

On the density of classes of closed convex sets with pointwise constraints in Sobolev spaces [☆]

M. Hintermüller^a, C.N. Rautenberg^a

^a*Institute for Mathematics, Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany*

Abstract

For a Banach space X of \mathbb{R}^M -valued functions on a Lipschitz domain, let $\mathbf{K}(X) \subset X$ be a closed convex set arising from pointwise constraints on the value of the function, its gradient or its divergence, respectively. The main result of the paper establishes, under certain conditions, the density of $\mathbf{K}(X_0)$ in $\mathbf{K}(X_1)$ where X_0 is densely and continuously embedded in X_1 . The proof is constructive, utilizes the theory of mollifiers and can be applied to Sobolev spaces such as $H_0(\operatorname{div}, \Omega)$ and $W_0^{1,p}(\Omega)$, in particular. It is also shown that such a density result cannot be expected in general.

Keywords: Closed convex sets, dense embedding, gradients, pointwise constraints, Sobolev spaces.

1. Introduction

Many problems in the calculus of variations involve, either directly or through (Fenchel) dualization, constraint sets of the type

$$\mathbf{K}(X) := \{\mathbf{f} \in X : |(\mathbf{G}\mathbf{f})(x)| \leq \alpha(x) \text{ a.e., } x \in \Omega\},$$

[☆]This research was supported by the Austrian Science Fund FWF through START-Projekt Y305 "Interfaces and Free Boundaries", the FWF-SFB F32 04-N18 "Mathematical Optimization and Its Applications in Biomedical Sciences", the DFG Research Center MATHEON through Project C28, and the DFG Priority Program SPP 1253 "Optimization with Partial Differential Equations". This research was conducted when Carlos N. Rautenberg was part of the Institute for Mathematics and Scientific Computing, University of Graz, Heinrichstrasse 36, 8010 Graz, Austria

Email addresses: hint@math.hu-berlin.de (M. Hintermüller), carlos.rautenberg@math.hu-berlin.de (C.N. Rautenberg)

where $\Omega \subset \mathbb{R}^N$ represents some underlying domain, $N \in \mathbb{N}$, X is a Banach space of functions on Ω , $|\cdot|$ stands for the Euclidian norm, α denotes a sufficiently regular function with $\alpha(x) \geq \underline{\alpha} > 0$ for $x \in \Omega$, and where ‘‘a.e.’’ stands for ‘‘almost everywhere’’. Further, the operator G takes one of the following choices:

$$G = \text{id}, \quad G = \nabla, \quad G = \text{div}.$$

Let X_1 denote, for instance, a Hilbert space of \mathbb{R}^M -valued functions over Ω and X_0 a Banach space which is continuously and densely embedded in X_1 . For approximation purposes it is often necessary to find an answer to the following question:

$$\text{Is } \mathbf{K}(X_0) \text{ dense in } \mathbf{K}(X_1) \text{ with respect to the norm in } X_1? \quad (1)$$

In general, the answer is not positive. In fact, $\mathbf{K}(X_0)$ might not even contain non-trivial elements as the following example¹ demonstrates.

A counter-example. Consider $X_1 = L^2(\Omega)^M$, where Ω is a domain in \mathbb{R}^N . Let $\{p_n\}_{n=1}^\infty$ be an enumeration of a dense set in Ω and $\phi_n(x) := |x - p_n|^{-1/4}$ for $x \neq p_n$. Note that since Ω is bounded, there exists $K > 0$ such that $|\phi_n|_{L^2(\Omega)} \leq K$ for all $n \in \mathbb{N}$. Then, $g := \sum_{k=1}^\infty k^{-2} \phi_k$ belongs to $L^2(\Omega)$, is strictly positive on $\overline{\Omega}$, and it is unbounded at each p_n , i.e., it is unbounded on a dense set.

Let X_0 be the linear space of functions $\mathbf{f} = \mathbf{h}g$ with $\mathbf{h} \in C(\overline{\Omega})^M$ and endowed with norm $|\mathbf{f}|_{X_0} = \sup_{x \in \Omega} |\mathbf{h}(x)|$, X_0 is a Banach space. Clearly, if $\mathbf{f} \in X_0$, then $\mathbf{f} \in X_1$ and $|\mathbf{f}|_{X_1} \leq |g|_{X_1} |\mathbf{h}|_{C(\overline{\Omega})^M} = |g|_{X_1} |\mathbf{f}|_{X_0}$, proving the embedding to be continuous. Let $\mathbf{f} \in C(\overline{\Omega})^M$ be arbitrary, and let $\mathbf{f}_n \in X_1$ be of the form $\mathbf{f}_n = \mathbf{h}_n g$ with $\mathbf{h}_n(x) = \mathbf{f}/g_n$ and $g_n = \sum_{k=1}^\infty k^{-2} \min(\phi_k, n) \in C(\overline{\Omega})$, where the min-operation is understood in a pointwise sense. Then it holds that

$$|\mathbf{f}_n - \mathbf{f}|_{X_1} = |\mathbf{h}_n g - \mathbf{f}|_{X_1} = |\mathbf{h}_n g - \mathbf{h}_n g_n|_{X_1} \leq |\mathbf{h}_n|_{C(\overline{\Omega})^M} |g - g_n|_{L^2(\Omega)}.$$

One readily observes that $|g - g_n|_{L^2(\Omega)} \rightarrow 0$ since we have $|\min(\phi_k, n) - \phi_k|_{L^2(\Omega)} \rightarrow 0$ for each k . Therefore, X_0 is dense in $C(\overline{\Omega})^M$ with respect to the X_1 -norm, and since the latter is dense in X_1 , X_0 is also dense in X_1 . Finally,

¹The construction is based on an idea of Martin Hairer, Department of Mathematics, University of Warwick.

