Multidimensional local skew-fields

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Contents

Introduction.

In this work we study *local skew fields*, which are natural generalisation of n-dimensional local fields, and their applications to the theory of central division algebras over henselian fields.

Local fields appear in a natural way in algebraic geometry and algebraic number theory if anyone try to find a connection between local and global properties of such objects like algebraic number fields, arithmetic schemes and algebraic varieties.

Historically the first examples of 1-dimensional local fields appeared in the 19 century in complex analisys and in algebraic number theory. These examples are known fields $\mathbf{C}((z))$ and $\mathbf{Q}_p$. Now we say that 1-dimensional local field is a quotient field of a complete discrete valuated ring.

A little bit later the first examples of local skew fields were found. They were finite dimensional division algebras over classical local fields, and they were completely studied by Hasse, Brauer, Noether and Albert. At the same time there were several works of Witt ([34]) about skew fields over discrete valuated fields, which opened researching of skew fields over henselian fields. Basic results about a structure of such skew fields were recently got by Jacob and Wandsworth in ([9]).

In the middle of 70-th A.N.Parshin introduced a notion of a multidimensional local field which generalised the notion of a usual local field ([19],[24], [7]).

n-dimensional local field is a complete discrete valuated field such that the residue field is a $n-1$-dimensional local field.

One of the typical examples of such a field is an iterated Laurent series field $k((z_1))((z_2))\dots((z_n))$. Elements $z_1, \dots, z_n$ are called local parameters of this field.

Multidimensional local fields appears also as natural generalisations of local objects on 1-dimensional scheme. As an example let us consider the following construction.

Consider an algebraic scheme X of dimension n. Let $Y_0 \supset \dots \supset Y_n$ be a flag of closed subschemes in X such that $Y_0 = X$, $Y_n = x$ is a closed point on X, Y_i is a codimension 1 subscheme in Y_{i-1} $(1 \le i \le n)$, x is a smooth point on all Y_i $(0 \le i \le n)$. Then there exists a construction which assign in canonical way to any given flag a n-dimensional local field. Moreover, let X be an algebraic variety over a field k, x be a rational point over k, $z_1, z_2, \dots, z_n \in k(X)$ be fixed local parameters such that $z_{n-i+1} = 0$ is an equation of Y_i on Y_{n-1} in a neiborhood of the point x $(1 \le i \le n)$. Then our n-dimensional local field can be identified with $k((z_1))((z_2))\dots((z_n))$ ([24], [7]).

Using this assignment a number of results known earlier only for the case of a curve was generalised to a higher dimensional case. These are such well-known results as multi-dimensional reciprocity lows of Parshin-Lomadze ([19], [15], [20], [7]).

During the last 25 years there was opened another direction in the theory of local fields. This is an application to the theory of integrable systems connected with the Krichever-Sato-Wilson correspondence on a curve (for more details on the Krichever correspondence see [6], [29], [16], [27]).

Recently there were issued several papers [17], [18], [23], where the ideas of the Krichever-Sato-Wilson correspondence on a curve were developed to the case of varieties of higher dimension. In particular, A.N. Parshin pointed out one class of non-commutative local fields arising in differential equations and showed that these skew fields possesses many features of commutative fields. He defined a skew field of formal pseudo-differential operators in n variables and studied some of their properties. He raised a problem of classifying non-commutative local skew fields. It was the first argument to begin to study such skew fields.

A generalisation of a notion "local field" looks very natural:
n-dimensional local skew field is a complete discrete valuated skew field such that the residue skew field is a $n-1$-dimensional local skew field.

In this work we try to study n-dimensional local skew fields bearing in mind only the definition. Unfortunately, there appear very hard obstructions already on the first steps which leads us to some restrictions. So, we study only skew fields with commutative residue skew field. By the way, a number of results valid in general case (see, for example, proposition 0.7 and corollary 1) and a number of results can be generalised to the case of skew fields with residue skew field finite dimensional over its centre (see, for example, section 1.4).

Some applications of developed theory to the Krichever correspondence we get in section 1.6. Namely, we get some generalisations of the classical KP-equations (hierarchy).

Surprisingly the studying of local skew fields leads to some new unexpected results in the valuation theory on finite dimensional division algebras. Using general formulas for splittable local skew fields (i.e. for skew fields such that the residue skew field can be embedded into the valuation ring) we get a decomposition theorem for a class of splittable wild division algebras over a Laurent series field with arbitrary residue field of characteristic greater than two. This theorem is a generalisation of the decomposition theorem for tame division algebras given by Jacob and Wadsworth in [9]. An extensive analysis of the wild division algebras of degree p over a field F with complete discrete rank 1 valuation with $char(\bar{F}) = p$ was given by Saltman in [28] (Tignol in [32] analysed more general case of the defectless division algebras of degree p over a field F with Henselian valuation). In his recent revue [33] Wadsworth pointed out that for most of the specific examples and applications it is suffice to consider Henselian valued fields like iterated Laurent series fields, that is n-dimensional local fields. So, we get in some sense the complete picture of a local structure of the Brauer group over such fields. As a corollary we get the positive answer on the following conjecture: the exponent of A is equal to its index for any division algebra A over a C_2-field $F = F_1((t_2))$, where F_1 is a C_1-field (see [26], 3.4.5.).

From the other hand, the problem of classification of local skew fields leads to the problem of classification of conjugacy classes in the automorphism group of an

n-dimensional local (commutative) field. We solve this problem for the group of continuous automorphisms.

We note that the automorphism group of a local field of positive characteristic is intensively studied now in the algebraic number theory (we mean recent applications to the problem of description the Galois group of an arithmetically profinite extension). Moreover, the automorphism group of the field $F_q((t))$ (so called Nottingham group) is now carefully studied in the group theory (for more details see papers [5], [3], [12], [8], [13], [14], [10], [11], [36]. We hope that our results on the automorphism group will be applied in the future to obtain some useful results about the Galois group of an arithmetically profinite extension.

Here is a brief overview of this thesis. It consists of two chapters.

The first chapter consists of five paragraphs. In §1 we give general definitions of a local skew field, of a splitness and of an isomorphism of local skew fields. Also we study some general properties of splittable skew fields.

Thereafter except §4 we study mostly two-dimensional local skew fields with commutative residue skew field. In §2 we give a sufficient condition for a skew field to be split. Namely, a local skew field splits if a canonical automorphism has infinite order. The canonical automorphism can be defined as a restriction of an inner automorphism $ad(z)$ on the residue field, where z is any local parameter. We show that there exist counterexamples when this condition does not hold. We note that this condition and counterexamples are true even in more general situation when the skew field is not two-dimensional skew field or the residue skew field is not commutative. We classify all the skew fields which possess this condition up to isomorphism. The results of §2 don't depend on the characteristic of a skew field.

In §3 we classify all the local splittable skew fields of characteristic 0 with commutative residue skew field and with the canonical automorphism of finite order.

In §4 we study splittable local skew fields of characteristic $p > 2$ with commutative residue skew field and with the canonical automorphism of finite order. We give a criterium when such a skew field is finite dimensional over its centre. Then we prove that every tame finite dimensional division algebra over a local complete field splits. Using this fact we prove the decomposition theorem for splittable algebras. As a corollary we get the proof of the conjecture mentioned above.

In §5 we study some properties of local skew fields described in §3. In particular, we give a criterium when two elements from such a skew field conjugate. This is a generalisation of analogous theorems from [23]. As a corollary we prove that almost all such skew fields are infinite dimensional over their centre. Also we prove that the Scolem-Noether theorem holds only in the case of the classical ring of pseudo-differential operators.

In §6 we get new KP-hierarchies for every class of isomorphic two-dimensional local skew fields, which were studied in section 3. We derive new equations of the KP-type for some hierarchies.

In the second chapter we classify conjugacy classes in the group of continuous automorphisms of a two-dimensional local field of characteristic zero with the residue field of the same characteristic. Some facts about automorphisms of a local field of characteristic $p > 0$ one can find in lemma 1.3. Also in this chapter we show how this classification can be generalised to the case of a n-dimensional local field, $n > 2$.

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Chapter 0

The structure of two-dimensional local skew fields.

0.1 General.

Definition 0.1 *Let K and k be arbitrary skew fields. A skew field K is called a complete discrete valuation skew field if K is complete with respect to a discrete valuation. A skew field K is called an n-dimensional local skew field if there are skew fields $K = K_n, K_{n-1}, \dots K_0 = k$ such that each K_i for $i > 0$ is a complete discrete valuation skew field with residue skew field K_{i-1}.*

The following properties are well known from the valuation theory of division algebras (see for ex. [31]).

Lemma 0.2 *Let K be a complete diskrete valuation skew field. Then the following properties hold:*
i) The valuation ring $\mathcal{O}$ is a topological group and a metric space under the natural topology;

ii) The ring $\mathcal{O}$ is a local ring and a principal ideal domain.

For every two-dimensional local skew field we have

$$K \supset \mathcal{O} \to \bar{K} \supset \bar{\mathcal{O}} \to k$$

where $\bar{\mathcal{O}}$ is a valuation ring in $\bar{K}$. There are two filtrations

$$K \supset \mathcal{O} \supset \wp \supset \wp^2 \supset \dots$$

$$\bar{K} \supset \bar{\mathcal{O}} \supset \bar{\wp} \supset \bar{\wp}^2 \supset \dots$$

where $\bar{\wp}$ is a maximal ideal in $\bar{\mathcal{O}}$, $\bar{\nu}$ is a discrete valuation on $\bar{K}$.

Definition 0.3 *Two two-dimensional local skew fields K and K' are isomorphic if there is an isomorphism which preserves the filtrations above, i.e. it maps $\mathcal{O}_K$ onto $\mathcal{O}_{K'}$, $\wp$ onto $.\wp'$ and $\mathcal{O}_{\bar{K}}$ onto $\mathcal{O}_{\bar{K}'}$, $\wp_{\bar{K}}$ onto $\wp_{\bar{K}'}$.*

Definition 0.4 *A two-dimensional skew field K is said to split if there is a section of the homomorphism $\mathcal{O}_K \to \bar{K}$.*

Elements $z \in \mathcal{O}$, $\nu(z) = 1$ and $u \in \bar{\mathcal{O}} \subset \bar{K}$, $\bar{\nu}(u) = 1$ are called local parameters (or variables) in K.

Proposition 0.5 *Suppose K splits. Fix some local parameters z and u. Then K is isomorphic to a two-dimensional local skew field $\bar{K}((z))$ where*

$$za = a^{\alpha} z + a^{\delta_1} z^2 + a^{\delta_2} z^3 + \dots$$

where $a \in \bar{K}$, α is an automorphism, $\delta_i : \bar{K} \to \bar{K}$ are linear maps.

Proof. Suppose $a \in K$, $\nu(a) = j$. Then we have $\nu(az^{-j}) = 0$ and $\overline{az^{-j}} := az^{-j} \mod \wp \in \bar{K}$. We will assume that the last element lies in $\mathcal{O}$, since there is a section. Then we have $\nu(az^{-j} - \overline{az^{-j}}) \geq 1$. Continuing this line of reasonings, we get $a = \sum_{i=j}^{\infty} a_i z^i$, $a_i \in \bar{K}$.

Now define $a^{\alpha} = zaz^{-1} \mod \wp$, where $a \in \bar{K}$. It's clear that α is an automorphism. Since $\nu(zaz^{-1}) = 0$, the element zaz^{-1} can be written as a series $\sum_{i=0}^{\infty} a_i z^i$, where $a_i \in \bar{K}$. Here we have $a_0 = a^{\alpha}$. Now put $a^{\delta_i} := a_i$ for $i \geq 1$. It is easy to see that δ_i are linear maps.

□

In fact, the maps δ_i satisfy some identities. To write them we need extra notation.

Consider the ring $\mathbb{Z} < \alpha, \delta >$ of noncommutative polynomials in two variables. Define the map

$$\sigma : \mathbb{Z} < \alpha, \sigma > \to \mathbb{Z} < \alpha, \delta, \delta_i ; i \geq 1 >,$$

$$\sigma(\alpha^{a_1} \delta^{b_1} \dots \alpha^{a_n} \delta^{b_n}) = \alpha^{a_1} \delta_{b_1} \dots \delta_{b_{n-1}} \alpha^{a_n - 1} \delta^{b_n},$$

where $a_1, b_n \geq 0$, $a_i, b_j \geq 1$, $i > 1$, $j < n$ for every word in $\mathbb{Z} < \alpha, \delta >$.

For example

$$\sigma(\alpha^k) = \alpha^k$$

$$\sigma(\alpha^k \delta^l \alpha^i) = \alpha^k \delta_l \alpha^{i-1}$$

where k, l, i are natural numbers, $i, l \geq 1$.

Let $S_i^k \in \mathbb{Z} < \alpha, \delta >$, $i \geq k$, $i \geq 1$ be polynomials given by the following formula:

$$S_i^k = \sum_{\tau \in S_i / G} \tau(\underbrace{\alpha \dots \alpha}_{i-k} \underbrace{\delta \dots \delta}_{k}),$$

where S_i is a permutation group and G is an isotropy subgroup.

Immediately from the definition we get the following lemma

Lemma 0.6 *The polynomials S_i^k satisfy the following property:-*

$$S_i^i = \delta^i, \quad S_i^0 = \alpha^i, \quad S_{i+1}^{k+1} = \alpha S_i^{k+1} + \delta S_i^k$$

Now we can define the identities for the maps δ_i:

Proposition 0.7 *Every map δ_i, $i \geq 1$ satisfy the identity*

$$\delta_i(ab) = \sum_{k=0}^{i} \sigma(\delta^{i-k}\alpha)(a)\sigma(S_i^k\alpha)(b), \quad a, b \in \bar{K}$$

Proof. For any $a, b \in \bar{K}$. We have

$$(*) \qquad (ab)^{\alpha} z + (ab)^{\delta_1} z^2 + \ldots = z(ab) = (a^{\alpha} z + a^{\delta_1} z^2 + \ldots)b$$

If we represent the right-hand side of $(*)$ as a series with coefficients shifted to the left and then compare the corresponding coefficients on the left-hand side and right-hand side, we get some formulas for $\delta_i(ab)$. We have to prove that these formulas are the same as in our proposition.

Let

$$z^{i+1-k} b = \alpha^{i+1-k}(b) z^{i+1-k} + \ldots + x_k z^{i+1} + \ldots$$

and

$$(\alpha(a)z + \delta_1(a)z^2 + \delta_2(a)z^3 + \ldots)b = \alpha(ab)z + y_2 z^2 + y_3 z^3 + \ldots$$

Then we have

$$y_{i+1} = \alpha(a)x_i + \sum_{k=0}^{i-1} \delta_{i-k}(a)x_k = \sum_{k=0}^{i} \sigma(\delta^{i-k}\alpha)(a)x_k$$

Note that x_k are polynomials which consist of monomials of the type

$$\alpha^{a_1}\delta_{b_1}\ldots\alpha^{a_n}\delta_{b_n}\alpha^{a_{n+1}}(b), \quad a_k, b_k \in \mathbb{Z}, \quad a_k, b_k \geq 0$$

(we put δ_0 to be equal to 1). It is easy to see that these polynomials have integral positive coefficients.

We claim that $x_k = \sigma(S_i^k\alpha)(b)$.

To prove this fact it suffice to show that x_k contains every monomial from

$\sigma(S_i^k\alpha)(b)$ and the sum of coefficients in x_k is equal to the sum of coefficients in $\sigma(S_i^k\alpha)(b)$.

By definition every coefficient of $\sigma(S_i^k\alpha)(b)$ is equal to 1. It is easy to see that the sum of coefficients is equal to $C_i^k = i!/(i-k)!k!$.

Let us show that x_k contains every monomial from $\sigma(S_i^k\alpha)(b)$. By definition, $\sigma(S_i^k\alpha)(b)$ consists of monomials $\sigma(\tau(\underbrace{\alpha\ldots\alpha}_{i-k}\underbrace{\delta\ldots\delta}_{k})\alpha)(b)$, where $\tau\in S_i$, i.e. it consists of monomials $\alpha^{a_1}\delta_{b_1}\ldots\alpha^{a_n}\delta_{b_n}\alpha^{a_{n+1}}(b)$, where $a_j\geq 0$, $b_j\geq 1$, $\sum_{j=1}^n b_j=k$, $\sum_{j=1}^{n+1}a_j=i-k+1-n$. We have

$$z^{i+1-k}b=z^{i+1-k-a_{n+1}}\alpha^{a_{n+1}}(b)z^{a_{n+1}}+\text{other terms},$$

$$z^{i+1-k-a_{n+1}}\alpha^{a_{n+1}}(b)z^{a_{n+1}}=z^{i+1-k-a_{n+1}-1}[\alpha^{a_{n+1}+1}(b)z+\ldots+\delta_{b_n}\alpha^{a_{n+1}}(b)z^{b_n+1}+\ldots]z^{a_{n+1}}=$$
$$z^{i+1-k-a_{n+1}-1}\alpha^{a_{n+1}+1}(b)z^{a_{n+1}+1}+z^{i+1-k-a_{n+1}-1}\delta_{b_n}\alpha^{a_{n+1}}(b)z^{b_n+1+a_{n+1}}+\ldots$$

Now put $d_1=\delta_{b_n}\alpha^{a_{n+1}}(b)$. Then we have

$$z^{i+1-k-a_{n+1}-1}d_1z^{b_n+1+a_{n+1}}=\ldots+z^{i+1-k-a_{n+1}-1-a_n-1}d_2z^{b_n+1+a_{n+1}+a_n+b_{n-1}+1}+\ldots,$$

where $d_2=\delta_{b_{n-1}}\alpha^{a_n}\delta_{b_n}\alpha^{a_{n+1}}(b)$. By induction we get

$$z^{i+1-k-\sum a_j-n}\alpha^{a_1}\delta_{b_1}\ldots\alpha^{a_n}\delta_{b_n}\alpha^{a_{n+1}}(b)z^{\sum b_j+n+\sum a_j}=\alpha^{a_1}\delta_{b_1}\ldots\alpha^{a_n}\delta_{b_n}\alpha^{a_{n+1}}(b)z^{i+1}+\ldots,$$

that is x_k contains any given monomial from $\sigma(S_i^k\alpha)(b)$.

Let us show that the sum of coefficients of x_k is equal to C_i^k.

Denote by s_n^l the sum of coefficients in y_l, where

$$z^naz^{-n}=\sum_{k=0}^{\infty}y_kz^k,\quad a\in\bar{K}$$

Then the sum of coefficients of x_k is equal to s_{i+1-k}^k. We claim that the following relation holds

$$s_n^d=\sum_{l=0}^{d}s_{n-1}^l$$

The proof is by induction on n. For $n=1$ we have $s_1^d=1$ for all $d\geq 0$, $s_0^l=0$ for $l>0$ and $s_0^0=1$.

For arbitrary n put

$$z^{n-1}az^{-n+1}=y_0+y_1z+\ldots,$$

where $y_0\in\bar{K}$. Then we have

$$z^naz^{-n}=zy_0z^{-1}+zy_1z^{-1}z+\ldots=[y_0^{\alpha}+y_0^{\delta_1}z+\ldots]+[y_1^{\alpha}+y_1^{\delta_1}z+\ldots]z+\ldots$$

Put

$$z^naz^{-n}=\sum_{k=0}^{\infty}w_kz^k.$$

Then we have

$$w_d = \sum_{j=1}^{d} \delta_j(y_{d-j}) + \alpha(y_d)$$

Since the sum of coefficients of y_j is equal to s_{n-1}^j, we get

$$s_n^d = \sum_{j=0}^{d} s_{n-1}^j$$

Now let us show that $s_{i+1-k}^k = C_i^k$ if $k < i+1$. The proof is by induction on i. For $i = 0$ we have $s_1^0 = 1 = C_i^0$. For arbitrary i we have

$$s_{i+1-k}^k = \sum_{l=0}^{k} s_{i-k}^l = C_i^k + C_{i-1}^{k-1} + \ldots + C_{i-k}^0 =$$
$$(\ldots(((C_{i-k}^0 + C_{i-k+1}^1) + C_{i-k+2}^2) + C_{i-k+3}^3) + \ldots + C_i^k) =$$
$$(\ldots(((C_{i-k+2}^1) + C_{i-k+2}^2) + C_{i-k+3}^3) + \ldots + C_i^k) = C_{i+1}^k$$

This completes the proof.
□

Corollary 1 *Suppose $\alpha = Id$. Then the following formula holds*

$$\delta_i(ab) = \delta_i(a)b + \sum_{k=1}^{i} \delta_{i-k}(a) \sum_{(j_1,\ldots,j_l)} C_{i-k+1}^l \delta_{j_1} \ldots \delta_{j_l}(b)$$

where $\delta_0 = \alpha$ and the second sum is taken over all the vectors $(j_1, \ldots, j_l)$ such that $0 < l \leq min\{i-k+1, k\}$, $j_m \geq 1$, $\sum j_m = k$.

In the sequel we will need the following definition.

Definition 0.8 *Let (α, β) be endomorphisms of a skew field L. A map $\delta : L \to L'$, where $L \subset L'$ is a subalgebra, is called a (α, β)-derivation if it is linear and satisfy the following identity*

$$\delta(ab) = \delta(a)b^\alpha + a^\beta \delta(b)$$

where $a, b \in L$.
We will say that $(\alpha, 1)$-derivation is an α-derivation.

For example δ_1 is an (α^2, α)-derivation.
If $\alpha = Id$, then δ_1 is an usual derivation; $\delta_2 = \delta_1^2 + \delta$, where δ is a derivation.

Corollary 2 *If $\delta_1 = \ldots = \delta_{k-1} = 0$, then δ_k is an (α^{k+1}, α)-derivation.*

The following corollary will be used in §3 of this chapter.

Corollary 3 *Let $\bar{K}$ be a field, $\bar{K} = k((u))$, $k \subset Z(K)$ and the maps δ_i, $i \geq 1$ be continuous if $chark = 0$. Then*

$$\delta_i(\sum_{j=N}^{\infty} x_j u^j) = \sum_{j=N}^{\infty} x_j \delta_i(u^j), \quad x_j \in k$$

So, for every i the map δ_i is completely defined by elements $\delta_i(u)$ and $\delta_j(u)$ for $j < i$.

Proof. If $chark = p \neq 0$ and $\alpha = id$ the maps δ_i, $i \geq 1$ are continuous, since $\delta_i(a^{p^i}) = 0$ for any $a \in \bar{K}$. Since a topology on a 1-dimensional local field is uniquely defined by its local structure, the continuity does not depend on the choice of local parameters (for more information about a relation between a topology and a parametrisation see [35]). If $\alpha \neq id$ one can use lemma 1.29 to reduce this case to the previous one.

Let us show that α is a continuous map. In our case it suffice to show that α preserves the valuation. Our proof will not depend on a characteristic.

It suffice to show that $\bar{\nu}(\alpha(u')) = 1$ for any u', $\bar{\nu}(u') = 1$. Consider the automorphism α':

$$\alpha'(a) := \overline{z^{-1}az}$$

where $a \in \bar{K}$ (we use the notation from proposition 1.7). It's clear that $\alpha' = \alpha^{-1}$.

