooth as a map from L^∞ to L^∞ [24]. By using the variation of constants formula it follows that the nonlinear problem is a small perturbation (in the L^∞ or C space sense) of the linear problem. This is needed in the proofs for the main Theorems on linearized stability and existence of center manifolds (Theorems 7.26, 8.15, 11.1 and 12.2).

The results of sections 5 and 6.2 are needed to prove the spectrum determined linearized stability Theorem 7.26 and the existence of smooth center manifolds in section 8 (Theorem 8.15) and sections 9, 11, 12. Since sections 5 and 6.2 yield the spectrum determined stability of the linearized semigroup in C space, and section 7 has shown that the solution map of the semilinear problem is continuously Fréchet differentiable, Theorem 7.26 follows by a standard argument. To prove the latter assertion on the existence of center manifolds using invariant manifold theory the method is roughly speaking the following [8, 7]: Starting from an exponentially attracting manifold (found by linearizing around a stationary state or setting a small parameter in the traveling wave equation to zero) one uses a geometric persistence argument based on Hadamard's graph transform to show that this manifold persists. For this persistence argument one needs to check certain exponential rates along and "normal" to the manifold implying the so called normal hyperbolicity of the invariant manifold. It is known that this normal hyperbolicity condition is not only sufficient but also necessary for the manifold to persist [46]. For a stationary state and the linear manifolds given by spectral projection normal hyperbolicity is equivalent to exponential dichotomy or the presence of a spectral gap near a circle of the linear semigroup, see Theorem 5.17. Moreover, for smoothness of the invariant manifold one needs to estimate the size of the spectral gap of the semigroup. However, the best one can do is detect a spectral gap only for the generator (the equations) and not for the semigroup which cannot be calculated analytically, in general. Here one applies my spectral gap mapping Theorems of sections 5 and 6.2, see the Theorems 5.4, 5.7, 5.5, 6.15 and 6.16, where I show that the presence of a spectral gap of the equations/generator implies a exponentially related spectral gap for the semigroup. In the first part of section 8 I recall in de-

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and use the results of Lopes, Neves and Ribeiro [48] because the model equations had an exceptional structure where the PDE was linear (neglecting for example nonlinear gain saturation effects) and only nonlinearly coupled to an ODE tail the notion of normal hyperbolicity and the required general persistence theorems for overflowing invariant manifolds for semiflows in Banach spaces obtained in [8].

In section 12 I explain how generically self pulsations are born via a Hopf bifurcation for the S^1 equivariant traveling wave equation under the assumption that a complex conjugated pair of eigenvalues crosses the imaginary axis transversally (of course this can be easily formulated for more general semilinear hyperbolic systems). Using the dichotomy / spectral gap mapping results of sections 5 and 6.2 it follows that for parameters near the bifurcation point there exists a three dimensional, in C space exponentially attracting center manifold. By using a suitable rotating coordinate frame and calculating the ODE on this center manifold one sees that the equations decouple and one can apply the standard Hopf theorem to obtain the equations for the self pulsations.

In section 9 I calculate a center manifold reduction for the autonomous general traveling wave equation exploiting slow fast structure. In section 11 I show that it is possible to perform the reduction also in the nonautonomous case. There I derive new mode-reduced nonautonomous equations which approximate the flow on the center manifold. These mode approximations extend the autonomous ones which have been used recently for bifurcation analysis using the path following software AUTO, see [66, 54]. The extension to the nonautonomous case is still explicit and simple enough that now it is possible to perform a numerical bifurcation analysis for applications when the laser is subject to external optical forcing (for example locking of selfpulsations [59]). The basic idea here to perform the center manifold reduction in the nonautonomous setting is to find a suitable boundary homogenization, which preserves the slow fast structure of the traveling wave equations, then make the system autonomous by adding a artificial time variable and prove for the resulting skew product semiflow the existence of a center manifold, which is similarly done as in section 9 for the nonautonomous model.

Finally, I have added an Appendix section 13. In subsection 13.1 I mention the Fejér Laplace and Fourier inversion formulas which are frequently used in the proof of Lemma 5.27 and Lemma 5.28 of section 5 where the growth rate for the linear system is calculated. In subsection 13.2 I note a well known regularity result for linear inhomogeneous evolution equations which is used in sections 7 and 10 and in the proof of Theorem 8.15.

In this work I have proposed to use C space as the phase space for a geometric dynamical systems approach in the context of semilinear hyperbolic system and I give a brief philosophical discussion. Of course there is no universal rule for selecting a suitable function space and the question arises if one could select a different setting allowing for a more simple or elegant treatment. From a mathematical point of view one is driven to select the space in such a way that certain good properties are available which are needed to prove the theorems one has in mind. In my case I sought a space that, first, the solution map becomes Fréchet differentiable, i.e. a space which is small enough but on the other hand large enough to allow general nonlinear Nemytskij operators which must not be compatible with boundary conditions, being the case in applications, and, second, the linearized system has a a spectral gap mapping property (or spectrum determined exponential dichotomy), needed to prove the main Theorems on linearized stability and existence of center manifolds, see Theorems 7.26, 8.15, 11.1 and 12.2. From this point of view and by noticing the resemblance to the well developed geometric theory of functional differential equations (see the examples in section 3.3), influenced by the work of Lunel and Hale [27], choosing C space for hyperbolic systems in one space dimension appears natural. I believe this choice to be close to optimal (even for Nemytskij operators generated by arbitrary smooth functions) because I do not put any compatibility restrictions to the Nemytskij operators with the boundary conditions.

I want to note that in related works for hyperbolic equations I know of in the literature the authors have considered very exceptional Nemytskij operators which are compatible with the boundary conditions so that they map a small space into itself: Renardy [62] and Haken/Renardy [26] considered not edge emitting, but ring lasers. There the spatial domain is not an interval, but a circle, and the Nemytskij operators map the "small" space of continuously differentiable functions on the circle into itself. Similarly Illner/Reed [34] and Vanderbauwhede/Iooss [75, Section 4, Example 3] considered exponential decay and center manifolds, respectively, for semilinear hyperbolic initial boundary value problems (not related to laser dynamics), where the nonlinearities are compatible with the boundary conditions. Very recently R. Racke and E. M. Rivera [63] proved exponential stability for a nonlinear wave equation with nondissipative damping. I roughly recall their procedure to point to the crucial technical difficulties of boundary compatibility: First they prove the exponential stability of the linearized system after bringing it to a first order hyperbolic system (of the type (H), see section 4) and then estimating the spectral bound in terms of the damping coefficient by using a fixed point argument. By applying the Gearhart, Herbst, Prss, Greiner Theorem they get exponential decay in the L^2 Hilbert space. By recursion they get exponential decay in $W^{k,2}$ space. Then they use the variation of constants formula to estimate the solution of the nonlinear equation in terms of the linear semigroup. Here they put a crucial assumption on the nonlinearity which guarantees that the nonlinear operator appearing in the variation of constants formula is compatible with the boundary data, so that they are allowed to put the H^2 norm under the integral [63, page 24, equation (3.24)].

Throughout my work I do not put any compatibility restrictions on the Nemytskij operator because this is usually required in applications unless one considers exceptional cases only. As a consequence I work in a large but still small enough C space including boundary conditions. The fact that C is only a Banach space and not a Hilbert space has put some significant length in the proof of the exponential dichotomy of the linear semigroup, the calculations in section 5.2 are due to this.

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Mark Lichtner

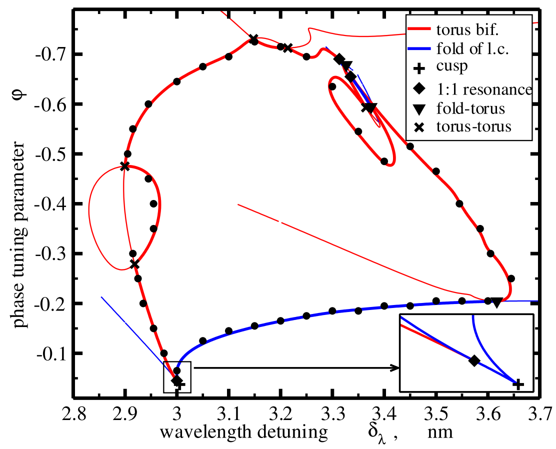

Figure 1.1: A two parameter numerical bifurcation analysis of a four mode approximation of the traveling wave model calculated with LDSL tool [54].

Chapter 2

An overview of hyperbolic systems and frequently used symbols

Let $n_1, n_2, n_3, n \in \mathbb{N}$ be natural numbers such that $n = n_1 + n_2 + n_3 > 0$ (each n_i is allowed to vanish).

The symbol (SH) will denote the following class of (degenerate) semilinear hyperbolic system with initial and boundary value conditions in normal form

$$\text{(SH)} \qquad \begin{cases} \dfrac{\partial}{\partial t}\begin{pmatrix} u(t,x) \\ v(t,x) \\ w(t,x) \end{pmatrix} + K(x)\dfrac{\partial}{\partial x}\begin{pmatrix} u(t,x) \\ v(t,x) \\ w(t,x) \end{pmatrix} + H(x, u(t,x), v(t,x), w(t,x)) = 0, \\ \dfrac{d}{dt}\left[v(t,l) - Du(t,l)\right] = F(u(t,\cdot), v(t,\cdot)), \\ u(t,0) = E\, v(t,0), \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x),\ w(0,x) = w_0(x). \end{cases}$$

Here $x \in\,]0,l[$, $l > 0$, and $t > 0$. The unknowns u, v, w are vectors of the following dimensions:

$$\begin{aligned} u(t,x) =&(u_1(t,x), \ldots, u_{n_1}(t,x)) \in \mathbb{R}^{n_1}, \\ v(t,x) =&(v_1(t,x), \ldots, v_{n_2}(t,x)) \in \mathbb{R}^{n_2}, \\ w(t,x) =&(w_1(t,x), \ldots, w_{n_3}(t,x)) \in \mathbb{R}^{n_3}. \end{aligned}$$

The symbol $K(x)$ denotes a real square $n \times n$ matrix,

$$K(x) = \operatorname{diag}(k_i(x))_{1 \le i \le n}, \tag{2.1}$$

where for $x \in [0,l]$

$$\begin{aligned} k_i(x) >& 0 \quad \text{for } i = 1, \dots, n_1 \\ k_i(x) <& 0 \quad \text{for } i = n_1 + 1, \dots n_1 + n_2, \\ k_i \equiv& 0 \quad \text{for } i = n_1 + n_2 + 1, \dots, n. \end{aligned}$$

We need that the nonlinearity H is smooth with respect to u,v,w and depends measurably on the space $x \in [0,l]$ satisfying a usual Carathéodory condition. The possibly nonlocal operator F is supposed to be a smooth map from $C([0,l], \mathbb{R}^n)$ into $\mathbb{R}^{n_2}$. The matrices E and D are of appropriate dimension. Initial data is denoted by u_0, v_0, w_0. The precise assumptions will be listed later in the text using Roman capital letters.

When formally linearizing (SH) we arrive to a degenerate linear hyperbolic system we denote with (DH)

$$\text{(DH)} \quad \begin{cases} \dfrac{\partial}{\partial t} \begin{pmatrix} u(t,x) \\ v(t,x) \\ w(t,x) \end{pmatrix} + K(x) \dfrac{\partial}{\partial x} \begin{pmatrix} u(t,x) \\ v(t,x) \\ w(t,x) \end{pmatrix} + C(x) \begin{pmatrix} u(t,x) \\ v(t,x) \\ w(t,x) \end{pmatrix} = 0, \\ \dfrac{d}{dt} [v(t,l) - Du(t,l)] = Fu(t,\cdot) + Gv(t,\cdot), \\ u(t,0) = Ev(t,0), \\ u(0,x) = u_0(x), \ v(0,x) = v_0(x), \ w(0,x) = w_0(x). \end{cases}$$

Here $C(x) = (c_{ij}(x))_{1 \le i,j \le n}$ is a square $n \times n$ matrix and F and G are linear operators.

If $n_3 = 0$ we have a nondegenerate linear hyperbolic system we denote with (H)

$$\text{(H)} \quad \begin{cases} \dfrac{\partial}{\partial t} \begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + K(x) \dfrac{\partial}{\partial x} \begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + C(x) \begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = 0, \\ \dfrac{d}{dt} [v(t,l) - Du(t,l)] = Fu(t,\cdot) + Gv(t,\cdot), \\ u(t,0) = Ev(t,0), \\ u(0,x) = u_0(x), \ v(0,x) = v_0(x), \ w(0,x) = w_0(x). \end{cases}$$

Corresponding to (DH) and (H) we consider different reduced systems which we explain now:
If the system has different speed everywhere, that is

$$\text{for } x \in [0,l] \text{ and } 1 \le i,j \le n_1 + n_2 \text{ with } i \ne j \text{ we have } k_i(x) \ne k_j(x), \quad (2.2)$$

then the reduced system is by definition obtained by first deleting the non-diagonal entries in C, then cancel the w equation (if it is present) and going over to static boundary conditions $u(t,0) = Ev(t,0)$ and $v(t,l) = Du(t,l)$. We denote this reduced system by (H_0):

$$(\mathrm{H}_0) \quad \begin{cases} \dfrac{\partial}{\partial t}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + K_0(x)\dfrac{\partial}{\partial x}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + C_0(x)\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = 0, \\ u(t,0) = Ev(t,0) \quad \text{and} \quad v(t,l) = Du(t,l). \end{cases}$$

where

$$C_0(x) := \operatorname{diag}(c_{11}(x), \dots, c_{n_1+n_2,n_1+n_2}(x))$$

is the diagonal part of $C(x)$ and

$$K_0(x) := \operatorname{diag}(k_1(x), \dots, k_{n_1+n_2}(x)). \tag{2.3}$$

(If $n_3 = 0$ then $K_0 = K$)
Assume that C satisfies the following property:

$$\begin{aligned} &\text{if } i \neq j,\ 1 \leq i,j \leq n_1+n_2 \text{ and } k_i(x) = k_j(x) \text{ for some x } \in [0,l] \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{then } c_{ij} \text{ vanishes on } [0,l]. \end{aligned} \tag{2.4}$$

Under assumption (2.4) the spectral properties of the full system can still be described in terms of the reduced diagonal system. Hence we define the reduced system to be the diagonal system (H_0).

However, if (2.4) and (2.2) are violated then one can show that the diagonal system (H_0) is a wrong choice for the reduced system, e.g. see the example [45, Example 6.8, p.326] which shows that the difference of the semigroups of the full system and the diagonal system is not compact anymore. Hence the essential spectrum of the semigroup of the full system can not be detected by the semigroup of the reduced diagonal system anymore. We will show in section 6.1 that if K is of the form (2.5) (containing identical entries, so that (2.1)+(2.2) and (2.1)+(2.4) are violated) one can still express the spectrum of both the generator and the semigroup in terms of spectral properties of a reduced block diagonal system. We now explain how we have to define the reduced block diagonal system:
Suppose

$$K = \begin{pmatrix} k_1 I_{d_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 I_{d_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_\alpha I_{d_\alpha} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_{\alpha+1} I_{d_{\alpha+1}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & k_{\alpha+\beta} I_{d_{\alpha+\beta}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \cdot I_{n_3} \end{pmatrix}, \tag{2.5}$$

where $d_i \in \mathbb{N}$, $d_i > 0$, $\alpha \in \mathbb{N}$, $\beta \in \mathbb{N}$, $\sum_{i=1}^{\alpha} d_i = n_1$, $\sum_{i=1}^{\beta} d_{\alpha+i} = n_2$, I_{d_i} denotes the identity matrix in $\mathbb{R}^{d_i \times d_i}$ and k_i satisfies (2.2) for $1 \leq i, j \leq \alpha+\beta$. Then write

$$C(x) =: (C_{ij}(x))_{1 \leq i,j \leq \alpha+\beta}, \quad C_{ij}(x) \in \mathbb{C}^{d_i \times d_j}$$

and define C_{b0} to be the block diagonal matrix containing the square matrices C_{ii} on its diagonal

$$C_{b0} := \text{blockdiag } (C_{ii})_{1 \leq i \leq \alpha+\beta}.$$

Then the reduced system, denoted again with the symbol (H_0), is per definitionem

$$(\mathrm{H}_0) \quad \begin{cases} \dfrac{\partial}{\partial t}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + K_0(x)\dfrac{\partial}{\partial x}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + C_{b0}(x)\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = 0, \\ u(t,0) = Ev(t,0), \qquad v(t,l) = Du(t,l), \\ u(0,x) = u_0(x), \ v(0,x) = v_0(x). \end{cases}$$

If K satisfies (2.5) but (2.2) is violated for $1 \leq i, j \leq \alpha + \beta$, $i \neq j$, we need to assume a condition analogous to (2.4):

$$\text{If } i \neq j,\ 1 \leq i, j \leq \alpha+\beta \text{ and } k_i(x) = k_j(x) \text{ for some x } \in [0,l] \\ \text{then } C_{ij} \text{ vanishes on } [0,l].$$

Remark 2.1. *The reduced system in blockdiagonal form becomes diagonal if $d_i = 1$, also the required assumptions on the coefficients become identical then. Hence the reduced blockdiagonal is a analogous generalization of the reduced diagonal system. This justifies that we use the same symbol* (H_0) *for different reduced systems. From the context it will be clear which system we mean.*

Remark 2.2. *Systems with identical speed are not just of academic interest. Important examples from applications are traveling wave models for the dynamics of semiconductor lasers. At a first glance (see the model equation* (3.10)*) it seems that condition* (2.2) *is satisfied. But in fact it is not: The linearized system is obtained in real space (since the nonlinearities are only real differentiable and not analytic). By realifying the complex part of the equations one sees that the system is of the above type containing identical speeds with* (2.1)+(2.2) *and* (2.1)+(2.4) *violated. But the realified and linearized system is of the form* (2.5)+(2.2) *(with $d_1 = d_2 = 2$). Hence the essential spectrum of the traveling wave model can not be described by the diagonal system, but by the reduced system in block diagonal form.*

$\mathbb{N}$	set of natural numbers including zero
$\mathbb{K}$	field of real $\mathbb{R}$ or complex $\mathbb{C}$ numbers
K_0	see (2.3)
$\mathfrak{Re}, \mathfrak{Im}$	real, imaginary part of a complex number
Im	Image of a linear operator
I, I_n	identity operator or matrix (n^2 matrix size)
$h_0(\lambda)$	characteristic function to (H_0), see (4.7)
$h(\lambda)$	characteristic function (for (H) or (DH)), see Def. 4.2
$H(\lambda)$	see (4.6)
Σ	residual or continuous spectrum due to degeneracies, see (6.5)
$\beta(\lambda), \beta_0(\lambda)$	see Proposition 4.3
γ_+	right spectral bound of the reduced system (H_0): $\gamma_+ := \sup\{\mathfrak{Re}\,\lambda \mid h_0(\lambda) = 0\}$, if (H_0) has empty spectrum then by definition $\gamma_+ := -\infty$
∂	derivative symbol
l	length of the spatial interval $[0, l]$
$L^p(]0, l[\,; \mathbb{K}^n)$	Banach space of equivalence classes of measurable functions $f :]0, l[\to \mathbb{K}^n$ such that $\int_0^l \|f(x)\|^p\, dx < \infty$
$C([0, l], \mathbb{K}^n)$	space of continuous functions on $[0, l]$ with values in $\mathbb{K}^n$
$W^{1,p}(]0, l[\,, \mathbb{K}^n)$	Sobolev space of functions $f \in L^p$ with $\partial f \in L^p$.
X_p	$X_p := L^p(]0, l[\,; \mathbb{K}^n) \times \mathbb{K}^{n_2},\ 1 \le p \le \infty$
Y	Phase space for (SH), $Y := \{(u, v, w, d) \in C([0, l]; \mathbb{K}^n) \times \mathbb{K}^{n_2} \mid$ $\quad u(0) = Ev(0),\ d = v(l) - Du(l)\}$ (if $n_3 = 0$, then $Y = \{(u, v, d) \in C([0, l]; \mathbb{K}^n) \times \mathbb{K}^{n_2} \mid$ $\quad u(0) = Ev(0),\ d = v(l) - Du(l)\})$
X	stands for the space Y or X_p, $1 \le p < \infty$, only in sections 5.1 and 8.1 X denotes a general Banach space
A	denotes a closed densely defined operator in X corresponding to the class of linear hyperbolic systems (H) or (DH), only in section 5.1 A denotes an arbitrary generator of a C_0 semigroup on a general Banach space X
σ, $\sigma(A)$	spectrum of A
ρ, $\rho(A)$	$\rho = \mathbb{C} \setminus \sigma$, resolvent set of A
$\mathfrak{P}$	see (10.10)

$\mathbb{C}_r$	$\mathbb{C}_r := \{\lambda \in \mathbb{C} \mid \lvert\mathfrak{Re}\lambda\rvert < r\}$
$\mathbb{C}_{\alpha,\beta}$	$\mathbb{C}_{\alpha,\beta} := \{\lambda \in \mathbb{C} \mid \alpha < \mathfrak{Re}\lambda < \beta\}$
$\overline{\mathbb{C}_{\alpha,\beta}}$	closure of $\mathbb{C}_{\alpha,\beta}$, i.e. $\overline{\mathbb{C}_{\alpha,\beta}} = \{\lambda \in \mathbb{C} \mid \alpha \leq \mathfrak{Re}\lambda \leq \beta\}$
$\mathcal{X}_T$	$\mathcal{X}_T := C([0,T],Y)$
Π	Projection in extended phase space by dropping the "right sided boundary component". If $(u,v,w) :]0,l[\to \mathbb{K}^{n_1} \times \mathbb{K}^{n_2} \times \mathbb{K}^{n_3}$ and $d \in \mathbb{K}^{n_2}$, then $\Pi(u,v,w,d) := (u,v,w)$ or $\Pi(u,v,d) := (u,v)$.
Δ	right sided boundary operator, if $(u,v) : [0,l] \to \mathbb{K}^{n_1+n_2}$, then $\Delta(u,v) := v(l) - Du(l)$.
$\mathfrak{F}$	Fourier transform, $(\mathfrak{F}\, g)\,(w) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iw\nu} g(\nu) d\nu$.
$(C,1) - \int_{-\infty}^{\infty}$	Integration by Cesaro means of order 1, see Def. 13.1.
$BV([0,l],\mathbb{C}^{d_i\times d_j})$	Space of matrix valued functions of bounded variation.

Chapter 3

Examples of semilinear hyperbolic systems

The semilinear hyperbolic systems of type (SH) studied in this work are very general and appear in many different applications. A large subclass of equations which can be considered as linear hyperbolic systems with nonlinear dynamic boundary conditions are retarded and neutral functional differential equations, see section 3.3, which are generalizations of ordinary differential equations with delay.

The first two examples we give are two interesting semilinear hyperbolic systems for which the whole theory of this work applies immediately. The stability and bifurcation scenarios of these two examples have been intensively studied in the applied literature [9, 66, 79, 79, 35, 39, 55, 33, 31]. The first example is a generalization of the classical Turing model, where the diffusion process or Brownian motion is replaced by a correlated random walk. This yields a more realistic model for reacting moving particles, whose mean free path length is not small, which occurs in mathematical biology. The second example is the traveling wave model for the longitudinal dynamics of semiconductor lasers which we already mentioned in the introduction.

According to chapter 6 of [45] hyperbolic systems often appear in counterflow heat exchanger processes, gas absorber processes, tubular reactor processes, connected vibrating strings and many other applications. Other examples of linearized hyperbolic system can be found in the work of Lopes, Neves and Ribeiro [48].

Closely related to first order hyperbolic systems are second order wave equations when brought to first order form, see e.g. (3.7) or [63] for another wave equation.

3.1 Turing model with correlated random walk

Reaction diffusion equations

$$\partial_t \rho = D\partial_{xx}\rho + f(\rho) \tag{3.1}$$

model the interaction of particles in space, where

$$\rho(t,x) = (\rho_1(t,x), \ldots, \rho_n(t,x))$$

is a vector of densities of n types of particles. The reaction is described by the ODE $\partial_t \rho = f(\rho)$, where $f = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ is smooth. Spatial spread is modeled by the diffusion equation, Brownian motion, $\partial_t \rho = D\triangle u$. The diffusion matrix D is diagonal with non negative diffusion coefficients $d_j \geq 0$, $1 \leq j \leq n$. System (3.1) is based on the assumption that the particle number is large and the mean free path length is small. A feature of (3.1) which is often criticized is that the speed of the particles can be arbitrarily large. The reason of this pathology is that for Brownian motion the direction of motion in successive time intervals is uncorrelated. This model may be appropriate for chemical reactions, but when modeling biological populations of microorganisms or bacteria, where the particle radius becomes larger, the assumption of finite speed of the particles is more realistic, see [33] and the references there. Hence other models of motion such as correlated random walks, which can be considered as a generalization of Brownian motion, are studied. Correlated random walks for species in one space dimension yield hyperbolic systems. They have been studied by Kac [37], Goldstein [23] and more recently by Hillen [31, 30] and Horsthemke [33] among many others. Kac found an equivalence with the telegraph equation. One assumes that particles with density ρ_j have constant speed γ_j and constant turning rate μ_j. One splits each particle density $\rho_j = u_j + v_j$ into particle densities u_j for right and v_j for left moving particles. Then one arrives to the following reaction random walk system [25, 31, 30, 33] on the interval $]0, l[$

$$\begin{aligned}
\partial_t u_j + \gamma_j \partial_x u_j =& \mu_j(v_j - u_j) + \frac{1}{2} f_j(u_1 + v_1, \ldots, u_n + v_n) \qquad (3.2)\\
\partial_t v_j - \gamma_j \partial_x v_j =& \mu_j(u_j - v_j) + \frac{1}{2} f_j(u_1 + v_1, \ldots, u_n + v_n).
\end{aligned}$$

If one introduces diagonal matrices

$$\Gamma := \mathrm{diag}\,(\gamma_1, \ldots, \gamma_n), \quad M := \mathrm{diag}\,(\mu_1, \ldots, \mu_n)$$

and vectors of the state variables $u := (u_1, \dots, u_n)$, $v := (v_1, \dots, v_n)$ then one gets the following compact notation for (3.2)

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \Gamma & 0 \\ 0 & -\Gamma \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} M & -M \\ -M & M \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} - \frac{1}{2}\begin{pmatrix} f(u+v) \\ f(u+v) \end{pmatrix} = 0. \tag{3.3}$$

Neumann boundary conditions are

$$u(t,0) = v(t,0), \qquad v(t,l) = u(t,l). \tag{3.4}$$

Other boundary conditions of Dirichlet, periodic or dynamic type are also used. Introducing the variables $\rho := u + v$ and $\sigma := u - v$ (σ describes the net particle flow of each species) system (3.3) with (3.4) can be equivalently written

$$\begin{aligned} \partial_t \rho + \Gamma \partial_x \sigma &= f(\rho), \\ \partial_t \sigma + \Gamma \partial_x \rho &= -2M\sigma \end{aligned} \tag{3.5}$$

with boundary conditions

$$\sigma(t,0) = \sigma(t,l) = 0. \tag{3.6}$$

Assume (ρ, σ) are two times continuous differentiable. If one differentiates the first equation of (3.5) with respect to t and the second equation of (3.5) with respect to x, eliminates the derivatives σ_{xt} and σ_x, then one gets the following reaction telegraph system

$$\rho_{tt} + (2M - Df(\rho))\rho_t - \Gamma^2 \rho_{xx} - 2Mf(\rho) = 0. \tag{3.7}$$

The Neumann boundary conditions (3.4) transform as follows. Because $\sigma(t,0) = 0$ and $\partial_t \sigma(t,0) = 0$ one gets from the second equation of (3.5) that $\partial_x \rho(t,0) = 0$, and similarly for $x = l$. Hence the transformed boundary conditions are

$$\partial_x \rho(t,0) = \partial_x \rho(t,l) = 0. \tag{3.8}$$

More precisely the relation of solutions of (3.7) and (3.5) is as follows (see [30, Satz 2.21-2.22]):

i) If (ρ, σ) is a (classical) solution of (3.5) then ρ is a (weak) solution of (3.7).
ii) If ρ is a (weak) solution of (3.7) then there exists a one parameter family $(\sigma_c)_{c \in \mathbb{R}}$ of functions so that (u, σ_c) is a (classical) solution of (3.5).
iii) If (ρ, σ) solves (3.5) with Neumann boundary condition (3.6) then ρ is a (weak) solution of (3.7) with boundary condition (3.8).
iv) If ρ is a (weak) solution of (3.7) with (3.8) then there exists a function σ such that (ρ, σ) is a (classical) solution of (3.5), (3.6) if and only

if the following compatibility condition for the initial data $\rho(0,\cdot)$ holds: $\int_0^l (f(\rho(0,x)) - \rho_t(0,x))\,dx = 0$.
Multiplying (3.7) by $(2M)^{-1}$ yields

$$(2M)^{-1}\rho_{tt} + (I - (2M)^{-1}Df(\rho))\rho_t - (2M)^{-1}\Gamma^2\rho_{xx} - f(\rho) = 0.$$

Hence, if one considers the formal limit

$$\gamma_j, \mu_j \to \infty \quad \text{such that} \quad \frac{\gamma_j^2}{2\mu_j} \to d_j,$$

then one obtains the reaction diffusion equation (3.1).

Hillen [31] studied the Turing instability for the two species reaction walk system (3.2) and found that via the identification $d_j = \frac{\gamma_j}{2\mu_j}$ the stability properties of (3.2) are identical to those of the reaction diffusion system (3.1) if $2\mu_1 > \partial_{\rho_1} f_1(\overline{\rho})$, where $\overline{\rho}$ denotes a homogeneous steady state, i.e. $f(\overline{\rho}) = 0$. He also found that loss of linearized stability, which does not appear for the reaction diffusion equation, may occur when $2\mu_1 > \partial_{\rho_1} f_1(\overline{\rho})$. Horsthemke [33] criticizes that (3.2) is unsound because the rate of decrease of particles of a given type must go to zero when the density of those particles tends to zero. Instead of (3.2) he proposes to use the following hyperbolic system

$$\begin{aligned}
\partial_t u_j + \gamma_j \partial_x u_j =& \mu_j(v_j - u_j) + \frac{1}{2} b_j(u_1 + v_1, \dots, u_n + v_n) \\
&- d_j(u_1 + v_1, \dots, u_n + v_n)u_j \\
\partial_t v_j - \gamma_j \partial_x v_j =& \mu_j(u_j - v_j) + \frac{1}{2} b_j(u_1 + v_1, \dots, u_n + v_n) \\
&- d_j(u_1 + v_1, \dots, u_n + v_n)v_j
\end{aligned} \tag{3.9}$$

with positive birth $b \geq 0$ and death $d \geq 0$ rates. Also the evolution equation (3.9) preserves positivity, whereas with (3.2) positivity is not guaranteed. In [33] Horsthemke performs a linearized stability analysis of (3.9) and finds different results than [31].

3.2 Traveling wave model for semiconductor laser dynamics

The traveling wave model used to describe the longitudinal dynamics of a DFB multisection laser with m sections S_k of length $|S_k|$, attached with a chosen physical unit of length, $1 \leq k \leq m$, is of the following form [3, 21, 47, 76, 60, 5, 50, 58, 57, 55, 66, 39, 67, 68, 9]:

$$\left\{\begin{array}{rcl} \partial_t \psi(t,x) &=& v_{gr} \begin{pmatrix} -\partial_x \psi_1(t,x) \\ \partial_x \psi_2(t,x) \end{pmatrix} + L(x, n(t,x))\psi(t,x) \\ && +K(x, n(t,x), \psi(t,x)), \\ && \\ \partial_t n(t,x) &=& \frac{I(x)+I_M(t,x)}{eV(x)} + \sum_{k=1}^m \frac{U_F^d}{eV_k r_{s,k}} \cdot \\ && \chi_{S_k}(x) \left(\frac{1}{|S_k|} \int_{S_k} n(t,y)\, dy - n(t,x) \right) \\ && -R(x, n(t,x)) - v_{gr} g(x, n(t,x)) \frac{\|\psi(t,x)\|^2}{1+\epsilon_G(x)\|\psi(t,x)\|^2}. \end{array}\right. \tag{3.10}$$

Here $\psi_1(t,x)$ and $\psi_2(t,x)$ denote the slowly varying complex amplitudes of the forward and backward traveling light wave at the spacial position $x \in]0, l[$ and time $t > 0$, and $n(t,x)$ is the carrier density at x and t. The linear operator L is defined through

$$L(x,n)\psi(t,x) \;=\; v_{gr} \begin{pmatrix} \beta(x,n) & \kappa(x) \\ -\overline{\kappa(x)} & \beta(x,n) \end{pmatrix} \psi(t,x),$$

where v_{gr} denotes group velocity, κ a coupling constant due to Bragg grating and β a field propagation constant which is modeled by the formula

$$\beta(x,n) \;=\; -i\delta(x) - i\frac{\beta_{th}(x)I(x)}{|S_k|} - \frac{\alpha(x)}{2} + \frac{1 - i\alpha_H}{2} g(x,n)$$

for $x \in S_k$ and $n \in \mathbb{R}$. There $g(x,n)$ is the field gain function, δ and β_{th} are detuning constants, $I(x)$ is current injection, α describes internal absorption and α_H is the so called linewidth enhancement factor. For the field gain g one commonly uses a linear or logarithmic model

$$g(x,n) = \left\{ \begin{array}{l} g_k^d(x)\,(n - n_{tr,k}) \\ g_k^d(x) n_{tr,k} \log\left(\frac{n}{n_{tr,k}}\right) \end{array} \right. \qquad \text{for } x \in S_k \text{ and } n \in \mathbb{R}.$$

Here $n_{tr,k}$ denotes transparency carrier density and g_k^d differential gain of the k-th section. Note that the logarithmic gain model has a singularity at $n = 0$ which does not exist physically. Thus this model is not appropriate for n close to zero (this is of importance when one derives apriori estimates for (3.10)).

The operator $K(x, n, \psi)$ accounts for nonlinear gain and index compression

$$
\begin{aligned}
K(x,n,\psi) &= v_{gr}\frac{1}{2}g(x,n)\left(\frac{1}{1+\epsilon_G(x)\left\|\psi\right\|^2}-1\right)\psi \\
&\quad -iv_{gr}\frac{\alpha_H}{2}g(x,n)\left(\frac{1}{1+\epsilon_I(x)\left\|\psi\right\|^2}-1\right)\psi \\
&= -\epsilon_G(x)\frac{1}{2}v_{gr}g(x,n)\frac{\left\|\psi\right\|^2}{1+\epsilon_G(x)\left\|\psi\right\|^2}\psi \\
&\quad +\epsilon_I(x)i\frac{\alpha_H}{2}v_{gr}g(x,n)\frac{\left\|\psi\right\|^2}{1+\epsilon_I(x)\left\|\psi\right\|^2}\psi.
\end{aligned}
$$

The time evolution for the carriers is described by a spatially extended ODE in (3.10). There $I(x)$ denotes current injection at position x, $I_M(t,x)$ current modulation, the sum describes current redistribution, R spontaneous recombination, which is modeled as

$$
R(x,n) = A(x)n + B(x)n^2 + C(x)n^3,
$$

and the remaining terms account for stimulated emission. The symbol $\chi_{S_k}(x)$ denotes the characteristic function for the interval S_k, i.e. $\chi_{S_k}(x) = 1$, if $x \in S_k$, and $\chi_{S_k}(x) = 0$ if $x \notin S_k$. The initial and boundary conditions are

$$
\left\{
\begin{aligned}
\psi(0,x) &= \psi_0(x), \\
n(0,x) &= n_0(x), \\
\psi_1(t,0) &= r_0\psi_2(t,0) + \alpha(t), \\
\psi_2(t,l) &= r_m\psi_1(t,l), \\
\psi_1(t,x_k+) &= r^+_{k-1,k}\psi_1(t,x_k-) + r^+_{kk}\psi_2(t,x_k+), \\
\psi_2(t,x_k-) &= r^-_{k-1,k}\psi_2(t,x_k+) + r^-_{kk}\psi_1(t,x_k-).
\end{aligned}
\right.
\tag{3.11}
$$

for $x \in]0,l[$, $t > 0$ and $k \in \mathbb{N}$, $1 < k < m$ (see Figure 3.1). Here x_k+ (x_k-) denote the trace at x_k from the right (left). The symbols r_0, r_m, $r^{+/-}_{k-1,k}$, $r^{+/-}_{kk}$ are complex reflexion coefficients and $\alpha(t)$ is a optical injection term.
In the autonomous case, i.e. $\alpha = 0$ and $I_M = 0$, the equations (3.10) and (3.11) can be written as a hyperbolic system in the form given by (SH) of (real) size $5m$ on the space interval $[0,1]$ when we write the equations separately for each section of the laser (see Figure):

$$
\begin{aligned}
u_1(t,x) &= \psi_1(t,x\cdot x_1) & v_1(t,x) &= \psi_1\left(t,x_2+x\left(x_1-x_2\right)\right) \\
v_m(t,x) &= \psi_2(t,x\cdot x_1) & u_m(t,x) &= \psi_2\left(t,x_2+x\left(x_1-x_2\right)\right) \\
w_1(t,x) &= n(t,x\cdot x_1) & w_2(t,x) &= n\left(t,x_2+x\left(x_1-x_2\right)\right)
\end{aligned}
$$

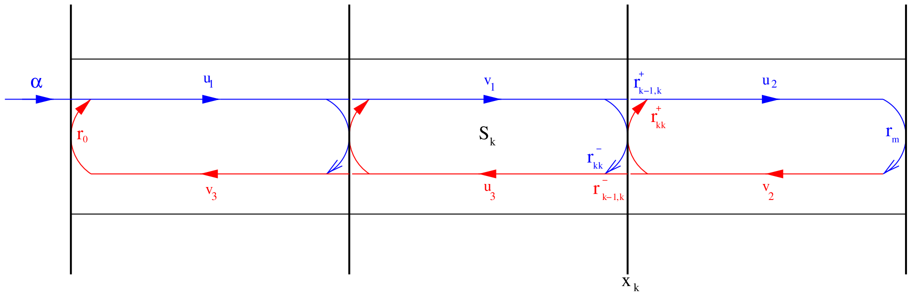

Figure 3.1: Boundary conditions at junction k=2 of a 3-section DFB laser

$$
\begin{array}{rcl}
u_2(t,x) & = & \psi_1\left(t, x_2 + x\left(x_3 - x_2\right)\right) \\
v_{m-1}(t,x) & = & \psi_2\left(t, x_2 + x\left(x_3 - x_2\right)\right) \qquad \ldots \\
w_3(t,x) & = & n\left(t, x_2 + x\left(x_3 - x_2\right)\right)
\end{array}
$$

Note that the operators of the original system (3.10), (3.11) of size 5 are generated by spatially discontinuous functions, whereas the expanded system of size $5m$ is composed of smooth functions only.
All appearing parameters together with their ranges and physical units are listed in table 3.1. Here $L(\cdot, n(t,\cdot))$ operates linearly on $\psi(t,\cdot)$, and K is nonlinear. The reason for the use of this splitting is, as we will see next, that the nonlinearity in ψ is small. Introducing dimensionless variables with suitable reference quantities as follows [1]

$$
\begin{array}{ll}
x \mapsto \dfrac{x}{|S_1|} =: \tilde{x}, & t \mapsto \frac{v_{gr}}{|S_1|} t =: \tilde{t}, \\
n \mapsto \dfrac{n}{n_{tr,1}} =: \tilde{n}, & \psi \mapsto \left(n_{tr,1}\epsilon\right)^{-\frac{1}{2}} \psi =: \tilde{\psi},
\end{array}
\tag{3.12}
$$

where $\epsilon > 0$ is an arbitrary scaling parameter, and writing the nondimensional parameters $\tilde{\kappa}$, l_k, $\tilde{\delta}$, $\tilde{\delta}_{th}$, $\tilde{\alpha}$, $\tilde{\epsilon}_G$, $\tilde{g}_k^d$, b_k, $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{I}$ (see table 3.1) and the intervals $\tilde{S}_k := [x_{k-1}, x_k]$, $x_k := \sum_{\nu=0}^{k} l_\nu$, $1 \le k \le m$, our model equations become

$$
\left\{
\begin{array}{rcl}
\partial_{\tilde{t}}\tilde{\psi}(\tilde{t},\tilde{x}) & = & \begin{pmatrix} -\partial_{\tilde{x}}\tilde{\psi}_1(\tilde{t},\tilde{x})) \\ \partial_{\tilde{x}}\tilde{\psi}_2(\tilde{t},\tilde{x})) \end{pmatrix} + \tilde{L}(\tilde{x},\tilde{n}(\tilde{t},\tilde{x}))\tilde{\psi}(\tilde{t},\tilde{x}) \\
 & & +\tilde{K}(\tilde{x},\tilde{n}(\tilde{t},\tilde{x}),\tilde{\psi}(\tilde{t},\tilde{x})), \\
\partial_{\tilde{t}}\tilde{n}(\tilde{t},\tilde{x}) & = & \tilde{I}(\tilde{x}) + \tilde{I}_M(\tilde{t},\tilde{x}) \\
 & & +\sum_{k=1}^{m} b_k \chi_{\tilde{S}_k}(\tilde{x}) \left(\frac{1}{l_k}\int_{\tilde{S}_k} \tilde{n}(\tilde{t},\tilde{y})\,d\tilde{y} - \tilde{n}(\tilde{t},\tilde{x})\right) \\
 & & -\tilde{R}(\tilde{x},\tilde{n}(\tilde{t},\tilde{x})) - \epsilon\cdot\tilde{g}(\tilde{x},\tilde{n}(\tilde{t},\tilde{x}))\frac{\left\|\tilde{\psi}(\tilde{t},\tilde{x})\right\|^2}{1+\tilde{\epsilon}_G(\tilde{x})\left\|\tilde{\psi}(\tilde{t},\tilde{x})\right\|^2},
\end{array}
\right.
$$

[1] the author would like to thank M. Radziunas and J. Sieber for providing (3.12)

with

$$\tilde{L}(\tilde{x},\tilde{n})\tilde{\psi}(\tilde{t},\tilde{x}) = \begin{pmatrix} \tilde{\beta}(\tilde{x},\tilde{n}) & \tilde{\kappa}(\tilde{x}) \\ -\overline{\tilde{\kappa}(\tilde{x})} & \overline{\tilde{\beta}(\tilde{x},\tilde{n})} \end{pmatrix} \tilde{\psi}(\tilde{t},\tilde{x}) \quad \text{for } x \in \,]0, x_m[\,, \tag{3.13}$$

$$\tilde{\beta}(\tilde{x},\tilde{n}) = -i\left(\tilde{\delta}(\tilde{x}) + \tilde{\delta}_{th}\right) - \tilde{\alpha}(\tilde{x}) + \frac{1-i\alpha_H}{2}\tilde{g}_l(\tilde{x},\tilde{n}) \quad \text{for } \tilde{x} \in \tilde{S}_k,$$

$$\tilde{g}(\tilde{x},\tilde{n}) = \begin{cases} \tilde{g}_k^d(\tilde{x})\left(\tilde{n} - \frac{n_{tr,k}}{n_{tr,1}}\right) \\ \tilde{g}_k^d(\tilde{x})\frac{n_{tr,k}}{n_{tr,1}}\log\left(\frac{n_{tr,1}}{n_{tr,k}}\tilde{n}\right) \end{cases} \quad \text{for } \tilde{x} \in \tilde{S}_k \text{ and } \tilde{n} \in \mathbb{R}, \tag{3.14}$$

$$\begin{aligned} \tilde{K}(\tilde{x},\tilde{n},\tilde{\psi}) &= -\tilde{\epsilon}_G(\tilde{x})\frac{1}{2}\tilde{g}(\tilde{x},\tilde{n})\frac{\left\|\tilde{\psi}\right\|^2}{1+\tilde{\epsilon}_G(\tilde{x})\left\|\tilde{\psi}\right\|^2}\tilde{\psi} \\ &\quad +\tilde{\epsilon}_I(x)i\frac{\alpha_H}{2}\tilde{g}(\tilde{x},\tilde{n})\frac{\left\|\tilde{\psi}\right\|^2}{1+\tilde{\epsilon}_I(\tilde{x})\left\|\tilde{\psi}\right\|^2}\tilde{\psi}, \end{aligned} \tag{3.15}$$

$$\tilde{R}(\tilde{x},\tilde{n}) = \tilde{A}(\tilde{x})\tilde{n} + \tilde{B}(\tilde{x})\tilde{n}^2 + \tilde{C}(\tilde{x})\tilde{n}^3. \tag{3.16}$$

We see that the variable n is two orders of magnitudes slower than ψ. Since $\tilde{I}, \tilde{A}, \tilde{B}, \tilde{C}, b_k$ are all of order $O(10^{-2})$ we can use a scaling $\epsilon \sim 10^{-2}$ and rewrite the equations, omitting the tildes, in the following form:

$$\begin{cases} \partial_t\psi(t,x) = \begin{pmatrix} -\partial_x\psi_1(t,x) \\ \partial_x\psi_2(t,x) \end{pmatrix} + L(x,n(t,x))\psi(t,x) \\ \qquad +\epsilon K(x,n(t,x),\psi(t,x)) \\ \\ \partial_t n(t,x) = \epsilon F\left(t,x,n(t,x),\psi(t,x)\right), \end{cases} \tag{3.17}$$

where

$$\begin{aligned} F(t,x,n(t,x),\psi(t,x)) &= I(t,x) + \sum_{k=1}^m b_k\chi_{S_k}(x)\left(⨍_{S_k} n(t,y)dy - n(t,x)\right) \\ &\quad -R(x,n(t,x)) \\ &\quad -g(x,n(t,x))\frac{\|\psi(t,x)\|^2}{1+\epsilon_G(x)\|\psi(t,x)\|^2}. \end{aligned}$$

Table 3.1: In the table we list typical parameter ranges for active laser sections together with their physical units and the formula describing the relation to the original dimensional quantity which follows from the scaling (3.12), the last column shows the order obtained. The author would like to thank M. Radziunas for providing the parameters.

parameter	typ. range	phys. unit	nondim. transform	new range	order
κ	130	$10^2 m^{-1}$	$\tilde{\kappa} := \kappa \lvert S_1 \rvert$	$3,25$	$O(1)$
$\lvert S_k \rvert$	250	$10^{-6} m$	$l_k := \lvert S_k \rvert / \lvert S_1 \rvert$	~ 1	$O(1)$
δ	400	$10^2 m^{-1}$	$\tilde{\delta} := \delta \lvert S_1 \rvert$	10	$O(10)$
β_{th}	40	A^{-1}			
I	70	$10^{-3} A$			
$\beta_{th} I / \lvert S_k \rvert$	$11,2$	$10^3 m^{-1}$	$\tilde{\delta}_{th} := (\beta_{th} I / \lvert S_k \rvert) \lvert S_1 \rvert$	$\sim 2,8$	$O(1)$
α	15	$10^2 m^{-1}$	$2\tilde{\alpha} := \alpha \lvert S_1 \rvert$	$0,75$	$O(1)$
α_H	-4				
$n_{tr,k}$	1	$10^{24} m^{-3}$	$n_{tr,k} / n_{tr,1}$	~ 1	$O(1)$
$\epsilon_G \Gamma$	$1,5$	$10^{-24} m^3$	$\tilde{\epsilon}_G := \epsilon_G \Gamma \epsilon n_{tr_1}$	$1,5\epsilon$	$O(\epsilon)$
g_k^d	10	$10^{-21} m^2$	$\tilde{g}_k^d := g_k^d n_{tr,1} \lvert S_1 \rvert$	$2,5$	$O(1)$
U_F^d	6	$10^{-26} V m^3$			
e	$1,6$	$10^{-19} As$			
V_k	150	$10^{-18} m^3$			
$r_{s,k}$	$2,5$	$\Omega = V/A$			
v_{gr}	$0,8$	$10^8 m/s$			
$U_F^d / (e V_k r_{s,k})$	10^{-2}	$10^{11} s^{-1}$	$b_k := \lvert S_1 \rvert U_F^d / (e V_k r_{s,k} v_{gr})$	$3,125 \cdot 10^{-3}$	$O(10^{-3})$
A	3	$10^8 s^{-1}$	$\tilde{A} := A \lvert S_1 \rvert / v_{gr}$	$9,375 \cdot 10^{-4}$	$O(10^{-3})$
B	1	$10^{-16} m^3 s^{-1}$	$\tilde{B} := B \lvert S_1 \rvert n_{tr,1} / v_{gr}$	$3,125 \cdot 10^{-4}$	$O(10^{-4})$
C	1	$10^{-40} m^6 s^{-1}$	$\tilde{C} := \lvert S_1 \rvert n_{tr,1}^2 / v_{gr}$	$3,125 \cdot 10^{-4}$	$O(10^{-4})$
$I/(eV)$	$0,292$	$10^{34} s^{-1} m^{-3}$	$\tilde{I} := \lvert S_1 \rvert I / (e V v_{gr} n_{tr,1})$	$9,115 \cdot 10^{-3}$	$O(10^{-2})$

We have used a slight abuse of notation by having written $I(t,x)$, b_k and $R(x,n(t,x))$ instead of $I(t,x)/\epsilon$, b_k/ϵ and $R(x,n(t,x))/\epsilon$. Note that F only contains terms of order $O(1)$. In the following we will ignore the dependence of I, b_k, R, on the fixed chosen scaling $\epsilon \sim 10^{-2}$ and treat ϵ as a sufficient small variable perturbation parameter.

Remark 3.1. *The careful reader would notice that here I have excluded polarization equations in the traveling wave equations which have been added in recent years visible in several publications already (although the model with polarization can be treated without difficulties and falls under my general setting for semilinear hyperbolic systems). The reason for this decision is that I consider the equations including polarization unsound because the carrier densities can not guaranteed to be positive anymore. In fact it is not difficult to construct initial data such that the densities become negative due to the added polarization term appearing in the carrier rate equation. Nevertheless, I have learned that it has benefits from the practical numerical point of view because it stabilizes modes.*

3.3 Neutral and retarded functional differential equations / linear hyperbolic systems with dynamic boundary conditions

Next we show that general functional differential equations are linear hyperbolic systems with nonlinear dynamic boundary conditions.

Let $n_1 = n_3 = 0$ and $n_2 = n > 0$. Put $K(x) = -I_n$, where I_n denotes the identity matrix of $\mathbb{K}^n$. Instead of the interval $[0,l]$ we chose the interval $[-r,0]$. Then (SH) becomes for $t > 0$ and $-r < x < 0$

$$\frac{\partial}{\partial t}v(t,x) = \frac{\partial}{\partial x}v(t,x) \tag{3.18}$$

$$\frac{d}{dt}v(t,0) = F(v(t,\cdot)). \tag{3.19}$$

Define $y : [-r,\delta[\to \mathbb{R}^n$ by $y(t) := v(t,0)$ for $t \in]0,\delta[$ and $y(t) := v(0,t)$ for $-r \le t \le 0$. From (3.18) it follows that $v(t,x) = y(t+x)$ for $t \ge 0$ and $-r \le x \le 0$. Substituting into (3.19) we get

$$\frac{d}{dt}y(t) = F(y(t+\cdot)), \tag{3.20}$$

where $F : C([-r,0],\mathbb{R}^n) \to \mathbb{R}^n$. Equation (3.20) is the usual retarded functional differential equation (see [27]).

Let $n_1 = n_2$, and $n_3 = 0$. Put $K(x) = \begin{pmatrix} I_{n_1} & 0 \\ 0 & -I_n \end{pmatrix}$. Let $E = I_{n_1}$. Instead of $[0, l]$ choose $[-\frac{r}{2}, 0]$ then (SH) reads

$$\begin{aligned} \frac{\partial}{\partial t} u(t,x) &= -\frac{\partial}{\partial x} u(t,x) && (3.21) \\ \frac{\partial}{\partial t} v(t,x) &= \frac{\partial}{\partial x} v(t,x) && (3.22) \\ \frac{d}{dt}(v(t,0) - Du(t,0)) &= F(v(t,\cdot), u(t,\cdot)) && (3.23) \\ u\left(t, -\frac{r}{2}\right) &= v\left(t, -\frac{r}{2}\right). \end{aligned}$$

Define $y : [-r, \delta[\to \mathbb{R}^{n_1}$ by $y(t) := v(t,0)$ for $t \in]0, \delta[$, $y(t) := v(0,t)$ for $-\frac{r}{2} \le t \le 0$ and $y(t) := u(0, -r-t)$ for $-r \le t \le -\frac{r}{2}$. From (3.22) we get $v(t,x) = y(t+x)$ for $t \ge 0$ and $-\frac{r}{2} \le x \le 0$ and from (3.21) $u(t,x) = y(t-x-r)$ for $t \ge 0$ and $-\frac{r}{2} \le x \le 0$. Substituting into (3.23) we get

$$\frac{d}{dt}\left(y(t) - Dy(t-r)\right) = F(y(t+\cdot), y(t-\cdot-r)), \tag{3.24}$$

where $F : C([-\frac{r}{2}, 0], \mathbb{R}^{n_1})^2 \to \mathbb{R}^{n_1}$. Rewrite (3.24) with $G : C([-r, 0], \mathbb{R}^{n_1}) \to \mathbb{R}^{n_1}$ and obtain the neutral functional differential equation

$$\frac{d}{dt}\left(y(t) - Dy(t-r)\right) = G(y(t+\cdot)).$$

We mention the mixed initial boundary value problem originating from a model for electronic circuit dynamics studied by Brayton and Miranker [12] which belongs to the class of linear hyperbolic systems with nonlinear boundary conditions.

3.4 Boltzmann systems

Discrete velocity models of the Boltzmann equations are of considerable interest in the kinetic theory of gases. There has been a lot of work on their mathematical and mechanical aspects. We refer to the review article [52] which contains a huge list of references until 1985. In one space dimension such models belong to the general class of semilinear hyperbolic systems treated in this work. The simplest discrete velocity approximation to the

Boltzmann equation is the so called Carleman model:

$$\begin{aligned} \frac{1}{\sqrt{2}}\left(\partial_t u + \partial_x u\right) =& v^2 - u^2, \\ \frac{1}{\sqrt{2}}\left(\partial_t v - \partial_x v\right) =& u^2 - v^2. \end{aligned} \tag{3.25}$$

One can consider x in some interval, $x \in [0, l]$, with boundary

$$u(t, 0) = r_0 v(t, 0), \quad u(t, l) = r_l v(t, l) \quad \text{for } t \geq 0$$

and initial conditions

$$u(0, x) = u_0(x) \quad v(0, x) = v_0(x).$$

Here $u(t, x) \in \mathbb{R}_+$ and $v(t, x) \in \mathbb{R}_+$ are the mass densities of particles with speeds plus and minus one. The boundary conditions mean that the number of particles hitting the wall with one speed is equal to r_0 or r_l times the number leaving the wall with the other speed.

The Carleman model belongs to the class of Boltzmann systems [17]:

Definition 3.2. The system of equations

$$\frac{\partial u_i}{\partial t} + v_i \frac{\partial u_i}{\partial x} = \sum_{1 \leq j,k \leq n} B^i_{jk} u_j u_k, \quad 1 \leq i \leq n,$$

is called an n-th order Boltzmann system with velocity states $v = (v_1, \ldots, v_n)$ and collision form $B = (B^1, \ldots, B^n)$, provided the matrices $B^i = \left(B^i_{jk}\right)$ satisfy the following: B^i is symmetric, $B^i_{jk} \geq 0$, $1 \leq j, k \leq n$, $j \neq i$, $k \neq i$, and $B^i_{jk} \leq 0$ for $j = i$ or $k = i$.

Chapter 4

Nondegenerate linear hyperbolic systems

4.1 Basic properties

We consider the class of nondegenerate hyperbolic systems (H) for $x \in \,]0,l[$ and $t > 0$

$$(\mathrm{H}) \quad \begin{cases} \dfrac{\partial}{\partial t}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + K(x)\dfrac{\partial}{\partial x}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + C(x)\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = 0, \\ \dfrac{d}{dt}\left[v(t,l) - Du(t,l)\right] = Fu(t,\cdot) + Gv(t,\cdot), \\ u(t,0) = Ev(t,0), \end{cases}$$

where
(HI) $K(x) = \mathrm{diag}\,(k_i(x))_{i=1,\dots,n}$ is a diagonal $n \times n$ matrix of functions $k_i \in C^1([0,l],\mathbb{R})$ which satisfy $k_i(x) > 0$ for $i = 1, \dots n_1$ and $k_j(x) < 0$ for $j = n_1 + 1, \dots n$ $(x \in [0,l])$.
(HII) $C(x) = (c_{ij}(x))_{i,j=1,\dots,n}$ is a $n \times n$ matrix with diagonal elements $c_{ii} \in L^\infty(]0,l[\,,\mathbb{C})$, $i = 1, \dots, n$, and nondiagonal elements $c_{ij} \in BV([0,l],\mathbb{C})$, $i,j = 1,\dots,n$ with $i \neq j$.
(HIII) If $i \neq j$ and $k_i(x) = k_j(x)$ for some $x \in [0,l]$ then c_{ij} vanishes completely on $]0,l[$.
(HIV) $u(t,x) = (u_1(t,x),\dots,u_{n_1}(t,x)) \in \mathbb{C}^{n_1}$ and $v(t,x) = (v_1(t,x),\dots,v_{n_2}(t,x)) \in \mathbb{C}^{n_2}$
(HV) $D \in \mathbb{C}^{n_2 \times n_1}$, $E \in \mathbb{C}^{n_1 \times n_2}$ and

$$F : C([0,l],\mathbb{C}^{n_1}) \to \mathbb{C}^{n_2}, \quad G : C([0,l],\mathbb{C}^{n_2}) \to \mathbb{C}^{n_2}$$

are linear continuous operators.

Remark 4.1. *In this section we do not need assumption* (HIII)*, but it will be required in sections 4.2 and 5.2. In section 6.1 we will relax condition* (HIII)*.*

System (H) can be written as an abstract evolution equation

$$\frac{d}{dt}w(t) = Aw(t)$$

in the complex space

$$X_p = L^p(]0, l[\,; \mathbb{C}^n) \times \mathbb{C}^{n_2} \tag{4.1}$$

for $1 \leq p < \infty$, where $w = (u, v, d)$ and A is a closed operator

$$A : \mathcal{D}(A) \subset X_p \to X_p,$$

$$A(u, v, d) := \left(-K(x)\frac{\partial}{\partial_x}\begin{pmatrix} u \\ v \end{pmatrix} - C(x)\begin{pmatrix} u \\ v \end{pmatrix}; \quad Fu + Gv \right),$$

on the dense domain

$$\mathcal{D}(A) := \Big\{ (u, v, d) \in X_p \,|(u, v) \in W^{1,p}(]0, l[\,; \mathbb{C}^n), \\ u(0) = Ev(0),\ d = v(l) - Du(l) \Big\}.$$

It is not difficult to verify that A generates a C_0 semigroup in Y, see Proposition 7.18. In special cases, for example if

$$Fu = F_0u(l) \quad \text{and} \quad Gv = G_0v(l),$$

where $F_0 \in \mathbb{C}^{n_2 \times n_1}$ and $G_0 \in \mathbb{C}^{n_2 \times n_2}$ are matrices, it is not difficult to see that A is the generator of a C_0 semigroup e^{At} in X_p for $1 \leq p < \infty$, see Proposition 7.20, the paper [48] or the book [45, Theorem 6.2, p. 312] for a detailed proof. Spectral properties of the semigroup in the space X_p, $1 \leq p < \infty$, have been studied in [48]. Hence the space

$$Y = \{(u, v, d) \in C([0, l]; \mathbb{C}^n) \times \mathbb{C}^{n_2} \mid u(0) = Ev(0),\ d = v(l) - Du(l)\},$$

where $C([0, l]; \mathbb{C}^n)$ is equipped with the sup-norm, is an A-admissible invariant subspace in the sense of [49, chapter 4.5]. This means that $e^{At}Y \subset Y$ for $t \geq 0$ and the restriction of e^{At} to Y is a C_0 semigroup in the stronger Y-norm. The generator of the restriction to Y is the operator $A_{|Y} : \mathcal{D}(A_{|Y}) \subset Y \to Y$, $A_{|Y}w := Aw$ for $w \in \mathcal{D}(A_{|Y})$ with domain

$$\mathcal{D}(A_{|Y}) = \Big\{ (u, v, d) \in W^{1,\infty}([0, l]; \mathbb{C}^n) \times \mathbb{C}^{n_2} \mid \\ u(0) = Ev(0),\ d = v(l) - Du(l) \text{ and } A(u, v, d) \in Y \Big\}.$$

Together with (H) we consider the reduced system (H_0)

$$(\mathrm{H}_0) \quad \begin{cases} \dfrac{\partial}{\partial t}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + K(x)\dfrac{\partial}{\partial x}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + C_0(x)\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = 0, \\ u(t,0) = Ev(t,0) \quad \text{and} \quad v(t,l) = Du(t,l). \end{cases}$$

where $C_0(x) = \operatorname{diag}(c_{11}(x), \ldots, c_{nn}(x))$ is the diagonal part of $C(x)$. A common choice of phase space for (H_0) is $L^p(]0,l[\,,\mathbb{C}^n)$, $1 \le p < \infty$, and it can be written as an abstract equation with an infinitesimal generator A_0 of a C_0 semigroup in an obvious way. It has the A_0 admissible invariant subspace

$$Y_0 := \{(u,v) \in C([0,l];\mathbb{C}^n) \mid u(0) = Ev(0),\ v(l) = Du(l)\}\,. \tag{4.2}$$

In the following we put either

$$X = X_p \quad \text{for } p \in [1,\infty[\qquad \text{or} \qquad X = Y \tag{4.3}$$

and consider $A : \mathcal{D}(A) \to X$ as a closed densely defined operator in the Banach space X.

Let $T(x,y,\lambda)$ denote the fundamental matrix satisfying

$$\begin{aligned} \frac{d}{dx}T(x,y,\lambda) &= -K(x)^{-1}\left(\lambda I + C(x)\right)T(x,y,\lambda) \quad \text{for } x,y \in [0,l], \quad (4.4) \\ T(y,y,\lambda) &= I \quad \text{for } y \in [0,l]. \end{aligned}$$

Let T_0 be the fundamental matrix (corresponding to the reduced system)

$$\begin{aligned} \frac{d}{dx}T_0(x,y,\lambda) &= -K^{-1}(x)\left(\lambda I + C_0(x)\right)T_0(x,y,\lambda) \quad \text{for } x,y \in [0,l], \quad (4.5) \\ T_0(y,y,\lambda) &= I \quad \text{for } y \in [0,l]. \end{aligned}$$

A formula for (4.5) is

$$T_0(x,y,\lambda) = \exp\left(-\int_y^x K^{-1}(z)\left(\lambda I + C_0(z)\right)\,dz\right).$$

Let $\sigma(A) := \{\lambda \in \mathbb{C} \mid \lambda I - A \text{ is not invertible}\}$ denote the spectrum of A and $\sigma_p(A) := \{\lambda \in \mathbb{C} \mid \exists_{v\in\mathcal{D}(A),v\neq 0} Av = \lambda v\}$ denote the point spectrum. We will see that the spectrum does not depend on the choice (4.3) of the Banach space X and hence we will just refer to the spectrum of (H).

Further let $I \in \mathbb{C}^{n_2\times n_2}$ denote the identity matrix and δ_l, $\delta_l f := f(l)$, the delta function at l. We put

$$H(\lambda) := (-\lambda D\delta_l - F, \lambda I\delta_l - G)T(\cdot,0,\lambda)\begin{pmatrix} E \\ I \end{pmatrix} \quad \left(\in \mathbb{C}^{n_2\times n_2}\right). \tag{4.6}$$

Definition 4.2. The function

$$h(\lambda) := \det H(\lambda)$$

is called the characteristic function to (H).

Further we put

$$\begin{aligned} h_0(\lambda) :=& \det H_0(\lambda), \\ H_0(\lambda) :=& (-D\delta_l, I\delta_l) T_0(\cdot, 0, \lambda) \begin{pmatrix} E \\ I \end{pmatrix} \quad \left(\in \mathbb{C}^{n_2 \times n_2}\right) \end{aligned} \tag{4.7}$$

and call h_0 the characteristic function to (H_0).