for $G = \text{id}$, we have that $\mathbf{K}(X_0) = \{0\}$ which is clearly not dense in $\mathbf{K}(X_1)$. Therefore, the dense and continuous embedding of X_0 in X_1 is not sufficient for

$$\overline{\mathbf{K}(X_0)}^{X_1} = \mathbf{K}(X_1). \quad (2)$$

Motivating applications. We briefly mention several motivating applications where (1) emerges from Fenchel dualization (Ekeland and Temam, 1976) and semismooth Newton solvers (Hintermüller et al., 2003). In a rather abstract setting, *regularized* total variation type image restoration (Hintermüller and Stadler, 2006), energies related to Bingham fluids (Hintermüller and de los Reyes, 2011), simplified friction problems or elastoplastic problems in material science (Duvaut and Lions, 1976; Johnson, 1976; Carstensen, 1997; Stadler, 2004; Hintermüller and Rösler, 2013) can be associated with the following problem:

$$\text{minimize } F(\mathbf{y}) + \alpha \int_S |C\mathbf{y}(s)| ds \quad \text{over } \mathbf{y} \in Y, \quad (3)$$

where Y denotes a real Banach space with topological dual Y^* , $F : Y \rightarrow \mathbb{R}$ is of the form $F(\mathbf{y}) = \frac{1}{2} \langle A(\mathbf{y} - f), \mathbf{y} - f \rangle_{Y^*, Y} + \langle a, \mathbf{y} \rangle_{Y^*, Y} + b$ where $\langle \cdot, \cdot \rangle_{Y^*, Y}$ denotes the duality pairing between Y and Y^* , $A \in \mathcal{L}(Y, Y^*)$ is invertible, $a \in Y^*$ and $b \in \mathbb{R}$. Furthermore, $\alpha > 0$ is fixed and C is a linear and continuous operator from Y to $L^2(S, \mathbb{R}^{L \times M})$, i.e., $C \in \mathcal{L}(Y, L^2(S, \mathbb{R}^{L \times M}))$, with $L \in \mathbb{N}$, $1 \leq L, M \leq N$, and $S \subseteq \Omega$ or $S \subseteq \partial\Omega$, where $\partial\Omega$ denotes the boundary of $\Omega \subset \mathbb{R}^N$. We emphasize here that the functional associated with $\alpha > 0$ in (3) changes in the context of the (non-regularized) total variation based image restoration: see (6) and (7) below, where $Y = BV(\Omega)$. The Fenchel dual problem of (3) is given by

$$\text{minimize } F^*(C^* \mathbf{p}) \quad \text{over } \mathbf{p} \in L^2(S, \mathbb{R}^{L \times M}) \quad (4a)$$

$$\text{subject to } |\mathbf{p}(x)| \leq \alpha \quad \text{a.e., } x \in S. \quad (4b)$$

Here, $F^* : Y^* \rightarrow \mathbb{R}$ denotes the convex conjugate of F , i.e., $F^*(\mathbf{z}) = \sup\{\langle \mathbf{z}, \mathbf{y} \rangle_{Y^*, Y} - F(\mathbf{y}) : \mathbf{y} \in Y\}$, and C^* is the adjoint of C . In particular, F^* has the form

$$F^*(\mathbf{z}) = \frac{1}{2} |\mathbf{z} - a - Af|_{A^{-1}}^2 - \frac{1}{2} |Af|_{A^{-1}}^2 - b,$$

where $|w|_{A^{-1}}^2 = \langle A^{-1}w, w \rangle_{Y^*, Y}$.

For the successful application of semismooth Newton solvers for (4), depending on the properties of C , additional regularization may be required. One way to achieve an associated regularity gain for \mathbf{y} is to add a smoothing term to F^* in (4) yielding

$$\text{minimize } F^*(C^*\mathbf{p}) + \frac{1}{2\gamma}|D\mathbf{p}|_{L^2(S, \mathbb{R}^{L \times M})^N}^2 \quad \text{over } \mathbf{p} \in X_0 \quad (5a)$$

$$\text{subject to } |\mathbf{p}(x)| \leq \alpha \quad \text{a.e., } x \in S, \quad (5b)$$

where $\gamma > 0$, X_0 is continuously and densely embedded in $L^2(S, \mathbb{R}^{L \times M})$, and $D \in \mathcal{L}(X_0, L^2(S, \mathbb{R}^{L \times M})^N)$ with $|D\mathbf{p}|_{L^2(S, \mathbb{R}^{L \times M})^N} \geq \beta_0|\mathbf{p}|_{X_0}$ for some $\beta_0 > 0$ for all $\mathbf{p} \in X_0$.

Setting $X_1 := L^2(S, \mathbb{R}^{L \times M})$ and $\mathbf{K}(X) = \{\mathbf{p} \in X : |\mathbf{p}(x)| \leq \alpha \text{ a.e., } x \in S\}$, with $X = X_i, i \in \{0, 1\}$, then we arrive at the framework introduced at the beginning of this section. Now, the relation of $\mathbf{K}(X_0)$ and $\mathbf{K}(X_1)$ is of relevance when the limit as $\gamma \rightarrow \infty$ is studied in (5).

In the context of the total variation (TV) regularization in image restoration (Rudin et al., 1992), in (Hintermüller and Kunisch, 2004) it is argued that the Fenchel dual problem of

$$\text{minimize } |\operatorname{div} \mathbf{p} + f|_{L^2(\Omega)^N}^2 \quad \text{over } \mathbf{p} \in H_0(\operatorname{div}, \Omega) \quad (6a)$$

$$\text{subject to } |\mathbf{p}(x)| \leq \alpha \quad \text{a.e., } x \in \Omega, \quad (6b)$$

is given by the TV-model

$$\text{minimize } \frac{1}{2}|y - f|_{L^2(\Omega)}^2 + \alpha \int_{\Omega} |Dy| \quad \text{over } y \in \operatorname{BV}(\Omega), \quad (7)$$

where $f \in L^2(\Omega)$ and $\alpha > 0$ are given. Further, $\operatorname{BV}(\Omega)$ denotes the space of functions of bounded variation (see Giusti (1984) for a definition),

$$\int_{\Omega} |Dy| = \sup \left\{ \int_{\Omega} y \operatorname{div} \mathbf{p} : \mathbf{p} \in C_0^1(\Omega), |\mathbf{p}|_{L^\infty(\Omega)} \leq 1 \right\}$$

is the total variation semi-norm, and $H_0(\operatorname{div}, \Omega) := \{\mathbf{p} \in L^2(\Omega)^N : \operatorname{div} \mathbf{p} \in L^2(\Omega), \mathbf{p} \cdot \boldsymbol{\nu} = 0 \text{ on } \partial\Omega\}$, where $\boldsymbol{\nu}$ denotes the outward unit normal on $\partial\Omega$. The corresponding proof in (Hintermüller and Kunisch, 2004, Thm. 2.2) requires the density result of this present paper for $\mathbf{K}(X) = \{\mathbf{p} \in X : |\mathbf{p}(x)|_2 \leq \alpha \text{ a.e. } x \in \Omega\}$ with $X = X_i, i \in \{0, 1\}$, where $X_1 = H_0(\operatorname{div}, \Omega)$

and $X_0 = C_0^1(\Omega)$.