Let u' be an arbitrary parameter. Put $\kappa = \bar{\nu}(\alpha(u'))$. We claim that $|\kappa| \leq 1$ or $|\kappa| = p^q$, $q \in \mathbb{N}$. Assume the converse. Then $\kappa = mp^q$, $(m,p) = 1$, $|m| \neq 1$ and there exist $c \in k$, $a \in \bar{K}$ such that $\alpha(u') = ca^m$. Therefore, we get

$$u' = \alpha^{-1}(\alpha(u')) = c(\alpha^{-1}(a))^m,$$

i.e.

$$\bar{\nu}(u') = 1 = \bar{\nu}(c(\alpha^{-1}(a))^m) = m\bar{\nu}(\alpha^{-1}(a)),$$

a contradiction.

Let us show that $\kappa \geq 0$. Assume the converse. Consider the element $u' + u'^2$ ($u' + u'^3$ if $chark = 2$). Then $\bar{\nu}(\alpha(u' + u'^2)) = 2\kappa < -1$. If $chark \neq 2$ we get a contradiction with the assertion $|\bar{\nu}(\alpha(u'))| = p^q$ or $|\bar{\nu}(\alpha(u'))| \leq 1$ for any parameter u'. If $chark = 2$ one can apply the same arguments to the element $u' + u'^3$.

Similarly, for $\kappa' := \bar{\nu}(\alpha^{-1}(u'))$ the property $0 \leq \kappa' \leq 1$ or $\kappa' = p^l$ holds.

Let us show that $\kappa \neq p^q$. Assume the converse. Consider the following two cases:

1) Suppose $\kappa' \leq 1$. There exist $r \in k$, $a_1 \in k((u))$ such that $\alpha(u') = c_2 u'^2 a_1^{p^q-2}$. Therefore,

$$1 = 2\bar{\nu}(\alpha^{-1}(u')) + (p^q - 2)\bar{\nu}(\alpha^{-1}(a_1)),$$

i.e. $(p^q-2)|1$. It is possible only if $p=3$, $q=1$. In this case one can use the same arguments with $\alpha(u')=c_3u'^5a_1^{-2}$. Then we get $\bar{\nu}(\alpha^{-1}(a_1))=2$, a contradiction (since $0\le\kappa'\le 1$ or $\kappa'=p^l$).

2) Suppose $\kappa'=p^l$. Let $\alpha(u')=cu'a^{p^q-1}$ for some $c\in k$, $a\in k((u))$, $\bar{\nu}(a)=1$. Then we have

$$\bar{\nu}(u')=1=\bar{\nu}(\alpha^{-1}(u'))+(p^q-1)\bar{\nu}.$$

But this contradicts with $\bar{\nu}(\alpha^{-1}(a))\ge 0$.

So, $\kappa=0$ or $\kappa=1$, i.e. for any parameter u' we have $\bar{\nu}(\alpha(u'))=0$ or $\bar{\nu}(\alpha(u'))=1$. Suppose $\kappa=0$. Consider the element $x=u'+c_1u'^2+c_2(u'^3+c_1u'^4))$, where $c_1=-w_0^{-1}$ if $\alpha(u')=w_0+\dots$ and c_2 is an element such that $\bar{\nu}(\alpha(x))>1$ (it always exists since $\bar{\nu}(\alpha(u'+c_1u'^2))>0$). But this contradicts with $\bar{\nu}(x)=1$. Therefore, $\kappa=1$ and α is a continuous map.

To complete the proof it suffice to show that the series $\sum_{j=N}^{\infty}x_j\delta_i(u^j)$ converges, because the topology on $k((u))$ is complete and separate. The proof is by induction on i. For $i=0$ we have $\bar{\nu}(\alpha(u^j))=j$ and the series converges. For $i=1$ we have $\bar{\nu}(\delta_1(u^j))=(j-1)\bar{\nu}(\delta_1(u))$ and again the series converges.

At last, by proposition 0.7 for $j>1$ we have $\delta_i(u^j)=\delta_i(u^{j-1})y_0+\sum_{k=0}^{i-1}\delta_k(u^{j-1})y_{i-k}$, where $\bar{\nu}(y_k)$ does not depend on j.
By induction we have $min\{\bar{\nu}(\delta_0(u^{j-1})y_i),\dots,\bar{\nu}(\delta_{i-1}(u^{j-1})y_1)\}>$
$min\{\bar{\nu}(\delta_0(u^{j-2})y_i),\dots,\bar{\nu}(\delta_{i-1}(u^{j-2})y_1)\}$ and $\bar{\nu}(y_0)=1$. So,
$min\{\bar{\nu}(\delta_i(u^{j-1})y_0),\bar{\nu}(\delta_0(u^{j-1})y_i),\dots,\bar{\nu}(\delta_{i-1}(u^{j-1})y_1)\}>$
$min\{\bar{\nu}(\delta_i(u^{j-2})y_0),\bar{\nu}(\delta_0(u^{j-2})y_i),\dots,\bar{\nu}(\delta_{i-1}(u^{j-2})y_1)\}$.
Therefore, the series converges.
□

0.2 Splittable skew fields.

In this section we will assume that $\bar{K}$ is a field.

For such a skew field one can define a notion of a canonical automorphism α.

By definition there exist the following exact sequences:

$$1\to\mathcal{O}^*\to K^*\xrightarrow{\nu}\mathbb{Z}\to 1$$

where $\mathcal{O}$ is a valuation ring;

$$1\to 1+\wp\to\mathcal{O}^*\to\bar{K}^*\to 1$$

where $\wp$ is a maximal ideal.

Consider the map

$$\phi: K^*\to Int(K),\quad \phi(x)=ad(x),\quad ad(x)(y)=x^{-1}yx$$

where $Int(K)$ is the group of inner automorphisms of the skew field K. Since inner automorphisms preserve the valuation, this group acts on the ring $\mathcal{O}$. Moreover, it preserve the ideal $\wp$. Therefore, there exists a map $\phi : K^* \to Aut(\mathcal{O}/\wp) = Aut(\bar{K})$. Let us show that the action of $\phi(\mathcal{O}^*)$ is trivial on $\bar{K}$. To show it we use the second exact sequence. Since $(1+\wp)^{-1}x(1+\wp) = x \mod \wp$ for any $x \in \mathcal{O}$, the action of $\phi(1+\wp)$ on $\bar{K}$ is trivial. Therefore, there exists an action of $\bar{K}$ on $\bar{K}$. Namely, an element $\bar{a} \in \bar{K}$ acts on $\bar{x} \in \bar{K}$ as $a^{-1}xa \mod \wp$, where a, x are any lifts of $\bar{a}, \bar{x}$ in $\mathcal{O}$. Since $\bar{K}$ is a commutative field, this action is trivial.

Definition 0.9 *An automorphism α of the field $\bar{K}$ defined by the formula*

$$\alpha = \phi(z)$$

where $z \in K^*$ and $\nu(z) = 1$, is called a canonical automorphism.

It does not depend on the choice of z.

We want to classify all splittable two-dimensional local skew fields which have isomorphic *last residue fields* up to isomorphism. Let K and K' be two splittable skew fields, $K \cong \bar{K}((z))$, $K' \cong \bar{K}'((z'))$. If $K \cong K'$, then one can represent an isomorphism $\varphi : K \to K'$ as a compositum of an isomorphism $\phi : K \longrightarrow K'$ such that $\phi(u) = u'$, $\phi(z) = z'$, and of an automorphism ψ of the skew field K. Since every isomorphism in our paper preserve the local structure, every automorphism of a splittable two-dimensional local skew field is defined by change of parameters

$$(z) \qquad u \mapsto u' = c_0 + c_1 z + c_2 z^2 + \ldots, \quad \bar{\nu}(c_0) = 1$$

$$z \mapsto z' = a_0 z + a_1 z^2 + \ldots, \quad a_0 \neq 0$$

where $a_i, c_i \in \bar{K}$.

It is easy to see that every change of parameters looks like above and can be decomposed into a sequence of changes $u \mapsto u'$, $z \mapsto z$; $u' \mapsto u'$, $z \mapsto z' = a'_0 z + a'_1 z^2 + \ldots$ (or in a backward order). Also $u \mapsto u'$ can be decomposed into a sequence of changes $u \mapsto u'_1 = c_0$, $u'_1 \mapsto u'_2 = u'_1 + c'_1 z, \ldots, u'_i \mapsto u'_{i+1} = u'_i + c'_i z^i, \ldots$ and $z \mapsto z'$ can be decomposed into a sequence of changes $z \mapsto z'_1 = a_0 z$, $z'_1 \mapsto z'_2 = z'_1 + a'_1 z^2, \ldots, z'_i \mapsto z'_{i+1} = z'_i + a'_i z^{i+1}, \ldots$.

Remark. We must note that any change of parameters (z) defines a map $f : K \to K$ which is not always an automorphism. Indeed, assume the converse. Consider a map which is given by $f(z) = z'$, $f(u) = u$, where z' is another parameter. Then we must have

$$f(zu) = f(z)f(u) = z'u = u^{\alpha'} z' + u^{\delta'_1} z'^2 + \ldots$$

$$f(zu) = f(u^{\alpha} z + u^{\delta_1} z^2 + \ldots) = u^{\alpha} z' + u^{\delta_1} z'^2 + \ldots$$

Hence, $\alpha = \alpha'$; $\delta_1 = \delta_1'$ and so on, i.e. $\delta_i' = \delta_i \quad \forall i$.
Consider the skew field "$((u))((z))$ with the relation $zu = (u+u^2)z$ and consider a change of parameters $z \mapsto z' = z + z^2$. Then

$$z'u = (z+z^2)u = (u+u^2)z + z(u+u^2)z = (u+u^2)z + [(u+u^2)z + (u+u^2)^2 z]z =$$

$$(u+u^2)z' + [u + 2u^2 + 2u^3 + u^4 - u - u^2]z^2 = (u+u^2)z' + [u^2 + 2u^3 + u^4]z'^2 + \ldots$$

So, $\delta_1 \neq \delta_1'$, a contradiction.

Proposition 0.10 *Let K be a splittable two-dimensional local skew field. Suppose the canonical automorphism α has infinite order.*

Then there exists a parameter z' such that $z'a = a^\alpha z'$ for any $a \in \bar{K}$.

Proof. We will show that there exists a sequence of parameters $\{z_k\}$ such that the equality $z_k a z_k^{-1} = a^\alpha \mod \wp^k$ holds and the sequence $\{z_k\}$ converges in K.

We need some additional lemmas.

Lemma 0.11 *Suppose the following relation holds:=*

$$zaz^{-1} = a^\alpha + a^{\delta_j} z^j + a^{\delta_{j+1}} z^{j+1} + \ldots, \quad a \in \bar{K}$$

where $\delta_1 = \ldots = \delta_{j-1} = 0$, $\delta_j \neq 0$. Then

(i) for $z' = z + bz^{q+1}$ we have

$$z'az'^{-1} = a^\alpha + \ldots + a^{\delta_{q-1}} z'^{q-1} + (a^{\delta_q} + ba^{\alpha^{q+1}} - a^\alpha b)z'^q + \ldots$$

i.e. $a^{\delta_q'} = a^{\delta_q} + ba^{\alpha^{q+1}} - a^\alpha b$.

(ii) Suppose $\alpha^n = id$, $n \geq 1$. Then for $z' = z + bz^{q+1}$, $n|q$ we have

$$z'az'^{-1} = a^\alpha + \ldots + a^{\delta_{q+j-1}} z'^{q+j-1} +$$

$$(a^{\delta_{q+j}} + b(a^{\delta_j})^{\alpha^q} - a^{\delta_j} b^{\alpha^j} + b\sum_{k=1}^{q}((a^{\alpha^k})^{\delta_j})^{\alpha^{q-k}} - a^{\delta_j}\sum_{k=0}^{j-1} b^{\alpha^k})z'^{q+j} + \ldots$$

(iii) for $z' = bz$, $b \in \bar{K}$, $b \neq 0$ we have

$$z'az'^{-1} = a^\alpha + a^{\delta_j}(b^{-1})^\alpha \ldots (b^{-1})^{\alpha^j} z'^j + \ldots$$

Corollary 4 *If $\alpha = Id$, then*

$$z'az'^{-1} = a + \ldots + a^{\delta_{q+j-1}} z'^{q+j-1} + (a^{\delta_{q+j}} + (q-j)a^{\delta_j} b)z'^{q+j} + \ldots$$

Proof of lemma.

(i) We have

$$z'az'^{-1}=(1+bz^q)zaz^{-1}(1+bz^q)^{-1}=(zaz^{-1}+bz^qzaz^{-1})(1-bz^q+bz^qbz^q-\dots)=$$

$$(zaz^{-1}-zaz^{-1}bz^q+\dots+bz^qzaz^{-1}-\dots)=$$

$$(zaz^{-1}-[a^\alpha+a^{\delta_j}z^j+\dots]bz^q+bz^q[a^\alpha+a^{\delta_j}z^j+\dots]+\dots)=$$

$$(zaz^{-1}-[a^\alpha b+a^{\delta_j}b^{\alpha^j}z^j+\dots]z^q+ba^{\alpha^{q+1}}z^q+\dots)=$$

$$(zaz^{-1}+(-a^\alpha b+ba^{\alpha^{q+1}})z^q+\dots)=a^\alpha+\dots+a^{\delta_{q-1}}z'^{q-1}+(a^{\delta_q}+ba^{\alpha^{q+1}}-a^\alpha b)z'^q+\dots$$

(ii) We have

$$z'az'^{-1}=(1+bz^q)zaz^{-1}(1+bz^q)^{-1}=(zaz^{-1}+bz^qzaz^{-1})(1+bz^q)^{-1}=$$

$$(a^\alpha+a^{\delta_j}z^j+\dots+a^{\delta_{q+j}}z^{q+j}+\dots+bz^q(a^\alpha+a^{\delta_j}z^j+\dots))(1+bz^q)^{-1}=$$

$$(a^\alpha+ba^{\alpha^{q+1}}z^q+a^{\delta_j}z^j+\dots+a^{\delta_{q+j}}z^{q+j}+\dots+b\sum_{k=1}^{q}((a^{\alpha^k})^{\delta_j})^{\alpha^{q-k}}z^{q+j}+b(a^{\delta_j})^{\alpha^q}z^{q+j}+\dots)(1+bz^q)^{-1}=$$

$$a^\alpha+(a^{\delta_j}z^j+\dots+a^{\delta_{q+j}}z^{q+j}+\dots+b\sum_{k=1}^{q}((a^{\alpha^k})^{\delta_j})^{\alpha^{q-k}}z^{q+j}+b(a^{\delta_j})^{\alpha^q}z^{q+j}+\dots)(1-bz^q+bz^qbz^q-\dots)=$$

$$a^\alpha+a^{\delta_j}z^j+\dots+a^{\delta_{q+j}}z^{q+j}+\dots+b\sum_{k=1}^{q}((a^{\alpha^k})^{\delta_j})^{\alpha^{q-k}}z^{q+j}+b(a^{\delta_j})^{\alpha^q}z^{q+j}+\dots-a^{\delta_j}b^{\alpha^j}z^{q+j}+\dots=$$

$$a^\alpha+\dots+a^{\delta_{q+j-1}}z'^{q+j-1}+(a^{\delta_{q+j}}+b(a^{\delta_j})^{\alpha^q}-a^{\delta_j}b^{\alpha^j}+b\sum_{k=1}^{q}((a^{\alpha^k})^{\delta_j})^{\alpha^{q-k}}-a^{\delta_j}\sum_{k=0}^{j-1}b^{\alpha^k})z'^{q+j},$$

because $z'^j=z^j+\sum_{k=0}^{j-1}b^{\alpha^k}z^{q+j}+\dots$.

(iii) We have

$$z'az'^{-1}=bzaz^{-1}b^{-1}=a^\alpha+ba^{\delta_j}(b^{-1})^{\alpha^j}z^j+\dots=a^\alpha+a^{\delta_j}(b^{-1})^\alpha\dots(b^{-1})^{\alpha^j}z'^j+\dots$$

□

Lemma 0.12 *Let δ be an (α,β)-derivation of a field $\bar{K}$ and $\alpha\neq\beta$.*

Then δ is an inner derivation, i.e. there exists $d\in\bar{K}$ such that

$$\delta(a)=da^\alpha-a^\beta d$$

for all $a\in\bar{K}$.

Proof. Put $d = \delta(a)/(a^\alpha - a^\beta)$, where a is any element such that $a^\alpha \neq a^\beta$. Put $\delta_{in}(x) = dx^\alpha - x^\beta d$. We claim that $\delta = \delta_{in}$. Indeed, consider the map $\bar{\delta} = \delta - \delta_{in}$. It is an (α, β)-derivation. Take arbitrary $b \in \bar{K}$. Then $\bar{\delta}(ab) = \bar{\delta}(ba)$. But we have

$$\bar{\delta}(ab) = \bar{\delta}(a)b^\alpha + a^\beta\bar{\delta}(b) = a^\beta\bar{\delta}(b),$$

and

$$\bar{\delta}(ba) = \bar{\delta}(b)a^\alpha + b^\beta\bar{\delta}(a) = a^\alpha\bar{\delta}(b)$$

Therefore, $\bar{\delta}(b) = 0$ for any b.

□

Proof of proposition.

Let

$$zaz^{-1} = a^\alpha + a^{\delta_1}z + a^{\delta_2}z^2 + \dots$$

By proposition 1.7 and corollary 1, δ_1 is an (α^2, α)-derivation. Since $\alpha^2 \neq \alpha$, by lemma 0.12 it is an inner derivation, say $\delta_1(a) = d_1a^{\alpha^2} - a^\alpha d_1$. By lemma 0.11, (i) for a parameter $z_2 = z - d_1z^2$ we have

$$z_2az_2^{-1} = a^\alpha + a^{\delta_2'}z_2^2 + \dots.$$

Note that $\delta_1' = 0$. By corollary 1, δ_2' is an (α^3, α)-derivation. Since $\alpha^3 \neq \alpha$, by lemma 0.12 it is an inner derivation. By lemma 0.11, (i) there exists a parameter $z_3 = z_2 - d_2z_2^3$ such that $z_3az_3^{-1} = a^\alpha \mod \wp^3$.

By induction for arbitrary $k \in \mathbb{N}$ we have

$$z_kaz_k^{-1} = a^\alpha + a^{\delta_k'}z_k^k + \dots$$

and $\delta_j' = 0$ for $j < k$. By corollary 1, δ_k' is an (α^{k+1}, α)-derivation. Since $\alpha^{k+1} \neq \alpha$, it is an inner derivation. By lemma 0.11, (i) there exists a parameter $z_{k+1} = z_k - d_kz_k^{k+1}$ such that $z_{k+1}az_{k+1}^{-1} = a^\alpha \mod \wp^{k+1}$.

It is clear that the sequence $\{z_n\}$: $z_{n+1} = z_n - d_nz_n^{n+1}$ converges in K. Since K is a complete and separate field, there exists a unique limit z. It is clear that $zaz^{-1} = a^\alpha$. The proposition is proved.

□

Theorem 0.13 *Let K be a two-dimensional local skew field. If $\alpha^n \neq id$ for all $n \in \mathbb{N}$ then*
(i) $charK = char\bar{K}$
(ii) K splits.

Proof.

If $char K \neq char \bar{K}$ then $char \bar{K} = p > 0$. Hence $\nu(p) = r > 0$. Then for any element $t \in K$ with $\nu(t) = 0$ we have $ptp^{-1} = \alpha^r(\bar{t}) \mod \wp$ where $\bar{t}$ is the image of t in $\bar{K}$. But on the other hand, $pt = tp$, a contradiction.

The proof of (ii) we will divide in three steps.

Step 1. Let π be the prime field in K. Since $char K = char \bar{K}$ the field π is a subring of $\mathcal{O}$.

Lemma 0.14 *There exists an element $c \in \bar{K}$ such that $c^{\alpha^k} \neq c$ for all $k \in \mathbb{N}$.*

Proof. We claim that there exists a sequence $\{c_{j_i}\}$, $j_i, i \in \mathbb{N}$, $c_{j_i} \in \bar{\mathcal{O}}$ such that

(i) $\bar{\nu}(c_{j_i}) > \bar{\nu}(c_{j_{i-1}}) \ \forall i$

(ii) if $k = 0 \mod j_2 \dots j_l$ and $k \neq 0 \mod j_2 \dots j_{l+1}$, then $\alpha^k(c_{j_1}) = c_{j_1}, \dots, \alpha^k(c_{j_{l-1}}) = c_{j_{l-1}}, \alpha^k(c_{j_l}) \neq c_{j_l}$ and

$$\bar{\nu}[(\alpha^k - Id)(c_{j_l})] < \bar{\nu}(c_{j_{l+1}})$$

Let us construct it. Take an element c_{j_1} such that $\alpha(c_{j_1}) \neq c_{j_1}$, and $\bar{\nu}(c_{j_1}) \geq 1$. Such an element always exists. Indeed, consider an element u with $\bar{\nu}(u) = 1$. If $\alpha(u) \neq u$, then one can put $c_{j_1} = u$. If $\alpha(u) = u$, then take any element $\tilde{c_{j_1}}$ such that $\alpha(\tilde{c_{j_1}}) \neq \tilde{c_{j_1}}$. If $\bar{\nu}(\tilde{c_{j_1}}) = 0$, then put $c_{j_1} = \tilde{c_{j_1}} u$. Then we have $\bar{\nu}(c_{j_1}) = 1$ and $\alpha(\tilde{c_{j_1}} u) = \alpha(\tilde{c_{j_1}}) u \neq \tilde{c_{j_1}} u$. Put $j_1 = 1$.

Let j_2 be a minimal positive integer such that $(\alpha^{j_1})^{j_2}(c_{j_1}) = c_{j_1}$, and let $k_1 = max\{\bar{\nu}[(\alpha^{j_1})^m(c_{j_1}) - c_{j_1}], m \in \{1, \dots, j_2 - 1\}\}$.
Take any $\tilde{c_{j_2}}$ such that $(\alpha^{j_1})^{j_2}(\tilde{c_{j_2}}) \neq \tilde{c_{j_2}}$. Put $c_{j_2} = \tilde{c_{j_2}} c_{j_1}^{k_1+1}$. Then $(\alpha^{j_1})^{j_2}(c_{j_2}) \neq c_{j_2}$ and $\bar{\nu}[(\alpha^{j_1})^m(c_{j_1}) - c_{j_1}] < \bar{\nu}(c_{j_2}) \ \forall m < j_2$.
By induction we get a sequence which satisfy (i) and (ii).