Proposition 4.3. *We have*

$$\sigma(A) = \sigma_p(A) = \{\lambda \in \mathbb{C} \mid h(\lambda) = 0\}. \tag{4.8}$$

For $\lambda \in \sigma(A)$ *the eigenspace is*

$$\mathrm{Eig}(A, \lambda) = \left\{ T(\cdot, 0, \lambda) \begin{pmatrix} E \\ I \end{pmatrix} v_0 \mid v_0 \in \mathrm{Ker}\, H(\lambda) \right\}.$$

In particular, the geometric multiplicity of λ *is less than or equal to* n_2*.*

Similarly for (H_0) *we have*

$$\sigma(A_0) = \sigma_p(A_0) = \{\lambda \in \mathbb{C} \mid h_0(\lambda) = 0\}.$$

For $\lambda \in \sigma(A_0)$

$$\mathrm{Eig}(A_0, \lambda) = \left\{ T_0(\cdot, 0, \lambda) \begin{pmatrix} E \\ I \end{pmatrix} v_0 \mid v_0 \in \mathrm{Ker}\, H_0(\lambda) \right\}.$$

For any λ *such that* $h(\lambda) \neq 0$ *the resolvent* $R(\lambda, A) = (\lambda I - A)^{-1}$ *is given by*

$$\left[R(\lambda, A) \begin{pmatrix} f \\ g \\ b \end{pmatrix} \right](x) = \begin{pmatrix} u(x) \\ v(x) \\ v(l) - D\,u(l) \end{pmatrix}, \tag{4.9}$$

where

$$\begin{pmatrix} u \\ v \end{pmatrix} = T(\cdot, 0, \lambda) \begin{pmatrix} E \\ I \end{pmatrix} H(\lambda)^{-1} \beta(\lambda)(f, g, b) + \int_0^{\cdot} T(\cdot, y, \lambda) K(y)^{-1} \begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy$$

and $\beta(\lambda) : X \to \mathbb{C}^{n_2}$ *denotes*

$$\beta(\lambda)(f,g,b) := b + (\lambda D\delta_l + F, G - \lambda I\delta_l) \int_0^{\cdot} T(\cdot,y,\lambda)K(y)^{-1} \begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy.$$

In particular, $R(\lambda, A) : X \to X$ *is compact for* $\lambda \notin \sigma(A)$.

For any λ *such that* $h_0(\lambda) \neq 0$ *the resolvent* $R(\lambda, A_0) = (\lambda I - A_0)^{-1}$ *is given by*

$$R(\lambda, A_0) \begin{pmatrix} f \\ g \end{pmatrix} = T_0(\cdot, 0, \lambda) \begin{pmatrix} E \\ I \end{pmatrix} H_0(\lambda)^{-1}\beta_0(\lambda)(f,g) + \int_0^{\cdot} T_0(\cdot,y,\lambda)K(y)^{-1} \begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy, \tag{4.10}$$

where

$$\beta_0(\lambda)(f,g) := (D, -I) \int_0^{l} T_0(l,y,\lambda)K(y)^{-1} \begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy. \tag{4.11}$$

Proof. We have $\lambda \in \sigma_p(A)$ iff there exists $v_0 \in \mathbb{C}^{n_2}$, $v_0 \neq 0$, such that

$$\begin{pmatrix} u \\ v \end{pmatrix}(x) = T(x,0,\lambda) \begin{pmatrix} E \\ I \end{pmatrix} v_0 \quad \text{and} \quad (-\lambda D\delta_l - F, \lambda I\delta_l - G) \begin{pmatrix} u \\ v \end{pmatrix} = 0.$$

This is equivalent to $H(\lambda)$ having a nontrivial kernel or $h(\lambda) = 0$. Hence

$$\sigma_p(A) = \{\lambda \in \mathbb{C} \mid h(\lambda) = 0\}.$$

The resolvent equation $R(\lambda, A)(f,g,b) = (u,v,d)$ for $(u,v,d) \in \mathcal{D}(A)$ and $(f,g,b) \in X$ is equivalent to

$$\begin{aligned} K\frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix} + (\lambda I + C)\begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} f \\ g \end{pmatrix} \\ (-\lambda D\delta_l - F,\ \lambda I\delta_l - G)\begin{pmatrix} u \\ v \end{pmatrix} &= b. \end{aligned}$$

And this in turn, since $(u,v,d) \in \mathcal{D}(A)$, is equivalent to

$$\begin{aligned} \begin{pmatrix} u \\ v \end{pmatrix}(x) &= T(x,0,\lambda)\begin{pmatrix} E \\ I \end{pmatrix} v(0) + \int_0^x T(x,y,\lambda)K(y)^{-1}\begin{pmatrix} f \\ g \end{pmatrix}(y)\,dy \\ b &= (-\lambda D\delta_l - F,\ \lambda I\delta_l - G)\begin{pmatrix} u \\ v \end{pmatrix}. \end{aligned}$$

If $\lambda \notin \sigma_p(A)$, by inserting the first equation into the second one, we get that the resolvent equation has a unique solution and $v(0) = H(\lambda)^{-1}\beta(\lambda)(f,g,b)$. This shows (4.9) and that $\sigma_p(A) = \sigma(A)$.

We note that $\lambda I - A$ is a compact perturbation of an isomorphism and hence a Fredholm operator of index zero. Indeed:

Denote $K_u(x) := \operatorname{diag}\,(k_i(x))_{i=1,\ldots,n_1}$ and $K_v(x) := \operatorname{diag}\,(k_i(x))_{i=n_1+1,\ldots,n}$. For $r \in \mathbb{R}$ consider the equation

$$r\begin{pmatrix} u \\ v \\ d \end{pmatrix} + \begin{pmatrix} K\frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix} \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{d} \end{pmatrix}, \tag{4.12}$$

where $\tilde{u} \in L^p([0,l];\mathbb{C}^{n_1})$, $\tilde{v} \in L^p([0,l];\mathbb{C}^{n_2})$ and $\tilde{d} \in \mathbb{C}^{n_2}$ are given and the unknown is $(u,v,d) \in \mathcal{D}(A)$. Here $p \in [1,\infty]$, i.e. the case $p = \infty$ is included. This equation has a solution $(u,v,d) \in \mathcal{D}(A)$ iff there exists $v_0 \in \mathbb{C}^{n_2}$ such that

$$\left\{\begin{array}{rcl} \begin{pmatrix} u \\ v \end{pmatrix}(x) & = & \exp(-r\int_0^x K^{-1}(z)\,dz)\begin{pmatrix} E \\ I \end{pmatrix} v_0 \\ & & + \int_0^x \exp(-r\int_y^x K^{-1}(z)\,dz)K^{-1}(y)\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}(y)\,dy, \\ \tilde{d} & = & r\,(-D,I)\,\delta_l\begin{pmatrix} u \\ v \end{pmatrix}. \end{array}\right. \tag{4.13}$$

It is unique iff v_0 is unique. Rewriting (4.13) we have

$$\begin{array}{rl} & r\left(\exp(-r\int_0^l K_v^{-1}(z)\,dz) - D\exp(-r\int_0^l K_u^{-1}(z)\,dz)E\right)v_0 \\ = & \tilde{d} - r\,(-D,I)\int_0^l \exp(-r\int_y^x K^{-1}(z)\,dz)K^{-1}(y)\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}(y)\,dy. \end{array} \tag{4.14}$$

By (HI) we have $\lim_{r\to\infty} re^{-r\int_0^l K_u^{-1}(z)\,dz} = 0$ and $\lim_{r\to\infty}\left\|re^{-r\int_0^l k_i^{-1}(z)\,dz}\right\| = \infty$ for $i = n_1+1,\ldots,n$. Therefore for sufficiently large $r > 0$ the matrix in the left side of equation (4.14) is invertible. Hence for large r the operator defined on the left hand side of (4.12) is an isomorphism from $\mathcal{D}(A)$ onto $L^p([0,l],\mathbb{C}^n) \times \mathbb{C}^{n_2}$ ($p \in [1,\infty]$). Since for $(u,v,d) \in \mathcal{D}(A)$

$$(\lambda I - A)\begin{pmatrix} u \\ v \\ d \end{pmatrix} = \left[r\begin{pmatrix} u \\ v \\ d \end{pmatrix} + \begin{pmatrix} K\frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix} \\ 0 \end{pmatrix}\right] + \left[(\lambda - r)\begin{pmatrix} u \\ v \\ d \end{pmatrix} + \begin{pmatrix} C\begin{pmatrix} u \\ v \end{pmatrix} \\ -Fu - Gv \end{pmatrix}\right]$$

and the imbedding $W^{1,p}([0,l],\mathbb{C}^n) \hookrightarrow L^p([0,l],\mathbb{C}^n)$ is compact for $p \in [1,\infty]$.

$\square$

Since $h_0(\lambda)$ is a finite exponential polynomial of the form $\sum_{n=1}^{m} a_n e^{b_n\lambda}$ with $b_n \in \mathbb{R}$ and $a_n \in \mathbb{C}$ we have the following

Proposition 4.4. *The zeros of h_0 are located in a strip, i.e. there exist $\gamma_0, \gamma_1 \in \mathbb{R}$ such that $\lambda \in \mathbb{C}$ with $h_0(\lambda) = 0$ implies $\lambda \in \mathbb{C}_{\gamma_0,\gamma_1}$.*

It is important to know the dimension of the spectral projection associated to an eigenvalue of A. For example, when studying the dynamics on a center manifold the knowledge of that dimension is essential. In [43] it has been shown that the dimension of the range of the spectral projection corresponding to a single characteristic root λ_0 is equal to the multiplicity of the spectral point λ_0 as root of the characteristic equation $h(\lambda) = 0$. More precisely the following theorem from [43, Theorem, p.343] holds:

Theorem 4.5. *If λ_0 is a root of $h(\lambda)$ of multiplicity m, then we have*
ı) $X_p = \mathrm{Ker}(\lambda_0 I - A)^m \oplus \mathrm{Im}(\lambda_0 I - A)^m$,
ıı) $\mathrm{Ker}(\lambda_0 I - A)^m = \pi(X_p)$, *where*

$$\pi = \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\delta} R(\lambda, A)\, d\lambda$$

and $\delta > 0$ is chosen such that $\sigma(A) \cap \{z \in \mathbb{C} \mid |z - \lambda_0| \le \delta\} = \{\lambda_0\}$,
ııı) the dimension of $\mathrm{Ker}(\lambda_0 I - A)^m$ *is m.*

4.2 Estimates for spectra and resolvents

In general the fundamental system T to (4.4) can not be calculated explicitely. And even in the case of constant coefficients it will be a complicated expression. In the following we will give a series expression for T in powers of λ^{-1}.

Denote the nondiagonal part of C by

$$C_1(x) := C(x) - C_0(x).$$

Define for $k \ge 1$

$$T_k(x, y, \lambda) := -\lambda T_0(x, y, \lambda)\int_y^x T_0(y, z, \lambda) K^{-1}(z) C_1(z) T_{k-1}(z, y, \lambda)\, dz. \tag{4.15}$$

Each T_k, $k \ge 1$, satisfies the initial value problem

$$\begin{aligned} \frac{d}{dx} T_k(x, y, \lambda) &= -K^{-1}(x)\left(\lambda I + C_0(x)\right) T_k(x, y, \lambda) \qquad (4.16)\\ &\quad -\lambda K^{-1}(x) C_1(x) T_{k-1}(x, y, \lambda),\\ T_k(x, x, \lambda) &= 0, \end{aligned}$$

and can be calculated recursively in terms of integrals of elementary functions. We will see that the series

$$\sum_{k=0}^{\infty} \lambda^{-k} T_k(x,y,\lambda) \tag{4.17}$$

converges in $W^{1,\infty}$ for sufficiently large $|\mathfrak{Im}\lambda|$. Therefore (4.17) is a representation of the fundamental Matrix T to (4.4). However, we already note here that the explicit expressions for T_k, $k \geq 2$, are not bounded for λ chosen from any stripe $\mathbb{C}_r$. Indeed we will see in the following that the expressions T_k, $k \geq 2$, contain some power terms λ^i with i up to the lower integer part of $k/2$ which will be due to successive failures in partial integration in the formula (4.15). After reordering terms for any finite $\kappa \in \mathbb{N}$ we will obtain an explicit representation of the form

$$T(x,y,\lambda) = \sum_{k=0}^{\kappa} \lambda^{-k} F_k(x,y,\lambda) + O(\lambda^{-(\kappa+1)}),$$

for λ in a stripe $\mathbb{C}_r$ and sufficiently large $|\mathfrak{Im}\lambda|$, where each F_k is of order 1 with respect to λ on stripes $\mathbb{C}_r$ (by this we mean that for any given $r > 0$ there exists $c > 0$ such that $\|F_k(x,y,\lambda)\| \leq c$ for $\lambda \in \mathbb{C}_r$, $x, y \in [0,l]$).

To see this we calculate the first two steps T_1 and T_2. Put

$$f_0(x,y,\lambda) := T_0(x,y,\lambda)(y_0^{(1)}, \ldots, y_0^{(n)})^t$$

with the arbitrary but fixed initial data $y_0^{(i)} \in \mathbb{C}$, $1 \leq i \leq n$. Define

$$f_k := -\lambda \int_y^x T_0(x,z,\lambda) K^{-1}(z) C_1(z) f_{k-1}(z,y,\lambda)\, dz$$

for $k \geq 1$. Then the i-th component $(1 \leq i \leq n)$ of f_k is

$$\begin{aligned} f_0^{(i)}(x,y,\lambda) &= \exp\left(-\textstyle\int_y^x \lambda k_i^{-1}(u)\, du\right) \exp\left(-\textstyle\int_y^x k_i^{-1}(u) c_{ii}(u)\, du\right) y_0^{(i)}, \\ f_k^{(i)}(x,y,\lambda) &= -\lambda \exp\left(-\textstyle\int_y^x \lambda k_i^{-1}(u)\, du\right) \exp\left(-\textstyle\int_y^x k_i^{-1}(u) c_{ii}(u)\, du\right) \\ &\qquad \sum_{\substack{1 \leq l \leq n \\ l \neq i}} \int_y^x \exp\left(\textstyle\int_y^z \left(\lambda k_i^{-1}(u) + \frac{c_{ii}(u)}{k_i(u)}\right) du\right) \frac{c_{il}(z)}{k_i(z)} f_{k-1}^{(l)}(z,y,\lambda)\, dz. \end{aligned}$$

If we choose $y_0^{(l)} = \delta_{lj}$ for $l = 1, \ldots, n$, where $\delta_{lj} = 1$ if $l = j$ and $\delta_{lj} = 0$ for $l \neq j$ denotes Kronecker's symbol, then $f_k^{(i)}$ is the entry in the i-th row and

j-th column of T_k. By assumptions (HII) and (HIII) we can perform partial integration and get rid of the λ factor appearing in the recursion formula for $f_k^{(i)}$:

$$
\begin{aligned}
f_1^{(i)}(x,y,\lambda) = &- \exp\left(- \textstyle\int_y^x \left(\lambda k_i^{-1}(u) + k_i^{-1}(u) c_{ii}(u)\right)\, du\right) \\
& \sum_{\substack{1 \le l \le n \\ l \ne i}} \int_y^x \lambda \left(k_i^{-1}(z) - k_l^{-1}(z)\right) \exp\left(\textstyle\int_y^z \lambda \left(k_i^{-1}(u) - k_l^{-1}(u)\right)\, du\right) \\
& \frac{c_{il}(z)}{k_i(z)} \frac{\exp(\int_y^z \left(k_i^{-1}(u)c_{ii}(u) - k_l^{-1}(u)c_{ll}(u)\right)\, du)}{k_i^{-1}(z) - k_l^{-1}(z)} y_0^{(l)}\, dz \\
= & \sum_{\substack{1 \le l \le n \\ l \ne i}} y_0^{(l)} \Bigg\{ - \exp\left(- \textstyle\int_y^x \lambda k_l^{-1}(u)\, du\right) \frac{c_{il}(x)}{k_i(x)} \frac{\exp\left(- \int_y^x \frac{c_{ll}(u)}{k_l(u)}\, du\right)}{k_i^{-1}(x) - k_l^{-1}(x)} \\
& + \exp\left(- \textstyle\int_y^x \lambda k_i^{-1}(u)\, du\right) \exp\left(- \textstyle\int_y^x k_i^{-1}(u) c_{ii}(u)\, du\right) \\
& \left[\frac{c_{il}(y)}{k_i(y)} \frac{1}{k_i^{-1}(y) - k_l^{-1}(y)} + \int_y^x \exp\left(\textstyle\int_y^z \lambda \left(k_i^{-1}(u) - k_l^{-1}(u)\right)\, du\right) \right. \\
& \left. \frac{d}{dz} \left(\frac{c_{il}(z)}{k_i(z)} \frac{\exp\left(\int_y^z \left(\frac{c_{ii}(u)}{k_i(u)} - \frac{c_{ll}(u)}{k_l(u)}\right)\, du\right)}{k_i^{-1}(z) - k_l^{-1}(z)} \right) dz \right] \Bigg\}.
\end{aligned}
$$

Note that for partial integration we used that in the sum for $l \ne i$ in the formula for $f_1^{(i)}$ the leading λ-exponential of $f_0^{(l)}$ is $e^{-\int_y^x \lambda k_l^{-1}(u)\, du}$. However, now $f_1^{(i)}$ not only contains $2(n-1)$ terms with λ-exponential $e^{-\int_y^x \lambda k_i^{-1}(u)\, du}$ but also $(n-1)$ terms of the form $e^{-\int_y^x \lambda k_l^{-1}(u)\, du}$, $1 \le l \le n$, $l \ne i$. Therefore, in the next step for f_2 we will not be able to get rid of all λ terms by partial integration as in the first step:

$$f_2^{(i)}(x,y,\lambda) = -\exp\left(-\int_y^x \lambda k_i^{-1}(u)\,du\right)\exp\left(-\int_y^x k_i^{-1}(u)c_{ii}(u)\,du\right)\sum_{\substack{1\le l_2,l_1\le n\\ l_2\ne i,l_1\ne l_2}} y_0^{(l_1)}\lambda$$
$$\int_y^x \Bigg\{\exp\left(\int_y^{z_2}\lambda\left(k_i^{-1}(u)-k_{l_1}^{-1}(u)\right)\,du\right)\frac{c_{il_2}(z_2)c_{l_2l_1}(z_2)}{k_i(z_2)k_{l_2}(z_2)}$$
$$\frac{\exp\left(\int_y^{z_2}\left(\frac{c_{ii}(u)}{k_i(u)}-\frac{c_{l_1l_1}(u)}{k_{l_1}(u)}\right)\,du\right)}{k_{l_2}^{-1}(z_2)-k_{l_1}^{-1}(z_2)} - \exp\left(\int_y^{z_2}\lambda\left(k_i^{-1}(u)-k_{l_2}^{-1}(u)\right)\,du\right)$$
$$\exp\left(\int_y^{z_2}\left(k_i^{-1}(u)c_{ii}(u)-k_{l_2}^{-1}(u)c_{l_2l_2}(u)\right)\,du\right)\frac{c_{il_2}(z_2)}{k_i(z_2)}\Bigg[\frac{c_{l_2l_1}(y)}{k_{l_2}(y)}$$
$$\frac{1}{k_{l_2}^{-1}(y)-k_{l_1}^{-1}(y)} + \int_y^{z_2}\exp\left(\int_y^{z_1}\lambda\left(k_{l_2}^{-1}(u)-k_{l_1}^{-1}(u)\right)\,du\right)$$
$$\frac{d}{dz_1}\left(\frac{c_{l_2l_1}(z_1)}{k_{l_2}(z_1)}\frac{\exp\left(\int_y^{z_1}\left(\frac{c_{l_2l_2}(u)}{k_{l_2}(u)}-\frac{c_{l_1l_1}(u)}{k_{l_1}(u)}\right)\,du\right)}{k_{l_2}^{-1}(z_1)-k_{l_1}^{-1}(z_1)}\right)\,dz_1\Bigg]\Bigg\}\,dz_2.$$

Partial integration is not possible for the terms in the sum corresponding to $l_1 = i$. Therefore we are forced to keep $(n-1)$ terms containing λ factors:

$$f_2^{(i)}(x,y,\lambda) = -\lambda\exp\left(-\int_y^x \lambda k_i^{-1}(u)\,du\right)\exp\left(-\int_y^x \frac{c_{ii}(u)}{k_i(u)}\,du\right)y_0^{(i)}$$
$$\sum_{\substack{1\le l_2\le n\\ l_2\ne i}}\int_y^x \frac{c_{il_2}(z_2)c_{l_2i}(z_2)}{k_i(z_2)k_{l_2}(z_2)}\frac{1}{k_{l_2}^{-1}(z_2)-k_i^{-1}(z_2)}\,dz_2$$
$$+\text{ terms of order } 1$$

However, in the next third step for these $(n-1)$ terms containing a λ factor partial integration can be done, so that in the third step there will be no λ^2 factors, only λ or 1 factors. Factors with λ^2 in the multisums will first appear in the fourth step. Thus, generally for $m \in \mathbb{N}$, terms containing λ^m factors appear for the first time in the $(2m)$-th recursion step. Besides these λ^m terms there only appear terms, which are bounded for $\lambda \in \mathbb{C}_r$, where the bound depends on r, C and K only. Thus, if we reorder the summands in the partial sums $\sum_{l=0}^p \lambda^{-l} f_k^{(i)}(x,y,\lambda)$ after partial integration - when possible - we see that there will be contributions for terms of type λ^{-m} only up to the $(2m)$-th recursion step. Since the number of terms in the m-th recursion step can not $(3(n-1))^m$ after partial integration and reordering we arrive at a series $\sum_{l=0}^\infty \lambda^{-l} g_l(x,y,\lambda)$ where each $g_l(x,y,\lambda)$ is of order 1 for $\lambda \in \mathbb{C}_r$ and the rough estimate

$$|g_l(x,y,\lambda)| \le l(3(n-1)\tilde{C})^{2l} \le \left(2\cdot 3(n-1)\tilde{C})^2\right)^l,$$

where $\tilde{C}$ is some constant depending only on K,C and r, is valid. Thus, if $r > 0$ is given, then for all $\lambda \in \mathbb{C}_r$ and $\mathfrak{Im}(\lambda)$ sufficiently large we see that the series $\sum_{l=0}^{\infty} |\lambda|^{-l}|g_l(x,y,\lambda)|$ is dominated by a convergent geometric series. We have proven the following

Lemma 4.6. *There exists a sequence $F_k(x,y,\lambda)$ of matrices, which has the following properties:*

ı) Each F_k can be calculated from T_n for $n = 1,\dots,2k$. We have $F_0 = T_0$ and F_1 is the matrix with the i-th diagonal element, $1 \le i \le n$,

$$\begin{aligned}(F_1(x,y,\lambda))_{ii} = -\exp&\left(-\int_y^x \lambda k_i^{-1}(u)\,du\right)\exp\left(-\int_y^x \tfrac{c_{ii}(u)}{k_i(u)}\,du\right) \\ &\sum_{\substack{1\le\nu\le n\\ \nu\ne i}} \int_y^x \frac{c_{i\nu}(z)}{k_i(z)}\rho_{\nu i}(z)\,dz,\end{aligned} \tag{4.18}$$

where

$$\rho_{lm}(z) := \frac{c_{lm}(z)}{k_l(z)}\frac{1}{k_l^{-1}(z)-k_m^{-1}(z)},\ z\in[0,l],\ 1\le l,m\le n,\ l\ne m,$$

and the i-th row and j-th column, $1 \le i,j \le n$, $i \ne j$,

$$\begin{aligned}(F_1(x,y,\lambda))_{ij} = &-\exp\left(-\int_y^x \lambda k_j^{-1}(u)\,du\right)\exp\left(-\int_y^x \tfrac{c_{jj}(u)}{k_j(u)}\,du\right)\rho_{ij}(x) \\ &+\exp\left(-\int_y^x \lambda k_i^{-1}(u)\,du\right)\exp\left(-\int_y^x \tfrac{c_{ii}(u)}{k_i(u)}\,du\right)\rho_{ij}(y) \\ &+\exp\left(-\int_y^x \lambda k_j^{-1}(u)\,du\right)\exp\left(-\int_y^x \tfrac{c_{jj}(u)}{k_j(u)}\,du\right) \\ &\int_y^x \exp\left(\int_y^z \lambda\left(k_i^{-1}(u)-k_j^{-1}(u)\right)\,du\right) \\ &\exp\left(\int_y^z \left(\tfrac{c_{ii}(u)}{k_i(u)}-\tfrac{c_{jj}(u)}{k_j(u)}\right)\,du\right) \\ &\left\{\rho_{ij}(z)\left(\frac{c_{ii}(z)}{k_i(z)}-\frac{c_{jj}(z)}{k_j(z)}\right)\,dz + d\rho_{ij}(z)\right\}.\end{aligned} \tag{4.19}$$

ıı) For $r > 0$ there exists a constant $c > 0$ such that

$$\|F_k(x,y,\lambda)\| \le c^k \quad \textit{for } \lambda\in\mathbb{C}_r \textit{ and } x,y\in[0,l] \textit{ and } k = 1,2,\dots.$$

ııı) For $r > 0$ there exists $d > 0$ such that for $\lambda \in \mathbb{C}_r$ with $|\mathfrak{Im}(\lambda)| > d$ the series $\sum_{k=0}^{\infty} \lambda^{-k} F_k(x, y, \lambda)$ converges absolutely (in $L^\infty([0, l] \times [0, l], \mathbb{C}^{n\times n})$) to $T(x, y, \lambda)$. For $r > 0$ there exist $c, d > 0$ such that for $\lambda \in \mathbb{C}_r$ and $|\mathfrak{Im}\lambda| > d$ we have

$$\left\| T(x, y, \lambda) - T_0(x, y, \lambda) - \frac{1}{\lambda} F_1(x, y, \lambda) \right\| \leq c \frac{1}{|\lambda|^2}.$$

From (4.16) and Lemma 4.6, ıı), we have

Remark 4.7. *For $r > 0$ there exists $d > 0$ such that for $\lambda \in \mathbb{C}_r$ with $|\mathfrak{Im}(\lambda)| > d$ the series $\sum_{k=0}^{\infty} \lambda^{-k} F_k(\cdot, y, \lambda)$ converges in $W^{1,\infty}([0, l], \mathbb{C}^{n\times n})$ to $T(\cdot, y, \lambda)$ for $y \in [0, l]$.*

Let Tr denote trace and Ad the adjugate of a square matrix, i.e. $\mathrm{Ad}(M) = (b_{ij})_{1\leq i,j\leq n}$, where $b_{ij} = (-1)^{i+j} \det(M_{ji})$ and M_{ji} is the matrix after deletion of the j-th row and i-th column of M.

Then Lemma 4.6, ııı), and Jacobi's Formula for the derivative $\mathrm{D}\det$ of the determinant of a matrix,

$$(\mathrm{D}\det)(B)H = \mathrm{Tr}(\mathrm{Ad}(B)\, H),$$

where B, H are matrices, imply the following Lemma:

Lemma 4.8. *For $r > 0$ there exist $c, d > 0$ such that for $\lambda \in \mathbb{C}_r$ and $|\mathfrak{Im}\lambda| > d$ we have*

$$\left\| \tilde{H}(\lambda) - H_0(\lambda) - \frac{1}{\lambda} H_1(\lambda) \right\| \leq c \frac{1}{|\lambda|^2},$$

where

$$\begin{aligned} \tilde{H}(\lambda) &:= \lambda^{-1} H(\lambda), \\ H_1(\lambda) &:= -(F, G) T_0(\cdot, 0, \lambda) \begin{pmatrix} E \\ I \end{pmatrix} - (D\delta_l, -I\delta_l) F_1(\cdot, 0, \lambda) \begin{pmatrix} E \\ I \end{pmatrix}, \end{aligned} \tag{4.20}$$

$$\left| \tilde{h}(\lambda) - h_0(\lambda) - \frac{1}{\lambda} \mathrm{Tr}\left(\mathrm{Ad}(H_0(\lambda)) H_1(\lambda)\right) \right| \leq c \frac{1}{|\lambda|^2}, \tag{4.21}$$

where

$$\tilde{h}(\lambda) := \det \tilde{H}(\lambda), \tag{4.22}$$

and

$$\left\| \tilde{H}(\lambda)^{-1} - H_0(\lambda)^{-1} + \frac{1}{\lambda} H_0(\lambda)^{-1} H_1(\lambda) H_0(\lambda)^{-1} \right\| \leq c \frac{1}{|\lambda|^2}.$$

By definition $\lambda^n \tilde{h}(\lambda) = h(\lambda)$. Therefore

$$\sigma(A) \setminus \{0\} = \{\lambda \in \mathbb{C} \mid \tilde{h}(\lambda) = 0\}.$$

It is important to note that for $\lambda \in \mathbb{C}_r$ the matrices $H_0(\lambda)$ and $H_1(\lambda)$ are bounded (the bound depending on r). Therefore (4.21) roughly states that for $|\mathfrak{Im}\lambda| \to \infty$ the eigenvalues of (H) are close to the eigenvalues of the reduced system (H_0), which will be stated in Lemma 4.14.

We need the following Lemma

Lemma 4.9. *Let f be an exponential polynomial of the form*

$$f(\lambda) = \sum_{j=1}^{r} a_j e^{b_j \lambda} \quad (\lambda, a_j \in \mathbb{C},\ b_j \in \mathbb{R}). \tag{4.23}$$

Let $Z = \{\lambda \in \mathbb{C} \mid f(\lambda) = 0\}$ denote the zero set of f. For all $\delta > 0$, $\alpha, \beta \in \mathbb{R}$ with $\alpha \leq \beta$ and $M \geq 0$ there exists a constant $m = m(\delta, \alpha, \beta, M) > 0$ such that for all $\lambda \in \mathbb{C}$, which satisfy $\operatorname{dist}(\lambda, Z) \geq \delta$, $\alpha \leq \mathfrak{Re}\lambda \leq \beta$ *and* $|\mathfrak{Im}\lambda| \geq M$, *we have*

$$|f(\lambda)| > m(\delta, \alpha, \beta, M).$$

Proof. We fix the proof of [13, Lemma 2.2]. Suppose the assertion was false. Then there exists a sequence $(z_m)_{m\in\mathbb{N}}$ in $\overline{\mathbb{C}_{\alpha,\beta}}$ so that

$$f(z_m) \to_{m\to\infty} 0, \quad \operatorname{dist}(z_m, Z) \geq \delta > 0.$$

Put

$$g_m(z) := f(z + z_m) = \sum_{j=1}^{r} a_j e^{b_j z} e^{b_j x_m} e^{i b_j y_m}, \quad \text{where } z_m =: x_m + i y_m.$$

Then there exists a subsequence so that g_m converges uniformly on $\mathbb{C}_{\alpha,\beta}$ to some function g. Indeed, by passing to a subsequence we can assume that $x_m \to x \in [\alpha, \beta]$ and $e^{i b_j y_m} \to s_j \in S^1$. Obviously g_m converges uniformly to $g = \sum_{j=1}^{r} a_j e^{b_j z} e^{b_j x} s_j$ on $\overline{\mathbb{C}_{\alpha,\beta}}$. Let $U := \left\{z \in \mathbb{C} \mid |z| < \frac{\delta}{2}\right\}$. By assumption $g_m(z) \neq 0$ for $z \in U$ and $g(0) = \lim g_m(0) = \lim f(z_m) = 0$. By Hurwitz theorem g must be identical to zero on U, and therefore on $\mathbb{C}$. Hence for all $z \in \mathbb{C}$ we get that $f(z) = g_m(z - z_m) \to_{m\to\infty} 0$. Which is a contradiction to the assumption that $\operatorname{dist}(z_m, Z) \geq \delta > 0$. $\square$

As a direct consequence of Lemma 4.9 (see also [10, Corollary 1, p. 145]) one has

Remark 4.10. *Let f be an exponential polynomial of the form* (4.23) *and let $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, be such that $f(\lambda) \neq 0$ for $\alpha \leq \mathfrak{Re}\lambda \leq \beta$. Then for any $0 < \delta \leq (\beta - \alpha)/2$*

$$\inf_{\lambda \in \mathbb{C}, \alpha+\delta \leq \mathfrak{Re}\lambda \leq \beta-\delta} |f(\lambda)| > 0.$$

An exponential polynomial belongs to the class of sine type functions:

Definition 4.11 (sine-type function)**.** An entire function $f : \mathbb{C} \to \mathbb{C}$ is called a sine type function if

- there exist constants $a, b > 0$ so that $|f(z)| \leq ae^{b|z|}$ for $z \in \mathbb{C}$ (f is of exponential type)
- the zeros of f lie in a strip $\mathbb{C}_{-h,h}$ for some $h > 0$
- there exist $x \in \mathbb{R}$ and $m, u > 0$ so that $m \leq |f(x + iy)| \leq u$ for all $y \in \mathbb{R}$.

We need the following proposition [13, Proposition 2.1] [1, Proposition II.1.28, p.61] on the distribution of zeros of a sine-type function

Proposition 4.12. *Let f be a sine-type function. Then its set Z of zeros is a finite union of separable sets, that is there exist $m < \infty$ sets Z_i with*

$$Z = \cup_{i=1}^{m} Z_i, \qquad \inf_{\lambda,\mu \in Z_i,\ \lambda \neq \mu} |\lambda - \mu| > 0.$$

Remark 4.13. *In Proposition 4.12 a multiple zero is repeated in a number of times of its multiplicity. Hence the multiplicities of zeros of a sine type function is uniformly upper bounded (by m).*

If (H_0) has nonempty spectrum we define

$$\gamma_- := \inf \{\mathfrak{Re}\lambda \mid h_0(\lambda) = 0\} \text{ and } \gamma_+ := \sup \{\mathfrak{Re}\lambda \mid h_0(\lambda) = 0\}.$$

From Proposition 4.4 γ_- and γ_+ are finite. If (H_0) has empty spectrum then we put by definition $\gamma_+ := -\infty$ and $\gamma_- := \infty$.

For $\lambda \in \mathbb{C}$ and $\epsilon > 0$ let $B_\epsilon(\lambda) := \{z \in \mathbb{C} \mid |z - \lambda| < \epsilon\}$ denote the ball around λ with radius ϵ.

Lemma 4.14. *For each $\gamma > \gamma_+$ there exist only finitely many eigenvalues λ of* (H) *that satisfy $\mathfrak{Re}\,\lambda \geq \gamma$.*

Proof. Let $\gamma > \gamma_+$. Denote

$$\sigma_+ := \{\lambda \in \mathbb{C} \mid \lambda \text{ is an eigenvalue of (H) with } \mathfrak{Re}\lambda \geq \gamma\}.$$

Suppose the set σ_+ was infinite. Because (H) generates a C_0 semigroup in the space Y, see Proposition 7.18, it follows that there exists an $\omega > \gamma$ so that $\sigma_+ \subset \overline{\mathbb{C}_{\gamma,\omega}}$. Because the characteristic function $h(\lambda)$ of (H) is analytic it follows that the infinitely many eigenvalues of σ_+ must accumulate at infinity within the closed stripe $\overline{\mathbb{C}_{\gamma,\omega}}$. Because $\gamma > \gamma_+$ it follows from Remark 4.10 that

$$\inf_{\lambda \in \overline{\mathbb{C}_{\gamma,\omega}}} |h_0(\lambda)| > 0,$$

where h_0 is the characteristic function to (H_0) defined in (4.7). Therefore (4.21) implies that for sufficiently large $d > 0$

$$\inf\{|\tilde{h}(\lambda)| \mid \lambda \in \overline{\mathbb{C}_{\gamma,\omega}},\ |\mathfrak{Im}\,\lambda| \geq d\} > 0.$$

Hence if we choose $d > 0$ sufficiently large we get a contradiction to the assumption that there existed infinitely many $\lambda \in \overline{\mathbb{C}_{\gamma,\omega}}$ with $|\mathfrak{Im}\,\lambda| \geq d$ and $\tilde{h}(\lambda) = \lambda^{-n}h(\lambda) = 0$. Hence σ_+ is finite. □

Lemma 4.15. *Suppose* (H_0) *has nonempty spectrum. Then the following hold:*
ı) For each $\delta > 0$ *there are only finitely many zeros* λ *of* h *which satisfy* $\mathfrak{Re}\lambda \leq \gamma_- - \delta$ *or* $\mathfrak{Re}\lambda \geq \gamma_+ + \delta$*.*
ıı) For $\epsilon > 0$ *there exists* $d > 0$ *such that*

$$\sigma(A) \cap \{\lambda \in \mathbb{C} \mid |\mathfrak{Im}\lambda| \geq d\} \subset \bigcup_{h_0(\lambda)=0} B_\epsilon(\lambda).$$

ııı) Suppose $\rho = \inf_{\lambda_1 \neq \lambda_2, h_0(\lambda_1)=h_0(\lambda_2)=0} |\lambda_1 - \lambda_2| > 0$*. Then for each* $\eta < \frac{\rho}{2}$ *there exists* $d > 0$ *such that for each* $\lambda_0 \in \mathbb{C}$ *with* $h_0(\lambda_0) = 0$ *and* $|\mathfrak{Im}\lambda_0| \geq d$ *there exists* $\lambda \in B_\eta(\lambda_0)$ *with* $h(\lambda) = 0$*. Both* h *and* h_0 *have the same number of zeros in each* $B_\eta(\lambda_0)$*. In particular, if* (H_0) *only possesses algebraically simple eigenvalues, then the eigenvalues* $\lambda \in B_\eta(\lambda_0)$ *of* (H) *are unique and algebraically simple.*

Proof. Because h_0 is an exponential polynomial (H_0) has infinitely many eigenvalues. Let $\delta > 0$ be arbitrary and fixed. For $\theta \in [0,1]$ consider the family of operators corresponding to a perturbation from the diagonal operator with $C = C_0$ to the nondiagonal one with $C = C_0 + C_1$

$$A_\theta(u,v,d) := \left(-K(x)\frac{\partial}{\partial_x}\begin{pmatrix} u \\ v \end{pmatrix} - (C_0(x) + \theta C_1(x))\begin{pmatrix} u \\ v \end{pmatrix};\quad \theta F u(\cdot) + \theta G v(\cdot)\right).$$

Let $h^\theta(\lambda)$ denote the corresponding characteristic function. Put $\tilde{h}^\theta(\lambda) := \frac{1}{\lambda^n} h^\theta(\lambda)$. Note that $\tilde{h}^0 = h_0$. According to Lemma 4.9 outside of balls of radius $\frac{\delta}{2}$ around each of the zeros of h_0 in the strip $\mathbb{C}_{\gamma_- - \delta, \gamma_+ + \delta}$ the function $|h_0|$ has an infimum $m > 0$. From (4.21) in Lemma 4.8 there exist $c, d > 0$ such that

$$|\tilde{h}^\theta(\lambda) - h_0(\lambda)| \leq c \frac{1}{\lambda} \tag{4.24}$$

for $\lambda \in \mathbb{C}_{\gamma_- - \delta, \gamma_+ + \delta}$ with $|\mathfrak{Im}(\lambda)| > d$. Therefore for

$$\lambda \in \mathbb{C}_{\gamma_- - \delta, \gamma_+ + \delta} \setminus \cup_{h_0(\lambda_0) = 0} B(\lambda_0, \delta/2)$$

with $|\mathfrak{Im}(\lambda)|$ sufficiently large it follows from (4.24) that $|\tilde{h}^\theta(\lambda)| \geq m/2 > 0$. And this holds true uniformly in $\theta \in [0, 1]$. Starting from $\theta = 0$ this shows that, as long as we increase θ up to 1 all but finitely many zeros of $h^\theta(\lambda)$ must stay in a $\delta/2$-ball of an zero of $h_0(\lambda)$. By the continuity of a finite system of zeros with respect to the perturbation parameter θ [38] it follows that $\{\lambda \in \mathbb{C} \mid \mathfrak{Re}\lambda \leq \gamma_- - \delta\} \cup \{\lambda \in \mathbb{C} \mid \mathfrak{Re}\lambda \geq \gamma_+ + \delta\}$ contains only finitely many eigenvalues. The remaining assertions follow by applying Lemma 4.9 and Rouchés Theorem. $\square$

In the following let Π denote the projection of X onto $L^p([0, l], \mathbb{C}^n)$ or $C([0, l], \mathbb{C}^n)$ by dropping the boundary component $\mathbb{C}^{n_2}$.
From the Lemmas 4.6 and 4.8 we get the following explicit approximation of the resolvent (4.9)

Lemma 4.16. *Suppose there exist $\alpha \in \mathbb{R}$, $\delta, \epsilon > 0$ such that for $\lambda \in \mathbb{C}$ with $|\mathfrak{Re}\lambda - \alpha| < \delta$ one has $|h_0(\lambda)| \geq \epsilon$. Then there exist constants $c, d > 0$ such that for all $\lambda \in \mathbb{C}$ with $|\mathfrak{Re}\lambda - \alpha| < \delta$ and $|\mathfrak{Im}\lambda| > d$ we have $\lambda \in \rho(A)$ and*

$$R(\lambda, A) \begin{pmatrix} f \\ g \\ b \end{pmatrix} = \begin{pmatrix} u \\ v \\ (-D, I)\delta_l \begin{pmatrix} u \\ v \end{pmatrix} \end{pmatrix},$$

where

$$\begin{aligned} \begin{pmatrix} u \\ v \end{pmatrix} =& R(\lambda, A_0) \begin{pmatrix} f \\ g \end{pmatrix} \\ &+ \frac{1}{\lambda} \left(R_1(\lambda) \begin{pmatrix} f \\ g \end{pmatrix} + R_2(\lambda) \begin{pmatrix} f \\ g \end{pmatrix} + R_3(\lambda) \begin{pmatrix} f \\ g \\ b \end{pmatrix} + R_4(\lambda) \begin{pmatrix} f \\ g \end{pmatrix} \right) \\ &+ \frac{1}{\lambda^2} \mathcal{E}(\lambda)(f, g, b), \end{aligned} \tag{4.25}$$

and each of the operators $R(\lambda, A_0)$, $R_1(\lambda)$, $R_2(\lambda)$, $\mathbb{R}_3(\lambda)$, $R_4(\lambda)$ *and* $\mathcal{E}(\lambda)$ *is bounded by c. Here* $R(\lambda, A_0)$ *denotes the resolvent of the reduced operator defined in* (4.10), β_0 *has been defined in* (4.11) *and the remaining terms are as follows:*

$$\begin{aligned}
R_1(\lambda)\begin{pmatrix} f \\ g \end{pmatrix} &:= F_1(\cdot, 0, \lambda)\begin{pmatrix} E \\ I \end{pmatrix} H_0(\lambda)^{-1}\beta_0(\lambda)(f,g), \\
R_2(\lambda)\begin{pmatrix} f \\ g \end{pmatrix} &:= -T_0(\cdot, 0, \lambda)\begin{pmatrix} E \\ I \end{pmatrix} H_0(\lambda)^{-1}H_1(\lambda)H_0(\lambda)^{-1}\beta_0(\lambda)(f,g), \\
R_3(\lambda)\begin{pmatrix} f \\ g \\ b \end{pmatrix} &:= T_0(\cdot, 0, \lambda)\begin{pmatrix} E \\ I \end{pmatrix} H_0(\lambda)^{-1}\beta_1(\lambda)(f,g,b), \\
R_4(\lambda)\begin{pmatrix} f \\ g \end{pmatrix} &:= \int_0^{\cdot} F_1(\cdot, y, \lambda)K(y)^{-1}\begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy, \\
\beta_1(\lambda)(f,g,b) &:= b + (D, -I)\int_0^l F_1(l, y, \lambda)K(y)^{-1}\begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy \\
&\quad + (F, G)\int_0^{\cdot} T_0(\cdot, y, \lambda)K(y)^{-1}\begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy.
\end{aligned}$$

Remark 4.17. *Suppose* $h_0(\lambda) \neq 0$ *for all* $\lambda \in \mathbb{C}$ *within the stripe* $|\mathfrak{Re}\lambda - \alpha| < r$*, with some* $\alpha \in \mathbb{R}$ *and* $r > 0$*. Because* h_0 *is an exponential polynomial of the form* (4.23) *Remark 4.10 implies that for* $\delta < r$

$$\inf_{\lambda \in \mathbb{C}, |\mathfrak{Re}\lambda - \alpha| \leq \delta} |h_0(\lambda)| > 0.$$

Hence the assumption of our previous Lemma is satisfied.

Remark 4.18. *Consider the problem with static boundary conditions*

$$(\mathrm{H}_1) \qquad \begin{cases} \dfrac{\partial}{\partial t}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + K(x)\dfrac{\partial}{\partial x}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + C(x)\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = 0, \\ \\ u(t,0) = Ev(t,0) \quad \text{and} \quad v(t,l) = Du(t,l) \end{cases}$$

for $x \in\,]0,l[$ *and* $t > 0$*. The phase space for* (H_1) *we consider are* $X_1 = L^p([0,l], \mathbb{C}^n)$ *or* $X_1 = C([0,l], \mathbb{C}^n)$ *as for* (H_0)*. Let* $T_1^t : X_1 \to X_1$ *denote the* C_0 *semigroup for* (H_1) *and* $T^t : X \to X$ *the semigroup corresponding to* (H) *with* $F = 0$ *and* $G = 0$ *in the extended phase space* X_p *or* Y*. Then for* $t \geq 0$ *we have* $T^t(u,v,0) = (T_1^t(u,v), 0)$*. The resolvent formula for* (H_1) *is identical to that of* (H_0) *in* (4.10) *with* T_0 *simply replaced by* T *and* H_0 *replaced by* $\tilde{H}$ *with* $F = 0$ *and* $G = 0$*. We define the characteristic function for* (H_1) *to be*

$\tilde{h}$ *which we have defined in* (4.22) *for* (H). *If* A_1 *denotes the generator for* (H_1) *and* A *the generator for problem* (H) *with* $F = 0$ *and* $G = 0$ *then*

$$R(\lambda, A_1)(f, g) = \Pi R(\lambda, A)(f, g, 0).$$

Example 4.19. *As a first example we consider a simple linear model for the pulsation of distributed feedback (DFB) semiconductor lasers: For* $t > 0$ *and* $x \in]0, l[$

$$\begin{cases} \partial_t \psi_1(t,x) = v_{gr}\left(-\partial_x \psi_1(t,x) + \beta(x)\psi_1(t,x) + \kappa_1(x)\psi_2(t,x)\right) \\ \partial_t \psi_2(t,x) = v_{gr}\left(\partial_x \psi_2(t,x) + \beta(x)\psi_2(t,x) + \kappa_2(x)\psi_1(t,x)\right) \\ \psi_1(t,0) = r_0\psi_2(t,0) \quad \psi_2(t,l) = r_l\psi_1(t,l) \end{cases}. \tag{4.26}$$

Recall from section 3.2 that ψ_1 *and* ψ_2 *denote the slowly varying complex amplitudes of the forward and backward traveling waves of the electric field,* $l > 0$ *is the length of the laser,* κ_1 *and* κ_2 *are complex coupling coefficients,* v_{gr} *is the constant group velocity,* β *is a propagation constant and* r_0, r_l *are complex reflection coefficients at the left and right facet of the laser. The semiconductor laser is composed of several (typically two or three) different laser sections. Hence the spatially dependent coefficients* $\kappa_1(x)$, $\kappa_2(x)$ *and* $\beta(x)$ *possess several discontinuities at the junctions of each laser section. For* $\kappa_1 \equiv \kappa_2 \equiv 0$ (4.26) *describes the so called Fabry Perot laser which corresponds to the reduced system* (H_0) *to* (4.26). *The characteristic equation for the reduced system is*

$$h_0(\lambda) = \frac{1}{2}\log(r_0 r_l) v_{gr} l^{-1} + l^{-1}\int_0^l \beta(x)\, dx.$$

The eigenvalues of the Fabry Perot laser (zeros of h_0*) are*

$$\lambda = \frac{1}{2}\log(r_0 r_l) l^{-1} v_{gr} + l^{-1}\int_0^l \beta(x)\, dx + 2\pi i z \quad (z \in \mathbb{Z}).$$

In dimensionless variables v_{gr}, l *and* β *become of order one, see section 3.2. According to condition* (HII) *we require that*

$$\beta \in L^\infty(]0, l[\,, \mathbb{C}) \quad \textit{and} \quad \kappa_1, \kappa_2 \in BV([0, l], \mathbb{C}).$$

According to Remark 4.18 system (4.26) *can be considered as a special case for* (H), *where the right boundary condition* $\psi_2(t,l) = r_l\psi_1(t,l)$ *is replaced by* $\partial_t(\psi_2(t,l) - r_l\psi_1(t,l)) = 0$ *(*$F = 0$ *and* $G = 0$*). Because the boundary conditions are static,* (4.26) *can be considered as an abstract evolution equation in* $L^p(]0, l[, \mathbb{C}^2)$, $1 \le p < \infty$, *or* $C([0, l], \mathbb{C}^2)$. *Let* $f(\lambda)$ *denote the characteristic equation to* (4.26) *and* $h(\lambda)$ *be the characteristic equation when* (4.26) *is*

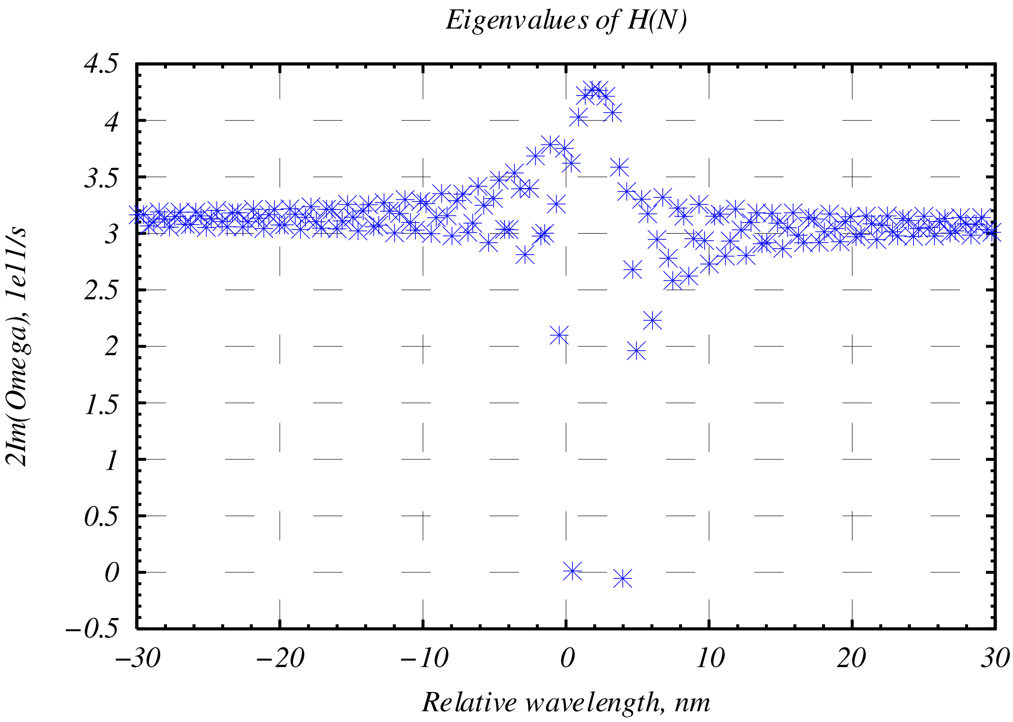

Figure 4.1: Spectrum of the traveling wave operator (4.26) calculated using LDSL. Here the horizontal axis corresponds to the imaginary axis and the vertical axis to the real axis. Two modes are close to the imaginary axis.

written in the augmented form (H)*. According to Lemma 4.14 the eigenvalues of* (4.26) *lie in a strip. Since* $\tilde{h}(\lambda) = f(\lambda)$ *Lemma 4.8 yields for sufficiently large* λ

$$|f(\lambda) - h_0(\lambda)| \leq \frac{c}{\lambda}.$$

In Figure 4.1 we have calculated the spectrum of (4.26) *using LDSL tool [53, 56, 80] under physical realistic parameter constellations.*

Example 4.20. *Consider the Carleman model* (3.25)*. For* $d \in \mathbb{R}$ (3.25) *has the homogenous equilibrium state* $u = v = d$*. We study the linearization of* (3.25) *in this equilibrium*

$$\begin{aligned}\frac{1}{\sqrt{2}}\left(\partial_t u + \partial_x u\right) =& 2dv - 2du,\\ \frac{1}{\sqrt{2}}\left(\partial_t v - \partial_x v\right) =& 2du - 2dv.\end{aligned}$$

Then after straightforward calculations we get the following expression for the characteristic function $\tilde{h}$:

$$\tilde{h}(\lambda) = \lambda \frac{\exp\left(l\sqrt{\lambda(\lambda+4\sqrt{2}d)}\right) - \exp\left(-l\sqrt{\lambda(\lambda+4\sqrt{2}d)}\right)}{\sqrt{\lambda(\lambda+4\sqrt{2}d)}}.$$

Here $\sqrt{\lambda(\lambda+4\sqrt{2}d)}$ denotes one of the complex square roots of $\lambda(\lambda+4\sqrt{2}d)$. It does not matter which one chooses because the expression for $\tilde{h}$ is not affected when one changes the sign of $\sqrt{\lambda(\lambda+4\sqrt{2}d)}$. The zeros of $\tilde{h}$ are given by the set

$$\sigma = \left\{ -2\sqrt{2}d \left(1 \pm \sqrt{1 - \tfrac{\pi^2}{8d^2l^2} z^2} \right) \mid z \in \mathbb{Z} \right\} \cup \{0\}.$$

The characteristic function h_0 of the reduced system is

$$h_0(\lambda) = \exp\left(l(\lambda + 2\sqrt{2}d) \right) - \exp\left(-l(\lambda + 2\sqrt{2}d) \right).$$

The spectrum σ_0 of the reduced system which is asymptotically close to σ according to (4.21) *is*

$$\sigma_0 = \left\{ -2\sqrt{2}d + i\frac{\pi}{l} z \mid z \in \mathbb{Z} \right\}.$$

Here we have $\gamma_+ = \gamma_- = -2\sqrt{2}d$ The spectrum σ is shown in figure 4.2. Since the root $\lambda = 0$ is of order one Theorem 4.5 implies that the eigenvalue 0 *has algebraic multiplicity one. Moreover we have that*

$$\sup_{\lambda \in \sigma \setminus \{0\}} \mathfrak{Re}\, \lambda < 0.$$

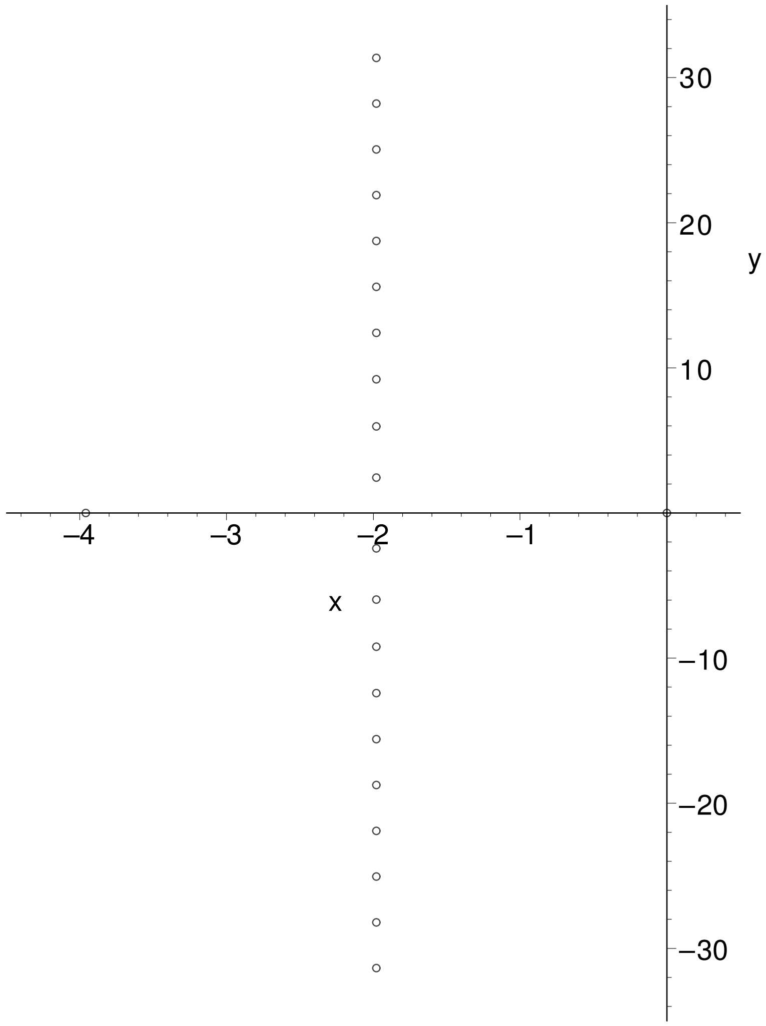

Figure 4.2: The Carleman spectrum for $d = 0.5$ and $l = 1$

Chapter 5

Exponential dichotomy / spectral gap mapping

In Lemma 4.14 we have seen that the closed, densely defined linear operator A corresponding to (H) always has a spectral gap near γ_+, that is for $\gamma > \gamma_+$ there exists $\eta > 0$ so that

$$\{\lambda \in \mathbb{C} \mid \gamma - \eta < \mathfrak{Re}\lambda < \gamma\} \subset \rho(A),$$

and there exist only finitely many eigenvalues (of finite algebraic multiplicity) λ with

$$\mathfrak{Re}\lambda \geq \gamma. \tag{5.1}$$

For such γ let π_1 denote the spectral projection of A corresponding to the finite system of eigenvalues satisfying (5.1) and denote $\pi_2 := I - \pi_1$. Recall that due to operational calculus the projections satisfy $\pi_1^2 = \pi_1$, $\pi_2^2 = \pi_2$, $\pi_1\pi_2 = \pi_2\pi_1 = 0$, $\pi_i\mathcal{D}(A) \subset \mathcal{D}(A)$ and $\pi_i Ax = A\pi_i x$, $x \in \mathcal{D}(A)$, $i = 1, 2$. This means that the chosen function space X decomposes into $X = X_1 \oplus X_2$, where $X_1 = \pi_1(X)$ is finite dimensional and $X_2 = \pi_2(X)$, the spaces X_1 and X_2 are invariant under A and the semigroup e^{At} of (H), i.e. $A(\mathcal{D}(A) \cap X_1) \subset X_1$, $e^{At}X_1 \subset X_1$ and $A(\mathcal{D}(A) \cap X_2) \subset X_2$, $e^{At}X_2 \subset X_2$. If A_i denotes the restriction of A on $\mathcal{D}(A) \cap X_i$, $i = 1, 2$, then it is easy to see that A_i is a closed, densely defined operator in X_i, that generates a C_0 semigroup of bounded operators on X_i (by the Hille Yosida theorem), and the semigroup e^{At} decomposes into $e^{At} = e^{A_1 t}\pi_1 + e^{A_2 t}\pi_2$.

Now the important question arises what information the location of the spectrum $\sigma(A)$ gives about the asymptotic behaviour of the solutions. In other words: How does the spectrum of A, which is estimated in Lemma 4.14 and can be easily calculated numerically from the zeros of h in a finite region of the complex plane, relate to the spectrum of the semigroup e^{At}. For example, if

$$\sup\{\mathfrak{Re}\lambda \mid h(\lambda) = 0\} < 0$$

one would want to conclude that

$$\sup\{|\lambda| \mid \lambda \in \sigma(e^{At})\} < 1 \quad (t > 0)$$

which yields exponential stability (by Proposition 5.11), i.e. there exist constants $M > 0$ and $\alpha < 0$ so that

$$\left\|e^{At}\right\| \leq Me^{\alpha t} \quad \text{for} \quad t \geq 0.$$

Moreover, one needs to relate a spectral gap for A (exponentially) to a spectral gap of e^{At}, so that one gets two exponential rates according to the location of the gap for decay or growth on X_1 and X_2. This is of basic importance for existence and smoothness of invariant manifolds, using a persistence theorem (see section 8.1), where the presence of a spectral gap for e^{At} (also called normal hyperbolicity) is required. For this one usually applies a spectral mapping property:

Definition 5.1 (Spectral mapping property)**.** The semigroup e^{At} has the spectral mapping property if for $t > 0$

$$\text{(SMP)} \qquad \sigma(e^{At}) \setminus \{0\} = e^{\sigma(A)t}.$$

But we note, that we only need a spectral gap mapping property:

Definition 5.2 (Spectral gap mapping property)**.** The semigroup e^{At} has the spectral gap mapping property if for $t > 0$ and $a < b$

$$\text{(SGM)} \qquad \mathbb{C}_{a,b} \subset \rho(A) \Leftrightarrow e^{\mathbb{C}_{a,b}t} \subset \rho(e^{At}).$$

Obviously (SMP) implies (SGM). Although in applications for invariant manifolds (SMP) is usually known, it does not hold in general for unbounded generators of C_0 semigroups. It is known that a strongly continuous semigroup has (SMP) if it belongs to one of the following classes of semigroups [20, 41]:

i) eventually norm continuous semigroups,
ii) eventually compact semigroups,
iii) eventually differentiable semigroups,
iv) analytic semigroups (parabolic equations),
v) uniformly continuous semigroups.

A main difficulty we are dealing with is that the hyperbolic system (H) is only C_0 and does not posses one of the regularizing properties $\imath$)-v) unless one restricts to trivial cases. Hence for (H) it is not obvious how the spectrum of A is related to the spectrum of e^{At}. Moreover, a remarkable counterexample found by M. Renardy [61, 41, 45], a lower order derivative perturbation of a two dimensional wave equation with periodic boundary condition, namely the system

$$\begin{cases} u_{tt} = u_{xx} + u_{yy} + e^{iy}u_x, & (x, y) \in \mathbb{R}^2, \\ u(x + 2\pi, y) = u(x, y), & u(x, y + 2\pi) = u(x, y), \end{cases}$$

is known, where the spectrum consists only of eigenvalues on the imaginary axis but the growth bound (see Def. 5.9) is greater or equal to $\frac{1}{2}$. In other words in two space dimensions both (SGM) and (SMP) fail.

It is well known that the failure of the spectral mapping property is completely determined by the continuous spectrum, see [49, Theorem 2.4, Theorem 2.5, p.46-47]. Hence in the counterexample of Renardy the semigroup has nonempty continuous spectrum which is not exponentially related to the spectrum of its generator (it contains point spectrum only).

This shows that one has to be extremely careful when one investigates the asymptotic behaviour of hyperbolic PDEs. Just the location of the spectrum does not give the sought information, in general.

Hence it is important to investigate the relation of the spectrum of A and e^{At} for the hyperbolic system (H). A result into this direction has been obtained by Neves, Ribeiro and Lopes [48] in the context of L^p spaces with $1 \leq p < \infty$. Their main result [48, Theorem A] is that for the semigroups e^{At} and $e^{A_0 t}$ in the space X_p (see (4.1)) with $1 \leq p < \infty$, where A_0 is the generator of the reduced system (H_0), the difference $e^{At} - e^{A_0 t} : X_p \to X_p$ is compact (here $e^{A_0 t}$ is trivially extended from L^p to X_p by setting the boundary $d = 0$). This implies from a well known fact that the essential spectral radii of e^{At} and $e^{A_0 t}$ must coincide. By showing that the essential spectral radius of $e^{A_0 t}$ is equal to $e^{\gamma_+ t}$ they conclude ([48, Theorem B])

Theorem 5.3. $\imath$) *For any* $\gamma > \gamma_+$ *the set* $\sigma(A) \cap \{\lambda \mid \mathfrak{Re}\lambda \geq \gamma\}$ *is finite,*
$\imath\imath$) *If* $|z| > e^{\gamma_+ t}$ *then* z *belongs to the spectrum of* e^{At} *if and only if* $z = e^{\lambda t}$ *for some* $\lambda \in \sigma(A)$*,*
$\imath\imath\imath$) *If* $\gamma > \gamma_+$ *and there is no solution of* $h(\lambda) = 0$ *satisfying* $\mathfrak{Re}(\lambda) = \gamma$ *then* $\|T(t)\pi_2\|_{X_p} \leq ce^{\gamma t}$, $t \geq 0$.