Situations with $G = \nabla$ (rather than $G = \text{id}$, as in the previous applications) arise, for instance, in problems of elasto-plastic material deformation and the study of the tangent cone to the set $\mathbf{K}(H_0^1(\Omega)) = \{y \in H_0^1(\Omega) : |\nabla y(x)| \leq \psi(x) \text{ a.e. } \Omega\}$; see (Hintermüller and Surowiec, 2011). Here, $H_0^1(\Omega) =: X_1$ denotes the usual Sobolev space (Adams and Fournier, 2003) and $\psi \in L^2(\Omega)$ with $\psi(x) \geq \underline{\psi} > 0$ for all $x \in \Omega$. In the study of the tangent cone to $\mathbf{K}(H_0^1(\Omega))$ at $\bar{y} \in H_0^1(\Omega)$ one needs X_0 such that $\nabla X_0 \subset L^\infty(\Omega)^N$ with X_0 continuously and densely contained in X_1 .

2. Notation

Suppose $\Omega \subset \mathbb{R}^N$ is a bounded domain with $N \in \mathbb{N}$. By bold letters $\mathbf{f}, \mathbf{g}, \mathbf{h}, \dots$ we denote \mathbb{R}^M -valued functions, with $M \in \mathbb{N}$, and for real-valued functions we sometimes also write f, g, h, \dots . The support of a function $\mathbf{p} : \Omega \rightarrow \mathbb{R}^M$ is denoted by $\text{supp } \mathbf{p}$ and it is defined as

$$\text{supp } \mathbf{p} := \overline{\{x \in \Omega : |\mathbf{p}(x)| \neq 0\}}.$$

For the linear space of infinitely differentiable functions with compact support in Ω we write $\mathcal{D}(\Omega)$. Further we define

$$\mathcal{D}(\bar{\Omega}) := \{v|_\Omega : v \in \mathcal{D}(\mathbb{R}^N)\}. \quad (8)$$

We denote by $C_0(\Omega)$ to the space of real-valued continuous functions vanishing at the boundary of Ω , i.e., $f \in C_0(\Omega)$ if it is continuous over Ω and for all $\epsilon > 0$ there exist a compact set K_ϵ such that $|f(x)| < \epsilon$ on $x \in \Omega \setminus K_\epsilon$. Sometimes $C_0(\Omega)$ is also called the space of real-valued continuous functions vanishing at infinity. Endowed with the norm $\|f\|_{C_0(\Omega)} := \sup_{x \in \Omega} |f(x)|$, $C_0(\Omega)$ becomes a Banach space. Note that $C_0(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in the $C_0(\Omega)$ -norm. Later we also use $C_c(\Omega)$ to denote the space of continuous real-valued functions with compact support in Ω .

We also make use of the usual real Lebesgue and Sobolev spaces $L^p(\Omega)$ and $W^{1,p}(\Omega)$, with $1 \leq p \leq \infty$, respectively, with norms $\|v\|_{L^p(\Omega)} = (\int_\Omega |v(x)|^p dx)^{1/p}$ and $\|w\|_{W^{1,p}(\Omega)} = \|w\|_{L^p(\Omega)} + \|\nabla w\|_{L^p(\Omega)^N}$. Additionally, $W_0^{1,p}(\Omega)$ denotes the subspace of $W^{1,p}(\Omega)$ of functions which are zero on $\partial\Omega$ in the sense of the trace, which is also a Banach space with norm $\|v\|_{W_0^{1,p}(\Omega)} = \|\nabla v\|_{L^p(\Omega)^N}$. It is

known that

$$\overline{\mathcal{D}(\Omega)}^{W^{1,p}(\Omega)} = W_0^{1,p}(\Omega), \quad \overline{\mathcal{D}(\bar{\Omega})}^{W^{1,p}(\Omega)} = W^{1,p}(\Omega).$$

For more information on Lebesgue and Sobolev spaces and their properties we refer the reader to (Adams and Fournier, 2003).

The *divergence* operator div is defined as the formal transpose of the operator $-\nabla$, i.e., for a general open set $\mathcal{O} \subset \mathbb{R}^N$,

$$\langle \operatorname{div} \mathbf{v}, w \rangle + \langle \mathbf{v}, \nabla w \rangle = 0, \quad \forall \mathbf{v} \in \mathcal{D}'(\mathcal{O})^N, w \in \mathcal{D}(\mathcal{O}), \quad (9)$$

where $\mathcal{D}'(\mathcal{O})$ is the dual space of $\mathcal{D}(\mathcal{O})$ (see (Dautray and Lions, 1999)); in our context, we consider $\mathcal{O} = \Omega$ or $\mathcal{O} = \mathbb{R}^N$. We now can define the Hilbert space $H(\operatorname{div}, \Omega) := \{\mathbf{v} \in L^2(\Omega)^N : \operatorname{div} \mathbf{v} \in L^2(\Omega)\}$ (sometimes also denoted by $H(\operatorname{div})$) and endow it with the inner product $(\mathbf{v}, \mathbf{w})_{H(\operatorname{div})} := (\mathbf{v}, \mathbf{w})_{L^2(\Omega)^N} + (\operatorname{div} \mathbf{v}, \operatorname{div} \mathbf{w})_{L^2(\Omega)}$. In an analogous fashion, the space $H(\operatorname{div}, \mathbb{R}^N)$ is also defined. The closure of $\mathcal{D}(\Omega)^N$ with respect to the $H(\operatorname{div}, \Omega)$ -norm is denoted by $H_0(\operatorname{div}, \Omega)$ (or simply $H_0(\operatorname{div})$ when the domain Ω is understood) and in the case Ω has a Lipschitz boundary it is equivalent to

$$H_0(\operatorname{div}, \Omega) = \{\mathbf{v} \in H(\operatorname{div}, \Omega) : \gamma \mathbf{v} := \mathbf{v} \cdot \boldsymbol{\nu}|_{\partial\Omega} = 0\}, \quad (10)$$

where $\boldsymbol{\nu}$ denotes the outer normal vector. The operator γ can be proven to be continuous from $H(\operatorname{div}, \Omega)$ to $H^{-1/2}(\partial\Omega)$.