Now put $c = \sum_{i=1}^{\infty} c_{j_i}$. Then for all k we have $\alpha^k(c) \neq c$. Indeed, let $k = 0 \mod j_2 \dots j_l$ and $k \neq 0 \mod j_2 \dots j_{l+1}$. By (ii), $\alpha^k(c) - c = \alpha^k(c_{j_l}) - c_{j_l} + \alpha^k(\sum_{i=l+1}^{\infty} c_{j_l}) - \sum_{i=l+1}^{\infty} c_{j_l}$. But $\bar{\nu}(\alpha^k(c_{j_l}) - c_{j_l}) < \bar{\nu}(c_{j_{l+1}}) \leq \bar{\nu}(\alpha^k(\sum_{i=l+1}^{\infty} c_{j_l}) - \sum_{i=l+1}^{\infty} c_{j_l})$. Therefore, $\alpha^k(c) - c \neq 0$.
□

Consider the field $\bar{F} = \pi(c) \subset K$. Let us show that this field can be embedded in $\mathcal{O}$.

Take any lift $c' \in \mathcal{O}$ of the element c: $c' \mod \wp = c$. It is clear that c' commute with any element from π. It is easy to see that c is a transcendental element over π. Indeed, assume the converse. Then its equation modulo $\wp$ must have infinite number of solutions, because $c^{\alpha^k} \neq c \ \forall k$, a contradiction. Therefore, $\pi[c'] \bigcap \wp = 0$. So, the field of fractions $\bar{F}$ can be embedded in $\mathcal{O}$.

Let $\bar{L}$ be a maximal field extension of $\bar{F}$ which can be embedded in $\mathcal{O}$. Denote by L its image in $\mathcal{O}$. Take $\bar{a} \in \bar{K}$, $\bar{a} \notin \bar{L}$. We claim that there exists a lifting $a \in \mathcal{O}$ of $\bar{a}$ such that a commutes with every element in L.

Step 2. Take any lifting a in $\mathcal{O}$ of $\bar{a}$. For every element $x \in L$ we have $axa^{-1} \mod \wp = x$. If z is a parameter of K we can write

$$axa^{-1} = x + x^{\delta_1'} z,$$

where $x^{\delta_1'} \in \mathcal{O}$. The map $\bar{\delta_1'} : x \in L \to \overline{\delta_1(x)} \in \bar{K}$ is an α-derivation. Indeed,

$$a(x_1 + x_2)a^{-1} = (x_1 + x_2) + (x_1 + x_2)^{\delta_1'} z$$

$$a(x_1 + x_2)a^{-1} = ax_1a^{-1} + ax_2a^{-1} = x_1 + x_1^{\delta_1'} z + x_2 + x_2^{\delta_1'} z = (x_1 + x_2) + (x_1^{\delta_1'} + x_2^{\delta_1'})z$$

Therefore, $\overline{(x_1 + x_2)^{\delta_1'}} = \overline{x_1^{\delta_1'}} + \overline{x_2^{\delta_1'}}$. Then, we have

$$a(x_1x_2)a^{-1} = (ax_1a^{-1})(ax_2a^{-1})$$

Hence

$$x_1x_2 + (x_1x_2)^{\delta_1'} z = (x_1 + x_1^{\delta_1'} z)(x_2 + x_2^{\delta_1'} z) = x_1x_2 + x_1x_2^{\delta_1'} z + x_1^{\delta_1'} z x_2 + x_1^{\delta_1'} z x_2^{\delta_1'} z$$

$$\equiv x_1x_2 + x_1x_2^{\delta_1'} z + x_1^{\delta_1'} x_2^{\alpha} z \mod \wp^2 = x_1x_2 + (x_1^{\delta_1'} x_2^{\alpha} + x_1x_2^{\delta_1'})z \mod \wp^2$$

Therefore,

$$\overline{(x_1x_2)^{\delta_1'}} = \overline{x_1^{\delta_1'}}\,\overline{x_2^{\alpha}} + \overline{x_1}\,\overline{x_2^{\delta_1'}} = \overline{x_1^{\delta_1'}} x_2^{\alpha} + x_1\overline{x_2^{\delta_1'}}$$

By lemma 0.12, $\bar{\delta_1'}$ is an inner α-derivation, say $\bar{\delta_1'}(x) = d(x^{\alpha} - x)$. Put $\tilde{a_1} := (1 + a_1 z)a$, where $a_1 \mod \wp = -d$. Using the same calculations as in lemma 0.11 we have

$$(1 + a_1 z)axa^{-1}(1 + a_1 z)^{-1} = x + (x^{\delta_1'} + a_1 x^{\alpha} - x a_1)z \mod \wp^2$$

Since $x^{\delta_1'} + a_1 x^{\alpha} - x a_1 = 0 \mod \wp$, we get $\tilde{a_1} x \tilde{a_1}^{-1} = x + x^{\delta_2'} z^2$. Using the same arguments as above one can check that $\bar{\delta_2'} : L \to \bar{K}$ is an α^2-derivation. By induction we can find an an element $\tilde{a_i} = (1 + a_i z^i) \dots (1 + a_1 z)a$ such that

$$\tilde{a_i} x \tilde{a_i}^{-1} = x + x^{\delta_{i+1}'} z^{i+1},$$

and $\bar{\delta_{i+1}'} : L \to \bar{K}$ is an α^{i+1}-derivation. By lemma 0.12, $\bar{\delta_{i+1}'}$ is an inner α^{i+1}-derivation. So there exists an element $\tilde{a_{i+1}} = (1 + a_{i+1} z^{i+1})\tilde{a_i}$ such that

$$\tilde{a_{i+1}} x \tilde{a_{i+1}}^{-1} = x + x^{\delta_{i+2}'} z^{i+2}$$

for any $x \in L$. It is clear that the sequence $\{\tilde{a_i}\}$ converges in K. Since $\tilde{a_i} \mod \wp = \bar{a}$, the limit of this sequence is a needed lifting.

Step 3. Now suppose $\bar{a}$ is a transcendental over $\bar{K}$. Then by step 2 there exists a lifting $a \in \mathcal{O}$ such that a commutes with every element in L. Then $L[a] \bigcap \wp = 0$ and the field of fractions $L(a)$ can be embedded in $\mathcal{O}$, which contradicts the maximality of L. So we can assume that $\bar{K}$ is algebraic over L. Suppose $\bar{a}$ is an algebraic and separable element over $\bar{L}$. Using a generalisation of Hensel's lemma (see below) we can find a lifting a' of a such that a' commutes with elements of L and a' is algebraic over L, which again leads to a contradiction.

Finally, let $\bar{a}$ be purely inseparable over $\bar{L}$, $\bar{a}^{p^k} = \bar{x}$, $x \in L$. Let a' be its lifting which commutes with every element of L. Then $a'^{p^k} - x$ commutes with every element of L. If $\nu(a'^{p^k} - x) = r \neq \infty$ then similarly to the beginning of this proof we deduce that the image of $(a'^{p^k} - x)c(a'^{p^k} - x)^{-1}$ in $\bar{K}$ is equal to $\alpha^r(c)$, where c is an element from lemma 0.14. Since $\alpha^r(c) \neq c$, we get a contradiction. Therefore, $a'^{p^k} = x$ and the field $L(a')$ can be embedded in $\mathcal{O}$, which contradicts the maximality of L. Thus, $\bar{L} = \bar{K}$. The theorem is proved.
□

Proposition 0.15 (Hensel's lemma) [1] *Let $\mathcal{O}$ be a complete valuation ring in K, I be the valuation ideal, $\bigcap I^n = 0$, and let F be a subfield in $\mathcal{O}$. Let $A \in \mathcal{O}$ be such that $\forall l \in F$ $Al = lA$. Let $f(X) \in F[X]$, $f'(A) \notin I$ and $f(A) \in I$.*

Then there exists an element $\hat{A} \in \mathcal{O}$ such that
a) $\hat{A}$ commutes with A,
b) $\hat{A} - A \in I$,
c) $f(\hat{A}) = 0$
d) $\hat{A}l = l\hat{A}$ $\forall l \in F$

Proof. If $\tilde{A}$ commutes with A, then

$$f(A + \tilde{A}) = f(A) + f'(A)\tilde{A} + P\tilde{A}^2$$

where $P \in F[A, \tilde{A}]$. We use Teilor's formula here. Put $\tilde{A} = -(f'(A))^{-1}f(A)$. It's clear that $\tilde{A} \in I$ and $\tilde{A}$ commutes with A. Moreover, $\tilde{A}$ commutes with every element in F. Thus, $f(A + \tilde{A}) = P\tilde{A}^2 \in I^2$ and $f'(A + \tilde{A}) = f'(A) + X\tilde{A} \notin I$, where $X \in F[A, \tilde{A}]$. Similarly we can find the element $\tilde{A}_2 = -(f'(A+\tilde{A}))^{-1}f(A+\tilde{A}) \in I^2$, which commutes with A, $\tilde{A}$ and with every element in F and such that

$$f(A + \tilde{A} + \tilde{A}_2) \in I^4$$

Continuing this line of reason we can find the element $\hat{A} = A + \tilde{A} + \tilde{A}_2 + \ldots$. The sum is converge because of completeness of $\mathcal{O}$.
□

Remark. If $\alpha^n = Id$, then the theorem is not true (see an example in §3).

[1]the idea of the proof of this lemma was offered by N.I.Dubrovin

Corollary 5 *Proposition 0.10 is true for any two-dimensional local skew field with $\alpha^n \neq id$ for all $n \in \mathbb{N}$.*

Theorem 0.16 *Let K, K' be two-dimensional local skew fields such that $\alpha^n \neq Id$, $\alpha'^n \neq Id$ for all $n \in \mathbb{N}$, $\bar{K}$, $\bar{K}'$ are commutative fields. Then*

(i) K is isomorphic to a two-dimensional local skew field $\bar{K}((z))$ where $za = a^\alpha z$, $a \in \bar{K}$.

(ii) K is isomorphic to K' iff $k \cong k'$ and there is an isomorphism $f : \bar{K} \mapsto \bar{K}'$ such that $\alpha = f^{-1}\alpha' f$.

Proof. The proof follows from corollary 5 and from the known classification of one-dimensional local fields (see for example [30]).
□

Definition 0.17 *Let $\bar{K}$ be a one-dimensional local field with residue field k, $char\bar{K} = chark$, let α be an automorphism of the field $\bar{K}$. Put $a_1 = \alpha(u)u^{-1} \mod \wp \in k$. Define $i_\alpha \in \mathbb{N}\bigcup\infty$ as follows:*
$i_\alpha = 1$ if a_1 is not a root of unity in k else
$i_\alpha = \bar{\nu}((\alpha^n - Id)(u))$, where $n \geq 1$: $a_1^n = 1$, $a_1^m \neq 1$ $\forall m < n$.

Lemma 0.18 *Let k be a field of characteristic 0. Any k-automorphism α of a field $k((u))$ with $\alpha(u) = \xi u + a_2u^2 + \ldots$, where $\xi^n = 1$, $n \geq 1$, $\xi^m \neq 1$ if $m < n$, is conjugate with an automorphism β: $\beta(u) = \xi u + xu^{i_\alpha} + yu^{2i_\alpha - 1}$, where $x \in k^*$, $y \in k$, x and y depend on α.*

Moreover, $i_\alpha = i_\beta$.

Proof. First we prove that $\alpha = f\beta' f^{-1}$ where

$$\beta'(u) = \xi u + xu^{in+1} + yu^{2in+1}$$

for some natural i. Then we prove that $i_\alpha = i_{\beta'}$.

Consider a set $\{\alpha_i : i \in \mathbb{N}\}$ where $\alpha_i = f_i\alpha_{i-1}f_i^{-1}$, $f_i(u) = u + x_iu^i$ for some $x_i \in k$, $\alpha_1 = \alpha$. Write

$$\alpha_i(u) = \xi u + a_{2,i}u^2 + a_{3,i}u^3 + \ldots$$

One can check that $a_{2,2} = x_2(\xi^2 - \xi) + a_{2,1}$ and hence there exists an element $x_2 \in k$ such that $a_{2,2} = 0$. Since $a_{j,j+1} = a_{j,i}$, we have $a_{2,j} = 0$ for all $j \geq 2$. Further, $a_{3,3} = x_3(\xi^3 - \xi) + a_{3,2}$ and hence there exists an element $x_3 \in k$ such that $a_{3,3} = 0$. Then $a_{3,j} = 0$ for all $j \geq 3$. Thus, any element $a_{k,k}$ can be made equal to zero if $n \not| (k-1)$ and so $\alpha = f\tilde{\alpha}f^{-1}$ where

$$\tilde{\alpha}(u) = \xi u + \tilde{a}_{in+1}u^{in+1} + \tilde{a}_{in+n+1}u^{in+n+1} + \ldots$$

for some $i, \tilde{a}_j \in k$. Notice that $\tilde{a}_{in+1}$ does not depend on x_i. Put $x = x(\alpha) = \tilde{a}_{in+1}$.

Now we replace α by $\tilde{\alpha}$. One can check that if $n|(k-1)$ then

$$a_{j,k} = a_{j,k-1} \quad \text{for} \quad 2 \le j < k + in$$

and

$$a_{k+in,k} = x_k x(k - in - 1) + a_{k+in} + \text{some polynomial which does not depend on } x_k$$

From this fact it immediately follows that $a_{2in+1,in+1}$ does not depend on x_i and for all $k \neq in+1$ $a_{k+in,k}$ can be made equal to zero. Then $y = y(\alpha) = a_{2in+1,in+1}$.

Now we prove that $i_\alpha = i_{\beta'}$. Using the formula

$$\beta'^n(u) = u + nx(\alpha)\xi^{-1}u^{in+1} + \dots$$

we get $i_{\beta'} = in+1$. Since $f^{-1}\alpha f = \beta'$, $f^{-1}(\alpha^n - Id)f = \beta'^n - Id$. Therefore, $\bar{\nu}(f^{-1}(\alpha^n - Id)f(u)) = \bar{\nu}((\beta'^n - Id)(u)) = i_{\beta'}$. Suppose $f(u) = u' = f_1 u + f_2 u^2 + \dots$, $f_1 \neq 0$. Let us show that $\bar{\nu} f^{-1}(\alpha^n - Id)(u') = i_\alpha$. It suffice to check that $\bar{\nu}(\alpha^n - Id)(u') = i_\alpha$. We have

$$(\alpha^n - Id)(u') = [f_1(u + a_{\bar{i_\alpha}} u^{i_\alpha} + \dots) + f_2(u + a_{\bar{i_\alpha}} u^{i_\alpha} + \dots)^2 + \dots] - [f_1 u + f_2 u^2 + \dots] =$$

$$[(f_1 u + f_1 a_{\bar{i_\alpha}} u^{i_\alpha} + . \;\; u^{>i_\alpha}) + (f_2 u^2 + . \;\; u^{>i_\alpha}) + (f_3 u^3 + . \;\; u^{>i_\alpha}) + \dots]$$

$$-[f_1 u + f_2 u^2 + \dots] = f_1 a_{\bar{i_\alpha}} u^{i_\alpha} + . \;\; u^{>i_\alpha}$$

The lemma is proved.
□

Proposition 0.19 *Let barK be a one-dimensional local field with the residue field k and char$\bar{K}$ = chark. Suppose k is algebraically closed and chark = 0. Let α, β be automorphisms of the field $\bar{K}$.*

Then $\bar{K} = k((u))$ and $\alpha = f^{-1}\beta f$ (where f is an automorphism of $\bar{K}$) iff $(a_1, i_\alpha, y(\alpha)) = (b_1, i_\beta, y(\beta))$.

Proof. The "only if" part is clear. We prove the "if" part.

It is easy to see that $a_1 = b_1$ if $\alpha = f^{-1}\beta f$.

If ξ is not a root of unity, then by lemma 0.18 α is conjugate with β: $\beta(u) = \xi u$. Therefore, the "if" part is proved for the case $i_\alpha = i_\beta = 1$.

Suppose now $i_\alpha = i_\beta \neq 1$ and $a_1 = b_1$ are roots of unity.

Lemma 0.20 *Let β, β' be k-automorphisms of the field $k((u))$: $\beta(u) = \xi u + x u^{in+1} + y u^{2in+1}$, $\beta'(u) = \xi u + \bar{x} u^{in+1} + \bar{y} u^{2in+1}$, where $\bar{x}/x \in (k^*)^{in}$, $\bar{y} = (\bar{x}/x)^2 y$.*

Then β and β' are conjugate.

Proof. Put $x_0 = (\bar{x}/x)^{(in)^{-1}}$. Let f be an automorphism such that $f(u) = x_0 u$. Then we have

$$f\beta(u) = \xi x_0 u + x(x_0 u)^{in+1} + y(x_0 u)^{2in+1} = x_0 \xi u + x_0 \bar{x} u^{in+1} + x_0 \bar{y} u^{2in+1} = \beta' f(u)$$

□

From this and previous lemmas we get the proof of the proposition.

□

Corollary 6 *In the conditions of the proposition suppose k is not algebraically closed field. Suppose $\alpha^n = Id$. Then there exists a parameter u' in $k((u))$ such that $\alpha(u') = a_1 u'$.*

Proof. The proof follows from lemma 0.18.

From the proposition we get also the following result:

Theorem 0.21 *Let K, K' be two-dimensional local skew fields with the last residue fields k and k' and with canonical automorphisms α, α'. Suppose $char K = char k$, $char K' = char k'$, $\alpha^n \neq Id$, $\alpha'^n \neq Id$ for all $n \in \mathbb{N}$, the fields k, k' are algebraically closed of characteristic 0.*

K is isomorphic to K' iff $k \cong k'$ and $(a_1, i_\alpha, y(\alpha)) = (a_1', i_{\alpha'}, y(\alpha'))$.

Now let us study skew fields with canonical automorphisms of finite order.

0.3 Classification of two-dimensional local splittable skew fields of characteristic 0.

In this part we assume that
a two-dimensional local skew field K splits,
$k \subset K$, $k \subset \bar{K}$, $k \subset Z(K)$,
$char(K) = char(k) = 0$,
$\alpha^n = id$ for some $n \geq 1$,
for any convergent sequence (a_j) in $\bar{K}$ the sequence $(z a_j z^{-1})$ converges in K (i.e. the maps δ_i, $i \geq 1$ are continuous, see corollary 3).

We note that the continuity of the maps δ_i, $i \geq 1$ does not depend on the choice of parameters, as it follows from lemma 0.11 and corollary 3.

0.3.1 The case $\alpha = Id$.

Definition 0.22 *Define*

$$i = \nu((\phi_z - 1)(u)) \quad \in \mathbb{N} \bigcup \infty$$

$$r = \bar{\nu}[((\phi_z - 1)(u))z^{-i} \quad mod \quad \wp] \quad mod \quad i \quad \in \mathbb{Z}/i\mathbb{Z}$$

where u, z are arbitrary local parameters of K, $\phi_z : K \to K$, $\phi_z(a) = ad(z)(a)$.

Proposition 0.23 *i and r do not depend on the choice of parameters u and z.*

Proof. We fix some parameters u, z: $K \cong k((u))((z))$. Let u', z' be other parameters. Then

$$u' = (x_0 u + x_1 u^2 + \ldots) + c_1 z + c_2 z^2 + \ldots \quad \text{where} \quad x_i \in k, \quad c_i \in k((u)), \quad , x_0 \neq 0;$$

$$z' = a_0 z + a_1 z^2 + \ldots, \quad a_i \in k((u)), \quad a_0 \neq 0$$

Put $z'' = a_0^{-1} z'$. It's clear that $\nu((\phi_{z''} - 1)(u)) = \nu((\phi_{z'} - 1)(u))$. From the other hand by corollary 4, $\nu((\phi_{z'} - 1)(u)) = \nu((\phi_z - 1)(u))$. So, i does not depend on the choice of parameter z.

Now we prove that $\nu((\phi_z - 1)(u')) = \nu((\phi_z - 1)(u))$. One can obtain this property from the following lemma.

Lemma 0.24 *Suppose the following relation in K holds:*

$$zuz^{-1} = u^\alpha + u^{\delta_j} z^j + \ldots,$$

where $\delta_1 = \ldots = \delta_{j-1} = 0$, $\delta_j \neq 0$. Then

(i) for $u' = u + bz^q$ we have

$$zu'z^{-1} = u'^\alpha + u'^{\delta_1} z + \ldots u'^{\delta_{q-1}} z^{q-1} + u'^{\delta_q} z^q + \ldots,$$

where $u'^{\delta_q} = u^{\delta_q} + b^\alpha - \partial/\partial u(u^\alpha)b$.

(ii) Suppose $\alpha(u) = \xi u$, $\xi \in k$, $\xi^n = 1$ for some natural n. Then for $u' = u + bz^q$, $n|q$ we have

$$zu'z^{-1} = \xi u' + \ldots + (u^{\delta_q} + b^\alpha - \xi b)z^q + \ldots + u'^{\delta_{q+j-1}} z^{q+j-1} + u'^{\delta_{q+j}} z^{q+j} + \ldots,$$

where $u'^{\delta_{q+j}} = u^{\delta_{q+j}} + b^{\delta_j} - \partial/\partial u(u^{\delta_j})b$

(iii) If $\alpha = id$, then for $u' = x_0 u + x_1 u^2 + \ldots$, where $x_q \in k$, $x_0 \neq 0$, we have

$$zu'z^{-1} = u' + (u^{\delta_j} \frac{\partial}{\partial u} u')z^j + \ldots$$

Proof. (i) We have

$$zu'z^{-1} = z(u + bz^q)z^{-1} = u^\alpha + u^{\delta_1}z + \ldots + (b^\alpha + b^{\delta_1}z + \ldots)z^q =$$

$$u^\alpha + u^{\delta_1}z + \ldots + (u^{\delta_q} + b^\alpha)z^q + \ldots = u'^\alpha + u'^{\delta_1} + \ldots + (u^{\delta_q} + b^\alpha - \partial/\partial u(u^\alpha)b)z^q + \ldots,$$

because $u'^{\delta_1} = (u + bz^q)^{\delta_1} = x_0(u + bz^q) + x_1(u + bz^q)^2 + \ldots = u^{\delta_1} + \partial/\partial u(u^{\delta_1})bz^q + \ldots$ if $u^{\delta_1} = x_0u + x_1u^2 + \ldots$.

(ii) We have

$$zu'z^{-1} = z(u + bz^q)z^{-1} = \xi u + u^{\delta_j}z^j + \ldots + (b^\alpha + b^{\delta_j}z^j + \ldots)z^q =$$

$$\xi u + u^{\delta_j}z^j + \ldots + (u^{\delta_q} + b^\alpha)z^q + u^{\delta_{q+1}}z^{q+1} + \ldots + u^{\delta_{q+j-1}}z^{q+j-1} + (u^{\delta_{q+j}} + b^{\delta_j})z^{q+j} + \ldots =$$

$$\xi u' + \ldots + (u^{\delta_q} + b^\alpha - \xi b)z^q + u'^{\delta_{q+1}}z^{q+1} + \ldots + u'^{\delta_{q+j-1}}z^{q+j-1} + (u^{\delta_{q+j}} + b^{\delta_j} - \frac{\partial}{\partial u}(u^{\delta_j})b)z^{q+j}$$

(iii) We have

$$zu'z^{-1} = x_0(u + u^{\delta_j}z^j + \ldots) + x_1(u + u^{\delta_j}z^j + \ldots)^2 + \ldots = u' + (u^{\delta_j}\frac{\partial}{\partial u}u')z^j + \ldots$$

□

Remark. Note that this lemma works also in characteristic $p > 0$.