Our main result in this section, which is more general than Theorem 5.3, is the following Theorem:

Theorem 5.4 (Spectral gap mapping theorem)**.** *Let $a < b$, $a, b \in \mathbb{R}$, $X = X_p$ for $p \in [1, \infty[$ or $X = Y$. Then for $t > 0$* (SGM) *holds for* (H)*:*

$$\mathbb{C}_{a,b} \subset \rho(A) \quad \textit{if and only if} \quad \{\lambda \in \mathbb{C} \mid e^{at} < |\lambda| < e^{bt}\} \subset \rho(e^{At}).$$

It will become obvious (see section 5.2) that Theorem 5.4 is equivalent to the following Theorem on exponential dichotomy (see Def. 5.14 for the notion of (α, β) exponential dichotomy):

Theorem 5.5 (Exponential dichotomy)**.** *Let $\alpha \leq \beta$, $\alpha, \beta \in \mathbb{R}$. System* (H) *is (α, β) exponentially dichotomous in the spaces Y and X_p, $p \in [1, \infty[$, if and only if there exists $\delta > 0$ so that $h(\lambda) \neq 0$ for $\lambda \in \mathbb{C}$ with $\alpha - \delta < \mathfrak{Re}\lambda < \beta + \delta$. In this case the exponential rates are independent on $p \geq 1$.*

As an immediate consequece of Theorem 5.4 we get

Corollary 5.6. *Growth bound and spectral bound of* (H) *coincide:*

$$\omega(A) = s(A).$$

As another special consequence we get the following improvement of Theorem 5.3 (we will see that in *iii*) the constant c is independent on p and the exponential rate also holds in L^∞ or Y.)

Theorem 5.7. *Let $X = X_p$ for $p \in [1, \infty]$ or $X = Y$. Let $\gamma > \gamma_+$ be such that $h(\gamma + is) \neq 0$, $s \in \mathbb{R}$. Let π_1 be the spectral projection according to the finite eigenvalues λ with $\mathfrak{Re}\lambda \geq \gamma$, $\pi_2 := I - \pi_1$ and A_i be the restriction of A to the invariant subspace $\pi_i(X)$, $i = 1, 2$. For all $\gamma_1 > \gamma$ and $\gamma_+ < \gamma_2 < \gamma$ such that $\{\lambda \in \mathbb{C} \mid \gamma_2 \leq \mathfrak{Re}\,\lambda \leq \gamma_1\} \subset \rho(A)$ there exist constants $c_1 > 0$ and $c_2 > 0$, which are independent on the choice of X (i.e. independent on $p \geq 1$), so that for $t \geq 0$*

$$\left\|e^{-A_1 t}\right\|_{\mathcal{L}(\pi_1(X))} \leq c_1 e^{-\gamma_1 t} \quad \textit{and} \quad \left\|e^{A_2 t}\right\|_{\mathcal{L}(\pi_2(X))} \leq c_2 e^{\gamma_2 t}.$$

In particular we have: If $|z| > e^{\gamma_+ t}$ then z belongs to the spectrum of e^{At} if and only if $z = e^{\lambda t}$ for some $\lambda \in \sigma(A)$.

As we have already pointed out in the introduction of this work our main Theorems 5.4 and 5.5 are obtained not only in the spaces X_p (L^p) ($1 \leq p < \infty$), but in the smaller space Y with the stronger sup norm, which is needed for nonlinear problems. Semilinear hyperbolic systems of class (SH) do not form a smooth semiflow in L^p or X_p for $1 \leq p < \infty$, but in the smaller admissible subspace Y equipped with the sup norm. The reason being that nonlinear Nemytskij operators are not differentiable from the large L^p space into itself. Because it is possible to expand the Nemytskij operator as a map from the "small" space Y into L^∞, we will be able to make conclusions on the asymptotic behaviour for the nonlinear system by locating spectral properties of the generator of the linearized system only. In sections 7-8 our main results Theorems 5.4 and 5.5 for the smaller space Y will be the basis to prove linearized stability and the existence of smooth center manifolds at equilibria for semilinear hyperbolic systems when the usual assumptions on the location of the spectrum of the generator for the linearization are known.

We prove Theorems 5.4 and 5.5 by using the theory of Kaashoek, Lunel and Latushkin [36, 42] together with the resolvent estimates (4.25). First we introduce some basic notions and propositions and recall the general results of Kaashoek, Lunel and Latushkin in section 5.1. Then we prove the main Theorems 5.4 and 5.5 in section 5.2.

5.1 General abstract theory: Growth rate, spectral gap, characterization of exponential dichotomy in terms of the resolvent (results of Kaashoek, Lunel and Latushkin)

In this section A will denote the infinitesimal generator of a C_0 semigroup $e^{At} = T(t)$ of bounded linear operators on a Banach space X.

We have the following spectral inclusion theorem [41, Theorem 2.6, p.25]

Theorem 5.8 (Spectral inclusion)**.** *For $t \geq 0$*

$$e^{t\sigma(A)} \subset \sigma(e^{At}).$$

Definition 5.9. The growth bound $\omega(A)$, also denoted $\omega(e^{At})$, is defined through

$$\omega(A) := \inf\Big\{\omega \in \mathbb{R} \mid \text{there exists a positive number } M = M(\omega)$$

$$\text{such that } \left\|e^{At}\right\| \leq Me^{\omega t} \text{ for } t \geq 0\Big\}.$$

Definition 5.10. The spectral bound $s(A)$, also denoted $s(e^{At})$, is defined by

$$s(A) := \sup\left\{\mathfrak{Re}\, z \mid z \in \sigma(A)\right\}.$$

By Gelfand's theorem for the spectral radius one has the following [73, Proposition 1.2.2.]

Proposition 5.11. *For all $t_0 > 0$ one has*

$$\omega(A) = \frac{\log r\left(e^{At_0}\right)}{t_0} = \lim_{t\to\infty} \frac{\log \left\|e^{At}\right\|}{t}.$$

Here $r\left(e^{At_0}\right)$ denotes the spectral radius of e^{At_0}.

Remark 5.12. *It follows from Theorem 5.8 and Proposition 5.11 that*

$$s(A) \leq \omega(A).$$

The counterexample of Renardy [61] shows that $s(A)$ must not equal $\omega(A)$ for hyperbolic PDEs.

Definition 5.13. We say that the C_0 semigroup $T(t)$ has an (α, β) gap, where $\alpha, \beta \in \mathbb{R}$ and $\alpha \leq \beta$, if there exists a continuous projection $P : X \to X$ so that for all $t \geq 0$ one has $PT(t) = T(t)P$, i.e. there exists a direct sum decomposition $X = X_1 \oplus X_2$, $X_2 = P(X)$, $X_1 = (I - P)(X)$, of $T(t)$ closed invariant subspaces, such that for the restrictions

$$T_1^t := T(t)_{|X_1} \quad \text{and} \quad T_2^t := T(t)_{|X_2}$$

the following properties hold

i) $\omega(T_1^t) < \alpha$,

ii) $(T_2^t)_{t\geq 0}$ extends to a C_0 group $(T_2^t)_{t\in\mathbb{R}}$ on X_2 so that $\omega((T_2^{-t})_{t\geq 0}) < -\beta$.

The next definition is a variant of Def. 5.13 using the generator: Let A be the generator of a C_0 semigroup in X. Suppose there exists a (bounded) projection $P : X \to X$ such that $P\mathcal{D}(A) \subset \mathcal{D}(A)$ and $PAx = APx$ for $x \in \mathcal{D}(A)$. This means that A is completely reduced by P, i.e. $X = X_1 \oplus X_2$, where $X_2 = P(X)$ and $X_1 = (I - P)(X)$, A maps $\mathcal{D}(A) \cap X_1$ into X_1 and $\mathcal{D}(A) \cap X_2$ into X_2, and $A = A_1(I - P) + A_2P$, where $A_1 = A_{|X_1}$, $A_2 = A_{|X_2}$, $\mathcal{D}(A_1) = \mathcal{D}(A) \cap X_1$, $\mathcal{D}(A_2) = \mathcal{D}(A) \cap X_2$, $\mathcal{D}(A) = \mathcal{D}(A_1) \oplus \mathcal{D}(A_2)$. From the Hille Yosida theorem it follows that A_1 and A_2 are generators of C_0 semigroups on X_1 and X_2, respectively.

Definition 5.14. We say A is (α, β) exponentially dichotomous, where $\alpha, \beta \in \mathbb{R}$ and $\alpha \leq \beta$, if $-A_2$ is the generator of a C_0 semigroup (i.e. e^{A_2t} extends to a group) and

i) $\omega(A_1) < \alpha$,

ii) $\omega(-A_2) < -\beta$.

Remark 5.15. *In the literature a semigroup $T(t)$ is called hyperbolic if it has a $(0,0)$-gap. A generator A is said to be exponentially dichotomous if it is $(0,0)$ exponentially dichotomous, see [36].*

Remark 5.16. *The projections P in Def. 5.13 and Def. 5.14 are unique and therefore they coincide. They are called the separating projections for A.*

Indeed, Remark 5.16 is readily verified:

Proof. Let P and $\tilde{P}$ be projections, $Q := I - P$, $\tilde{Q} := I - \tilde{P}$, satisfying the conditions of Def. 5.13 or Def. 5.14, i.e. $X = X_1 \oplus X_2 = \tilde{X}_1 \oplus \tilde{X}_2$, where $X_2 = P(X)$, $\tilde{X}_2 = \tilde{P}(X)$, $X_1 = Q(X)$, $\tilde{X}_1 = \tilde{Q}(X)$ are all invariant under e^{At} and both $T_2^t = e^{At}_{|X_2}$ and $\tilde{T}_2^t = e^{At}_{|\tilde{X}_2}$ extend to groups on X_2 and $\tilde{X}_2$, respectively, where $\omega((T_2^{-t})_{t\geq 0}) < -\beta$, $\omega((\tilde{T}_2^{-t})_{t\geq 0}) < -\beta$ and $\omega((T_1^t)_{t\geq 0}) < \alpha$, $\omega((\tilde{T}_1^t)_{t\geq 0}) < \alpha$. Thus there exists a constant $K > 0$ such that

$$\left\|T_1^t Qx\right\| \leq Ke^{\alpha t}\left\|Qx\right\|, \ \left\|T_2^t Px\right\| \geq K^{-1}e^{\beta t}\left\|Px\right\|,$$

$$\left\|T_2^{-t} Px\right\| \leq Ke^{-\beta t}\left\|Px\right\| \qquad \text{for } x \in X,\ t \geq 0,$$

and the same relations hold for $\tilde{T}_1^t$, $\tilde{T}_2^t$, $\tilde{P}$, $\tilde{Q}$ instead of T_1^t, T_2^t, P and Q. This implies $X_1 = \tilde{X}_1$. Indeed, suppose there would exist $x \in \tilde{X}_1 \setminus X_1$, i.e. $\tilde{P}x = 0$ but $Px \neq 0$. This yields the contradiction

$$\begin{aligned} Ke^{\alpha t}\left\|x\right\| \geq & \left\|\tilde{T}_1^t x\right\| = \left\|e^{At}x\right\| = \left\|T_2^t Px + T_1^t(I-P)x)\right\| \\ \geq & \left|\left\|T_2^t Px\right\| - \left\|T_1^t(I-P)x\right\|\right| \\ \geq & K^{-1}e^{\beta t}\left\|Px\right\| - Ke^{\alpha t}\left\|(I-P)x\right\| \quad \text{for } t \text{ sufficiently large.} \end{aligned}$$

To verify $X_2 = \tilde{X}_2$, suppose there existed $x \in \tilde{X}_2 \setminus X_2$, i.e. $Qx \neq 0$ and $\tilde{Q}x = 0$. Since e^{At} is invertible on $\tilde{X}_2$ and X_2 we can define

$$\left(e^{-At}Qx\right) := \tilde{T}_2^{-t}x - T_2^{-t}Px \quad \text{for } t \geq 0,$$

which satisfies $e^{At}\left(e^{-At}Qx\right) = Qx$ and $\left\|e^{-At}Qx\right\| \leq 2Ke^{-\beta t}\left(\left\|x\right\| + \left\|Px\right\|\right)$. But $\left(e^{-At}Qx\right) \in X_1$ (from $PQx = Pe^{At}\left(e^{-At}Qx\right) = e^{At}P\left(e^{-At}Qx\right) = T_2^t P\left(e^{-At}Qx\right)$ the assumption $P\left(e^{-At}Qx\right) \neq 0$ would imply the contradiction $0 < \left\|P\left(e^{-At}Qx\right)\right\| \leq \left\|T_2^{-t}\right\|\left\|PQx\right\| = 0$) and therefore letting $t \to \infty$ in

$$\left\|Qx\right\| = \left\|T_1^t e^{-At}Qx\right\| \leq Ke^{\alpha t}\left\|e^{-At}Qx\right\| \leq 2K^2 e^{(\alpha-\beta)t}\left(\left\|x\right\| + \left\|Px\right\|\right)$$

yields a contradiction. □

We have the following observation

Theorem 5.17. *Let $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$. Then the following statements are equivalent:*
ı) A is (α, β) exponentially dichotomous,
ıı) e^{At} has a (α, β)-gap,
ııı) for all $t > 0$ the set $G(t) := \{\lambda \in \mathbb{C} \mid e^{\alpha t} \leq |\lambda| \leq e^{\beta t}\}$ is contained in the resolvent set of e^{At},
ıv) there exists $t_0 > 0$ so that $\{\lambda \in \mathbb{C} \mid e^{\alpha t_0} \leq |\lambda| \leq e^{\beta t_0}\}$ is contained in the resolvent set of e^{At_0}.
If one of the conditions $\imath) - \imath v)$ holds true, then the splitting projection P is given by the Riesz projection

$$(I - P) = \frac{1}{2\pi i} \int_{|z|=r} \left(zI - e^{At}\right)^{-1} dz,$$

where $r \in [e^{\alpha t}, e^{\beta t}]$.

Proof. ı) $\Leftrightarrow$ ıı) is plain.
ıı) $\Rightarrow$ ııı): By Proposition 5.11 we have $r(T_1^t) = \sup\{|z| \mid z \in \sigma(T_1^t)\} = e^{t\omega(T_1^t)} < e^{\alpha t}$ and $r(T_2^{-t}) = e^{t\omega(T_2^{-t})}$, which yields

$$\inf\left\{|z| \mid z \in \sigma(T_2^t)\right\} = \frac{1}{r(T_2^{-t})} = e^{-t\omega(T_2^{-t})} > e^{\beta t}.$$

Because P completely reduces e^{At} we have $\sigma(e^{At}) = \sigma(T_1^t) \cup \sigma(T_2^t)$ which implies ııı).
ıv) $\Rightarrow$ ıı) Put $Q = \frac{1}{2\pi i} \int_\gamma \left(zI - e^{At_0}\right)^{-1} dz$, where γ is a simple closed loop in $G(t_0)$, and $P = I - Q$. Then for $x \in X$ and $t \geq 0$

$$Qe^{At}x = \frac{1}{2\pi i} \int_\gamma \left(zI - e^{At_0}\right)^{-1} e^{At}x\, dz = \frac{1}{2\pi i} \int_\gamma e^{At} \left(zI - e^{At_0}\right)^{-1} x\, dz = e^{At}Qx.$$

Thus $X_1 = Q(X)$ and $X_2 = P(X)$ are e^{At} invariant and $r(T_1^{t_0}) < e^{\alpha t_0}$ if $Q \neq 0$ and $r((T_2^{t_0})^{-1}) < e^{-\beta t_0}$ if $P \neq 0$, where $T_1^t := e^{At}_{|X_1}$ and $T_2^t := e^{At}_{|X_2}$. From Proposition 5.11 we get $\omega((T_1^t)_{t\geq 0}) = \frac{\log r(T_1^{t_0})}{t_0} < \alpha$ and $\omega((T_2^{-t})_{t\geq 0}) = \frac{\log r((T_2^{t_0})^{-1})}{t_0} < -\beta$, if $(T_2^t)_{t\geq 0}$ extends to a C_0 group on X_2: For $\theta \in [0, 1]$ we put $T_2^{-\theta t_0} := \left(T_2^{t_0}\right)^{-1} T_2^{t_0(1-\theta)}$. Then $T_2^{-\theta t_0} T_2^{\theta t_0} = T_2^{\theta t_0} T_2^{-\theta t_0} = I$, i.e. $T_2^{\theta t_0}$ is invertible. Thus for each $n \in \mathbb{N}$ and $\theta \in [0, 1]$ the linear map $T_2^{n\theta t_0}$ is invertible which implies that T_2^t extends to a group on X_2. $\square$

Since the semigroup e^{At} is usually unknown in applications it is an important question how to characterize the (α,β) gap condition on e^{At} in terms of the known generator A only. If

$$\{\lambda \in \mathbb{C} \mid \alpha \le \mathfrak{Re}\lambda \le \beta\} \subset \rho(A), \tag{5.2}$$

and (SMP) is known, then Theorem 5.17 implies that A is (α,β) exponentially dichotomous. If (SMP) is not known and if X is a Hilbert space then the Gearhart-Herbst theorem implies that A is (α,β) exponentially dichotomous if and only if (5.2) holds and the resolvent of A is bounded on the stripe $\{\lambda \in \mathbb{C} \mid \alpha \le \mathfrak{Re}\lambda \le \beta\}$. If X is not a Hilbert space then the boundedness of the resolvent is necessary but not sufficient to guarantee that e^{At} has an (α,β)-gap [41, Example 2.22]. The theory of Kaashoek, Lunel and Latushkin gives necessary and sufficient conditions on the resolvent of A which characterize the (α,β)-gap. The basic idea is to use the Laplace inversion formula to characterize the growth rate of the semigroup in terms of the resolvent. For this recall that the resolvent is given by the Laplace transform of the semigroup [49]:

Theorem 5.18. *Let $M > 0$ and $\omega \in \mathbb{R}$ be such that $\left\|e^{At}\right\| \le Me^{\omega t}$. Then*

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} e^{At}\, dt \qquad \text{for} \quad \mathfrak{Re}\lambda > \omega.$$

By inverting the Laplace transform one has the following (see [49, Lemma 7.1] and [49, Corollary 7.5])

Theorem 5.19. *Suppose $\left\|e^{At}\right\| \le Me^{\omega t}$. Let $\rho > \max(0, \omega(A))$. If $x \in \mathcal{D}(A^2)$, then*

$$e^{At}x = \frac{1}{2\pi i}\int_{\rho - i\infty}^{\rho + i\infty} e^{st} R(s, A)x\, ds. \tag{5.3}$$

The Laplace inversion formula (5.3) still holds for all $x \in X$ if one replaces the integral by weaker integration using Cesaro means of order 1 (see Def. 13.1):

Theorem 5.20. *[29, Theorem 11.6.2, p. 350] For each $x \in X$, $t > 0$ and $\rho > \max(0, \omega(A))$*

$$\begin{aligned} e^{At}x &= \frac{1}{2\pi i}(C,1) - \int_{\rho - i\infty}^{\rho + i\infty} e^{st} R(s, A)x\, ds \qquad (5.4)\\ &= \frac{1}{2\pi}\lim_{\tau\to\infty}\int_{-\tau}^{\tau} e^{(\rho + i\nu)t} R(\rho + i\nu, A)x\left(1 - \frac{|\nu|}{\tau}\right) d\nu. \end{aligned}$$

Remark 5.21. *The representation formulas* (5.4) *and* (5.3) *are valid for* $\rho > \omega(A)$ *even if* $\omega(A) < 0$ *(apply Theorem 5.20 to* $\alpha I + A$ *for* $\alpha > 0$ *sufficiently large).*

We next state the main result of [36, 0.2 Theorem]. For this we need to explain some notations first. The symbol $\mathcal{S}$ denotes the Schwartz space of rapidly decreasing functions equipped with the family of seminorms

$$\sup_{x\in\mathbb{R}} |x^k \varphi^{(q)}(x)|, \quad k, q \in \mathbb{N},$$

that makes $\mathcal{S}$ a locally convex topological Hausdorff space. Let $\mathcal{S}^*$ denote the topological dual of $\mathcal{S}$ and $\langle \cdot, \cdot \rangle$ the dual pairing on $\mathcal{S}^* \times \mathcal{S}$, i.e. for $s^* \in \mathcal{S}^*$ and $\varphi \in \mathcal{S}$ $\langle s^*, \varphi \rangle := s^*(\varphi)$. The space $\mathcal{S}^*$ is called the space of tempered distributions. Any polynomially bounded equivalence class of measurable functions $f : \mathbb{R} \to \mathbb{C}$ can be identified uniquely with an element in $\mathcal{S}^*$ by $\langle f, \varphi \rangle = \int_{-\infty}^{\infty} f(\nu)\varphi(\nu)\, d\nu$. Let $\mathfrak{F}$ denote the Fourier transform on the Schwartz space,

$$(\mathfrak{F}\, g)\,(w) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iw\nu} g(\nu) d\nu \quad \text{for} \quad g \in \mathcal{S},$$

which is continuous and bijective from $\mathcal{S}$ onto $\mathcal{S}$ with inverse

$$\left(\mathfrak{F}^{-1} g\right)(t) = \int_{-\infty}^{\infty} e^{-i\omega t} g(\omega)\, d\omega.$$

Theorem 5.22. *The semigroup* e^{At} *is hyperbolic if and only if*

$\imath$) *There exists an* $\omega > 0$ *such that* $\{\lambda \in \mathbb{C} \mid |\mathfrak{Re}\lambda| < \omega\} \subset \rho(A)$,

$\imath\imath$) $\sup_{|\mathfrak{Re}\lambda| < \omega} \|R(\lambda, A)\| < \infty$,

$\imath\imath\imath$) $(\mathrm{C}, 1) - \int_{-\infty}^{\infty} R(\rho + i\nu, A)x\, d\nu$ *exists for each* $x \in X$ *and* $|\rho| < \omega$,

$\imath v$) *For each* $|\rho| < \omega$ *there exists a constant* $K_\rho > 0$ *such that for all* $x \in X, x^* \in X^*$, *the function* $r(\cdot, \rho, x, x^*) : \mathbb{R} \to \mathbb{C}$, *defined by*

$$r(\nu, \rho, x, x^*) = x^* R(\rho + i\nu, A)x,$$

satisfies

$$|\langle r(\cdot, \rho, x, x^*), \varphi \rangle_{\mathcal{S}^*}| \leq K_\rho \|x\| \|x^*\| \left\|\mathfrak{F}^{-1}\varphi\right\|_{L^1(\mathbb{R})} \quad \textit{for all } \varphi \in \mathcal{S}. \tag{5.5}$$

Since $r(\cdot, \rho, x, x^*)$ is bounded it can be identified with a tempered distribution in $\mathcal{S}^*$. The Fourier transform on $\mathcal{S}^*$ is defined as the adjoint of the Fourier transform $\mathfrak{F} : \mathcal{S} \to \mathcal{S}$ of Schwartz functions and denoted with the same symbol $\mathfrak{F}$. Since $\mathcal{S}$ is dense and $(L^1)^* \simeq L^\infty$ (5.5) means that the Fourier transform $\mathfrak{F}r(\cdot, \rho, x, x^*)$ of r in the sense of distributions can be identified with a bounded measurable function and the inequality

$$\|\mathfrak{F}r(\cdot, \rho, x, x^*)\|_{L^\infty} \leq K_\rho \|x\| \|x^*\| \tag{5.6}$$

holds (from the inversion formula (5.4) it follows that $t \mapsto \mathfrak{F}r(\cdot, \rho, x, x^*)(t)$ is continuous).
Theorem 5.22 is a consequence of the following characterization of the growth bound [36, 2.1 Theorem]:

Theorem 5.23. *The growth bound $\omega(A)$ is the infimum of the real numbers ρ satisfying the following conditions:*

$\imath)\sigma(A) \subset \{\lambda \in \mathbb{C} \mid \mathfrak{Re}\lambda < \rho\}$,

$\imath\imath)\ \sup\limits_{\mathfrak{Re}\lambda \geq \rho} \|R(\lambda, A)\| < \infty$,

$\imath\imath\imath)(\mathrm{C}, 1) - \displaystyle\int_{-\infty}^{\infty} R(\rho + i\nu, A)x\, d\nu$ *exists for each* $x \in X$,

$\imath v)$*for each* $x \in X, x^* \in X^*$,

the function $r(\cdot, \rho, x, x^*) : \mathbb{R} \to \mathbb{C}$, *defined by*

$$r(\nu, \rho, x, x^*) = x^* R(\rho + i\nu, A)x,$$

satisfies

$$|\langle r(\cdot, \rho, x, x^*), \varphi\rangle_{\mathcal{S}^*}| \leq K_\rho \|x\| \|x^*\| \left\|\mathfrak{F}^{-1}\varphi\right\|_{L^1(\mathbb{R})} \quad \textit{for all } \varphi \in \mathcal{S}.$$

Remark 5.24. *Conditions $\imath) - \imath\imath\imath)$ imply (see [36, 2.3. Lemma]) the representation formula* (5.4) *for the semigroup. Therefore*

$$e^{-\rho t} e^{At} x = \frac{1}{2\pi}(C, 1) - \int_{-\infty}^{\infty} e^{i\nu t} R(\rho + i\nu, A)x\, d\nu.$$

Hence $\imath v)$ and (5.6) *imply (it is not difficult to see that the Fourier transform in the $(C, 1)$ sense coincides with the Fourier transform on $\mathcal{S}^*$)*

$$e^{-\rho t} \left\|e^{At} x\right\| = \sup_{x^* \in X^*, \|x^*\|=1} |(\mathfrak{F}r(\cdot, \rho, x, x^*))(t)| \leq K_\rho \|x\|.$$

Thus $\imath v)$ yields the growth bound

$$\left\|e^{At}\right\| \leq K_\rho e^{\rho t}.$$

Remark 5.25. *Latushkin and Shvydkoy have shown recently [42, Theorem 2.7] that the integrability condition* $\imath\imath\imath$) *is a consequence of* $\imath v$). *Thus* $\imath\imath\imath$) *can be dropped.*

Next we give a characterization of (α, β) exponential dichotomy which is slightly more general than Theorem 5.22. The proof is basically the same as for Theorem 5.22.

Theorem 5.26. *A is (α, β) exponentially dichotomous, $\alpha \leq \beta$, if and only if there exists an $\delta > 0$ such that*

$\imath$) $\rho(A) \supset \mathbb{C}_{\alpha-\delta,\beta+\delta}$,

$\imath\imath$) $\sup_{\lambda \in \mathbb{C}_{\alpha-\delta,\beta+\delta}} \|R(\lambda, A)\| < \infty$,

$\imath\imath\imath$) $(\mathrm{C}, 1) - \int_{-\infty}^{\infty} R(\rho + i\nu, A)x \, d\nu$ *exists for* $x \in X$ *and* $\alpha - \delta < \rho < \beta + \delta$,

$\imath v$) *For each* $\alpha - \delta < \rho < \beta + \delta$ *there exists a constant* $K_\rho > 0$ *such that for all* $x \in X, x^* \in X^*,$ *the function* $r(\cdot, \rho, x, x^*) : \mathbb{R} \to \mathbb{C},$ *defined by*

$$r(\nu, \rho, x, x^*) = x^* R(\rho + i\nu, A)x,$$

satisfies

$$|\langle r(\cdot, \rho, x, x^*), \varphi \rangle_{\mathcal{S}^*}| \leq K_\rho \|x\| \|x^*\| \left\|\mathfrak{F}^{-1}\varphi\right\|_{L^1(\mathbb{R})} \quad \textit{for all } \varphi \in \mathcal{S}.$$

5.2 Proof of the spectral gap mapping / exponential dichotomy theorem for hyperbolic systems

In this section we prove Theorem 5.5 by showing that the conditions of Theorem 5.26 are fulfilled under the assumptions of Theorem 5.5. Hence we assume the following:

(A): $\alpha \leq \beta$ and $\delta > 0$ are such that $h(\lambda) \neq 0$ for $\lambda \in \mathbb{C}_{\alpha-\delta,\beta+\delta}$.

Under this assumption we have to show that the conditions of Theorem 5.26 are fulfilled when X is the Banach space $X = Y$ and $X = X_p$, and A is the generator corresponding to system (H).
From (A) it follows that

$$h_0(\lambda) \neq 0 \quad \text{for} \quad \lambda \in \mathbb{C}_{\alpha-\delta,\beta+\delta}.$$

Indeed, if there existed $\lambda_0 \in \mathbb{C}_{\alpha-\delta,\beta+\delta}$ with $h_0(\lambda_0) = 0$ then h_0 would have infinitely many zeros λ with $\mathfrak{Re}\,\lambda = \mathfrak{Re}\,\lambda_0$. From this we would conclude similar as we deed for Lemmas 4.8 and 4.15 that h had a zero in $\mathbb{C}_{\alpha-\delta,\beta+\delta}$.

Relation (4.8) directly implies condition $\imath$) of Theorem 5.26. By possibly making δ smaller and applying Remark 4.10 we can assume without loss of generality

$$\inf_{\alpha-\delta<\mathfrak{Re}\lambda<\beta+\delta} |h_0(\lambda)| > 0. \tag{5.7}$$

Then Lemma 4.16 implies condition $\imath\imath$) of Theorem 5.26. The remaining condition we have to check is $\imath v$), it implies condition $\imath\imath\imath$) by the results of Latushkin and Shvydkoy [42]. In the rest of this chapter we will verify condition $\imath v$). This will finish the proof of Theorem 5.5 (all calculations will not depend on $p \geq 1$). Before we start we show how Theorem 5.7 follows from Theorem 5.5 and how one sees the equivalence of Theorem 5.4 and Theorem 5.5:

We prove Theorem 5.7: If γ, γ_1 and γ_2 are chosen as in Theorem 5.7, then by Lemma 4.14 there exist $\delta > 0$ so that $\mathbb{C}_{\gamma_2-\delta,\gamma_1+\delta} \subset \rho(A)$. Hence Theorems 5.5 yields (γ_2, γ_1) dichotomy. Let $\tilde{\pi}_1$ denote the separating projection and $\tilde{\pi}_2 := I - \tilde{\pi}_1$. By Theorem 5.17

$$\tilde{\pi}_2 = \frac{1}{2\pi i} \int_{|z|=e^{\gamma_2 t}} (zI - e^{At})^{-1}\, dz.$$

We have that $e^{At}_{|\tilde{\pi}_1(X)}$ extends to a group and there exist constants $c_2 > 0$ and $c_1 > 0$ so that for $t \geq 0$

$$\left\| e^{At}_{|\tilde{\pi}_2(X)} \right\|_{\mathcal{L}(\tilde{\pi}_2(X))} \leq c_2 e^{\gamma_2 t} \quad \text{and} \quad \left\| e^{A(-t)}_{|\tilde{\pi}_1(X)} \right\|_{\mathcal{L}(\tilde{\pi}_1(X))} \leq c_1 e^{\gamma_1(-t)}.$$

We have to show that $\tilde{\pi}_1 = \pi_1$, where

$$\pi_1 = \int_\beta (zI - A)^{-1}\, dz,$$

is the spectral projection accorting to the finite eigenvalues with $\mathfrak{Re}\,\lambda \geq \gamma$ (β is a closed rectifiable simple loop around these finite eigenvalues in the half plane $\mathfrak{Re}\,z \geq \gamma_1$): Indeed, for $x \in X$

$$\begin{aligned}
\pi_1 x &= \pi_1(\tilde{\pi}_1 x + \tilde{\pi}_2 x) \\
&= \int_\beta (zI - A)^{-1}_{|\tilde{\pi}_1(X)}\, dz\, \tilde{\pi}_1 x + \int_\beta (zI - A)^{-1}_{|\tilde{\pi}_2(X)}\, dz\, \tilde{\pi}_2 x \\
&= I_{\tilde{\pi}_1(X)} \tilde{\pi}_1 x + 0_{\tilde{\pi}_2(X)}\, \tilde{\pi}_2 x \\
&= \tilde{\pi}_1 x.
\end{aligned}$$

Let $z \in \mathbb{C}$, $|z| > e^{\gamma_+ t}$, belong to the spectrum of e^{At}. Choose γ_1, γ_2 and γ so that $\gamma_+ < \gamma_2 < \gamma < \gamma_1 < \frac{\log|z|}{t}$. We have either $z \in \sigma(e^{A_1 t})$ or $z \in \sigma(e^{A_2 t})$. By Theorems 5.5 and Theorem 5.17

$$\sigma(e^{A_2 t}) \subset \{z \in \mathbb{C} \mid |z| < e^{\gamma_2 t}\}.$$

Hence $z \in \sigma(e^{A_1 t})$. Because the spectral mapping theorem holds for the bounded operator A_1 there exists $\lambda \in \sigma(A_1) \subset \sigma(A)$ so that $z = e^{\lambda t}$. If $z = e^{\lambda t}$ for some $\lambda \in \sigma(A)$ then by Theorem 5.8 $z \in \sigma(e^{At})$. □

The equivalence of Theorem 5.4 and Theorem 5.5 can be seen as follows: Assume Theorem 5.5 holds. Let $a < b$ and $\mathbb{C}_{a,b} \subset \rho(A)$. Then for any $\alpha \leq \beta$ that satisfy $a < \alpha \leq \beta < b$ we have that A is (α, β) exponentially dichotomous. By Theorem 5.17 we get that $\{\lambda \in \mathbb{C} \mid e^{\alpha t} \leq |\lambda| \leq e^{\beta t}\} \subset \rho(e^{At})$. This shows that $\{\lambda \in \mathbb{C} \mid e^{at} < |\lambda| < e^{bt}\} \subset \rho(e^{At})$. If $\{\lambda \in \mathbb{C} \mid e^{at} < |\lambda| < e^{bt}\} \subset \rho(e^{At})$ then by Theorem 5.8 it follows that $\mathbb{C}_{a,b} \subset \rho(A)$. Conversely suppose Theorem 5.4 holds. Let $\alpha \leq \beta$, $\alpha, \beta \in \mathbb{R}$ and suppose there exists $\delta > 0$ so that $\mathbb{C}_{\alpha-\delta,\beta+\delta} \subset \rho(A)$. Then by Theorem 5.4

$$\{\lambda \in \mathbb{C} \mid e^{\alpha t} \leq |\lambda| \leq e^{\beta t}\} \subset \{\lambda \in \mathbb{C} \mid e^{(\alpha-\delta)t} < |\lambda| < e^{(\beta+\delta)t}\} \subset \rho(e^{At}).$$

Theorem 5.17 implies that A is (α, β) exponentially dichotomous. Finally, if A is (α, β) exponentially dichotomous then from Theorem 5.17 it follows that

$$\operatorname{dist}\big(\{\lambda \in \mathbb{C} \mid e^{\alpha t} \leq |\lambda| \leq e^{\beta t}\}, \sigma(e^{At})\big) > 0.$$

Hence Theorem 5.8 implies that there exists $\delta > 0$ so that $\mathbb{C}_{\alpha-\delta,\beta+\delta} \subset \rho(A)$. □

Now we return back to the proof of Theorem 5.5. We assume (A), (5.7) and $\rho \in\,]\alpha - \delta, \beta + \delta[$. We will show that condition *iv)* holds:

Since it will be used frequently, we denote with $\tau_{ij}(\rho + i\nu)$, $1 \leq i, j \leq n$, the i-th row and j-th column of the matrix

$$\begin{pmatrix} E \\ I \end{pmatrix} H_0^{-1}(\rho + i\nu)(D, -I).$$

Since $H_0^{-1}(\rho + i\nu) = \frac{1}{h_0(\rho+i\nu)} \operatorname{Ad} H_0(\rho + i\nu)$, and both $h_0(\rho + i\nu)$ and the elements of $\operatorname{Ad} H_0(\rho + i\nu)$ are exponential polynomials, where $h_0(\rho + i\nu)$ is bounded away from zero by (5.7), it follows from the $1/f$ theorem of Bochner, Wiener, Pitt and Cameron [11, 78, 51, 14] that the elements of H_0^{-1} belong to the algebra

$$\mathfrak{A} = \left\{ f \mid f(x) = \sum_{n=1}^{\infty} a_n e^{ib_n x}, a_n \in \mathbb{C}, b_n \in \mathbb{R}, \sum_{n=1}^{\infty} |a_n| < \infty \right\}$$

of absolutely convergent exponential series. This implies that the Fourier transforms of the entries of H_0^{-1} are of the form $\sum_{n=1}^{\infty} a_n \delta_{-b_n}$, where $\sum_{n=1}^{\infty} |a_n| < \infty$ and δ_{-b_n} denotes the delta distribution at $-b_n$. In other words the transform is a measure of countable Dirac masses on $\mathbb{R}$ with bounded variation. Further we put

$$h := (h_j)_{j=1,\ldots,n} := (f, g) \quad \text{and}$$

$$I_{mj}(\nu) := \int_0^l \exp\left(-i\nu \int_y^l k_m^{-1}(z)\, dz\right) \exp\left(-\int_y^l \frac{\rho + c_{jj}(z)}{k_m(z)}\, dz\right) k_m^{-1}(y) h_m(y)\, dy$$

for $\nu \in \mathbb{R}$ and $1 \leq m, j \leq n$.

For $(f, g, 0) \in X$, $x^* \in X^*$, $\nu \in \mathbb{R}$ define the scalar matrix function

$$r_0(\nu, \rho, (f, g, 0), x^*) := \left\langle x^*, \left(R(\rho + i\nu, A_0)(f, g); \left(-D, I\right) R(\rho + i\nu, A_0)(f, g)\right)\right\rangle .$$

To prepare the proofs we recall that by Riesz's representation theorem the dual space C^* of $C = C([0, l], \mathbb{C}^n)$ is isometrically isomorphic to the space of countable additive $\mathbb{C}^n$ valued Radon measures on the Borel sigma algebra $\mathfrak{B}$ on $[0, l]$ with the finite total variation norm. That is for $x^* \in C^*$ there exists a Radon measure $\alpha = (\alpha_1, \ldots, \alpha_n) : \mathfrak{B} \to \mathbb{C}^n$ such that for $\varphi = (\varphi_1, \ldots, \varphi_n) \in C([0, l]; \mathbb{C}^n)$

$$\langle x^*, \varphi \rangle = \sum_{j=1}^{n} \int_{[0,l]} \varphi_j \, d\alpha_j .$$

The dual of $L^p([0, l]; \mathbb{C}^n)$ is $L^q([0, l]; \mathbb{C}^n)$, where $q \in \,]1, \infty]$ satisfies $\frac{1}{q} + \frac{1}{p} = 1$: for $x^* \in (L^p([0, l]; \mathbb{C}^n))^*$ there exists a unique $f \in L^q([0, l]; \mathbb{C}^n)$ such that for $\varphi \in L^p([0, l]; \mathbb{C}^n)$ we have

$$\langle x^*, \varphi \rangle = \int_{[0,l]} \langle f, \varphi \rangle_{\mathbb{C}^n} d\lambda,$$

where λ denotes Lebesque's measure on $\mathbb{R}$.

Lemma 5.27. *Suppose* (A), (5.7) *and* $\rho \in \,]\alpha - \delta, \beta + \delta[$. *Then there exists* $\kappa > 0$ *such that for* $x = (f, g, 0) \in X$, $x^* \in X^*$,

$$\mathfrak{F}[r_0(\cdot, \rho, x, x^*)] \in L^\infty(\mathbb{R}) \text{ and } \|\mathfrak{F}[r_0(\cdot, \rho, x, x^*)]\|_{L^\infty} \leq \kappa \, \|(f, g, 0)\|_X \, \|x^*\| . \tag{5.8}$$

Proof. First assume $X = Y$, so $(f,g) \in C([0,l];\mathbb{C}^n)$. Corresponding to $x^* \in X^*$ there exist bounded Radon measures α_i, $1 \le i \le n$, on $[0,l]$ and $x_1,\dots,x_{n_2} \in \mathbb{C}$ such that

$$r_0(\nu,\rho,(f,g,0),x^*) = \sum_{j=1}^{n} r_{0j}(\nu,\rho,(f,g,0),\alpha_j) + \sum_{j=1}^{n_2} \tilde{r}_{0j}(\nu,\rho,(f,g,0),x_j),$$

where for $j = 1,\dots,n$

$$r_{0j}(\nu,\rho,(f,g,0),\alpha_j) := \int_0^l R^{(j)}(\rho+i\nu,A_0)(f,g)\,d\alpha_j.$$

and for $j = 1,\dots,n_2$

$$\tilde{r}_{0j}(\nu,\rho,(f,g,0),x_j) := x_j\left((-D,I)\delta_l R(\rho+i\nu,A_0)(f,g)\right)_j.$$

Here $R^{(j)}(\rho+i\nu,A_0)(f,g)$ denotes the j-th component, $1 \le j \le n$, of the resolvent $R(\rho+i\nu,A_0)(f,g)$ and $\left((-D,I)\delta_l R(\rho+i\nu,A_0)(f,g)\right)_j$ denotes the j-th component, $1 \le j \le n_2$, of the $\mathbb{C}^{n_2}$ vector $(-D,I)R(\rho+i\nu,A_0)(f,g)$.

We show (5.8) for r_{0j} (we omit $\tilde{r}_{0j}$ because it is even simpler). For $j = 1,\dots,n$ we have from (4.10)

$$r_{0j} = \left(\sum_{m=1}^{n} \tau_{jm}(\rho+i\nu) r_{0jm}\right) + r_{0j0},$$

where for $m = 1,\dots,n$

$$r_{0jm} := \int_0^l \exp\left(-\textstyle\int_0^y (\rho+i\nu+c_{jj}(r))\,k_j^{-1}(r)\,dr\right)\,d\alpha_j(y)\cdot I_{mm}(\nu)$$

and

$$r_{0j0} := \int_0^l \left(\int_0^y \exp\left(-\textstyle\int_z^y (\rho+i\nu+c_{jj}(r))\,k_j^{-1}(r)\,dr\right) k_j^{-1}(z) h_j(z)\,dz\right)\,d\alpha_j(y).$$

By Fubini's Theorem, the Fejer Laplace inversion Theorem, Lebesgue's dominated convergence and the change of variables $x = \int_z^l k_m^{-1}(r)\,dr$ we have

$$\frac{1}{2\pi}(\mathrm{C},1)\!-\!\int_{-\infty}^{\infty} e^{i\omega\nu} r_{0jm}(\nu) d\nu$$

$$=\int_0^l \left(\frac{1}{2\pi}(\mathrm{C},1)\!-\!\int_{-\infty}^{\infty} \exp\left(i\nu\left(\omega - \textstyle\int_0^y k_j^{-1}(r)\,dr\right)\right) I_{mm}(\nu)\,d\nu\right)$$

$$\exp\left(-\textstyle\int_0^y \frac{\rho+c_{jj}(r)}{k_j(r)}\,dr\right)\,d\alpha_j(y)$$

$$=\int_0^l \left(\frac{1}{2\pi}(\mathrm{C},1)\!-\!\int_{-\infty}^{\infty} \exp\left(i\nu\left(\omega - \textstyle\int_0^y k_j^{-1}(r)\,dr\right)\right) \int_0^{\int_0^l k_m^{-1}(r)\,dr} e^{-i\nu x}\right.$$

$$\left. h_m(z(x)) \exp\left(-\textstyle\int_{z(x)}^l k_m^{-1}(r)(\rho + c_{mm}(r))\,dr\right)\,dx\,d\nu\right)$$

$$\exp\left(-\textstyle\int_0^y k_j^{-1}(r)(\rho + c_{jj}(r))\,dr\right)\,d\alpha_j(y)$$

$$=\mathrm{sgn}(k_j)\int_0^l \left(\frac{1}{2\pi}(\mathrm{C},1)\!-\!\int_{-\infty}^{\infty} \exp\left(i\nu\left(\omega - \textstyle\int_0^y k_j^{-1}(r)\,dr\right)\right) \int_{-\infty}^{\infty} e^{-i\nu x}\tilde\zeta(x)\,dx\,d\nu\right)$$

$$\exp\left(-\textstyle\int_0^y k_j^{-1}(r)(\rho + c_{jj}(r))\,dr\right)\,d\alpha_j(y)$$

$$=\int_0^l \zeta\left(\omega - \textstyle\int_0^y k_j^{-1}(r)\right) \exp\left(-\textstyle\int_0^y k_j^{-1}(r)(\rho + c_{jj}(r))\,dr\right)\,d\alpha_j(y)$$

where

$$\chi(x) := \begin{cases} 1 & \text{if } x \in [0, \int_0^l k_m^{-1}(r)\,dr] \cup [\int_0^l k_m^{-1}(r)\,dr, 0] \\ 0 & \text{elsewhere} \end{cases},$$

$$\tilde\zeta(x) := \chi(x) h_m(z(x)) \exp\left(-\textstyle\int_{z(x)}^l k_m^{-1}(r)(\rho + c_{mm}(r))\,dr\right) \quad (\tilde\zeta : \mathbb{R} \to \mathbb{C}),$$

$$\zeta(x) := \frac{1}{2}\left(\tilde\zeta(x+) + \tilde\zeta(x-)\right).$$

Since ζ has compact support we have proven

$$\mathfrak{F}r_{0jm} \in L^\infty \text{ with compact support and (5.8) holds for } r_{0jm}. \tag{5.9}$$

Hence for $m = 1, \ldots, n$

$$\begin{aligned} &\mathfrak{F}\left(r_{0jm} \cdot \tau_{jm}(\rho + i\cdot)\right) = \mathfrak{F}r_{0jm} * \mathfrak{F}\left(\tau_{jm}(\rho + i\cdot)\right) \in L^\infty \\ \text{and} \quad &\left\|\mathfrak{F}\left(r_{0jm} \cdot \tau_{jm}(\rho + i\cdot)\right)\right\|_{L^\infty} \le \left\|\mathfrak{F}r_{0jm}\right\|_{L^\infty} \left\|\mathfrak{F}\tau_{jm}(\rho + i\cdot)\right\|_{Var}, \end{aligned}$$

where $\|\mathfrak{F}\tau_{jm}(\rho + i\cdot)\|_{Var}$ denotes the total variation of the measure $\mathfrak{F}\tau_{jm}(\rho + i\cdot)$ on $\mathbb{R}$.

Suppose $X = X_p$. Then $h = (f, g) \in L^p([0,l];\mathbb{C}^n)$. Since $C([0,l];\mathbb{C}^n)$ is dense in $L^p([0,l];\mathbb{C}^n)$ $(1 \le p < \infty)$ we can choose a sequence $(h_i)_{i\in\mathbb{N}}$ in $C([0,l];\mathbb{C}^n)$ which converges in L^p to h. Then the above calculation is valid for h_i instead of h. The integration with respect to the bounded measure $d\alpha_j$ is replaced with Lebesgue integration with respect to some L^q density corresponding to $x^* \in (L^p)^* \simeq L^q$, where $q \in]1,\infty]$, $q^{-1}+p^{-1} = 1$, is the conjugated exponent to p. By Hï¿½der's inequality (5.9) holds uniformly in i. Since $r_{0jm}(h_i) \to r_{0jm}(h)$ in $\mathcal{S}^*$ (even in L^∞) we have $\mathfrak{F}r_{0jm}(h) = \lim_{i\to\infty,\mathcal{S}^*} \mathfrak{F}r_{0jm}(h_i)$. Since $\mathfrak{F}r_{0jm}(h_i)$ is bounded in L^∞, by weak-$*$ compactness of L^∞, after possibly passing to a subsequence, we see that $\mathfrak{F}r_{0jm}(h) \in L^\infty$ and (5.9) holds for the limit also.

In the following we will assume $X = Y$, the case $X = X_p$ follows similarly as just explained.

Using the change of variable $x = -\int_0^z k_j^{-1}(r)\,dr$ we have

$$\frac{1}{2\pi}(\mathrm{C},1)\!-\!\int_{-\infty}^{\infty} e^{i\omega\nu} r_{0j0}(\nu)d\nu$$

$$= \int_0^l \exp\left(-\textstyle\int_0^y k_j^{-1}(r)(\rho + c_{jj}(r))\,dr\right) \Bigg(\frac{1}{2\pi}(\mathrm{C},1)\!-\!\int_{-\infty}^{\infty} \exp\left(i\nu(\omega - \textstyle\int_0^y k_j^{-1}(r)\,dr)\right)$$

$$\int_0^y \exp\left(i\nu \textstyle\int_0^z k_j^{-1}(r)\,dr\right) \exp\left(\textstyle\int_0^z k_j^{-1}(r)(\rho + c_{jj}(r))\,dr\right) \frac{h_j(z)}{k_j(z)}\,dz\,d\nu\Bigg)\,d\alpha_j(y)$$

$$= \int_0^l \exp\left(-\textstyle\int_0^y k_j^{-1}(r)(\rho + c_{jj}(r))\,dr\right)$$

$$\Bigg(\frac{1}{2\pi}(\mathrm{C},1)\!-\!\int_{-\infty}^{\infty} \exp\left(i\nu(\omega - \textstyle\int_0^y k_j^{-1}(r)\,dr)\right) \int_{-\infty}^{\infty} e^{-i\nu x}\tilde{\zeta}(x,y)\,dx\,d\nu\Bigg)\,d\alpha_j(y)$$

$$= \int_0^l \exp\left(-\textstyle\int_0^y k_j^{-1}(r)(\rho + c_{jj}(r))\,dr\right) \zeta\left(\omega - \textstyle\int_0^y k_j^{-1}(r)\,dr\right) d\alpha_j(y),$$

where $\tilde{\zeta}(x,y) = (-1)^{s(j)}\chi_y(x)\exp\left(\int_0^{z(x)} k_j^{-1}(r)(\rho + c_{jj}(r))\,dr\right) h_j(z(x))$, $s(j) := 0$ if $1 \le j \le n_1$, $s(j) := 1$ if $n_1 + 1 \le j \le n$, χ_y is the characteristic set function to $[0, -\int_0^y k_j^{-1}(r)\,dr] \cup [-\int_0^y k_j^{-1}(r)\,dr, 0]$ and $\zeta(x,y) := \frac{1}{2}\left(\tilde{\zeta}(x+,y) + \tilde{\zeta}(x-,y)\right)$. Thus we have

$$\mathfrak{F}r_{0j0} \in L^\infty \text{ with compact support and (5.8) holds for } r_{0j0}.$$

□

We continue verifying condition iv) of Theorem 5.26 using estimate (4.25). Note that (4.25) is valid on stripes if $|\mathfrak{Im}\,\lambda|$ is sufficiently large. But we need an estimate of type (4.25) on the whole stripe $\mathbb{C}_{\alpha-\delta,\beta+\delta}$. Such is easily obtained: let $-s < \alpha - \delta$. Then for $\lambda \in \mathbb{C}_{\alpha-\delta,\beta+\delta}$ we have

$$R(\lambda, A)\begin{pmatrix} f \\ g \\ b \end{pmatrix} = \begin{pmatrix} u \\ v \\ (-D, I)\delta_l \begin{pmatrix} u \\ v \end{pmatrix} \end{pmatrix},$$

where

$$\begin{aligned}\begin{pmatrix} u \\ v \end{pmatrix} =& R(\lambda, A_0)\begin{pmatrix} f \\ g \end{pmatrix} \qquad (5.10)\\ &+ \frac{1}{\lambda + s}\left(R_1(\lambda)\begin{pmatrix} f \\ g \end{pmatrix} + R_2(\lambda)\begin{pmatrix} f \\ g \end{pmatrix} + R_3(\lambda)\begin{pmatrix} f \\ g \\ b \end{pmatrix} + R_4(\lambda)\begin{pmatrix} f \\ g \end{pmatrix}\right)\\ &+ \frac{1}{1+|\lambda|^2}\tilde{\mathcal{E}}(\lambda)(f, g, b),\end{aligned}$$

and $\tilde{\mathcal{E}}$, R_1, R_2, R_3, R_4 are bounded for $\lambda \in \mathbb{C}_{\alpha-\delta,\beta+\delta}$.

Hence define the scalar 'matrix elements' corresponding to the nondiagonal terms of (5.10). Put

$$\begin{aligned} r_1^s(\nu, \rho, (f, g, 0), x^*) :=& \tfrac{1}{\rho+s+i\nu}\left\langle x^*, \left(R_1(\rho+i\nu)\begin{pmatrix} f \\ g \end{pmatrix}; (-D, I)\delta_l R_1(\rho+i\nu)\begin{pmatrix} f \\ g \end{pmatrix}\right)\right\rangle,\\ r_2^s(\nu, \rho, (f, g, 0), x^*) :=& \tfrac{1}{\rho+s+i\nu}\left\langle x^*, \left(R_2(\rho+i\nu)\begin{pmatrix} f \\ g \end{pmatrix}; (-D, I)\delta_l R_2(\rho+i\nu)\begin{pmatrix} f \\ g \end{pmatrix}\right)\right\rangle,\\ r_3^s(\nu, \rho, (f, g, b), x^*) :=& \tfrac{1}{\rho+s+i\nu}\left\langle x^*, \left(R_3(\rho+i\nu)\begin{pmatrix} f \\ g \\ b \end{pmatrix}; (-D, I)\delta_l R_3(\rho+i\nu)\begin{pmatrix} f \\ g \\ b \end{pmatrix}\right)\right\rangle,\\ r_4^s(\nu, \rho, (f, g, 0), x^*) :=& \tfrac{1}{\rho+s+i\nu}\left\langle x^*, \left(R_4(\rho+i\nu)\begin{pmatrix} f \\ g \end{pmatrix}; (-D, I)\delta_l R_4(\rho+i\nu)\begin{pmatrix} f \\ g \end{pmatrix}\right)\right\rangle,\end{aligned}$$

Lemma 5.28. *Suppose* (A), (5.7), $\rho \in]\alpha-\delta, \beta+\delta[$ *and* $s \in \mathbb{R}$, $s > -\alpha + \delta$. *Then there exists* $\kappa > 0$ *such that for* $(f, g, b) \in X$ *and* $x^* \in X^*$

$$\begin{aligned}&\mathfrak{F}[r_i^s(\cdot, (f, g, 0), x^*)] \in L^\infty(\mathbb{R}), \|\mathfrak{F}r_i^s\|_{L^\infty} \le \kappa \|(f, g, 0)\|_X \|x^*\| \ (i = 1, 2, 4), \qquad (5.11)\\ &\mathfrak{F}[r_3^s(\cdot, (f, g, b), x^*)] \in L^\infty(\mathbb{R}), \|\mathfrak{F}r_3^s\|_{L^\infty} \le \kappa \|(f, g, b)\|_X \|x^*\|.\end{aligned}$$

Proof. Let $R_i^{(j)}$ denote the j-th component of R_i, $1 \le j \le n$, $1 \le i \le 4$. Corresponding to $x^* \in X^*$ there exist bounded Radon measures α_i, $1 \le i \le n$, on $[0,l]$ and $x_1, \dots, x_{n_2} \in \mathbb{C}$ such that

$$r_1^s(\nu, \rho, (f, g, 0), x^*) = \sum_{j=1}^{n} r_{1j}^s(\nu, \rho, (f, g, 0), \alpha_j) + \sum_{j=1}^{n_2} \tilde{r}_{1j}^s(\nu, \rho, (f, g, 0), x_j),$$

where

$$\begin{aligned} r_{1j}^s(\nu, \rho, (f, g, 0), \alpha_j) :=& \frac{1}{\rho + s + i\nu} \int_0^l R_1^{(j)}(\rho + s + i\nu) \begin{pmatrix} f \\ g \end{pmatrix} \, d\alpha_j, \\ \tilde{r}_{1j}^s(\nu, \rho, (f, g, 0), x_j) :=& \frac{1}{\rho + s + i\nu} \left((-D, I)\delta_l R_1(\rho + s + i\nu) \begin{pmatrix} f \\ g \end{pmatrix} \right)_j \cdot x_j. \end{aligned}$$

We verify (5.11) for r_{1j}^s, the expression $\tilde{r}_{1j}^s$ will be omitted because it is treated in the same manner. By definition of R_1 (see Lemma 4.16) we have

$$\begin{aligned} r_{1j}^s =& \frac{1}{\rho + s + i\nu} \sum_{1 \le p,m \le n} \int_0^l (F_1(\cdot, 0, \rho + i\nu))_{jp} \, d\alpha_j \cdot \tau_{pm}(\rho + i\nu) \cdot I_{mm}(\nu) \\ =& \frac{1}{\rho + s + i\nu} \Bigg(\sum_{1 \le m \le n} \tau_{jm}(\rho + i\nu) r_{1jjm}(\nu, \rho, (f, g, 0), \alpha_j) \\ &+ \sum_{\substack{1 \le p,m \le n \\ p \ne j}} \sum_{q=1}^{3} \tau_{pm}(\rho + i\nu) r_{1jpmq}(\nu, \rho, (f, g, 0), \alpha_j) \Bigg), \end{aligned} \tag{5.12}$$

where (see the expressions (4.18) and (4.19) for F_1)

$$\begin{aligned} r_{1jjm} := - \int_0^l & \exp\left(-i\nu \textstyle\int_0^x k_j^{-1}(u)\, du\right) \exp\left(- \textstyle\int_0^x \left(\frac{\rho + c_{jj}(u)}{k_j(u)}\right) \, du\right) \\ & \sum_{\substack{1 \le \sigma \le n \\ \sigma \ne j}} \int_0^x \frac{c_{j\sigma}(z)}{k_j(z)} \rho_{\sigma j}(z) \, dz \, d\alpha_j(x) \cdot I_{mm}(\nu), \end{aligned}$$

and for $p \neq j$

$$r_{1jpm1} := -\int_0^l \exp\left(-i\nu \int_0^x k_p^{-1}(u)\,du\right) \exp\left(-\int_0^x \tfrac{\rho+c_{pp}(u)}{k_p(u)}\,du\right)$$
$$\rho_{jp}(x)\,d\alpha_j(x) \cdot I_{mm}(\nu),$$
$$r_{1jpm2} := \int_0^l \exp\left(-i\nu \int_0^x k_j^{-1}(u)\,du\right) \exp\left(-\int_0^x \tfrac{\rho+c_{jj}(u)}{k_j(u)}\,du\right)$$
$$\rho_{jp}(0)\,d\alpha_j(x) \cdot I_{mm}(\nu),$$
$$r_{1jpm3} := \int_0^l \exp\left(-i\nu \int_0^x k_p^{-1}(u)\,du\right) \exp\left(-\int_0^x \tfrac{\rho+c_{pp}(u)}{k_p(u)}\,du\right)$$
$$\int_0^x \exp\left(i\nu \int_x^z (k_j^{-1}(u) - k_p^{-1}(u))\,du\right)$$
$$\exp\left(\int_x^z \left(\rho(k_j^{-1}(u) - k_p^{-1}(u)) + \tfrac{c_{jj}(u)}{k_j(u)} - \tfrac{c_{pp}(u)}{k_p(u)}\right)\,du\right)$$
$$\left\{\rho_{jp}(z)\left(\frac{c_{jj}(z)}{k_j(z)} - \frac{c_{pp}(z)}{k_p(z)}\right)\,dz + \,d\rho_{jp}(z)\right\} d\alpha_j(x) \cdot I_{mm}(\nu).$$

We calculate the Fourier Transform of r_{1jpm3}. For $x, z \in [0, l]$ we have by the Fejér Fourier inversion theorem (Corollary 13.3) and the change of variable $w = \int_y^l k_m^{-1}(z)\,dz$:

$$\frac{1}{2\pi}(\mathrm{C},1) - \int_{-\infty}^{\infty} \exp\left(i\nu\left(\omega - \int_0^x k_p^{-1}(u)\,du + \int_x^z \left(k_j^{-1}(u) - k_p^{-1}(u)\right)\,du\right)\right)$$
$$I_{mm}(\nu)\,d\nu$$
$$= \frac{1}{2\pi}(\mathrm{C},1) - \int_{-\infty}^{\infty} \exp\left(i\nu\left(\omega - \int_0^x k_p^{-1}(u)\,du + \int_x^z \left(k_j^{-1}(u) - k_p^{-1}(u)\right)\,du\right)\right)$$
$$\int_0^{\int_0^l k_m^{-1}(z)\,dz} e^{-i\nu w} \exp\left(-\int_{y(w)}^l \tfrac{\rho+c_{mm}(z)}{k_m(z)}\,dz\right) h_m(y(w))\,dw\,d\nu$$
$$= \frac{1}{2\pi}(\mathrm{C},1) - \int_{-\infty}^{\infty} \exp\left(i\nu\left(\omega - \int_0^x k_p^{-1}(u)\,du + \int_x^z \left(k_j^{-1}(u) - k_p^{-1}(u)\right)\,du\right)\right)$$
$$\int_{-\infty}^{\infty} e^{-i\nu w}\tilde{\zeta}(w)\,dw\,d\nu$$
$$= \zeta\left(\omega - \int_0^x k_p^{-1}(u)\,du + \int_x^z \left(k_j^{-1}(u) - k_p^{-1}(u)\right)\right),$$

where

$$\zeta : \mathbb{R} \to \mathbb{C}, \zeta(w) := \frac{1}{2}\left(\tilde{\zeta}(w+) + \tilde{\zeta}(w-)\right)$$

is compactly supported,

$$\tilde{\zeta}(w) := (-1)^{s(m)}\chi(w) \exp\left(-\int_{y(w)}^l \tfrac{\rho+c_{mm}(z)}{k_m(z)}\,dz\right) h_m(y(w)),$$

χ is the characteristic function of the interval $[0, \int_0^l k_m^{-1}(z)\,dz] \cup [\int_0^l k_m^{-1}(z)\,dz, 0]$ and $s(m) := 0$, if $1 \le m \le n_1$, $s(m) := 1$, if $n_1 + 1 \le m \le n$.

Therefore by Fubini and Lebesgue's dominated convergence using Remark 13.4 for passing to the limit we have

$$\frac{1}{2\pi}(\mathrm{C},1) - \int_{-\infty}^{\infty} e^{i\nu\omega} r_{1jpm3}(\nu, \rho, (f,g,0), \alpha_j)\,d\nu$$
$$= \int_0^l \exp\left(-\int_0^x \tfrac{\rho + c_{pp}(u)}{k_p(u)}\,du\right) \int_0^x \exp\left(\int_x^z \left(\rho(k_j^{-1}(u) - k_p^{-1}(u)) + \tfrac{c_{jj}(u)}{k_j(u)} - \tfrac{c_{pp}(u)}{k_p(u)}\right) du\right)$$
$$\zeta\left(\omega - \int_0^x k_p^{-1}(u)\,du + \int_x^z \left(k_j^{-1}(u) - k_p^{-1}(u)\right)\right)$$
$$\left\{\rho_{jp}(z)\left(\frac{c_{jj}(z)}{k_j(z)} - \frac{c_{pp}(z)}{k_p(z)}\right) dz + d\rho_{jp}(z)\right\} d\alpha_j(x).$$

Because the measure $d\rho_{jp}$ is bounded this shows the existence of a constant κ such that

$$\begin{gathered} \mathfrak{F} r_{1jpm3} \in L^\infty \quad \text{with compact support and} \\ \|\mathfrak{F} r_{1jpm3}\|_{L^\infty} \le \kappa \,\|\alpha_j\|\, \|(f,g,0)\|_X\,. \end{gathered} \tag{5.13}$$

The Fourier transforms of the simpler expressions r_{1jjm}, r_{1jpm1} and r_{1jpm2} can be calculated analogously. We get the same estimate (5.13).

To verify (5.11) for r_{1j}^s we see from (5.12), since $\mathfrak{F}(\tau_{jm}(\rho + i\cdot))$ is a bounded measure, that we only have to show that the Fourier transform of $\frac{1}{\rho+s+i\cdot}$ is in $L^1(\mathbb{R})$. For this let

$$\eta(x) := \begin{cases} e^{-x} & , \quad 0 \le x < \infty \\ 0 & , \quad -\infty < x < 0 \end{cases}.$$

Then $(\mathfrak{F}^{-1}\eta)(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x}\eta(x)\,dx = \frac{1}{1+i\omega}$. Hence Corollary 13.3 implies

$$\frac{1}{2\pi}(\mathrm{C},1) - \int_{\infty}^{\infty} e^{i\omega x}\frac{1}{1+i\omega}\,d\omega = \begin{cases} e^{-x} & , \quad 0 < x < \infty \\ \frac{1}{2} & , \quad x = 0 \\ 0 & , \quad -\infty < x < 0 \end{cases}.$$

From this it follows easily that

$$\frac{1}{2\pi}(\mathrm{C},1) - \int_{\infty}^{\infty} e^{i\omega x}\frac{1}{\rho + s + i\omega}\,d\omega = \begin{cases} e^{-(\rho+s)x} & , \quad 0 < x < \infty \\ \frac{1}{2} & , \quad x = 0 \\ 0 & , \quad -\infty < x < 0 \end{cases},$$

which is in $L^1(\mathbb{R})$.