3. Main Result

The main density result of this paper is the following one.

Theorem 1. *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set with Lipschitz boundary. Suppose that $\alpha \in C(\bar{\Omega})$ with $\alpha(x) > 0$ for all $x \in \bar{\Omega}$. Consider the set valued mapping \mathbf{K}*

$$\mathbf{K}(X) := \{\mathbf{p} \in X : |\mathbf{p}(x)| \leq \alpha(x) \text{ a.e., } x \in \Omega\}, \quad (11)$$

where X is one of the following: $X_0 := \mathcal{D}(\Omega)^M$, $X_1 := H_0(\operatorname{div}, \Omega)$, $X_2 := (W_0^{1,p})^M$, $X_3 := C_0(\Omega)^M$ and $X_4 := L^p(\Omega)^M$. Then,

$$\overline{\mathbf{K}(X_0)}^{X_i} = \mathbf{K}(X_i), \quad \text{for } i = 1, 2, 3, 4. \quad (12)$$

The proof of this theorem relies on the theory of mollifiers and rescaling arguments. For this purpose, let $\Omega \subset \mathbb{R}^N$ be open and bounded. We say that Ω is *strictly star-shaped with respect to the point* $y_0 \in \Omega$ if, for each $y \in \Omega$, the segment $[y_0, y]$ is contained in Ω . In other words, taking y_0 as the origin, we obtain $\theta\bar{\Omega} \subset \Omega$ for all $\theta \in (0, 1)$. If Ω has a Lipschitz boundary it, of course, need not be strictly star-shaped. But there exists a finite collection of strictly star-shaped open sets $\{\Omega_j\}_{j=1}^J$ with Lipschitz boundaries such that $\Omega = \cup_{j=1}^J \Omega_j$; see for example (Boyer and Fabrie, 2012) for a proof. This fact justifies to call each bounded Lipschitz domain *locally strictly star-shaped*.

Our proof technique makes use of the standard theory of mollifiers (see Adams and Fournier (2003) for example) to obtain smooth approximanting functions. Henceforth, we assume that $\rho \in \mathcal{D}'(\mathbb{R}^N)$, $\rho \geq 0$, $\rho(x) = 0$ for $|x| \geq 1$ and $\int_{\mathbb{R}^N} \rho(x) dx = 1$. Defining $\rho_n(x) := n^N \rho(nx)$, we get $\rho_n(x) = 0$ for $|x| \geq 1/n$, $\rho_n(x) \geq 0$, $\int_{\mathbb{R}^N} \rho_n(x) dx = 1$. Moreover, ρ_n converges to Dirac's delta in the sense of $\mathcal{D}'(\mathbb{R}^N)$ as $n \rightarrow \infty$.

Definition 1. Let $\Omega \subset \mathbb{R}^N$ be open, bounded, strictly star-shaped with respect to the origin and with Lipschitz boundary. Let $\mathbf{p} \in L^p(\Omega)^M$ with $1 \leq p < \infty$ and $M \in \mathbb{N}$. We define the sequence $\{S_n(\mathbf{p}, \Omega)\}$, or $\{S_n(\mathbf{p})\}$ for short, when the domain is understood, as follows:

$$S_n(\mathbf{p}, \Omega)(x) := \rho_n * \tilde{\mathbf{p}}^{\theta_n} = \int_{\mathbb{R}^N} \tilde{\mathbf{p}}(y/\theta_n) \rho_n(x - y) dy, \quad x \in \mathbb{R}^N, n \in \mathbb{N}. \quad (13)$$

Here, $\tilde{\mathbf{p}}$ denotes the extension of \mathbf{p} by zero outside Ω , $\{\theta_n\}$ is a non-decreasing sequence in $(0, 1)$ such that $\theta_n \uparrow 1$ and

$$\text{supp } S_n(\mathbf{p}, \Omega) \subset \Omega, \quad (14)$$

for all $n \geq K$ and some sufficiently large K . Further, we use $\tilde{\mathbf{p}}^\theta(x) := \tilde{\mathbf{p}}(x/\theta)$.

Note that each $S_n(\mathbf{p})$ has compact support and it is an element of $C^\infty(\Omega)^M$. Hence, $\{S_n(\mathbf{p})\}$ is in $\mathcal{D}'(\Omega)^M$. A few words on Definition 1 are still in order. Since $\mathbf{p} \in L^p(\Omega)^M$, it holds that $\tilde{\mathbf{p}} \in L^p(\mathbb{R}^N)^M$ and $\text{supp } \tilde{\mathbf{p}} \subset \bar{\Omega}$. Also, $\Omega \subset \mathbb{R}^N$ is strictly star-shaped with respect to the origin and thus $\theta\bar{\Omega} \subset \Omega$, for all $\theta \in (0, 1)$. For $\tilde{\mathbf{p}}^{\theta_n}$ we observe that

$$\text{supp } \tilde{\mathbf{p}}^{\theta_n} \subset \theta_n \bar{\Omega} \subset \Omega. \quad (15)$$

Additionally, we have that $S_n(\mathbf{p}) = \rho_n * \tilde{\mathbf{p}}^{\theta_n}$ and from standard properties of the convolution and $\text{supp } \tilde{\mathbf{p}} \subset \bar{\Omega}$, we infer

$$\text{supp } S_n(\mathbf{p}) \subset \text{supp } \tilde{\mathbf{p}}^{\theta_n} + \text{supp } \rho_n \subset \theta_n \bar{\Omega} + \overline{B(0, 1/n)}. \quad (16)$$

Here the sequence $\{\theta_n\}$ has to satisfy $\theta_n \uparrow 1$ and $\theta_n \bar{\Omega} + \overline{B(0, 1/n)} \subset \Omega$ for all $n \geq K$ and for some sufficiently large K .

Lemma 2. *Let $\Omega \subset \mathbb{R}^N$ be open, bounded, strictly star-shaped with respect to the origin and with Lipschitz boundary. Let $X_1 := H_0(\text{div}; \Omega)$, $X_2 = W_0^{1,p}(\Omega)^M$, $X_3 = C_0(\Omega)^M$ and $X_4 = L^p(\Omega)^M$, then*

$$\mathbf{p} \in X_i \implies S_n(\mathbf{p}) \xrightarrow[X_i]{n \rightarrow \infty} \mathbf{p}, \quad \text{for } i = 1, 2, 3, 4. \quad (17)$$

PROOF. We split the proof into steps. Let $\mathbf{p}_i \in X_i$ for $i \in \{1, \dots, 4\}$.