So, i does not depend on the choice of parameters u and z.

Now we prove it for r. Recall that in our proposition $\alpha = id$ (because i and r were defined only for $\alpha = id$). By lemma 0.24 for any parameter u' we have

$$zu'z^{-1} = u' + (u^{\delta_i}\frac{\partial}{\partial u}u')z^i + \ldots$$

Therefore, $\bar{\nu}[((\phi_z - 1)(u'))z^{-i}] = \bar{\nu}(u^{\delta_i}) = \bar{\nu}[((\phi_z - 1)(u'))z^{-i}]$

If we change z by z' we get

$$z'uz'^{-1} = zuz^{-1} \mod \wp^i$$

Hence

$$\bar{\nu}[((\phi_{z'} - 1)(u))z'^{-i} \mod \wp] = \bar{\nu}[((\phi_z - 1)(u))z'^{-i} \mod \wp] =$$

$$\bar{\nu}[((\phi_z - 1)(u))z^{-i} \mod \wp] + \bar{\nu}(a_0^{-i}) = \bar{\nu}[((\phi_z - 1)(u))z^{-i} \mod \wp] \mod i$$

□

Definition 0.25 *Define*

$$a = res_u \left\{ \frac{u^{\delta_{2i} - \frac{i+1}{2}\delta_i^2}}{(u^{\delta_i})^2} du \right\} \quad \in k$$

Proposition 0.26 *$a = a(u^{\delta_{i+1}}, \ldots, u^{\delta_{2i-1}})$, i.e. a depends only on the maps $u^{\delta_{i+1}}, \ldots, u^{\delta_{2i-1}}$.*

Proof. We comment on the statement first. The maps δ_j are uniquely defined by parameters u, z and they depend on the choice of these parameters. So it suffice to show that a does not depend on the on the choice of parameters which preserve the maps $\delta_{i+1}, \ldots, \delta_{2i-1}$. We can assume that $\delta_{i+1} = 0, \ldots, \delta_{2i-1} = 0$, because we can change the parameters to make this maps to be equal to zero (see lemma 0.11).

First we show that any change of the type $u \mapsto u' = u + c_1 z + \ldots + c_i z^i$ is equivalent to a change of parameters as follows: $z \mapsto z' = z + a_1 z^2 + \ldots$, $u \mapsto u' = u + c_i' z^i + \ldots$, i.e. we get the same maps δ_j in both cases. The proof is by induction.

One can decompose the change $u \mapsto u' = u + c_1 z + \ldots + c_i z^i$ in a finite number of changes $u \mapsto u_1 = u + c_i z^i$, $u_1 \mapsto u_2 = u_1 + c_{i-1} z^{i-1}, \ldots, u_{i-1} \mapsto u_i = u_{i-1} + c_1 z$. So it suffice to prove our assertion for any change of the type $u_j \mapsto u_{j+1} = u_j + c_{i-j} z^{i-j}$.

For $j = 1$ the assertion is trivial. Consider an arbitrary case. By lemma 0.24, δ_{2i-j} is the first map which is not invariant under this change. By lemma 0.11,(ii) there exists a parameter $z' = z + a_{i-j} z^{i-j+1}$ such that the compositum of $u_j \mapsto u_{j+1} = u_j + c_{i-j} z^{i-j}$ and $z \mapsto z' = z + a_{i-j} z^{i-j+1}$ does not change this map. To use the induction hypothesis and complete the proof we have to show that there exists a parameter $u' = u + bz^i$ such that the compositum of $u_j \mapsto u_{j+1} = u_j + c_{i-j} z^{i-j}$, $z \mapsto z' = z + a_{i-j} z^{i-j+1}$ and $u \mapsto u' = u + bz^i$ does not change the map δ_{2i}. Denote by δ''_{2i} the map which is given by the compositum $u_j \mapsto u_{j+1} = u_j + c_{i-j} z^{i-j}$, $z \mapsto z' = z + a_{i-j} z^{i-j+1}$. By lemma 0.24 there exists such a parameter u' iff

$$res_{u_{j+1}} \frac{(\delta''_{2i} - \delta_{2i})(u_{j+1})}{(u_{j+1}^{\delta_i})^2} du_{j+1} = 0.$$

We have $u_{j+1} = c_{i-j} z^{i-j} + \ldots + c_i z^i$. One can decompose the change $u \mapsto u_{j+1}$ in two changes: $u \mapsto u' = u + c_{i-j} z^{i-j}$ and $u' \mapsto u'' = u' + c_{i-j+1} z^{i-j+1} + \ldots + c_i z^i$. The second change does not change the map δ_{2i-j}, so by the induction hypothesis it suffice to prove that the residue is equal to zero for the compositum of $u \mapsto u' = u + c_{i-j} z^{i-j}$ and $z \mapsto z' = z + a_{i-j} z^{i-j+1}$.

Using lemma 0.24, we can calculate a_{i-j}: $a_{i-j} = \frac{\partial}{\partial u}((ju^{\delta_i})^{-1} c_{i-j}) u^{\delta_i}$. Note that if $\bar{\nu}(c_{i-j}) = r$ is big enough then the residue is equal to zero. One can show it with help of lemmas 0.11 and 0.24. We denote by r the minimal positive integer which satisfy this property.

Let $c_{i-j} = \sum_{h=N}^{r} x_h u^h + \sum_{h=r+1}^{\infty} x_h u^h$. Then we can decompose the change $u \mapsto u' = u + c_{i-j} z^{i-j}$ in finite number of changes $u \mapsto u_1' = u + x_N u^N z^{i-j}, \ldots, u'_{r-N-1} \mapsto u'_{r-N} = u'_{r-N-1} + \sum_{h=r+1}^{\infty} x_h u^h z^{i-j}$. It is clear that it suffice to prove our assertion for each change. So we have to prove it for an arbitrary change $u \mapsto u + xu^h z^j$, $x \in k$. We have to check that the compositum of $u \mapsto u + xu^h z^j$ and $z \mapsto z - (j-i)^{-1}(h -$

$r)xu^{h-1}z^{j+1}$ does not change the map δ_{2i}, i.e. the residue above is equal to zero. Put $-(j-i)^{-1}(h-r)xu^{h-1} = b$, $xu^h = b'$.

Let us show that such a compositum change only the maps δ_{i+qj}, $q \in \mathbb{N}$. Moreover, we claim that $\delta_{i+qj}(u) = const \cdot u^{r+q(h-1)}$. Indeed, if $u' = u + b'z^j$ we have

$$zu'z^{-1} = u + u^{\delta_i}z^i + u^{\delta_{2i}}z^{2i} + \ldots + (b' + b'^{\delta_i} + b'^{\delta_{2i}}z^{2i} + \ldots)z^j =$$

$$u' + u^{\delta_i}z^i + b'^{\delta_i}z^{i+j} + u^{\delta_{2i}}z^{2i} + \ldots =$$

$$u' + u'^{\delta_i}z^i + (\frac{\partial}{\partial u}(b')u^{\delta_i} - \frac{\partial}{\partial u}(u^{\delta_i})b')z^{i+j} - \frac{1}{2!}\frac{\partial^2}{\partial u^2}(u^{\delta_i})b'^2 z^{i+2j} - \frac{1}{3!}\frac{\partial^3}{\partial u^3}(u^{\delta_i})b'^3 z^{i+3j} - \ldots$$

$$+(u^{\delta_{2i}} - \frac{1}{e!}\frac{\partial^e}{\partial u^e}(u^{\delta_i})b'^e)z^{2i}$$

where $ej = i$ if $j|i$. If $j \not| i$, $u^{\delta_{2i}}$ does not change.

Therefore,

$$u'^{\delta_{i+j}} = \frac{\partial}{\partial u}(b')u^{\delta_i} - \frac{\partial}{\partial u}(u^{\delta_i})b',$$

and $\bar{\nu}(u'^{\delta_{i+j}}) = r + (h-1)$.

Then

$$u'^{\delta_{i+2j}} = -\frac{\partial}{\partial u}(u'^{\delta_{i+j}})b' - \frac{1}{2!}\frac{\partial^2}{\partial u^2}(u^{\delta_i})b'^2$$

and $\bar{\nu}(u'^{\delta_{i+2j}}) = r + 2(h-1)$,

$$u'^{\delta_{i+qj}} = -\frac{\partial}{\partial u}(u'^{\delta_{i+(q-1)j}})b' - \frac{1}{2!}\frac{\partial^2}{\partial u^2}(u'^{\delta_{i+(q-2)j}})b'^2 - \ldots - \frac{1}{q!}\frac{\partial^q}{\partial u^q}(u'^{\delta_i})b'^q$$

and $\bar{\nu}(u'^{\delta_{i+qj}}) = r + q(h-1)$.

If $z \mapsto z' = z + bz^{j+1}$ we have

$$z'u = (z + bz^{j+1})u = uz + u^{\delta_i}z^{i+1} + u^{\delta_{i+j}}z^{i+j+1} + \ldots$$

$$+u^{\delta_{2i}}z^{2i+1} + \ldots + buz^{j+1} + (j+1)bu^{\delta_i}z^{i+j+1} + \ldots + (j+1)bu^{\delta_{2i-j+1}} + \text{m. with} z^{>2i+1} =$$

$$uz' + u^{\delta_i}z'^{i+1} + u^{\delta'_{i+j}}z'^{i+j+1} + \ldots + u^{\delta'_{2i}}z'^{2i+1} + \ldots =$$

$$u(z + bz^{j+1}) + u^{\delta_i}(z + bz^{j+1})^{i+1} + u^{\delta'_{i+j}}(z + bz^{j+1})^{i+j+1} + \ldots + u^{\delta'_{2i}}(z + bz^{j+1})^{2i+1} + \ldots$$

Hence,

$$u^{\delta'_{i+j}} = u^{\delta_{i+j}} + b(j-i)u^{\delta_i}$$

and $\bar{\nu}(u^{\delta'_{i+j}}) = r + (h-1)$,

$$u^{\delta'_{i+2j}} = u^{\delta_{i+2j}} - C^2_{i+1}b^2u^{\delta_i} - C^1_{i+j+1}bu^{\delta'_{i+j}} + (j+1)bu^{\delta_{i+j}}$$

and $\bar{\nu}(u^{\delta'_{i+2j}}) = r + 2(h-1)$,

$$u^{\delta'_{i+qj}} = u^{\delta_{i+qj}} - C^q_{i+1} b^q u^{\delta_i} - C^{q-1}_{i+j+1} b^{q-1} u^{\delta'_{i+j}} - \ldots - C^1_{i+(q-1)j+1} b u^{\delta'_{i+(q-1)j}} + (j+1) b u^{\delta_{i+(q-1)j}}$$

and $\bar{\nu}(u^{\delta'_{i+qj}}) = r + q(h-1)$.

So if $j \not| i$, $u^{\delta_{2i}}$ does not change. If $j|i$ but $e(h-1) - r \neq -1$, then the residue is equal to zero (note that $e(h-1)-r \neq -1$ if $(r-1, i) = 1$). At last, if $e(h-1)-r = -1$, then one can check the assertion by direct calculations.

So we have shown that the change $u \mapsto u' = u + c_1 z + \ldots + c_i z^i$ is equivalent to the change $z \mapsto z' = z + a_1 z^2 + \ldots$, $u \mapsto u' = u + c'_i z^i + \ldots$. By lemma 0.24 the change $u \mapsto u' = u + c'_i z^i + \ldots$ does not change a. By lemma 0.11 the change $z \mapsto z' = z + a_1 z^2 + \ldots$ does not change $\delta_{i+1}, \ldots, \delta_{2i-1}$ only if $a_1 = a_2 = \ldots = a_{i-1} = 0$. But then it does not change also a. Therefore, any change of the type $z \mapsto z' = z + a_1 z^2 + \ldots$, $u \mapsto u' = u + c_1 z + \ldots$, which does not change $\delta_{i+1}, \ldots, \delta_{2i-1}$, does not change also a.

To complete the proof we only have to show that changes of the type $u \mapsto u' = x_0 u + x_1 u^2 + \ldots$, $x_j \in k$ and $z \mapsto z' = a_0 z$, $a_0 \neq 0 \in k((u))$ don't change a. It is clear for the first change. For the second change we have

$$u^{\delta'_{2i}} = a_0^{-2i}[u^{\delta_{2i}} + i a_0 (a_0^{-1})^{\delta_i} u^{\delta_i} - a_0^{-i}(a_0^{i-1} a_0^{\delta_i} + \ldots + a_0 (a_0^{i-1})^{\delta_i})] =$$

$$a_0^{-2i}[u^{\delta_{2i}} + i(i+1)/2 a_0^{-1} a_0^{\delta_i} u^{\delta_i}]$$

$$u^{(\delta'_i)^2} = a_0^{-2i} u^{\delta_i^2} - i a_0^{-2i-1} a_0^{\delta_i} u^{\delta_i}$$

Therefore,

$$\frac{u^{\delta'_{2i}} - (i+1)/2 u^{(\delta'_i)^2}}{(u^{\delta'_i})^2} = \frac{u^{\delta_{2i}} - (i+1)/2 u^{\delta_i^2}}{(u^{\delta_i})^2} = a$$

The proposition is proved.

□

Remark. If a two-dimensional skew field K does not split, then the numbers i, r, a can not be defined as the following example shows (cf. also the remark after theorem 0.13).

Example.[2] Let ”$((u)) < x_1, x_2 >$ be a free associative algebra over ”$((u))$ with generators x_1, x_2. Let $I =< [[x_1, x_2], x_1], [[x_1, x_2], x_2] >$. It is easy to see that the quotient

$$S = ”((u)) < x_1, x_2 > /I$$

is a ”-algebra which has no non-trivial zero divisors, and in which $z = [x_1, x_2] + I$ is a central element. The elements z, $u_i = x_i + I$ $(i = 1, 2)$ are algebraically independent. Any element of S can be uniquely represented in the form

$$f_0 + f_1 z + f_2 z^2 + \ldots + f_m z^m$$

[2] I thank N.I. Dubrovin for showing me this example

where $f_0, \dots f_m$ are polynomials in the variables u_1, u_2:

$$a + bu_1 + cu_2 + d_1u_1^2 + d_2u_1u_2 + d_3u_2^2 + \dots$$

S is an Ore domain (see [4]), and the quotient skew field K has a discrete valuation ν such that $\nu(u_i) = 0$, $\nu(\text{"}((u))) = 0$, $\nu(z) = 1$. The completion of K with respect to ν is a two-dimensional local skew field which does not split as the following lemma shows.

Lemma 0.27 *Suppose there exist elements u_1, u_2 in the valuation ring $\mathcal{O}$ of a two-dimensional local skew field K such that the element $z = u_1u_2 - u_2u_1$ is a parameter and for any $m \in z\mathcal{O} \setminus z^2\mathcal{O}$ the elements $[u_i, m] = u_im - mu_i$ $(i = 1, 2)$ belong to $z^2\mathcal{O}$.*

Then K does not split.

Proof. Assume the converse. Let $\pi : \bar{K} \mapsto K$ be an embedding. Consider elements $f \in \pi(u_1)$, $g \in \pi(u_2)$. Then $m_1 = f - u_1$, $m_2 = g - u_2 \in z\mathcal{O}$ and

$$0 = [u_1 + m_1, u_2 + m_2] = [u_1, u_2] + [m_1, u_2] + [u_1, m_2] + [m_1, m_2] =$$

$$z + [m_1, u_2] + [u_1, m_2] + [m_1, m_2]$$

Note that the second and the third summands belong to $z^2\mathcal{O}$ and $[m_1, m_2] \in z^2\mathcal{O}$, because $m_1m_2, m_2m_1 \in z^2\mathcal{O}$. But then

$$\infty = \nu(0) = \nu(z + [m_1, u_2] + [u_1, m_2] + [m_1, m_2]) = \nu(z) = 1,$$

a contradiction.

□

In this skew field we have $i = \infty$, and r, a are not defined. From the other hand, if we consider the change $z \mapsto u_1z$, then i become equal to 1, $r = 0$, $a = 0$.

So these numbers depend on the choice of parameters in this unsplittable skew field.

Proposition 0.28 *Let K be a skew field which satisfy the conditions in the beginning of this paragraph. Let char $k = 0$, $\alpha = 1$ and $i \geq 1$. Then K is isomorphic to a skew field $k((u))((z))$ with $zuz^{-1} = u + u^{\delta'_i}z^i + u^{\delta'_{2i}}z^{2i}$, where $\delta'_i(u) = cu^r$, $c \in k^*/(k^*)^e$, where $e = (r-1, i)$; $\delta'_{2i}(u) = (a(0, \dots, 0) + r(i+1)/2)u^{-1}(\delta'_i(u))^2$ and $\delta'_j(u) = 0$ for $j \neq i, 2i$.*

Proof. Consider the change $z \mapsto z' = a_0z$. By lemma 0.11, (iii) we have $u^{\delta'_i} = a_0^{-i}u^{\delta_i}$. So, $u^{\delta'_i}$ can be made to be equal to

$$u^{\delta'_i} = c_0u^{\bar{\nu}(u^{\delta_i})} \mod {}^{i},$$

where $c_0 \in k^*/(k^*)^i$. By lemmas 0.24 and 0.11, c_0 depend only on changes of the type $z \mapsto z' = a_0z$, $u \mapsto u' = x_0u$, where $a_0, x_0 \in k$. So, c_0 can be made to be equal to $c = c_0a_0^{-i}x_0^{-r+1}$. Therefore, $c \in k^*/(k^*)^e$, where $e = (r-1, i)$.

Let us show that there exists a change $z \mapsto z' = z + a_1z^2 + \dots$ such that $\delta'_j(u) = 0$ for $2i > j > i$. Indeed, it can be done by a sequence of changes of the type $z \mapsto z' = z + bz^{j+1}$. By corollary 4, for any such j there exists b such that $\delta'_j(u) = 0$.

Let us show that there exists a change such that it changes the map δ_{2i} as follows: $\delta'_{2i}(u') = (a + r(i+1)/2)u'^{-1}(u'^{\delta_i})^2$. To show it we use lemma 0.24, (ii). By this lemma it suffice to show that there exists an element b such that

$$u^{\delta_{2i}} - (a + r(i+1)/2)u^{-1}(u^{\delta_i})^2 + b^{\delta_i} - (u^{\delta_i})'b = 0$$

, where the prime ' denote the derivation by u. By corollary 2, δ_i is a derivation. Therefore, we can rewrite the equation above as follows

$$u^{\delta_{2i}} - (a + r(i+1)/2)u^{-1}(u^{\delta_i})^2 + b'u^{\delta_i} - (u^{\delta_i})'b = 0$$

One can find a solution of this equation in the form $b = u^{\delta_j}\tilde{b}$. The equation has a solution if $\tilde{b}' + u^{\delta_{2i}}(u^{\delta_i})^{-2} - (a + r(i+1)/2)u^{-1} = 0$. The last equation holds, because $res_u \frac{\delta_i^2(u)}{(\delta_i(u))^2} du = r$.

Using now the same arguments as in the previous paragraph, we can complete the proof.

□

0.3.2 The general case.

Consider now the case $\alpha^n = Id$ for some natural $n > 1$.

Lemma 0.29 *Suppose the canonical automorphism α of a local skew field K satisfy the property $\alpha^n = 1$ for some natural $n > 1$. Then there exists a parameter $z' = z + a_1z^2 + \dots$ such that*

$$z'uz'^{-1} = u^\alpha + u^{\delta'_n}z'^n + u^{\delta'_{2n}}z'^{2n} + \dots$$

Here $\delta'_j = 0$ if $n \not| j$.

Proof. Let

$$zuz^{-1} = u^\alpha + u^{\delta_1}z + u^{\delta_2}z^2 + \dots$$

By corollary 2, δ_1 is a (α^2, α)-derivation.

Since $n > 1$, $\alpha^2 \neq \alpha$. Therefore, by lemma 0.12, δ_1 is an inner derivation and $\delta_1(u) = du^{\alpha^2} - u^\alpha d$. Put $z' = z - dz^2$. By lemma 0.11, (i) we have

$$z'uz'^{-1} = u^\alpha + u^{\delta'_2}z'^2 + \dots$$

By corollary 2, δ'_2 is a (α^3, α)-derivation. If $n \neq 2$ then it is inner and we can apply lemma 0.11. By induction we get that there exists a parameter z' such that

$$z'uz'^{-1} = u^\alpha + u^{\delta'_n}z'^n + u^{\delta'_{n+1}}z'^{n+1} + \dots$$

where δ'_n is a $(\alpha^{n+1},\alpha)=(\alpha,\alpha)$-derivation, i.e. $\delta'_n\alpha^{-1}$ is a derivation.

Note that δ'_{n+1} is a (α^2,α)-derivation. Indeed, by proposition 0.7 we have

$$\delta'_{n+1}(ab)=\sum_{k=0}^{n+1}\delta'_{n+1-k}(a)\sigma(S^k_{n+1}\alpha)(b),\quad a,b\in\bar{K}$$

But $\delta'_j=0$ if $j<n$. Therefore,

$$\delta'_{n+1}(ab)=\delta'_{n+1}(a)\alpha^{n+2}(b)+\delta'_n(a)(\sum_{k=0}^{n}\alpha^k\delta'_1\alpha^{n-k})(b)+\alpha(a)\delta'_{n+1}(b)=\delta'_{n+1}(a)\alpha^2(b)+\alpha(a)\delta'_{n+1}(b)$$

and by lemma 0.12, δ'_{n+1} is an inner derivation. Using lemma 0.11 with $z'\mapsto z''=z'+bz'^{n+2}$ for an appropriate b, we have

$$z''uz''^{-1}=u^\alpha+u^{\delta'_n}z''^n+u^{\delta'_{n+2}}z''^{n+2}+\dots$$

with $\delta'_{n+1}=0$. By induction we can assume that there exists a parameter z' such that

$$z'uz'^{-1}=u^\alpha+u^{\delta'_n}z'^n+u^{\delta'_{2n}}z'^{2n}+\dots+u^{\delta'_{k+1}}z'^{k+1}+\dots$$

Then if $n\not|k+1$, δ'_{k+1} is a (α^{k+2},α)-derivation. Indeed,

$$\delta'_{k+1}(ab)=\sum_{l=0}^{k+1}\sigma(\delta'_{k+1-l}\alpha)(a)\sigma(S^l_{k+1}\alpha)(b)=$$

$$\delta'_{k+1}(a)\alpha^{k+2}(b)+\sum_{m=1}^{x}\delta'_{mn}(a)\sigma(S^{k+1-mn}_{k+1}\alpha)(b)+\alpha(a)\delta'_{k+1}(b)$$

where $x\in\mathbb{N}:xn\le k+1$, $(x+1)n>k+1$, because $\delta'_j=0$ if $j<k+1$ and $n\not|j$.

Every monomial $\sigma(S^{k+1-mn}_{k+1}\alpha)$ contain an element δ'_j with $j<k+1$ and $n\not|j$. It follows from the definition of S^l_{k+1} together with $n\not|k+1-mn$. Therefore, $\sigma(S^{k+1-mn}_{k+1}\alpha)(b)=0$ $\forall m$ and δ'_{k+1} is a (α^{k+2},α)-derivation.