Next we calculate the Fourier transform of r_4^s. Recall that in the expansion for the fundamental solution T through our recursion we arrived in the first step to the matrix F_1 with nondiagonal entries $(i \neq j)$

$$\begin{aligned}(F_1(x,y,\lambda))_{ij} = & -\lambda \exp\left(-\int_y^x \lambda k_j^{-1}(u)\,du\right)\exp\left(-\int_y^x \tfrac{c_{jj}(u)}{k_j(u)}\,du\right) \\ & \int_y^x \exp\left(\lambda \int_x^z (k_i^{-1}(u) - k_j^{-1}(u))\,du\right) \\ & \exp\left(\int_x^z \left(\tfrac{c_{ii}(u)}{k_i(u)} - \tfrac{c_{jj}(u)}{k_j(u)}\right)du\right)\frac{c_{ij}(z)}{k_i(z)}\,dz.\end{aligned} \tag{5.14}$$

After partial integration we got expression (4.19). A formula for the diagonal entries of F is given by (4.18) which we got in the second recursion step. Therefore we have

$$\begin{aligned} r_4^s(\nu,\rho,(f,g,0),x^*) = & \frac{\rho+i\nu}{\rho+s+i\nu}\sum_{\substack{m,j=1\\ m\neq j}}^{n} r_{4mj}^s(\nu,\rho,(f,g,0),\alpha_m) + \\ & \frac{1}{\rho+s+i\nu}\sum_{m=1}^{n} r_{4mm}^s(\nu,\rho,(f,g,0),\alpha_m) + \\ & \tilde{r}_4^s\left(\nu,\rho,(f,g,0),(x_j)_{1\leq j\leq n_2}\right),\end{aligned}$$

where for $1 \leq m, j \leq n$, $m \neq j$,

$$\begin{aligned} r_{4mj}^s := & -\int_0^l \int_0^x \exp\left(-\int_y^x (\rho+i\nu)k_j^{-1}(u)\,du\right)\exp\left(-\int_y^x \tfrac{c_{jj}(u)}{k_j(u)}\,du\right) \\ & k_j^{-1}(y)h_j(y)\int_y^x \exp\left(-\int_z^x (\rho+i\nu)(k_m^{-1}(u)-k_j^{-1}(u))\,du\right) \\ & \exp\left(-\int_z^x \left(\tfrac{c_{mm}(u)}{k_m(u)} - \tfrac{c_{jj}(u)}{k_j(u)}\right)du\right)\frac{c_{mj}(z)}{k_m(z)}\,dz\,dy\,d\alpha_m(x)\end{aligned}$$

and for $j = 1, \ldots, n$

$$\begin{aligned} r_{4jj}^s := & -\int_0^l \int_0^x \exp\left(-\int_y^x (\rho+i\nu)k_j^{-1}(u)\,du\right)\exp\left(-\int_y^x \tfrac{c_{jj}(u)}{k_j(u)}\,du\right) \\ & \sum_{\substack{\nu=1\\ \nu\neq j}}^{n}\int_y^x \frac{c_{j\nu}(z)}{k_j(z)}\rho_{\nu j}(z)\,dz\,k_j^{-1}(y)h_j(y)\,dy\,d\alpha_j(x).\end{aligned}$$

Again $\tilde{r}_4^s\left(\nu,\rho,(f,g,0),(x_j)_{1\leq j\leq n_2}\right)$ is very similar to its preceding terms in the sum and we do not consider it. As for r_{0j0} the transform of r_{4jj}^s is in L^∞

with compact support and estimate (5.13) holds for r^s_{4jj}.
Using the change of variable $r(z,x) := -\int_z^x \left(k_m^{-1}(u) - k_j^{-1}(u)\right) du$ we can write for $m \neq j$ (recall the definition of ρ_{mj} in Lemma 4.6)

$$\begin{aligned} r^s_{4mj} &= -\int_0^l \int_0^x \exp\left(-i\nu \int_y^x k_j^{-1}(u)\,du\right) \exp\left(-\int_y^x \tfrac{c_{jj}(u)+\rho}{k_j(u)}\,du\right) \\ &\quad \frac{h_j(y)}{k_j(y)} \int_{-\infty}^{\infty} e^{-i\nu r}\tilde{\zeta}(y,x,r)\,dr\,dy\,d\alpha_m(x), \end{aligned}$$

where

$$\tilde{\zeta}(y,x,r) := e^{-\rho r} \exp\left(-\int_{z(r,x)}^x \left(\tfrac{c_{mm}(u)}{k_m(u)} - \tfrac{c_{jj}(u)}{k_j(u)}\right) du\right) \rho_{mj}(z(r,x))\chi(y,x,r)$$

and $\chi(y,x,\cdot)$ is the characteristic function of the interval

$$\left[-\int_y^x (k_m^{-1}(u) - k_j^{-1}(u))\,du, 0\right] \cup \left[0, -\int_y^x (k_m^{-1}(u) - k_j^{-1}(u))\,du\right].$$

Therefore for $m \neq j$

$$\frac{1}{2\pi}(C,1)\!-\!\int_{-\infty}^{\infty} e^{i\nu\omega} r^s_{4mj}(\nu)\,d\nu =$$

$$\int_0^l \int_0^x \exp\left(-\int_y^x \tfrac{c_{jj}(u)+\rho}{k_j(u)}\,du\right) \frac{h_j(y)}{k_j(y)} \zeta\left(y,x,\omega - \int_y^x k_j^{-1}(u)\right) dy\,d\alpha_m(x),$$

where $\zeta(y,x,r) := (\tilde{\zeta}(y,x,r+) + \tilde{\zeta}(y,x,r-))/2$. Hence

$$\mathfrak{F} r^s_{4mj} \in L^\infty \quad \text{with compact support for } 1 \leq m,j \leq n.$$

Considering the Fourier transform of r^s_2 it follows from (4.20), (4.18), (5.14) and the previous arguments that the transform of

$$S(\rho+i\nu) := -\begin{pmatrix} E \\ I \end{pmatrix} H_0(\rho+i\nu)^{-1} H_1(\rho+i\nu) H_0(\rho+i\nu)^{-1}(D,-I)$$

is a bounded measure. Since

$$r^s_2 = \frac{1}{\rho+s+i\nu} \int_0^l T_0(x,0,\rho+i\nu) S(\rho+i\nu) \int_0^l T_0(l,y,\lambda) K(y)^{-1} \begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy\,d\alpha(x)$$

it follows as above that there exists a constant κ such that (5.11) is satisfied for $i = 2$.

Finally we look at r_3^s: We write

$$\begin{aligned} r_3^s(\nu,\rho,(f,g,b),x^*) = &\frac{1}{\rho+s+i\nu}\left(r_{31}^s(\nu,\rho,(f,g,b),\alpha)+r_{32}^s(\nu,\rho,(f,g),\alpha)\right) \\ &+\tilde{r}_3^s(\nu,\rho,(f,g,b),(x_j)_{1\le j\le n_2}), \end{aligned}$$

where

$$\begin{aligned} r_{31}^s := &\int_0^l T_0(x,0,\rho+i\nu)\begin{pmatrix} E \\ I \end{pmatrix} H_0(\rho+i\nu)^{-1} \\ &\left(b+(D,-I)\int_0^l F_1(l,y,\lambda)K(y)^{-1}\begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy\right) d\alpha(x), \\ r_{32}^s := &\int_0^l T_0(x,0,\rho+i\nu)\begin{pmatrix} E \\ I \end{pmatrix} H_0(\rho+i\nu)^{-1} \\ &(F,G)\int_0^{\cdot} T_0(\cdot,y,\lambda)K(y)^{-1}\begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy\, d\alpha(x) \end{aligned}$$

and $\tilde{r}_3^s$ is similar to its preceding terms. We see that r_{31}^s is composed of terms similar to the ones we have already treated. The term r_{32}^s differs slightly from the previous terms since it contains the $n_2 \times n_1$ matrix of measures (F,G). However, the arguments above still work (only an additional integral with a bounded measure from (F,G) appears and one uses Fubini once more, the $(C,1)$-Fourier transform is taken in the first inner integrals as we did above). Thus one shows similarly that

$$\frac{1}{2\pi}(\mathrm{C},1)-\int_{-\infty}^{\infty} e^{i\nu\omega} r_{32}^s(\nu)\,d\nu \in L^\infty$$

and (5.11) holds for $i=3$. $\square$

Chapter 6

Systems containing identical speed and degeneracies

In section 6.1 we extend the previous results obtained for nondegenerate hyperbolic systems to nondgenerate systems containing identical speed where condition (HIII) of section 4 can be violated. We allow the occurence of identical entries (speeds) in the matrix K with possibly full coupling C.

6.1 Nondegenerate linear hyperbolic systems with full coupling containing identical speed

In analogy with the previous sections we will keep the same notation. This will cause no confusion because all assumptions and estimates are analogous to section 4.

We consider the following class of nondegenerate hyperbolic systems containing identical speeds: For $x \in]0, l[$ and $t > 0$

$$\text{(H)} \quad \begin{cases} \dfrac{\partial}{\partial t}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + K(x)\dfrac{\partial}{\partial x}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + C(x)\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = 0, \\ \dfrac{d}{dt}\left[v(t,l) - Du(t,l)\right] = Fu(t,\cdot) + Gv(t,\cdot), \\ u(t,0) = Ev(t,0), \end{cases}$$

where

(HI) K is a diagonal $n \times n$ matrix of the form

$$K = \begin{pmatrix} k_1 I_{d_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 I_{d_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_\alpha I_{d_\alpha} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_{\alpha+1} I_{d_{\alpha+1}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & k_{\alpha+\beta} I_{d_{\alpha+\beta}} \end{pmatrix},$$

where $d_i \in \mathbb{N}$, $d_i > 0$, $\alpha \in \mathbb{N}$, $\beta \in \mathbb{N}$, $\sum_{i=1}^{\alpha} d_i = n_1$, $\sum_{i=1}^{\beta} d_{\alpha+i} = n_2$, I_{d_i} denotes the identity matrix in $\mathbb{R}^{d_i \times d_i}$ and $k_i \in C^1([0,l], \mathbb{R})$ satisfy for $x \in [0,l]$

$$\begin{aligned} k_i(x) &> 0 \text{ for } i = 1, \ldots, \alpha, \\ k_j(x) &< 0 \text{ for } j = \alpha+1, \ldots \alpha+\beta. \end{aligned}$$

(HII) $C(x) = (C_{ij}(x))_{1 \le i,j \le \alpha+\beta} \in \mathbb{C}^{n \times n}$ with $C_{ij}(x) \in \mathbb{C}^{d_i \times d_j}$ and

$$\begin{aligned} C_{ii} &\in L^\infty\left(]0,l[, \mathbb{C}^{d_i \times d_i}\right), \quad i = 1, \ldots, \alpha+\beta, \\ C_{ij} &\in \mathrm{BV}\left([0,l], \mathbb{C}^{d_i \times d_j}\right), \quad i,j = 1, \ldots, \alpha+\beta \quad \text{with } i \neq j. \end{aligned}$$

(HIII) Either

$$k_i(x) \neq k_j(x) \text{ for } 1 \le i,j \le \alpha+\beta, i \neq j, x \in [0,l],$$

or, if $i \neq j$ and $k_i(x) = k_j(x)$ for some $x \in [0,l]$, then C_{ij} vanishes completely on $[0,l]$.
(HIV) same as in section 4
(HV) same as in section 4

Let C_{b0} to be the block diagonal matrix containing the square matrices C_{ii}

$$C_{b0} := \text{blockdiag } (C_{ii})_{1 \le i \le \alpha+\beta}. \tag{6.1}$$

The reduced system is per definitionem

$$(\mathrm{H}_0) \quad \begin{cases} \dfrac{\partial}{\partial t}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + K(x)\dfrac{\partial}{\partial x}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + C_{b0}(x)\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = 0, \\ u(t,0) = Ev(t,0), \qquad v(t,l) = Du(t,l), \\ u(0,x) = u_0(x), \ v(0,x) = v_0(x). \end{cases}$$

Let A and A_0 denote the closed, densely defined operator corresponding to (H) and (H_0), respectively. Then A generates a C_0 semigroup in X_p, $1 \le p < \infty$, and Y and A_0 generates a C_0 semigroup in $L^p([0,l],\mathbb{C}^n)$ or Y_0 (defined in (4.2)). Let T be the fundamental matrix satisfying formula (4.4) and T_0 be the fundamental system satisfying

$$\begin{aligned} \frac{d}{dx}T_0(x,y,\lambda) &= -K^{-1}(x)\,(\lambda I + C_{b0}(x))\,T_0(x,y,\lambda) \quad \text{for } x,y \in [0,l], \\ T_0(y,y,\lambda) &= I \quad \text{for } y \in [0,l]. \end{aligned} \tag{6.2}$$

Because (6.2) here is not in diagonal, but only in blockdiagonal form we do not have an explicit formula for T_0. But we have the following

Proposition 6.1. *For $1 \le i \le \alpha+\beta$ there exist F_i depending only on C_{b0} and K, $F_i : [0,l]^2 \to \mathbb{C}^{d_i \times d_i}$, $F_i(\cdot,y) \in W^{1,\infty}([0,l],\mathbb{C}^{d_i\times d_i})$ for $y \in [0,l]$, so that for $F := (\text{blockdiag}\, F_i)_{1\le i\le\alpha+\beta}$ we have*

$$T_0(x,y,\lambda) = \exp\left(-\lambda \int_y^x K^{-1}(z)\,dz\right) F(x,y). \tag{6.3}$$

Moreover for $x \ge y \ge z$ we have $F_i(x,z) = F_i(x,y)F_i(y,z)$.

Proof. Define F_i to be the solution to

$$\frac{d}{dx}F_i(x,y) = -k_i^{-1}(x)C_{ii}(x)F_i(x,y), \quad F_i(y,y) = I_{d_i}.$$

From (6.1) and (HI) it follows that K, C_{b0} and $\exp\left(-\lambda\int_y^x K^{-1}(z)\,dz\right)$ commute. And this shows that the right hand side of (6.3) solves (6.2). □

Proposition 6.2. *Proposition 4.3 holds literally.*

Let h_0 denote the characteristic function to (H_0) defined literally as in formula (4.7) (but where T_0 is the fundamental system to the blockdiagonal system (6.2) of this section of course). Again h_0 is an exponential polynomial. Let h denote the characteristic function to (H) defined as in Definition 4.2.

Using our definition of the reduced blockdiagonal system (H_0) we will see in the remaining of this section that spectra and resolvents can be estimated as in section 4.2. The resulting expressions will be still explicit enough so that the growth rate of the semigroup can be calculated as we did in 5.2.

Define

$$C_1(x) := C(x) - C_{b0}(x) \text{ and } T_k \text{ as in (4.15) and (4.16).}$$

We will check again that $\sum_{k=0}^{\infty} \lambda^{-k} T_k(x,y,\lambda)$ converges in $W^{1,\infty}$ for sufficiently large $|\mathfrak{Im}\lambda|$. After reordering terms we will obtain for any finite $\kappa \in \mathbb{N}$ an explicit representation of the form

$$T(x,y,\lambda) = \sum_{k=0}^{\kappa} \lambda^{-k} F_k(x,y,\lambda) + O(\lambda^{-(\kappa+1)}),$$

for λ in a stripe $\mathbb{C}_r$ and sufficiently large $|\mathfrak{Im}\lambda|$, where each F_k is of order 1 with respect to λ on stripes $\mathbb{C}_r$ (by this we mean that for any given $r > 0$ there exists $c > 0$ such that $\|F_k(x,y,\lambda)\| \leq c$ for $\lambda \in \mathbb{C}_r$, $x,y \in [0,l]$).

To see this we calculate the first two steps T_1 and T_2. Put

$$f_0(x,y,\lambda) := T_0(x,y,\lambda)(y_0^{(1)}, \ldots, y_0^{(\alpha+\beta)})^t$$

with the arbitrary but fixed initial data $y_0^{(i)} \in \mathbb{C}^{d_i}$, $1 \leq i \leq \alpha+\beta$. Define

$$f_k := -\lambda \int_y^x T_0(x,z,\lambda) K^{-1}(z) C_1(z) f_{k-1}(z,y,\lambda)\, dz \text{ for } k \geq 1.$$

Then according to Proposition 6.1 the i-th component, $1 \leq i \leq \alpha+\beta$ of f_k is

$$\begin{aligned} f_0^{(i)}(x,y,\lambda) &= \exp\left(-\textstyle\int_y^x \lambda k_i^{-1}(u)\, du\right) F_i(x,y) y_0^{(i)}, \\ f_k^{(i)}(x,y,\lambda) &= -\lambda \exp\left(-\textstyle\int_y^x \lambda k_i^{-1}(u)\, du\right) F_i(x,y) \\ & \sum_{\substack{1 \leq l \leq \alpha+\beta \\ l \neq i}} \int_y^x \exp\left(\lambda \textstyle\int_y^z k_i^{-1}(u)\, du\right) F_i(y,z) \frac{C_{il}(z)}{k_i(z)} f_{k-1}^{(l)}(z,y,\lambda)\, dz. \end{aligned}$$

By assumptions (HII) and (HIII) we can perform partial integration and get rid of the λ factor appearing in the recursion formula for $f_k^{(i)}$:

$$
\begin{aligned}
f_1^{(i)}(x,y,\lambda) = & - \exp\left(-\lambda \textstyle\int_y^x k_i^{-1}(u)\,du\right) F_i(x,y) \\
& \sum_{\substack{1\le l\le \alpha+\beta \\ l\neq i}} \int_y^x \lambda\left(k_i^{-1}(z)-k_l^{-1}(z)\right) \exp\left(\textstyle\int_y^z \lambda\left(k_i^{-1}(u)-k_l^{-1}(u)\right)\,du\right) \\
& F_i(y,z)\frac{C_{il}(z)}{k_i(z)}\frac{F_l(z,y)}{k_i^{-1}(z)-k_l^{-1}(z)} y_0^{(l)}\,dz \\
= & \sum_{\substack{1\le l\le \alpha+\beta \\ l\neq i}} \Bigg\{ - \exp\left(-\lambda \textstyle\int_y^x k_l^{-1}(u)\,du\right) \frac{C_{il}(x)}{k_i(x)}\frac{F_l(x,y)}{k_i^{-1}(x)-k_l^{-1}(x)} \\
& + \exp\left(-\textstyle\int_y^x \lambda k_i^{-1}(u)\,du\right) F_i(x,y) \Bigg[\frac{C_{il}(y)}{k_i(y)}\frac{1}{k_i^{-1}(y)-k_l^{-1}(y)} \\
& + \int_y^x \exp\left(\textstyle\int_y^z \lambda\left(k_i^{-1}(u)-k_l^{-1}(u)\right)\,du\right) \\
& \frac{d}{dz}\left(F_i(y,z)\frac{C_{il}(z)}{k_i(z)}\frac{F_l(z,y)}{k_i^{-1}(z)-k_l^{-1}(z)}\right)\,dz \Bigg] \Bigg\} y_0^{(l)}.
\end{aligned}
$$

Note that for partial integration we used that in the sum for $l \neq i$ in the formula for $f_1^{(i)}$ the leading λ-exponential of $f_0^{(l)}$ is $e^{-\int_y^x \lambda k_l^{-1}(u)\,du}$. However, now $f_1^{(i)}$ not only contains $2(\alpha+\beta-1)$ terms with λ-exponential $e^{-\int_y^x \lambda k_i^{-1}(u)\,du}$ but also $(\alpha+\beta-1)$ terms of the form $e^{-\int_y^x \lambda k_l^{-1}(u)\,du}$, $1 \le l \le \alpha+\beta$, $l \neq i$. Therefore, in the next step for f_2 we will not be able to get rid of all λ terms by partial integration as in the first step:

$$f_2^{(i)}(x,y,\lambda) = -\exp\left(-\lambda \int_y^x k_i^{-1}(u)\,du\right) F_i(x,y) \sum_{\substack{1\le l_2,l_1\le\alpha+\beta \\ l_2\neq i, l_1\neq l_2}} \lambda$$

$$\int_y^x \Bigg\{ -\exp\left(\int_y^{z_2} \lambda\left(k_i^{-1}(u)-k_{l_1}^{-1}(u)\right)\,du\right) F_i(y,z) \frac{C_{il_2}(z_2)C_{l_2l_1}(z_2)}{k_i(z_2)k_{l_2}(z_2)}$$

$$\frac{F_{l_1}(z_2,y)}{k_{l_2}^{-1}(z_2)-k_{l_1}^{-1}(z_2)} + \exp\left(\lambda\int_y^{z_2}\left(k_i^{-1}(u)-k_{l_2}^{-1}(u)\right)\,du\right) F_i(y,z_2)$$

$$\frac{C_{il_2}(z_2)}{k_i(z_2)} F_{l_2}(z_2,y) \Bigg[\frac{C_{l_2l_1}(y)}{k_{l_2}(y)} \frac{1}{k_{l_2}^{-1}(y)-k_{l_1}^{-1}(y)}$$

$$+ \int_y^{z_2} \exp\left(\lambda\int_y^{z_1}\left(k_{l_2}^{-1}(u)-k_{l_1}^{-1}(u)\right)\,du\right)$$

$$\frac{d}{dz_1}\left(F_{l_2}(y,z_1)\frac{C_{l_2l_1}(z_1)}{k_{l_2}(z_1)}\frac{F_{l_1}(z_1,y)}{k_{l_2}^{-1}(z_1)-k_{l_1}^{-1}(z_1)}\right)\,dz_1\Bigg]\Bigg\} y_0^{(l_1)}\,dz_2.$$

Partial integration is not possible for the terms in the sum corresponding to $l_1 = i$. Therefore we are forced to keep $(\alpha+\beta-1)$ terms containing λ factors:

$$f_2^{(i)}(x,y,\lambda) = -\lambda\exp\left(-\lambda\int_y^x k_i^{-1}(u)\,du\right) F_i(x,y)$$

$$\sum_{\substack{1\le l_2\le\alpha+\beta \\ l_2\neq i}} \int_y^x \frac{C_{il_2}(z_2)C_{l_2i}(z_2)}{k_i(z_2)k_{l_2}(z_2)} \frac{F_i(z_2,y)}{k_{l_2}^{-1}(z_2)-k_i^{-1}(z_2)}\,dz_2 \cdot y_0^{(i)}$$

$$+ \text{ terms of order } 1$$

However, in the next third step for these $(\alpha+\beta-1)$ terms containing a λ factor partial integration can be done, so that in the third step there will be no λ^2 factors, only λ or 1 factors. Factors with λ^2 in the multisums will first appear in the fourth step. Thus, generally for $m \in \mathbb{N}$, terms containing λ^m factors appear for the first time in the $(2m)$-th recursion step. Besides these λ^m terms there only appear terms, which are bounded for $\lambda \in \mathbb{C}_r$, where the bound depends on r, C and K only. After reordering terms we have proven the following

Lemma 6.3. *There exists a sequence $F_k(x,y,\lambda)$ of matrices, which has the following properties:*

ı) Each F_k can be calculated from T_n for $n = 1,\ldots,2k$. We have $F_0 = T_0$

and F_1 is the matrix with the i-th blockdiagonal element, $1 \leq i \leq \alpha+\beta$,

$$
\begin{aligned}
(F_1(x,y,\lambda))_{ii} = &- \exp\left(-\lambda \int_y^x k_i^{-1}(u)\,du\right) F_i(x,y) \\
&\sum_{\substack{1\leq \nu \leq n \\ \nu \neq i}} \int_y^x \frac{C_{i\nu}(z)}{k_i(z)} \rho_{\nu i}(z) F_i(z,y)\,dz,
\end{aligned}
$$

where

$$
\rho_{lm}(z) := \frac{C_{lm}(z)}{k_l(z)} \frac{1}{k_l^{-1}(z) - k_m^{-1}(z)},\ z \in [0,l],\ 1 \leq l, m \leq n,\ l \neq m,
$$

and the i-th blockrow and j-th blockcolumn, $1 \leq i, j \leq n$, $i \neq j$,

$$
\begin{aligned}
(F_1(x,y,\lambda))_{ij} = &- \exp\left(-\lambda \int_y^x k_j^{-1}(u)\,du\right) \rho_{ij}(x) F_j(x,y) \\
&+ \exp\left(-\lambda \int_y^x k_i^{-1}(u)\,du\right) F_i(x,y)\rho_{ij}(y) \\
&+ \exp\left(-\lambda \int_y^x k_j^{-1}(u)\,du\right) F_i(x,y) \\
&\int_y^x \exp\left(\int_y^z \lambda \left(k_i^{-1}(u) - k_j^{-1}(u)\right)\,du\right) \\
&\frac{d}{dz}\left(F_i(y,z)\rho_{ij}(z)F_j(z,y)\right)\,dz.
\end{aligned}
$$

ii) For $r > 0$ there exists a constant $c > 0$ such that

$$
\|F_k(x,y,\lambda)\| \leq c^k \quad \text{for } \lambda \in \mathbb{C}_r \text{ and } x, y \in [0,l] \text{ and } k = 1, 2, \ldots .
$$

iii) For $r > 0$ there exists $d > 0$ such that for $\lambda \in \mathbb{C}_r$ with $|\mathfrak{Im}(\lambda)| > d$ the series $\sum_{k=0}^{\infty} \lambda^{-k} F_k(x,y,\lambda)$ converges absolutely (in $L^\infty([0,l]\times[0,l], \mathbb{C}^{n\times n})$) to $T(x,y,\lambda)$. For $r > 0$ there exist $c, d > 0$ such that for $\lambda \in \mathbb{C}_r$ and $|\mathfrak{Im}\lambda| > d$ we have

$$
\left\| T(x,y,\lambda) - T_0(x,y,\lambda) - \frac{1}{\lambda} F_1(x,y,\lambda) \right\| \leq c \frac{1}{|\lambda|^2}.
$$

As a consequence we have:

Lemma 6.4. *Lemma 4.8 holds literally.*

If (H_0) has nonempty spectrum we define again

$$
\gamma_- := \inf \{\mathfrak{Re}\lambda \mid h_0(\lambda) = 0\} \text{ and } \gamma_+ := \sup \{\mathfrak{Re}\lambda \mid h_0(\lambda) = 0\}.
$$

Lemma 6.5. *Lemmas 4.14 and 4.15 hold literally.*

Lemma 6.6. *Lemma 4.16 holds literally. (in the definition of R_i one has to use T_0 and F_1 of this section of course)*

By proceeding as in section 5.2 we prove

Theorem 6.7. *Theorems 5.4, 5.5 and 5.7 hold literally.*

6.2 Degenerate linear hyperbolic systems

In this section we will extend previous results obtained only for nondegenerate hyperbolic systems to degenerate systems. We will express spectra and resolvents for degenerate systems in terms of the nondegenerate system we have already studied.

The degenerate system is of the form

$$\text{(DH)}\quad \begin{cases} \dfrac{\partial}{\partial t}\begin{pmatrix} u(t,x) \\ v(t,x) \\ w(t,x)\end{pmatrix} + K(x)\dfrac{\partial}{\partial x}\begin{pmatrix} u(t,x) \\ v(t,x) \\ w(t,x)\end{pmatrix} + C(x)\begin{pmatrix} u(t,x) \\ v(t,x) \\ w(t,x)\end{pmatrix} = 0, \\ \dfrac{d}{dt}\left[v(t,l) - Du(t,l)\right] = Fu(t,\cdot) + Gv(t,\cdot), \\ u(t,0) = Ev(t,0), \end{cases}$$

where $x \in \,]0,l[$ and $t > 0$. We put the following assumptions on (DH):

(DHI) K is a diagonal matrix of the form

$$K = \begin{pmatrix} k_1 I_{d_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 I_{d_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_\alpha I_{d_\alpha} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_{\alpha+1} I_{d_{\alpha+1}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & k_{\alpha+\beta} I_{d_{\alpha+\beta}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \cdot I_{n_3} \end{pmatrix},$$

where $d_i \in \mathbb{N}$, $d_i > 0$, $\alpha \in \mathbb{N}$, $\beta \in \mathbb{N}$, I_{d_i} denotes the identity matrix in $\mathbb{R}^{d_i \times d_i}$,

$$\sum_{i=1}^{\alpha} d_i = n_1, \quad \sum_{i=1}^{\beta} d_{\alpha+i} = n_2.$$

We assume

$$\begin{aligned} k_i \in C^1\left([0,l],\mathbb{R}\right), &\quad 1 \le i \le \alpha+\beta, \\ k_i(x) > 0 &\quad \text{for } i = 1,\dots,\alpha \\ k_j(x) < 0 &\quad \text{for } j = \alpha+1,\dots \alpha+\beta. \end{aligned}$$

(DHII) $C(x)$ is a $n \times n$ matrix of the form

$$C(x) = \begin{pmatrix} C_\alpha(x) & C_\beta(x) \\ C_\gamma(x) & C_\delta(x) \end{pmatrix}$$

with

$$C_\alpha \in \mathbb{C}^{n_1+n_2\times n_1+n_2}, \quad C_\alpha = (C_{ij})_{1\le i,j\le\alpha+\beta}, \quad C_{ij} \in \mathbb{C}^{d_i\times d_j}.$$

Denote

$$\left(\tilde{C}_{ij}\right)_{1\le i,j\le\alpha+\beta} := K_0^{-1} C_\beta C_\gamma.$$

We assume

$$\begin{array}{rcl} C_{ii} & \in & L^\infty\left([0,l],\mathbb{C}^{d_i\times d_i}\right), \quad i = 1,\dots,\alpha+\beta \\ C_{ij} & \in & BV\left([0,l],\mathbb{C}^{d_i\times d_j}\right), \quad i,j = 1,\dots,\alpha+\beta,\ i\neq j, \\ C_\beta & \in & L^\infty([0,l],\mathbb{C}^{(n_1+n_2)\times n_3}), \\ C_\gamma & \in & C([0,l],\mathbb{C}^{n_3\times(n_1+n_2)}), \\ C_\delta & \in & C([0,l],\mathbb{C}^{n_3\times n_3}), \\ \tilde{C}_{ij} & \in & BV\left([0,l],\mathbb{C}^{d_i\times d_j}\right), \text{ for } i\neq j. \end{array}$$

(DHIII) If $i \neq j$, $1 \le i,j \le \alpha+\beta$, and $k_i(x) = k_j(x)$ for some $x \in [0,l]$ then both C_{ij} and $\tilde{C}_{ij}$ vanish completely on $[0,l]$.
(DHIV) $u(t,x) = (u_1(t,x),\dots,u_{n_1}(t,x)) \in \mathbb{C}^{n_1}$ and $v(t,x) = (v_1(t,x),\dots,v_{n_2}(t,x)) \in \mathbb{C}^{n_2}$ and $w(t,x) = (w_1(t,x),\dots,w_{n_3}(t,x)) \in \mathbb{C}^{n_3}$.
(DHV) $D \in \mathbb{C}^{n_2\times n_1}$, $E \in \mathbb{C}^{n_1\times n_2}$ and

$$F : C([0,l],\mathbb{C}^{n_1}) \to \mathbb{C}^{n_2}, \quad G : C([0,l],\mathbb{C}^{n_2}) \to \mathbb{C}^{n_2}$$

are linear continuous operators.

Write (DH) as an abstract evolution equation

$$\frac{d}{dt}z(t) = Az(t) \quad (z = (u,v,w,d)),$$

in the complex spaces X_p or Y with the closed densely defined operator $A : \mathcal{D}(A) \subset X \to X$

$$A(u,v,w,d) := \left(-\left(K(x)\frac{\partial}{\partial_x} + C(x)\right)\begin{pmatrix} u \\ v \\ w \end{pmatrix}; \quad Fu + Gv \right),$$

$$\mathcal{D}(A) := \{z \in X \mid Az \in X,\ u(0) = Ev(0),\ d = v(l) - Du(l)\},$$

where X denotes X_p or Y. It is not difficult to see that A generates a C_0 semigroup in Y, see Proposition 7.18, and X_p for $p \in [1,\infty[$ and special choices of F and G, see Proposition 7.20, [45, Theorem 6.2, p. 312] or [48].

Then for given $(f,g,h,b) \in X$ the resolvent equation

$$(\lambda I - A)(u,v,w,d) = (f,g,h,b), \quad (u,v,w,d) \in \mathcal{D}(A)$$

reads

$$\begin{cases} \left(\lambda I_{n_1+n_2} + K_0(x)\frac{\partial}{\partial x} + C_\alpha(x)\right)\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} + C_\beta(x)w(x) &= \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} \\ \left(\lambda I_{n_3} + C_\delta(x)\right) w(x) + C_\gamma(x)\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} &= h \\ \lambda\left(v(l) - Du(l)\right) - Fu - Gv &= b. \end{cases} \tag{6.4}$$

The spectrum may not only consist of point spectrum, but also continuous or residual spectrum, depending on the choice of the underlying Banach space (continuous spectrum when $X = X_p$, $p \in [1,\infty[$ and residual spectrum when $X = Y$). Let

$$\Sigma := \{\lambda \in \mathbb{C} \mid \exists x \in [0,l] : \det(\lambda I_{n_3} + C_\delta(x)) = 0\}. \tag{6.5}$$

Then Σ is compact consisting of a finite union of closed curves. If $\lambda \notin \Sigma$ then (6.4) is equivalently written as:

$$\begin{cases} \left(\lambda I_{n_1+n_2} + K_0\frac{\partial}{\partial x} + C_\alpha - C_\beta J(\lambda) C_\gamma\right)\begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} f \\ g \end{pmatrix} - C_\beta J(\lambda) h \\ w &= J(\lambda)\left(-C_\gamma\begin{pmatrix} u \\ v \end{pmatrix} + h\right) \\ \lambda\left(v(l) - Du(l)\right) - Fu - Gv &= b, \end{cases} \tag{6.6}$$

with

$$J(\lambda)(x) := \left(\lambda I_{n_3} + C_\delta(x)\right)^{-1}.$$

As in Proposition 4.3 it follows that the operator

$$(u,v,d) \longmapsto \begin{cases} \left(\lambda I_{n_1+n_2} + K_0\frac{\partial}{\partial x} + C_\alpha - C_\beta J(\lambda) C_\gamma\right)\begin{pmatrix} u \\ v \end{pmatrix} \\ \lambda d - Fu - Gv \end{cases} \tag{6.7}$$

is an index 0 Fredholm operator from

$$\left\{(u,v,d) \in W^{1,p}([0,l],\mathbb{C}^{n_1+n_2}) \times \mathbb{C}^{n_2} \mid u(0) = Ev(0),\ d = v(l) - Du(l)\right\}$$

into $L^p([0,l],\mathbb{C}^{n_1+n_2}) \times \mathbb{C}^{n_2}$ for $p \in [1,\infty]$.
In particular, if $\lambda \in \sigma(A) \setminus (\Sigma \cup \sigma_p(A))$ then (6.7) is injective and hence

bijective, so that (6.6) has a unique solution, i.e. $\lambda \in \rho(A)$. Hence $\sigma(A) \setminus \Sigma$ only contains pointspectrum

$$\sigma(A) \setminus \Sigma \subset \sigma_p(A).$$

Let $T(x, y, \lambda)$ denote the fundamental system to

$$\frac{d}{dx}\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = -K_0^{-1}(x)\left(\lambda I_{n_1+n_2} + C_\alpha(x) - C_\beta(x)J(\lambda)(x)C_\gamma(x)\right)\begin{pmatrix} u(x) \\ v(x) \end{pmatrix}. \tag{6.8}$$

Define $h(\lambda)$ and $H(\lambda)$ as in Def. 4.2 and (4.6) [1]:

$$\begin{aligned} h(\lambda) &:= \det H(\lambda), \\ H(\lambda) &:= (-\lambda D\delta_l - F, \lambda I_{n_2}\delta_l - G)T(\cdot, 0, \lambda)\begin{pmatrix} E \\ I_{n_2} \end{pmatrix}. \end{aligned}$$

Then

$$\sigma(A) \setminus \Sigma = \{\lambda \in \mathbb{C} \mid h(\lambda) = 0\}$$

and $\sigma(A) \setminus \Sigma$ is discrete since $h(\lambda)$ is analytic. We have

$$\begin{cases} \left[R(\lambda, A)\begin{pmatrix} f \\ g \\ h \\ b \end{pmatrix}\right](x) &= \begin{pmatrix} u(x) \\ v(x) \\ w(x) \\ \Delta(u, v) \end{pmatrix}, \\ \begin{pmatrix} u \\ v \end{pmatrix} &= T(\cdot, 0, \lambda)\begin{pmatrix} E \\ I \end{pmatrix} H(\lambda)^{-1}\beta(\lambda)(f, g, h, b) \\ & \quad + \int_0^{\cdot} T(\cdot, y, \lambda)\theta(f, g, h, \lambda, y)\, dy, \\ w &= J(\lambda)\left(-C_\gamma\begin{pmatrix} u \\ v \end{pmatrix} + h\right), \end{cases} \tag{6.9}$$

$$\theta(f, g, h, \lambda, y) := K_1(y)^{-1}\left[\begin{pmatrix} f(y) \\ g(y) \end{pmatrix} - C_\beta(y)J(\lambda)(y)h(y)\right]$$

and $\beta(\lambda) : X \to \mathbb{C}^{n_2}$ denotes

$$\beta(\lambda)(f, g, h, b) := b + (\lambda D\delta_l + F, G - \lambda I_{n_2}\delta_l)\int_0^{\cdot} T(\cdot, y, \lambda)\theta(y)\, dy.$$

We want to express the characteristic functions and resolvents for (DH) in terms of the nondegenerate system (H) and in powers of λ^{-1} for λ on stripes $\mathbb{C}_r$ with sufficiently large imaginary part. For this let T_H denote the

[1] the reader will not be confused that by analogy we use the same symbols h, H and T, as we did for the nondegenerate system (H) in section 4

fundamental system corresponding to the nondegenerate problem (H), i.e. T_H is the fundamental system of the initial value problem

$$\frac{d}{dx}\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = -K_0^{-1}(x)\left(\lambda I_{n_1+n_2} + C_\alpha(x)\right)\begin{pmatrix} u(x) \\ v(x) \end{pmatrix}.$$

According to Lemma 6.3 (or 4.6) we have that for $r > 0$ there exist $c, d > 0$ such that for $\lambda \in \mathbb{C}_r$ and $|\mathfrak{Im}\lambda| > d$ we have

$$\left\| T_H(x,y,\lambda) - T_0(x,y,\lambda) - \frac{1}{\lambda}F_1(x,y,\lambda) \right\| \le c\frac{1}{|\lambda|^2}, \tag{6.10}$$

where T_0 and F_1 are bounded for $\lambda \in \mathbb{C}_r$ and only depend on the nondegenerate system (obtained by deleting w).

Because we only require a expansion of T up to order λ^{-2} we will instead of T estimate the fundamental solution to

$$\frac{d}{dx}\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = -K_0^{-1}(x)\left(\lambda I_{n_1+n_2} + C_\alpha(x) - \lambda^{-1}C_\beta(x)C_\gamma(x)\right)\begin{pmatrix} u(x) \\ v(x) \end{pmatrix}. \tag{6.11}$$

Denote the fundamental solution to (6.11) by $\tilde{T}$. Because for $|\lambda| > \|C_\delta\|$

$$J(\lambda) = \frac{1}{\lambda}I_{n_3} - \frac{1}{\lambda^2}C_\delta \sum_{i=0}^{\infty}\left(-\frac{C_\delta}{\lambda}\right)^i$$

it follows from Grownwall's inequality that T is a λ^{-2} perturbation of $\tilde{T}$ for λ in a neighbourhood of ∞.

Define

$$\begin{aligned}
\tilde{T}_0(x,y,\lambda) &:= T_H(x,y,\lambda) \\
\tilde{T}_k(x,y,\lambda) &:= \int_y^x T_H(x,z,\lambda)K_0^{-1}(z)C_\beta(z)C_\gamma(z)\tilde{T}_{k-1}(z,y,\lambda)\,dz \quad \text{for } k \ge 1.
\end{aligned}$$

Then

$$\tilde{T}(x,y,\lambda) = \sum_{k=0}^{\infty}\frac{1}{\lambda^k}\tilde{T}_k(x,y,\lambda) = T_H(x,y,\lambda) + \frac{1}{\lambda}\tilde{T}_1(x,y,\lambda) + \sum_{k=2}^{\infty}\frac{1}{\lambda^k}\tilde{T}_k(x,y,\lambda)$$

for $|\lambda|$ sufficiently large. Hence we only have to estimate $\tilde{T}_1$,

$$\tilde{T}_1(x,y,\lambda) = \int_y^x T_H(x,z,\lambda)K_0^{-1}(z)C_\beta(z)C_\gamma(z)T_H(z,y,\lambda)\,dz.$$

From (6.10) it follows that

$$\frac{1}{\lambda}\tilde{T}_1(x,y,\lambda) = \frac{1}{\lambda}T_0(x,y,\lambda)\int_y^x T_0(y,z,\lambda)K_0^{-1}(z)C_\beta(z)C_\gamma(z)T_0(z,y,\lambda)\,dz + O\left(\frac{1}{\lambda^2}\right).$$

By Proposition 6.1 we have

$$T_0(x,y,\lambda) = \exp\left(-\lambda\int_y^x K_0^{-1}(z)\,dz\right)F(x,y),$$

where $F = \text{blockdiag}\,(F_i)_{1\le i\le \alpha+\beta}$ and for $1 \le i \le \alpha+\beta$ $F_i : [0,l]^2 \to \mathbb{C}^{d_i\times d_i}$ only depends on the reduced blockdiagonal nondegenerate system obtained from (DH) after canceling w.

Proposition 6.8.

$$\begin{aligned}&\int_y^x T_0(y,z,\lambda)K_0^{-1}(z)C_\beta(z)C_\gamma(z)T_0(z,y,\lambda)\,dz\\ =&\int_y^x \text{blockdiag}(F_i(y,z)\tilde{C}_{ii}(z)F_i(z,y))_{1\le i\le\alpha+\beta}\,dz + O\left(\frac{1}{\lambda}\right).\end{aligned}$$

Proof. For $1 \le i,j \le \alpha+\beta$ we have

$$\begin{aligned}&\left(T_0(y,z,\lambda)K_0^{-1}(z)C_\beta(z)C_\gamma(z)T_0(z,y,\lambda)\right)_{ij}\\ =&\exp\left(\lambda\int_y^z \left(k_i^{-1}(u)-k_j^{-1}(u)\right)\,du\right)F_i(y,z)\tilde{C}_{ij}(z)F_j(z,y).\end{aligned}$$

If we integrate this equation from y to x then for $i \neq j$ we can perform partial integration by assumptions (DHII) and (DHIII). We get for $i \neq j$

$$\int_y^x \left(T_0(y,z,\lambda)K_0^{-1}(z)C_\beta(z)C_\gamma(z)T_0(z,y,\lambda)\right)_{ij}\,dz = O\left(\frac{1}{\lambda}\right).$$

□

Hence we have proven the following

Lemma 6.9. *Let $T(x,y,\lambda)$ denote the fundamental system to* (6.8)*. For $r > 0$ there exist constants $c,d > 0$ such that for $\lambda \in \mathbb{C}$ with $|\mathfrak{Re}\lambda| < r$ and $|\mathfrak{Im}\lambda| > d$ we have*

$$\left\|T(x,y,\lambda) - T_0(x,y,\lambda) - \frac{1}{\lambda}\tilde{F}_1(x,y,\lambda)\right\| \le c\frac{1}{|\lambda|^2}.$$

$\tilde{F}_1$ is the matrix with the i-th blockdiagonal element, $1 \le i \le \alpha+\beta$,

$$\left(\tilde{F}_1(x,y,\lambda)\right)_{ii} = (F_1(x,y,\lambda))_{ii} + \exp\left(-\lambda \int_y^x k_i^{-1}(u)\,du\right) F_i(x,y) \int_y^x F_i(y,z)\tilde{C}_{ii}(z)F_i(z,y)\,dz,$$

and for the i-th blockrow and j-th blockcolumn, $1 \le i,j \le \alpha+\beta$, $i \neq j$,

$$\left(\tilde{F}_1(x,y,\lambda)\right)_{ij} = (F_1(x,y,\lambda))_{ij}$$

Remark 6.10. *The expansion differs from the nondegenerate case only in an additional term on the (block)diagonal of F_1.*

Lemma 6.11. *Lemma 4.8 holds true literally if we replace F_1 with $\tilde{F}_1$.*

We have the following two Lemmas which are proved similar as Lemmas 4.14 and 4.15

Lemma 6.12. *For each $\gamma > \gamma_+$ there exist only finitely many eigenvalues λ of* (DH) *in the complement of Σ that satisfy $\mathfrak{Re}\,\lambda \ge \gamma$.*

Lemma 6.13. *Suppose* (H_0) *has nonempty spectrum. Then the following hold:*
ı) For each $\delta > 0$ there are only finitely many eigenvalues of (DH) *in the complement of Σ which satisfy $\mathfrak{Re}\lambda \le \gamma_- - \delta$ or $\mathfrak{Re}\lambda \ge \gamma_+ + \delta$.*
ıı) For $\epsilon > 0$ there exists $d > 0$ such that

$$\sigma(\mathrm{DH}) \cap \{\lambda \in \mathbb{C} \mid |\mathfrak{Im}\lambda| \ge d\} \subset \bigcup_{h_0(\lambda)=0} B_\epsilon(\lambda).$$

ııı) Suppose $\rho = \inf_{\lambda_1 \neq \lambda_2, h_0(\lambda_1)=h_0(\lambda_2)=0} |\lambda_1 - \lambda_2| > 0$. Then for each $\eta < \frac{\rho}{2}$ there exists $d > 0$ such that for each $\lambda_0 \in \mathbb{C}$ with $h_0(\lambda_0) = 0$ and $|\mathfrak{Im}\lambda_0| \ge d$ there exists $\lambda \in B_\eta(\lambda_0)$ which is an eigenvalue of (DH)*, i.e. $h(\lambda) = 0$. Both h and h_0 have the same number of zeros in each $B_\eta(\lambda_0)$. In particular, if* (H_0) *only possesses algebraically simple eigenvalues, then the eigenvalues $\lambda \in B_\eta(\lambda_0)$ of* (DH) *are unique and algebraically simple.*

Let

$$\Delta F_1 := \tilde{F}_1 - F_1.$$

Let $H_1^{(\mathrm{H})}$ be defined as in the formula (4.20), and let $H_1^{(\mathrm{DH})}$ be defined as in formula (4.20) but using $\tilde{F}_1$ instead of F_1. Denote

$$\Delta H_1 := H_1^{(\mathrm{DH})} - H_1^{(\mathrm{H})}.$$

Formulas for ΔF_1 and ΔH_1 are

$$\begin{aligned}
\Delta F_1(x,y,\lambda) &= \text{blockdiag}\Bigg(\exp\left(-\lambda \textstyle\int_y^x k_i^{-1}(u)\,du\right) \\
&\qquad F_i(x,y) \int_y^x F_i(y,z)\tilde{C}_{ii}(z)F_i(z,y)\,dz \Bigg)_{1\le i\le \alpha+\beta}, \\
\Delta H_1 &= -(D,-I)\,\Delta F_1(l,0,\lambda)\begin{pmatrix} E \\ I \end{pmatrix}.
\end{aligned}$$

Lemma 6.14. *Let*

$$\begin{aligned}
R_{nd}^{\mathrm{ap}}\begin{pmatrix} f \\ g \\ b \end{pmatrix} :=& R(\lambda, A_0)\begin{pmatrix} f \\ g \end{pmatrix} + \\
& \frac{1}{\lambda}\left(R_1(\lambda)\begin{pmatrix} f \\ g \end{pmatrix} + R_2(\lambda)\begin{pmatrix} f \\ g \end{pmatrix} + R_3(\lambda)\begin{pmatrix} f \\ g \\ b \end{pmatrix} + R_4(\lambda)\begin{pmatrix} f \\ g \end{pmatrix}\right)
\end{aligned}$$

denote the resolvent approximation for the nondegenerate hyperbolic system which we obtain from (DH) *by deleting terms including w (formulas for $R_1,\dots,R_4$ are given below* (4.25)*). Suppose there exist $h\in\mathbb{R}$, $\delta,\Delta,\epsilon>0$ such that for $\lambda\in\mathbb{C}$ with $|\mathfrak{Re}\lambda - h|<\delta$ and $|\mathfrak{Im}\lambda|>\Delta$ the inequality $|h_0(\lambda)|\ge\epsilon$ and the relation $\lambda\notin\Sigma$ hold. Then there exist constants $c,d>0$ such that for all $\lambda\in\mathbb{C}$ with $|\mathfrak{Re}\lambda-h|<\delta$ and $|\mathfrak{Im}\lambda|>d$ we have $\lambda\in\rho(A)$ and*

$$R(\lambda,A)\begin{pmatrix} f \\ g \\ h \\ b \end{pmatrix} = \begin{pmatrix} u \\ v \\ w \\ \Delta(u,v) \end{pmatrix} + \frac{1}{\lambda^2}\mathcal{E}(\lambda)(f,g,h,b),$$

where

$$
\begin{aligned}
\begin{pmatrix} u \\ v \end{pmatrix} =& R_{nd}^{\text{app}} \begin{pmatrix} f \\ g \\ b \end{pmatrix} \\
&+ \frac{1}{\lambda}\Bigg(-T_0(\cdot, 0, \lambda) \begin{pmatrix} E \\ I \end{pmatrix} H_0(\lambda)^{-1} \Delta H_1(\lambda) H_0(\lambda)^{-1} \beta_0(\lambda)(f, g) \\
&- T_0(\cdot, 0, \lambda) \begin{pmatrix} E \\ I \end{pmatrix} H_0(\lambda)^{-1}(D, -I) \int_0^l T_0(l, y, \lambda) K_1(y)^{-1} C_\beta(y) h(y)\, dy \\
&+ \Delta F_1(\cdot, 0, \lambda) \begin{pmatrix} E \\ I \end{pmatrix} H_0(\lambda)^{-1} \beta_0(\lambda)(f, g) \\
&- \int_0^{\cdot} T_0(\cdot, y, \lambda) K_1(y)^{-1} C_\beta(y) h(y)\, dy \\
&+ T_0(\cdot, 0, \lambda) \begin{pmatrix} E \\ I \end{pmatrix} H_0(\lambda)^{-1}(D, -I) \int_0^l \Delta F_1(l, y, \lambda) K_1(y)^{-1} \begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy \\
&+ \int_0^{\cdot} \Delta F_1(\cdot, y, \lambda) K_1(y)^{-1} \begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy \Bigg), \\
w =& \frac{1}{\lambda} \left(-C_\gamma R(\lambda, A_0)(f, g) + h\right)
\end{aligned}
$$

and the error term $\mathcal{E}$ is bounded by c,

$$\|\mathcal{E}(\lambda)\|_{\mathcal{L}(X)} \leq c.$$

Proof. By Lemma 6.9 and Remark 6.10 we have

$$T = T_0 + \frac{1}{\lambda} F_1 + \frac{1}{\lambda} \Delta F_1 + O\left(\frac{1}{\lambda^2}\right).$$

From this we get (Lemma 6.11)

$$
\begin{aligned}
\frac{1}{\lambda} H(\lambda) &= H_0(\lambda) + \frac{1}{\lambda} H_1(\lambda) + \frac{1}{\lambda} \Delta H_1(\lambda) + O\left(\frac{1}{\lambda^2}\right), \\
\lambda H(\lambda)^{-1} &= H_0(\lambda)^{-1} - \frac{1}{\lambda} H_0(\lambda)^{-1} H_1(\lambda) H_0(\lambda)^{-1} - \frac{1}{\lambda} H_0(\lambda)^{-1} \Delta H_1(\lambda) H_0(\lambda)^{-1} \\
&\quad + O\left(\frac{1}{\lambda^2}\right).
\end{aligned}
$$

After we plug these into (6.9) we get the stated estimate for the resolvent of (DH). $\square$

Using the resolvent approximation of Lemma 6.14 it is not difficult to verify condition *iv*) of Theorem 5.26 for the additional $\frac{1}{\lambda}$ terms appearing above in the expansion for the nondegenerate system) (as done in section 5.2). Thus we have

Theorem 6.15 (Exponential dichotomy for (DH))**.** *Let $\alpha \leq \beta$, $\alpha, \beta \in \mathbb{R}$. System* (DH) *is (α, β) exponentially dichotomous in the spaces Y and X_p, $p \in [1, \infty[$, if and only if there exists $\delta > 0$ so that*

$$h(\lambda) \neq 0 \quad \textit{and} \quad \lambda \notin \Sigma$$

for $\lambda \in \mathbb{C}$ with $\alpha - \delta < \mathfrak{Re}\lambda < \beta + \delta$. In this case the exponential rates are independent on $p \geq 1$.

Theorem 6.16 (Spectral gap mapping Theorem for (DH))**.**
Theorem 5.4 holds for (DH)*.*

Chapter 7

Semilinear hyperbolic systems: Fréchet differentiability of the solution map and stability by linearization

In this section we define weak solutions, show local existence and uniqueness and regularity for the class of semilinear hyperbolic systems (SH). We show that (SH) generates a smooth semiflow in the phase space Y and prove the stability Theorem 7.26.

We consider the class of semilinear hyperbolic systems

$$
\text{(SH)} \quad \begin{cases} \dfrac{\partial}{\partial t}\begin{pmatrix} u(t,x) \\ v(t,x) \\ w(t,x) \end{pmatrix} + K(x)\dfrac{\partial}{\partial x}\begin{pmatrix} u(t,x) \\ v(t,x) \\ w(t,x) \end{pmatrix} \\ +H(x,u(t,x),v(t,x),w(t,x)) = 0, \\ \dfrac{d}{dt}\left[v(t,l) - Du(t,l)\right] = F(u(t,\cdot),v(t,\cdot)), \\ u(t,0) = E\,v(t,0), \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x),\ w(0,x) = w_0(x). \end{cases}
$$

for $x \in]0,l[$ and $t > 0$ with the following assumptions:

(SHI) $K(x) = \operatorname{diag}\,(k_i(x))_{i=1,\dots,n}$ is a diagonal $n \times n$ matrix of functions $k_i \in C^1([0,l],\mathbb{R})$ which satisfy $k_i(x) > 0$ for $i = 1,\dots,n_1$ and $k_i(x) < 0$ for $i = n_1+1,\dots n_1+n_2$ ($x \in [0,l]$) and $k_i \equiv 0$ for $i = n_1+n_2+1,\dots,n_1+n_2+n_3 = n$.
(SHII) $H :]0,l[\times \mathbb{R}^n \to \mathbb{R}^n$ is a C^k Carathéodory function, $k \geq 1$ (see

Def. 10.11). The last n_3 components $H_w :]0,l[\times \mathbb{R}^n \to \mathbb{R}^{n_3}$ of H satisfy $H_w(\cdot, z) \in C([0,l], \mathbb{R}^{n_3})$ for $z \in \mathbb{R}^n$. We denote with H_{uv} the first $n_1 + n_2$ components of H, i.e. $H = (H_{uv}, H_w)$.
(SHIII) $F : C([0,l], \mathbb{R}^{n_1+n_2}) \to \mathbb{R}^{n_2}$ is C^k and has bounded and uniformly continuous derivatives on bounded sets (for each $b > 0$ and $\epsilon > 0$ there exists $\delta > 0$ so that $\left\|\partial^k F(u_1, v_1) - \partial^k F(u_2, v_2)\right\| \leq \epsilon$ for $(u_1, v_1), (u_2, v_2) \in C([0,l], \mathbb{R}^{n_1+n_2})$, $\|(u_1, v_1) - (u_2, v_2)\| \leq \delta$, $\|(u_1, v_1)\| \leq b$).
(SHIV) $u(t,x) = (u_1(t,x), \ldots, u_{n_1}(t,x)) \in \mathbb{R}^{n_1}$, $v(t,x) = (v_1(t,x), \ldots, v_{n_2}(t,x)) \in \mathbb{R}^{n_2}$ and $w(t,x) = (w_1(t,x), \ldots, w_{n_3}(t,x)) \in \mathbb{R}^{n_3}$
(SHV) $D \in \mathbb{R}^{n_2 \times n_1}$, $E \in \mathbb{R}^{n_1 \times n_2}$

Let $\mathfrak{H} : C([0,l], \mathbb{R}^n) \to L^\infty([0,l], \mathbb{R}^n)$,

$$\mathfrak{H}(u,v,w)(x) := H(x, u(x), v(x), w(x)), \quad \text{a.a. } x \in [0,l],$$

denote the superposition operator generated by H. We denote the (u,v) and w component of $\mathfrak{H}$ with $\mathfrak{H}_{uv}$ and $\mathfrak{H}_w$, respectively.

Remark 7.1. *By* (SHII) *the superposition operator $\mathfrak{H}$ maps $L^\infty([0,l], \mathbb{R}^n)$ C^k-smoothly into itself and has locally bounded derivatives [24]. In particular $\mathfrak{H}$ is locally Lipschitz from $L^\infty([0,l], \mathbb{R}^n)$ into itself, i.e. for $b > 0$ there exists $L > 0$ so that for $z_1 = (u_1, v_1, w_1) \in L^\infty([0,l], \mathbb{R}^n)$ and $z_2 = (u_2, v_2, w_2) \in L^\infty([0,l], \mathbb{R}^n)$ with $\|z_1\|_{L^\infty} \leq b$ and $\|z_2\|_{L^\infty} \leq b$ one has $\|\mathfrak{H}(z_1) - \mathfrak{H}(z_2)\|_{L^\infty} \leq L \|z_1 - z_2\|_{L^\infty}$. By* (SHIII) *also F is locally Lipschitz from its domain into $\mathbb{R}^{n_2}$.*

Let $T(t)$ denote the semigroup to

$$\left\{ \begin{array}{l} \dfrac{\partial}{\partial t} \begin{pmatrix} u(t,x) \\ v(t,x) \\ w(t,x) \end{pmatrix} + K(x) \dfrac{\partial}{\partial x} \begin{pmatrix} u(t,x) \\ v(t,x) \\ w(t,x) \end{pmatrix} = 0, \\ \dfrac{d}{dt} \left[v(t,l) - Du(t,l) \right] = 0, \\ u(t,0) = E\, v(t,0), \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x),\ w(0,x) = w_0(x). \end{array} \right. \tag{7.1}$$

System (7.1) can be written as an abstract evolution equation

$$\frac{d}{dt} z(t) = A_0 z(t) \quad (z = (u,v,w,d)),$$

in the space X_p or Y with the closed densely defined operator

$$A_0 : \mathcal{D}(A_0) \subset X \to X, \quad A_0(u,v,w,d) := \left(-K(x) \frac{\partial}{\partial_x} \begin{pmatrix} u \\ v \\ w \end{pmatrix} ; \quad 0 \right), \tag{7.2}$$

$$\mathcal{D}(A_0) := \{(u,v,w,d) \in X \mid A_0(u,v,w,d) \in X, \qquad u(0) = Ev(0),\ d = \Delta(u,v)\}, \tag{7.3}$$

where X denotes X_p or Y.

By integrating along characteristics one can derive an explicit formula for the semigroup $T(t)$. We do not need such a formula, we only need the following

Proposition 7.2. *The semigroup $T(t)$ is strongly continuous on the spaces X_p for $1 \le p < \infty$, Y and $\mathcal{D}(A_0)$ (but not on X_∞, see Remark 7.4 and [44]). For $T > 0$ there exists $c > 0$ such that for $(u_0, v_0, w_0) \in L^\infty([0,l], \mathbb{R}^n)$ and $d_0 \in \mathbb{R}^{n_2}$ we have*

$$\|T(t)(u_0, v_0, w_0, d_0)\|_{X_\infty} \le c\,\|(u_0, v_0, w_0, d_0)\|_{X_\infty} \quad \textit{for} \quad 0 \le t \le T.$$

In particular Proposition 7.2 states that $T(t)$ is a semigroup of bounded operators on X_∞ (which is not C_0, even not Bochner measurable according to Remark 7.4). Our choice of phase space for (SH) will be Y. For $T > 0$ denote

$$\mathcal{X}_T := C([0,T], Y). \tag{7.4}$$

Definition 7.3. Let $T > 0$. The triplet $(u, v, w, \Delta(u,v)) \in \mathcal{X}_T$ is called a weak (or mild) solution of (SH) up to T for the initial data $(u_0, v_0, w_0, \Delta(u_0, v_0)) \in Y$ if for all $t \in [0,T]$

$$(u(t), v(t), w(t), \Delta(u(t), v(t))) = \mathcal{G}(u, v, w, \Delta(u,v))(t),$$

where

$$\mathcal{G}(u,v,w,\Delta(u,v))(t) := T(t)\begin{pmatrix} u_0 \\ v_0 \\ w_0 \\ \Delta(u_0, v_0) \end{pmatrix} + \int_0^t T(t-s)\begin{pmatrix} -\mathfrak{H}(u(s), v(s), w(s)) \\ F(u(s), v(s)) \end{pmatrix} ds. \tag{7.5}$$

We need to add a remark and explain in which sense the integral in (7.5) has to be understood:

Remark 7.4. *It does not make sense to define the Bochner integral*

$$\int_0^t T(t-s)\begin{pmatrix} -\mathfrak{H}(u(s), v(s), w(s)) \\ F(u(s), v(s)) \end{pmatrix} ds$$

in the space X_∞ *because the integrand* $s \mapsto T(t-s)\begin{pmatrix} -\mathfrak{H}(u(s),v(s),w(s)) \\ F(u(s),v(s)) \end{pmatrix}$ *will not be measurable in the sense of Bochner in* X_∞.

Indeed, consider a real valued step function on $[0,l]$ *which has a jump (shock) at* $\frac{l}{2}$. *Then translation of this function is not measurable on a time interval with values into the Banach space* $L^\infty([0,l],\mathbb{R})$ *because the image is not separable with respect to the strong* L^∞ *norm* [1]. *Now the Nemytskij operator* $\mathfrak{H}$ *will not be compatible with the boundary conditions (even if the generating function is arbitrary smooth with respect to all variables or linear with constant coefficients, in general), so that shocks will travel along the characteristics due to incompatibilities at the boundary when the translation semigroup* $T(t-s)$ *is applied.*

Hence the integrand will not be measurable in X_∞, *so the integral in* (7.5) *can not be defined in* X_∞. *But it is well defined in the Banach space* X_p *for* $1 \le p < \infty$: *Because* T *is a strongly continuous semigroup on* X_p *for* $1 \le p < \infty$ *it follows that the integrand is measurable with values in the larger space* X_p. *Moreover, we are allowed to estimate the* X_∞ *norm of the integral: Let* $f : [0,T] \to X_\infty$ *be measurable and bounded. Then* $s \mapsto T(t-s)f(s)$ *is measurable on* $[0,t]$ *with values in* X_p, $1 \le p < \infty$, *and we have*

$$\left\| \int_0^t T(t-s)f(s)\,ds \right\|_{X_p} \le c \int_0^t \|f(s)\|_{X_p}\,ds, \tag{7.6}$$

where c *does not depend on* $1 \le p < \infty$. *By letting* $p \to \infty$ *we get*

$$\left\| \int_0^t T(t-s)f(s)\,ds \right\|_{X_\infty} \le c \int_0^t \|f(s)\|_{X_\infty}\,ds. \tag{7.7}$$

In this work we will do such L^∞ *estimates many times without any comments even though the integrand will not be measurable with values in the Banach space* L^∞ *(or* X_∞*). In section 10 we will consider even weaker solutions where it will happen that* f *will not be measurable on* $[0,t]$ *with values in* $L^\infty([0,l])$, *but* f *will be measurable on the time space product space* $[0,t]\times[0,l]$, $f \in L^\infty([0,t]\times[0,l])$. *For almost all* $s \in [0,t]$ *we have that* $f(s,\cdot) \in L^\infty([0,l])$ *and it follows that* $s \mapsto \|f(s,\cdot)\|_{L^\infty([0,l])}$ *is measurable (because the map is obtained as a limit for* $p \to \infty$ *of the measurable map* $s \mapsto \|f(s,\cdot)\|_{L^p([0,l])}$*). Again the integral* $\int_0^t T(t-s)f(s)\,ds$ *will be well defined in* L^p, $p < \infty$ *and we are allowed to perform norm estimates as* (7.6), (7.7).