Step 1. It follows immediately that the extensions by zero outside Ω of $\mathbf{p}_2, \mathbf{p}_3$ and \mathbf{p}_4 ($\tilde{\mathbf{p}}_2, \tilde{\mathbf{p}}_3$ and $\tilde{\mathbf{p}}_4$) belong to $W_0^{1,p}(\mathbb{R}^N)^M$, $C_0(\mathbb{R}^N)^M$ and $L^p(\mathbb{R}^N)^M$, respectively. Now we prove that $\tilde{\mathbf{p}}_1 \in H(\text{div}, \mathbb{R}^N)$. Since $\mathbf{p}_1 \cdot \boldsymbol{\nu} = 0$ on $\partial\Omega$ by Green's formula (see Girault and Raviart (1986, Chapter I. Section 2.2. (2.17))) we observe

$$(\mathbf{p}_1, \nabla \varphi)_{L^2(\Omega)^N} + (\text{div } \mathbf{p}_1, \varphi)_{L^2(\Omega)} = \langle \mathbf{p}_1 \cdot \boldsymbol{\nu}, \varphi \rangle_{\partial\Omega} = 0, \quad \forall \varphi \in \mathcal{D}(\bar{\Omega}), \quad (18)$$

where $\langle \mathbf{p}_1 \cdot \boldsymbol{\nu}, \varphi \rangle_{\partial\Omega}$ is shorthand for the duality pairing $\langle \mathbf{p}_1 \cdot \boldsymbol{\nu}, \varphi \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}$. Note that (18) holds for $\varphi \in H^1(\Omega)$ and hence it further holds for $\varphi \in \mathcal{D}(\bar{\Omega})$.

Define $v_0 := \text{div } \mathbf{p}_1$ and let \tilde{v}_0 and $\tilde{\mathbf{p}}_1$ denote the extensions by zero outside Ω of v_0 and \mathbf{p}_1 , respectively. Then, (18) can be written as

$$(\tilde{\mathbf{p}}_1, \nabla \varphi)_{L^2(\mathbb{R}^N)^N} + (\tilde{v}_0, \varphi)_{L^2(\mathbb{R}^N)} = 0, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^N). \quad (19)$$

Recalling (9) this implies $\tilde{v}_0 = \text{div } \tilde{\mathbf{p}}_1$ in $\mathcal{D}'(\mathbb{R}^N)$. Moreover, since $\tilde{v}_0 \in L^2(\mathbb{R}^N)$, we have that $\text{div } \tilde{\mathbf{p}}_1 \in L^2(\mathbb{R}^N)$. Therefore $\tilde{\mathbf{p}}_1 \in H(\text{div}, \mathbb{R}^N)$ with $\text{supp } \tilde{\mathbf{p}}_1 \subset \bar{\Omega}$.

Step 2. Define $\tilde{\mathbf{p}}_i^\theta$, for $i = 1, 2, 3, 4$, as $\tilde{\mathbf{p}}_i^\theta(x) := \tilde{\mathbf{p}}_i(x/\theta)$, then $\text{supp } \tilde{\mathbf{p}}_i^\theta \subset \theta \bar{\Omega} \subset \Omega$. Since $C_c(\Omega)$ is dense in $L^p(\Omega)$ for $1 \leq p < \infty$, we have that $\tilde{\mathbf{p}}_1^\theta$ converges in the $L^2(\mathbb{R}^N)^M$ -sense to $\tilde{\mathbf{p}}_1$: Let $\epsilon > 0$ be arbitrary and $\mathbf{h} \in C_c(\Omega)^M$ be such that $\|\tilde{\mathbf{p}}_1 - \tilde{\mathbf{h}}\|_{L^2(\Omega)^M} \leq \epsilon$ (with $\tilde{\mathbf{h}}$ being the extension by zero of \mathbf{h}) so that $\|\tilde{\mathbf{p}}_1^\theta - \tilde{\mathbf{h}}^\theta\|_{L^2(\theta\Omega)^M} \leq \epsilon\theta^{1/2}$ with $\tilde{\mathbf{h}}^\theta(x) = \tilde{\mathbf{h}}(x/\theta)$. Note that

by Lebesgue's dominated convergence theorem, $\lim_{\theta \uparrow 1} |\tilde{\mathbf{h}}^\theta - \tilde{\mathbf{h}}|_{L^2(\Omega)^M} = 0$. Therefore, by taking the limit as $\theta \uparrow 1$ in

$$|\tilde{\mathbf{p}}_1^\theta - \tilde{\mathbf{p}}_1|_{L^2(\mathbb{R}^N)^M} \leq |\tilde{\mathbf{p}}_1 - \tilde{\mathbf{h}}|_{L^2(\mathbb{R}^N)^M} + |\tilde{\mathbf{p}}_1^\theta - \tilde{\mathbf{h}}^\theta|_{L^2(\mathbb{R}^N)^M} + |\tilde{\mathbf{h}}^\theta - \tilde{\mathbf{h}}|_{L^2(\mathbb{R}^N)^M},$$

we observe that $\lim_{\theta \uparrow 1} |\tilde{\mathbf{p}}_1^\theta - \tilde{\mathbf{p}}_1|_{L^2(\mathbb{R}^N)^M} \leq 2\epsilon$. As $\epsilon > 0$ was arbitrary, the assertion is proven.