If $n|k+1$, we can apply the same arguments and conclude that δ'_{k+2} is a (α^{k+2},α)-derivation. Therefore, by lemma 0.11 there exists a parameter $z''=z'+bz'^{k+2}$ ($z'+bz'^{k+3}$ if $n|k+1$) such that

$$z''uz''^{-1}=u^\alpha+u^{\delta'_n}z''^n+u^{\delta'_{2n}}z''^{2n}+\dots+u^{\delta'_{k+2}}z''^{k+2}+\dots$$

$$(\text{or}\quad u^{\delta'_{k+1}}z''^{k+1}+u^{\delta'_{k+3}}z''^{k+3}+\dots\quad\text{if } n|k+1)$$

Since $z_{l+1}=(1+z_l^l)z_l$, the sequence $\{z_l\}_{l=1}^\infty$ converges in K. Therefore, we get the proof of the lemma.

□

Lemma 0.30 *There exists a parameter $u \in K$ such that $\alpha(u) = \xi u$, where $\xi^n = 1$, and for all j $\delta'_{jn}(u) = u(\sum_k y_{jk} u^{nk}) \in uk((u^n))$, where $y_{jk} \in k$, and for k not divisible by n $\delta'_k = 0$.*

Proof We can assume that the relation from lemma 0.29 holds. We will do changes of the form $u \mapsto u' = u + b_{jn} z^{jn}$. We have seen in the proof of lemma 0.24 that the maps δ_k, $n \not| k$ don't change under such substitutions. By lemma 0.24, (i) we can see that $u'^{\delta'_{jn}} = u^{\delta'_{jn}} + b^\alpha - \partial/\partial u(u^\alpha)b$. By corollary 6 we can assume $\alpha(u) = \xi u$, where $\xi^n = 1$. Therefore, $u'^{\delta'_{jn}} = u^{\delta'_{jn}} + b^\alpha - \xi b$ and we can find an element b to satisfy the conditions of the lemma.
□

As in the case $\alpha = Id$ we can define i_n, r_n and a_n.

Definition 0.31 *Put*

$$i_n = \nu((\phi_{z^n} - 1)(u)) \quad \in \mathbb{N} \bigcup \infty$$

$$r_n = \bar{\nu}[((\phi_{z^n} - 1)(u))z^{-i_n} \quad mod \quad \wp] \quad mod \quad i_n \quad \in \mathbb{Z}/i_n\mathbb{Z}$$

$$a_n = res_u \left\{ \frac{u^{\delta_{2i_n} - \frac{i_n+1}{2}\delta^2_{i_n}}}{(u^{\delta_{i_n}})^2} du \right\} \quad \in k$$

where u, z are arbitrary local parameters in K, $\phi_z : K \to K$, $\phi_z(a) = ad(z)(a)$.

From the previous two lemmas we can derive that if z is a local parameter from lemma 0.29 then $i_n \in n\mathbb{N}$ and $r_n = 1 \mod n$. It is easy to see that the number i_n is the number of the first non-zero map δ_{i_n} in lemma 0.30. In the same way as in the proof of proposition 0.26 we can get the following result:

Proposition 0.32 *We have $i_n = i_n(u^{\delta_j}, j \notin n\mathbb{N})$, $r_n = r_n(i_n)$, $a_n = a_n(u^{\delta_{i_n+1}}, \dots, u^{\delta_{2i_n-1}})$.*

Proposition 0.33 *Let K be a two-dimensional local skew field which satisfy the conditions in the beginning of this paragraph.*

Let $char k = 0$ and $\alpha^n = Id$ for some natural n.

Then K is isomorphic to a skew field $k((u))((z))$ with the relation

$$zuz^{-1} = \xi u + u^{\delta'_{i_n}} z^{i_n} + u^{\delta'_{2i_n}} z^{2i_n},$$

where $\xi^n = 1$, $i_n = i_n(0, \dots, 0)$,
$\delta'_{i_n}(u) = cu^{r_n}$, $c \in k^*/(k^*)^e$, $e = (r_n - 1, i)$,
$\delta'_{2i_n}(u) = (a_n(0, \dots, 0) + r_n(i_n + 1)/2)u^{-1}(\delta'_{i_n}(u))^2$.

Proof. We can assume that the conditions of lemma 0.30 hold. Then, because of special choice of the element $u^{\delta'_{jn}}$ it suffice to repeat the proof of proposition 0.28. $\square$

Combining all these results we get:

Theorem 0.34 *Let K and K' be local skew fields of characteristic zero. Suppose they satisfy the conditions in the beginning of this chapter. Then K is isomorphic K' iff $k \cong k'$ and the sets $(n, \xi, i_n, r_n, c, a_n)$, $(n', \xi', i'_n, r'_n, c', a'_n)$ coinside.*

Remark. If $n = 1$ and $i_n = \infty$, then K is a commutative two-dimensional local field $k((u))((z))$.

Let us now summarise all the classification results we have got above.

Theorem 0.35 *(I) Let K be a two-dimensional local skew field with a commutative residue skew field.*

It splits if the canonical automorphism α satisfy the condition $\alpha^n \neq Id$ for all n. If this condition does not hold, there are examples of non-splittable skew fields.

(II) Let K, K' be skew fields as in (I). Assume $\alpha^n \neq Id$, $\alpha'^n \neq Id$ for all n. Then

(a) K is isomorphic to a two-dimensional local skew field $\bar{K}((z))$ where $za = a^{\alpha}z$, $a \in \bar{K}$ and $\bar{K}$ is a one-dimensional local field with the residue field k.

(b) K and K' are isomorphic iff $k \cong k'$ and there exists an isomorphism $f : \bar{K} \mapsto \bar{K}'$ such that $\alpha = f^{-1}\alpha' f$.

(c) If $charK = chark$, $charK' = chark'$ and k, k' are algebraically closed fields of characteristic 0, then K is isomorphic to K' iff $k \cong k'$ and $(a_1, i_{\alpha}, y(\alpha)) = (a'_1, i_{\alpha'}, y(\alpha'))$.

(III) Let K, K' be two-dimensional splittable local skew fields of characteristic 0, $k \subset Z(K)$, $k' \subset Z(K')$, and $\alpha^n = Id$, $\alpha'^{n'} = Id$ for some natural $n, n' \geq 1$. Then (a) K is isomorphic to a two-dimensional local skew field $k((u))((z))$ where

$$zuz^{-1} = \xi u + u^{\delta'_{i_n}} z^{i_n} + u^{\delta'_{2i_n}} z^{2i_n},$$

where $\xi^n = 1$, $i_n = i_n(0, \ldots, 0)$,
$\delta'_{i_n}(u) = cu^{r_n}$, $c \in k^*/(k^*)^e$, $e = (r_n - 1, i)$,
$\delta'_{2i_n}(u) = (a_n(0, \ldots, 0) + r_n(i_n + 1)/2)u^{-1}(\delta'_{i_n}(u))^2$
(i_n, r_n, a_n were defined in 0.31).
If $n = 1$, $i_n = \infty$, then K is commutative.

(b) K is isomorphic to K' iff $k \cong k'$ and the sets $(n, \xi, i_n, r_n, c, a_n)$, $(n', \xi', i'_n, r'_n, c', a'_n)$ coinside.

Corollary 7 *Every two-dimensional local skew field K with the ordered set*

$$(n, \xi, i_n, r_n, c, a_n)$$

is a finite-dimensional extension of a skew field with the ordered set $(1, 1, 1, 0, 1, a)$ if $i_n < \infty$.

Remark. It's easy to see from the corollary that skew fields in the theorem above are almost always infinite dimensional over the centre. Namely, the only finite dimensional skew fields are the skew fields with $i_n = \infty$. In the case of skew fields of positive characteristic the situation is much more complicated.

0.4 Splittable skew fields of characteristic $p > 0$.

It is difficult to classify all splittable two-dimensional skew fields with the canonical automorphism of finite order in positive characteristic even if we consider only skew fields with $\alpha = id$, at least because there are infinitely many maps δ_j which can not be removed by any change of parameters. Nevertheless, our methods give some useful tools for studying splittable skew fields finite dimensional over their centre.

For splittable skew fields in positive characteristic one can define an invariant which is in some sense a replacement of the invariant a_n for skew fields of characteristic 0. Certainly, there are infinitely many of other invariants.

Namely, if K is a splittable two-dimensional local skew field of positive characteristic with $\bar{K}$ commutative, $k = \bar{\bar{K}} \subset K$, $k \subset \bar{K}$, $k \subset Z(K)$, α of finite order *we define*

$$d_K = \max_{u,z} \nu(zuz^{-1} - \alpha(u) - \delta_{i_n}^{(z)}(u) z^{i_n}),$$

where $\delta_{i_n}^{(z)}$ is a map defined by a parameter z, and i_n is a number defined in 0.31.

In the case of a skew field of characteristic 0 we have $d_K = 2i_n(0, \dots, 0)$ or $d_K = \infty$, that is why it is in some sense a replacement of a_n: in characteristic 0 it reflects the property of a_n to be zero or not.

In this section we will prove the following theorem:

Theorem 0.36 *Suppose that a two-dimensional local skew field K splits, $\bar{K}$ is a field, $k = \bar{\bar{K}} \subset Z(K)$, $char(K) = char(\bar{\bar{K}}) = p > 2$, $\alpha = id$, and $d_K \leq 2i = 2i_1$ or $d_K = \infty$.*

Then K is a finite dimensional vector space over its centre if and only if K is isomorphic to a two-dimensional local skew field $k((u))((z))$, where

$$z^{-1}uz = u + xz^i$$

with $x \in \bar{K}^p$, $(i, p) = 1$.

To prove this theorem we prove more general result about finite dimensional algebras, which generalises some known results of Jacob and Wadsworth in [9] and Saltman [28]. As a corollary we get the positive answer on the conjecture about exponent and index of a finite dimensional division algebra over a C_2-field for some types of C_2-fields. These results will be proved in the subsection below. Now we prove only the "if" part. Indeed, since $x \in \bar{K}^p$, we have $\delta_i^2(u) = 0$. Hence, by corollary 1 we have $\delta_j = c\delta_i^{j/i}$, $c \in k$ if $i|j$, and $\delta_j = 0$ if $i \not| j$. But then $za^pz^{-1} = a^p$ for any $a \in \bar{K}$, so K is a finite dimensional skew field over its centre and the index $indK = p$.

To prove the "only if" part we need results from the following subsection:

0.4.1 Wild division algebras over Laurent series fields

In this subsection we prove a decomposition theorem for some class of wild division algebras over a Laurent series field with arbitrary residue field of characteristic greater than two. Namely, we prove this theorem for wild division algebras which satisfy the following condition: there exists a section $\bar{D} \hookrightarrow D$ of the residue homomorphism $D \to \bar{D}$, where D is a central division algebra. This theorem is a generalisation of the decomposition theorems for tame division algebras given by Jacob and Wadsworth in [9]. An extensive analysis of the wild division algebras of degree p over a field F with complete discrete rank 1 valuation with $char(\bar{F}) = p$ was given by Saltman in [28] (Tignol in [32] analysed more general case of the defectless division algebras of degree p over a field F with Henselian valuation).

The main result of this subsection is Theorem 0.55; it is a corollary of Theorem 0.43 and propositions 0.51-0.54. Theorem 0.43 is a key tool in the proof of Theorem 0.55. As a corollary we get the positive answer on the following conjecture: the exponent of A is equal to its index for any division algebra A over a C_2-field $F = F_1((t_2))$ (corollary 8) (see also [37], corollary 4, §8.3.2.), where F_1 is a C_1-field. We note that the proof of the conjecture does not depend on the statement of theorem 0.55, but uses only several lemmas from it's proof. In fact, these lemmas are most important technical statements about the maps δ_m and their generalisations.

We change the notation in this subsection and use the notation of [9], because it is more convenient for applications in the valuation theory. We always denote here by D a division algebra finite dimensional over its centre $F = Z(D)$. Recall that any Henselian valuation on F has a unique extension to a valuation on D.

Given a valuation v on D, we denote by Γ_D its value group, by V_D its valuation ring, by M_D its maximal ideal and by $\bar{D} = V_D/M_D$ its residue division ring.

By [31], p.21 one has the fundamental inequality

$$[D : F] \geq |\Gamma_D : \Gamma_E| \cdot [\bar{D} : \bar{F}].$$

D is called defectless over F if equality holds and defective otherwise. It is known that D is defectless if it has a discrete valuation of rank 1.

Jacob and Wadsworth in [9] introduced the basic homomorphism

$$\theta_D : \Gamma_D/\Gamma_F \to Gal(Z(\bar{D})/\bar{F})$$

induced by conjugation by elements of D. They showed that θ_D is surjective and $Z(\bar{D})$ is the compositum of an Abelian Galois and a purely inseparable extension of $\bar{F}$.

We say D is tame division algebra if $char(\bar{F}) = 0$ or $char(\bar{F}) = q \neq 0$, D is defectless over F, $Z(\bar{D})$ is separable over $\bar{F}$, and $q \not| |ker(\theta_D)|$. We say D is wild division algebra if it is non tame.

We call a division algebra D *inertially split* if $Z(\bar{D})$ is separable over $\bar{F}$, the map θ_D is an isomorphism, and D is defectless over F.

0.4.2 Cohen's theorem

There is a natural question if there exists a generalisation of Cohen's theorem, i.e. is any central division algebra splittable or not. It is not true if a division algebra is not finite dimensional over its centre, as Dubrovin's example shows. It is not true also for finite dimensional division algebras, as we will see in Wadsworth's example below. But it is true for tame division algebras over complete discrete valued fields. This easily follows from results of Jacob and Wadsworth [9].

Theorem 0.37 *Let (F, v) be a valued field which is complete with respect to a discrete rank 1 valuation v. Suppose $char F = char \bar{F}$. Let D be a tame division algebra with $Z(D) = F$ and $[D : F] < \infty$.*

Then there exists a section $\bar{D} \hookrightarrow D$ of the residue homomorphism $D \to \bar{D}$.

Proof. Since F is a complete field, F is a Henselian field and v extends uniquely to a valuation w on D. Since D is tame, $Z(\bar{D})/\overline{Z(D)}$ is a cyclic Galois extension. There exists an inertial lift Z of $Z(\bar{D})$ over F, Z is Galois over F, and by classical Cohen's theorem there exists a section $\tilde{Z}(\bar{D}) \hookrightarrow Z$.

Consider the centraliser $C = C_D(Z)$ of Z in D. Then we have $\bar{C} = \bar{D}$.

Indeed, by Double Centraliser Theorem we have $[D : F] = [C : F][Z : F]$ and $[Z : F] = |Gal(Z(\bar{D})/\bar{F})|$. By [9], prop.1.7 a homomorphism $\theta_D : \Gamma_D/\Gamma_F \to Gal(Z(\bar{D})/\bar{F})$ is surjective, so for any parameter z we have $\theta_D(w(z)) = \sigma$, where $< \sigma >= Gal(Z(\bar{D})/\bar{F})$. It is clear that $z \notin C$. Now let $u_1, \dots, u_{[C:F]}$ be a F-basis of C. It is easy to see that the elements $u_j, zu_j, \dots, z^{n-1}u_j$, $j = 1, \dots, [C : F]$, where $n = ord(\sigma)$, the order of σ, are linearly independent, so form a basis for D over F. Since

$$w(F < zu_j, \dots, z^{n-1}u_j, j = 1, \dots, [C : F] >) \cap \Gamma_C = 0,$$

where $F < zu_j, \dots, z^{n-1}u_j, j = 1, \dots, [C : F] >$ denote a vector space in D over F generated by elements $u_j z^i$, this implies that for any element $x \in D$ with $w(x) = 0$ we can find elements $r_1, \dots r_{[C:F]} \in F$ such that $x = r_1 u_1 + \dots + r_{[C:F]} u_{[C:F]} \mod M_D$. Hence $\bar{C} = \bar{D}$.

Fix an embedding $\bar{F} \hookrightarrow F$ and consider the algebra $A = \bar{C} \otimes_F Z(C)$. It is easy to see that A is an unramified division algebra with $\bar{A} = \bar{C} = \bar{D}$. Therefore by [2], Th.31, $A \cong C$; so there exists a section $\bar{D} \hookrightarrow C$.

The theorem is proved.

□

Example (Wadsworth). Let p be any prime number, let $k = Z/pZ$, the field with p elements, and let r, s be independent indeterminates over k. Let $F = k(r, s)((t))$, the formal Laurent series field in t over the rational function field $k(r, s)$. F has its complete discrete t-adic valuation with residue field $\bar{F} = k(r, s)$. Let $f = x^p + tx - r$ in $F[x]$. The derivative test shows that f has no repeated roots. Let θ be a root of f, and let $K = F(\theta)$, which is separable over F.

Let M be the separable closure of K over F. So, the Galois group $Gal(M/F)$ is isomorphic to a subgroup of the symmetric group S_p. Let L be the fixed field of a p-Sylow subgroup of $Gal(M/F)$, and let σ be a generator of $Gal(M/L)$, a cyclic group of order p. The valuation on F extends uniquely to complete discrete valuations on L and on M. Note that $\bar{L}$ doesn't contain $r^{1/p}$, since $[\bar{L} : \bar{F}]$ divides $[L : F]$, which is prime to p. (For the same reason, $\bar{L}$ doesn't contain a p-th root of s.) But $\bar{M}$ contains $\bar{\theta}$, which is a p-th root of r. So, $[\bar{M} : \bar{L}] = [M : L] = p$, and $\bar{M} = \bar{L}(r^{1/p})$, which is purely inseparable over $\bar{L}$. Since σ acts trivially on $\bar{M}$, the norm map from M to L induces the p-th power map from $\bar{M}$ to $\bar{L}$. So, s is not in the image of the norm from M to L. Therefore, the cyclic algebra $D = (M/L, \sigma, s)$ is nonsplit of degree p, so it is a division algebra. With respect to the unique extension of the valuation on L to D, we have $\bar{D}$ contains a pth root of r and also of s, so $p^2 = [D : L] \geq [\bar{D} : \bar{L}] \geq [\bar{L}(r^{1/p}, s^{1/p})] = p^2$.

This shows that $\bar{D}$ is the field $\bar{L}(r^{1/p}, s^{1/p})$, which is purely inseparable over $\bar{L}$. Hence also, the ramification index of D over L must be 1.

0.4.3 Decomposition theorem

In this part we prove a generalisation of Jacob-Wadsworth's decomposition theorem ([9], Th.6.3., lemma 6.2) for finite dimensional splittable division algebras over a Laurent series field $k((t))$, $char k > 2$.
So, *in this section we consider only splittable division algebras*. Moreover, we will need more strong condition:

Definition 0.38 *A division algebra D is called good splittable if there exists a section $\bar{D} \hookrightarrow D$ compatible with an embedding $\overline{Z(D)} \hookrightarrow Z(D)$, i.e. $Z(D) \supset \overline{Z(D)} \subset \bar{D}$.*

It's easy to see that all tame division algebras are good splittable, because by Cohen's theorem any embedding $\overline{Z(D)} \hookrightarrow Z(D)$ can be uniquely extended to any separable extension of $Z(D)$.

We note that the skew field K from theorem 0.36 is good splittable if K is a finite dimensional division algebra over its center. Indeed, because of the condition of the theorem, we can assume k is an algebraically closed field. Then the center of K is a C_2-field by Tsen's theorem (see the definition and the properties of C_2-fields below, at the end of this subsection). Then it will be shown in corollary 8 that all division algebras over C_2-fields are good splittable.

For division algebras of index p over a Laurent series field the condition to be splittable is equivalent to the condition to be good splittable, see the end of this subsection.

Let D be a finite dimensional division algebra over a complete valued field $F = k((t))$. Let w be a unique extension of the valuation v to D. We will denote by z any parameter of D, i.e. any element with $< w(z) >= \Gamma_D$.

Proposition 0.39 *D is isomorphic to a local skew-field $\bar{D}((z))$, where*

$$zaz^{-1} = \alpha(a) + \delta_1(a)z + \delta_2(a)z^2 + \dots, \quad a \in \bar{D};$$

here $\alpha : \bar{D} \to \bar{D}$ is an automorphism and $\delta_i : \bar{D} \to \bar{D}$ are linear maps such that the map δ_i satisfy the identity

$$\delta_i(ab) = \sum_{k=0}^{i} \sigma(\delta^{i-k}\alpha)(a)\sigma(S_i^k\alpha)(b), \quad a, b \in \bar{D}$$

The proof is an easy combination of the proofs of propositions in section 1 and Cohen's theorem 0.37.

Definition 0.40 *Let us define maps ${}_m\delta_i : \bar{D} \to \bar{D}$, $m \in \mathbb{Z}$, $i \in \mathbb{N}$ as follows.*

$$z^m a z^{-m} = \alpha^m(a) + {}_m\delta_1(a)z + {}_m\delta_2(a)z^2 + \dots, \quad a \in \bar{D}.$$

If $m = 0$, put ${}_m\delta_i = 0$.

Note that if $\alpha = id$, then ${}_m\delta_i = 0$ for $m = p^k$, where k is sufficiently large, k depends on i. Moreover, ${}_m\delta_i = {}_{m+p^k}\delta_i$ for k sufficiently large. We will use also the following notation:

$${}_m\tilde{\delta}_i = {}_{-m}\delta_i.$$

Proposition 0.41 *(i) The maps ${}_m\delta_i$ satisfy the following identities:*

$${}_m\delta_i(ab) = {}_m\delta_i(a)\alpha^{i+m}(b) + \alpha^m(a){}_m\delta_i(b) + \sum_{k=1}^{i-1} {}_m\delta_{i-k}(a)_{i-k+m}\delta_k(b)$$

(ii) Suppose $\alpha = id$. Then the maps ${}_m\delta_i$ satisfy the following identities:

$$ {}_m\delta_i(ab) = {}_m\delta_i(a)b + a_m\delta_i(b) + \sum_{k=1}^{i-1} {}_m\delta_{i-k}(a) \sum_{(j_1,\dots,j_l)} C^l_{i-k+m}\delta_{j_1}\dots\delta_{j_l}(b) $$

where the second sum is taken over all the vectors $(j_1,\dots,j_l)$ such that $0 < l \le min\{i-k+m, k\}$, $j_m \ge 1$, $\sum j_m = k$; $C^k_j = 0$ if $j = 0$, and $C^k_j = C^k_{j+p^q}$ for $q >> 0$ if $j \le 0$.

Proof. For any $a, b \in \bar{D}$ we have

$$ \alpha^m(ab)z^m + {}_m\delta_1(ab)z^{m+1} + {}_m\delta_2(ab)z^{m+2} + \ldots = z^m(ab) = $$

$$ (*) \qquad (\alpha^m(a)z^m + {}_m\delta_1(a)z^{m+1} + {}_m\delta_2(a)z^{m+2} + \ldots)b $$

If we represent the right-hand side of $(*)$ as a series with coefficients shifted to the left and then compare the corresponding coefficients on the left-hand side and right-hand side, we get some formulas for ${}_m\delta_i(ab)$. We have to prove that these formulas are the same as in our proposition.