Theorem 7.5. *Weak solutions of* (SH) *are unique.*

[1] the author would like to thank Prof. A. Mielke for pointing to this technical difficulty

Proof. The proof is standard and uses Gronwall's Lemma: Let

$$z_1 = (u_1, v_1, w_1, \Delta(u_1, v_1)),\ z_2 = (u_2, v_2, w_2, \Delta(u_2, v_2)) \in \mathcal{X}_T$$

be solutions of (SH). By Remark 7.1 and Proposition 7.2 there exist constants $c > 0$ and $L > 0$ so that for $t \in [0, T]$

$$\begin{aligned}
&\|z_1 - z_2\|_Y \\
\leq &\left\| \int_0^t T(t-s) \begin{pmatrix} -\mathfrak{H}(u_1(s), v_1(s), w_1(s)) + \mathfrak{H}(u_2(s), v_2(s), w_2(s)) \\ F(u_1(s), v_1(s)) - F(u_2(s), v_2(s)) \end{pmatrix} ds \right\|_{X_\infty} \\
\leq &\int_0^t c \left\| \begin{pmatrix} \mathfrak{H}(u_1(s), v_1(s), w_1(s)) - \mathfrak{H}(u_2(s), v_2(s), w_2(s)) \\ F(u_1(s), v_1(s)) - F(u_2(s), v_2(s)) \end{pmatrix} \right\|_{X_\infty} ds \\
\leq &cL \int_0^t \|z_1 - z_2\|_Y \, ds.
\end{aligned}$$

Gronwall's inequality yields

$$\|(z_1 - z_2)(t)\|_Y = 0 \quad \text{for } t \in [0, T].$$

□

Proposition 7.6. *Suppose*

$$\rho \in C\left([0, T], L^\infty(]0, l[\,, \mathbb{R}^{n_1+n_2}) \times C([0, l], \mathbb{R}^{n_3}) \times \mathbb{R}^{n_2}\right).$$

Then

$$\int_0^{\cdot} T(\cdot - s)\rho(s)\, ds \in C([0, T], Y).$$

Proof. Denote $X_{\infty C} := L^\infty(]0, l[\,, \mathbb{R}^{n_1+n_2}) \times C([0, l], \mathbb{R}^{n_3}) \times \mathbb{R}^{n_2}$. By mollification there exists a sequence $\rho_k \in C^1([0, T], X_{\infty C})$ such that ρ_k converges uniformly to ρ in $C([0, T], X_{\infty C})$. Let $\mathcal{D}(A_0)$ be as in (7.3) with $X = X_p$ for a fixed $1 \leq p < \infty$. Since $\rho_k \in C^1([0, T], X_p)$ it follows from Proposition 13.6 that

$$\int_0^{\cdot} T(\cdot - s)\rho_k(s)\, ds \in C([0, T], D(A_0)).$$

The domain $\mathcal{D}(A_0)$ is continuously embedded in

$$\{(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{d}) \in C([0, l], \mathbb{R}^{n_1+n_2}) \times L^p([0, l], \mathbb{R}^{n_3}) \times \mathbb{R}^{n_2} \mid \\ \tilde{u}(0) = E\tilde{v}(0),\ \tilde{d} = \tilde{v}(l) - D\tilde{u}(l)\}.$$

Moreover, the third component of ρ_k lies in $C([0,l],\mathbb{R}^{n_3})$ and the semigroup acts trivially on the third component. Therefore

$$\int_0^{\cdot} T(\cdot - s)\rho_k(s)\,ds \in C([0,T],Y).$$

For $t \in [0,T]$

$$\begin{aligned}\left\|\int_0^t T(t-s)(\rho(s)-\rho_k(s))\,ds\right\|_{X_\infty} \leq & T \sup_{s\in[0,T]} \|T(s)\|_{\mathcal{L}(X_\infty)} \|\rho-\rho_k\|_{C([0,T],X_\infty)} \\ \leq & c\,\|\rho-\rho_k\|_{C([0,T],X_\infty C)}\,.\end{aligned}$$

Hence it follows that

$$\int_0^{\cdot} T(\cdot - s)\rho(s)\,ds \in C([0,T],Y).$$

□

As an immediate consequence we have:

Corollary 7.7. *If* $(u_0,v_0,w_0,\Delta(u_0,v_0)) \in Y$ *and* $(u,v,w,\Delta(u,v)) \in \mathcal{X}_T$, *then*

$$T(\cdot)\begin{pmatrix} u_0 \\ v_0 \\ w_0 \\ \Delta(u_0,v_0)\end{pmatrix} + \int_0^{\cdot} T(\cdot - s)\begin{pmatrix} -\mathfrak{H}(u(s),v(s),w(s)) \\ F(u(s),v(s))\end{pmatrix} ds \in C([0,T],Y).$$

The following Proposition is a direct consequence of Definition 7.3 and Corollary 7.7:

Proposition 7.8. *If* $(u,v,w,\Delta(u,v)) \in \mathcal{X}_T$ *is a weak solution to* (SH), *then*

$$\Delta(u,v) \in C^1([0,T],\mathbb{R}^{n_2}) \quad \textit{and} \quad \frac{d}{dt}\Delta(u,v)(t) = F(u(t),v(t)).$$

Proposition 7.9. *Let* $z = (u,v,w,\Delta(u,v)) \in \mathcal{X}_T$ *be a weak solution of* (SH) *with initial data* $z(0) = (u_0,v_0,w_0,\Delta(u_0,v_0)) \in Y$. *Suppose*

$$(u_0,v_0) \in W^{1,\infty}(]0,l[\,,\mathbb{R}^{n_1+n_2}).$$

Then for all $p \in\,]1,\infty[$

$$\begin{aligned}(u,v,w,\Delta(u,v)) \in & C([0,T],W^{1,p}(]0,l[\,,\mathbb{R}^{n_1+n_2}) \times C([0,l],\mathbb{R}^{n_3}) \times \mathbb{R}^{n_2}) \\ & \cap C^1([0,T],L^p(]0,l[\,,\mathbb{R}^{n_1+n_2}) \times C([0,l],\mathbb{R}^{n_3}) \times \mathbb{R}^{n_2})\end{aligned} \tag{7.8}$$

and (SH) *holds in a classical sense.*

Proof. Let $h > 0$ and $0 \leq t < t+h \leq T$. We have

$$\begin{aligned} z(t+h) - z(t) =& (T(h) - I)T(t)z(0) \\ &+ \int_0^t T(t-s) \begin{pmatrix} -\mathfrak{H}((u,v,w)(h+s)) + \mathfrak{H}((u,v,w)(s)) \\ F(u(h+s), v(h+s)) - F(u(s), v(s)) \end{pmatrix} ds \\ &+ \int_0^h T(t+h-s) \begin{pmatrix} -\mathfrak{H}((u,v,w)(s)) \\ F(u(s), v(s)) \end{pmatrix} ds. \end{aligned} \tag{7.9}$$

By (SHII), (SHIII) and Proposition 7.2 there exists $c > 0$ so that

$$\begin{aligned} \|z(t+h) - z(t)\|_Y \leq& \|(T(h) - I)T(t)z(0)\|_{X_\infty} + ch \\ &+ c \int_0^t \|z(s+h) - z(s)\|_Y \, ds. \end{aligned}$$

Moreover we have

$$(T(h) - I)T(t)z(0) = \int_0^h T(s)T(t)(A_0 z(0)) \, ds. \tag{7.10}$$

And because $A_0 z(0) \in X_\infty$ by our assumption on the initial data we have (the constant c will differ from each line)

$$\|(T(h) - I)T(t)z(0)\|_{X_\infty} \leq ch.$$

Hence

$$\|z(t+h) - z(t)\|_Y \leq ch + c \int_0^t \|z(s+h) - z(s)\|_Y \, ds.$$

Gronwall's Lemma implies

$$\|z(t+h) - z(t)\|_Y \leq hc.$$

Hence $\omega : [0,T] \to Y$ is Lipschitz continuous. This shows that

$$\begin{pmatrix} -\mathfrak{H}((u,v,w)(\cdot)) \\ F(u(\cdot), v(\cdot)) \end{pmatrix} : [0,T] \to X_\infty \subset X_p$$

is Lipschitz continuous. Because X_p is reflexive for $1 < p < \infty$ it follows that

$$\begin{pmatrix} -\mathfrak{H}((u,v,w)(\cdot)) \\ F(u(\cdot), v(\cdot)) \end{pmatrix} \in W^{1,\infty}([0,T], X_p)$$

and Proposition 13.6 yields the assertion. $\square$

Remark 7.10. *Suppose* $F : C([0,l], \mathbb{R}^{n_1+n_2}) \to \mathbb{R}^{n_2}$ *satisfies a* L^p *Lipschitz condition for a fixed* $p \in]1, \infty[$ *on bounded subsets of* $C([0,l], \mathbb{R}^{n_1+n_2})$*, i.e. for any bounded subset* B *of* $C([0,l], \mathbb{R}^{n_1+n_2})$ *there exists a constant* $L > 0$ *so that for* $(u_1, v_1), (u_2, v_2) \in B$ *the relation*

$$\|F(u_1, v_1) - F(u_2, v_2)\| \leq L \, \|(u_1, v_1) - (u_2, v_2)\|_{L^p([0,l])}$$

holds. If

$$(u_0, v_0) \in W^{1,p}(]0, l[\,, \mathbb{R}^{n_1+n_2}),$$

then (7.8) *holds.*

Proof. The proof is similar to Proposition 7.9. Indeed, the generating function of the Nemytskij operator $\mathfrak{H}$ is locally Lipschitz with respect to the unknown variables (uniformly for almost all $x \in]0, l[$, see the Definition 10.11 of C^k Carathéodory function), which implies - since z is bounded with values in X_∞ - that there exists a constant c so that

$$\begin{aligned} &\|\mathfrak{H}((u,v,w)(h+s)) - \mathfrak{H}((u,v,w)(s))\|_{L^p} \\ &\quad \leq c \, \|(u,v,w)(h+s) - (u,v,w)(s)\|_{L^p} \, . \end{aligned} \tag{7.11}$$

Since F satisfies a L^p Lipschitz condition, we get from (7.9) and (7.10) using the assumption on $z(0)$ that

$$\begin{aligned} \|z(t+h) - z(t)\|_{X_p} \leq& \|(T(h) - I)T(t)z(0)\|_{X_p} + ch \\ &+ c \int_0^t \|z(h+s) - z(s)\|_{X_p} \, ds \\ \leq& ch + c \int_0^t \|z(s+h) - z(s)\|_{X_p} \, ds. \end{aligned}$$

Gronwall yields that $z : [0,T] \to X_p$ is Lipschitz.
And (7.11) implies that $\begin{pmatrix} -\mathfrak{H}((u,v,w)(\cdot)) \\ F(u(\cdot), v(\cdot)) \end{pmatrix} : [0,T] \to X_p$ is Lipschitz. From the reflexivity of X_p it follows that $\begin{pmatrix} -\mathfrak{H}((u,v,w)(\cdot)) \\ F(u(\cdot), v(\cdot)) \end{pmatrix} \in W^{1,\infty}([0,T], X_p)$ and we apply Proposition 13.6 again. □

Theorem 7.11 (local existence)**.** *For any* $(u_0, v_0, w_0, \Delta(u_0, v_0)) \in Y$ *there exists a* $\delta > 0$*, depending only on* $\|(u_0, v_0, w_0, \Delta(u_0, v_0))\|_Y$*, such that* (SH) *has a weak solution up to* δ*.*

Proof. Corollary 7.7 shows that $\mathcal{G}$ maps $\mathcal{X}_T$ into itself. Let $0 < \delta < 1$. Define the closed subspace of $\mathcal{X}_\delta$ (see (7.4))

$$\begin{aligned} B_\delta := &\Big\{(u, v, w, \Delta(u, v)) \in \mathcal{X}_\delta \mid \text{for } t \in [0, \delta] \\ &\|(u(t), v(t), w(t), \Delta(u(t), v(t))) - T(t)(u_0, v_0, w_0, \Delta(u_0, v_0))\|_Y \leq 1\Big\}. \end{aligned}$$

By Remark 7.1 there exists $L > 0$, depending only on $\|(u_0, v_0, w_0, \Delta(u_0, v_0))\|_Y$, such that if $z_1, z_2 \in B_\delta$ then

$$\|\mathcal{G}(z_1)(t) - \mathcal{G}(z_2)(t)\|_Y \leq \delta L \|z_1 - z_2\|_{\mathcal{X}_\delta}. \tag{7.12}$$

Moreover, since $\mathfrak{H}$ and F are locally bounded it follows from the definition of B_δ that there exists a bound $M > 0$, depending only on $\|(u_0, v_0, w_0, \Delta(u_0, v_0))\|_Y$, such that for $z = (u, v, w, \Delta(u, v)) \in B_\delta$

$$\begin{aligned} &\|\mathcal{G}(z)(t) - T(t)(u_0, v_0, w_0, \Delta(u_0, v_0))\|_Y \\ \leq &\left\| \int_0^t T(t-s) \begin{pmatrix} -\mathfrak{H}((u, v, w)(s)) \\ F(u(s), v(s)) \end{pmatrix} ds \right\|_{X_\infty} \\ \leq &M\delta \qquad \text{for } t \in [0, \delta]. \end{aligned} \tag{7.13}$$

Therefore (7.12) and (7.13) imply that for sufficiently small $\delta > 0$ the operator $\mathcal{G}$ maps B_δ into itself and becomes a contraction. By Banachs contraction mapping theorem G has a fixed point in $B_\delta \subset \mathcal{X}_\delta$. □

For $z_0 \in Y$ let $\omega = \omega(z_0) \in \,]0, \infty]$ denote the maximal time up to which the solution exists, i.e.

$$\omega(z_0) := \sup\{t \in \mathbb{R} \mid \text{there exists a weak solution up to } t \text{ with } z(0) = z_0\}. \tag{7.14}$$

We have the following consequence of Theorem 7.11

Corollary 7.12. *For any $z_0 \in Y$ either*
ı) $\omega(z_0) = \infty$
or
ıı) $\omega(z_0) < \infty$ and $\lim_{t \uparrow \omega(z_0)} \|z(t)\|_Y = \infty$, where $z : [0, \omega(z_0)[\, \to Y$ denotes the weak solution with $z(0) = z_0$.

Proof. Suppose $\omega(z_0) < \infty$ and the assertion $\lim_{t \uparrow \omega(z_0)} \|z(t)\|_Y = \infty$ was false. Then there would exist a sequence $(t_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$, $0 < t_n < \omega(z_0)$, converging to $\omega(z_0)$, such that $\|z(t_n)\|_\infty$ were bounded. Since in Theorem 7.11 $\delta > 0$ only depended on the norm of the initial data we would find a $\tilde{\delta} > 0$ and construct a solution $z : \left[0, \omega(z_0) + \tilde{\delta}\right] \to Y$ with $z(0) = z_0$ contradicting definition (7.14) (here we have used that the concatenation of two solutions is a solution which follows directly from Def. 7.3). □

Theorem 7.13. *Let $z \in \mathcal{X}_T$ be a weak solution of* (SH) *up to T. Then there exists a neighborhood U of $z(0)$ in Y such that for all $y_0 \in U$ there is a weak solution $y \in \mathcal{X}_T$ of* (SH) *up to T satisfying $y(0) = y_0$.*
There exists a constant $c > 0$ such that for all $y_0 \in U$

$$\|z(t) - y(t)\|_Y \leq c\,\|z(0) - y_0\|_Y .$$

Proof. Proceed similar as in the proof of [71, Theorem 11.15, p. 117]. □

For a given solution $z = (u, v, w, \Delta(u, v))$ of (SH) we consider the z-linearized equation of (SH)

$$\text{(LH)} \quad \begin{cases} \dfrac{\partial}{\partial t}\begin{pmatrix} u_L(t,x) \\ v_L(t,x) \\ w_L(t,x) \end{pmatrix} + K(x)\dfrac{\partial}{\partial x}\begin{pmatrix} u_L(t,x) \\ v_L(t,x) \\ w_L(t,x) \end{pmatrix} \\ +\partial_{(u,v,w)} H(x, (u,v,w)(t,x)) \begin{pmatrix} u_L(t,x) \\ v_L(t,x) \\ w_L(t,x) \end{pmatrix} = 0, \\ \dfrac{d}{dt}\left[v_L(t,l) - D u_L(t,l)\right] = \langle \partial F(u(t,\cdot), v(t,\cdot)),\, (u_L(t,\cdot), v_L(t,\cdot))\rangle, \\ u_L(t,0) = E\, v_L(t,0), \\ u_L(0,x) = h_u(x),\ v_L(0,x) = h_v(x),\ w_L(0,x) = h_w(x). \end{cases}$$

Definition 7.14. Let $z = (u, v, w, \Delta(u, v)) \in \mathcal{X}_T$ be a given weak solution of (SH). The quadruplet $z_L = (u_L, v_L, w_L, \Delta(u_L, v_L)) \in \mathcal{X}_T$ is called a weak (or mild) solution of (LH) to the initial data $z_L(0) = (h_u, h_v, h_w, \Delta(h_u, h_v)) \in Y$ iff for all $t \in [0, T]$

$$z_L(t) = \mathcal{G}_L(z, z_L)(t),$$

where

$$\begin{aligned} \mathcal{G}_L(z, z_L)(t) :=& T(t)\begin{pmatrix} h_u \\ h_v \\ h_w \\ \Delta(h_u, h_v) \end{pmatrix} \\ &+ \int_0^t T(t-s)\begin{pmatrix} \langle -\partial\mathfrak{H}((u,v,w)(s)), (u_L(s), v_L(s), w_L(s))\rangle \\ \langle \partial F(u(s), v(s)), (u_L(s), v_L(s))\rangle \end{pmatrix} ds. \end{aligned}$$

($\partial\mathfrak{H}$ denotes the total derivative of the Nemytskij operator $\mathfrak{H} : L^\infty(]0, l[\,, \mathbb{R}^n) \to L^\infty(]0, l[\,, \mathbb{R}^n)$)

Theorem 7.15. *Let $T > 0$. For any weak solution $z = (u, v, w, \Delta(u, v)) \in \mathcal{X}_T$ of* (SH) *and $(h_u, h_v, h_w, \Delta(h_u, h_v)) \in Y$ there exists a unique weak solution*

$z_L = (u_L, v_L, w_L, \Delta(u_L, v_L)) \in \mathcal{X}_T$ *of the corresponding linearized problem* (LH)*. There is a constant* $c > 0$*, depending only on* $\|z\|_{\mathcal{X}_T}$*, such that*

$$\|z_L(t)\|_Y \leq c \,\|(h_u, h_v, h_w, \Delta(h_u, h_v))\|_Y .$$

Proof. First we note that by Proposition 7.6 we have that the operator $\mathcal{G}_L(z, \cdot)$ maps $\mathcal{X}_T$ into itself. We have to show that $z_L = \mathcal{G}_L(z, z_L)$ has a unique solution $z_L \in \mathcal{X}_T$. As in Theorem 7.11 one shows local existence, i.e. that a unique solution exists in $\mathcal{X}_\delta$ for $\delta > 0$ sufficiently small. Then we need an a-priori estimate to show that δ can be chosen arbitrary large ($\delta = T$): By assumptions (SHII) and (SHIII) the derivatives $\partial\mathfrak{H}$ and ∂F are bounded on bounded subsets. Hence it follows from the variation of constants formula in Definition 7.14 that there exists a constant c depending on $\|z\|_{\mathcal{X}_T}$ such that $\|z_L(t)\|_Y \leq c \,\|z_L(0)\|_Y + \int_0^t c \,\|z_L(s)\|_Y \, ds$. Gronwall's inequality implies $\|z_L(t)\|_Y \leq c \,\|z_L(0)\|_Y \, e^{ct}$. □

Suppose there exists a weak solution $\tilde{z} \in \mathcal{X}_T$ of (SH) up to T. Then according to Theorem 7.13 there exists an open neighborhood U of $\tilde{z}(0)$ in Y so that for any $z_0 \in U$ there exists a unique solution $z \in \mathcal{X}_T$ with $z(0) = z_0$. Define the solution map

$$S^t : U \to Y, \quad S^t(z_0) := z(t) \quad (t \in [0, T]). \tag{7.15}$$

For $z_0 \in U$ and $h = (h_u, h_v, h_w, \Delta(h_u, h_v)) \in Y$ define the linearized solution operator

$$S^t_{\mathcal{L}}(z_0) : Y \to Y, \quad S^t_{\mathcal{L}}(z_0)h := z_L(t) \quad (t \in [0, T]), \tag{7.16}$$

where $z_L \in \mathcal{X}_T$ denotes the solution of the, along the given solution $z(t) = S^t(z_0)$ of (SH), linearized system (LH) with initial data h.

Theorem 7.16. *For each* $t \in [0, T]$ *the map* $S^t : U \to Y$ *is* C^k *smooth. Moreover,*

$$\partial S^t(z_0) = S^t_{\mathcal{L}}(z_0).$$

Proof. For $z = (u, v, w, \Delta(u, v)) \in \mathcal{X}_T$ and initial data $z_0 = (u_0, v_0, w_0, \Delta(u_0, v_0)) \in Y$ the operator $\mathcal{G}(z)$ has been defined in Definition 7.3. To emphasize the dependence on z_0 we write $\mathcal{G}(z, z_0)$. Define the operator $\mathcal{F}$

$$(\mathcal{F}(z, z_0))\,(t) := (\mathcal{G}(z, z_0))\,(t) - z(t).$$

By Corollary 7.7 for each $z_0 \in Y$ the operator $\mathcal{G}(\cdot, z_0)$ maps $\mathcal{X}_T$ into itself. Thus $\mathcal{F} : \mathcal{X}_T \times Y \to \mathcal{X}_T$. By assumption for each $z_0 \in U$ the equation $\mathcal{F}(z, z_0) = 0, z \in \mathcal{X}_T$, has a unique solution $z = \gamma(z_0)$.
It follows from (SHII), (SHIII) and the definition of $\mathcal{G}$ (see (7.5)) that $\mathcal{G}$ is C^k

from $\mathcal{X}_T \times Y$ into $\mathcal{X}_T$ and that we have for $h_j = (h_{uj}, h_{vj}, h_{wj}, \Delta(h_{uj}, h_{vj})) \in \mathcal{X}_T$, $1 \le j \le k$, $t \in [0,T]$

$$\begin{aligned}
&\left(\frac{\partial^j \mathcal{G}}{\partial z^j}(z, z_0) h_1 \dots h_j\right)(t) \qquad (7.17)\\
&= \int_0^t T(t-s) \begin{pmatrix} -\partial^j \mathfrak{H}(u(s), v(s), w(s)) \, (h_{ui}(s), h_{vi}(s), h_{wi}(s))_{1 \le i \le j} \\ \partial^j F(u(s), v(s)) \, (h_{ui}(s), h_{vi}(s))_{1 \le i \le j} \end{pmatrix} ds.
\end{aligned}$$

Indeed, for $j = 1$ we have

$$\begin{aligned}
&\mathcal{G}(z+h_1)(t) - \mathcal{G}(z)(t)\\
&- \int_0^t T(t-s) \begin{pmatrix} -\partial \mathfrak{H}((u,v,w)(s))(h_{u1}(s), h_{v1}(s), h_{w1}(s)) \\ \partial F(u(s), v(s)) \, (h_{u1}(s), h_{v1}(s)) \end{pmatrix} ds\\
&= \int_0^t T(t-s) \Bigg[- \begin{pmatrix} -\partial \mathfrak{H}((u,v,w)(s))(h_{u1}(s), h_{v1}(s), h_{w1}(s)) \\ \partial F(u(s), v(s))(h_{u1}(s), h_{v1}(s)) \end{pmatrix}\\
&\quad + \begin{pmatrix} -\mathfrak{H}(u(s)+h_{u1}(s), v(s)+h_{v1}(s), w(s)+h_{w1}(s)) + \mathfrak{H}((u,v,w)(s)) \\ F(u(s)+h_{u1}(s), v(s)+h_{v1}(s)) - F(u(s), v(s)) \end{pmatrix} \Bigg] ds\\
&= \int_0^t T(t-s) \int_0^1 \Bigg[- \begin{pmatrix} -\partial \mathfrak{H}((u,v,w)(s))(h_{u1}(s), h_{v1}(s), h_{w1}(s)) \\ \partial F(u(s), v(s))(h_{u1}(s), h_{v1}(s)) \end{pmatrix}\\
&\begin{pmatrix} -\partial \mathfrak{H}((u,v,w)(s) + \theta(h_{u1}(s), h_{v1}(s), h_{w1}(s)))(h_{u1}(s), h_{v1}(s), h_{w1}(s)) \\ \partial F(u(s)+\theta h_{u1}(s), v(s)+\theta h_{v1}(s))(h_{u1}(s), h_{v1}(s)) \end{pmatrix} \Bigg] d\theta \, ds
\end{aligned}$$

Therefore by the uniform continuity of the derivative stated in conditions (SHII) and (SHIII) we have

$$\begin{aligned}
&\|h_1\|_{\mathcal{X}_T}^{-1} \Bigg\| \mathcal{G}(z+h_1) - \mathcal{G}(z)\\
&- \int_0^{\cdot} T(\cdot - s) \begin{pmatrix} -\partial \mathfrak{H}((u,v,w)(s))(h_{u1}(s), h_{v1}(s), h_{w1}(s)) \\ \partial F(u(s), v(s)) \, (h_{u1}(s), h_{v1}(s)) \end{pmatrix} ds \Bigg\|_{\mathcal{X}_T} \overset{\|h_1\|_{\mathcal{X}_T} \downarrow 0}{\longrightarrow} 0.
\end{aligned}$$

By induction one obtains (7.17) for $1 \le j \le k$.
It follows from a generalization of Banachs fixed point theorem that $\frac{\partial \mathcal{F}}{\partial z}$ is an isomorphism from $\mathcal{X}_T$ onto itself. Indeed, assume $w \in \mathcal{X}_T$ is given. Then for $h = (h_u, h_v, h_w, \Delta(h_u, h_v)) \in \mathcal{X}_T$ the equation $\frac{\partial \mathcal{F}}{\partial z}(z, z_0) h = w$ is equivalent to $\mathcal{P}h = h$, where $\mathcal{P} : \mathcal{X}_T \to \mathcal{X}_T$,

$$(\mathcal{P}h)(t) = \int_0^t T(t-s) \begin{pmatrix} -\partial \mathfrak{H}((u,v,w)(s))(h_u, h_v, h_w)(s) \\ \partial F(u(s), v(s)) \, (h_u(s), h_v(s)) \end{pmatrix} ds - w(t).$$

There exists a constant $M > 0$, depending only on $T, \mathfrak{H}, F, z$, so that for $h_1, h_2 \in \mathcal{X}_T$

$$\|\mathcal{P}h_1(t) - \mathcal{P}h_2(t)\|_Y \leq Mt \|h_1 - h_2\|_{\mathcal{X}_T}.$$

Proceeding with $\mathcal{P}^2 = \mathcal{P} \circ \mathcal{P}$ we get $\|(P^2 h_1)(t) - (P^2 h_2)(t)\|_Y \leq \frac{(Mt)^2}{2} \|h_1 - h_2\|_{\mathcal{X}_T}$. By induction

$$\left\|\mathcal{P}^i h_1 - \mathcal{P}^i h_2\right\|_{\mathcal{X}_T} \leq \frac{(MT)^i}{i!} \|h_1 - h_2\|_{\mathcal{X}_T}.$$

Thus for i sufficiently large $\mathcal{P}^i$ is a contraction on $\mathcal{X}_T$.
From the implicit function theorem it follows that γ is a C^k smooth map from U into $\mathcal{X}_T$. Hence $S^t : U \to Y$ is C^k.

From (7.5) and (7.17) it follows that ∂S^t is the solution to (LH) in the sense of Definition 7.14. □

Remark 7.17. *The map $S^\cdot : U \to \mathcal{X}_T, u \mapsto S^\cdot u$ is C^k smooth.*

Suppose H and F are linear. i.e.

$$H_{uv}(x, u, v, w) = C_{uv}(x)(u, v, w)^t, \quad H_w(x, u, v, w) = C_w(x)(u, v, w)^t,$$

where $\cdot^t$ denotes transpose, $C_{uv} \in L^\infty([0, l], \mathbb{K}^{(n_1+n_2)\times n})$, $C_w \in C([0, l], \mathbb{K}^{n_3 \times n})$ and $F \in \mathcal{L}(C([0, l], \mathbb{K}^{n_1+n_2}), \mathbb{K}^{n_2})$. Then the weak solutions exist for all $t \geq 0$. We denote the corresponding linear semigroup by $T_1(t)$. Further denote $C := \begin{pmatrix} C_{uv} \\ C_w \end{pmatrix}$.

Proposition 7.18. *$T_1(t) : Y \to Y$ is a C_0 semigroup in Y with infinitesimal generator $A_1 : \mathcal{D}(A_1) \to Y$,*

$$A_1 \begin{pmatrix} u \\ v \\ w \\ d \end{pmatrix} = \begin{pmatrix} (-K(x)\partial_x - C(x)) \begin{pmatrix} u(x) \\ v(x) \\ w(x) \end{pmatrix} \\ F(u, v) \end{pmatrix},$$

$$\begin{aligned} \mathcal{D}(A_1) = \Big\{(u, v, w, d) \in Y \;|(u, v, w, d) &\in W^{1,\infty}([0, l], \mathbb{K}^{n_1+n_2}) \\ &\times C([0, l], \mathbb{K}^{n_3}) \times \mathbb{K}^{n_2}, \\ &A_1(u, v, w, d) \in Y\Big\}. \end{aligned} \tag{7.18}$$

Proof. By our definition of weak solutions to (SH) in the space $\mathcal{X}_T$ we have that T_1 is a strongly continuous semigroup on Y. We verify that A_1 is its infinitesimal generator:

Let $(u(s), v(s), w(s), \Delta(u(s), v(s))) = T_1(s)(u_0, v_0, w_0, \Delta(u_0, v_0))$. Then we have

$$\frac{T_1(h)-I}{h}\begin{pmatrix}u_0\\ v_0\\ w_0\\ d_0\end{pmatrix} = \frac{T(h)-I}{h}\begin{pmatrix}u_0\\ v_0\\ w_0\\ d_0\end{pmatrix} + \frac{1}{h}\int_0^h T(h-s)\begin{pmatrix}-C(\cdot)\begin{pmatrix}u(s)\\ v(s)\\ w(s)\end{pmatrix}\\ F(u(s),v(s))\end{pmatrix} ds. \tag{7.19}$$

Since $T(t)$ is a C_0 semigroup on X_p for $1 \le p < \infty$ (recall the definition of T and A_0 in (7.1), (7.2), (7.3)) we have the following convergence in X_p

$$\frac{1}{h}\int_0^h T(h-s)\begin{pmatrix}-C(\cdot)\begin{pmatrix}u(s)\\ v(s)\\ w(s)\end{pmatrix}\\ F(u(s),v(s))\end{pmatrix} ds = \frac{1}{h}\int_0^h T(s)\begin{pmatrix}-C(\cdot)\begin{pmatrix}u(h-s)\\ v(h-s)\\ w(h-s)\end{pmatrix}\\ F(u(h-s),v(h-s))\end{pmatrix} ds \xrightarrow[h\downarrow 0]{X_p} \begin{pmatrix}-C(\cdot)\begin{pmatrix}u_0\\ v_0\\ w_0\end{pmatrix}\\ F(u_0,v_0)\end{pmatrix}. \tag{7.20}$$

Hence, if $y_0 = (u_0, v_0, w_0, d_0) \in Y$ and the limit

$$\lim_{h\downarrow 0}\frac{T_1(h)-I}{h}y_0 = A_1 y_0 \in Y$$

exists in $Y \subset X_p$, then it follows from (7.19) and (7.20) that the limit $\lim_{h\downarrow 0}\frac{T(h)-I}{h}y_0$ exists in X_p. Therefore, $(u_0, v_0) \in W^{1,p}([0,l], \mathbb{K}^{n_1+n_2})$ and

$$A_0 y_0 = A_1 y_0 - \begin{pmatrix}-C(\cdot)\begin{pmatrix}u_0\\ v_0\\ w_0\end{pmatrix}\\ F(u_0,v_0)\end{pmatrix} \in X_\infty.$$

This shows y_0 belongs to the right hand side of equation (7.18).

Conversely suppose y_0 belongs to the right hand side of (7.18). If we plug

$$(T(h) - I)(u_0, v_0, w_0, d_0)^t = \int_0^h T(s) A_0 (u_0, v_0, w_0, d_0)^t$$

into (7.19) we get

$$\frac{T_1(h) - I}{h} \begin{pmatrix} u_0 \\ v_0 \\ w_0 \\ d_0 \end{pmatrix} = h^{-1} \int_0^h T(s) \begin{pmatrix} -K(\cdot)\partial_x \begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix} - C(\cdot) \begin{pmatrix} u(h-s) \\ v(h-s) \\ w(h-s) \end{pmatrix} \\ F(u(h-s), v(h-s)) \end{pmatrix} ds. \tag{7.21}$$

We have to show that the right hand side of (7.21) converges in Y for $h \downarrow 0$ to

$$\tilde{A}_1 \begin{pmatrix} u_0 \\ v_0 \\ w_0 \\ d_0 \end{pmatrix} := \begin{pmatrix} -K(\cdot)\partial_x \begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix} - C(\cdot) \begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix} \\ F(u_0, v_0) \end{pmatrix}.$$

(Note that the integrand in (7.21) belongs to X_∞ and not to Y, this would be true even if C and C_w would have constant coefficients, in general. Also $T(t)$ is not C_0 on X_∞, but on X_p.)

$$\begin{aligned}
&\frac{T_1(h) - I}{h} \begin{pmatrix} u_0 \\ v_0 \\ w_0 \\ d_0 \end{pmatrix} - \tilde{A}_1 \begin{pmatrix} u_0 \\ v_0 \\ w_0 \\ d_0 \end{pmatrix} \\
=&h^{-1} \int_0^h T(s) \begin{pmatrix} -C(\cdot) \begin{pmatrix} u(h-s) - u_0 \\ v(h-s) - v_0 \\ w(h-s) - w_0 \end{pmatrix} \\ F(u(h-s), v(h-s)) - F(u_0, v_0) \end{pmatrix} ds \\
&+ h^{-1} \int_0^h \left\{ T(s)\tilde{A}_1 \begin{pmatrix} u_0 \\ v_0 \\ w_0 \\ d_0 \end{pmatrix} - \tilde{A}_1 \begin{pmatrix} u_0 \\ v_0 \\ w_0 \\ d_0 \end{pmatrix} \right\} ds \\
=:&I + II.
\end{aligned}$$

The first term I tends to zero in Y for $h \downarrow 0$ because $(u, v, w, \Delta(u, v)) \in \mathcal{X}_T$. And II goes to zero because $\tilde{A}_1(u_0, v_0, w_0, d_0) \in Y$ and T is a C_0 semigroup on Y. $\square$

Remark 7.19. *The question arises if the semigroup $T_1(t)$ on Y can be extended to a C_0 semigroup on the larger space X_p for $p \in [1, \infty[$ or a semigroup on X_∞ (which will not be C_0). For many cases, see Proposition 7.20 and Remark 7.21 this is not difficult to verify. The space Y then can be considered as an admissible subspace in the sense of [49, p.122, Definition. 5.3]. On these larger spaces the functions do not have to satisfy the boundary conditions pointwise and hence it makes sense to write the variation of constants formula $z(t) = T_1(t)z(0) + \int_0^t T_1(t-s)r(z(s))\,ds$ where r is a Nemytskij operator which is not compatible with boundary conditions and hence maps out of the space Y. This will face us when we will discuss the existence of center manifolds for equilibria, where we expand the Nemytskij operator at the equilibrium state (T_1 will correspond to the linear part and r the remainder containing terms of order two).*

A detailed proof of the next Proposition by using the Lumer-Phillips theorem (and an equivalent weighted norm on X_p) can be found in [45, Theorem 6.2, p.312]

Proposition 7.20. *Suppose*

$$F(u,v) = F_0 u(l) + G_0 v(l) \quad \text{with } F_0 \in \mathbb{K}^{n_2 \times n_1} \text{ and } G_0 \in \mathbb{K}^{n_2 \times n_2}.$$

Then T_1 can be extended to a C_0 semigroup on X_p, $p \in [1, \infty[$, with infinitesimal generator

$$\tilde{A}_1(u,v,w,d) = \left((-K(x)\partial_x - C(x))\,(u(x), v(x), w(x))^t ; F(u,v) \right),$$

$$\mathcal{D}(\tilde{A}_1) = \{(u,v,w,d) \in W^{1,p}([0,l], \mathbb{K}^{n_1+n_2}) \times L^p([0,l], \mathbb{K}^{n_3}) \times \mathbb{K}^{n_2} \mid \\ u(0) = Ev(0),\ d = \Delta(u,v)\}.$$

Remark 7.21. *Proposition 7.20 can be seen directly by solving the equation for $C = 0$. One can verify Proposition 7.20 for more general choices of F including "delays".*

Definition 7.22. We call $a \in Y$ a stationary or equilibrium solution of (SH) if the constant function $z(t) := a$ is a weak solution of (SH) in the sense of Definition 7.3.

Proposition 7.23. *A state $a = (a_u, a_v, a_w, \Delta(a_u, a_v)) \in Y$ is an equilibrium solution if and only if there exists $p \in [1, \infty[$ so that $(a_u, a_v) \in W^{1,p}([0,l], \mathbb{R}^{n_1+n_2})$ and both $K\partial_x(a_u, a_v, a_w)^t + \mathfrak{H}(a_u, a_v, a_w) = 0$ and $F(a_u, a_v) = 0$ vanish. In this case $(a_u, a_v) \in \bigcap_{1 \le p < \infty} W^{1,p}([0,l], \mathbb{R}^{n_1+n_2})$.*

Proof. If a is an equilibrium, then the constant solution $z(t) := a$ is differentiable with values in X_p for $1 \leq p \leq \infty$. Because z satisfies the variation of constants formula (7.5), and T is a C_0 semigroup on X_p for $1 \leq p < \infty$, and the constant map $s \mapsto \begin{pmatrix} -\mathfrak{H}(a_u, a_v, a_w) \\ F(a_u, a_v) \end{pmatrix}$ is differentiable into X_p, it follows from Proposition 13.6 that $t \mapsto T(t)a$ is differentiable into X_p for $1 \leq p < \infty$. Hence a is in the domain of the generator A_0 of the semigroup $T(t) : X_p \to X_p$ which means $(a_u, a_v) \in W^{1,p}([0, l], \mathbb{R}^{n_1+n_2})$. □

Definition 7.24. Let $a \in Y$ be an equilibrium of (SH). Then a is called (exponentially) stable if there exists a neighborhood U of a in Y and constants $c > 0$, $\beta > 0$, such that if z is a mild solution of (SH) with $z(0) \in U$ then z exists for all $t \geq 0$ and

$$\|z(t) - a\|_Y \leq ce^{-\beta t} \|z(0) - a\|_Y \quad \text{for } t \geq 0.$$

Definition 7.25. Let $a = (a_u, a_v, a_w, \Delta(a_u, a_v))$ be an equilibrium of (SH). Define the linearized operator $\mathfrak{A}_\mathfrak{a}$ in Y:

$$\mathfrak{A}_a \begin{pmatrix} u \\ v \\ w \\ \Delta(u, v) \end{pmatrix} := \begin{pmatrix} A_a \begin{pmatrix} u \\ v \\ w \end{pmatrix} \\ \partial F(a_u, a_v) \begin{pmatrix} u \\ v \end{pmatrix} \end{pmatrix}, \tag{7.22}$$

$$A_a := -K(\cdot)\partial_x - \partial\mathfrak{H}(a_u, a_v, a_w).$$

Theorem 7.26. *Let $a = (a_u, a_v, a_w, \Delta(a_u, a_v))$ be an equilibrium of* (SH) *such that there exists $\alpha > 0$ with the property that the spectrum of $\mathfrak{A}_a$ lies in the left half-space $\mathfrak{Re}\lambda \leq -\alpha$. Suppose that (the complexification of) $\mathfrak{A}_a$ belongs to the class* (DH)*, satisfying the conditions* (DHI) – (DHIII)*. Then a is a stable equilibrium of* (SH) *in the sense of Def. 7.24.*

Proof. Theorem 6.16 and Proposition 5.11 imply that the solution operator $S_\mathcal{L}$ for the linearization (LH) in a satisfies $\|S_\mathcal{L}(t)\|_{\mathcal{L}(Y)} \leq ce^{-\gamma t}$ for some $c > 0$ and $\gamma \in]0, \alpha[$. With this and Theorem 7.16 the proof is in the line with the proof of [71, Theorem 11.22, p.121-122]. □

Remark 7.27. *In all applications we have encountered linearization $\mathfrak{A}_a$ belongs to the class* (DH) *and satisfies conditions* (DHI) – (DHIII).

Chapter 8

Smooth center manifolds for semilinear hyperbolic systems

In this section we will show that near an nonhyperbolic equilibrium of (SH) there exists a smooth C^k center manifold which is exponentially attracting with respect to the C norm in the phase space Y. For this and later applications to model reduction of the traveling wave model and Hopf bifurcation of rotating waves into modulated waves (selfpulsations of the laser) we will need the persistence results of P. Bates, K. Lu and C. Zeng [8] for overflowing manifolds in the context of a smooth semiflow on a Banach space. First we summarize the required main results of [8] in section 8.1 without proofs and then we show the existence of center manifolds for semilinear hyperbolic systems in 8.2.

8.1 Persistence of overflowing manifolds for semiflows in Banach spaces (theory of P. W. Bates, K. Lu, C. Zeng)

Let X be a Banach space and $T \in C^1(X, X)$ a map. Suppose M is a C^1 Banach manifold without boundary and $\psi : M \to X$ is an C^1 immersion from M into X, i.e. ψ is C^1 and locally injective (so M is allowed to penetrate itself).

For a subset $S \subset X$ and $a > 0$ put $B(S, a) := \{x \in X \mid d(x, S) < a\}$. For each $m_0 \in M$ let $B_c(m_0, a)$ denote the connected component of $\psi^{-1}(B(\psi(m_0), a))$ which contains m_0.

Definition 8.1. The manifold M is said to be overflowing with respect to the map T if the following conditions hold:

$\imath$) There exist an open subset $M_1 \subset M$ and a homeomorphism $u : M \to M_1$ such that

$$\psi(m) = T(\psi(u(m))) \quad \text{for all } m \in M.$$

$\imath\imath$) There exists an $r > 0$ such that for any $m_0 \in M_1$ the set $\psi(\overline{B_c(m_0, r)})$ is closed in X.

Condition $\imath$) means that the image of $\psi(M_1)$ under T covers $\psi(M)$, condition $\imath\imath$) roughly says that the distance from $\psi(M_1)$ to the boundary of $\psi(M)$ is at least r.

The overflowing manifold is required to be normally hyperbolic. More precisely assume the following:

(H1) For each $m \in M$ there is a decomposition

$$X = X^c_m \oplus X^s_m$$

of closed subspaces with X^s_m being transversal to $\partial\psi(m)(T_m M)$, i.e.

$$X = \partial\psi(m)(T_m M) \oplus X^s_m$$

(here $T_m M$ denotes the tangent space of M at m). Furthermore, for any $m_1 \in M$

$$\Pi^c_{m_1} \partial T(\psi(m_0)) : X^c_{m_0} \to X^c_{m_1}$$

is an isomorphism, where $m_0 = u(m_1) \in M_1$ and Π^c_m is the projection onto X^c_m with kernel X^s_m. There exists $\lambda \in]0, 1[$ such that

$$\left\| \Pi^s_{m_1} \partial T(\psi(m_0))_{|X^s_{m_0}} \right\| < \lambda \min \left\{ 1, m\left(\Pi^c_{m_1} \partial T(\psi(m_0))_{|X^c_{m_0}} \right) \right\}. \tag{8.1}$$

Here $\Pi^s_m := I - \Pi^c_m$ and

$$m\left(\Pi^c_{m_1} \partial T(\psi(m_0))_{|X^c_{m_0}} \right) := \inf \left\{ \left\| \Pi^c_{m_1} \partial T(\psi(m_0)) x^c \right\| \mid x^c \in X^c_{m_0}, \|x^c\| = 1 \right\} \tag{8.2}$$

denotes the minimum norm of $\Pi^c_{m_1} \partial T(\psi(m_0))_{|X^c_{m_0}}$.

Condition (H1) means that $\psi(M)$ is exponentially stable and that ∂T contracts along the normal direction and does so more strongly than it does along the tangential direction. Hypothesis (H1) differs slightly from the standard definition of normal hyperbolicity, see for example [32], where X^c_m is usually required to be equal to the tangent space of $\psi(M)$ which is invariant under ∂T. Here X^c_m is only required to be an approximation of the tangent space of $\psi(M)$, see the forthcoming condition (H3).

In order to establish tubular neighbourhoods with a uniform size the following assumption is needed:

(H2) For any $m_0 \in M$, $m_1, m_2 \in B_c(m_0, r)$, $m_1 \neq m_2$,

$$\left\|\Pi^c_{m_1} - \Pi^c_{m_2}\right\| \leq L \left\|\psi(m_1) - \psi(m_2)\right\|$$

and

$$\frac{\left\|\psi(m_1) - \psi(m_2) - \Pi^c_{m_0}(\psi(m_1) - \psi(m_2))\right\|}{\left\|\psi(m_1) - \psi(m_2)\right\|} \leq \epsilon_1, \tag{8.3}$$

where L, ϵ_1 are constants that satisfy $1 \leq L < \frac{\sqrt{2}-1}{r}$ and $\epsilon_1 < 1$.

This means that the projection Π^c_m is Lipschitz in m and that M does not "twist" too much.
Moreover, the following uniformity assumptions on T, Π^c_m and Π^s_m are needed (note that M is not required to be locally compact)

(H3) $\imath$) There exists a constant $B > 0$ such that $\|\Pi^c_m\| \leq B$ and $\|\Pi^s_m\| \leq B$ for all $m \in M$.
$\imath\imath$) There exist a constant $\mu_0 > 0$ and for any $m \in M$ a $\Lambda_m \in \mathcal{L}(X^c_m, X^s_m)$ with

$$\|\Lambda_m\| \leq \mu_0, \quad \partial\psi(m)\,(T_m M) = (I + \Lambda_m)X^c_m.$$

$\imath\imath\imath$) For any $\eta > 0$ there exists $\epsilon > 0$ such that for any $x_1, x_2 \in B(\psi(M_1), \epsilon)$, $\|x_1 - x_2\| < \epsilon$,

$$\|\partial T(x_1) - \partial T(x_2)\| \leq \eta.$$

$\imath v$) There exist constants $a > 0$ and $B_1 > 0$ such that ($m(\cdot)$ denotes the minimum norm (8.2))

$$m\left(\Pi^c_{m_1}\partial T(\psi(m_0))_{|X^c_{m_0}}\right) \geq a \quad \text{and} \quad \left\|\partial T_{|B(\psi(M_1,r))}\right\| \leq B_1$$

for $m_1 \in M$, $m_0 = u(m_1) \in M_1$.

Condition $\imath\imath$) implies that X^c_m is an approximation of the tangent space $(I + \Lambda_m)X^c_m$ of $\psi(M)$ at $\psi(m)$ with an error bounded by μ_0. The following main Theorem 8.6 states the persistence of the overflowing manifold if X^c_m is a good approximation of the tangent bundle of $\psi(M)$. The following conditions for μ_0 have been shown to be sufficient: Denote

$$\mu_1 := \frac{(1 + \epsilon_1)\mu_0}{(1 - \epsilon_1)(1 - \mu_0)}.$$

Then μ_0 satisfies

$$\mu_0 < \frac{(1-\lambda)^2}{16B^2B_1^2}\min\{1,a^2\} \quad \text{and} \quad \mu_1 < \frac{(1-\lambda)^2 a}{4BB_1}\min\{1,\frac{a}{6}\}. \tag{8.4}$$

If X_m^c is equal to the tangent bundle of $\psi(M)$ then $\mu_0 = 0$ and (8.4) is of course satisfied automatically.

The following fourth hypothesis is needed for proving the C^1-closeness of the perturbed manifold to the original manifold from which it follows that condition $\imath\imath$) in (H3) also holds for the perturbed manifold. This is needed for obtaining higher order smoothness of the perturbed manifold stated in Corollary 8.9. For this we need the following straightforward Lemma which is contained in [7, Lemma 4.1, p.20]:

Lemma 8.2. *If $\pi_i : X \to X$, $i = 1,2$, are two continuous linear projections from X into itself that satisfy $\|\pi_1 - \pi_2\| \le \eta < \sqrt{2}-1$, then $\pi_{1|\pi_2 X}$ is an isomorphism from $\pi_2 X$ onto $\pi_1 X$ and $\pi_{2|\pi_1 X}$ is an isomorphism from $\pi_1 X$ onto $\pi_2 X$. Moreover we have*

$$\left\|\pi_{1|\pi_2 X}\right\|, \left\|\pi_{2|\pi_1 X}\right\| \le 1+\eta \quad \textit{and} \quad \left\|\left(\pi_{1|\pi_2 X}\right)^{-1}\right\|, \left\|\left(\pi_{2|\pi_1 X}\right)^{-1}\right\| \le \frac{1}{1-\eta}.$$

We recall the following standard estimate which is easy to verify

Lemma 8.3. *If $A : V \to W$, $B : V \to W$ are bounded linear maps from the Banach space V into the Banach space W, A is invertible and $\|A - B\| < \|A^{-1}\|^{-1}$, then B is invertible and*

$$\left\|B^{-1}\right\| \le \frac{\|A^{-1}\|}{1 - \|A^{-1}\|\,\|A-B\|}.$$

Proof. Let $I : V \to V$ be the identity operator. Then we have

$$\left\|I - A^{-1}B\right\| = \left\|A^{-1}(A-B)\right\| \le \left\|A^{-1}\right\|\|A-B\| < 1.$$

Hence a standard argument using Neumann series implies that $A^{-1}B = I - (I - A^{-1}B)$ is invertible with

$$\left\|\left(A^{-1}B\right)^{-1}\right\| \le \frac{1}{1-\|I - A^{-1}B\|} \le \frac{1}{1-\|A^{-1}\|\,\|A-B\|}.$$

Because $A^{-1}B$ and A are invertible it follows that B is invertible and we have

$$\left\|B^{-1}\right\| = \left\|\left(A^{-1}B\right)^{-1}A^{-1}\right\| \le \left\|\left(A^{-1}B\right)^{-1}\right\|\left\|A^{-1}\right\| \le \frac{\|A^{-1}\|}{1-\|A^{-1}\|\,\|A-B\|}.$$

$\square$

From (H2), since $Lr < \sqrt{2}-1$, and Lemma 8.2 we have that for $m_0 \in M$ and $m \in B_c(m_0, r)$ the projection $\Pi^c_{m_0}$ is an isomorphism from X^c_m to $X^c_{m_0}$ with

$$\left\| \left(\Pi^c_{m_0 | X^c_m} \right)^{-1} \right\|^{-1} \geq 1 - \left\| \Pi^c_m - \Pi^c_{m_0} \right\|. \tag{8.5}$$

Let $m_0 \in M$, $m \in B_c(m_0, r)$ and $K \in \mathcal{L}(X^c_m, X^s_m)$. Then define for sufficiently small $\|K\|$

$$W_c(K, m_0) := \left(\Pi^c_{m_0} (I + K)_{|X^c_m} \right)^{-1}$$

and

$$W(K, m_0) := \Pi^s_{m_0} (I + K) W_c(K, m_0) \in \mathcal{L}(X^c_{m_0}, X^s_{m_0}).$$

Since

$$\left\| \Pi^c_{m_0} K \right\| \leq \left\| \Pi^c_m - \Pi^c_{m_0} \right\| \|K\|$$

it follows from Lemma 8.3 and (8.5) that $W_c(K, m_0)$ and $W(K, m_0)$ are well defined, when

$$\|K\| < \left\| \Pi^c_m - \Pi^c_{m_0} \right\|^{-1} - 1. \tag{8.6}$$

From (H2) a sufficient condition for (8.6) is $\|K\| \leq \sqrt{2}$.
By definition of $W(K, m_0)$ one sees that $W(K, m_0)$ is just the representation of the subspace $(I + K)(X^c_m)$ in the coordinate system $X^c_{m_0} \oplus X^s_{m_0}$, i.e.

$$(I + K) X^c_m = (I + W(K, m_0)) X^c_{m_0}.$$

We are now able to define condition

(H4) For any $\eta > 0$ there exists $\delta > 0$ such that for any $m_0 \in M_1$ and $m \in B_c(m_0, r) \cap \psi^{-1}(B(\psi(m_0), \delta))$

$$\|W(\Lambda_m, m_0) - \Lambda_{m_0}\| \leq \eta.$$

Remark 8.4. *By definition $(I + \Lambda_M)(X^c_m)$ is the tangent space of $\psi(M)$ at m so that $W(\Lambda_m, m_0)$ is the representation of the tangent space at m with respect to the coordinate system $X^c_{m_0} \oplus X^s_{m_0}$.*
It can be shown (see [8, Lemma 3.6, Lemma 3.7, Lemma 4.1]) that for any $m_0 \in M$ there is a local coordinate representation $f : \{x^c \in X^c_{m_0} \mid \|x_c\| \leq \delta\} \to X^s_{m_0}$ ($\delta > 0$ sufficiently small), i.e. for m sufficiently close to m_0 and x_c sufficiently small

$$\psi(m) = \psi(m_0) + x^c + f(x^c).$$

Since $\partial f(x^c) = W(\Lambda_m, m_0)$ and $W(\Lambda_{m_0}, m_0) = \Lambda_{m_0}$ hypothesis (H4) *is equivalent to assuming that ∂f is uniformly continuous in some sense.*

If ψ can be extended to an immersion on a compact manifold, i.e. there exists a compact manifold M_0 with $M \subset M_0$ and an immersion $\psi_0 : M_0 \to X$ so that $\psi_{0|M} = \psi$, then (H4) *holds.*

In order to state the main result of [8] precisely we briefly explain the "tubular neighbourhoods" of $\psi(M)$ used in the construction of the perturbed manifold.

Definition 8.5. For each $m_0 \in M$ and $\epsilon > 0$ define

$$N(m_0, \epsilon) := \{\psi(m) + x^s \mid m \in B_c(m_0, r),\ x^s \in X^s_m,\ \|x^s\| < \epsilon\},$$

$$N(M, \epsilon) := \cup_{m_0 \in M} N(m_0, \epsilon).$$

It can be shown (see Lemma 3.4 and Lemma 3.5 in [8]) that there exists $\epsilon_0 > 0$ (depending only on r, ϵ_1, B, L) such that for $0 < \epsilon < \epsilon_0$ the set $N(m_0, \epsilon)$ is open in X. Thus $N(M, \epsilon)$ is an open tubular neighbourhood containing $\psi(M)$.

Now we can state the main theorems on persistence and smoothness of overflowing manifolds (which is a slightly more detailed summary of Theorem A, Proposition 4.8 and Theorem B in [8]):

Theorem 8.6. *Suppose M is an overflowing invariant manifold for the map T and* (H1)$-$(H3) *together with* (8.4) *are satisfied. Then there exist constants ϵ^* and $\sigma(\epsilon)$, which depend only on the constants given in* (H1) $-$ (H3)*, such that if $\epsilon < \epsilon^*$ and $\tilde{T} \in C^1(X, X)$ satisfies $\left\|\tilde{T} - T\right\|_{C^1(B(\psi(M_1), r))} < \sigma$, then there exists a C^1 immersion $h_0 : M \to X$ such that $h_0(M)$ is an overflowing manifold within $\overline{N(M, \epsilon)}$ and* (H1), (H2) *and conditions $\imath), \imath\imath\imath), \imath v)$ in* (H3) *hold for $\tilde{T}$ and $h_0(M)$ with possibly larger ϵ_1, B, B_1, L, λ and smaller a and r. Under the C^0 norms for h_0 and $\tilde{T}$, h_0 is Lipschitz with respect to $\tilde{T}$. If the spectral gap condition* (8.7) *in Theorem 8.8 holds for $\psi(M)$ then it also holds for $h_0(M)$ with possibly larger $\lambda < 1$.*
If (H4) *holds for $\psi(M)$ then h_0 is close to ψ in the C^1 topology, i.e. for $\sigma \to 0$ we have $\|\psi - h_0\|_{C^1(M)} \to 0$. In particular* (H3) *is satisfied for $h_0(M)$ when* (H4) *is true for $\psi(M)$.*
Moreover, there exists a constant $\delta > 0$, depending only on ϵ and the constants of (H1) $-$ (H3) *such that for any $m_0 \in M$ and $x^s_0 \in \overline{X^s_{m_0}(\epsilon)}$ one has the following characterization of h_0: The relation $\psi(m_0) + x^s_0 = h_0(m_0)$ holds if and only if there exists a sequence*

$$(m_i, x^s_i)_{i \in \mathbb{N}} \quad \text{with } m_i \in M,\ x^s_i \in X^s_{m_i},\ \|x^s_i\| \le \epsilon \quad \text{for } i \in \mathbb{N},$$

such that

$$m_i \in B_c\left(u(m_{i-1}, r) \cap \psi^{-1}\left(B(\psi(u(m_{i-1})), \delta\right)\right)$$

and

$$\tilde{T}(\psi(m_i) + x_i^s) = \psi(m_{i-1}) + x_{i-1}^s \quad \text{for } i = 1, 2, \dots.$$

Remark 8.7. *If ψ is an embedding (i.e. injective) and* (8.3) *holds for $m_1, m_2 \in \psi^{-1}(B(\psi(m_0), r))$ (m_1 and m_2 are not chosen only from the connected component in contrast to condition* (H2)*) then h_0 is an embedding.*

If T is C^k with uniformly bounded i-th order derivatives, μ_0 is chosen smaller than in (8.4) and a higher order spectral gap condition holds, then the following theorem states that a C^1 overflowing manifold is automatically C^k.

Theorem 8.8. *Suppose $T \in C^k(X, X)$ and M is an overflowing invariant manifold for T and* (H1) – (H3) *hold. Assume there exist constants $B_1 \geq 1$ and $d > 0$ such that $\|\partial^i T(x)\| \leq B_1$ for $x \in B(\psi(M_1), d)$ and $1 \leq i \leq k$. Furthermore let*

$$\left\| \Pi_{m_1}^s \partial T(\psi(m_0))_{|X_{m_0}^s} \right\| < \lambda \left(m \left(\Pi_{m_1}^c \partial T(\psi(m_0))_{|X_{m_0}^c} \right) \right)^i \tag{8.7}$$

for $m_1 \in M$, $m_0 = u(m_1) \in M_1$, $i = 1, 2, \dots, k$ and some $\lambda \in]0, 1[$. If

$$\mu_0 < \min \left\{ \frac{(1-\lambda)a}{2BB_1(k + a^{1-k})}, \frac{(1-\lambda)^2 a^2}{16B^2 B_1^2}, \frac{(1-\lambda)^2 a^3}{16B^2 B_1^2} \right\} \tag{8.8}$$

then $\psi(M)$ is C^k. When $T \in C^{k,1}(X, X)$, then $\psi(M)$ is also $C^{k,1}$.

Corollary 8.9. *Under the assumptions of Theorem 8.8 it follows for h_0 in Theorem 8.6 that $h_0 \in C^k$ ($h_0 \in C^{k,1}$) if condition* (H4) *is satisfied.*

Finally, we state the persistence for semiflows. Let $T^t, \tilde{T}^t \in C([0, \infty] \times X, X)$ be a semiflow, i.e.

$$T^0 = I, \ T^{t+s} = T^t \circ T^s, \quad \text{for } t, s \geq 0.$$

A semiflow T is called C^k smooth if $T^t \in C^k(X, X)$ for all $t \geq 0$. The following hypothesis is required

(H5) For all $\eta > 0$ there exists $\zeta > 0$ such that for all $x \in B(\psi(M), r)$ and $t \in [0, \zeta]$ we have

$$\left\| \tilde{T}^t(x) - x \right\| < \eta.$$

Then we have the following

Theorem 8.10. *Let $\tilde{T}, T$ be C^1 semiflows so that condition* (H5) *is satisfied for $\tilde{T}$. Suppose $t_0 > 0$ is such that $\psi(M)$ is overflowing invariant with respect to the map T^{t_0} and* (H1) − (H3) *together with* (8.4) *hold for T^{t_0}. Let ϵ^* and $\sigma(\epsilon)$, $\epsilon < \epsilon^*$ be the constants from Theorem 8.6. Further, assume $\left\|\tilde{T}^{t_0} - T^{t_0}\right\|_{C^1(B(\psi(M_1),r))} < \sigma$. Let h_0 be the immersion for $\tilde{T}^{t_0}$. Then, for any $m \in M$ there exists $t_1 > 0$ such that $\tilde{T}^t(h_0(m)) \in h_0(M)$ for all $t \in [0, t_1]$. If $\gamma :]-\infty, 0] \to N(M, \epsilon)$ is a trajectory that satisfies $T^{\Delta}\gamma(-t) = \gamma(-t + \Delta)$ for $t, \Delta \geq 0$, $\Delta \leq t$, then $\gamma(-t) \in h_0(M)$ for all $t \geq 0$.*

Remark 8.11. *If $\tilde{T}, T$ are C^k semiflows and the conditions of Theorem 8.8 together with* (H4) *are satisfied for T^{t_0}, then $h_0(M)$ is a C^k smooth invariant manifold for $\tilde{T}$ when σ is sufficiently small.*

Remark 8.12. *It can be shown that t_1 only depends on the size of r_2 which satisfies*

$$\{x^c \in X_m^c \mid \|x^c\| \leq r_2\} \subset P_m(\psi(B_c(m, r))).$$

Here P_m denotes the map from X to X_m^c that is given by $P_m(x) = \Pi_m^c(x - \psi(m))$. Note that $P_m \circ \psi$ is a diffeomorphism from $B_c(m, r)$ to its image, which is an open subset of X_m^c [8, Lemma 3.4].

Remark 8.13 (Uniqueness)**.** *Suppose the conditions of Theorem 8.6 are satisfied. It should be stated that the perturbed manifold $h_0(M)$ is unique. From the proofs in [8] this uniqueness is obvious within the category of certain Lipschitz graphs (Γ) contained in the tubular neighbourhood $N(M, \epsilon)$, where $\epsilon < \epsilon^*$. However, it is desirable to state the uniqueness within the larger class of continuous manifolds in $N(M, \epsilon)$, i.e. for $C^0(M, X, \epsilon) := \{h \in C^0(M, X) \mid h(m) - \psi(m) \in \overline{X_m^s(\epsilon)}$ for $m \in M\}$. From a correspondence with Chongchun Zeng the following statement follows from [8, Lemma 4.5]:*
Suppose
ı) $h_1 \in C^0(M, X, \epsilon)$
ıı) $h_1(M)$ is overflowing invariant with respect to $\tilde{T}$, i.e. there exists a homeomorphism $u_1 : M \to u_1(M)$ such that $\tilde{T}(h_1(u_1(m))) = h_1(m)$ for $m \in M$ and there exists $r_1 > 0$ such that for any $m_0 \in u_1(M)$, $\psi(\overline{B_c(m_0, r)})$ is closed in X.
ııı) For any $m_0 \in M$, $m_1 = u_1(m_0) \in B_c(u(m_0), r) \cap \psi^{-1}B(\psi(u(m_0)), \delta)$ (here $\delta = \epsilon/(2\mu)$ and μ satisfies (3.13), (4.1)-(4.3) in [8]).
Then $h_1 = h_0$.
To see this take h_0 as the h in [8, Lemma 4.5] and $x_1^s = h_1(m_0) - \psi(m_0)$, $m_1 = u_1(m_0)$, $\bar{x}_1^s = h_1(m_1) - \psi(m_1)$. Then it follows from [8, Lemma 4.5] that $\|h_1 - h_0\| \leq \lambda_1 \|h_1 - h_0\|$ where $\lambda_1 < 1$, i.e. $h_1 = h_0$.