Similarly, $\operatorname{div} \tilde{\mathbf{p}}_1^\theta = \frac{1}{\theta} \operatorname{div} \tilde{\mathbf{p}}_1(\cdot/\theta)$ converges in the $L^2(\mathbb{R}^N)$ -sense to $\operatorname{div} \tilde{\mathbf{p}}_1$ as $\theta \uparrow 1$, $\tilde{\mathbf{p}}_2^\theta$ converges in the $L^p(\mathbb{R}^N)^M$ -sense to $\tilde{\mathbf{p}}_2$ (and also $\tilde{\mathbf{p}}_4^\theta$ converges in the $L^p(\mathbb{R}^N)^M$ -sense to $\tilde{\mathbf{p}}_4$) and $\nabla \tilde{\mathbf{p}}_2^\theta = \frac{1}{\theta} \nabla \tilde{\mathbf{p}}_2(\cdot/\theta)$ converges in the $L^p(\mathbb{R}^N)^{N \times M}$ -sense to $\nabla \tilde{\mathbf{p}}_2$. Also, $\tilde{\mathbf{p}}_3^\theta$ and $\tilde{\mathbf{p}}_3$ have compact support, and it follows that $\tilde{\mathbf{p}}_3^\theta \rightarrow \tilde{\mathbf{p}}_3$ uniformly on \mathbb{R}^N and $\tilde{\mathbf{p}}_3^\theta \rightarrow \mathbf{p}$ uniformly on Ω as $\theta \uparrow 1$. Summarizing,

$$\tilde{\mathbf{p}}_i^\theta \xrightarrow[\hat{X}_i]{\theta \uparrow 1} \tilde{\mathbf{p}}_i \quad \text{and} \quad \tilde{\mathbf{p}}_i^\theta \xrightarrow[X_i]{\theta \uparrow 1} \mathbf{p}_i, \quad \text{for } i = 1, 2, 3, 4, \quad (20)$$

where $\hat{X}_1 := H_0(\operatorname{div}; \mathbb{R}^N)$, $\hat{X}_2 = W_0^{1,p}(\mathbb{R}^N)^M$, $\hat{X}_3 = C_0(\mathbb{R}^N)^M$ and $\hat{X}_4 = L^p(\mathbb{R}^N)^M$.

Step 3. We have $\operatorname{div} S_n(\mathbf{p}_1) = \operatorname{div}(\rho_n * \tilde{\mathbf{p}}_1^{\theta_n}) = \rho_n * \operatorname{div} \tilde{\mathbf{p}}_1^{\theta_n}$ and that $\nabla S_n(\mathbf{p}_2) = \nabla(\rho_n * \tilde{\mathbf{p}}_2^{\theta_n}) = \rho_n * \nabla \tilde{\mathbf{p}}_2^{\theta_n}$ (see the proof of Lemma 3.16 in Adams and Fournier (2003) for this property of the convolution). It is known (see Adams and Fournier (2003, Theorem 2.29)) that $\rho_n * \mathbf{v} \rightarrow \mathbf{v}$ in $L^p(\Omega)^M$ for all $\mathbf{v} \in L^p(\Omega)^M$ as $n \rightarrow \infty$ and $|\rho_n * \mathbf{v}|_{L^p(\Omega)^M} \leq C|\mathbf{v}|_{L^p(\Omega)^M}$ for some constant C independent of \mathbf{v} and M for all $n \in \mathbb{N}$. Additionally, since $\tilde{\mathbf{p}}_3 \in C(\bar{\Omega})^N$ it follows that $\rho_n * \tilde{\mathbf{p}}_3 \rightarrow \tilde{\mathbf{p}}_3$ uniformly on Ω (see Adams and Fournier (2003), for example). Therefore, by the triangle inequality and (20), it is straightforward to observe that

$$S_n(\mathbf{p}_i) \xrightarrow[X_i]{n \rightarrow \infty} \mathbf{p}_i, \quad \text{for } i = 1, 2, 3, 4. \quad (21)$$

Remark. We note that in the proof of Lemma 2 more effort is devoted to the case $H_0(\operatorname{div}, \Omega)$ than to the other the cases, which is due to the type of boundary condition incorporated in $H_0(\operatorname{div}, \Omega)$.

We are now ready to prove our main result. The proof consists in scaling the sequence $\{S_n(\mathbf{p})\}$, based on the constraint $|\mathbf{p}(x)| \leq \alpha(x)$ a.e. $x \in \Omega$, in order for it to satisfy the same constraint while retaining the convergence properties of the previous lemma.

PROOF (OF THEOREM 1). Again we split the proof into several steps.

Step 1. Suppose that Ω is strictly star-shaped, i.e., there exists a point x in Ω such that the domain is star-shaped with respect to x . For convenience we translate the origin to that point. Then, for $\mathbf{p}_i \in \mathbf{K}(X_i)$ we have that $S_n(\mathbf{p}_i, \Omega)$ is well defined (with $i = 1, 2, 3, 4$), and $S_n(\mathbf{p}_i, \Omega) \in \mathcal{D}(\Omega)^M$ with $\text{supp } S_n(\mathbf{p}_i, \Omega) \subset \Omega$. Further, (21) holds true.

Let D be a bounded open subset of \mathbb{R}^N such that $\Omega \subset D$ and $d(\overline{\Omega}, \mathbb{R}^N \setminus D) > 0$, where $d(A, B) := \inf_{x \in A, y \in B} |x - y|$ denotes the distance between two sets in \mathbb{R}^N . Then, by Urysohn's lemma, there exists a continuous function $f : \mathbb{R}^N \rightarrow [0, 1]$ such that $f(\overline{\Omega}) = 1$ and $f(\mathbb{R}^N \setminus D) = 0$. Additionally, by the Tietze extension theorem there exists an extension $\hat{\alpha} \in C(\mathbb{R}^N)$ of $\alpha \in C(\overline{\Omega})$ such that $\sup_{x \in \overline{\Omega}} |\alpha(x)| = \sup_{x \in \mathbb{R}^N} |\hat{\alpha}(x)|$. We extend α to the entire \mathbb{R}^N by $\tilde{\alpha} = f \max(\hat{\alpha}, 0)$. Then, we have $\tilde{\alpha} \in C_c(\mathbb{R}^N)$, $\tilde{\alpha}(x) = \alpha(x)$ for $x \in \overline{\Omega}$, $\tilde{\alpha}(x) = 0$ for $x \in \mathbb{R}^N \setminus D$ and $\tilde{\alpha}(x) \geq 0$ for all $x \in \mathbb{R}^N$.