Let

$$ z^{i+m-k}b = \alpha^{i+m-k}(b)z^{i+m-k} + \ldots + x'_k z^{i+m} + \ldots $$

and

$$ (\alpha^m(a)z^m + {}_m\delta_1(a)z^{m+1} + {}_m\delta_2(a)z^{m+2} + \ldots)b = \alpha^m(ab)z^m + y_{m+1}z^{m+1} + y_{m+2}z^{m+2} + \ldots $$

Then we have

$$ y_{i+m} = \alpha^m(a)x'_i + \sum_{k=0}^{i-1} {}_m\delta_{i-k}(a)x'_k $$

In the proof of prop. 0.7 we have shown that

$$ z^{i+1-k}b = \alpha^{i+1-k}(b)z^{i+1-k} + \ldots + \sigma(S^k_i\alpha)(b)z^{i+1} + \ldots $$

Hence $x'_k = \sigma(S^k_{i+m-1}\alpha)(b)$ for $k < i$. It is easy to see that $x'_i = {}_m\delta_i(b)$, $x'_0 = \alpha^{i+m}(b)$ and $\sigma(S^k_{i+m-1}\alpha) = {}_{i+m-k}\delta_k$, which proves (i).

For $\alpha = id$, by corollary 1,

$$ \sigma(S^k_{i+m-1}\alpha)(b) = \sum_{(j_1,\dots,j_l)} C^l_{i-k+m}\delta_{j_1}\dots\delta_{j_l}(b), $$

where $l, j_1, \dots, j_l$ were defined in our proposition. This proves (ii).
The proposition is proved.
□

Definition 0.42 *Let D be a splittable division algebra. For any element $a \in \bar{D}$ define the numbers*

$$i(a) = \max_{j(a),z} w(zaz^{-1} - \alpha(a)) \in \mathbb{N} \cup \infty,$$

where $j : \bar{D} \hookrightarrow D$, z — parameter in D;

$$d_D(a) = \max_{j(a),z} w(zaz^{-1} - \alpha(a) - \delta^{(z)}_{i_j(a)} z^{i_j(a)}) \in \mathbb{N} \cup \infty,$$

where $i_j(a) = w(zaz^{-1} - \alpha(a))$ for a given embedding j. It does not depend on the choice of z as lemma 0.11 shows.

The following theorem is a main technical result of this subsection.

Theorem 0.43 *Let D be a good splittable division algebra. Let $u \in Z(\bar{D})$ be a purely inseparable element over $\overline{Z(D)}$ and $\mathrm{char} F > 2$. Then $d_D(u) = 2i(u) + np$, where $n > 0$, and $u^p \in \bar{F}$ only if $d_D(u) = \infty$.*

Proof. By proposition 0.39 we can assume that $u \in \bar{D} \subset D$. Without loss of generality we can assume that $Z(\bar{D})/\bar{F}$ is a purely inseparable extension. Moreover, it can be assumed that $Z(\bar{D}) = \bar{F}(u)$. Then by Scolem-Noether theorem we can choose a parameter z such that $\alpha = id$. Suppose

$$zuz^{-1} = u + \delta_i(u)z^i + \dots,$$

i.e. $\delta_1|_{F(u)} = \dots = \delta_{i-1}|_{F(u)} = 0$, $\delta_i(u) \neq 0$. Suppose $u^{p^k} \in F$.

Without loss of generality it can be assumed that the following property holds:

ψ) let δ_j, $j > i$ be the first map such that $\delta_j \neq 0$ if j is not divisible by i and $\delta_j \neq c_{j/i}\delta_i^{j/i}$ for some $c_{j/i} \in \bar{D}$ otherwise; then $j = 2i \mod p$.

Indeed, let δ_j be the first map such that $\delta_j \neq 0$ and $i \not| j$, $j \neq 2i \mod p$. Then by lemma 0.11, (ii) there exists a parameter z' such that $\delta'_j(u) = 0$. Therefore by corollary 1, $\delta'_j|_{F(u)} = 0$. By Scolem-Noether theorem, δ'_j is an inner derivation. Therefore by lemma 0.11 (ii) there exists a parameter z such that $\delta_j = 0$.

If δ_j is the first map such that $i|j$, $\delta_j|_{F(u)} \neq c_{j/i}\delta_i^{j/i}|_{F(u)}$ and $j \neq 2i \mod p$, then by lemma 0.11, (ii) there exists a parameter z' such that $\delta'_j(u) = c_{j/i}\delta_i'^{j/i}(u) = c_{j/i}\delta_i^{j/i}(u)$, where

$$c_{j/i} = \frac{(i+1)\dots(i(j/i-1)+1)}{(j/i)!} \tag{1}$$

One can easily show that $\delta'_j|_{F(u)} = c_{j/i}\delta_i'^{j/i}|_{F(u)}$. By Scolem-Noether theorem, $(\delta'_j - c_{j/i}\delta_j'^{j/i})$ is an inner derivation. Therefore by lemma 0.11 (ii) there exists a parameter z such that $\delta_j = c_{j/i}\delta_i^{j/i}$.

Let's divide our proof in two steps.

Step 1. First we prove that $(i,p)=1$.

In this Step we don't use the condition D is a good splittable algebra. We use only a condition that D is splittable.

Lemma 0.44 *Let j be the minimal positive integer such that $\delta_j|_{F(u^{p^l})} \neq 0$, $l \geq 0$. Then the maps ${}_n\delta_m$, $kj \leq m < (k+1)j$, $k \in \{1,\ldots,p-1\}$ satisfy the following properties:*

i) there exist elements $c_{m,k} \in \bar{D}$ such that

$$({}_n\delta_m - c_{m,1}\delta - \ldots - c_{m,k}\delta^k)|_{F(u^{p^l})} = 0,$$

where $\delta : \bar{D} \to \bar{D}$ is any F-linear map such that $\delta|_{F(u^{p^l})}$ is a derivation, $\delta(u^j) = 0$ for $j \notin p^l\mathbb{N}$, $\delta(u^{p^l}) = 1$, $c_{kj,k} = c(\delta_j(u^{p^l}))^k$, $c \in \mathbb{F}_p$.

ii) $c_{kj,k} \neq 0$ iff $n, n+j, \ldots, n+(k-1)j \neq 0 \mod p$.

Proof. i) The proof is by induction on k. Let $a, b \in F(u^{p^l})$. Put $t = u^{p^l}$. For $k=1$, by proposition 0.41, (ii) we have

$${}_n\delta_m(ab) = {}_n\delta_m(a)b + a{}_n\delta_m(b)$$

because all the maps δ_q, $q < j$ are equal to zero on $F(u^{p^l})$. Hence, ${}_n\delta_m$ is a derivation on $F(u^{p^l})$ and $c_{j,1} = {}_n\delta_j(t) = n\delta_j(t)$.

For arbitrary k, by proposition 0.41, (i) and by the induction hypothesis we have (**)

$${}_n\delta_m(t^q) = q{}_n\delta_m(t)t^{q-1} + {}_n\delta_j(t)(\sum_{l=0}^{q-2}(c_1\delta + \ldots + c_{k-1}\delta^{k-1})(t^{q-1-l})t^l) + \ldots + {}_n\delta_s(t)(\sum_{l=0}^{q-2}(g_1\delta)(t^{q-1-l})t^l),$$

where $c_j, g_j \in \bar{D}$, $s > m-2j$. Therefore, ${}_n\delta_m(t^p) = 0$, because $k \leq p-1$ and $\sum_{l=0}^{p-2} c_j\delta^j(t^{p-1-l})t^l = 0$ for $j \leq p-2$. Hence, ${}_n\delta_m|_{F(t)} = c_{m,1}\delta + \ldots + c_{m,p-1}\delta^{p-1}$ and we only have to show that $c_{m,q} = 0$ for $q > k$.

Using (**) we can calculate $c_{m,j}$. We have

$$c_{m,1} = {}_n\delta_m(t);$$

$$c_{m,2} = \frac{1}{2!}{}_n\delta_m(t^2) - 2c_{m,1}t = \frac{1}{2}{}_n\delta_j(t)(c_1\delta(t)) + \ldots + {}_n\delta_s(t)(g_1\delta(t))$$

$$\ldots$$

$$c_{m,q} = \frac{1}{q!}({}_n\delta_j(t)(\sum_{l=0}^{q-2} c_{q-1}\delta^{q-1}(t^{q-1-l})t^l) + \ldots + {}_n\delta_{m-q+1}(t)(\sum_{l=0}^{q-2} g_{q-1}\delta^{q-1}(t^{q-1-l})t^l))$$

$$= \frac{1}{q}({}_n\delta_j(t)c_{q-1} + \ldots + {}_n\delta_{m-q+1}(t)g_{q-1})$$

Hence, $c_{m,k+1} = \ldots = c_{m,p-1} = 0$ and $c_{kj,k} = c\delta_j(t)c_{k-1} = \tilde{c}(\delta_j(t))^k$, $c, \tilde{c} \in \mathbb{F}_p$. Note that $c_{(p-1)j,p} = ({}_j\delta_j(t))^{p-1}$.

ii) Suppose $n = 0 \mod p$. For $k = 1$ we have ${}_n\delta_j|_{F(t)} = n\delta_j|_{F(t)} = 0$. For arbitrary k we have

$$ {}_n\delta_{kj}(t^q) = q_n\delta_{kj}(t)t^{q-1} + {}_n\delta_j(t)\sum_{r=0}^{q-2} {}_{n+j}\delta_{(k-1)j}(t^{q-1-r})t^r + \ldots + \sum_{l=j+1}^{kj-1} {}_n\delta_l(t)\sum_{r=0}^{q-2} {}_{n+l}\delta_{kj-l}(t^{q-1-r})t^r $$

Since ${}_n\delta_j(t) = 0$ and ${}_{m_1}\delta_h|_{F(t)} = c_1\delta + \ldots + c_{k-2}\delta^{k-2}$ for $h < (k-1)j$, the same arguments as in i) show that $c_{kj,k} = 0$.

Suppose $n + rj \neq 0 \mod p$, $r < k-1$. The same arguments as above show that in this case $c_{kj,k}({}_n\delta_{kj}) = 0$ iff $c_{(k-1)j,k-1}({}_{n+j}\delta_{(k-1)j}) = 0$. So, by induction, $c_{kj,k}({}_n\delta_{kj}) = 0$ iff $c_{(k-r)j,k-r}({}_{n+rj}\delta_{(k-r)j}) = 0$, which proves ii).

The lemma is proved.

□

Lemma 0.45 *If $p|i$, then there exists a map δ_j such that $\delta_j(u^{p^k}) \neq 0$.*

Proof. We claim that $\delta_{p^q i}$ is the first map such that $\delta_{p^q i}|_{F(u^{p^q})} \neq 0$. The proof is by induction on q. For $q = 0$, there is nothing to prove. For arbitrary q, put $t = u^{p^{q-1}}$. By proposition 0.41 we have

$$ \delta_{p^q i}(t^p) = \delta_{p^{q-1}i}(t)\sum_{r=0}^{p-2} {}_{1+p^{q-1}i}\delta_{p^{q-1}i(p-1)}(t^{p-1-r})t^r + \sum_{l=p^{q-1}i+1}^{p^q i-1} \delta_l(t)\sum_{r=0}^{p-2} {}_{1+l}\delta_{p^q i-l}(t^{p-1-r})t^r $$

By induction and lemma 0.44, ${}_{1+l}\delta_{p^q i-l}|_{F(t)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2}$ for $l > p^{q-1}i$. Therefore, $\sum_{r=0}^{p-2} {}_{1+l}\delta_{p^q i-l}(t^{p-1-r})t^r = 0$. By lemma 0.44, (ii), ${}_{1+p^{q-1}i}\delta_{p^{q-1}i(p-1)}|_{F(t)} = c_1\delta + \ldots + c_{p-1}\delta^{p-1}$ with $c_{p-1} \neq 0$. Hence, $\delta_{p^q i}(t^p) = -c_{p-1}\delta_{p^{q-1}i}(t) \neq 0$.

The same arguments show that $\delta_j(t^p) = 0$ for $j < p^q i$. So, $\delta_{p^q i}$ is the first non-zero map on $F(u^{p^q})$.

□

Step 2.

From now on $(i, p) = 1$. Note that $\delta_i(u) \in Z(\bar{D})$. Indeed, for any $a \in \bar{D}$ we have

$$ \delta_i(au) = \delta_i(a)u + a\delta_i(u) = \delta_i(ua) = u\delta_i(a) + \delta_i(u)a $$

Therefore, $a\delta_i(u) = \delta_i(u)a$. Since $(i, p) = 1$, there exists $k_1 \in \mathbb{N}$ such that p^k divides $1 + k_1 i$. Therefore by lemma 0.11 (iii) there exists a parameter z' such that $\delta'_i(u) = (\delta_i(u))^{1+k_1 i} \in Z(\bar{D})$.

So we can assume that $\delta_i(u) \in Z(\bar{D})$ and $\psi)$ holds.

Assume that $d_D(u) \leq 2i(u)$.

Then to prove our theorem it is sufficient to show that there exists a parameter z such that the maps δ_j satisfy the following property:

$(*)$ If j is not divisible by i, then $\delta_j|_{F(u)} = 0$. If j is divisible by i, then $\delta_j|_{F(u)} = c_{j/i}\delta_i^{j/i}|_{F(u)}$ with some $c_{j/i} \in \bar{D}$.

To show it we prove that if property $(*)$ does not hold, then there exists a map δ_j such that $\delta_j(u^{p^k}) \neq 0$.

Suppose $(*)$ *does not hold and* δ_{2i+mp} *is the first map which does not satisfy* $(*)$. *So,* $\delta_{2i+mp}(u) \neq 0$. *Note that* $\delta_q(u) = 0$ *for* $i < q < 2i + mp$.

Note that $\delta_{2i+mp}(u), \tilde{\delta}_{2i+mp}(u) \in Z(\bar{D})$. Indeed, by proposition 0.41, ${}_i\tilde{\delta}_{2i+mp}|_{F(u)}$ is a derivation. Therefore, ${}_i\tilde{\delta}_{2i+mp}(u) \in Z(\bar{D})$. Since $\delta_i(u) \in Z(D)$ and $\delta_q(u) = 0$ for $i < q < 2i + mp$, ${}_i\tilde{\delta}_{2i+mp}(u) = i\tilde{\delta}_{2i+mp}(u)$ and $\tilde{\delta}_{2i+mp}(u) \in Z(\bar{D})$. Therefore, $\delta_{2i+mp}(u) \in Z(\bar{D})$.

First we prove that there exists a parameter $\bar{z}$ such that $\bar{\delta}_q = \delta_q$ for $q \leq 2i + mp$ and ${}_{2i+mp+(p-1)i}\tilde{\bar{\delta}}_q(u) = 0$ for $q \neq 2i \mod p$, $q > 2i + mp$; here $\bar{\delta}_q$ are maps given by the parameter $\bar{z}$. Put $j(1) = 2i + mp + (p-1)i$.

Suppose ${}_{j(1)}\tilde{\delta}_q(u) \neq 0$, $q > 2i + mp$ and $q \neq 2i \mod p$. By definition,

$$ {}_{j(1)}\tilde{\delta}_q(u) = -j(1)\delta_q(u) + \sum \delta_{k_1} \ldots \delta_{k_l}(u), $$

where $k_i < q$. By lemma 0.11, (ii) for any $a \in \bar{D}$ there exists a parameter $\bar{z}_q$ such that

$$ \bar{z}_q u \bar{z}_q^{-1} = u + \delta_i(u)\bar{z}_q^i + \ldots + \delta_{q-1}(u)\bar{z}_q^{q-1} + a\bar{z}_q^q + \ldots $$

Therefore, there exists an element $a \in \bar{D}$ such that ${}_{j(1)}\tilde{\bar{\delta}}_q(u) = 0$. It is easy to see that the sequence $\{\bar{z}_q\}$ converges in D. So, $\bar{z} = lim\bar{z}_q$.

Lemma 0.46 *Put* $\kappa = j(1) = 2i + mp + (p-1)i$. *Then there exists a parameter* z *such that the following properties hold:*

(i) ${}_\kappa\tilde{\delta}_{2i+mp+(p-1)i}$ *is the first map such that* ${}_\kappa\tilde{\delta}_{2i+mp+(p-1)i}|_{F(u^p)} \neq 0$.

(ii) ${}_\kappa\tilde{\delta}_{2i+mp+(p-1)i+i+mp}(u^p) \neq 0$ *and* ${}_\kappa\tilde{\delta}_r|_{F(u^p)} = 0$ *for* $j(1) < r < j(1) + i + mp$.

(iii) ${}_\kappa\tilde{\delta}_{2i+mp+(p-1)i}(u^p) \in Z(\bar{D})$, ${}_\kappa\tilde{\delta}_{2i+mp+(p-1)i+i+mp}(u^p) \in Z(\bar{D})$.

Proof.

i) Put $w := 2i + mp + (p-1)i$. By proposition 0.41 we have

$$ {}_\kappa\tilde{\delta}_w(u^p) = {}_\kappa\tilde{\delta}_i(u) \sum_{q=0}^{p-2} {}_{i-\kappa}\delta_{w-i}(u^{p-1-q})u^q + {}_\kappa\tilde{\delta}_{2i+mp}(u) \sum_{q=0}^{p-2} {}_{2i+mp-\kappa}\delta_{(p-1)i}(u^{p-1-q})u^q + $$

$$\sum_{k=2i+mp+1}^{w-1} {}_{\kappa}\tilde{\delta}_k(u) \sum_{q=0}^{p-2} {}_{k-\kappa}\delta_{w-k}(u^{p-1-q})u^q$$

By lemma 0.44, ${}_{k-\kappa}\delta_{w-k}|_{F(u^p)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2}$ for $w - k < (p-1)i$ and ${}_{2i+mp-\kappa}\delta_{(p-1)i}|_{F(u^p)} = c_1\delta + \ldots + c_{p-1}\delta^{p-1}$ with $c_{p-1} = ({}_i\delta_i(u))^{p-1} \neq 0$.

By proposition 0.41 we have

$${}_{i-\kappa}\delta_{w-i}(u^q) = q_{i-\kappa}\delta_{w-i}(u)u^{q-1} + {}_{i-\kappa}\delta_i(u) \sum_{r=0}^{q-2} {}_{m_1}\delta_{w-2i}(u^{q-1-r})u^r +$$

$$\sum_{s=2i+mp}^{w-i-1} {}_{i-\kappa}\delta_s(u) \sum_{r=0}^{q-2} {}_{s+i-\kappa}\delta_{w-i-s}(u^{q-1-r})u^r$$

By lemma 0.44, ${}_{s+i-\kappa}\delta_{w-i-s}|_{F(u)} = c_1\delta + \ldots + c_{p-3}\delta^{p-3}$ for $w-i-s < (p-2)i$. Since $i-\kappa = 0 \mod p$, ${}_{i-\kappa}\delta_i(u) = 0$ and ${}_{i-\kappa}\delta_{2i+mp}(u) = 0$. So, ${}_{i-\kappa}\delta_{w-i}|_{F(u)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2}$. Hence,

$${}_{\kappa}\tilde{\delta}_w(u^p) = -{}_{\kappa}\tilde{\delta}_{2i+mp}(u)({}_i\delta_i(u))^{p-1} = -{}_i\tilde{\delta}_{2i+mp}(u)({}_i\delta_i(u))^{p-1} \neq 0$$

and ${}_{\kappa}\tilde{\delta}_w(u^p) \in Z(\bar{D})$.

The same arguments show that ${}_{\kappa}\tilde{\delta}_w$ is the first map such that ${}_{\kappa}\tilde{\delta}_w|_{F(u^p)} \neq 0$.

ii) For $j(1) < w \leq 2i + mp + (p-1)i + i + mp$, by proposition 0.41 we have

$${}_{\kappa}\tilde{\delta}_w(u^p) = {}_{\kappa}\tilde{\delta}_i(u) \sum_{q=0}^{p-2} {}_{i-\kappa}\delta_{w-i}(u^{p-1-q})u^q + {}_{\kappa}\tilde{\delta}_{2i+mp}(u) \sum_{q=0}^{p-2} {}_{2i+mp-\kappa}\delta_{w-2i-mp}(u^{p-1-q})u^q + \ldots +$$

$${}_{\kappa}\tilde{\delta}_{w-(p-1)i}(u) \sum_{q=0}^{p-2} {}_{w-(p-1)i-\kappa}\delta_{(p-1)i}(u^{p-1-q})u^q + \sum_{k=w-(p-1)i+1}^{w-1} {}_{\kappa}\tilde{\delta}_k(u) \sum_{q=0}^{p-2} {}_{k-\kappa}\delta_{w-k}(u^{p-1-q})u^q$$

By lemma 0.44, ${}_{k-\kappa}\delta_{w-k}|_{F(u)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2}$ for $w - k < (p-1)i$.

Let us prove that ${}_{2i+mp-\kappa}\delta_{\zeta}|_{F(u)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2}$ for $(p-1)i < \zeta < (p-1)i + i + mp$.

If $(p-1)i < \zeta < 2i + mp$, then it is clear that ${}_{2i+mp-\kappa}\delta_{\zeta}|_{F(u)} = 0$. By proposition 0.41, for $\zeta \geq 2i + mp$ and $q < p$ we have

$${}_{2i+mp-\kappa}\delta_{\zeta}(u^q) = q_{2i+mp-\kappa}\delta_{\zeta}(u)u^{q-1} +$$

$${}_{2i+mp-\kappa}\delta_i(u) \sum_{r=0}^{q-2} {}_{m}\delta_{\zeta-i}(u^{q-1-r})u^r + {}_{2i+mp-\kappa}\delta_{2i+mp}(u) \sum_{r=0}^{q-2} {}_{m_1}\delta_{\zeta-2i-mp}(u^{q-1-r})u^r + \ldots$$

Since $\zeta - 2i - mp < (p-2)i$, ${}_{m_1}\delta_s|_{F(u)} = c_1\delta + \ldots + c_{p-3}\delta^{p-3}$ for $s \leq \zeta - 2i - mp$.

To show that ${}_m\delta_{\zeta-i}|_{F(u)} = c_1\delta+\ldots+c_{p-3}\delta^{p-3}$ we use induction on r, where $i+mp \le \zeta-(p-2-r)i < 2i+mp+ri$. For arbitrary r we can use the same calculations, so we only have to prove that ${}_m\delta_{\zeta-(p-2-r)i}|_{F(u)} = 0$ for some $r \ge 0$.

There exists $r \ge 0$ such that $i+mp \le \zeta-(p-2-r)i < 2i+mp$. If $2i+mp > 2i$, then ${}_m\delta_{\zeta-(p-2-r)i} = c\delta_i^k$, $k > p$ if $i|\zeta$ and ${}_m\delta_{\zeta-(p-2-r)i} = 0$ otherwise. So, ${}_m\delta_{\zeta-(p-2-r)i}|_{F(u)} = 0$. If $2i+mp \le 2i$, then $(p-1)i < \zeta < pi$. So, $i < \zeta-(p-2)i < 2i+mp$ and ${}_m\delta_{\zeta-(p-2)i} = 0$.