8.2 Center manifolds for semilinear hyperbolic systems

Let $a \in Y$ be an equilibrium of (SH). Suppose that the (complexification of the) linearization $\mathfrak{A}_a$, defined in (7.22), belongs to (DH) and satisfies (DHI)−(DHIII). Moreover suppose for the spectral set Σ, defined in (6.5),

$$\sup \{\mathfrak{Re}\, \lambda \mid \lambda \in \Sigma\} < 0,$$

and $\gamma_+ < 0$ for the reduced system (H_0). Assume that the spectrum σ of $\mathfrak{A}_a$ is on the left side of the imaginary axis,

$$\sigma \subset \{\lambda \in \mathbb{C} \mid \mathfrak{Re}\lambda \leq 0\},$$

and that a is nonhyperbolic, i.e.

$$E_c := \sigma \cap i\mathbb{R} \neq \emptyset.$$

Then E_c is finite and only contains eigenvalues of finite algebraic multiplicities and we have a spectral gap: There exists a $\delta > 0$ so that

$$\mathbb{C}_{-\delta,\delta} \cap \sigma = E_c.$$

The critical set of eigenvalues E_c is of the form

$$E_c = \{\lambda_1, \overline{\lambda_1}, \ldots, \lambda_p, \overline{\lambda}_p\}.$$

Let

$$T_a(t) : Y \to Y$$

denote the C_0 semigroup generated by $\mathfrak{A}_a$, see Proposition 7.18. As in (7.15) and (7.16) let S^t denote the solution map and $S^t_{\mathcal{L}}(a) = \partial S^t(a)$ its linearization in the equilibrium a. We have

$$T_a(t) = S^t_{\mathcal{L}}(a).$$

Define the spectral projection

$$\pi_c := \int_\gamma (\lambda I - \mathfrak{A}_a)^{-1}\, d\lambda, \quad \pi_s := \mathrm{Id} - \pi_c. \tag{8.9}$$

where γ is a simple positive oriented loop in $\mathbb{C}_{-\delta,\delta}$ around E_c. Let $X_p^{\mathbb{C}}$, $X_\infty^{\mathbb{C}}$ and $Y^{\mathbb{C}}$ denote the complexifications of the (real) spaces X_p, X_∞ and Y. Since the resolvent in (8.9) is analytic in λ with values in the space of bounded linear maps on $X_p^{\mathbb{C}}$, $X_\infty^{\mathbb{C}}$ and $Y^{\mathbb{C}}$, the projections π_c and π_s are well defined on $X_p^{\mathbb{C}}$, $X_\infty^{\mathbb{C}}$ and $Y^{\mathbb{C}}$. We have that π_c maps $X_p^{\mathbb{C}}$ continuously into the n_c dimensional subspace

$$X_c^{\mathbb{C}} := \operatorname{Im} \pi_c \subset \mathcal{D}(\mathfrak{A}_a),$$

where $\mathcal{D} = \mathcal{D}(\mathfrak{A}_a)$ is the domain of $\mathfrak{A}_a$. We have $I = \pi_c + \pi_s$, $\pi_c \pi_s = \pi_s \pi_c = 0$, $\pi_c \mathcal{D} \subset \mathcal{D}$, $\pi_s \mathcal{D} \subset \mathcal{D}$, $\mathfrak{A}_a(\mathcal{D} \cap X_c) \subset X_c$, $\mathfrak{A}_a(\mathcal{D} \cap X_s) \subset X_s$. Denote

$$X_s^{\mathbb{C}} := \operatorname{Im} \pi_s.$$

The linear spaces $X_c^{\mathbb{C}}$ and $X_s^{\mathbb{C}}$ decompose the space

$$Y^{\mathbb{C}} = X_c^{\mathbb{C}} \oplus X_s^{\mathbb{C}}$$

($Y^{\mathbb{C}}$ denotes the complexification of Y) into the direct sum of two closed linear subspaces which are invariant with respect to the complexification of the linearized semigroup $S_{\mathcal{L}}^t(a)$. Both $X_c^{\mathbb{C}}$ and $X_s^{\mathbb{C}}$ are invariant under complex conjugation. Hence it follows that

$$X_c := X_c^{\mathbb{C}} \cap Y \quad \text{and} \quad X_s := X_s^{\mathbb{C}} \cap Y$$

decompose the real space Y

$$Y = X_s \oplus X_c$$

in, with respect to the a-linearized flow $S_{\mathcal{L}}^t(a)$, invariant closed subspaces.

The spectral gap mapping Theorem 6.15 implies that there exist constants $\alpha > 0$ and $c > 0$ such that

$$\left\| S_{\mathcal{L}}^t(a)_{|X_s} \right\|_{\mathcal{L}(X_s)} \leq c e^{-\alpha t} \quad (t \geq 0).$$

Hence there exists $0 < \lambda < 1$ so that for sufficiently large $t > 0$

$$\left\| S_{\mathcal{L}|X_s}^t \right\|_{\mathcal{L}(X_s)} < \lambda \min \left\{ 1, \inf\{ \left\| S_{\mathcal{L}}^t x_c \right\| \mid x_c \in X_c, \|x_c\| = 1 \} \right\}. \tag{8.10}$$

Because the operator $(\mathfrak{H}, F)$ will map out of the space Y (the Nemytskij operator $\mathfrak{H}$ is not restricted to be compatible with the boundary conditions), we will need that the semigroup T_a can be extended to a larger space in order to be able to write the variation of constants formula. Hence we assume that T_a can be extended to the space X_p, see Proposition 7.20 and Remark 7.21.

Moreover we require that F can be truncated in the following sense: Let

$$F(a_u + u, a_v + v) = F(a_u, a_v) + \partial F(a_u, a_v)(u, v) + r_F(u, v) \tag{8.11}$$

with $r_F(u,v) = o(\|(u,v)\|_\infty)$. We suppose that for any truncation parameter $\delta > 0$ there exists a C^k smooth map $r_{F\delta} : C([0,l],\mathbb{R}^{n_1+n_2}) \to \mathbb{R}^{n_2}$ having the following properties:
ı) $r_{F\delta}(u,v) = r_F(u,v)$ for $(u,v) \in C([0,l],\mathbb{R}^{n_1+n_2})$ with $\|(u,v)\|_\infty \le \delta$
ıı) there exists a positive function $\tilde{\delta} = \tilde{\delta}(\delta)$ with $\lim_{\delta\downarrow 0} \tilde{\delta}(\delta) = 0$ so that

$$\|r_{F\delta}(u,v)\|_\infty \le \tilde{\delta}(\delta)\delta \quad \text{and} \quad \|\partial r_{F\delta}(u,v)\|_\infty \le \tilde{\delta}(\delta) \tag{8.12}$$

for all $(u,v) \in C([0,l],\mathbb{R}^{n_1+n_2})$.

Example 8.14. *Let $x_k \in [0,l]$, $1 \le k \le m$, and $F_k : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_2}$ be C^k. Suppose F is of the form $F(\tilde{u},\tilde{v}) = \sum_{k=1}^m F_k(\tilde{u}(x_k),\tilde{v}(x_k))$. Then F has the above truncation property.*
Indeed, we have $r_F(u,v) = \sum_{k=1}^m r_k(u(x_k),v(x_k))$, where

$$\begin{aligned} r_k(u(x_k),v(x_k)) &= F_k(a_u(x_k)+u(x_k),a_v(x_k)+v(x_k)) - F_k(a_u(x_k),a_v(x_k)) \\ &\quad -\partial F_k(a_u(x_k),a_v(x_k))(u(x_k),v(x_k)). \end{aligned}$$

Then $r_{F\delta}(u,v) = \sum_{k=1}^m r_k(u(x_k),v(x_k))\chi_\delta(u(x_k),v(x_k))$ does it, where $\chi_\delta(\cdot) := \chi(\delta^{-1}\cdot)$ and $\chi : \mathbb{R}^{n_1+n_2} \to \mathbb{R}$ is a C^∞ cut off function so that $\chi(x) = 1$ for $\|x\| \le 1$, $\chi(x) = 0$ for $\|x\| \ge 2$ and $0 \le \chi(x) \le 1$ for $x \in \mathbb{R}^{n_1+n_2}$.

Theorem 8.15 (Existence of C^k smooth center manifolds)**.** *Let $k \ge 1$. There exists an open neighbourhood Ω of zero in Y and a graph $\gamma \in C^k(\Omega \cap X_c, X_s)$ such that*
ı) $\gamma(0) = 0$, $\partial\gamma(0) = 0$;
ıı) *the manifold*

$$W := \{a + x_c + \gamma(x_c) \mid x_c \in \Omega \cap X_c\}$$

is locally invariant for (SH), *i.e. for $t \ge 0$ we have $S^t(W) \cap \Omega \subset W$;*
ııı) *if $z :]-\infty,0] \to a+\Omega$ is a solution of* (SH) *then $z(t) \in W$ for $t \in]-\infty,0]$.*
ıv) *For $p \in [1,\infty[$ we have*

$$\begin{aligned} \gamma(\Omega \cap X_c) &\subset X_s \cap \left(W^{1,p}(]0,l[\,,\mathbb{R}^{n_1+n_2}) \times C([0,l],\mathbb{R}^{n_3}) \times \mathbb{R}^{n_2}\right), \\ W &\subset Y \cap \left(W^{1,p}(]0,l[\,,\mathbb{R}^{n_1+n_2}) \times C([0,l],\mathbb{R}^{n_3}) \times \mathbb{R}^{n_2}\right). \end{aligned}$$

If $z : [0,\delta] \to W$ $(\delta > 0)$ is a solution of (SH) *then*

$$z \in C^k([0,\delta],Y).$$

The flow on W is given by the ordinary differential equation

$$\frac{d}{dt}x_c = \mathfrak{A}_a x_c + f(x_c),$$

where $f : X_c \to X_c$ is C^k smooth, $f(0) = 0$ and $\partial f(0) = 0$.

Proof. Let $z(t)$ be a weak solution with

$$z(0) \in W^{1,\infty}(]0,l[\,,\mathbb{R}^{n_1+n_2}) \times C([0,l],\mathbb{R}^{n_3}) \times \mathbb{R}^{n_2}$$

close (in the sense of the space Y) to $a = (a_u, a_v, a_w, \Delta(a_u, a_v))$.
We expand $\mathfrak{H}$ in (a_u, a_v, a_w) and F in (a_u, a_v) and denote the remainders r and r_F, respectively: We write (8.11) and for $\tilde{z} \in L^\infty([0,l],\mathbb{R}^n)$

$$\mathfrak{H}((a_u, a_v, a_w) + \tilde{z}) = \mathfrak{H}(a_u, a_v, a_w) + \partial\mathfrak{H}(a_u, a_v, a_w)\tilde{z} - r(\tilde{z}),$$

where $r(\tilde{z}) = o(\|\tilde{z}\|_{L^\infty})$ (here $\partial\mathfrak{H} \in \mathcal{L}(L^\infty([0,l],\mathbb{R}^n))$ denotes the Fréchet derivative of $\mathfrak{H}$). Put

$$x(t) := (x_u, x_v, x_w, \Delta(x_u, x_v))(t) := z(t) - a.$$

Then

$$\frac{d}{dt}x(t) = \mathfrak{A}_a x(t) + \begin{pmatrix} r(x_u(t), x_v(t), x_w(t)) \\ r_F(x_u(t), x_v(t)) \end{pmatrix}. \tag{8.13}$$

By assumption $T_a(t)$ can be extended to a C_0 semigroup on X_p with some $p \in [1,\infty[$. Hence we can write

$$x(t) = T_a(t)x(0) + \int_0^t T_a(t-s) \begin{pmatrix} r(x_u(s), x_v(s), x_w(s)) \\ r_F(x_u(s), x_v(s)) \end{pmatrix} ds. \tag{8.14}$$

Note that $(r(x_u(s), x_v(s), x_w(s)), r_F(x_u(s), x_v(s)))$ is in $X_\infty \setminus Y$, in general.
Define

$$x_c := \pi_c x \quad \text{and} \quad x_s := \pi_s x.$$

Then we have by projecting (8.14) or (8.13)

$$\begin{cases} x_c(t) = & T_a(t)\, x_c(0) + \int_0^t T_a(t-s) r_c(x_c(s), x_s(s))\, ds, \\ x_s(t) = & T_a(t)\, x_s(0) + \int_0^t T_a(t-s) r_s(x_c(s), x_s(s))\, ds, \end{cases} \tag{8.15}$$

$$r_c(x_c, x_s) := \pi_c \tilde{r}(x_c + x_s), \quad r_s(x_c, x_s) := \pi_s \tilde{r}(x_c + x_s),$$

$$\tilde{r}(x_c + x_s) := \begin{pmatrix} r(x_{cu} + x_{su}, x_{cv} + x_{sv}, x_{cw} + x_{sw}) \\ r_F(x_{cu} + x_{su}, x_{cv} + x_{sv}) \end{pmatrix}.$$

or, equivalently,

$$\begin{cases} \frac{d}{dt}x_c(t) = & \mathfrak{A}_a\, x_c(t) + r_c(x_c(t), x_s(t)), \\ \frac{d}{dt}x_s(t) = & \mathfrak{A}_a\, x_s(t) + r_s(x_c(t), x_s(t)). \end{cases} \tag{8.16}$$

Denote the restriction of $\mathfrak{A}_a$ to X_c as

$$A_c := \mathfrak{A}_{a|X_c}.$$

The projections x_c and x_s form a smooth semiflow in $X_c \times X_s$. For x_c and x_s close to zero (8.16) is a small C^1 perturbation of the linear flow $T_a(t)$ which has the invariant linear center manifold $X_c \times \{0\}$. Inequality (8.10) means that the center is normally hyperbolic.

In the following we will modify the linear flow $e^{A_c t}$ on $X_c \simeq X_c \times \{0\}$ and make it overflowing at the border of an ellipsoid. This will be our starting overflowing normally hyperbolic center manifold. Then, after modifying (8.15) on X_c outside a small neighbourhood of zero we will see that this modified system will be a small smooth C^1 perturbation of the modified linear flow. Thus we see that the assumptions of Theorem 8.10 and Remark 8.11 (see (H1) – (H5) and (8.7)) are satisfied, so that we obtain the existence of a C^k invariant smooth center manifold for the modified system. Since the modified system coincides locally near zero with the original one (8.15) this will prove existence of a local C^k invariant smooth center manifold for (SH) near the equilibrium a as stated in our theorem.

Let $\sigma > 0$ be a small parameter. Then $A_c + \sigma I$ has only eigenvalues with real part equal to $\sigma > 0$ and it follows from Jordans normal form theorem that one can find an overflowing invariant ellipsoid $\mathcal{E}$ for the flow $e^{(A_c+\sigma I)t}$. This ellipsoid is independent on σ because the Jordan basis does not depend on σ. According to $\mathcal{E}$ let $e : X_c \to \mathbb{R}$ be a C^∞ bump function with the property that $e(x) = 1$ for each $x \in \partial\mathcal{E}$, $0 \le e(x) \le 1$ for $x \in X_c$ and $e(x) = 0$ for all $x \in X_c$ with $\mathrm{dist}(x, \partial\mathcal{E}) > \frac{r}{2}$, where

$$r := \mathrm{dist}(\partial\mathcal{E}, \{0\}).$$

Consider the small perturbation of the flow $e^{A_c t}$ given by

$$\frac{d}{dt} x_c(t) = (A_c + \sigma e(x_c(t)) I) x_c(t)$$

and denote the corresponding flow by S_c^t. For each $\sigma > 0$ the flow S_c^t is identical to $e^{A_c t}$ within a small neighbourhood of 0 in X_c (whose size depends on t) and it has the invariant overflowing manifold $\mathcal{E}$ for $t \ge 0$.

Next we verify that for sufficiently small $\sigma > 0$ the manifold $\mathcal{E} \times \{0\} \subset X_c \times X_s$ is normally hyperbolic with respect to the flow

$$\left\{ \begin{array}{ll} x_c(t) = & e^{A_c t} x_c(0) + \int_0^t e^{A_c(t-s)} \sigma e(x_c(s)) x_c(s)\, ds, \\ x_s(t) = & T_a(t)\, x_s(0). \end{array} \right. \tag{8.17}$$

We have to check (8.1) and (8.7). For this we need to verify that there exist $\lambda < 1$ and $t > 0$ such that for all $m \in \mathcal{E}$

$$\lambda \min\left\{1, \inf\{\left\|\partial S_c^t(m) x_c\right\| \mid x_c \in X_c, \|x_c\| = 1\}\right\} > \|T_a(t)\|_{\mathcal{L}(X_s)} \tag{8.18}$$

and for all $1 \leq i \leq k$

$$\lambda \left(\inf\{\left\|\partial S_c^t(m)x_c\right\| \mid x_c \in X_c, \|x_c\| = 1\}\right)^i > \|T_a(t)\|_{\mathcal{L}(X_s)}. \tag{8.19}$$

By (8.10) we have that there exist $\alpha > 0$ and $C > 0$ such that

$$\|T_a(t)\|_{\mathcal{L}(X_s)} \leq Ce^{-\alpha t} \quad \text{for } t \geq 0. \tag{8.20}$$

Now $\partial S_c^t(m)x_c$ solves

$$\frac{d}{dt}y(t) = (A_c + \sigma D(t))\, y(t) \tag{8.21}$$

with initial condition $y(0) = x_c$, where $D(t) := S_c^t(m)\partial e(S_c^t(m)) + e(S_c^t(m))I$. If y is a solution then y satisfies the variation of constants formula

$$y(t) = e^{A_c(t-t_0)}y(t_0) + \int_{t_0}^t e^{A_c(t-s)}\sigma D(s)y(s)\, ds.$$

Let D be a bound for $\sup_{s\in\mathbb{R}} \|D(s)\|$ for all $m \in \mathcal{E}$. Then for all $\epsilon > 0$ there exists a constant $M(\epsilon)$ so that

$$\|y(t)\| \leq M(\epsilon)e^{\epsilon|t-t_0|} \|y(t_0)\| + \left|\int_{t_0}^t \sigma M(\epsilon)e^{\epsilon|t-s|}D \|y(s)\| \, ds\right|.$$

Multiplying with $e^{-\epsilon|t-t_0|}$ yields

$$\left\|e^{-\epsilon|t-t_0|}y(t)\right\| \leq M(\epsilon) \|y(t_0)\| + \left|\int_{t_0}^t \sigma M(\epsilon)De^{-\epsilon|s-t_0|} \|y(s)\|\right|.$$

Gronwall's inequality implies

$$e^{-\epsilon|t-t_0|} \|y(t)\| \leq M(\epsilon) \|y(t_0)\| \, e^{\sigma M(\epsilon)D|t-t_0|}$$

or

$$\|y(t)\| \leq M(\epsilon) \|y(t_0)\| \, e^{(\epsilon+\sigma M(\epsilon)D)|t-t_0|}.$$

In other words, if $T(t, t_0)$ denotes the fundamental solution to (8.21), then we have

$$\|T(t, t_0)\| \leq M(\epsilon)e^{(\epsilon+\sigma M(\epsilon)D)|t-t_0|}.$$

Thus for $x_c \in X_c$ with $\|x_c\| = 1$ we get from

$$1 = \|x_c\| \leq \|T(0, t)\| \, \|T(t, 0)x_c\| \leq M(\epsilon)e^{(\epsilon+\sigma M(\epsilon)D)t} \|T(t, 0)x_c\| \quad (t \geq 0)$$

the estimate

$$\left\|\partial S_c^t(m)x_c\right\| = \|T(t,0)x_c\| \geq \frac{1}{M(\epsilon)}e^{-(\epsilon+\sigma M(\epsilon)D)t} \quad (t \geq 0).$$

Now choose $\epsilon < \alpha$ and σ so small that $\epsilon + \sigma M(\epsilon)D < \alpha$. Then it follows that for each $\lambda < 1$ condition (8.18) holds if t is chosen sufficiently large. If ϵ and σ are chosen so that $i\,(\epsilon + \sigma M(\epsilon)D) < \alpha$ for $1 \leq i \leq k$, then condition (8.19) is also satisfied for t sufficiently large.

We have seen that $\mathcal{E}$ is an overflowing invariant normally hyperbolic manifold for (8.17) if $\sigma > 0$ is taken sufficiently small and $t > 0$ sufficiently large. In a small neighbourhood close to zero (8.15) is, roughly speaking, a small perturbation of (8.17). Let S^t denote the flow of (8.15) and U^t the flow generated by (8.17). We will modify (8.15) outside a small δ-neighbourhood of 0 and close to the boarder $\partial\mathcal{E}$ of the ellipsoid $\mathcal{E}$. Thus we will construct a perturbed semiflow P^t of U^t in such a way that for any given neighbourhood B (which contains a tubular neighbourhood of the with respect to U^t overflowing, normally hyperbolic invariant manifold $\mathcal{E}$), and any given $\eta > 0$ and $t_1 > 0$ there exists $\delta_0 > 0$ such that for $0 < \delta < \delta_0$ the relations

$$\left\|P^t - U^t\right\|_1 < \eta \text{ for } 0 \leq t \leq t_1 \quad \text{and} \tag{8.22}$$

$$\left\|P^t - U^t\right\|_0 < \eta \text{ for } 0 \leq t \leq t_1 \tag{8.23}$$

hold. Here

$$\left\|P^t - U^t\right\|_0 := \sup_{x\in B}\left\|P^t(x) - U^t(x)\right\|$$

and

$$\left\|P^t - U^t\right\|_1 := \sup_{x\in B}\left\|P^t(x) - U^t(x)\right\| + \sup_{x\in B}\left\|\partial P^t(x) - \partial U^t(x)\right\|.$$

Then Theorem 8.10 implies that $\mathcal{E}$ persists uniquely for the perturbed semiflow P^t. Since P^t coincides in a small neighbourhood of 0 with the original unmodified flow (8.15) this will prove the existence of local center-manifolds.

Let $r_{F\delta}$ be a truncation of r_F as explained before Example 8.14. As in Example 8.14 we truncate the remainder r of $\mathfrak{H}$: Let $\chi : \mathbb{R}^n \to \mathbb{R}$ be a C^∞ cut off function such that $\chi(x) = 1$ for $\|x\| \leq 1$, $\chi(x) = 0$ for $\|x\| \geq 2$ and $0 \leq \chi(x) \leq 1$ for $x \in \mathbb{R}^n$. Put $\chi_\delta(\cdot) := \chi\left(\delta^{-1}\cdot\right)$. Define the truncated remainder r_δ

$$r_\delta(\tilde{z})(x) := r(x, \tilde{z}(x))\chi_\delta(\tilde{z}(x)).$$

Due to this type of truncation we have

$$r_\delta(\tilde z) = r(\tilde z) \quad \text{for } \|\tilde z\|_{L^\infty([0,l],\mathbb{R}^n)} \le \delta$$

and

$$\|r_\delta(\tilde z)\| \le \tilde\delta(\delta)\delta, \quad \|\partial r_\delta(\tilde z)\| \le \tilde\delta(\delta) \quad \text{for all } \tilde z \in L^\infty \tag{8.24}$$

with some positive function $\tilde\delta$ satisfying $\lim_{\delta\downarrow 0}\tilde\delta(\delta) = 0$.

Define P^t to be the flow generated by the following modification of (8.15)

$$\begin{cases} \tilde x_c(t) = & e^{A_c t}\, x_c(0) + \int_0^t e^{A_c(t-s)}\left[r_{c\delta}(\tilde x_c(s), \tilde x_s(s)) + \sigma e(\tilde x_c(s))\tilde x_c(s)\right] ds, \\ \tilde x_s(t) = & T_a(t)\, x_s(0) + \int_0^t T_a(t-s) r_{s\delta}(\tilde x_c(s), \tilde x_s(s))\, ds, \end{cases}$$

$$r_{c\delta}(x_c, x_s) := \pi_c \tilde r_\delta(x_c + x_s), \quad r_{s\delta}(x_c, x_s) := \pi_s \tilde r_\delta(x_c + x_s), \tag{8.25}$$

$$\tilde r_\delta(x_c + x_s) := \begin{pmatrix} r_\delta(x_{cu} + x_{su}, x_{cv} + x_{sv}, x_{cw} + x_{sw}) \\ r_{F\delta}(x_{cu} + x_{su}, x_{cw} + x_{sw}) \end{pmatrix}.$$

Choose $\delta_1 > 0$ so small that $\|x\|_Y < \delta_1$ implies $\|z_c\|_Y < \frac{r}{2}$. Then for $\delta < \delta_1$ the flow of (8.25) coincides with (8.15) locally within a subset of the ball $B_\delta \subset Y$ (depending on t). Subtracting (8.25) and (8.17) we get

$$\begin{aligned} \tilde x_c(t) - x_c(t) = & \int_0^t e^{A_c(t-s)}[r_{c\delta}(\tilde x_c(s), \tilde x_s(s)) + \\ & \sigma\left(e(\tilde x_c(s))\tilde x_c(s) - e(x_c(s))x_c(s)\right)] ds, \\ = & \int_0^t e^{A_c(t-s)}[r_{c\delta}(\tilde x_c(s), \tilde x_s(s)) \\ & +\sigma\left(\left(e(\tilde x_c(s)) - e(x_c(s))\right)\tilde x_c(s) + e(x_c(s))\left(x_c(s) - \tilde x_c(s)\right)\right)] ds, \\ \tilde x_s(t) - x_s(t) = & \int_0^t T_a(t-s) r_{s\delta}(\tilde x_c(s), \tilde x_s(s))\, ds, \end{aligned}$$

From (8.24) and (8.12) it follows that

$$\begin{aligned} \|r_{c\delta}(x_c, x_s)\|_\infty \le \tilde\delta(\delta)\delta, \quad & \|r_{s\delta}(x_c, x_s)\|_\infty \le \tilde\delta(\delta)\delta, \\ \|\partial r_{c\delta}(x_c, x_s)\|_{\mathcal{L}(Y,Y)} \le \tilde\delta(\delta), \quad & \|\partial r_{s\delta}(x_c, x_s)\|_{\mathcal{L}(Y,X_\infty)} \le \tilde\delta(\delta). \end{aligned}$$

If Λ_e denotes a Lipschitz constant for the bump function e, B_e is a bound for e, B_c is a bound for $\sup_{0\le s\le t_1}\|\tilde x_c(s)\|$ and $\sup_{0\le s\le t_1}\|x_c(s)\|$ (depending only on t_1 and B) and

$$D := \sup_{\theta\in[0,t_1]}\left(\left\|e^{A_c\theta}\right\|_{\mathcal{L}(X_c)} + \|T_a(\theta)\|_{\mathcal{L}(X_\infty)}\right),$$

then for $0 \le t \le t_1$

$$\begin{aligned} \|\tilde x_c(t) - x_c(t)\|_Y \le \; & D t_1 \tilde\delta(\delta)\delta + \int_0^t \sigma D(\Lambda_e B_c + B_e)\, \|\tilde x_c(s) - x_c(s)\|\, ds, \\ \|\tilde x_s(t) - x_s(t)\|_Y \le \; & D t_1 \tilde\delta(\delta)\delta. \end{aligned}$$

Gronwall yields

$$\|\tilde{x}_c(t) - x_c(t)\|_Y \leq Dt_1\tilde{\delta}(\delta)e^{\sigma D(\Lambda_e B_c + B_e)t} \quad \text{for } 0 \leq t \leq t_1. \tag{8.26}$$

Thus there exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ condition (8.23) is satisfied.
By taking the norm of

$$\begin{aligned}
& \partial\tilde{x}_c(t) - \partial x_c(t) \\
= \; & \int_0^t e^{A_c(t-s)}\partial r_{c\delta}(\tilde{x}_c(s), \tilde{x}_s(s))(\partial\tilde{x}_c(s) + \partial\tilde{x}_s(s))\, ds \\
& +\sigma \int_0^t e^{A_c(t-s)}[\partial e(\tilde{x}_c(s))\partial\tilde{x}_c(s)\tilde{x}_c(s) - \partial e(x_c(s))\partial x_c(s)x_c(s) \\
& +e(\tilde{x}_c(s))\partial\tilde{x}_c(s) - e(x_c(s))\partial x_c(s)]\, ds \\
= \; & \int_0^t e^{A_c(t-s)}\partial r_{c\delta}(\tilde{x}_c(s), \tilde{x}_s(s))(\partial\tilde{x}_c(s) + \partial\tilde{x}_s(s))\, ds \\
& +\sigma \int_0^t e^{A_c(t-s)}[(\partial e(\tilde{x}_c(s)) - \partial e(x_c(s)))\partial\tilde{x}_c(s)\tilde{x}_c(s) \\
& +\partial e(x_c(s))\,(\partial\tilde{x}_c(s)(\tilde{x}_c(s) - x_c(s)) + (\partial\tilde{x}_c(s) - \partial x_c(s))x_c(s)) \\
& + (e(\tilde{x}_c(s)) - e(x_c(s)))\,\partial\tilde{x}_c(s) + e(x_c(s))\,(\partial\tilde{x}_c(s) - \partial x_c(s))]\, ds
\end{aligned}$$

$$\begin{aligned}
& \partial\tilde{x}_s(t) - \partial x_s(t) \\
= \; & \int_0^t T_a(t-s)\partial r_{s\delta}(\tilde{x}_c(s), \tilde{x}_s(s))\,(\partial\tilde{x}_c(s) + \partial\tilde{x}_s(s))\, ds.
\end{aligned}$$

we get for $0 \leq t \leq t_1$ because of (8.26)

$$\begin{aligned}
& \|\partial\tilde{x}_c(t) - \partial x_c(t)\|_{\mathcal{L}(Y,X_c)} \\
\leq \; & Dt_1 2E\tilde{\delta}(\delta) + \sigma D \int_0^t [\|\partial e(\tilde{x}_c(s)) - \partial e(x_c(s))\|\,\|\partial\tilde{x}_c(s)\|\,\|\tilde{x}_c(s)\| \\
& + \|\partial e(x_c(s))\|\,(\|\partial\tilde{x}_c(s)\|\,\|\tilde{x}_c(s) - x_c(s)\| + \|\partial\tilde{x}_c(s) - \partial x_c(s)\|\,\|x_c(s)\|) \\
& + \|e(\tilde{x}_c(s)) - e(x_c(s))\|\,\|\partial\tilde{x}_c(s)\| + \|e(x_c(s))\|\,\|\partial\tilde{x}_c(s) - \partial x_c(s)\|]\, ds \\
\leq \; & Dt_1 2E\tilde{\delta}(\delta) + \sigma D \int_0^t [(E_e B_c + B_e)\,\|\partial\tilde{x}_c(s) - \partial x_c(s)\| + \\
& E(\tilde{\Lambda}B_c + E_e + \Lambda_e)\,\|\tilde{x}_c(s) - x_c(s)\|]\, ds \\
\leq \; & Dt_1 2E\tilde{\delta}(\delta) + \sigma D^2 t_1^2 E\left(\tilde{\Lambda}B_c + E_e + \Lambda_e\right)\tilde{\delta}(\delta)\delta e^{\sigma D(\Lambda_e B_c + B_e)t_1} \\
& + \int_0^t \sigma D\,(E_e B_c + B_e)\,\|\partial\tilde{x}_c(s) - \partial x_c(s)\|\, ds
\end{aligned}$$

and

$$\|\partial\tilde{x}_s(t) - \partial x_s(t)\|_{\mathcal{L}(Y,X_s)} \leq Dt_1 2E\tilde{\delta}(\delta).$$

Here E is a bound for $\sup_{0\leq s\leq t_1}\|\partial\tilde{x}_c(s)\|$ and $\sup_{0\leq s\leq t_1}\|\partial\tilde{x}_s(s)\|$ which only depends on B and t_1, E_e is a bound for $\|\partial e\|$, and $\tilde{\Lambda}$ is a Lipschitz constant for ∂e.
Gronwall implies that $\delta_0 > 0$ can be chosen sufficiently small such that (8.22) holds for $0 < \delta < \delta_0$.

Next we prove $\imath$):
The relation $\gamma(0) = 0$ is plain from the construction of the center manifold

because 0 is a fixed point of (8.25) for any $\delta > 0$.
Since $\gamma : \mathcal{E} \to X_s$ is invariant unter the flow P^t we have

$$\gamma(\xi) = \pi_s P^t(x_c + \gamma(x_c)) \quad \text{with} \quad \xi = \pi_c P^t(x_c + \gamma(x_c)) \tag{8.27}$$

for all $x_c \in \mathcal{E}$ and $t \geq 0$ such that $\xi \in \mathcal{E}$. Deriving (8.27) at $x_c = 0$ yields

$$\partial\gamma(0)\pi_c \partial P^t(0)(I + \partial\gamma(0)) = \pi_s \partial P^t(0)(I + \partial\gamma(0)). \tag{8.28}$$

Here $I : X_c \to X_c \times X_s$ denotes simple inclusion. Because $\partial P^t(0) = \big(e^{A_c t},\ T_a(t)\pi_s\big)$ we have $\pi_c \partial P^t(0)\partial\gamma(0) = 0$ and $\pi_s \partial P^t(0) = T_a(t)\pi_s$. Multiplying (8.28) from the right with $(\pi_c \partial P^t(0) I)^{-1} = e^{-A_c t}$ yields

$$\partial\gamma(0) = T_a(t)\pi_s \partial\gamma(0) e^{-A_c t}.$$

Letting $t \to \infty$ equation (8.20) implies

$$\partial\gamma(0) = 0.$$

Note that (8.27) makes sense for arbitrary large t for x_c in a neighbourhood of zero (whose size depends on t). Therefore (8.28) holds for all $t \geq 0$.

Statement *iii*) follows from the final sentence of Theorem 8.10.

Finally we prove *iv*). Let $x_c^0 \in \Omega \cap X_c$. Let $z : [0, \delta] \to Y$ be a local trajectory with $z(0) = a + x_c^0 + \gamma(x_c^0)$. Then $z(t) = a + x_c(t) + x_s(t)$, where $x_s(t) = \gamma(x_c(t))$. Because $\gamma \in C^k(\Omega \cap X_c, X_s)$ and $x_s = \gamma(x_c)$ it follows from (8.15) that $x_c(t)$ is the solution of an ODE in the unknown x_c with a C^k smooth vectorfield. Hence $z \in C^k([0, \delta], Y)$ and

$$\begin{pmatrix} -\mathfrak{H}((u, v, w)) \\ F(u, v) \end{pmatrix} \in C^1([0, \delta], X_p).$$

Since T is a C_0 semigroup on X_p Proposition 13.6 yields

$$\int_0^t T(t - s) \begin{pmatrix} -\mathfrak{H}((u, v, w)(s)) \\ F(u(s), v(s)) \end{pmatrix} ds \in C^1([0, \delta[\,, X_p)$$

for $1 \leq p < \infty$. Hence $T(\cdot)z(0) \in C^1([0, \delta], X_p)$ which implies

$$z(0) \in W^{1,p}(]0, l[\,, \mathbb{R}^{n_1+n_2}) \times L^p(]0, l[\,, \mathbb{R}^{n_3}) \times \mathbb{R}^{n_2}.$$

Hence $\gamma(x_c^0) = z(0) - a - x_c^0 \in W^{1,p}(]0, l[\,, \mathbb{R}^{n_1+n_2}) \times L^p(]0, l[\,, \mathbb{R}^{n_3}) \times \mathbb{R}^{n_2}$. $\square$

Remark 8.16 (Nonuniqueness). *Center manifolds are not unique (not even in a sufficiently small neighbourhood of the equilibirium). This does not contradict the uniqueness stated in Remark 8.13. In the proof of Theorem 8.15 we have obtained a local center manifold which coincides with a unique center manifold in a neigbourhood (size δ) of the origin of equation* (8.25). *We have obtained this unique manifold after modifying the nonlinearities H and F outside such a neighbourhood. Many different persisting manifolds can coexist each depending uniquely on the type of modification one has performed, see [70, 74].*

Remark 8.17 (Center manifolds including parameters,bifurcations).
Suppose H and F depend on a parameter $\lambda \in \mathbb{R}^d$, i.e. $H :]0, l[\times\big(\mathbb{R}^n \times \mathbb{R}^d\big) \to \mathbb{R}^n$ is a C^k Caratheodory function and $F : C([0, l], \mathbb{R}^{n_1+n_2}) \times \mathbb{R}^d \to \mathbb{R}^{n_2}$ is C^k with bounded and uniformly continuous derivatives on bounded subsets. We explain how one obtains a center manifold depending smoothly on λ. Write

$$\begin{aligned}\mathfrak{H}((u,v,w),\lambda) &= \mathfrak{H}((a_u,a_v,a_w),0)\\ &\quad +\partial_{(u,v,w)}\mathfrak{H}((a_u,a_v,a_w),0)(u-a_u,v-a_v,w-a_w)\\ &\quad +\partial_\lambda\mathfrak{H}(a_u,a_v,a_w,0)\lambda - r(u-a_u,v-a_v,w-a_w,\lambda),\end{aligned}$$

$$\begin{aligned}F(u,v,\lambda) = F(a_u,a_v,0) &+ \partial_{(u,v)}F(a_u,a_v,0)(u-a_u,v-a_v)\\ &+\partial_\lambda F(a_u,a_v,0)\lambda + r_F(u-a_u,v-a_v,\lambda),\end{aligned}$$

where r and r_F are of second order in all variables. Then (8.14) *becomes*

$$x(t) = T_a(t)x(0)+\int_0^t T_a(t-s)\begin{pmatrix}-\partial_\lambda\mathfrak{H}(a_u,a_v,a_w,0)\lambda + r(x_u(s),x_v(s),x_w(s),\lambda)\\ \partial_\lambda F(a_u,a_v,0)\lambda + r_F(x_u(s),x_v(s),\lambda)\end{pmatrix} ds.$$

and (8.15) *changes to*

$$\begin{cases} x_c(t) = & T_a(t)\,x_c(0) + \int_0^t T_a(t-s)\,(r_c(x_c(s),x_s(s),\lambda)+\alpha\lambda)\,ds,\\ x_s(t) = & T_a(t)\,x_s(0) + \int_0^t T_a(t-s)\,(r_s(x_c(s),x_s(s),\lambda)+\beta\lambda)\,ds,\end{cases} \tag{8.29}$$

$$r_c(x_c,x_s,\lambda) := \pi_c\tilde r(x_c+x_s,\lambda),\ r_s(x_c,x_s,\lambda) := \pi_s\tilde r(x_c+x_s,\lambda),\ \alpha := \pi_c\zeta,$$

$$\beta := \pi_s\zeta, \tilde r(x_c+x_s,\lambda) := \begin{pmatrix} r(x_{cu}+x_{su},x_{cv}+x_{sv},x_{cw}+x_{sw},\lambda)\\ r_F(x_{cu}+x_{su},x_{cv}+x_{sv},\lambda)\end{pmatrix},$$

$$\zeta := \begin{pmatrix}-\partial_\lambda\mathfrak{H}(a_u,a_v,a_w,0)\\ \partial_\lambda F(a_u,a_v,0)\end{pmatrix}.$$

Let A_s denote the restriction of $\mathfrak{A}_a$ to X_s. Since $0 \notin \sigma(A_s)$ the operator A_s is an isomorphism from its domain of definition (equipped with the graph *norm) onto X_s. A further transformation of the form $\tilde{x}_s = x_s + A_s^{-1}\beta\lambda$ makes $\beta = 0$, i.e.* (8.29) *can be written as*

$$\begin{array}{rcl} \partial_t x_c &=& A_c x_c + \alpha\lambda + \tilde{r}_c(x_c, \tilde{x}_s, \lambda) \\ \partial_t \lambda &=& 0 \\ \partial_t \tilde{x}_s &=& A_s \tilde{x}_s + \tilde{r}_s(x_c, \tilde{x}_s, \lambda), \end{array} \tag{8.30}$$

where $\tilde{r}_{c/s}(x_c, \tilde{x}_s, \lambda) = r_{c/s}(x_c, \tilde{x}_s - A_s^{-1}\beta\lambda, \lambda)$, $\tilde{r}_c$ and $\tilde{r}_s$ are of second order in all variables. Following the above proof we get that (8.30) *has a local center manifold of the form $M = \{(x_c, \lambda, \gamma(x_c, \lambda)) \mid (x_c, \lambda) \in \Omega\}$, where Ω is a neighbourhood of 0 in $X_c \times \mathbb{R}^d$ and $\gamma(0,0) = 0$, $\partial_{x_c}\gamma(0,0) = 0$, $\partial_\lambda \gamma(0,0) = 0$ and for each $k > 1$ one can choose Ω sufficiently small such that $\gamma : \Omega \to X_s$ is of class C^k. The manifold M contains all sufficiently small bounded time reversible solutions (for example equilibria and periodic solutions). The intersection of M with the planes $\lambda = const$ are invariant under the flow of* (8.30)*. On such intersections the equation is given by the ODE*

$$\partial_t x_c = A_c x_c + \alpha\lambda + f(x, \lambda),$$

where $f(x, \lambda)$ is of second order in both variables.

If $0 \notin \sigma(\mathfrak{A}_a)$, which occurs e.g. for Hopf bifurcations, then we can solve $-K\partial_x(a_u, a_v, a_w) - \mathfrak{H}(a_u, a_v, a_w, \lambda) = 0$ and $F(a_u, a_v, \lambda) = 0$ for

$$\begin{aligned} a &= (a_u, a_v, a_w, \Delta(a_u, a_v)) \\ &= a^*(\lambda) \in Y \cap W^{1,\infty}([0,l], \mathbb{R}^{n_1+n_2}) \times C([0,l], \mathbb{R}^{n_3}) \times \mathbb{R}^{n_2}. \end{aligned}$$

We then translate $a^*(\lambda)$ to the origin,

$$x(t) = z(t) - a^*(\lambda),$$

and get the following equation for x

$$\begin{array}{rcl} \dfrac{d}{dt} x(t) &=& \begin{pmatrix} -K\partial_x \begin{pmatrix} a_u(\lambda) + x_u(t) \\ a_v(\lambda) + x_v(t) \\ a_w(\lambda) + x_w(t) \end{pmatrix} - \mathfrak{H} \begin{pmatrix} a_u(\lambda) + x_u(t) \\ a_v(\lambda) + x_v(t) \\ a_w(\lambda) + x_w(t) \\ \lambda \end{pmatrix} \\ F(a_u(\lambda) + x_u(t), a_v(\lambda) + x_v(t), \lambda) \end{pmatrix} \\ \dfrac{d}{dt}\lambda &=& 0. \end{array}$$

There are two ways to proceed then: We can expand $-K\partial_x(a_u(\lambda) + x_u, a_v(\lambda) + x_v, a_w(\lambda) + x_w) - \mathfrak{H}(a_u(\lambda) + x_u, a_v(\lambda) + x_v, a_w(\lambda) + x_w, \lambda)$ in $x_u = 0, x_v =$

$0, x_w = 0, \lambda = 0$ *(similarly for F) and arrive at an equation which has the same form as* (8.30)*, but with $\alpha = 0$, and get the existence of a smooth center manifold M as above in X_c and X_s coordinates.*

Another possibility is to expand around $a^*(\lambda)$ for λ near zero and use λ dependent coordinates: Denote

$$\mathfrak{A}(\lambda)x := \begin{pmatrix} -K\partial_x \begin{pmatrix} x_u \\ x_v \\ x_w \end{pmatrix} - \partial_{(u,v,w)}\mathfrak{H} \begin{pmatrix} a_u(\lambda) \\ a_v(\lambda) \\ a_w(\lambda) \\ \lambda \end{pmatrix} \begin{pmatrix} x_u \\ x_v \\ x_w \end{pmatrix} \\ \partial_{(u,v)} F(a_u(\lambda), a_v(\lambda), \lambda) \begin{pmatrix} x_u \\ x_v \end{pmatrix} \end{pmatrix}$$

($\mathfrak{A}(0) = \mathfrak{A}_a$). Then we have

$$\begin{aligned} \frac{d}{dt}x(t) &= \mathfrak{A}(\lambda)x(t) + r(x(t), \lambda) \\ \frac{d}{dt}\lambda &= 0. \end{aligned} \tag{8.31}$$

with

$$r(0, \lambda) = 0,\ \partial_{(u,v,w)} r(0, \lambda) = 0 \text{ for } \lambda \text{ in a neighbourhood of zero,}$$

and

$$\partial_{(u,v,w,\lambda)} r(0, 0) = 0.$$

For λ in a sufficiently small neighbourhood of zero let $\pi_c(\lambda)$ denote the spectral projection for the critical eigenvalues of $\mathfrak{A}(\lambda)$ near the imaginary axis, $\pi_s(\lambda) := \mathrm{Id} - \pi_c(\lambda)$, $X_c(\lambda) := \mathfrak{Re}\,\mathrm{Im}\,\pi_c(\lambda)$, $X_s(\lambda) := \mathfrak{Re}\,\mathrm{Im}\,\pi_s(\lambda)$, $X_c = X_c(0)$, $X_s = X_s(0)$. Let $B_c(\lambda) : X_c \to X_c(\lambda)$ and $B_s(\lambda) : X_s \to X_s(\lambda)$ be smooth linear bases. By using the coordinates

$$x_c := B_c^{-1}(\lambda)\pi_c(\lambda)x, \quad x_s := B_s(\lambda)^{-1}\pi_s(\lambda)x$$

($\pi_c(\lambda)$ leaves real space invariant, because $\overline{\pi_c(\lambda)x} = \pi_c(\lambda)\overline{x}$) (8.31) *recasts as*

$$\begin{aligned} \frac{d}{dt}x_c(t) &= B_c^{-1}(\lambda)\mathfrak{A}(\lambda)B_c(\lambda)x_c(t) + \\ &\quad B_c^{-1}(\lambda)\pi_c(\lambda)r(B_c(\lambda)x_c(t) + B_s(\lambda)x_s(t), \lambda) \\ \frac{d}{dt}x_s(t) &= B_s^{-1}(\lambda)\mathfrak{A}(\lambda)B_s(\lambda)x_s(t) + \\ &\quad B_s^{-1}(\lambda)\pi_s(\lambda)r(B_c(\lambda)x_c(t) + B_s(\lambda)x_s(t), \lambda) \\ \frac{d}{dt}\lambda &= 0. \end{aligned} \tag{8.32}$$

The linear part

$$\begin{aligned}\frac{d}{dt}x_c(t) &= B_c^{-1}(\lambda)\mathfrak{A}(\lambda)B_c(\lambda)x_c(t) \\ \frac{d}{dt}x_s(t) &= B_s^{-1}(\lambda)\mathfrak{A}(\lambda)B_s(\lambda)x_s(t) \\ \frac{d}{dt}\lambda &= 0.\end{aligned} \tag{8.33}$$

has the invariant manifold $x_s = 0$, which is normally hyperbolic due to the presence of the spectral gap and the mapping Theorem 6.15 (we have to linearize the flow generated by (8.33) *with respect to all variables x_c, x_s and λ in $x_s = 0$; the λ derivatives do not cause any difficulties). For x_c, x_s and λ near zero* (8.32) *is a small C^1 perturbation of* (8.33)*. By modifying the equations we can obtain an overflowing manifold. Then it follows from Theorem 8.10 that locally this manifold persists: There exists a smooth graph $\gamma : X_c \times \mathbb{R}^d \to X_s$ and a $\delta > 0$ so that $x_s = \gamma(x_c, \lambda)$, $\|x_c\| < \delta$, $\|\lambda\| < \delta$, is a C^k smooth invariant center manifold for* (8.32)*. On the center manifold the equations are governed by the following ODE*

$$\frac{d}{dt}x_c(t) = B_c^{-1}(\lambda)\mathfrak{A}(\lambda)B_c(\lambda)x_c(t) + f(x_c(t), \lambda),$$

where $f(x_c, \lambda) = B_c^{-1}(\lambda)\pi_c(\lambda)r(B_c(\lambda)x_c + B_s(\lambda)\gamma(x_c, \lambda), \lambda)$ is of second order.

Suppose $d = 1$ and a pair of complex conjugated eigenvalues $\lambda_1 = \lambda_1(\lambda)$, $\lambda_2 = \lambda_2(\lambda) = \overline{\lambda_1}(\lambda)$ crosses the imaginary axis at $\lambda = 0$, the remaining spectrum being separated to the left with a spectral gap. Then we can choose a basis $B_c(\lambda)$ so that

$$B_c^{-1}(\lambda)\mathfrak{A}(\lambda)B_c(\lambda) = \begin{pmatrix} \mathfrak{Re}\,\lambda_1(\lambda) & -\mathfrak{Im}\,\lambda_1(\lambda) \\ \mathfrak{Im}\,\lambda_1(\lambda) & \mathfrak{Re}\,\lambda_1(\lambda) \end{pmatrix}.$$

Remark 8.18. *By taking higher derivatives of* (8.27) *in $x_c = 0$ one can prove the following: Suppose*

$$\pi_s \begin{pmatrix} \partial^j \mathfrak{H}(a_u, a_v, a_w) \\ \partial^j F(a_u, a_v) \end{pmatrix} z^j = 0 \quad \textit{for } z \in X_c \textit{ and } 2 \le j \le k.$$

Then we have

$$\partial^j \gamma(0) = 0 \quad \textit{for } 1 \le j \le k.$$

Chapter 9

Center manifold / model reduction for the autonomous traveling wave model

In section 3.2 we have seen that the traveling wave model has a slow fast structure. The carriers n are two orders of magnitude slower than the optical field ψ. In this section we consider the autonomous case when $\alpha = 0$ which we will generalize later to the nonautonomous case $\alpha \neq 0$. Hence, here we consider the static boundary conditions for the optical field ψ:

$$\psi_1(0) = r_0\psi_2(0), \quad \psi_2(l) = r_l\psi_1(l).$$

The slow-fast structure is expressed in (3.17) within the small variable ϵ. For better readability we rewrite (3.17) in the following operator form: For fixed carriers $n(\cdot)$ denote the linear differential operator of the ψ equation in (3.17) with $\mathfrak{A}(n)$,

$$\begin{cases} \mathfrak{A} := \mathfrak{A}(n) & := & (\mathfrak{A}_0 + L(x, n(x)))\,\psi(t,x), \\ \mathfrak{A}_0 & := & \begin{pmatrix} -\partial_x & 0 \\ 0 & \partial_x \end{pmatrix}, \end{cases} \tag{9.1}$$

where L is given in dimensionless form by (3.13). Then (3.17) can be written as:

$$\begin{cases} \partial_t\psi(t) & = & \mathfrak{A}(n(t))\psi(t) + \epsilon\mathfrak{K}(n(t), \psi(t)) \\ \partial_t n(t) & = & \epsilon\mathfrak{F}\left(t, n(t), \psi(t)\right). \end{cases} \tag{9.2}$$

Here $\mathfrak{K}$ is a Nemytskij operator generated by the nonlinear function K defined in (3.15) and the operator $\mathfrak{F}$ is composed of Nemytskij operators and a nonlocal term as in the definition of F below (3.17). Without the nonlocal term (9.2) falls under the setting for degenerate semilinear hyperbolic systems introduced in section 7 after expanding the system size to obtain completely smooth operators as explained in section 3.2. So we obtain a smooth semiflow in a function space setting of systems of continuous functions including boundary conditions. For a more compact (and equivalent) formulation here we prefer not to expand the system size and consider (9.2) as a smooth semiflow in the space $\mathcal{W} \times C_P$, where

$$\mathcal{W} := \{\psi = (\psi_1, \psi_2) \in C([0,l], \mathbb{C}^2) \mid \psi_1(0) = r_0\psi_2(0), \psi_2(l) = r_l\psi_1(l)\}$$

and C_P denotes the space of section wise uniformly continuous functions exactly as we have done in section 10. We denote the smooth semiflow by the symbol T^t, $t \geq 0$.

We have seen that the spectrum of $\mathfrak{A}(n)$ always possesses a gap at γ_+, where γ_+ denotes the supremum of the real part of the spectrum of the reduced diagonal operator. That is for $\alpha > \gamma_+$ the set $\sigma(\mathfrak{A}) \cap \{\lambda \in \mathbb{C} \mid \mathfrak{Re}\, \lambda \geq \alpha\}$ is finite. Under physical realistic parameters $\gamma_+ < 0$ is always satisfied and there are only a few critical modes (typically one to four) which are close to the imaginary axis. We have seen in section 4.2 that this splitting of the eigenvalues holds for a (probably small) open neighbourhood $\mathcal{U}$ for n because all but finitely many eigenvalues can be controlled (see Lemmas 4.8 and 4.15) and the remaining finite eigenvalues must depend continuously on n. Therefore, since $\epsilon > 0$ is small one expects that the ψ dynamics is appropriately described by those few leading finite critical eigenvalues. If $\mathcal{U}$ is sufficiently small then one has a uniform spectral gap of the generator $\mathfrak{A}(n)$. According to this spectral splitting we get uniform exponential dichotomy for $n \in \mathcal{U}$ by Theorem 5.5.

If we assume that $\mathcal{U}$ is a starshaped neighbourhood $\mathcal{U} \subset C_P$ (for example choose $\mathcal{U}$ to be a small ball), then one can choose bases corresponding to the spectral splitting for $n \in \mathcal{U}$ in the space $\mathcal{W}$ so that the bases depend smoothly on $n \in C_P$. More precisely, there exist smooth maps

$$\mathfrak{B} : \mathcal{U} \to \mathcal{L}(\mathbb{C}^q, L^p) \quad \text{and} \quad \mathfrak{C} : \mathcal{U} \to \mathcal{L}(\mathcal{Y}, L^p),$$

where $\mathcal{Y} \subset L^p$ is a closed codimension q subspace of L^p, $L^p = \mathrm{Im}\mathfrak{B}(n) \oplus \mathrm{Im}\mathfrak{C}(n)$, $\mathrm{Im}\mathfrak{B}(n) = \mathrm{Im}\mathfrak{P}(n)$, $\mathfrak{P}(n) := \int_\gamma (\lambda\mathfrak{I} - \mathfrak{A}(n))^{-1}\, d\lambda$, $\mathfrak{P}\mathfrak{A} = \mathfrak{A}\mathfrak{P}$,$\mathfrak{Q}(n) := \mathfrak{I} - \mathfrak{P}(n)$, $\mathrm{Im}\mathfrak{C}(n) = \mathrm{Im}\mathfrak{Q}(n)$, γ denoting a positively oriented path in the resolvent set of $\mathfrak{A}$ in a neighbourhood of $i\mathbb{R}$ enclosing the finite critical eigenvalues uniformly for $n \in \mathcal{U}$. Here $\mathcal{L}$ denotes the space of bounded linear operators, Im denotes the image of a linear operator, $\mathfrak{I}$ denotes the identity operator in L^p. Let $\mathcal{Y}_\mathcal{W} := \mathcal{Y} \cap \mathcal{W}$ and $\mathfrak{C}_\mathcal{W}$ denote the restriction of $\mathfrak{C}$ to $\mathcal{Y}_\mathcal{W}$, $\mathfrak{C}_\mathcal{W}(n)(y) := \mathfrak{C}(n)(y)$ for $y \in \mathcal{Y}_\mathcal{W}$. We have $\mathfrak{C}_\mathcal{W} \in C^k(\mathcal{U}, \mathcal{L}(\mathcal{Y}_\mathcal{W}, \mathcal{W}))$. We are interested in solutions with $n(t) \in \mathcal{U}$ for $0 \leq t < \infty$ (for example (relative) periodic solutions bifurcating from (relative) equilibria). Using the n-dependent coordinate transformation

$$(\psi, n) \mapsto (x_c, x_s, n), \quad \psi = \mathfrak{B}(n)x_c + \mathfrak{C}_{\mathcal{W}}(n)x_s$$

the transformed smooth semiflow $S^t \in C^k(\mathbb{C}^q \times \mathcal{Y}_{\mathcal{W}} \times \mathcal{U})$ in the phase space $\mathbb{C}^q \times \mathcal{Y}_{\mathcal{W}} \times \mathcal{U}$,

$$S^t(x_c, x_s, n) := \begin{pmatrix} \mathfrak{B}(T_n^t(\psi, n))^{-1}\mathfrak{P}(n)T_\psi^t(\psi, n) \\ \mathfrak{C}_{\mathcal{W}}(T_n^t(\psi, n))^{-1}\mathfrak{Q}(n)T_\psi^t(\psi, n) \\ T_n^t(\psi, n) \end{pmatrix},$$

where $T_\psi^t \in \mathcal{W}$ denotes the ψ-component and $T_n^t \in C_P$ the n-component of the flow T^t, is described by the following set of equations:

$$\left\{ \begin{array}{rcl} \partial_t x_c(t) & = & \mathfrak{A}_c(n)x_c + \epsilon\mathfrak{G}_c(n, x_c, x_s) \\ \partial_t x_s(t) & = & \mathfrak{A}_s(n)x_s + \epsilon\mathfrak{G}_s(n, x_c, x_s) \\ \partial_t n(t) & = & \epsilon\mathfrak{F}(t, n, \mathfrak{B}(n)x_c + \mathfrak{C}(n)x_s), \\ & & \\ x_c(0) & = & x_{c0}, \\ x_s(0) & = & x_{s0}, \\ n(0) & = & n_0, \\ \psi_0 & = & \mathfrak{B}(n_0)x_{c0} + \mathfrak{C}(n_0)x_{s0}, \end{array} \right. \tag{9.3}$$

where

$$\begin{array}{rcl} \mathfrak{A}_c(n) & := & (\mathfrak{B}(n))^{-1}\,\mathfrak{A}(n)\mathfrak{B}(n), \\ \mathfrak{A}_s(n) & := & (\mathfrak{C}(n))^{-1}\,\mathfrak{A}(n)\mathfrak{C}(n), \end{array}$$

$$\begin{array}{rcl} \mathfrak{G}_c(n, x_c, x_s) & := & (\mathfrak{B}(n))^{-1}\,\mathfrak{P}(n)\mathfrak{K}(n, \mathfrak{B}(n)x_c + \mathfrak{C}(n)x_s) \\ & & -\,(\mathfrak{B}(n))^{-1}\,\mathfrak{P}(n)\left(\partial\mathfrak{B}(n)\mathfrak{F}(t, n, \mathfrak{B}(n)x_c + \mathfrak{C}(n)x_s)\right)x_c \\ & & -\,(\mathfrak{B}(n))^{-1}\,\mathfrak{P}(n)\left(\partial\mathfrak{C}(n)\mathfrak{F}(t, n, \mathfrak{B}(n)x_c + \mathfrak{C}(n)x_s)\right)x_s, \end{array}$$

$$\begin{array}{rcl} \mathfrak{G}_s(n, x_c, x_s) & := & (\mathfrak{C}(n))^{-1}\,\mathfrak{Q}(n)\mathfrak{K}(n, \mathfrak{B}(n)x_c + \mathfrak{C}(n)x_s) \\ & & -\,(\mathfrak{C}(n))^{-1}\,\mathfrak{Q}(n)\left(\partial\mathfrak{B}(n)\mathfrak{F}(t, n, \mathfrak{B}(n)x_c + \mathfrak{C}(n)x_s)\right)x_c \\ & & -\,(\mathfrak{C}(n))^{-1}\,\mathfrak{Q}(n)\left(\partial\mathfrak{C}(n)\mathfrak{F}(t, n, \mathfrak{B}(n)x_c + \mathfrak{C}(n)x_s)\right)x_s. \end{array}$$

The above set of equations can be understood as follows: If $\psi_0 \in W^{1,2}(]0, l[, \mathbb{C}^2)$ then $x_c \in C^1([0, \infty[, \mathbb{C}^q)$, $x_s \in C^1([0, \infty[, \mathcal{Y})$ and $n \in C^1([0, \infty[, L^\infty(]0, L[, \mathbb{R}))$ and (9.3) holds in a classical sense. For $\epsilon = 0$ S^t has the (not locally compact) invariant Banach-manifold

$$\mathrm{IM}_0 := \mathbb{C}^q \times \{0\} \times \mathcal{U} \subset \mathbb{C}^q \times \mathcal{Y}_{\mathcal{W}} \times C_P.$$

Theorem 5.5 implies that the semigroup generated by $\mathfrak{A}_s(n)$ on $\mathcal{Y}_{\mathcal{W}}$ decays exponentially in the C topology, this decay is uniform for $n \in \mathcal{U}$ and faster than the decay of the exponential $e^{\mathfrak{A}_c(n)t}$. This means that the manifold IM_0 is normally hyperbolic. Hence it persists smoothly in our chosen function space $\mathbb{C}^q \times \mathcal{Y}_{\mathcal{W}} \times C_P \simeq \mathcal{W} \times C_P$ as a nonlinear smooth manifold IM_ϵ for sufficiently small $0 < \epsilon < \epsilon_0$ and can be represented as a C^k smooth graph

$$\gamma : \mathbb{C}^q \times \mathcal{U} \times]0, \epsilon_0[\to \mathcal{Y}_{\mathcal{W}}.$$

(In order to prove the persistence we would modify the flows to obtain an overflowing manifold and apply invariant manifold theory similarly as we did in Theorem 8.6. We omit the details.)
Therefore the flow on IM_ϵ in the coordinates $\mathbb{C}^q \times \mathcal{U}$ is given by the equations

$$\begin{cases} \partial_t x_c(t) &= \mathfrak{A}_c(n)x_c + \epsilon\mathfrak{G}_c\left(n, x_c, \gamma(x_c, n, \epsilon)\right) \\ \partial_t n(t) &= \epsilon\mathfrak{F}(t, n, \mathfrak{B}(n)x_c + \mathfrak{C}(n)\gamma(x_c, n, \epsilon)). \end{cases} \tag{9.4}$$

Here the operators on the right hand side are C^k-smooth in $n \in \mathcal{U}$, $x_c \in \mathbb{C}^q$, $\epsilon \in \mathbb{R}$ and we have arrived to a Banach ODE. In (9.4) the nonlinearities can be expanded in terms of powers of ϵ. In a first order approximation, dropping the $O(\epsilon^2)$ terms, the unknown graph γ disappears, the resulting equation is called mode approximation. We will calculate the first order approximation on the center manifold in more detail for the more general nonlinear and nonautonomous traveling wave model in section 11.

Mode approximations have been first derived formally by physicists [3], the first rigorous derivation in the context of Laser equations modeled by ODEs has been obtained in [72] (by using persistence theory of invariant manifolds for ODEs [22, 32]) and then was extended to a special linear autonomous traveling wave model in [65, 66] by using a L^2 phase space and the spectral theory of Lopes, Neves, Ribeiro [48] which works for L^2. This was only possible because of the exceptional structure of the model in [65, 66] that the PDE was linear and only nonlinearly coupled to an ODE (this simplified model in particular neglects relevant effects such as spacial hole burning or nonlinear gain and index compression). For this special model the mild solutions generate a smooth semiflow in a L^2 space (plus some finite dimensional components for the nonlinear carrier rate ODEs) which will not be the case for nonlinear hyperbolic systems such as the general traveling wave model.

Chapter 10

Nonautonomous traveling wave models

10.1 Assumptions and results

The system we consider is of the following form:

$$\begin{cases} \partial_t\psi(t,x) &=& (-\partial_x\psi_1(t,x), \partial_x\psi_2(t,x)) + G\left(x,\psi(t,x),n(t,x)\right) \\ \partial_t n(t,x) &=& I(t,x) + H\left(x,\psi(t,x),n(t,x)\right) \\ && +\sum_{k=1}^m b_k\chi_{S_k}(x)\left(\fint_{S_k} n(t,y)\,dy - n(t,x)\right) \end{cases} \tag{10.1}$$

with the inhomogeneous, dynamic boundary conditions

$$\begin{cases} \psi_1(t,0) &=& r_0\psi_2(t,0)+\alpha(t) \\ \psi_2(t,l) &=& r_l\psi_1(t,l) \end{cases} \tag{10.2}$$

and the initial values

$$\psi(0,x)=\psi^0(x), \quad n(0,x)=n^0(x). \tag{10.3}$$

The function n is real valued, $\psi=(\psi_1,\psi_2)$ is $\mathbb{C}^2$ valued. They depend on the time $t\in\mathbb{R}$ and space variable $x\in[0,l]$. The interval $[0,l]=\cup_{k=1}^m\overline{S_k}$ is divided into m subsectional intervals $S_k :=]x_{k-1},x_k[$, $x_{k-1}<x_k$, $k=1,...,m$. By χ_{S_k} we denote the characteristic function of S_k, that is $\chi_{S_k}(x):=1$ for $x\in S_k$, $\chi_{S_k}(x):=0$ if $x\notin S_k$. The symbol $\fint_{S_k} := \frac{1}{x_k-x_{k-1}}\int_{S_k}$ denotes the integral average on the subinterval S_k. The nonlinearities $G:]0,l[\times\mathbb{C}^2\times\mathbb{R}\to\mathbb{C}^2$ and $H:]0,l[\times\mathbb{C}^2\times\mathbb{R}\to\mathbb{R}$ are differentiable with respect to the phase variables

(ψ, n), but only measurable and bounded with respect to the spacial variable $x \in [0, l]$. We now list the general assumptions required:

(I) The functions G and H are C^k-Carathéodory functions (see Def. 10.11) on $]0, l[$ from $\mathbb{C}^2 \times \mathbb{R}$ into $\mathbb{C}^2$ and $\mathbb{R}$, respectively.