Let $i \in \{1, \dots, 4\}$. Since $|\mathbf{p}_i(x)| \leq \alpha(x)$ on Ω , the extension by zero of \mathbf{p}_i outside Ω satisfies $|\tilde{\mathbf{p}}_i(x)| \leq \tilde{\alpha}(x)$ over \mathbb{R}^N . Given that $S_n(\mathbf{p}_i, \Omega) = \rho_n * \tilde{\mathbf{p}}_i^{\theta_n}$, we observe

$$|S_n(\mathbf{p}_i, \Omega)(x)| \leq \int_{\mathbb{R}^N} \tilde{\alpha}(y/\theta_n) \rho_n(x - y) dy =: \tilde{\alpha}_n(x), \quad x \in \mathbb{R}^N. \quad (22)$$

We now prove that $\tilde{\alpha}_n \rightarrow \tilde{\alpha}$ uniformly on \mathbb{R}^N . Since $\tilde{\alpha}$ has compact support ($\text{supp } \tilde{\alpha} \subset \overline{D}$), it is uniformly continuous. Moreover, for all $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that $|\tilde{\alpha}(y/\theta_n) - \tilde{\alpha}(y)| < \epsilon$ for all $y \in \mathbb{R}^N$ if $|\frac{1}{\theta_n} - 1| < \delta(\epsilon)$. Then,

$$|(\tilde{\alpha}_n - \rho_n * \tilde{\alpha})(x)| \leq \int_{\mathbb{R}^N} |\tilde{\alpha}(y/\theta_n) - \tilde{\alpha}(y)| \rho_n(x - y) dy \leq \sup_{y \in \mathbb{R}^N} |\tilde{\alpha}(y/\theta_n) - \tilde{\alpha}(y)|,$$

which implies that $\sup_{x \in \mathbb{R}^N} |\tilde{\alpha}_n(x) - (\rho_n * \tilde{\alpha})(x)| \rightarrow 0$ as $n \rightarrow \infty$. However, since $\tilde{\alpha}$ is continuous with compact support, it is known (see Adams and Fournier (2003) or Attouch et al. (2006)) that $\rho_n * \tilde{\alpha} \rightarrow \tilde{\alpha}$ uniformly on \mathbb{R}^N as $n \rightarrow \infty$. Therefore, we have $\tilde{\alpha}_n \rightarrow \tilde{\alpha}$ uniformly on \mathbb{R}^N .

Define,

$$\beta_n := \left(1 + \frac{\sup_{x \in \mathbb{R}^N} |\tilde{\alpha}_n(x) - \tilde{\alpha}(x)|}{\underline{\alpha}} \right)^{-1}, \quad (23)$$

where $\underline{\alpha} := \min_{x \in \overline{\Omega}} \alpha(x) > 0$. We have that $\beta_n \rightarrow 1$ as $n \rightarrow \infty$ and it is straightforward to see that $\beta_n \tilde{\alpha}_n(x) \leq \tilde{\alpha}(x)$ for any $x \in \overline{\Omega}$ by noting that

$\tilde{\alpha}|_{\bar{\Omega}} = \alpha \geq \underline{\alpha}$. Then, for $i = 1, 2, 3, 4$, it follows that $\beta_n S_n(\mathbf{p}_i, \Omega) \in \mathcal{D}(\Omega)^N$,

$$\beta_n S_n(\mathbf{p}_i, \Omega) \xrightarrow[X_i]{n \rightarrow \infty} \mathbf{p}_i \quad \text{and} \quad |\beta_n S_n(\mathbf{p}_i, \Omega)(x)| \leq \alpha(x), \quad x \in \bar{\Omega}. \quad (24)$$

Since $\mathbf{p}_i \in \mathbf{K}(X_i)$ was arbitrary the result is proven in the case of a strictly star-shaped domain since $\beta_n S_n(\mathbf{p}_i, \Omega) \in \mathbf{K}(\mathcal{D}(\Omega)^M)$.

Step 2. If Ω is not strictly star-shaped, then since Ω is bounded with Lipschitz boundary, it can be covered by a finite number of strictly star-shaped open sets with Lipschitz boundary $\{\Omega_j\}_{j=1}^J$ such that $\Omega = \cup_{j=1}^J \Omega_j$. Let $\{\sigma_j\}_{j=1}^J$ be a partition of the unity subordinated to this covering, i.e., $\sigma_j \in \mathcal{D}(\Omega_j)$, $0 \leq \sigma_j(x) \leq 1$ and $\sum_{j=1}^J \sigma_j(x) = 1$, $x \in \Omega$. The proof of the existence of $\{\sigma_j\}_{j=1}^J$ can be found in (Yosida, 1968, I. 12. The Partition of Unity). Then, $\tilde{\mathbf{p}}_i = \sum_{j=1}^J \sigma_j \tilde{\mathbf{p}}_i$, on \mathbb{R}^N . Since each Ω_j is strictly star-shaped, we can apply the argument of step 1 using the approximating sequence $S_n(\sigma_j \mathbf{p}_i, \Omega_j)$ and the proof follows by the application of the argument for each $j \in J$.

It should be noted that since the $W^{1,p}$ -norm is equivalent to the $W_0^{1,p}$ -norm on $W_0^{1,p}(\Omega)$ and the $H(\text{div})$ -norm is exactly the $H_0(\text{div})$ -norm, the density results above for $W_0^{1,p}(\Omega)$ and $H_0(\text{div})$ also read

$$\overline{\mathbf{K}(\mathcal{D}(\Omega)^M)}^{H(\text{div})} = \mathbf{K}(H_0(\text{div})) \quad \text{and} \quad \overline{\mathbf{K}(\mathcal{D}(\Omega)^M)}^{W^{1,p}(\Omega)} = \mathbf{K}(W_0^{1,p}(\Omega)).$$

Additionally, note that since $\mathcal{D}(\Omega) \subset C_0^k(\Omega)$ and $C_0^k(\Omega)$ is a subset of $H_0(\text{div})$ and $W_0^{1,p}(\Omega)$, we have the following corollary.

Corollary 3. *Under the assumptions of Theorem 1, for each $k = 0, 1, 2, \dots$, we have that*

$$\overline{\mathbf{K}(C_0^k(\Omega)^M)}^{H(\text{div})} = \mathbf{K}(H_0(\text{div})) \quad \text{and} \quad \overline{\mathbf{K}(C_0^k(\Omega)^M)}^{W^{1,p}(\Omega)} = \mathbf{K}(W_0^{1,p}(\Omega)).$$

4. Extensions

The proof of Theorem 1 can be extended to handle more complicated sets than the one (11). In fact, it can be used to prove the density of sets with pointwise constraints on the gradient or the divergence.