Let us prove that ${}_{2i+mp-\kappa}\delta_{(p-2)i+2i+mp}|_{F(u)} = c_1\delta+\ldots+c_{p-1}\delta^{p-1}$ with $c_{p-1} \ne 0$. Note that ${}_{2i+mp-\kappa}\tilde{\delta}_{2i+mp+ri}|_{F(u)} = c_1\delta+\ldots+c_{r+1}\delta^{r+1}$ with $c_{r+1} \ne 0$ for any $0 \le r \le p-2$.

Indeed, by proposition 0.41 we have

$$ {}_{2i+mp-\kappa}\tilde{\delta}_{2i+mp+ri}(u^q) = q\,{}_{2i+mp-\kappa}\tilde{\delta}_{2i+mp+ri}(u)u^{q-1} + {}_{2i+mp-\kappa}\tilde{\delta}_i(u)\sum_{t=0}^{q-2} $$

$$ \sum_{(j_1,\ldots,j_l),l\ge p} C^l_{\kappa-i-mp}\delta_{j_1}\ldots\delta_{j_l}(u^{q-1-t})u^t + {}_{2i+mp-\kappa}\tilde{\delta}_{2i+mp}(u)\sum_{t=0}^{q-2} {}_\kappa\delta_{ri}(u^{q-1-t})u^t + \ldots $$

By lemma 0.44, ${}_\kappa\delta_{ri}|_{F(u)} = c_1\delta+\ldots+c_r\delta^r$ with $c_r \ne 0$ and ${}_m\delta_s|_{F(u)} = c_1\delta+\ldots+c_{r-1}\delta^{r-1}$ for $s < ri$.

If there exists $j_k \ge 2i+mp$, then $j_1+\ldots+\hat{j_k}+\ldots+j_l \le ri$; so there exists $j_t < i$ and $\delta_{j_1}\ldots\delta_{j_l} = 0$. If there are no $j_k \ge 2i+mp$, then $\delta_{j_k} = c\delta_i^k$, $c \in$ " and $\delta_{j_1}\ldots\delta_{j_l}|_{F(u)} = 0$, because $l \ge p$.

Hence by lemma 0.44, ${}_{2i+mp-\kappa}\tilde{\delta}_{2i+mp+ri}|_{F(u)} = c_1\delta+\ldots+c_{r+1}\delta^{r+1}$ with $c_{r+1} = \frac{1}{r+1}{}_{2i+mp-\kappa}\tilde{\delta}_{2i+mp}(u)({}_i\delta_i(u))^r = \frac{1}{r+1}{}_i\tilde{\delta}_{2i+mp}(u)({}_i\delta_i(u))^r \ne 0$.

We have

$$ {}_{2i+mp-\kappa}\delta_{(p-2)i+2i+mp} + {}_{2i+mp-\kappa}\tilde{\delta}_{(p-2)i+2i+mp} + {}_{2i+mp-\kappa}\delta_i \cdot {}_{2i+mp-\kappa}\tilde{\delta}_{(p-3)i+2i+mp} + \ldots + $$

$$ {}_{2i+mp-\kappa}\delta_{(p-2)i} \cdot {}_{2i+mp-\kappa}\tilde{\delta}_{2i+mp} + {}_{2i+mp-\kappa}\delta_{(p-3)i+2i+mp} \cdot {}_{2i+mp-\kappa}\tilde{\delta}_i = 0 $$

We have ${}_{2i+mp-\kappa}\delta_{ri} \cdot {}_{2i+mp-\kappa}\tilde{\delta}_{(p-2-r)i+2i+mp}|_{F(u)} = c_1\delta+\ldots+c_{p-1}\delta^{p-1}$ with $c_{p-1} = \frac{1}{p-1-r}{}_i\tilde{\delta}_{2i+mp}(u)({}_i\delta_i(u))^{p-2}$.

Since $({}_{2i+mp-\kappa}\tilde{\delta}_i)^p|_{F(u)} = 0$, and using induction, we get ${}_{2i+mp-\kappa}\delta_{(p-2)i+2i+mp}|_{F(u)} = c_1\delta+\ldots+c_{p-1}\delta^{p-1}$ with

$$ c_{p-1} = -(1+\ldots+\frac{1}{p-1}){}_i\tilde{\delta}_{2i+mp}(u)({}_i\delta_i(u))^{p-2}- $$

$$ (1+\ldots+\frac{1}{p-2}){}_i\tilde{\delta}_{2i+mp}(u)({}_i\delta_i(u))^{p-2}-\ldots-{}_i\tilde{\delta}_{2i+mp}(u)({}_i\delta_i(u))^{p-2} = -{}_i\tilde{\delta}_{2i+mp}(u)({}_i\delta_i(u))^{p-2} \ne 0 $$

Note that ${}_\kappa\tilde{\delta}_{w-(p-1)i}(u)\sum_{q=0}^{p-2} {}_{w-(p-1)i-\kappa}\delta_{(p-1)i}(u^{p-1-q})u^q \ne 0$ only if $w = i \mod p$.

Indeed, suppose $w-(p-1)i-\kappa \ne i \mod p$. Therefore by lemma 0.44, (ii), ${}_{w-(p-1)i-\kappa}\delta_{(p-1)i}|_{F(u)} = c_1\delta+\ldots+c_{p-2}\delta^{p-2}$.

Let us prove that $\sum_{q=0}^{p-2} {}_{i-\kappa}\delta_{w-i}(u^{p-1-q})u^q = 0$. By proposition 0.41 we have

$$_{i-\kappa}\delta_{w-i}(u^q) = q_{i-\kappa}\delta_{w-i}(u)u^{q-1} + {}_{i-\kappa}\delta_i(u)\sum_{r=0}^{q-2} {}_{2i-\kappa}\delta_{w-2i}(u^{q-1-r})u^r +$$

$$\sum_{s=2i+mp}^{w-i-1} {}_{i-\kappa}\delta_s(u)\sum_{r=0}^{q-2} {}_{s+i-\kappa}\delta_{w-i-s}(u^{q-1-r})u^r$$

Since $i-\kappa = 0 \mod p$, $_{i-\kappa}\delta_s(u) = 0$ for $s < 2i+mp+(p-1)i$. For $s \geq 2i+mp+(p-1)i$ we have $w-i-s \leq mp$ and $_{s+i-\kappa}\delta_{w-i-s} = c\delta_i^k$, $c \in$ ". But $m \leq 0$ by our assumption in the beginning of Step 2, so $_{s+i-\kappa}\delta_{w-i-s} = 0$.

So, we have $_\kappa\tilde{\delta}_w(u^p) \neq 0$ only if $w = i \mod p$ or $w = 2i+mp+(p-1)i+i+mp$. By lemma 0.11, (ii),(see the same arguments before this lemma, for example) there exists a parameter z such that the map $_\kappa\tilde{\delta}_w(u^p)$ becomes equal to zero on u^p if $w = i \mod p$. Since $2i+mp+(p-1)i+i+mp-w \leq i$ by our assumption, the change from lemma 0.11 does not change the map $_\kappa\tilde{\delta}_{2i+mp+(p-1i+i+mp}$. So, we get the proof of (ii).

Now we have $_\kappa\tilde{\delta}_{2i+mp+(p-1i+i+mp}(u^p) = -_\kappa\tilde{\delta}_{2i+mp}(u)(-_i\tilde{\delta}_{2i+mp}(u)(_i\delta_i(u))^{p-2}) \in Z(\bar{D})$, which proves (iii).
The lemma is proved.
□

Consider the following two cases.

Case 1. $\delta_i(\tilde{\delta}_{2i+mp}(u)) = 0$ or $i+mp < i$. In this case we have shown that $\delta_i(_i\tilde{\delta}_{j(1)}(u^p)) = 0$ and $\delta_i(_i\tilde{\delta}_{j(1)+i+mp}(u^p)) = 0$.

Lemma 0.47 *Let $\delta_{j(n+1)}$ be the first map such that $\delta_{j(n+1)}|_{F(u^{p^{n+1}})} \neq 0$. Suppose the following conditions hold:*

i') $_{j(n)}\tilde{\delta}_{j(n+1)+i+mp}(u^{p^{n+1}})|_{F(u^{p^{n+1}})} \neq 0$ and $_{j(n)}\tilde{\delta}_r|_{F(u^{p^{n+1}})} = 0$ for $j(n+1) < r < j(n+1)+i+mp$;

ii') $\delta_i(_{j(n)}\tilde{\delta}_{j(n+1)+i+mp}(u^{p^{n+1}})) = 0$ and $\delta_i(_{j(n)}\tilde{\delta}_{j(n+1)}(u^{p^{n+1}})) = 0$.

Then there exists a parameter z such that the following conditions hold:

i) $_{j(n+1)}\tilde{\delta}_{j(n+1)+i+mp+(p-1)j(n+1)}$ is the first map such that $_{j(n+1)}\tilde{\delta}_{j(n+1)+i+mp+(p-1)j(n+1)}|_{F(u^{p^{n+2}})} \neq 0$;

ii) $_{j(n+1)}\tilde{\delta}_{j(n+1)+i+mp+(p-1)j(n+1)+i+mp}(u^{p^{n+2}}) \neq 0$ and $_{j(n+1)}\tilde{\delta}_r|_{F(u^{p^{n+2}})} = 0$ for $j(n+1)+i+mp+(p-1)j(n+1) < r < j(n+1)+i+mp+(p-1)j(n+1)+i+mp$;

iii) $\delta_i({}_{j(n+1)}\tilde{\delta}_{j(n+2)}(u^{p^{n+2}})) = 0$ *and* $\delta_i({}_{j(n+1)}\tilde{\delta}_{j(n+2)+i+mp}(u^{p^{n+2}})) = 0$, *where* $j(n+2) = j(n+1) + i + mp + (p-1)j(n+1)$.

Proof. First we prove that there exists a parameter $\bar{z}$ such that $\bar{\delta}_q|_{F(u^{p^{n+1}})} = \delta_q|_{F(u^{p^{n+1}})}$ for $q \le j(n+1)+i+mp$ and ${}_{j(n+1)}\tilde{\bar{\delta}}_q(u^{p^{n+1}}) \neq 0$ only if $q = 2i \mod p$ for $q > j(n+1)+i+mp$; here $\bar{\delta}_q$ are the maps given by the parameter $\bar{z}$.

Suppose ${}_{j(n+1)}\tilde{\bar{\delta}}_q(u^{p^{n+1}}) \neq 0$, $q > j(n+1)+i+mp$ and $q \neq 2i \mod p$. By definition, ${}_{j(n+1)}\tilde{\delta}_q(u^{p^{n+1}}) = -j(n+1)\delta_q(u^{p^{n+1}}) + \sum \delta_{k_1}\dots\delta_{k_l}(u^{p^{n+1}})$, where $k_i < q$. By lemma 0.11, (ii), for any $a \in \bar{D}$ there exists a parameter $\bar{z}_q$ such that

$$\bar{z}_q u^{p^{n+1}} \bar{z}_q^{-1} = u^{p^{n+1}} + \delta_{j(n+1)}(u^{p^{n+1}})\bar{z}_q^{j(n+1)} + \dots + \delta_{q-1}(u^{p^{n+1}})\bar{z}_q^{q-1} + a\bar{z}_q^q + \dots$$

Therefore there exists an element $a \in \bar{D}$ such that ${}_{j(n+1)}\tilde{\delta}_q(u^{p^{n+1}}) = 0$. It is easy to see that the sequence $\{\bar{z}_q\}$ converges in D. So, $\bar{z} = lim\bar{z}_q$.

Now we prove that ${}_{j(n+1)}\tilde{\delta}_{j(n+1)+i+mp}(u^{p^{n+1}}) \neq 0$ and ${}_{j(n+1)}\tilde{\delta}_r|_{F(u^{p^{n+1}})} = 0$ for $j(n+1) < r < j(n+1)+i+mp$ and $\delta_i({}_{j(n+1)}\tilde{\delta}_{j(n+1)+i+mp}(u^{p^{n+1}})) = 0$.

We have $j(n+1) = j(n) \mod p$. Therefore,

$$z^{-j(n+1)}u^{p^{n+1}}z^{j(n+1)} = z^{-pk}(z^{-j(n)}u^{p^{n+1}}z^{j(n)})z^{pk} = z^{-pk}(u^{p^{n+1}} + {}_{j(n)}\tilde{\delta}_{j(n+1)}(u^{p^{n+1}})z^{j(n+1)} +$$

$${}_{j(n)}\tilde{\delta}_{j(n+1)+i+mp}(u^{p^{n+1}})z^{j(n+1)+i+mp} + \dots)z^{pk} =$$

$$z^{-pk}u^{p^{n+1}}z^{pk} + z^{-pk}{}_{j(n)}\tilde{\delta}_{j(n+1)}(u^{p^{n+1}})z^{pk}z^{j(n+1)} + {}_{j(n)}\tilde{\delta}_{j(n+1)+i+mp}(u^{p^{n+1}})z^{j(n+1)+i+mp} + \dots =$$

$$u^{p^{n+1}} + {}_{j(n)}\tilde{\delta}_{j(n+1)}(u^{p^{n+1}})z^{j(n+1)} + {}_{j(n)}\tilde{\delta}_{j(n+1)+i+mp}(u^{p^{n+1}})z^{j(n+1)+i+mp} + \dots,$$

because ii') provide $\delta_i(\delta_r(u^{p^{n+1}})) = 0$ for $j(n+1) < r < j(n+1)+i+mp$. So, $z^{-pk}u^{p^{n+1}}z^{pk} = u^{p^{n+1}} \mod M_D^{j(n+1)+2i+mp}$.

i) Put $w = j(n+1)+i+mp+(p-1)j(n+1)$, $t = u^{p^{n+1}}$. By proposition 0.41 we have

$${}_{j(n+1)}\tilde{\delta}_w(t^p) = {}_{j(n+1)}\tilde{\delta}_{j(n+1)+i+mp}(t)\sum_{q=0}^{p-2} {}_{i+mp}\delta_{(p-1)j(n+1)}(t^{p-1-q})t^q +$$

$$\sum_{k=j(n+1)+i+mp+1}^{w-1} {}_{j(n+1)}\tilde{\delta}_k \sum_{q=0}^{p-2} {}_{k-j(n+1)}\delta_{w-k}(t^{p-1-q})t^q$$

By lemma 0.44, ${}_{k-j(n+1)}\delta_{w-k}|_{F(t)} = c_1\delta + \dots + c_{p-2}\delta^{p-2}$ for $w-k < (p-1)j(n+1)$ and ${}_{i+mp}\delta_{(p-1)j(n+1)}|_{F(t)} = c_1\delta + \dots + c_{p-1}\delta^{p-1}$ with $c_{p-1} \neq 0$. Therefore, ${}_{j(n+1)}\tilde{\delta}_w(t^p) = -{}_{j(n+1)}\tilde{\delta}_{j(n+1)+i+mp}(t)c_{p-1} \neq 0$.

The same arguments show that ${}_{j(n+1)}\tilde{\delta}_w$ is the first map such that ${}_{j(n+1)}\tilde{\delta}_w(t^p) \neq 0$.

ii) Put $t = u^{p^{n+1}}$. Using the same arguments as above, we can find a parameter $\bar{z}$ such that $\bar{\delta}_q = \delta_q$ for $q \le j(n+1)+i+mp+(p-1)j(n+1)$ and ${}_{j(n+1)}\tilde{\bar{\delta}}_q(t^p) = 0$

for $q > j(n+1)+i+mp+(p-1)j(n+1)$, $q \neq 2i \mod p$. Since ${}_{j(n+1)}\tilde{\tilde{\delta}}_r|_{F(t^p)}$, $j(n+1)+i+mp+(p-1)j(n+1) < r < j(n+1)+i+mp+(p-1)j(n+1)+i+mp$ are derivations (see lemma 0.44), ${}_{j(n+1)}\tilde{\tilde{\delta}}_r|_{F(t^p)} = 0$ for $j(n+1)+i+mp+(p-1)j(n+1) < r < j(n+1)+i+mp+(p-1)j(n+1)+i+mp$.

For $j(n+1)+i+mp+(p-1)j(n+1) < w \leq j(n+1)+i+mp+(p-1)j(n+1)+i+mp$, $w = 2i \mod p$, by proposition 0.41 we have

$$ {}_{j(n+1)}\tilde{\delta}_w(t^p) = {}_{j(n+1)}\tilde{\delta}_{j(n+1)+i+mp}(t)\sum_{q=0}^{p-2} {}_{i+mp}\delta_{(p-1)j(n+1)+i+mp}(t^{p-1-q})t^q + \ldots + $$

$$ {}_{j(n+1)}\tilde{\delta}_{w-(p-1)j(n+1)}(t)\sum_{q=0}^{p-2} {}_{w-(p-1)j(n+1)-j(n+1)}\delta_{(p-1)j(n+1)}(t^{p-1-q})t^q + $$

$$ \sum_{k=w-(p-1)j(n+1)+1}^{w-1} {}_{j(n+1)}\tilde{\delta}_k(t)\sum_{q=0}^{p-2} {}_{k-j(n+1)}\delta_{w-k}(t^{p-1-q})t^q $$

By lemma 0.44, ${}_{k-j(n+1)}\delta_{w-k}|_{F(t)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2}$ if $w-k < (p-1)j(n+1)$. Therefore, $\sum_{q=0}^{p-2} {}_{k-j(n+1)}\delta_{w-k}(t^{p-1-q})t^q = 0$.

Note that ${}_{j(n+1)}\tilde{\delta}_{w-(p-1)j(n+1)}(t)\sum_{q=0}^{p-2} {}_{w-(p-1)j(n+1)-j(n+1)}\delta_{(p-1)j(n+1)}(t^{p-1-q})t^q = 0$.

Indeed, $w-(p-1)j(n+1)-j(n+1) = 2i \mod p$. Therefore by lemma 0.44 (ii), ${}_{w-(p-1)j(n+1)-j(n+1)}\delta_{(p-1)j(n+1)}|_{F(t)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2}$.

Let us prove that ${}_{k-j(n+1)}\delta_\zeta|_{F(t)} = c_1\delta + \ldots + c_r\delta^r$ for $(r+1)j(n+1) < \zeta < (r+1)j(n+1)+i+mp$, $r \leq p-2$.

The proof is by induction on r. By ii') and i'), $\delta_s(t) = 0$ for $j(n+1) < s < j(n+1)+i+mp$. Therefore for $r=0$, ${}_{k-j(n+1)}\delta_\zeta|_{F(t)} = 0$.

For arbitrary r we have

$$ {}_{k-j(n+1)}\delta_\zeta(t^q) = q_{k-j(n+1)}\delta_\zeta(t) + {}_{k-j(n+1)}\delta_{j(n+1)}(t)\sum_{r=0}^{q-2} {}_k\delta_{\zeta-j(n+1)}(t^{q-1-r})t^r + $$

$$ \sum_{s=j(n+1)+i+mp}^{\zeta-1} {}_{k-j(n+1)}\delta_s(t)\sum_{r=0}^{q-2} {}_{s+(k-j(n+1))}\delta_{\zeta-s}(t^{q-1-r})t^r $$

By lemma 0.44, ${}_{s+(k-j(n+1))}\delta_{\zeta-s}|_{F(t)} = c_1\delta + \ldots + c_{r-1}\delta^{r-1}$ for $\zeta - s < rj(n+1)$.

For any m ${}_m\delta_{\zeta-j(n+1)}|_{F(t)} = 0$ if $r=1$, and ${}_m\delta_{\zeta-j(n+1)}|_{F(t)} = c_1\delta + \ldots + c_{r-1}\delta^{r-1}$ by induction and lemma 0.44, because

$$ {}_m\delta_{\zeta-j(n+1)}(t^q) = q_m\delta_{\zeta-j(n+1)}(t)t^{q-1} + {}_m\delta_{j(n+1)}(t)\sum_{l=0}^{q-2} {}_{m_1}\delta_{\zeta-2j(n+1)}(t^{q-1-l})t^l + $$

$$\sum_{s=j(n+1)+i+mp}^{\zeta-j(n+1)-1} {}_{m}\delta_s \sum_{l=0}^{q-2} {}_{m_s}\delta_{\zeta-j(n+1)-s}(t^{q-1-l})t^l,$$

and by lemma 0.44, ${}_{m_s}\delta_{\zeta-j(n+1)-s}|_{F(t)} = c_1\delta + \ldots + c_{r-2}\delta^{r-2}$ for $s \geq j(n+1)+i+mp$.

The same arguments show that ${}_{k-j(n+1)}\tilde{\delta}_\zeta|_{F(t)} = c_1\delta + \ldots + c_r\delta^r$ for $(r+1)j(n+1) < \zeta < (r+1)j(n+1)+i+mp$.

Let us show that ${}_{i+mp}\delta_{(p-1)j(n+1)+i+mp}|_{F(t)} = c_1\delta + \ldots + c_{p-1}\delta^{p-1}$, $c_{p-1} \neq 0$. Put $\zeta = (p-1)j(n+1)+i+mp$. We have

$${}_{i+mp}\delta_\zeta + {}_{i+mp}\tilde{\delta}_\zeta + \sum_{s=1}^{w-k-1} {}_{i+mp}\delta_{\zeta-s} \cdot {}_{i+mp}\tilde{\delta}_s = 0$$

First we prove that ${}_{i+mp}\tilde{\delta}_{rj(n+1)+i+mp}|_{F(t)} = c_1\delta + \ldots + c_r\delta^r$ with $c_r = \frac{1}{r}{}_{i+mp}\tilde{\delta}_{j(n+1)+i+mp}(t)({}_{j(n+1)}\delta_{j(n+1)}(t))^{r-1} \neq 0$. We use the same arguments as above. The proof is by induction on r. For $r = 0$, since $i+mp < j(n+1)$, ${}_{i+mp}\tilde{\delta}_{i+mp}|_{F(t)} = 0$.

Put $w = rj(n+1)+i+mp$. For arbitrary r we have

$${}_{i+mp}\tilde{\delta}_w(t^q) = q{}_{i+mp}\tilde{\delta}_w(t)t^{q-1} + {}_{i+mp}\tilde{\delta}_{j(n+1)}(t)\sum_{r=0}^{q-2} {}_{j(n+1)-i-mp}\delta_{w-j(n+1)}(t^{q-1-r})t^r +$$

$$\sum_{s=j(n+1)+i+mp}^{w-1} {}_{i+mp}\tilde{\delta}_s(t)\sum_{r=0}^{q-2} {}_{s-i-mp}\delta_{w-s}(t^{q-1-r})t^r$$

By lemma 0.44, ${}_{s-i-mp}\delta_{w-s}|_{F(t)} = c_1\delta + \ldots + c_{r-2}\delta^{r-2}$ for $w-s < (r-1)j(n+1)$ and ${}_{j(n+1)}\delta_{(r-1)j(n+1)}|_{F(t)} = c_1\delta + \ldots + c_{r-1}\delta^{r-1}$ with $c_{r-1} = ({}_{j(n+1)}\delta_{j(n+1)}(t))^{r-1} \neq 0$.