(II) There exist constants $0 < \nu_1 < \nu_2$ and $c_1, c_2, d_1, d_2 > 0$ such that for all $\psi \in \mathbb{C}^2$ and $a.a.$ $x \in]0, l[$ the relations
$$H(x, \psi, n) \geq -c_1 n, \text{ if } n \leq \nu_1,$$
$$H(x, \psi, n) \leq -c_2 n, \text{ if } n \geq \nu_2,$$
$$H(x, \psi, n) + d_1 \mathfrak{Re} \langle G(x, \psi, n), \psi \rangle \leq -d_2 \left(n + |\psi|^2 \right) \text{ for } n \in \mathbb{R}$$
hold.

(III) For every compact $K \subset \mathbb{R}$ there exists $M > 0$ such that for all $n \in K, \psi \in \mathbb{C}^2$ and $a.a$ $x \in]0, l[$ we have
$$\|G(x, \psi, n)\| \leq M \left(\|\psi\| + 1 \right).$$

(IV) $I \in L^\infty \left( ]0, T[\times]0, l[, \mathbb{R} \right), I(t, x) \geq 0$ for $a.a.$ $(t, x) \in]0, T[\times]0, l[$.

(V) $\alpha \in L^\infty \left( ]0, T[; \mathbb{C} \right)$.

(VI) $r_0, r_l \in \mathbb{C},\ |r_0| < 1,\ |r_l| \leq 1$.

(VII) $n^0 \in L^\infty(]0, l[; \mathbb{R}),\ n^0(x) \geq 0$ for $a.a.$ $x \in]0, l[,\ \psi^0 \in L^\infty(]0, l[; \mathbb{C}^2)$.

(VIII) $b_k \in \mathbb{R}, b_k \geq 0$ for $1 \leq k \leq m$.

Remark 10.1. *Comparing with the results in [35] I have added two new conditions. Condition* (I) *roughly is a smoothness assumption of the nonlinearities with respect to the unknown state variables (but not the space variable) which is needed to prove the smooth dependence on the initial data. The third relation in condition* (II) *implies the apriori estimate* (10.7) *which allows to treat the nonlocal term appearing in the carrier rate equation. When the nonlocal term vanishes, as in [35, 50], this condition can be dropped.*

We assume that $T > 0$ is arbitrarily chosen but fixed. The abbreviation "$a.a.$" stands for "almost all" in the sense of Lebesgue's measure, $\mathfrak{Re}$ denotes the real part of a complex number, $\langle \cdot, \cdot \rangle$ the canonical scalar product in $\mathbb{C}^2$ and $\|\cdot\|$ its corresponding norm.

Definition 10.2. A pair $(\psi, n) \in L^\infty \left( ]0, T[\times]0, l[; \mathbb{C}^2 \times \mathbb{R} \right)$ is a weak solution to (10.1), (10.2), (10.3) if

$$\begin{aligned}
&\int_0^l \left\langle \psi(t, x) - \psi^0(x), \varphi(x) \right\rangle dx \\
&= \int_0^t \left\{ \int_0^l \left[\psi_1(s, x) \overline{(\partial_x \varphi_1)(x)} - \psi_2(s, x) \overline{(\partial_x \varphi_2)(x)} \right.\right. \\
&\quad \left.\left. + \langle G(x, \psi(s, x), n(s, x)), \varphi(x) \rangle \right] dx + \alpha(s) \overline{\varphi_1(0)} \right\} ds
\end{aligned}$$

for all $t \in [0,T]$ and all $\varphi \in W^{1,2}\left(]0,l[,\mathbb{C}^2\right)$ with $\varphi_2(0) = \overline{r_0}\varphi_1(0)$ and $\varphi_1(l) = \overline{r_l}\varphi_2(l)$ and if

$$\begin{aligned} n(t,x) &= n^0(x) + \int_0^t \Bigg\{ I(s,x) + H(x,\psi(s,x),n(s,x)) \\ &\qquad + \sum_{k=1}^m b_k \chi_{S_k}(x) \left[\fint_{S_k} n(s,y)\,dy - n(s,x) \right] \Bigg\} ds \end{aligned} \tag{10.4}$$

for all $t \in [0,T]$ and a.a. $x \in]0,l[$.

Theorem 10.3 (Existence, Uniqueness and smooth Dependence)**.** *Assume* (I) – (VIII)*. There exists a unique weak solution* (ψ,n) *to* (10.1), (10.2), (10.3)*. Moreover, the map*

$$\begin{aligned} &(\psi_0, n_0, I, \alpha) \in L^\infty\left(]0,l[\,;\mathbb{C}^2 \times \mathbb{R}\right) \times L^\infty\left(]0,T[\,\times\,]0,l[\,,\mathbb{R}\right) \times L^\infty\left(]0,T[\,;\mathbb{C}\right) \\ &\mapsto (\psi,n) \in L^\infty\left(]0,T[\,\times\,]0,l[\,;\mathbb{C}^2 \times \mathbb{R}\right) \end{aligned}$$

is C^k-smooth.

We denote the closed subspace in $L^\infty(]0,l[\,,\mathbb{R})$ of section-wise uniformly continuous functions

$$C_P := \left\{ n \in L^\infty(]0,l[;\mathbb{R}) \mid n_{|S_k} \text{ uniformly continuous for } k = 1,2,\ldots,m \right\}. \tag{10.5}$$

Theorem 10.4 (Solution Regularity I)**.** *Assume* (I) – (VIII)*. Let* (ψ,n) *be the weak solution. Then the following holds:*

ı) $\psi \in C\left([0,T]\,;L^2\left(]0,l[\,;\mathbb{C}^2\right)\right)$, $n \in W^{1,\infty}\left(]0,T[\,;L^\infty\left(]0,l[\,;\mathbb{R}\right)\right)$.

ıı) For $t \in [0,T]$ denote $\tilde\psi(t) := \int_0^t \psi(s)\,ds$.
Then for all $t \in [0,T]$ we have $\tilde\psi(t) \in W^{1,2}\left(]0,l[\,;\mathbb{C}^2\right)$ and

$$\tilde\psi_1(t)(0) = r_0\tilde\psi_2(t)(0) + \int_0^t \alpha(s)ds,\ \tilde\psi_2(t)(l) = r_l\tilde\psi_1(t)(l).$$

ııı) Let $\alpha \in W^{1,2}\left(]0,T[\,;\mathbb{C}\right)$, $\psi^0 \in W^{1,2}(]0,l[;\mathbb{C}^2)$ and suppose

$$\psi_1^0(0) = r_0\psi_2^0(0) + \alpha(0),\ \psi_2^0(l) = r_l\psi_1^0(l). \tag{10.6}$$

Then

$$\psi \in C\left([0,T];W^{1,2}\left(]0,l[\,;\mathbb{C}^2\right)\right)\cap C^1\left([0,T]\,;L^2\left(]0,l[\,;\mathbb{C}^2\right)\right)$$

and (10.1), (10.2) *hold for* $t\in[0,T]$ *in the classical sense.*
If $I\in C\left([0,T];L^\infty(]0,T[\,;\mathbb{R})\right)$ *then* $n\in C^1([0,T];L^\infty(]0,l[;\mathbb{R}))$.

iv) Suppose $\psi^0\in C\left([0,l]\,;\mathbb{C}^2\right)$, $\alpha\in C\left([0,T]\,;\mathbb{C}\right)$ *and* (10.6). *Then*

$$\psi\in C\left([0,T]\times[0,l]\,;\mathbb{C}^2\right)\ \text{and (10.2) is satisfied pointwise.}$$

Further assume $n^0\in C_P$, $I(t)\in C_P$ *for a.a.* $t\in[0,T]$ *and*

(IX) $\qquad H(\cdot,\psi,n)\in C_P$ *for all* $\psi\in\mathbb{C}^2$ *and* $n\in\mathbb{R}$.

Then $n\in C([0,T];C_P)$. *If* $I\in C([0,T];C_P)$, *then* $n\in C^1([0,T];C_P)$.

Theorem 10.5 (A priori estimates)**.** *Suppose* (I) $-$ (VIII). *Let* (ψ,n) *denote the weak solution.*

For all $t\in[0,T]$

$$\int_0^l n(t,x)dx+\frac{d_1}{2}\left\|\psi(t)\right\|_{L^2}^2\le\mu+\max\left\{\int_0^l n^0(x)dx+\frac{d_1}{2}\left\|\psi^0\right\|_{L^2}^2-\mu,0\right\}e^{-ct},\tag{10.7}$$

where

$$c:=\min\left\{d_2,\frac{2d_2}{d_1}\right\},\qquad \mu:=c^{-1}\left(\frac{d_1}{2(1-|r_0|^2)}\left\|\alpha\right\|_{L^\infty}^2+L\left\|I\right\|_{L^\infty}\right).$$

Moreover, for all $t\in[0,T]$ *and a.a.* $x\in\,]0,l[$

$$\min\left\{n^0(x),\nu_1\right\}e^{-(c_1+b)t}\le n(t,x)\le N+\max\left\{n^0(x)-N,0\right\}e^{-c_2t},\tag{10.8}$$

where

$$N:=\max\left\{\nu_2,c_2^{-1}\left(\left\|I\right\|_{L^\infty}+\max_{1\le k\le m}\left(\frac{b_k}{|S_k|}\right)\cdot\max\left\{\mu,\int_0^l n^0(x)dx+\frac{d_1}{2}\left\|\psi^0\right\|_{L^2}^2\right\}\right)\right\}$$

and

$$b:=\max_{1\le k\le m}(b_k).$$

If the data ψ^0 and α are $W^{1,2}$-smooth, then Theorem 10.4, *iii*), states that the weak solution ψ will be $W^{1,2}$-smooth with respect to the spacial variable x. Of course, under assumptions of piecewise smoothness for the data entering the equation for n, this smoothness of ψ carries over to n via the coupling of ψ and n in (10.1). Theorem 10.6 states this precisely. Let

$$W_P^{1,2}:=\left\{n\in L^\infty(]0,l[;\mathbb{R})\mid n_{|S_k}\in W^{1,2}\left(S_k;\mathbb{R}\right)\ k=1,2,\dots,m\right\}\tag{10.9}$$

denote the Hilbert space of piecewise $W^{1,2}$ functions.

Theorem 10.6 (Solution Regularity II, piecewise smoothness of n)**.** *Suppose* (I) – (VIII) *and*

(X) $H_{|S_k\times\mathbb{C}^2\times\mathbb{R}} \in C^1\left(\overline{S_k}\times\mathbb{C}^2\times\mathbb{R};\mathbb{R}\right)$ *for* $1\le k\le m$.

(XI) *For all compact* $K\subset\mathbb{R}$ *there exists* $\Lambda>0$ *such that* $\|\partial H(x,\psi,n_1)-\partial H(x,\psi,n_2)\|\le\Lambda|n_1-n_2|$ *for* $x\in S_k$, $\psi\in\mathbb{C}$, $n_1,n_2\in K$.

(XII) *There exists a constant* $\tau>0$ *such that for all compact* $K\subset\mathbb{R}$ *there exists* $R>0$ *with*

$$\begin{aligned}\partial_x H(x,\psi,n)\tilde{n}+\partial_n H(x,\psi,n)\tilde{n}^2+\partial_\psi H(x,\psi,n)\tilde{\psi}\tilde{n}\\ \le R\left(1+|\tilde{n}|+\left\|\tilde{\psi}\right\|+\left\|\tilde{\psi}\right\||\tilde{n}|+\left\|\tilde{\psi}\right\|^2\right)-\tau\tilde{n}^2\end{aligned}$$

for all $x\in S_k$, $1\le k\le m$, $\psi\in\mathbb{C}^2$, $\tilde{\psi}\in\mathbb{C}^2$, $n\in K$ *and* $\tilde{n}\in\mathbb{R}$.

If $\alpha\in W^{1,2}(]0,T[;\mathbb{C})$, $\psi^0\in W^{1,2}(]0,l[;\mathbb{C}^2)$, (10.6) *is satisfied,* $n^0\in W^{1,2}_P$ *and* $I\in C\left([0,T];W^{1,2}_P\right)$, *then*

$$n\in C^1([0,T];W^{1,2}_P).$$

In (XI) the ∂H denotes the total derivative of H with respect to all variables (x,ψ,n). We note that all assumptions (I) – (XII) are fulfilled in applications, see Section 3.2.

Define the phase space

$$\mathfrak{P}:=\{\psi\in C([0,l],\mathbb{C}^2)\mid\psi_1(0)=r_0\psi_2(0),\psi_2(l)=r_l\psi_1(l)\}\times C_P. \quad (10.10)$$

The following Theorems 10.7-10.8 are a direct consequence of Theorems 10.3-10.5:

Theorem 10.7 (C^k-Semiflow property)**.** *Suppose* (I) – (IX).
In the autonomous case, that is $\alpha=0$ *and* $I=0$*, the weak solutions generate a smooth semiflow in the function space* $\mathfrak{P}$*. The operator* $S^t:\mathfrak{P}\to\mathfrak{P}$*, defined through*

$$S^t\left(\psi^0,n^0\right):=(\psi(t),n(t))$$

for $t\ge0$ *and* $(\psi^0,n^0)\in\mathfrak{P}$*, where* $(\psi(t),n(t))$ *denotes the weak solution corresponding to the initial values* (ψ^0,n^0)*, has the following properties*

i) $(t,\psi,n)\mapsto S^t(\psi,n)$ *is continuous from* $[0,\infty[\times\mathfrak{P}$ *into* $\mathfrak{P}$,
ii) $S^t:\mathfrak{P}\to\mathfrak{P}$ *is* C^k *smooth,*
iii) $S^{t+s}=S^t\circ S^s$, $t,s\in\mathbb{R}$, $t,s\ge0$,
iv) S^0 *is the identity operator on* $\mathfrak{P}$.

Finally consider the nonautonomous case: assume $\alpha \in C(\mathbb{R};\mathbb{C})$ and $I \in L^\infty(\mathbb{R};C_P)$. Let $G \in C(\mathbb{R}_+ \times C_p; C([0,l],\mathbb{C}^2))$ be such that G satisfies the inhomogeneous boundary condition $G(t,n)_1(0) = r_0 G(t,n)_2(0) + \alpha(t)$ and $G(t,n)_2(l) = r_l G(t,n)_1(l)$ for $t \geq 0$. For $t \geq s$ define $X\left(t,s,(\psi^0,n^0)\right) := (\psi(t-s), n(t-s))$, where (ψ,n) is the weak solution in the sense of Definition 10.2 to the initial data ψ^0, n^0 and $\alpha(s+\cdot)$. Then $X\left(t,s,(\psi^0,n^0)\right)$, $s \leq t$, can be interpreted as the weak solution at time $t+s$ corresponding to the initial condition $\psi(s,x) = \psi^0(x)$, $n(s,x) = n^0(x)$ for *a.a.* $x \in\,]0,l[$ at time s. Define the operator $Y(t,s) : \mathfrak{P} \to \mathfrak{P}$, through

$$\begin{aligned} Y\left(t,s\right)\left(\psi^0,n^0\right) :=& X\left(t,s,\left(\psi^0 + G(s,n^0),n^0\right)\right) \\ & - \begin{pmatrix} G(t,\Pi_n X\left(t,s,(\psi^0 + G(s,n^0),n^0)\right)) \\ 0 \end{pmatrix} \end{aligned}$$

for $t \geq s$ and $(\psi^0,n^0) \in \mathfrak{P}$ ($\Pi_n X$ denotes the n-component of X). The operator $Y(t,s)$ maps $\mathfrak{P}$ into itself, hence the function G homogenizes the boundary condition (10.2). From the definition of Y one verifies that Y has the process property stated in Theorem 10.8.
Suppose that for $t \geq 0$ the map

$$n \in C_p \mapsto G(t,n) \in C([0,l],\mathbb{C}^2) \quad \text{is } C^k \text{ smooth.}$$

Theorem 10.8. *The operator $Y\left(t,s\right)$ is a C^k smooth two parameter nonautonomous process satisfying*

- *i) for $t \geq s$ the map $p \in \mathfrak{P} \mapsto Y(t,s,p) \in \mathfrak{P}$ is C^k smooth,*
- *ii) the map $(t,s,p) \mapsto Y(t,s,p)$ is continuous from $\{(t,s) \in \mathbb{R}^2 \mid s \leq t\} \times \mathfrak{P}$ into $\mathfrak{P}$,*
- *iii) $Y(s,s,\cdot)$ is the identity operator on $\mathfrak{P}$,*
- *iv) for $t \geq s \geq r$ the process property $Y(t,s,Y(s,r,p)) = Y(t,r,p)$ holds.*

Example 10.9. *In applications one has to choose an appropriate homogenization G. We give two examples of choices of G:*
(i) $G(t,n)(x) = \frac{l-x}{l}\begin{pmatrix}\alpha(t)\\ 0\end{pmatrix}$,
(ii) For each $n \in C_p$ $G(t,n)$ solves

$$\left\{\begin{array}{rcl} \partial_t G(t,n) &=& \mathfrak{A}(n)G(t,n), \\ G_1(t,n)_{|x=0} &=& r_0 G_2(t,n)_{|x=0} + \alpha(t), \\ G_2(t,n)_{|x=l} &=& r_l G_1(t,n)_{|x=l} \end{array}\right.$$

with suitable initial data.
The simple choice (i) has been used by Sandstede and Peterhof in [50]. Choice (ii) is used in section 11 to perform a center manifold reduction for the traveling wave equation with optical injection.

The process Y can be equivalently written as a skew product semiflow Z^t on the trivial Banach bundle $\mathfrak{P} \times [0, \infty[$ if one defines for $(p, \theta) \in \mathfrak{P} \times [0, \infty[$

$$Z^t\,(p, \theta) := (Y(\theta + t, \theta, p), \theta + t)\,,\ p \in \mathfrak{P},\ (\theta, t \geq 0).$$

We extend Z^t onto the Banach space $\mathfrak{P}_e := \mathfrak{P} \times \mathbb{R}$ by setting

$$Z^t(p, \theta) = \begin{cases} Z^t\,(p, \theta) & , \theta \geq 0 \\ \left(\Pi_p Z^{t+\theta}\,(p, 0)\,, \theta + t\right) & , \theta < 0, \theta + t \geq 0 \\ (p, \theta + t) & , \theta < 0, \theta + t < 0. \end{cases}$$

Then we can state the following

Theorem 10.10. *If $\alpha \in C^k([0, \infty[\,, \mathbb{R})$ and $G(t, n)$ is of class C^k in both variables (t, n), then the operator Z^t is a C^k smooth semiflow on $\mathfrak{P}_e$.*

In assumption (I) we require that both G and H are C^k-Carathéodory functions, which we define next.

Definition 10.11 (C^k-Carathéodory functions)**.** Let V, W be finite dimensional vector spaces and $k \in \mathbb{N}$. A function $S :]0, l[\times V \to W$, $S = S(x, v)$, $x \in]0, l[$, $v \in V$, is called a C^k Carathéodory function iff S satisfies the following three conditions:

i) For *a.a.* $x \in]0, l[$ $S(x, \cdot) \in C^k(V; W)$ and $S(\cdot, v)$ is measurable for all $v \in V$.

ii) For all compact $K \subset V$ there exists a constant $M > 0$ such that $\left\| \frac{\partial^i S(x,v)}{\partial v^i} \right\| \leq M$ for $0 \leq i \leq k$, all $v \in K$ and *a.a.* $x \in]0, l[$.

iii) For all compact $K \subset V$ and $\epsilon > 0$ there exists a $\delta > 0$ such that for all $v_1 \in K$, $v_2 \in V$ with $\|v_1 - v_2\| < \delta$ and *a.a.* $x \in]0, l[$ we have $\left\| \frac{\partial^k S(x,v_1)}{\partial v^k} - \frac{\partial^k S(x,v_2)}{\partial v^k} \right\| < \epsilon$.

10.2 Variation of constants formula and proofs for the truncated problem

Let $S :]0, l[\times V \to W$ be a C^k Carathéodory function. Denote the corresponding superposition operator

$$\mathfrak{S} : \mathcal{M}(]0, l[\,; V) \to \mathcal{M}(]0, l[\,; W),\ \mathfrak{S}(v)(x) := S(x, v(x)),\ a.a.\, x \in]0, l[\,, \tag{10.11}$$

where $\mathcal{M}(]0, l[\,; V)$ denotes the linear space of measurable functions defined almost everywhere on $]0, l[$ with values in V. We need the following easy to prove differentiability property of $\mathfrak{S}$.

Proposition 10.12. *(see [24]) The superposition operator $\mathfrak{S}$ maps $L^\infty(]0,l[\,;V)$ C^k-smoothly into $L^\infty(]0,l[\,;W)$.*

In the following we will frequently make use of the superposition operators

$$\begin{aligned} \mathfrak{G} &\in C^k(L^\infty(]0,l[\,;\mathbb{C}^2\times\mathbb{R}), L^\infty(]0,l[\,;\mathbb{C}^2)) \\ \mathfrak{H} &\in C^k(L^\infty(]0,l[\,;\mathbb{C}^2\times\mathbb{R}), L^\infty(]0,l[\,;\mathbb{R})) \end{aligned}$$

generated by G and H through (10.11). Also the following operators

$$\mathfrak{B}\in\mathcal{L}\left(L^\infty(]0,l[\,;\mathbb{R})\right),\ \mathfrak{I}\in L^\infty(]0,T[\,,L^\infty(]0,l[;\mathbb{R}))$$

will appear which are defined through

$$\begin{aligned} \mathfrak{B}(n)(x) &:= \sum_{k=1}^m b_k\chi_{S_k}(x)\left(\fint_{S_k} n(y)\,dy - n(t,x)\right) \text{ for a.a. } x\in\,]0,l[\\ \mathfrak{I}(t)(x) &:= I(t,x) \text{ for } x\in\,]0,l[\,. \end{aligned}$$

Here $\mathcal{L}(L^\infty(]0,l[;\mathbb{R}))$ denotes the space of bounded linear mappings of $L^\infty(]0,l[\,;\mathbb{R})$ into itself.

For establishing the variation of constants formula for our notion of weak solution we first need some definitions:

For $\eta\in\mathbb{R}$ let

$$L^2_\eta(]0,\infty[,\mathbb{C}) := \left\{ f:]0,\infty[\to\mathbb{C} \mid f \text{ measurable } \int_0^\infty |f(x)|^2(1+x^2)^\eta dx<\infty\right\}$$

denote the Hilbert space of complex valued weighted square integrable functions on $]0,\infty[$ with weight $(1+x^2)^\eta$ with respect to the Lebesque measure on $]0,\infty[$. We denote its scalar product by $\langle f,g\rangle_{L^2_\eta} := \int_0^\infty f(x)\overline{g(x)}(1+x^2)^\eta dx$. Let $W^{1,2}_\eta$ denote the corresponding Sobolev space of functions $f\in L^2_\eta(]0,\infty[,\mathbb{C})$ with distributional derivative in $L^2_\eta(]0,\infty[,\mathbb{C})$. Define the extended space

$$X_e := L^2(]0,l[;\mathbb{C}^2)\times L^2(]0,l[;\mathbb{R})\times L^2_\eta(]0,\infty[;\mathbb{C}) \tag{10.12}$$

with some fixed $\eta<-0.5$. This choice of η guarantees that $L^\infty(]0,\infty[;\mathbb{C})$ is continuously embedded in $L^2_\eta(]0,\infty[;\mathbb{C})$. Put

$$T_e(t)\left(\psi_1^0,\psi_2^0,n^0,a\right) := \left(\psi_1(t),\psi_2(t),n^0,\tau_t a\right),$$

where $\tau_t a(x) := a(t+x)$ denotes the left translation of a by t and ψ_1, ψ_2 are given by

$$\begin{aligned}
\psi_1(t,x) \quad &:= \quad \begin{cases} \psi_1^0(x-t) & , \text{ for } a.a.\ x \in \,]t,l[\\ r_0\psi_2^0(t-x) + a(t-x) & , \text{ for } a.a.\ x \in \,]0,t[\end{cases} \\
\psi_2(t,x) \quad &:= \quad \begin{cases} \psi_2^0(x+t) & , \text{ for } a.a.\ x \in \,]0,l-t[\\ r_l\psi_1^0(2l-x-t) & , \text{ for } a.a.\ x \in \,]l-t,l[\,. \end{cases}
\end{aligned} \tag{10.13}$$

Extend $T_e(t), t \in [0,l]$ to the whole positive axis $[0,\infty[$ by defining for $t > l$ inductively $T_e(t) := T_e(t-l)T_e(l)$. Then it is easy to verify that $T_e(\cdot)$ is a C_0 semigroup of bounded operators in X_e with infinitesimal generator

$$A_e := \operatorname{diag}(-\partial_x, \partial_x, 0, \partial_x)$$

having the domain

$$\mathfrak{D}(A_e) := \{(\psi, n, a) \in W^{1,2}(]0,l[;\mathbb{C}^2) \times L^2(]0,l[;\mathbb{R}) \times W^{1,2}_\eta(]0,\infty[;\mathbb{C}) \mid$$

$$\psi_1(0) = r_0\psi_2(0) + a(0), \psi_2(l) = r_l\psi_1(l)\}.$$

Set

$$T(t)(\psi^0) := \Pi_\psi T_e(t)(\psi^0, 0, 0)$$

for $t \geq 0$ and $\psi \in L^2(]0,l[\,,\mathbb{C}^2)$, where Π_ψ is the projection onto the first variable ψ. Then $T(t)$ is a C_0 semigroup of contractions in $L^2(]0,l[\,,\mathbb{C}^2)$ with infinitesimal generator

$$A := \operatorname{diag}(-\partial_x, \partial_x)$$

and domain

$$\mathfrak{D}(A) := \left\{\psi \in W^{1,2}(]0,l[\,;\mathbb{C}^2) \mid \psi_1(0) = r_0\psi_2(0),\ \psi_2(l) = r_l\psi_1(l)\right\}.$$

Let $\prod_{(\psi,n)}$ denote the projection of X_e onto $L^2(]0,l[;\mathbb{C}^2 \times \mathbb{R})$ by dropping the trivial last component. Then the following Lemma holds

Lemma 10.13. *The pair (ψ, n) is a weak solution to* (10.1), (10.2), (10.3) *iff (ψ, n) satisfies the variation of constants formula*

$$\begin{pmatrix} \psi(t) \\ n(t) \end{pmatrix} = \prod_{(\psi,n)} T_e(t) \begin{pmatrix} \psi^0 \\ n^0 \\ \alpha \end{pmatrix} + \int_0^t \begin{pmatrix} T(t-s)\mathfrak{G}(\psi(s), n(s)) \\ \mathfrak{I}(s) + \mathfrak{B}n(s) + \mathfrak{H}(\psi(s), n(s)) \end{pmatrix} ds \tag{10.14}$$

for all $t \in [0,T]$.

Proof. Straightforward calculations yield that the adjoint A_e^* of A_e is the closed densely defined operator

$$A_e^*(\psi, n, a) = (\partial_x\psi_1, -\partial_x\psi_2, 0, -(1+x^2)^{-\eta}\partial_x(a(x)\cdot(1+x^2)^\eta)) =: (A_I^*, 0, A_a^*)$$

with the domain

$$\begin{aligned}\mathfrak{D}(A_e^*) &= \Big\{(\psi, n, a) \in W^{1,2}(]0,l[;\mathbb{C}^2) \times L^2(]0,l[;\mathbb{R}) \times W_\eta^{1,2}(]0,\infty[;\mathbb{C}) \mid \\ &\qquad \psi_2(0) = \overline{r_0}\psi_1(0), \psi_1(l) = \overline{r_l}\psi_2(l), a(0) = \psi_1(0)\Big\}.\end{aligned}$$

We trivially extend α on the whole axis $[0,\infty[$ by setting α to zero on $[T,\infty[$. Then define $a \in C([0,\infty[; L_\eta^2([0,\infty[;\mathbb{C}))$, $a(t) := \tau_t\alpha, t \in [0,\infty[$. By definition (ψ, n) is a weak solution iff $(\psi, n) \in L^\infty(]0,T[\times]0,l[;\mathbb{C}^2\times\mathbb{R})$ and for all $(\varphi, 0, \varphi_a) \in D(A_e^*)$ the equation

$$\begin{aligned}&\langle\psi(t)-\psi_0,\varphi\rangle_{L^2} + \langle a(t)-a(0),\varphi_a\rangle_{L_\eta^2} \\ = \;& \lim_{\rho\to 0}\Big\{\int_0^t \Big(\langle\psi(s), A_I^*\varphi\rangle_{L^2} + \langle\mathfrak{G}(\psi(s),n(s)),\varphi\rangle_{L^2} + \alpha_\rho(s)\overline{\varphi_1(0)}\Big)\,ds \\ &+ \int_0^t \langle(\partial_x\alpha_\rho)(s+\cdot),\varphi_a\rangle_{L_\eta^2} ds\Big\} \\ = \;& \lim_{\rho\to 0}\Big\{\int_0^t \Big(\langle\psi(s), A_I^*\varphi\rangle_{L^2} + \langle\mathfrak{G}(\psi(s),n(s)),\varphi\rangle_{L^2} + \langle\alpha_\rho(s+\cdot), A_a^*\varphi_a\rangle_{L_\eta^2}\Big)ds\Big\} \\ = \;& \int_0^t \Big(\langle\psi(s), A_I^*\varphi\rangle_{L^2} + \langle\mathfrak{G}(\psi(s),n(s)),\varphi\rangle_{L^2} + \langle a(s), A_a^*\varphi_a\rangle_{L_\eta^2}\Big)ds\end{aligned}$$

holds and (10.4) is satisfied for n. Here

$$\alpha_\rho(x) := \int_0^T m_\rho(x-y)\alpha(y)dy, \quad m_\rho(y) := \frac{m_0(\rho y)}{\rho} \quad (x, y \in \mathbb{R})$$

denotes the mollification of α with parameter $\rho > 0$ with respect to some mollifier $m_0 \in C^\infty(\mathbb{R})$, $m_0 \geq 0$, supp $m_0 \subset B_1$, $\int_{-\infty}^\infty m_0(y)dy = 1$. It was used above in order to perform partial integration. For the first equality one should note that for $\alpha \in L_\eta^2, \alpha_\rho \in W_\eta^{1,2}$ and $\lim_{x\to\infty}\alpha_\rho(x)\,(1+x^2)^\eta = 0$. The above calculations together with [2] proves: (ψ, n) is a weak solution iff (10.14) holds for $t \in [0,T]$. □

We now define the truncated problem to (10.1)-(10.3):

Definition 10.14. Let $\delta \in\,]0,\infty[$ be arbitrary. Let $T_1^\delta : \mathbb{R} \to \mathbb{R}$ be a C^∞ function with $T_1^\delta(n) = n$ for $|n| \leq \delta^{-1}$ and $T_1^\delta(n) = 2\delta^{-1}|n|^{-1}n$ for

$|n| \geq 2\delta^{-1}$. Similarly let $T_2^\delta : \mathbb{C}^2 \to \mathbb{C}^2$ be C^∞ with $T_2^\delta(v) = v$ for $\|v\| \leq \delta^{-1}$ and $T_2^\delta(v) = 2\delta^{-1} \|v\|^{-1} v$ for $\|v\| \geq 2\delta^{-1}$. Define the truncated nonlinearities

$$G^\delta :]0, l[\times \mathbb{C}^2 \times \mathbb{R} \to \mathbb{C}^2, \quad G^\delta(x, \psi, n) := G(x, T_2^\delta(\psi), T_1^\delta(n)),$$

$$H^\delta :]0, l[\times \mathbb{C}^2 \times \mathbb{R} \to \mathbb{R}, \quad H^\delta(x, \psi, n) := H(x, T_2^\delta(\psi), T_1^\delta(n)).$$

Then G^δ, H^δ are C^k-smooth Carathéodory functions generating the smooth superposition operators $\mathfrak{G}^\delta, \mathfrak{H}^\delta$. The truncated problem reads:

$$\begin{cases} \partial_t \psi^\delta(t,x) &= \left(-\partial_x \psi_1^\delta(t,x), \partial_x \psi_2^\delta(t,x)\right) + G^\delta(x, \psi^\delta(t,x), n^\delta(t,x)) \\ \partial_t n^\delta(t,x) &= I(t,x) + H^\delta(x, \psi^\delta(t,x), n^\delta(t,x)) \\ & \qquad + \sum_{k=1}^m b_k \chi_{S_k}(x) \left(\displaystyle\fint_{S_k} n^\delta(t,y)\, dy - n^\delta(t,x) \right) \end{cases} \tag{10.15}$$

with the same boundary conditions and initial values:

$$\psi_1^\delta(t,0) = r_0 \psi_2^\delta(t,0) + \alpha(t), \; \psi_2^\delta(t,l) = r_l \psi_1^\delta(t,l) \tag{10.16}$$

$$\psi^\delta(0,x) = \psi^0(x), \; n^\delta(0,x) = n^0(x). \tag{10.17}$$

Weak solutions to (10.15)-(10.17) are defined analogously to Def. 10.2.

Remark 10.15. *After truncation G^δ and H^δ satisfy condition ıı) of Definition 10.11 globally. In particular G^δ and H^δ become globally Lipschitz uniformly with respect to $x \in]0, l[$, that is for each $\delta > 0$ there exists a constant Λ such that for all $\psi_1, \psi_2 \in \mathbb{C}^2$, $n_1, n_2 \in \mathbb{R}$ and a.a. $x \in]0, l[$*

$$\begin{aligned} \left\|G^\delta(x, \psi_1, n_1) - G^\delta(x, \psi_2, n_2)\right\| + |H^\delta(x, \psi_1, n_1) - H^\delta(x, \psi_2, n_2)| \\ \leq \Lambda \left(\|\psi_1 - \psi_2\| + |n_1 - n_2| \right). \end{aligned}$$

The superposition operators $\mathfrak{G}^\delta$ and $\mathfrak{H}^\delta$ become globally Lipschitz from $L^p(]0, l[\,; \mathbb{C}^2 \times \mathbb{R})$ into $L^p(]0, l[\,; \mathbb{C}^2)$ and $L^p(]0, l[\,; \mathbb{R})$, respectively, for any $p \in [1, \infty]$.

Lemma 10.16. *For each $\delta > 0$ the Theorems 10.3 and 10.4 hold for the weak solution $\left(\psi^\delta, n^\delta\right)$ to the truncated problem* (10.15)-(10.17)

Proof. Denote the weak solution space

$$\mathfrak{X} := L^\infty \left( ]0, T[\times]0, l[\,; \mathbb{C}^2 \times \mathbb{R} \right).$$

Extend it to

$$\mathfrak{X}_e := \mathfrak{X} \times L^\infty(]0, l[; \mathbb{C}^2 \times \mathbb{R}) \times L^\infty(]0, T[; \mathbb{C}) \times L^\infty(]0, T[\times]0, l[; \mathbb{R})$$

by attaching the corresponding spaces of the initial data ψ^0, n^0 and the dynamic data α, I. Both $\mathfrak{X}$ and $\mathfrak{X}_e$ are equipped with the corresponding L^∞ norms. Define the operator $\mathfrak{F} : \mathfrak{X}_e \to \mathfrak{X}$,

$$\mathfrak{F}\begin{pmatrix}\psi\\ n\\ \psi^0\\ n^0\\ \alpha\\ I\end{pmatrix}(t) \;:=\; \begin{pmatrix}\psi(t)\\ n(t)\end{pmatrix} - \prod_{(\psi,n)}\left\{T_e(t)\begin{pmatrix}\psi^0\\ n^0\\ \alpha\end{pmatrix} + \int_0^t T_e(t-s)\begin{pmatrix}\mathfrak{G}^\delta(\psi(s),n(s))\\ \mathfrak{I}(s)+\mathfrak{B}n(s)+\mathfrak{H}^\delta(\psi(s),n(s))\\ 0\end{pmatrix} ds\right\}.$$

For fixed ψ^0, n^0, α, I denote $\mathfrak{F}_0 : \mathfrak{X} \to \mathfrak{X}$,

$$\mathfrak{F}_0(\psi,n)(t) := (\psi(t),n(t)) - (\mathfrak{F}(\psi,n,\psi^0,n^0,\alpha,I))(t).$$

By Lemma 10.13 the truncated problem (10.15)-(10.17) has a unique weak solution (ψ^δ, n^δ) corresponding to the data ψ^0, n^0, α, I iff $\mathfrak{F}_0$ has a unique fixed point in $\mathfrak{X}$. By Remark 10.15 $\mathfrak{G}^\delta$ and $\mathfrak{H}^\delta$ are globally Lipschitz from $L^\infty(]0,l[\,;\mathbb{C}^2\times\mathbb{R})$ into $L^\infty(]0,l[\,,\mathbb{C}^2)$ and $L^\infty(]0,l[\,,\mathbb{R})$, respectively, with some Lipschitz constant Λ depending on the truncation parameter δ. Thus from the explicit formula (10.13) for the semigroup $T_e(t)$ it follows by induction that for $l\in\mathbb{N}$, $(\psi_a,n_a),(\psi_b,n_b)\in\mathfrak{X}$

$$\left\|\mathfrak{F}_0^l(\psi_a,n_a)-\mathfrak{F}_0^l(\psi_b,n_b)\right\|_{\mathfrak{X}} \le \frac{(\Lambda T)^l}{l!}\left\|(\psi_a,n_a)-(\psi_b,n_b)\right\|_{\mathfrak{X}}.$$

Hence, for l sufficiently large $\mathfrak{F}_0^l$ is a contraction in the Banach space $\mathfrak{X}$. By a generalization of Banachs fixed point theorem $\mathfrak{F}_0$ has a unique fixed point (ψ^δ, n^δ) in $\mathfrak{X}$. This proves the existence and uniqueness part of Theorem 10.3.

From the assumptions that G, H are C^k Caratheódory functions (Definition 10.11) and Proposition 10.12 we get that $\mathfrak{F}$ maps $\mathfrak{X}_e$ C^k-smoothly into $\mathfrak{X}$. The existence and uniqueness of the weak solutions just proved is equivalent to saying that for any ψ^0, n^0, α, I there exists a unique $(\psi,n)\in\mathfrak{X}$ such that $\mathfrak{F}(\psi,n,\psi^0,n^0,\alpha,I)=0$. The partial derivative of $\mathfrak{F}$ with respect to

(ψ, n) operating on $v = (v_\psi, v_n) \in \mathfrak{X}$ satisfies the formula

$$\left(\frac{\partial \mathfrak{F}}{\partial(\psi, n)}(\psi, n, \psi^0, n^0, \alpha, I)(v_\psi, v_n)\right)(t)$$

$$= \begin{pmatrix} v_\psi(t) \\ v_n(t) \end{pmatrix} - \prod_{(\psi,n)} \int_0^t T_e(t-s) \begin{pmatrix} \left(\partial \mathfrak{G}^\delta(\psi(s), n(s))\right) v(s) \\ \mathfrak{B} v_n(s) + \partial \mathfrak{H}^\delta(\psi(s), n(s)) v(s) \\ 0 \end{pmatrix} ds.$$

Again it follows by Banachs fixed point theorem that for any $w \in \mathfrak{X}$ there exists a unique $v \in \mathfrak{X}$ such that

$$v(t) = \prod_{(\psi,n)} \int_0^t T_e(t-s) \begin{pmatrix} \left(\partial \mathfrak{G}^\delta\left(\psi(s), n(s)\right)\right) v(s) \\ \mathfrak{B} v_n(s) + \partial \mathfrak{H}^\delta\left(\psi(s), n(s)\right) v(s) \\ 0 \end{pmatrix} ds + w(t) \quad (t \in [0, T]).$$

Banachs open mapping theorem implies that $\partial_{(\psi,n)} \mathfrak{F}$ is an isomorphism from $\mathfrak{X}$ onto $\mathfrak{X}$. Hence Theorem 10.3 is a consequence of the implicit function theorem.

Statement $\imath)$ of Theorem 10.4 follows directly from Definition 10.2 and the variation of constants formula.

We now prove $\imath\imath)$: As in the proof of Lemma 10.13 trivially extend α to the whole $[0, \infty[$ by setting α almost everywhere to zero on $[T, \infty[$ and define

$$a \in C\left(]0, \infty[\,; L^2_\eta(]0, \infty[\,; \mathbb{C})\right), \; a(s)(x) := \tau_s \alpha(x),$$

for $s \geq 0$ and $a.a.\ x \in]0, \infty[$, where τ_s denotes the left translation of α again. Integrating the variation of constants formula (10.14) with respect to time yields

$$\int_0^t \begin{pmatrix} \psi(s) \\ n(s) \\ a(s) \end{pmatrix} ds = \int_0^t T_e(s) \begin{pmatrix} \psi^0 \\ n^0 \\ \alpha \end{pmatrix} ds$$

$$+ \int_0^t \int_0^s T_e(s-r) \begin{pmatrix} \mathfrak{G}^\delta(\psi(r), n(r)) \\ \mathfrak{I}(r) + \mathfrak{B} n(r) + \mathfrak{H}^\delta(\psi(r), n(r)) \\ 0 \end{pmatrix} dr ds \;\; (t \in [0, T]).$$

From this formula and the uniform continuity $(t, p) \mapsto T_e(t)p$ of the C_0 semigroup T_e one easily proves that the limit

$$\lim_{h \downarrow 0} \frac{T_e(h) - I}{h} \int_0^t (\psi(s), n(s), a(s)) ds$$

exists in X_e (see (10.12)) for each $t \in [0,T]$. This is equivalent to

$$\int_0^t (\psi(s), n(s), a(s))ds \in \mathfrak{D}(A_e)$$

or statement $\imath\imath$).

Now assume $\alpha \in W^{1,2}(]0,T[\,;\mathbb{C})$, $\psi^0 \in W^{1,2}(]0,l[;\mathbb{C}^2)$ and (10.6). Extend α to the whole $]0,\infty[$ such that the extension lies in $W^{1,2}_\eta(]0,\infty[\,;\mathbb{C})$. Then (ψ^0, n^0, α) belongs to $\mathfrak{D}(A_e)$. Since X_e is reflexive it follows from Proposition 4.3.9 in [15] that

$$(\psi, n, \tau_t\, \alpha) \in C([0,T]; \mathfrak{D}(A_e)) \cap C^1([0,T]; X_e),$$

which proves $\imath\imath\imath$).

We prove Theorem 10.4, $\imath v$). Choose sequences $\psi_i^0 \in W^{1,2}(]0,l[;\mathbb{C}^2)$, $\alpha_i \in W^{1,2}(]0,T[\,;\mathbb{C})$, $i \in \mathbb{N}$, which satisfy the boundary condition $\psi^0_{i\,1}(0) = r_0\psi^0_{i\,2}(0) + \alpha_i(0)$ and $\psi^0_{i\,2}(l) = r_l\psi^0_{i\,1}(l)$, and have the property that $\psi_i^0 \to \psi^0$ in $L^\infty(]0,l[\,;\mathbb{C}^2)$ and $\alpha_i \to \alpha$ in $L^\infty(]0,T[\,;\mathbb{C})$. By Theorem 10.4 $\imath\imath\imath$) $\psi_i \in C([0,T]\times[0,l];\mathbb{C}^2)$, and by Theorem 10.3 the solution sequences (ψ_i, n) converge to (ψ, n) in $\mathfrak{X}$. Thus $\psi \in C([0,T]\times[0,l];\mathbb{C}^2))$ and ψ satisfies (10.2) pointwise in $[0,T]$. By assumption (IX) on H the superposition operator $\mathfrak{H}^\delta$ keep the space C_P invariant. The ψ-part of the fixed point (ψ, n) of the operator $\mathfrak{F}_0$ is uniformly continuous on $[0,T]\times[0,l]$. Since $n^0 \in C_P$ and the part n can be obtained by a fixed point iteration in the space $C([0,T];C_P)$ alone, keeping ψ unchanged, we obtain that $n \in C([0,T];C_P)$. The relation $n \in C^1([0,T],C_P)$ follows directly from (10.4) if $I \in C([0,T];C_P)$. $\square$

Remark 10.17. *(Lipschitz dependence of solutions with respect to L^2) Because of Remark 10.15 Gronwall's Lemma applied to* (10.14) *easily shows that there exists a constant $C = C(\delta, T)$ such that*

$$\left\|(\psi, n) - (\tilde{\psi}, \tilde{n})\right\|_{C([0,T];L^2(]0,l[;\mathbb{C}^2\times\mathbb{R})} \leq$$

$$C\left(\left\|(\psi^0, n^0) - (\tilde{\psi}^0, \tilde{n}^0)\right\|_{L^2(]0,l[;\mathbb{C}^2\times\mathbb{R})} + \|\alpha - \tilde{\alpha}\|_{L^2(]0,T[;\mathbb{C})}\right)$$

where (ψ, n) and $(\tilde{\psi}, \tilde{n})$ denote the weak solution with initial data (ψ^0, n^0, α) and $\left(\tilde{\psi}^0, \tilde{n}^0, \tilde{\alpha}\right)$, respectively.

10.3 A priori estimates

We will use the following elementary inequality:

Proposition 10.18. *Let $u : [0,b] \to \mathbb{R}$ be absolutely continuous and $u^* \in \mathbb{R}$. Suppose there are constants $r_1, r_2 > 0$ such that $u'(t) \leq -r_1 u(t) + r_2$ for a.a. $t \in [0,b]$ with $u(t) \geq u^*$. Then $u(t) \leq \bar{u} + \max\{u(0) - \bar{u}, 0\} e^{-r_1 t}$ for $t \in [0,b]$ with $\bar{u} := \max\left\{\frac{r_2}{r_1}, u^*\right\}$.*

Proof. Define $h : \mathbb{R} \to \mathbb{R}, h(x) := (\max\{x - \bar{u}, 0\})^2$. Set $f(t) := h(u(t))$. Then f is absolutely continuous and

$$f'(t) = h'(u(t))u'(t) \leq -h'(u(t))r_1 \left(u(t) - \frac{r_2}{r_1}\right) \leq -2r_1 f(t)$$

for *a.a.* $t \in [0,b]$. Therefore $f(t) \leq e^{-2r_1 t} f(0)$ for $t \in [0,b]$ and taking the square root yields the inequality. □

Lemma 10.19. *Let (ψ^δ, n^δ) be the weak solution to the truncated problem (10.15), (10.16), (10.17). There exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ estimate* (10.7) *holds for $t \in [0,T]$ and the bounds* (10.8) *are satisfied for $t \in [0,T]$ and a.a. $x \in \,]0,l[$. Moreover, there exists a constant B not depending on $\delta > 0$ such that*

$$\left\|\psi^\delta(t)\right\|_{L^\infty} \leq B \quad \text{for all } t \in [0,T].$$

Proof. Let $t_0 \in [0,T]$ be arbitrary and assume first that $\fint_{S_k} n^\delta(t,y)\,dy \geq 0$ for all $t \in [0,t_0]$ and all $1 \leq k \leq m$. Let $k \in \mathbb{N}$, $1 \leq k \leq m$. Suppose $0 < \delta \leq \nu_1^{-1}$. Then for *a.a.* $x \in S_k$ assumptions (II), (IV), (VIII) imply that for *a.a.* $t \in [0,t_0]$ which satisfy $n^\delta(t,x) \leq \nu_1$ the inequality

$$\frac{d}{dt} n^\delta(t,x) \geq (-c_1 - b_k)\, n^\delta(t,x)$$

holds. Put

$$h(t,x) := \min\left\{n^\delta(t,x), \nu_1\right\} \quad \text{and} \quad \tau_k(n) := \begin{cases} 1 & , n \leq \nu_1 \\ 0 & , n > \nu_1 \end{cases}.$$

Then for *a.a.* $x \in S_k$ and *a.a.* $t \in [0,t_0]$

$$\begin{aligned} \frac{d}{dt} h(t,x) &= \tau_k\left(n^\delta(t,x)\right) \frac{d}{dt} n^\delta(t,x) \\ &\geq (-c_1 - b_k)\, \tau_k\left(n^\delta(t,x)\right) n^\delta(t,x) \\ &\geq (-c_1 - b_k)\, h(t,x). \end{aligned}$$

Therefore for $a.a.\ x \in S_k$ and all $t \in [0, t_0]$

$$n^\delta(t,x) \geq h(t,x) \geq h(0,x)e^{-(c_1+b_k)t} = \min\left\{n^0(x), \nu_1\right\} e^{-(c_1+b_k)t} \quad (\geq 0). \tag{10.18}$$

Now we show that $⨍_{S_k} n^\delta(t,y)\,dy \geq 0$ for all $t \in [0,T]$ and all $1 \leq k \leq m$. Assume the contrary. Then there exists a $k \in \mathbb{N}$, $1 \leq k \leq m$, such that

$$t_0 := \sup\left\{t \in [0,T] \mid ⨍_{S_k} n^\delta(s,y)dy \geq 0 \text{ for } s \in [0,t]\right\} < T. \tag{10.19}$$

By (10.18) we have $n^\delta(t_0,x) \geq 0$ for $a.a.\ x \in]0,l[$ and by (10.19) $\int_{S_k} n^\delta(t_0,y)\,dy = 0$. Therefore $n^\delta(t_0,x) = 0$ for $a.a\ x \in S_k$. Hence, by continuity, there exists $0 < \epsilon < T - t_0$ such that for all $t \in [t_0, t_0+\epsilon[$ and $a.a.\ x \in S_k$ we have $n^\delta(t,x) \leq \nu_1$. Thus from the assumptions (II) and (IV), definition of H^δ and due to the choice $\delta \leq \nu_1^{-1}$ we have for $a.a\ t \in [t_0, t_0+\epsilon[$

$$\frac{d}{dt}⨍_{S_k} n^\delta(t,y)dy \;=\; ⨍_{S_k}\big(I(t,y) + H(y,\psi^\delta(t,y),n^\delta(t,y))\big)\,dy \geq -c_1 ⨍_{S_k} n^\delta(t,y)dy.$$

This yields $⨍_{S_k} n^\delta(t,y)dy \geq ⨍_{S_k} n^\delta(t_0,y)\,dy \cdot e^{-c_1(t-t_0)} = 0$ for $t \in [t_0, t_0+\epsilon[$ which contradicts the choice of t_0 from which there exist infinitely many points $s \in]t_0, t_0+\epsilon[$ with $⨍_{S_k} n^\delta(s,y)dy < 0$ accumulating in t_0. This proves (10.18) for all $t \in [0,T]$ and the lower bound for n^δ in (10.8).

Now define

$$T_\delta := \sup\left\{t \in [0,T] \mid \left\|\psi^\delta(s)\right\|_{L^\infty} \leq \delta^{-1} \text{ and } \left\|n^\delta(s)\right\|_{L^\infty} \leq \delta^{-1} \text{ for } s \in [0,t]\right\}.$$

Suppose $\delta > 0$ is sufficiently small such that $T_\delta > 0$. Assume $\alpha \in W^{1,2}(]0,T[;\mathbb{C})$ and $\psi_0 \in W^{1,2}(]0,l[;\mathbb{C}^2)$ together with (10.6). Denote

$$h(t) := \int_0^l n^\delta(t,x)\,dx + \frac{d_1}{2}\int_0^l \left\|\psi^\delta(t,x)\right\|^2\,dx.$$

From (I), (II), (VI) and Theorem 10.4 *iii*), proved for the truncated problem in Lemma (10.16), it follows by partial integration that for $a.a\ t \in [0, T_\delta]$

$$\begin{aligned}
&\frac{d}{dt}h(t) \\
= \;& d_1\, \mathfrak{Re} \int_0^l \Big[-\partial_x \psi_1^\delta(t,x)\overline{\psi_1^\delta(t,x)} + \partial_x \psi_2^\delta(t,x)\overline{\psi_2^\delta(t,x)} \Big]\, dx \\
& + \int_0^l \Big[I(t,x) + H(x, \psi^\delta(t,x), n^\delta(t,x)) \\
& + d_1 \mathfrak{Re} \left\langle G(x, \psi^\delta(t,x), n^\delta(t,x)), \psi^\delta(t,x) \right\rangle \Big] dx \\
\leq \;& \frac{d_1}{2} \left(- \left|\psi_1^\delta(t,l)\right|^2 + \left|\psi_1^\delta(t,0)\right|^2 + \left|\psi_2^\delta(t,l)\right|^2 - \left|\psi_2^\delta(t,0)\right|^2 \right) \\
& + \int_0^l I(t,x)\, dx - d_2 \left(\int_0^l n^\delta(t,x)\, dx + \int_0^l \left\|\psi^\delta(t,x)\right\|^2\, dx \right) \\
\leq \;& \frac{d_1}{2} \Big(\left(|r_0|^2 - 1\right) |\psi_2^\delta(t,0)|^2 + |\alpha(t)|^2 + 2|r_0||\psi_2^\delta(t,0)||\alpha(t)| \\
& + \left(|r_l|^2 - 1\right) |\psi_1^\delta(t,l)|^2 \Big) + l \left\|I\right\|_{L^\infty} - c \cdot h(t) \\
\leq \;& l \left\|I\right\|_{L^\infty} + \frac{d_1}{2} \left\|\alpha\right\|_{L^\infty}^2 + d_1 \max_{\rho \in \mathbb{R}} \left(\frac{|r_0|^2 - 1}{2} \rho^2 + |r_0| \left\|\alpha\right\|_{L^\infty} \rho \right) - c \cdot h(t) \\
= \;& \frac{d_1}{2(1 - |r_0|^2)} \left\|\alpha\right\|_{L^\infty}^2 + l \left\|I\right\|_{L^\infty} - c \cdot h(t).
\end{aligned}$$

Therefore the δ-independent estimate (10.7) for $\left(\psi^\delta, n^\delta\right)$ and $t \in [0, T_\delta]$ follows from Proposition 10.18. Because of Remark 10.17 this remains valid by density if $\alpha \in L^\infty(]0,T[\,;\mathbb{C}) \setminus W^{1,2}(]0,T[\,;\mathbb{C})$ or $\psi_0 \in L^\infty(]0,l[\,;\mathbb{C}^2) \setminus W^{1,2}(]0,l[\,;\mathbb{C}^2)$. By Definition 10.2 $n^\delta(\cdot, x)$ is absolutely continuous on $[0,T]$ for $a.a\ x \in]0,l[$. From assumption (II) it follows that for $a.a\ t \in [0, T_\delta]$ with $n^\delta(t,x) \geq \nu_2$ the inequality

$$\frac{d}{dt} n^\delta(t,x) \leq \left\|I\right\|_{L^\infty} + \max_{1 \leq k \leq m} \left(\frac{b_k}{|S_k|} \right) \cdot \max \left\{ \mu, \int_0^l n^0(x) dx + \frac{d_1}{2} \left\|\psi^0\right\|_{L^2}^2 \right\} - c_2 n^\delta(t,x)$$

holds. Proposition 10.18 yields the δ-independent upper bound for n^δ and $t \in [0, T_\delta]$ in (10.8).

From the explicit formula (10.13) we have the following decay rates for the semigroups T and T_e: For $t \geq 0$

$$\left\| \Pi_\psi T_e(t) \begin{pmatrix} \psi^0 \\ n^0 \\ \alpha \end{pmatrix} \right\|_{L^\infty} \leq D_0 e^{-\gamma t} \left\|\psi^0\right\|_{L^\infty} + 2 \left(1 - |r_0 r_l|\right)^{-1} \left\|\alpha\right\|_{L^\infty}, \quad (10.20)$$

where $D_0 := \begin{cases} |r_0 r_l|^{-1} & , r_0 r_l \neq 0 \\ e & , r_0 r_l = 0 \end{cases}$ and $\gamma := \begin{cases} -(2l)^{-1} \log |r_0 r_l| & , r_0 r_l \neq 0 \\ (2l)^{-1} & , r_0 r_l = 0 \end{cases}$.
Let M_0 be a constant in assumption (III) for $K = [0, N + \|n^0\|_{L^\infty}]$. From (10.20), (10.14), (10.8) and (III) we get for $t \in [0, T_\delta]$

$$\begin{aligned} \left\|\psi^\delta(t)\right\|_{L^\infty} &\leq \left\| \Pi_\psi T_e(t) \begin{pmatrix} \psi^0 \\ n^0 \\ \alpha \end{pmatrix} \right\|_{L^\infty} + \int_0^t \left\|T(t-s)\mathfrak{G}^\delta(\psi^\delta(s), n^\delta(s))\right\|_{L^\infty} ds \\ &\leq D_0 e^{-\gamma t} \left\|\psi^0\right\|_{L^\infty} + 2\left(1 - |r_0 r_l|\right)^{-1} \|\alpha\|_{L^\infty} \\ &\quad + M_0 T + \int_0^t M_0 \left\|\psi^\delta(s)\right\|_{L^\infty} ds. \end{aligned}$$

Gronwall's Lemma yields the existence of a constant B independent on $\delta > 0$ such that $\left\|\psi^\delta(t)\right\|_{L^\infty} \leq B$ for $t \in [0, T_\delta]$.
Moreover, since assumption (III) is valid also for the truncated nonlinearity G^δ and n^δ is continuous from $[0, T]$ to L^∞ by choosing a possibly larger M_0 corresponding to a larger set K than above we can find a constant B independent of $\delta > 0$ such that for each $\delta > 0$ there exists a neighbourhood U_δ of T_δ so that $\left\|\psi^\delta(t)\right\|_{L^\infty} \leq B$ for $t \in [0, T_\delta] \cup U_\delta$. This proves that $T_\delta = T$ if δ is chosen sufficiently small. $\square$

We have shown that for sufficiently small $\delta > 0$ the weak solutions of the truncated problem coincide with the original weak solutions of the nontruncated problem. Hence the proof of Theorems 10.3-10.5 is complete. We are left with the proofs of Theorem 10.6 and 10.10.

of Theorem 10.10. From the assumption that α and G are of class C^k it follows that the map

$$\begin{pmatrix} \psi^0 \\ n^0 \\ \theta \end{pmatrix} \in \mathfrak{P}_e \mapsto \begin{pmatrix} \psi^0 + G(\theta, n^0) \\ n^0 \\ \alpha(\theta + \cdot) \end{pmatrix} \in L^\infty(]0, l[\,, \mathbb{C}^2 \times \mathbb{R}) \times L^\infty(]0, T[\,, \mathbb{C})$$

is C^k. Hence Theorems 10.3 and 10.4 imply that

$$\begin{pmatrix} \psi^0 \\ n^0 \\ \theta \end{pmatrix} \in \mathfrak{P}_e \mapsto X(\theta + t, \theta, \begin{pmatrix} \psi^0 + G(\theta, n^0) \\ n^0 \end{pmatrix}) \in C([0, l], \mathbb{C}^2) \times C_P$$

is C^k. This shows that for $t \geq 0$ the map $(p, \theta) \in \mathfrak{P}_e \mapsto Y(\theta + t, \theta, p) \in \mathfrak{P}$ is of class C^k. Hence Z^t is a C^k smooth semiflow on $\mathfrak{P}_e$. $\square$

of Theorem 10.6. Let (ψ, n) be the weak solution. From the differentiability assumption (X) on H the map $w \mapsto \mathfrak{H}(\psi(s), w)$ is well defined from $W_P^{1,2}$ into itself for $s \in [0,T]$ since $\psi \in C([0,T], W^{1,2})$. Furthermore condition (XI) implies that this map is Lipschitz on bounded subsets of $W_P^{1,2}$ uniformly in $s \in [0,T]$. By truncation we can make it globally Lipschitz: for $\eta > 0$ let $T_\eta : W_P^{1,2} \to W_P^{1,2}$ be globally Lipschitz with $T_\eta(w) = w$, if $\|w\|_{W_P^{1,2}} \le \eta^{-1}$, $T_\eta(w) = 2\eta^{-1} w \|w\|_{W_P^{1,2}}^{-1}$, if $\|w\|_{W_P^{1,2}} \ge 2\eta^{-1}$. Define the following truncated operators

$$\mathfrak{H}_\eta(p, w) := \mathfrak{H}(p, T_\eta(w)) \text{ for } p \in W^{1,2} \text{ and } w \in W_P^{1,2}.$$

Then for all $p \in W^{1,2}$ the map $w \mapsto \mathfrak{H}_\eta(p, w)$ is globally Lipschitz in $W_P^{1,2}$ where the Lipschitz constant depends only on η and $\|p\|_{W^{1,2}}$.

Define $\mathfrak{F} : C([0,T], W_P^{1,2}) \to C([0,T], W_P^{1,2})$,

$$(\mathfrak{F}m)(t) := n^0 + \int_0^t (\mathfrak{I}(s) + \mathfrak{B}m(s) + \mathfrak{H}_\eta(\psi(s), m(s)))\, ds \quad (t \in [0,T]).$$

Then $\mathfrak{F}$ has a unique fixed point n_η in $C([0,T], W_P^{1,2})$ by a generalization of Banachs fixed point theorem since sufficient high iterates of $\mathfrak{F}$ become contractive. In particular $n_\eta \in C^1([0,T], W_P^{1,2})$.

Set $T_\eta := \sup\left\{ t \in [0,T] \mid \|n_\eta(s)\|_{W_P^{1,2}} \le \eta^{-1} \text{ for } 0 \le s \le t \right\}$. By (XII) and the Hï¿½der-Young inequalities we have for all $t \in [0, T_\eta]$

$$\begin{aligned}
&\partial_t \frac{1}{2} \|\partial_x n_\eta(t)\|_{L^2(S_k)}^2 \\
= \ & \int_{S_k} \partial_x \left(I(t,x) - b_k n_\eta(t,x) + H(x, \psi(t,x), n_\eta(t,x)) \right) \partial_x n_\eta(t,x)\, dx \\
\le \ & \int_{S_k} |\partial_x I(t,x) \partial_x n_\eta(t,x)|\, dx + \int_{S_k} \Big(\partial_x H(x, \psi(t,x), n_\eta(t,x)) \partial_x n_\eta(t,x) \\
& + \partial_\psi H(x, \psi(t,x), n_\eta(t,x)) \partial_x \psi(t,x) \partial_x n_\eta(t,x) \\
& + \partial_n H(x, \psi(t,x), n_\eta(t,x)) \left(\partial_x n_\eta(t,x)\right)^2 \Big)\, dx \\
\le \ & \frac{3}{2\tau} \|\partial_x I(t)\|_{L^2(S_k)}^2 - \tau \frac{5}{6} \|\partial_x n_\eta(t)\|_{L^2(S_k)}^2 \\
& + R_0 \Big(\|1\|_{L^1(S_k)} + \|\partial_x n_\eta(t)\|_{L^1(S_k)} + \|\partial_x n_\eta(t)\|_{L^2(S_k)} \|\partial_x \psi(t)\|_{L^2(S_k)} \\
& + \|\partial_x \psi(t)\|_{L^2(S_k)}^2 \Big) \\
\le \ & \frac{3}{2\tau} \sup_{t \in [0,T]} \|\partial_x I(t)\|_{L^2}^2 + R_0 l + \frac{3}{2\tau} R_0^2 l + \left(\frac{3R_0^2}{2\tau} + 1 \right) \|\partial_x \psi(t)\|_{L^2(S_k)}^2 \\
& - \tau \tfrac{1}{2} \|\partial_x n_\eta(t)\|_{L^2}^2 .
\end{aligned}$$

Hence (see Prop. 10.18) we get the following η independent bound

$$\begin{aligned} \|\partial_x n_\eta(t)\|^2_{L^2(S_k)} &\leq \frac{3}{2\tau^2}\sup_{t\in[0,T]}\|\partial_x I(t)\|^2_{L^2(S_k)} + \frac{R_0 l}{\tau} + \frac{3R_0^2 l}{2\tau^2} \\ &\quad + \left(\frac{3R_0^2}{2\tau^2} + \frac{1}{\tau}\right)\sup_{s\in[0,T_\eta]}\|\partial_x \psi(s)\|^2_{L^2} \end{aligned}$$

which is valid for $t \in [0, T_\eta]$.