Theorem 4. *Suppose that X is either $X_1 := H_0(\text{div}, \Omega)$, or $X_2 := W_0^{1,p}(\Omega)^M$, α is defined as in Theorem 1 and consider an operator G_i given by $G_1 = \text{div}$ or $G_2 = \nabla$. Define the sets*

$$\mathbf{K}_G(X) := \{\mathbf{p} \in X(\Omega) : |(G\mathbf{p})(x)| \leq \alpha(x) \text{ a.e., } x \in \Omega\}.$$

Then,

$$\overline{\mathbf{K}_{G_i}(X_0)}^{X_i} = \mathbf{K}_{G_i}(X_i), \quad \text{for } i = 1, 2, \quad (25)$$

with $X_0 = \mathcal{D}(\Omega)^M$.

PROOF. If Ω is star-shaped with respect to some point, we translate the origin into that point and $S_n(\mathbf{p}_i, \Omega) \in \mathcal{D}(\Omega)^M$ with $\text{supp } S_n(\mathbf{p}_i, \Omega) \subset \Omega$ is well-defined for $\mathbf{p}_i \in \mathbf{K}_G(X_i)$, with $i = 1, 2$ and satisfies (17). Since $S_n(\mathbf{p}_i, \Omega) = \rho_n * \tilde{\mathbf{p}}_i^{\theta_n}$, we have $GS_n(\mathbf{p}_i) = \rho_n * (G\tilde{\mathbf{p}}_i^{\theta_n})$ by the properties of the convolution and given that $G\tilde{\mathbf{p}}_i^{\theta_n} = G(\tilde{\mathbf{p}}_i(\cdot/\theta_n)) = \frac{1}{\theta_n}(G\tilde{\mathbf{p}}_i)(\cdot/\theta_n)$. Then, the analogous inequality to (22) is given by

$$|GS_n(\mathbf{p}_i, \Omega)(x)| \leq \int_{\mathbb{R}^N} \frac{1}{\theta_n} \tilde{\alpha}(y/\theta_n) \rho_n(x-y) dy =: \tilde{\alpha}_n(x), \quad x \in \mathbb{R}^N, \quad (26)$$

where $\tilde{\alpha}$ is the one defined in the proof of Theorem 1. A slight modification of the argument there shows also that $\tilde{\alpha}_n \rightarrow \tilde{\alpha}$ uniformly on \mathbb{R}^N . Then, for β_n defined in (23) and $i = 1, 2$, it follows that $\beta_n S_n(\mathbf{p}_i, \Omega) \in \mathcal{D}(\Omega)^M$,

$$\beta_n S_n(\mathbf{p}_i, \Omega) \xrightarrow[X_i]{n \rightarrow \infty} \mathbf{p} \quad \text{and} \quad |G(\beta_n S_n(\mathbf{p}_i, \Omega))(x)| \leq \alpha(x), \text{ a.e.,} \quad (27)$$

with $x \in \bar{\Omega}$. Since $\mathbf{p}_i \in \mathbf{K}(X_i)$ was arbitrary and since $\beta_n S_n(\mathbf{p}_i, \Omega) \in \mathbf{K}_G(\mathcal{D}(\Omega)^N)$, the result is proven in the case of a strictly star-shape domain.

If Ω is not strictly-star shaped, then step 2 in the proof of Theorem 1 can be applied as it is and the initial statement follows.

References

- Adams, R.A., Fournier, J.J.F., 2003. Sobolev Spaces. second ed., Academic Press.
- Attouch, H., Buttazzo, G., Michaille, G., 2006. Variational Analysis in Sobolev and BV Spaces. MPS-SIAM.

- Boyer, F., Fabrie, P., 2012. *Mathematical Tools for the Study of the Incompressible Navier-Stokes Equations and*. Applied mathematical sciences, Springer.
- Carstensen, C., 1997. Domain decomposition for a non-smooth convex minimization problem and its application to plasticity. *Numerical Linear Algebra with Applications* 4, 177–190.
- Dautray, R., Lions, J., 1999. *Mathematical Analysis and Numerical Methods for Science and Technology: Volume 3 Spectral Theory and Applications*. *Mathematical Analysis and Numerical Methods for Science and Technology*, Springer Berlin Heidelberg.
- Duvaut, G., Lions, J.P., 1976. *Inequalities in Mechanics and Physics*. Springer-Verlag, Berlin, Paris.
- Ekeland, I., Temam, R., 1976. *Convex Analysis and Variational Problems*. North-Holland and American Elsevier.
- Girault, V., Raviart, P.A., 1986. *Finite element methods for Navier-Stokes equations: theory and algorithms*. volume 5. Springer-Verlag, Berlin.
- Giusti, E., 1984. *Minimal Surfaces and Functions of Bounded Variation*. Birkhäuser.
- Hintermüller, M., Ito, K., Kunisch, K., 2003. The primal-dual active set strategy as a semismooth Newton method. *SIAM J. Optim.* 13, 865–888.
- Hintermüller, M., Kunisch, K., 2004. Total bounded variation regularization as a bilaterally constrained optimization problem. *SIAM Journal of Applied Mathematics* 64, 1311–1333.
- Hintermüller, M., de los Reyes, J., 2011. A duality-based semismooth Newton framework for solving variational inequalities of the second kind. *Interfaces and Free Boundaries* 13, 437–462.
- Hintermüller, M., Rösler, S., 2013. A duality-based path-following semismooth newton methods for elasto-plastic contact problems. IFB-Report No. 70, Institute of Mathematics and Scientific Computing, University of Graz .

- Hintermüller, M., Stadler, G., 2006. An infeasible primal-dual algorithm for total bounded variation-based inf-convolution-type image restoration. *SIAM Journal on Scientific Computing* 28, 1–23.
- Hintermüller, M., Surowiec, T., 2011. First-order optimality conditions for elliptic mathematical programs with equilibrium constraints via variational analysis. *SIAM Journal on Optimization* 21, 1561–1593.
- Johnson, C., 1976. Existence theorems for plasticity problems. *J. Math. pures et appl.* 55, 431–444.
- Rudin, L.I., Osher, S., Fatemi, E., 1992. Nonlinear total variation based noise removal algorithms. *Physica D: Nonlinear Phenomena* 60, 259–268.
- Stadler, G., 2004. Semismooth Newton and Augmented Lagrangian methods for a simplified friction problem. *SIAM Journal on Optimization* 15, 39–62.
- Yosida, K., 1968. *Functional Analysis. Classics in mathematics.* second ed., Springer.