Since the a priori estimates of Theorem 10.5 must hold for n_η as long as $t \in [0, T_\eta]$ we see that $T_\eta = T$ and $n_\eta = n$ if η is chosen sufficiently small. $\square$

Chapter 11

Center manifold / model reduction for the traveling wave equation in the nonautonomous case

We now perform the center manifold reduction for (3.17) in the case of nonautonomous boundary conditions, $\alpha \neq 0$ in (3.11). Assume $\alpha \in C^k\left([0,\infty[\,;\mathbb{C}\right)$ and there exists

$$G \in C^k\left([0,\infty[\, \times L^\infty\left(]0,l[\,,\mathbb{R}\right); C\left([0,l]\,;\mathbb{C}^2\right)\right)$$

which satisfies for all $n \in L^\infty\left(]0,l[\,,\mathbb{R}\right)$

$$\left\{\begin{array}{rcl} \partial_t G(t,n) & = & \mathfrak{A}(n)G(t,n), \\ G_1(t,n)_{|x=0} & = & r_0 G_2(t,n)_{|x=0} + \alpha(t), \\ G_2(t,n)_{|x=l} & = & r_l G_1(t,n)_{|x=l}. \end{array}\right.$$

Let Z^t be the smooth skew product semiflow on $\mathfrak{P}_e$ defined in section 10.1. Then Z^t is generated by the equations

$$\left\{\begin{array}{rcl} \partial_t\psi(t) & = & \mathfrak{A}(n(t)) + \epsilon\Big[\mathfrak{K}\left(n(t),\psi(t)+G(s,n(t))\right) \\ & & -\partial_n G(s,n(t))\mathfrak{F}(s,n(t),\psi(t))\Big] \\ \partial_t n(t) & = & \epsilon\,\mathfrak{F}\left(s,n(t),\psi(t)+G(s,n(t))\right) \\ \partial_t s & = & 1, \end{array}\right.$$

with boundary condition

$$\begin{cases} \psi(0) &=& \psi^0 - G(0,n^0), \\ n(0) &=& n^0, \\ s(0) &=& \theta, \\ \psi_1(t,0) &=& r_0\psi_2(t,0), \\ \psi_2(t,l) &=& r_l\psi_1(t,l). \end{cases}$$

Here $\mathfrak{K}$ and $\mathfrak{F}$ denote operators generated by K and F in (3.17), respectively. Note that the boundary conditions are homogeneous and time independent now, the nonautonomous time dependence appear through the variable s in the terms $G(s,n(t))$ and $\partial_n G(s,n(t))$ in both equations for ψ and n. Under a spectral gap assumption for $\mathfrak{A}(n)$ we can locally make a change of coordinates,

$$\psi = B(n)x_c + C(n)x_s,$$

as we have done in section 9 and arrive at the following set of equations (compare it with (9.3))

$$\begin{cases} \partial_t x_c &=& \mathfrak{A}_c(n)x_c + \epsilon\mathfrak{G}_c(s,n,x_c,x_s) \\ \partial_t x_s &=& \mathfrak{A}_s(n)x_s + \epsilon\mathfrak{G}_s(s,n,x_c,x_s) \\ \partial_t n &=& \epsilon\mathfrak{F}(s,n,\mathfrak{B}(n)x_c + \mathfrak{C}(n)x_s + G(s,n)) \\ \partial_t s &=& 1 \\ x_c(0) &=& \mathfrak{B}(n^0)^{-1}\mathfrak{P}(n^0)\left(\psi^0 - G(\theta,n^0)\right), \\ x_s(0) &=& \mathfrak{C}(n^0)^{-1}\mathfrak{Q}(n^0)\left(\psi^0 - G(\theta,n^0)\right), \\ n(0) &=& n^0, \\ s(0) &=& \theta, \end{cases}$$

where

$$\begin{aligned} \mathfrak{A}_c(n) &:= (\mathfrak{B}(n))^{-1}\,\mathfrak{A}(n)\mathfrak{B}(n), \\ \mathfrak{A}_s(n) &:= (\mathfrak{C}(n))^{-1}\,\mathfrak{A}(n)\mathfrak{C}(n), \end{aligned}$$

$$\begin{aligned} \mathfrak{G}_c(s,n,x_c,x_s) &:= (\mathfrak{B}(n))^{-1}\,\mathfrak{P}(n)\mathfrak{G}(s,n,x_c,x_s), \\ \mathfrak{G}_s(s,n,x_c,x_s) &:= (\mathfrak{C}(n))^{-1}\,\mathfrak{Q}(n)\mathfrak{G}(s,n,x_c,x_s), \end{aligned}$$

and

$$\begin{aligned} \mathfrak{G}(s,n,x_c,x_s) := {} & \mathfrak{K}(n,\mathfrak{B}(n)x_c + \mathfrak{C}(n)x_s + G(s,n)) \\ & - \left(\partial\mathfrak{B}(n)\mathfrak{F}(s,n,\mathfrak{B}(n)x_c + \mathfrak{C}(n)x_s + G(s,n))\right)x_c \\ & - \left(\partial\mathfrak{C}(n)\mathfrak{F}(s,n,\mathfrak{B}(n)x_c + \mathfrak{C}(n)x_s + G(s,n))\right)x_s \\ & - \partial_n G(s,n)\mathfrak{F}(s,n,\mathfrak{B}(n)x_c + \mathfrak{C}(n)x_s + G(s,n)). \end{aligned}$$

Theorem 5.5 implies that for $\epsilon = 0$ the smooth semiflow Z^t in the Banach Space $\mathbb{C}^q \times \mathcal{Y}_{\mathcal{W}} \times C_P \times \mathbb{R}$ has the (non locally compact) normally hyperbolic invariant Banach-manifold $\mathbb{C}^q \times \{0\} \times \mathcal{U} \times \mathbb{R} \subset \mathbb{C}^q \times \mathcal{Y}_{\mathcal{W}} \times C_P \times \mathbb{R}$. Using a cut off modification we can construct overflowing manifolds

$$\mathrm{IM}_0^r := \{\psi_c \in \mathbb{C}^q \mid |\psi_c| < r\} \times \{0\} \times \mathcal{U} \times \{s \in \mathbb{R} \mid |s| < r\}$$

for any given $r > 0$, so that the modified equation coincides with the original one within a radius of $\frac{r}{2}$. By applying persistence theory for semiflows in Banach spaces [8] we get

Theorem 11.1. *For any $r > 0$ there exists an $\epsilon_0 > 0$ so that for $0 < \epsilon < \epsilon_0$ the manifold IM_0^r persists as a nonlinear exponentially attracting smooth invariant manifold IM_ϵ^r, which can be represented as a C^k smooth graph $x_s = \gamma(x_c, n, s, \epsilon)$,*

$$\gamma : \mathrm{IM}_0^r \times]0, \epsilon_0[\to \mathcal{Y}_{\mathcal{W}}.$$

The flow on IM_ϵ^r is given by the equations

$$\begin{cases} \partial_t x_c &= \mathfrak{A}_c(n) x_c + \epsilon \mathfrak{G}_c\left(s, n, x_c, \gamma(s, x_c, n, \epsilon)\right) \\ \partial_t n &= \epsilon \mathfrak{F}(s, n, \mathfrak{B}(n) x_c + \mathfrak{C}(n)\gamma(s, x_c, n, \epsilon) + g(s, n)) \\ \partial_t s &= 1. \end{cases}$$

If $z : I \to \mathfrak{P}_e$ is a trajectory on IM_ϵ^r then $z \in C^k(I, \mathfrak{P}_e)$.

Rewriting the equations without the time substitute variable s we arrive to the following C^k-smooth ordinary nonautonomous differential equation in the Banach space $\mathbb{C}^q \times \mathcal{U}$:

$$\begin{cases} \partial_t x_c &= \mathfrak{A}_c(n) x_c + \epsilon \mathfrak{G}_c\left(t, n, x_c, \gamma(t, x_c, n, \epsilon)\right) \\ \partial_t n &= \epsilon \mathfrak{F}(t, n, \mathfrak{B}(n) x_c + \mathfrak{C}(n)\gamma(t, x_c, n, \epsilon) + G(t, n)). \end{cases} \tag{11.1}$$

Since γ is smooth and $\gamma(t, x_c, n, 0) = 0$ we have $\gamma(t, x_c, n, \epsilon) = \epsilon\overline{\gamma}(t, x_c, n, \epsilon)$, where $\overline{\gamma}$ is smooth. Next we expand (11.1) in powers of ϵ. According to the Table in Section 3.2, (3.17) and (3.15) the generator of the Nemytskij operator $\mathfrak{K}$ is

$$K(x, n, \psi) = -\overline{\epsilon}_G(x)\frac{1}{2}g(x, n)\frac{\|\psi\|^2}{1 + \epsilon\overline{\epsilon}_G(x)\|\psi\|^2} + \overline{\epsilon_I}(x) i \frac{\alpha_H}{2} g(x, n)\frac{\|\psi\|^2}{1 + \epsilon\overline{\epsilon}_I(x)\|\psi\|^2},$$

where g is given in (3.14) and $\overline{\epsilon}_G(x) = \frac{\tilde{\epsilon}_G(x)}{\epsilon} = \epsilon_G \Gamma n_{tr,1} \sim 1,5$ (and similarly $\overline{\epsilon}_I(x)$) is of order 1. In the following we frequently use the expansion

$$\frac{\|\psi\|^2}{1 + \epsilon\overline{\epsilon}_G(x)\|\psi\|^2} = \|\psi\|^2 + O(\epsilon).$$

Then

$$K(x,n,\psi) = -\bar{\epsilon}_G(x)\frac{1}{2}g(x,n)\left\|\psi\right\|^2 + \overline{\epsilon_I}(x)i\frac{\alpha_H}{2}g(x,n)\left\|\psi\right\|^2 + O(\epsilon).$$

Therefore we can expand the terms of $\mathfrak{G}_c$ as follows:

$$\begin{aligned}
&\epsilon\left(\mathfrak{B}(n)\right)^{-1}\mathfrak{P}(n)\mathfrak{K}(n,\mathfrak{B}(n)x_c + \mathfrak{C}(n)\gamma(t,x_c,n,\epsilon) + G(t,n)) \\
=& -\epsilon\left(\mathfrak{B}(n)\right)^{-1}\mathfrak{P}(n)\Big[\bar{\epsilon}_G(\cdot)\frac{1}{2}g(\cdot,n(\cdot))\left\|\mathfrak{B}(n)x_c + G(t,n)\right\|^2\left(\mathfrak{B}(n)x_c + G(t,n)\right)\Big] \\
&+ \epsilon\left(\mathfrak{B}(n)\right)^{-1}\mathfrak{P}(n)\Big[\overline{\epsilon_I}(\cdot)i\frac{\alpha_H}{2}g(\cdot,n(\cdot))\left\|\mathfrak{B}(n)x_c + G(t,n)\right\|^2\left(\mathfrak{B}(n)x_c + G(t,n)\right)\Big] \\
&+ O(\epsilon^2),
\end{aligned}$$

$$\begin{aligned}
&-\epsilon\left(\mathfrak{B}(n)\right)^{-1}\mathfrak{P}(n)\left[\partial\mathfrak{B}(n)\mathfrak{F}(t,n,\mathfrak{B}(n)x_c + \mathfrak{C}(n)\gamma(t,x_c,n,\epsilon) + G(t,n))x_c\right] \\
=& -\epsilon\left(\mathfrak{B}(n)\right)^{-1}\mathfrak{P}(n)\Big[\partial\mathfrak{B}(n)\mathfrak{h}(t,n,x_c)x_c\Big] + O(\epsilon^2),
\end{aligned}$$

where

$$\begin{aligned}
\mathfrak{h}(t,n,x_c) :=& I(t,\cdot) + \sum_{k=1}^{m} b_k\chi_{S_k}(\cdot)\left(\fint_{S_k} n(y)dy - n(\cdot)\right) \\
&- R(\cdot,n(\cdot)) - g(\cdot,n(\cdot))\left\|\mathfrak{B}(n)x_c + G(t,n)\right\|^2,
\end{aligned}$$

$$\begin{aligned}
-\epsilon\left(\mathfrak{B}(n)\right)^{-1}\mathfrak{P}(n)\left(\partial\mathfrak{C}(n)\mathfrak{F}(t,n,\mathfrak{B}(n)x_c + \mathfrak{C}(n)\gamma(t,x_c,n,\epsilon) + G(t,n))\right)& \\
\gamma(t,x_c,n,\epsilon) = O(\epsilon^2),&
\end{aligned}$$

$$\begin{aligned}
&-\epsilon\left(\mathfrak{B}(n)\right)^{-1}\mathfrak{P}(n)\partial_n G(t,n)\mathfrak{F}(t,n,\mathfrak{B}(n)x_c + \mathfrak{C}(n)\gamma(t,x_c,n,\epsilon) + G(t,n)) \\
=& -\epsilon\left(\mathfrak{B}(n)\right)^{-1}\mathfrak{P}(n)\partial_n G(t,n)\mathfrak{h}(t,n,x_c) + O(\epsilon^2).
\end{aligned}$$

Suppose $\mathfrak{A}(n)$ has q critical eigenvalues $\lambda_1,\ldots,\lambda_q$ near $i\mathbb{R}$ of algebraic multiplicity one for $n \in \mathcal{U}$. Then, see Proposition 4.3, each eigenvector is a scalar multiple of

$$b_i(n) = T(\cdot,0,\lambda_i,n)\begin{pmatrix} r_0 \\ 1 \end{pmatrix} \qquad i = 1,\ldots,q,$$

where $T(x,y,\lambda,n)$ denotes the fundamental system to the nonautonomous ODE

$$\frac{d}{dx}\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}(x) = \begin{pmatrix} -\lambda + \beta(x,n(x)) & \kappa(x) \\ \dfrac{\hphantom{-\lambda + \beta(x,n(x))}}{\kappa(x)} & \lambda - \beta(x,n(x)) \end{pmatrix}\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}(x).$$

Note that if n is a piecewise constant function - for example obtained after a Galerkin projection using Steklov average step functions - then there is an explicit expression for T and b_i in terms of elementary functions. A natural choice for $\mathfrak{B}(n)$ then is

$$\mathfrak{B}(n)x_c = \sum_{i=1}^{q} x_{c,i} b_i(n) \quad \text{for } x_c = (x_{c,i})_{1\le i\le q} \in \mathbb{C}^q.$$

The inverse $\mathfrak{B}(n)^{-1}\mathfrak{P}(n)$ can be expressed in terms of adjoint eigenfunctions b_i^* which are simply related to b_i. To see this one calculates that the adjoint operator $\mathfrak{A}^*(n)$ to $\mathfrak{A}(n)$ is

$$\begin{cases} \mathfrak{A}^*(n) &= \begin{pmatrix} \partial_x + \beta(\cdot, n(\cdot)) & -\kappa(\cdot) \\ \overline{\kappa} & -\partial_x + \beta(\cdot, n(\cdot)) \end{pmatrix}, \\ \psi_2(0) &= \overline{r}_0 \psi_1(0) \\ \psi_1(l) &= \overline{r}_l \psi_2(l). \end{cases}$$

Hence there is a one to one correspondence of (generalized) eigenvectors of $\mathfrak{A}$ and $\mathfrak{A}^*$ via the map

$$J\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}\overline{\psi_2}\\ \overline{\psi_1}\end{pmatrix}.$$

If b_i is a eigenvector of $\mathfrak{A}$ to the eigenvalue λ then $b_i^* = Jb_i$ is the corresponding eigenvector of $\mathfrak{A}^*$ with the eigenvalue $\overline{\lambda}$. It has been shown in [60] that the system of root functions corresponding to the eigenvalues of $\mathfrak{A}$ is complete in $L^2([0,l],\mathbb{C}^2)$. From the completeness it follows that $\langle x_k^*, x_k\rangle \neq 0$ because $\langle c^*, d\rangle = 0$ if c and d are generalized eigenvectors to different eigenvalues. Therefore

$$\mathfrak{B}(n)^{-1}\mathfrak{P}(n)\psi = \left(\frac{b_i^*(n)\psi}{b_i^*(n)b_i(n)}\right)_{1\le i\le q}.$$

Summarizing we arrive at the following expansion of (11.1)

$$\begin{cases} \partial_t x_{c,i} &= \mathfrak{P}_i(t, x_c, n) + O(\epsilon^2), \quad 1 \le i \le q, \\ \partial_t n &= \epsilon \mathfrak{h}(t, n, x_c) + O(\epsilon^2), \\ & \\ \mathfrak{P}_i(t, x_c, n) &= \lambda_i x_{c,i} - \epsilon \left(b_i^* b_i\right)^{-1} b_i^* \Big[\\ & \quad \overline{\epsilon}_G(\cdot)\frac{1}{2} g(\cdot, n(\cdot)) \left\|\mathfrak{B}(n)x_c + G(t,n)\right\|^2 \left(\mathfrak{B}(n)x_c + G(t,n)\right) \\ & \quad + \overline{\epsilon_I}(\cdot) i \frac{\alpha_H}{2} g(\cdot, n(\cdot)) \left\|\mathfrak{B}(n)x_c + G(t,n)\right\|^2 \left(\mathfrak{B}(n)x_c + G(t,n)\right) \\ & \quad + D\mathfrak{B}(n)\mathfrak{h}(t,n,x_c)x_c - D_n G(t,n)\mathfrak{h}(t,n,x_c)\Big], \end{cases} \tag{11.2}$$

Remark 11.2 (Explicit formula for G)**.** *It is possible to give an explicit expression for G in the case of simple α. For example, if α is quasiperiodic, that is can be represented as $\alpha(t) = \sum_{k=1}^{m} a_k e^{i\omega_k t}$ with $\omega_k \in \mathbb{R}$, $a_k \in \mathbb{C}$, and if the frequencies ω_k do not coincide with a point in the spectrum of $\mathfrak{A}(n)$ for $n \in \mathcal{U}$, that is*

$$\{i\omega_k \mid k = 1, 2, \ldots, m\} \cap \{\lambda \in \sigma(\mathfrak{A}(n)) \mid n \in \mathcal{U}\} = \emptyset,$$

then

$$G(s,n) = \sum_{k=1}^{m} \left(a_k e^{i\omega_k s} \zeta_k(n)\right),$$

where

$$\begin{array}{rcl} \mathfrak{A}(n)\zeta_k &=& i\omega_k \zeta_k, \\ \zeta_{k,1}(0) &=& r_0 \zeta_{k,2}(0) + 1, \\ \zeta_{k,2}(l) &=& r_l \zeta_{k,1}(l) \end{array}$$

is an explicit expression for G. For each ζ_k we have

$$\zeta_k(n) = T(\cdot, 0, i\omega_k, n) \begin{pmatrix} r_0 c + 1 \\ c \end{pmatrix},$$

where

$$c = -\frac{\begin{pmatrix} r_l & -1 \end{pmatrix} T(l, 0, i\omega_k, n) \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{h(i\omega_k, n)}$$

and

$$h(\lambda, n) = \begin{pmatrix} r_l & -1 \end{pmatrix} T(l, 0, \lambda, n) \begin{pmatrix} r_0 \\ 1 \end{pmatrix}$$

is the characteristic function to $\mathfrak{A}(n)$.

In the system (11.2) the unknown graph γ only appears in $O(\epsilon^2)$ terms. The system resulting from (11.2) when neglecting all $O(\epsilon^2)$ terms, i.e.

$$\left\{ \begin{array}{rcl} \partial_t x_{c,i} &=& \mathfrak{P}_i(t, x_c, n), \quad 1 \le i \le q, \\ \partial_t n &=& \epsilon \mathfrak{h}(t, n, x_c). \end{array} \right. \tag{11.3}$$

is called mode approximation. Thus mode approximations can be regarded as a small perturbation of the flow on certain local center manifolds. Mode approximations, first derived formally by physicists [3], for the traveling wave equation have been studied by several authors, see [55, 68, 66, 5]. They have been rigorously derived first in the thesis [65] of J. Sieber for a simplified model (averaged carrier densities, no optical injection, no nonlinearities due to gain or index compression in the PDE). We note that the mode approximations calculated by Sieber follows from (11.3) when one uses a lowest order Galerkin approximation, where on each section $S_k = [x_{k-1}, x_k]$ the distribution $n(x)$ is projected to the average $⨍_{S_k} n(y)\,dy$ (indeed, the formulas coincide when one sets $\alpha = 0$ and $\epsilon_G = \epsilon_I = 0$ and ignores gain dispersion terms $g_P = 0$; note that gain dispersion can be easily incorporated to (11.2), but we have not done this because the model including gain dispersion allows for negative carrier densities as we have mentioned in remark 3.1). Such kind of low dimensional projection of course neglects the spacial variation of n in a active laser section totally, so that important physical effects such as spacial hole burning get lost. Of course, using a higher order Galerkin approximation scheme of (11.2) for n using piecewise constant step functions we get a refined scale of ODEs which converges to the full model (11.2), we will remark this next more precisely. Thus, if one wants to include spacial hole burning effects a higher dimensional Galerkin projection should be used. The advantage of using piecewise constant step functions is that the vector field of the resulting ODE can be expressed explicitely in terms of elementary functions where only a few critical eigenvalues λ_i have to be traced numerically, and, moreover, the Galerkin approximations converge uniformly to the solution of (11.2). Of course there is no natural global choice of basis functions for n. Let $m \in \mathbb{N}$ be the number of laser sections and $k, r, s, \in \mathbb{N}$ be indices, where $1 \leq k \leq m$ runs trough all sections S_k, s denotes the order of the Galerkin approximation and $1 \leq r \leq s$. Then the basis functions b_{ksr} we use are the following characteristic (step) functions

$$b_{ksr} := \chi_{A_{ksr}}, \quad A_{ksr} := [x_{k-1} + (r-1)\theta_{ks}, x_{k-1} + r\theta_{ks}], \quad \theta_{ks} := \frac{x_k - x_{k-1}}{s}.$$

Then, by approximating $n \sim \sum_{\substack{1 \leq k \leq m \\ 1 \leq r \leq s}} n_{ksr} b_{ksr}$, the Galerkin projection of order s for (11.3) is given by

$$\begin{cases} \partial_t x^{\pi}_{c,i} &= \mathfrak{P}_i\left(t, x^{\pi}_c, \sum_{\substack{1 \leq k \leq m \\ 1 \leq r \leq s}} n_{ksr}(t) b_{ksr}(\cdot)\right), \quad 1 \leq i \leq q, \\ \partial_t n_{ksr} &= \epsilon\, \pi_{ksr}\, \mathfrak{h}\left(t, \sum_{\substack{1 \leq k \leq m \\ 1 \leq r \leq s}} n_{ksr}(t) b_{ksr}(\cdot), x^{\pi}_c\right), \quad 1 \leq k \leq m, 1 \leq r \leq s, \end{cases}$$

where $\pi_{ksr} n := \frac{1}{|A_{ksr}|} \int_{A_{ksr}} n(y)\,dy$.

It follows readily, by using Theorem 10.4 *iv*), Theorem 10.5, Lemma 10.19 and Gronwall's Lemma, that for $T > 0$ and any initial data $x_c(0)$ and $n_0 \in C_P$ corresponding to a $\psi_0 \in C([0, l], \mathbb{C}^2)$ satisfying the boundary condition (10.2) and for the corresponding projected data $x^{\pi}_c(0) = x_c(0)$ and $n_{ksr}(0) = \pi_{ksr} n_0$

we have

$$\|x_c - x_c^\pi\|_{C([0,T]\times[0,l],\mathbb{C}^2)} + \left\|n - \sum n_{ksr} b_{ksr}\right\|_{C([0,T],C_P)} \to_{s\to\infty} 0.$$

Chapter 12

Hopf bifurcation of rotating waves into selfpulsations for the traveling wave equation

We consider Hopf bifurcation of rotating waves (relative equilibria) for the S^1 equivariant traveling wave equation (3.17), (3.11) which we write in operator form

$$\left\{\begin{array}{rcl} \partial_t\psi(t) &=& \mathfrak{A}(n(t))\psi(t) + \epsilon\mathfrak{K}(n(t),\psi(t)) \\ \partial_t n(t) &=& \epsilon\mathfrak{F}(n(t),\psi(t)) \\ \psi(t) &\in& Y_l(r_0,r_m) \\ n(t) &\in& C_P \\ \psi(0) &=& \psi_0 \\ n(0) &=& n_0, \end{array}\right. \tag{12.1}$$

where

$$Y_l(r_0,r_m) := \left\{(\varphi_1,\varphi_2) \in C([0,l],\mathbb{C}^2) \mid \varphi_1(0) = r_0\varphi_2(0),\ \varphi_2(l) = r_m\varphi_1(l)\right\},$$

$$\mathfrak{A}(n) = \mathfrak{A}_0 + \mathfrak{L}(n), \quad \mathfrak{A}_0 := \begin{pmatrix} -\partial_x & 0 \\ 0 & \partial_x \end{pmatrix},$$

has been defined in (9.1), $\mathfrak{K}$ is generated by (3.15) and C_P is the space of piecewise (on each S_k) uniformly continuous functions, see (10.5). Let $T(\theta)$, $\theta \in S^1 = \mathbb{R}/(2\pi\mathbb{Z})$, be the group of linear isomorphisms acting on (ψ,n) via

$$T(\theta)(\psi,n) := (e^{i\theta}\psi,n).$$

The system (12.1) is equivariant with respect to T, i.e.

$$\mathfrak{G}(T(\theta)(\psi,n)) = T(\theta)\mathfrak{G}(\psi,n) \quad \text{for } \theta \in S^1 \text{ and } \psi \in \mathcal{D}(\mathfrak{A}), n \in C_P,$$

where

$$\mathfrak{G}(\psi, n) := \begin{pmatrix} \mathfrak{A}(n)\psi + \epsilon\mathfrak{K}(n,\psi) \\ \epsilon\mathfrak{F}\,(n,\psi)) \end{pmatrix}.$$

We consider the autonomous case, where the pump term I (injection current) does not depend on t and no light is injected into the laser ($\alpha = 0$). Typical bifurcation parameters are I or the phase φ of the reflection coefficient $r_m = |r_m|e^{i\varphi}$ at the right facet of the laser [9, 66] (for simplicity neglect internal reflections, i.e. put $r^+_{k-1,k} = r^-_{k-1,k} = 1$ and $r^+_{kk} = r^-_{kk} = 0$ in (3.11)). Note that the reflection coefficients r_0 and r_m are hidden in the function space Y_l. Therefore, when considering them as bifurcation parameters, one has to normalize the boundary conditions first [60]: Let a be a complex number satisfying $e^{2al} = r_0 r_m$ and $U : [0,l] \to \mathcal{L}(\mathbb{C}^2)$ be the diagonal matrix function $U(x) := \operatorname{diag}(r_0 e^{-ax}, e^{ax})$ generating the Nemytskij operator $\mathfrak{U}$, $(\mathfrak{U}\psi)(x) := U(x)\psi(x)$. Then the linear transformation

$$\psi \mapsto \mathfrak{U}^{-1}\psi$$

normalizes the boundary condition, i.e. (12.1) recasts as

$$\begin{cases} \partial_t\psi(t) &=& \mathfrak{H}(n)\psi(t) + \epsilon\mathfrak{U}^{-1}\mathfrak{K}(n(t), \mathfrak{U}\psi(t)) \\ \partial_t n(t) &=& \epsilon\mathfrak{F}\,(n(t), \mathfrak{U}\psi(t)) \\ \psi(t) &\in& Y_l(1,1) \\ n(t) &\in& C_P \\ \psi(0) &=& \mathfrak{U}^{-1}\psi_0 \\ n(0) &=& n_0, \end{cases} \tag{12.2}$$

where $\mathfrak{H}(n) := \mathfrak{U}^{-1}\mathfrak{A}(n)\mathfrak{U}$,

$$\mathfrak{U}^{-1}\begin{pmatrix} -\partial_x & 0 \\ 0 & \partial_x \end{pmatrix}\mathfrak{U} = \begin{pmatrix} -\partial_x + a & 0 \\ 0 & \partial_x + a \end{pmatrix}$$

and

$$\mathfrak{U}^{-1}\mathfrak{L}(n)\mathfrak{U} = \begin{pmatrix} \beta(\cdot, n(\cdot)) & r_0^{-1}e^{2a\cdot}\kappa(\cdot) \\ -r_0 e^{-2a\cdot}\overline{\kappa}(\cdot) & \beta(\cdot, n(\cdot)) \end{pmatrix}.$$

(note that if $\psi \in Y_l(1,1)$ then $\mathfrak{U}^{-1}\mathcal{L}\mathfrak{U}\psi \notin Y_l(1,1)$ unless $r_0 = r_m = 1$) In the T-equivariant system (12.2) the bifurcation parameters r_0, r_m appear in the operators on the right hand side and the function space $Y_l(1,1)$ stays constant.

A solution to (12.2) of the form

$$(\psi(t), n(t)) = T(\omega t)(\hat{\psi}, \hat{n}) \quad \text{for some } \omega \in \mathbb{R} \setminus \{0\},\ \hat{\psi} \neq 0 \text{ and } \hat{n}$$

is called a rotating wave or relative equilibrium, because with respect to the moving time frame obtained by the change of variable

$$(\psi(t), n(t)) \mapsto T(-\omega t)(\psi(t), n(t))$$

system (12.2), where $\mathfrak{H}(n)$ is replaced with $(\mathfrak{H}(n) - i\omega)$, has the family (or group orbit) of equilibria $T(\theta)(\hat{\psi}, \hat{n})$, $\theta \in S^1$.
In the linear case $\mathfrak{K} = 0$ in (12.2) there exists a rotating wave solution if and only if there exists $\hat{n} \in C_p$, $i\omega \in \sigma(\mathfrak{H}(\hat{n}))$ and an eigenfunction $\hat{\psi}$ to the eigenvalue $i\omega$ so that $\mathfrak{F}(\hat{n}, \mathfrak{U}\hat{\psi}) = 0$. This is equivalent to the solvability of

$$h(i\omega, \hat{n}) = 0 \quad \text{and} \quad \mathfrak{F}(\hat{n}, |S|^2 \mathfrak{U}\hat{\psi}_0(\omega, \hat{n})) = 0 \tag{12.3}$$

for some $S, \omega \in \mathbb{R}$ and $\hat{n}$, in this case $\hat{\psi} = |S|^2 \hat{\psi}_0$. Here $\hat{\psi}_0(\omega, \hat{n})$ is a normalized eigenfunction with a certain fixed boundary (recall that all eigenvalues are geometrically simple), e.g. $\hat{\psi}_0(\omega, \hat{n})(0) = (r_0, 1)$, and $h(\lambda, n)$ is the characteristic function to $\mathfrak{H}(n)$.
If $\mathfrak{K} \neq 0$ then (12.2) has a rotating wave solution iff there exist $S \in \mathbb{R}$, $\omega \in \mathbb{R}$ and $\hat{n}$ so that

$$\begin{pmatrix} r_m & -1 \end{pmatrix} R(\omega, \hat{n}) S \begin{pmatrix} r_0 \\ 1 \end{pmatrix} = 0 \quad \text{and} \quad \mathfrak{F}(\hat{n}, \mathfrak{U} R(\omega, \hat{n}) S \begin{pmatrix} r_0 \\ 1 \end{pmatrix}) = 0, \tag{12.4}$$

where $R(\omega, \hat{n}) d = \psi$, $d \in \mathbb{C}^2$, is the solution to

$$(\mathfrak{H}(n) - i\omega)\, \psi + \epsilon \mathfrak{U}^{-1} \mathfrak{K}(\hat{n}, \mathfrak{U}\psi) = 0 \quad \text{with initial value} \quad \psi(0) = d.$$

We assume that such a rotating wave exists and that - by means of the implicit function theorem applied to (12.3) or (12.4) - it (i.e. the parameters ω, S and $\hat{n}$) depends smoothly on some bifurcation parameter.

Hence we suppose that there exist $\hat{\psi}(\lambda) \in Y_l(1,1) \cap W^{1,2}([0,l], \mathbb{C}^2)$ and $\hat{n}(\lambda) \in W_P^{1,2}$ (see (10.9)), which depend smoothly on a bifurcation parameter λ in some neighbourhood of zero in $\mathbb{R}$, such that $T(\theta)(\hat{\psi}(\lambda), \hat{n}(\lambda))$ is a family of equilibria for the system

$$\begin{cases} \partial_t \begin{pmatrix} \psi(t) \\ n(t) \end{pmatrix} & = & \mathfrak{G}(\psi(t), n(t), \lambda) \\ \psi(t) & \in & Y_l(1,1) \\ n(t) & \in & C_P \\ \psi(0) & = & \psi_0 \\ n(0) & = & n_0. \end{cases} \tag{12.5}$$

Here $\mathfrak{G}$ is T equivariant and of the form $\mathfrak{G}(\psi, n, \lambda) = (\mathfrak{A}_0\psi, 0) + \mathfrak{Q}(\psi, n, \lambda)$, where $\mathfrak{Q}(\psi, n, \lambda)(x) = Q(x, \psi(x), n(x), \lambda)$ is a Nemytskij operator generated by a C^k Caratheodory function $Q :]0, l[\times \mathbb{C}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{C}^2 \times \mathbb{R}$ (in the sense of Definition 10.11) which is piecewise smooth on finite sections of the interval $[0, l]$. We consider the linearization $\partial\mathfrak{G}(\hat{\psi}(\lambda), \hat{n}(\lambda), \lambda)_{|\lambda=0}$ which is a closed densely defined degenerate hyperbolic system with equal speed in the real Banach space $\mathfrak{P} \subset C([0, l], \mathbb{R}^4) \times C_P$ (see (10.10)). For spectral properties we have to complexifiy the operator and Banach space. It is not difficult to check that the linearization belongs to the class (DH) with equal speeds which we have treated in section 6.2 (for checking condition (DHII) we use that $\hat{n}(\lambda)$ is piecewise smooth).

Note that here $n \in C_P$ is only piecewise continuous although we assumed for (SH) and (DH) in sections 7 and 6.2 that the degenerate part of the equation is continuous on the whole interval $[0, l]$. We give the following answer to this observation: First, it is not difficult to extend the results to piecewise continuous data in the degenerate equation. Second, one can avoid the space of piecewise continuous functions (due to appearing discontinuities) if one writes the traveling wave equation as a hyperbolic system of extended size with everywhere smooth Nemytskij operators as explained in section 3.2 (this moreover has the advantage that one can treat internal reflectivities). The extended system then belongs to the class (SH) and the linearization belongs to (DH). We assume that $b_k = 0$ in the model (3.17).

We assume that the reduced system satisfies $\gamma_+ < 0$. It follows from the model of the gain function (3.14) and recombination function (3.16) and the positivity of the constants A, B, C, g_d^k, see table 3.1 in section 3.2, that there exists $\delta > 0$ so that the spectral set Σ (see (6.5)) satisfies

$$\Sigma \subset \mathbb{C}_{-\infty,-\delta}.$$

From the T-equivariance it follows that zero is an eigenvalue of $\partial\mathfrak{G}(T(\theta)(\hat{\psi}(\lambda), \hat{n}(\lambda)))$ with eigenvector

$$T'(\theta)(\hat{\psi}(\lambda), \hat{n}(\lambda)) = (ie^{i\theta}\hat{\psi}(\lambda), 0).$$

We suppose that zero has algebraic multiplicity one, there is a pair of simple complex conjugate eigenvalues $\pm i\sigma$, $\sigma \neq 0$ and the remaining eigenvalues in $\mathbb{C} \setminus \Sigma$ have real part strictly less then zero.
Note that

$$\partial\mathfrak{G}(T(\theta)(\psi, n), \lambda) = T(\theta)\partial\mathfrak{G}((\psi, n), \lambda)T(\theta)^{-1}.$$

Hence

$$\sigma(\partial\mathfrak{G}(T(\theta)(\psi, n), \lambda)) = \sigma(\partial\mathfrak{G}((\psi, n), \lambda))$$

and $T(\theta)$ maps the generalized eigenspaces of $\partial\mathfrak{G}((\psi, n), \lambda)$ to the corresponding eigenspaces of $\partial\mathfrak{G}(T(\theta)(\psi, n), \lambda))$. Along the group orbit $T(\theta)(\hat{\psi}(0), \hat{n}(0))$,

$\theta \in S^1$, the linearized system has a vector bundle of two dimensional (real) invariant center subspaces associated with the eigenvalues $\pm i\sigma$. Theorem 6.15 implies that this bundle is exponentially attracting, more precisely it is a normally hyperbolic three dimensional center manifold with respect to the C topology. Theorem 8.10 implies that close to the relative equilibrium $T(\theta)(\hat{\psi}(0), \hat{n}(0))$ this manifold persists. Hence we have that there exists a local three dimensional exponentially attracting normally hyperbolic center manifold for (12.5) in a neighbourhood of $T(\theta)(\hat{\psi}, \hat{n})$. In the following we will calculate the flow on the center manifold and prove Theorem 12.2.

We have the following assumption: For λ in a small neighbourhood Λ of zero we have curves $\rho_1(\lambda)$ and $\rho_2(\lambda) = \overline{\rho_1}(\lambda)$ of complex conjugated simple eigenvalues of $\partial\mathfrak{G}(\hat{\psi}(\lambda), \hat{n}(\lambda), \lambda)$ with $\rho_1(0) = i\sigma$, $\sigma > 0$, a constant simple eigenvalue zero due to the T symmetry, and a set of discrete eigenvalues in $\mathbb{C} \setminus \Sigma$ separated uniformly for $\lambda \in \Lambda$. Hence there exists $\alpha < 0$ so that

$$\sigma(\partial\mathfrak{G}(\hat{\psi}(\lambda), \hat{n}(\lambda), \lambda)) \setminus \{\rho_1(\lambda), \rho_2(\lambda), 0\} \subset \{c \in \mathbb{C} \mid \mathfrak{Re}\, c \leq \alpha\} \quad \text{for } \lambda \in \Lambda. \tag{12.6}$$

Let $w(\lambda)$ be a smooth family of eigenfunctions to the eigenvalue $\rho_1(\lambda)$. Put

$$v_1(\lambda) := \mathfrak{Re}\, w(\lambda) \quad \text{and} \quad v_2(\lambda) := -\mathfrak{Im}\, w(\lambda)$$

for $\lambda \in \Lambda$. Let $\pi_c(\lambda)$ be the spectral projection corresponding to the three critical eigenvalues $\rho_1(\lambda)$, $\rho_2(\lambda)$ and zero. Put

$$\begin{aligned} \pi_s(\lambda) &:= (I - \pi_c(\lambda)), \\ X_s(\lambda) &:= \mathfrak{Re}\,\mathrm{Im}\,\pi_s(\lambda), \\ X_s &:= X_s(0). \end{aligned}$$

We have $\overline{\pi_s(\lambda)u} = \pi_s(\lambda)\overline{u}$. Hence $\mathrm{Im}\,\pi_s(\lambda)$ is invariant under complex conjugation and the complexification of $X_s(\lambda)$ is $\mathrm{Im}\,\pi_s(\lambda)$. It follows from Lemma 8.2 that the projection $\pi_s(\lambda)$ maps X_s isomorphically onto $X_s(\lambda)$ for λ close to zero. We assume that this holds true for $\lambda \in \Lambda$ (if necessary make Λ smaller).
Introduce the coordinates

$$N(x_1, x_2, x_s, \theta, \lambda) : \mathbb{R} \times \mathbb{R} \times X_s \times S^1 \times \Lambda \mapsto \mathfrak{P},$$

$$N(x_1, x_2, x_s, \theta, \lambda) := T(\theta) \left[\begin{pmatrix} \hat{\psi}(\lambda) \\ \hat{n}(\lambda) \end{pmatrix} + v_1(\lambda)x_1 + v_2(\lambda)x_2 + \pi_s(\lambda)x_s \right].$$

Equation (12.5) then recasts as [1]

$$\partial_t \begin{pmatrix} x_1 \\ x_2 \\ x_s \\ \theta \end{pmatrix} = Q(x_1, x_2, x_s, \theta, \lambda), \tag{12.7}$$

where [2]

$$Q(x_1, x_2, x_s, \theta, \lambda) \ := \ \partial N^{-1}(N(x_1, x_2, x_s, \theta, \lambda))\mathfrak{G}(N(x_1, x_2, x_s, \theta, \lambda), \lambda).$$

Because

$$\partial N^{-1}(T(\sigma)p)T(\sigma) = \partial N^{-1}(p)$$

and $\mathfrak{G}$ is T equivariant we have the following important

Remark 12.1. *The operator Q does not depend on $\theta \in S^1$:*

$$Q(x_1, x_2, x_s, \theta, \lambda) = Q(x_1, x_2, x_s, 0, \lambda) \quad \textit{for } \theta \in S^1.$$

Hence we write

$$Q(x_1, x_2, x_s, \lambda) := Q(x_1, x_2, x_s, 0, \lambda).$$

Expanding $\mathfrak{G}$ and ∂N^{-1} around $N(0,0,0,0,\lambda) = \begin{pmatrix} \hat{\psi}(\lambda) \\ \hat{n}(\lambda) \end{pmatrix}$ we get [3]:

$$\begin{aligned} Q(x_1, x_2, x_s, \lambda) \ &= \ \partial N^{-1} \begin{pmatrix} \hat{\psi}(\lambda) \\ \hat{n}(\lambda) \end{pmatrix} \left[\partial \mathfrak{G} \begin{pmatrix} \hat{\psi}(\lambda) \\ \hat{n}(\lambda) \\ \lambda \end{pmatrix} (v_1(\lambda)x_1 + v_2(\lambda)x_2 + \pi_s(\lambda)x_s) \right] \\ &\quad + r(x_1, x_2, x_s, \lambda) \\ &= \ \partial N^{-1} \begin{pmatrix} \hat{\psi}(\lambda) \\ \hat{n}(\lambda) \end{pmatrix} \big[(\mathfrak{Re}\, \rho_1(\lambda)\, x_1 - \mathfrak{Im}\, \rho_1(\lambda)\, x_2) v_1(\lambda) \qquad (12.8) \\ &\quad + (\mathfrak{Im}\, \rho_1(\lambda)\, x_1 + \mathfrak{Re}\, \rho_1(\lambda)\, x_2) v_2(\lambda) \\ &\quad + \partial \mathfrak{G} \begin{pmatrix} \hat{\psi}(\lambda) \\ \hat{n}(\lambda) \\ \lambda \end{pmatrix} \pi_s(\lambda) x_s \big] + r(x_1, x_2, x_s, \lambda) \\ &= \begin{pmatrix} \mathfrak{Re}\, \rho_1(\lambda)\, x_1 - \mathfrak{Im}\, \rho_1(\lambda)\, x_2 \\ \mathfrak{Im}\, \rho_1(\lambda)\, x_1 + \mathfrak{Re}\, \rho_1(\lambda)\, x_2 \\ \pi_s(\lambda)^{-1} \partial \mathfrak{G}(\hat{\psi}(\lambda), \hat{n}(\lambda), \lambda) \pi_s(\lambda) x_s \\ 0 \end{pmatrix} + r(x_1, x_2, x_s, \lambda) \end{aligned}$$

---

[1] N is locally diffeomorphic also in the L^p completion, $1 \le p < \infty$, for each λ, compare with proposition 7.9

[2] the dependence of ∂N^{-1} on λ is not indicated in the notation

[3] $\partial N^{-1} \begin{pmatrix} \hat{\psi}(\lambda) \\ \hat{n}(\lambda) \end{pmatrix} (v_1(\lambda)x_1 + v_2(\lambda)x_2 + \pi_s(\lambda)x_s + T'(0) \begin{pmatrix} \hat{\psi}(\lambda) \\ \hat{n}(\lambda) \end{pmatrix} \theta) = (x_1, x_2, x_s, \theta)$

with

$$r(0,0,0,\lambda) = 0, \quad \partial_{(x_1,x_2,x_s,\lambda)} r(0,0,0,\lambda) = 0 \quad \text{for } \lambda \in \Lambda.$$

We use a standard "trick" and consider the parameter λ as an additional variable in state space by adding the equation

$$\frac{d}{dt}\lambda = 0 \tag{12.9}$$

to system (12.7)/(12.8). It follows that system (12.8) without the nonlinearity r has the invariant manifold $x_s = 0$ for x_1, x_2 and λ sufficiently small and $\theta \in S^1$. The spectral gap (12.6) for the generator and the spectral gap mapping Theorem 6.15 imply that it is normally hyperbolic: Indeed, we have the following formula for the flow S_0^t generated by (12.8) without r and (12.9)

$$S_0^t \begin{pmatrix} x_1 \\ x_2 \\ \theta \\ \lambda \\ x_s \end{pmatrix} = \begin{pmatrix} e^{A_c(\lambda)t} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \theta \\ \lambda \\ \pi_s(\lambda)^{-1} e^{A_s(\lambda)t} \pi_s(\lambda) \end{pmatrix},$$

where

$$A_c(\lambda) := \begin{pmatrix} \mathfrak{Re}\,\rho_1(\lambda) & -\mathfrak{Im}\,\rho_1(\lambda) \\ \mathfrak{Im}\,\rho_1(\lambda) & \mathfrak{Re}\,\rho_1(\lambda) \end{pmatrix}, \quad A_s(\lambda) := \mathfrak{G}(\hat{\psi}(\lambda), \hat{n}(\lambda), \lambda).$$

To verify condition (8.1) we have to linearize the flow in $x_s = 0$ with respect to all variables x_1, x_2, x_s, θ and λ, and use Theorem 6.15 and (12.6), the λ derivatives do not cause any difficulties. Because in a sufficiently small neighbourhood of $x_1^* = 0$, $x_2^* = 0$, $x_s^* = 0$, $\lambda^* = 0$ for $\theta \in S^1$ we have that (12.8) is a small C^1 perturbation of the system (12.8) without r, and because we can modify the equations so that the starting center manifold for $r = 0$ becomes overflowing Theorem 8.10 and Remark 8.11 imply that this manifold persists for (12.7) (we omit the details, the proof can be done similar as for Theorem 8.15): There exists a $\delta > 0$ and a C^k smooth graph $\gamma :]-\delta,\delta[^3 \times S^1 \to X_s$, $\gamma = \gamma(x_1, x_2, \lambda, \theta)$ so that the persistent manifold is given as the graph of γ. Because Q does not depend on θ the graph γ also does not depend on θ. Hence we have proven the following:

Theorem 12.2. *There exists $\delta > 0$ and a smooth function $\gamma :]-\delta,\delta[^3 \to X_s$, which satisfies $\gamma(0,0,\lambda) = 0$ for $|\lambda| < \delta$, such that for $|\lambda| < \delta$*

$$M := \{N(x_1, x_2, \gamma(x_1, x_2, \lambda), \theta, \lambda) \mid |x_1|, |x_2| < \delta\}$$

is a locally invariant exponentially attracting center manifold for (12.5). *Any global time reversible trajectory which stays in a sufficiently small tubular neighbourhood of M lies on M.*
The flow on M is given by the ordinary differential equation

$$\partial_t \begin{pmatrix} x_1 \\ x_2 \\ \theta \end{pmatrix} = \begin{pmatrix} \mathfrak{Re}\rho_1(\lambda) & -\mathfrak{Im}\rho_1(\lambda) & 0 \\ \mathfrak{Im}\rho_1(\lambda) & \mathfrak{Re}\rho_1(\lambda) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \theta \end{pmatrix} + h(x_1, x_2, \lambda), \tag{12.10}$$

where

$$h(x_1, x_2, \theta, \lambda) := \pi_{12\theta} r(x_1, x_2, \gamma(x_1, x_2, \lambda), \lambda),$$

$\pi_{12\theta}$ *denotes the projection onto x_1, x_2 and θ by dropping x_s. The nonlinearity h is of order two, i.e.*

$$h(0,0,\lambda) = 0 \quad \textit{and} \quad \partial_{(x_1,x_2,\lambda)} h(0,0,\lambda) = 0 \quad \textit{for } \lambda \in \Lambda.$$

In (12.10) the (x_1, x_2) variables are decoupled from θ. If we assume

$$\frac{d}{d\lambda} \mathfrak{Re}\, \rho_1(0) \neq 0, \tag{12.11}$$

then generically, as the parameter λ crosses zero, a Hopf bifurcation will occur. Hence for λ near zero there exists a periodic solution $x_\pi(t,\lambda) = (x_{1\pi}(t,\lambda), x_{2\pi}(t,\lambda))$ with period $\pi_0(\lambda)$ close to $\frac{2\pi}{\mathfrak{Im}\,\rho_1(0)}$. Solving for θ we have

$$\theta(t) = \theta_0 + \int_0^t h(x_{1\pi}(s,\lambda), x_{2\pi}(s,\lambda), \lambda)\, ds,$$

where θ_0 is the phase at $t = 0$. We can write

$$\theta(t) = ct + \rho_\pi(t),$$

where

$$\begin{aligned} c &:= \frac{1}{\pi_0(\lambda)} \int_0^{\pi_0(\lambda)} h(x_{1\pi}(s,\lambda), x_{2\pi}(s,\lambda), \lambda)\, ds, \\ \rho_\pi(t) &:= \theta_0 + \int_0^t h(x_{1\pi}(s,\lambda), x_{2\pi}(s,\lambda), \lambda)\, ds - ct \end{aligned}$$

and ρ_π is periodic with period π_0. When we go back to (ψ, n) coordinates we have

Corollary 12.3. *Under assumption* (12.11) *for λ near zero there generically exists a solution* $(\psi(t), n(t))$ *on the center manifold* M *for* (12.5) *of the form*

$$(\psi(t), n(t)) = T(ct)p(t), \tag{12.12}$$

where

$$\begin{aligned} p(t) \quad &= \quad T(\rho_\pi(t))\Big[\begin{pmatrix} \hat{\psi}(\lambda) \\ \hat{n}(\lambda) \end{pmatrix} + v_1(\lambda)x_{1\pi}(t) + v_2(\lambda)x_{2\pi}(t) \\ &\qquad + \pi_s(\lambda)\gamma(x_{1\pi}(t), x_{2\pi}(t), \lambda)\Big] \end{aligned}$$

is $\pi_0(\lambda)$ periodic.

Remark 12.4. *Solutions of the form* (12.12) *are called self pulsations or modulated waves, c is called the optical frequency and $\frac{2\pi}{\pi_0}$ is called the power frequency.*

Figure 12.1 shows a space time plot of a self pulsation calculated numerically with LDSL [53, 56, 80].

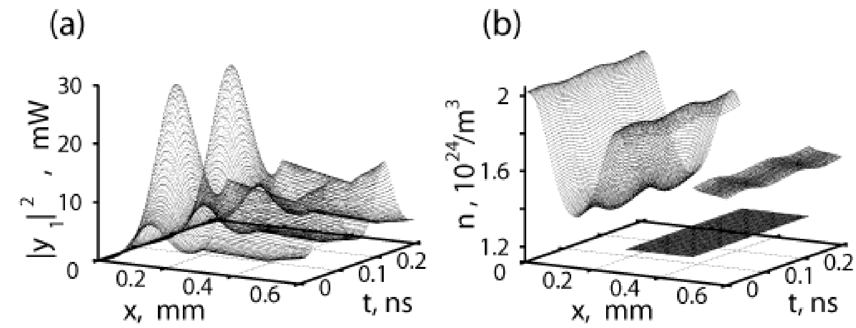

Figure 12.1: A space time plot of optical field amplitutes (left) and carrier densities (right) of a self pulsating 3-section DFB semiconductor laser calculated with LDSL

Chapter 13

Appendix

13.1 The Fejér Laplace and Fourier inversion formulas

In most cases the Fourier transform of an integrable function, even if it is bounded with compact support, will not be integrable. For example the Fourier transform of the characteristic function $\chi_{[-1,1]}$ is the nonintegrable function $2\frac{\sin\omega}{\omega}$. Therefore it is of interest to have generalizations of the Fourier inversion formula when the Fourier transform is not integrable. In our work we need the inversion formula for compactly supported discontinuous functions for which both $f(t+) := \lim_{h\downarrow 0} f(t+h)$ and $f(t-) := \lim_{h\downarrow 0} f(t-h)$ exist at each $t \in \mathbb{R}$. One can show that such functions have a Fourier transform which is integrable in the weaker Cesaro sense and the Fourier inversion theorem holds. A precise statement of this is given in Corollary 13.3. This generalized inversion theorem does not seem to be well established in Fourier Analysis textbooks. A proof of it can be found for the Laplace transformation in the classical book [77], the Fourier version follows immediately from the Laplace version.

In Remark 13.4 we note that if f is bounded measurable then the Cesaro integrals in the inversion formula have $\|f\|_{L^\infty}$ as a uniform bound. This fact which we have not found in the literature is of importance if one has to deal with multiple integrals containing Fourier integrals and wants to pass to the limit using Lebesgue's dominated convergence theorem. We will use this Remark several times when we prove the spectral gap of the semigroup generated by linear hyperbolic evolution equations needed for the existence of smooth center manifolds.

Definition 13.1 (Cesaro integrability)**.** The function $f \in L^1_{loc}(\mathbb{R}, \mathbb{C})$ is integrable by Cesaro means of order 1 if the limit

$$\lim_{R\to\infty} \int_{-R}^{R} \left(1 - \frac{|x|}{R}\right) f(x)\, dx$$

exists. If f is Cesaro integrable we denote this limit by

$$(C,1) - \int_{-\infty}^{\infty} f(x)\, dx.$$

By partial integration we have

$$\begin{aligned} \int_{-R}^{R} \left(1 - \frac{|x|}{R}\right) f(x)\, dx &= \int_{-R}^{R} f(x)\, dx - \frac{1}{R}\int_0^R x\left(f(x) + f(-x)\right)\, dx \\ &= \frac{1}{R}\int_0^R \int_{-t}^{t} f(x)\, dx\, dt. \end{aligned} \tag{13.1}$$

From (13.1) it is easy to see that if Cauchy's principal value

$$\mathrm{PV} \int_{-\infty}^{\infty} f(x)\, dx := \lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx$$

exists then f is $(C,1)$ integrable and

$$(C,1) - \int_{-\infty}^{\infty} f(x)\, dx = PV \int_{-\infty}^{\infty} f(x)\, dx.$$

Therefore Cesaro integrability is a weaker notion than integrability in the sense of Cauchy's principal value which is weaker than the usual notion of L^1 (absolute) integrability.

Let $f \in L^1_{loc}([0,\infty[, \mathbb{C})$. Then the Laplace transform $\mathfrak{L}f$ of f is defined by

$$\mathfrak{L}f(s) := \int_0^{\infty} e^{-st} f(t)\, dt := \lim_{R\to\infty} \int_0^R e^{-st} f(t)\, dt$$

whenever this integral exists as a convergent improper integral. If this integral converges for some $s_0 \in \mathbb{C}$ then it converges for all $s \in \mathbb{C}$ with $\mathfrak{Re} s > \mathfrak{Re} s_0$ (see Chapter II in [77]). Hence three possibilities arise:
$\imath$) the integral converges for no point
$\imath\imath$) it converges for every point
$\imath\imath\imath$) there exists $\sigma_c \in \mathbb{R}$ such that the integral converges for all $s \in \mathbb{C}$ with $\mathfrak{Re}\, s > \sigma_c$ and diverges for $\mathfrak{Re}\, s < \sigma_c$.
The real number σ_c is called the abscissa of convergence. In case $\imath$) one sets $\sigma_c := \infty$, in case $\imath\imath$) $\sigma_c := -\infty$.

Theorem 13.2 (Fejér Laplace Inversion Theorem)**.** *(see [77] Theorem 9.2 p.77) For* $c \in]\sigma_c, \infty[$

$$\frac{1}{2\pi i}(C,1) - \int_{c-i\infty}^{c+i\infty} e^{st} \left(\mathfrak{L}f\right)(s) ds = \begin{cases} 0 & , t < 0 \\ \frac{1}{2} f(0+) & , t = 0 \\ \frac{1}{2}\left(f(t+) + f(t-)\right) & , t > 0 \end{cases} . \quad (13.2)$$

That is the Cesaro integral

$$\begin{aligned} (C,1) - \int_{c-i\infty}^{c+i\infty} e^{st} \left(\mathfrak{L}f\right)(s) ds &:= \lim_{R\to\infty} \frac{1}{R} \int_0^R \int_{-\tau}^{\tau} e^{(c+i\nu)t} \left(\mathfrak{L}f\right)(c+i\nu) i \, d\nu \, d\tau \\ &= \lim_{R\to\infty} \int_{-R}^{R} e^{(c+i\nu)t} \left(\mathfrak{L}f\right)(c+i\nu) \left(1 - \frac{|\nu|}{R}\right) i \, d\nu \end{aligned}$$

converges to $2\pi i$ *times the right side of* (13.2) *whenever the right side has a meaning.*

As an immediate consequence we have the general Fourier Inversion formula

Corollary 13.3 (Fejér Fourier Inversion Formula)**.** *Let* $f \in L^1_{loc}(\mathbb{R}, \mathbb{C})$ *and* $t \in \mathbb{R}$ *be a point where both the limit from the right* $f(t+)$ *and left* $f(t-)$ *exist. Assume* $f_1(s) := f(s)$, $f_2(s) := f(-s)$, $s \geq 0$, *have both* $\sigma_c < 0$. *Suppose that the Fourier transform of* f *in Cauchy's principal value sense exists, that is* $\mathcal{F}^{-1} f = \lim_{R\to\infty} \int_{-R}^{R} e^{-i\omega\tau} f(\tau) \, d\tau$ *converges. Then*

$$\frac{1}{2\pi}(C,1) - \int_{-\infty}^{\infty} e^{i\omega t} \mathcal{F}^{-1} f(\omega) \, d\omega = \frac{1}{2}\left(f(t+) + f(t-)\right). \quad (13.3)$$

Instead of $f \in L^1_{loc}(\mathbb{R}, \mathbb{C})$, $\sigma_c < 0$, we can simply assume $f \in L^1(\mathbb{R}, \mathbb{C})$. This guarantees the uniform convergence of $\int_0^\infty e^{-iuy} f(y) \, dy$ for $-R \leq u \leq R$ and allows for interchanging the order of integration by Fubini, see the proof of Theorem 9.2 in [77, p.77] for the details.

Remark 13.4. *If* $f \in L^\infty(\mathbb{R}, \mathbb{C}) \cap L^1(\mathbb{R}, \mathbb{C})$ *the limit* (13.3) *has* $\|f\|_{L^\infty}$ *as a uniform majorant.*

Proof. Since $f \in L^1$ the order of integration can be exchanged due to Fubinis theorem and we have

$$
\begin{aligned}
& \left|\frac{1}{2\pi}\int_{-R}^{R} e^{i\omega t}\left(1-\frac{|\omega|}{R}\right)\int_{-\infty}^{\infty} e^{-i\omega y} f(y)\,dy\,d\omega\right| \\
= & \left|\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2\pi R i(t-y)}\left(\int_{0}^{R} e^{i\omega(t-y)}\,d\omega-\int_{-R}^{0} e^{i\omega(t-y)}\,d\omega\right) f(y)\,dy\right| \\
= & \left|\frac{2}{\pi R}\int_{-\infty}^{\infty}\frac{\sin^2((t-y)R/2)}{(t-y)^2} f(y)\,dy\right| \\
\leq & \frac{\|f\|_{L^\infty}}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2 u}{u^2}\,du \\
= & \|f\|_{L^\infty}
\end{aligned}
$$

Here we have used the identity $\int_{-\infty}^{\infty}\frac{\sin^2 u}{u^2}\,du=\pi$, which can be calculated as follows: For $0<r_1<r_2$ let γ be the positively oriented loop around the origin 0 by first going along the path $\gamma_2(s):=r_2e^{i\pi s}$ $(0\leq s\leq 1)$, then along $[-r_2,-r_1]$, $\gamma_1(s):=r_1e^{i\pi(s-1)}$ $(0\leq s\leq 1)$ and $[r_1,r_2]$. By the residue theorem $\int_\gamma \frac{1}{2}\frac{1-e^{2iz}}{z^2}\,dz=2\pi$. Further we have $\lim_{r_1\downarrow 0}\int_{\gamma_1}\frac{1}{2}\frac{1-e^{2iz}}{z^2}\,dz=-\lim_{r_1\downarrow 0}\int_{\gamma_1} iz^{-1}\,dz=\pi$. Hence

$$
\begin{aligned}
\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}\,dx &= \lim_{r_1\downarrow 0, r_2\uparrow\infty}\left(\int_{-r_2}^{-r_1}\frac{\sin^2 x}{x^2}\,dx+\int_{r_1}^{r_2}\frac{\sin^2 x}{x^2}\,dx\right) \\
&= \lim_{r_1\downarrow 0, r_2\uparrow\infty}\mathfrak{Re}\left(\int_\gamma\frac{1}{2}\frac{1-e^{2iz}}{z^2}\,dz-\int_{\gamma_1}\frac{1}{2}\frac{1-e^{2iz}}{z^2}\,dz\right) \\
&= \pi.
\end{aligned}
$$

□

Proposition 13.5. *Let $a\in\mathbb{R}$ and δ_{-a} be the delta distribution at $-a$. Then*

$$\mathfrak{F}e^{ia\cdot}=\delta_{-a}\quad \textit{in } \mathcal{S}^*.$$

13.2 Regularity for linear inhomogeneous evolution equations

In this short section we state and proof a well known regularity result (see for example [15, Proposition 4.1.6, p.51]).

Let X be a Banach space and A be a closed densely defined operator in X with domain $\mathcal{D}(A)$ generating a C_0 semigroup $(T(t))_{t\geq 0}$ of bounded linear operators on X.

Proposition 13.6. *Let $f \in W^{1,1}(]0,T[\,,X)$ and*

$$v(t) := \int_0^t T(t-s)f(s)\,ds.$$

Then

$$v \in C([0,T],\mathcal{D}(A)) \cap C^1([0,T],X)$$

and $\frac{d}{dt}v(t) = Av(t) + f(t)$.

Proof. We follow the proof of [15, Proposition 4.1.6, p.51]. Let $t \in [0,T[$ and $h \in [0,T-t]$. We have

$$\frac{v(t+h)-v(t)}{h} = \int_0^t T(s)\frac{f(t+h-s)-f(t-s)}{h}\,ds + \frac{T(h)}{h}\int_0^h T(t-s)f(s)\,ds.$$

Because

$$\frac{f(t+h-\cdot)-f(t-\cdot)}{h} \overset{L^1([0,t],X)}{\underset{h\downarrow 0}{\longrightarrow}} f'(t-\cdot)$$

it follows that $\frac{d^+v}{dt}(t) = \int_0^t T(s)f'(t-s)\,ds + T(t)f(0)$ for all $t \in [0,T[$. Hence $\frac{d^+v}{dt} \in C([0,T],X)$. Similarly one shows for $t \in]0,T]$ that $\frac{d^-v}{dt} \in C(]0,T]\,,X)$ and $\frac{d^-v}{dt}(t) = \frac{d^+v}{dt}(t)$ for $t \in]0,T[$. So $v \in C^1([0,T],X)$.
Let $t \in [0,T[$ and $h \in [0,T-t]$. We have

$$\begin{aligned} \frac{T(h)-I}{h}v(t) &= \frac{1}{h}\int_0^t T(t+h-s)f(s)\,ds - \frac{1}{h}\int_0^t T(t-s)f(s)\,ds \\ &= \frac{v(t+h)-v(t)}{h} - \frac{1}{h}\int_t^{t+h} T(t+h-s)f(s)\,ds. \end{aligned}$$

By letting $h \downarrow 0$ it follows that for $t \in [0,T[$ we have $v(t) \in \mathcal{D}(A)$ and $Av(t) = v'(t) - f(t)$. Since A is closed this remains true for $t = T$. Hence $v \in C([0,T],\mathcal{D}(A))$. $\square$

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