By proposition 0.41 we have

$${}_{j(n+1)-i-mp}\delta_{w-j(n+1)}(t^q) = q{}_{j(n+1)-i-mp}\delta_{w-j(n+1)}(t)t^{q-1}+$$

$${}_{j(n+1)-i-mp}\delta_{j(n+1)}(t)\sum_{r=0}^{q-2} {}_{m_1}\delta_{w-2j(n+1)}(t^{q-1-r})t^r+$$

$$\sum_{s=j(n+1)+i+mp}^{w-j(n+1)-1} {}_{j(n+1)-i-mp}\delta_s(t)\sum_{r=0}^{q-2} {}_{s+j(n+1)-i-mp}\delta_{w-j(n+1)-s}(t^{q-1-r})t^r$$

By lemma 0.44, ${}_{s+j(n+1)-i-mp}\delta_{w-j(n+1)-s}|_{F(t)} = c_1\delta + \ldots + c_{r-3}\delta^{r-3}$ for $w-j(n+1)-s < (r-2)j(n+1)$. Since $j(n+1)-i-mp = 0 \mod p$, ${}_{j(n+1)-i-mp}\delta_{j(n+1)}(t) = 0$ and ${}_{j(n+1)-i-mp}\delta_{j(n+1)+i+mp}(t) = 0$ (here we use also i') and ii')). Therefore, ${}_{j(n+1)-i-mp}\delta_{w-j(n+1)}|_{F(t)} = c_1\delta + \ldots + c_{r-2}\delta^{r-2}$ and lemma 0.44 completes the proof.

The same arguments show that ${}_{i+mp}\tilde{\delta}_{rj(n+1)}|_{F(t)} = c_1\delta + \ldots + c_{r-1}\delta^{r-1}$. Therefore, ${}_{i+mp}\delta_{\zeta-s} \cdot {}_{i+mp}\tilde{\delta}_s|_{F(t)} = c_1\delta + \ldots + c_{p-1}\delta^{p-1}$ only if $s = j(n+1)$ or $s > j(n+1)$ and $s = rj(n+1) + i + mp$. Note that $c_{p-1} = \frac{1}{r} {}_i\tilde{\delta}_{j(n+1)+i+mp}(t)({}_i\delta_{j(n+1)}(t))^{p-2}$ if $s = rj(n+1) + i + mp$.

Using the same arguments for the maps ${}_{i+mp}\delta_{\zeta-rj(n+1)}$, we get ${}_{i+mp}\delta_\zeta|_{F(t)} = c_1\delta + \ldots + c_{p-1}\delta^{p-1}$, where

$$c_{p-1} = -(1 + \frac{1}{2} + \ldots + \frac{1}{p-1}) {}_i\tilde{\delta}_{j(n+1)+i+mp}(t)({}_i\delta_{j(n+1)}(t))^{p-2} -$$

$$(1 + \ldots + \frac{1}{p-2}) {}_i\tilde{\delta}_{j(n+1)+i+mp}(t)({}_i\delta_{j(n+1)}(t))^{p-2} - \ldots - {}_i\tilde{\delta}_{j(n+1)+i+mp}(t)({}_i\delta_{j(n+1)}(t))^{p-2} =$$

$$-{}_i\tilde{\delta}_{j(n+1)+i+mp}(t)({}_i\delta_{j(n+1)}(t))^{p-2} \neq 0.$$

This completes the proof of ii) and iii).
The lemma is proved.
□

Case 2. $\delta_i(\tilde{\delta}_{2i+mp}(u)) \neq 0$.

Lemma 0.48 *Suppose $i + mp \geq i$. Put $t = u^p$. Then there exists a parameter z such that the following properties hold:*

i) ${}_{j(1)}\tilde{\delta}_{j(1)+i+mp+(p-1)j(1)}$ is the first map such that ${}_{j(1)}\tilde{\delta}_{j(1)+i+mp+(p-1)j(1)}|_{F(t^p)} \neq 0$.

ii) ${}_{j(1)}\tilde{\delta}_r|_{F(t^p)} = 0$ for $j(2) < r < j(2) + i$ and ${}_{j(1)}\tilde{\delta}_{j(2)+i}(t^p) \neq 0$, where $j(2) = j(1) + i + mp + (p-1)j(1)$.

iii) ${}_{j(1)}\tilde{\delta}_{j(2)+i}(t^p) \in Z(\bar{D})$, ${}_{j(1)}\tilde{\delta}_{j(2)}(t^p) \in Z(\bar{D})$.

Proof. To prove i) one can repeat the proof of i) in lemma 0.47. Note that ${}_{j(1)}\tilde{\delta}_{j(2)}(t^p) = -{}_{j(1)}\tilde{\delta}_{j(1)+i}({}_i\delta_{j(1)}(t))^{p-2} \in Z(\bar{D})$.

ii) As in the case 1 we can find a parameter $\bar{z}$ such that $\bar{\delta}_q|_{F(t)} = \delta_q|_{F(t)}$ for $q \leq j(1) + i + mp$, $\tilde{\bar{\delta}}_q(t^p) = 0$ for $q \neq 2i \mod p$, $q > j(2)$.

For $r = 2i \mod p$, by proposition 0.41 we have

$${}_{j(1)}\tilde{\delta}_r(t^p) = {}_{j(1)}\tilde{\delta}_{j(1)+i+mp}(t) \sum_{q=0}^{p-2} {}_{i+mp}\delta_{r-j(1)-i-mp}(t^{p-1-q})t^q +$$

$$\sum_{k=j(1)+i+mp+1}^{r-1} {}_{j(1)}\tilde{\delta}_k(t) \sum_{q=0}^{p-2} {}_{k-j(1)}\delta_{r-k}(t^{p-1-q})t^q$$

By lemma 0.44, ${}_{k-j(1)}\delta_{r-k}|_{F(t)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2}$ if $r-k < (p-1)j(1)$.

Note that ${}_{j(1)}\tilde{\delta}_{r-(p-1)j(1)}(t)\sum_{q=0}^{p-2} {}_{r-pj(1)}\delta_{(p-1)j(1)}(t^{p-1-q})t^q = 0$.

Indeed, $r - pj(1) = 2i \mod p$. Therefore by lemma 0.44 (ii), ${}_{r-pj(1)}\delta_{(p-1)j(1)}|_{F(t)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2}$.

The same arguments as in lemma 0.47 (ii) show that ${}_m\delta_s|_{F(t)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2}$ for $(p-1)j(1) < s < (p-1)j(1) + i$.

Let us prove that ${}_{i+mp}\delta_{(p-1)j(1)+i}|_{F(t)} = c_1\delta + \ldots + c_{p-1}\delta^{p-1}$ with $c_{p-1} \neq 0$.

First let us show that ${}_{i+mp}\tilde{\delta}_\zeta|_{F(t)} = c_1\delta + \ldots + c_r\delta^r$ for $\zeta = (r+1)j(1)+i$, $r \le p-2$. For $r = 0$, ${}_{i+mp}\tilde{\delta}_\zeta|_{F(t)}$ is a derivation. Since ${}_{j(1)}\tilde{\delta}_\zeta(t) = 0$ for $\zeta = j(1)+i$ and $j(1) = i + mp \mod p$ and $p > 2$, we have ${}_{i+mp}\tilde{\delta}_\zeta(t) = 0$ and ${}_{i+mp}\tilde{\delta}_\zeta|_{F(t)} = 0$.

For arbitrary r we have

$$ {}_{i+mp}\tilde{\delta}_\zeta(t^q) = q\,{}_{i+mp}\tilde{\delta}_\zeta(t)t^{q-1} + {}_{i+mp}\tilde{\delta}_{j(1)}(t)\sum_{r=0}^{q-2} {}_{j(1)-i-mp}\delta_{\zeta-j(1)}(t^{q-1-r})t^r + $$

$$ \sum_{k=j(1)+i+1}^{\zeta-1} {}_{i+mp}\tilde{\delta}_k(t)\sum_{r=0}^{q-2} {}_{k-i-mp}\delta_{\zeta-k}(t^{q-1-r})t^r $$

By lemma 0.44, ${}_{k-i-mp}\delta_{\zeta-k}|_{F(t)} = c_1\delta + \ldots + c_{r-1}\delta^{r-1}$ for $\zeta - k < rj(1)$, i.e. $k \ge j(1)+i+1$.

By definition,

$$ {}_{j(1)-i-mp}\delta_{\zeta-j(1)} = \sum_{(j_1,\ldots,j_l),l\ge p} C^l_{j(1)-i-mp}\delta_{j_1}\ldots\delta_{j_l}, $$

because $j(1) - i - mp = 0 \mod p$. Since $l \ge p$, there exist j_k, j_{k_1} such that $j_k < j(1)$ and $j_{k_1} < j(1)$. Thus, $j_1 + \ldots + \hat{j_k} + \ldots + \hat{j_{k_1}} + \ldots + j_l < rj(1)$ and $\delta_{j_1}\ldots\delta_{j_l}|_{F(t)} = c_1\delta + \ldots + c_{r-1}\delta^{r-1}$. Hence by lemma 0.44, ${}_{i+mp}\tilde{\delta}_\zeta|_{F(t)} = c_1\delta + \ldots + c_r\delta^r$.

Now we have

$$ {}_{i+mp}\delta_{(p-1)j(1)+i} + {}_{i+mp}\tilde{\delta}_{(p-1)j(1)+i} + \sum_{k=i}^{(p-1)j(1)} {}_{i+mp}\delta_k \cdot {}_{i+mp}\tilde{\delta}_{(p-1)j(1)+i-k} = 0 $$

with ${}_{i+mp}\delta_k \cdot {}_{i+mp}\tilde{\delta}_{(p-1)j(1)+i-k}|_{F(t)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2}$ for $k \neq (p-2)j(1)+i$.

Therefore,

$$ {}_{i+mp}\delta_{(p-1)j(1)+i}|_{F(t)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2} - {}_{i+mp}\delta_{(p-2)j(1)+i} \cdot {}_{i+mp}\tilde{\delta}_{j(1)}|_{F(t)} $$

So by induction,

$$ {}_{i+mp}\delta_{(p-1)j(1)+i}|_{F(t)} = \tilde{c}_1\delta + \ldots + \tilde{c}_{p-2}\delta^{p-2} + {}_{i+mp}\delta_i({}_{i+mp}\delta_{j(1)})^{p-1}|_{F(t)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2} + $$

$${}_{i+mp}\delta_i(({}_i\delta_{j(1)}(t))^{p-1})\delta^{p-1}$$

Since ${}_i\delta_{j(1)}(t) = -{}_{j(1)}\tilde{\delta}_{j(1)}(t) = {}_i\tilde{\delta}_{2i+mp}(u)({}_i\delta_i(u))^{p-1}$, we have ${}_{i+mp}\delta_i({}_i\delta_{j(1)}(t)) \neq 0$, which completes the proof of ii).

Finally, ${}_{j(1)}\tilde{\delta}_{j(2)+i}(t^p) = -{}_{j(1)}\tilde{\delta}_{j(1)+i+mp}(t)_{i+mp}\delta_i({}_i\delta_{j(1)}(t))({}_i\delta_{j(1)}(t))^{p-2} \in Z(\bar{D})$, because ${}_{i+mp}\delta_i$ is a derivation and ${}_i\delta_{j(1)}(t) \in Z(\bar{D})$. This proves iii).
□

The following lemma completes the proof of Case 2 and of Theorem.

Lemma 0.49 *Suppose the following conditions hold:*

i') ${}_{j(1)}\tilde{\delta}_{j(n)+i}|_{F(u^{p^n})} \neq 0$ and ${}_{j(1)}\tilde{\delta}_r|_{F(u^{p^n})} = 0$ for $j(n) < r < j(n)+i$, $n \geq 1$;

ii') ${}_{j(1)}\tilde{\delta}_{j(n)}(u^{p^n}), {}_{j(1)}\tilde{\delta}_{j(n)+i}(u^{p^n}) \in Z(\bar{D})$.

Then there exists a parameter z such that

i) ${}_{j(1)}\tilde{\delta}_{j(n)+i+(p-1)j(n)}$ is the first map such that ${}_{j(1)}\tilde{\delta}_{j(n)+i+(p-1)j(n)}|_{F(u^{p^{n+1}})} \neq 0$;

ii) ${}_{j(1)}\tilde{\delta}_{j(n+1)+i}(u^{p^{n+1}}) \neq 0$ and ${}_{j(1)}\tilde{\delta}_r|_{F(u^{p^{n+1}})} = 0$ for $j(n+1) < r < j(n+1)+i$, where $j(n+1) = j(n)+i+(p-1)j(n)$;

iii) ${}_{j(1)}\tilde{\delta}_{j(n+1)}(u^{p^{n+1}}), {}_{j(1)}\tilde{\delta}_{j(n+1)+i}(u^{p^{n+1}}) \in Z(\bar{D})$.

Proof. By induction, $j(n) = i \mod p$. Put $a = {}_{j(1)}\tilde{\delta}_{j(n)}(u^{p^n})$. We have $a^{kj(n)+1} = a^{p^l} \in Z(D)$ for some $k \in \mathbb{Z}$. Put $z' = (a^{-kj(1)}z^{j(1)})^{1/j(1)}$. We claim that i'), ii') hold for ${}_{j(1)}\tilde{\delta'}_{j(n)+i}$, i.e. ${}_{j(1)}\tilde{\delta'}_{j(n)+i}|_{F(u^{p^n})} \neq 0$, and ${}_{j(1)}\tilde{\delta'}_r|_{F(u^{p^n})} = 0$ for $j(n) < r < j(n)+i$, and ${}_{j(1)}\tilde{\delta'}_{j(n)+i}(u^{p^n}), {}_{j(1)}\tilde{\delta'}_{j(n)}(u^{p^n}) \in Z(\bar{D})$. Moreover, ${}_{j(1)}\tilde{\delta'}_{j(n)}(u^{p^n}) \in Z(D)$.

Note that $\alpha' = id$, because $a \in Z(\bar{D})$. We have

$$z'^{-j(1)}u^{p^n}z'^{j(1)} = z^{-j(1)}u^{p^n}z^{j(1)} = u^{p^n} + az^{j(n)} + {}_{j(1)}\tilde{\delta}_{j(n)+i}(u^{p^n})z^{j(n)+i} + \ldots$$

Let us show that

$$z'^{j(n)} = a^{-kj(n)}z^{j(n)} \mod \wp^{j(n)+i+1}$$

We have

$$z'^{j(n)} = z'^{j(n)-j(1)}a^{-kj(1)}z^{j(1)}.$$

It is easy to see that $z' = a^{-k}z + xz^{i+1} + \ldots$, $x \in \bar{D}$. Since $j(n) - j(1) = 0 \mod p$, $z'^{j(n)-j(1)} = (a^{-k}z)^{j(n)-j(1)} \mod \wp^{j(n)-j(1)+i+1}$. Now we have

$$(a^{-k}z)^{j(n)-j(1)}a^{-kj(1)} = a^{-kj(n)}z^{j(n)-j(1)} + xz^{j(n)-j(1)+i} + \ldots,$$

where $x = [-kj(1) - k(j(1)+1) - \ldots - k(j(n)-1)]a^{-kj(n)-1}\delta_i(a) = 0$. Therefore,

$$z'^{-j(1)}u^{p^n}z'^{j(1)} = u^{p^n} + a^{p^l}z'^{j(n)} + {}_{j(1)}\tilde{\delta}_{j(n)+i}(u^{p^n})a^{k(j(n)+i)}z'^{j(n)+i} + \ldots$$

and ${}_{j(1)}\tilde{\delta}'_{j(n)+i}(u^{p^n}) = {}_{j(1)}\tilde{\delta}_{j(n)+i}(u^{p^n})a^{k(j(n)+i)} \in Z(\bar{D})$, ${}_{j(1)}\tilde{\delta}'_{j(n)}(u^{p^n}) = a^{p^l} \in Z(D)$. So i'), ii') hold.

Now to prove i) one can repeat the proof of i) in lemma 0.47. Note that ${}_{j(1)}\tilde{\delta}_{j(n+1)}(t^p) = -{}_{j(1)}\tilde{\delta}_{j(n)+i}({}_i\delta_{j(n)}(t))^{p-2} \in Z(\bar{D})$.

ii) We use the same arguments as in ii) of lemma 0.48. Put $t = u^{p^n}$.

For $r = 2i \mod p$, by proposition 0.41 we have

$${}_{j(1)}\tilde{\delta}_r(t^p) = {}_{j(1)}\tilde{\delta}_{j(n)}(t)\sum_{q=0}^{p-2} {}_{j(n)-j(1)}\delta_{r-j(n)}(t^{p-1-q})t^q + {}_{j(1)}\tilde{\delta}_{j(n)+i}(t)\sum_{q=0}^{p-2} {}_{j(n)-j(1)+i}\delta_{r-j(n)-i}(t^{p-1-q})t^q +$$

$$\sum_{k=j(n)+i+1}^{r-1} {}_{j(1)}\tilde{\delta}_k(t)\sum_{q=0}^{p-2} {}_{k-j(1)}\delta_{r-k}(t^{p-1-q})t^q$$

By lemma 0.44, ${}_{k-j(1)}\delta_{r-k}|_{F(t)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2}$ if $r - k < (p-1)j(n)$.

Note that ${}_{j(n)-j(1)}\delta_{r-j(n)}|_{F(t)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2}$. Indeed, by proposition 0.41 we have

$${}_{j(n)-j(1)}\delta_{r-j(n)}(t^q) = q\,{}_{j(n)-j(1)}\delta_{r-j(n)}(t)t^{q-1} + {}_{j(n)-j(1)}\delta_{j(n)}(t)\sum_{s=0}^{q-2} {}_{2j(n)-j(1)}\delta_{r-2j(n)}(t^{q-1-s})t^s +$$

$${}_{j(n)-j(1)}\delta_{j(n)+i}(t)\sum_{s=0}^{q-2} {}_{2j(n)-j(1)+i}\delta_{r-2j(n)-i}(t^{q-1-s})t^s + \ldots +$$

$${}_{j(n)-j(1)}\delta_{j(n)+2i}(t)\sum_{s=0}^{q-2} {}_{2j(n)-j(1)+2i}\delta_{r-2j(n)-2i}(t^{q-1-s})t^s + \ldots$$

Recall that $r \le pj(n) + 2i$. By lemma 0.44, ${}_m\delta_s|_{F(t)} = c_1\delta + \ldots + c_{p-3}\delta^{p-3}$ if $s < (p-2)j(n)$. Since ${}_{j(1)}\tilde{\delta}_{j(n)}(t) \in Z(D)$, we have $\delta_{j(n)}(t) \in Z(D)$. Since $j(n) - j(1) = 0 \mod p$ and $\delta_{j(n)}(t) \in Z(D)$ and $char F > 2$, we have ${}_{j(n)-j(1)}\delta_{j(n)+ei}(t) = 0$ for $e \le p-1$, which completes the proof.

Note that ${}_{j(1)}\tilde{\delta}_{r-(p-1)j(n)}(t)\sum_{q=0}^{p-2} {}_{r-(p-1)j(n)-j(1)}\delta_{(p-1)j(n)}(t^{p-1-q})t^q = 0$.

Indeed, $r - (p-1)j(n) - j(1) = 2i \mod p$. Therefore by lemma 0.44 (ii), ${}_{r-(p-1)j(n)-j(1)}\delta_{(p-1)j(n)}|_{F(t)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2}$.

The same arguments as in lemma 0.47 (ii) show that ${}_m\delta_s|_{F(t)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2}$ for $(p-1)j(n) < s < (p-1)j(n) + i$.

Let us prove that ${}_{j(n)-j(1)+i}\delta_{(p-1)j(n)+i}|_{F(t)} = c_1\delta + \ldots + c_{p-1}\delta^{p-1}$ with $c_{p-1} \neq 0$.

Note that ${}_{j(n)-j(1)+i}\tilde{\delta}_{rj(n)+i}|_{F(t)} = c_1\delta + \ldots + c_r\delta^r$ with $c_r \neq 0$ for any $1 \le r \le p-1$.

Indeed, by proposition 0.41 we have

$$_{j(n)-j(1)+i}\tilde{\delta}_{rj(n)+i}(t^q) = q_{j(n)-j(1)+i}\tilde{\delta}_{rj(n)+i}(t)t^{q-1} + {}_{j(n)-j(1)+i}\tilde{\delta}_{j(n)}(t)\sum_{s=0}^{q-2}$$

$$_{j(1)-i}\delta_{(r-1)j(n)+i}(t^{q-1-s})t^s + {}_{j(n)-j(1)+i}\tilde{\delta}_{j(n)+i}(t)\sum_{s=0}^{q-2}{}_{j(1)}\delta_{(r-1)j(n)}(t^{q-1-s})t^s + \ldots$$

By lemma 0.44, $_{j(1)}\delta_{(r-1)j(n)}|_{F(t)} = c_1\delta + \ldots + c_{r-1}\delta^{r-1}$ with $c_{r-1} \neq 0$ and $_m\delta_s|_{F(t)} = c_1\delta + \ldots + c_{r-2}\delta^{r-2}$ for $s < (r-1)j(n)$.

Let us prove that $_{j(1)-i}\delta_{(r-1)j(n)+i}|_{F(t)} = c_1\delta + \ldots + c_{r-2}\delta^{r-2}$. By proposition 0.41 we have

$$_{j(1)-i}\delta_{(r-1)j(n)+i}(t^q) = q_{j(1)-i}\delta_{(r-1)j(n)+i}(t)t^{q-1} + {}_{j(1)-i}\delta_{j(n)}(t)\sum_{s=0}^{q-2}{}_{j(n)+j(1)-i}\delta_{(r-2)j(n)+i}(t^{q-1-s})t^s +$$

$$_{j(1)-i}\delta_{j(n)+i}(t)\sum_{s=0}^{q-2}{}_{j(n)+j(1)}\delta_{(r-2)j(n)}(t^{q-1-s})t^s + \ldots$$

Since $_{j(1)}\tilde{\delta}_{j(n)}(t) \in Z(D)$, we have $\delta_{j(n)}(t) \in Z(D)$. Since $j(1) - i = 0 \mod p$, $_{j(1)-i}\delta_{j(n)}(t) = 0$ and $_{j(1)-i}\delta_{j(n)+i}(t) = 0$. By lemma 0.44, $_m\delta_s|_{F(t)} = c_1\delta + \ldots + c_{r-3}\delta^{r-3}$ for $s < (r-2)j(n)$. So, $_{j(1)-i}\delta_{(r-1)j(n)+i}|_{F(t)} = c_1\delta + \ldots + c_{r-2}\delta^{r-2}$.

Hence by lemma 0.44, $_{j(n)-j(1)+i}\tilde{\delta}_{rj(n)+i}|_{F(t)} = c_1\delta + \ldots + c_r\delta^r$ with $c_r = \frac{1}{r}{}_{j(n)-j(1)+i}\tilde{\delta}_{j(n)+i}(t)({}_i\delta_{j(n)}(t))^{r-1} = \frac{1}{r}{}_i\tilde{\delta}_{j(n)+i}(t)({}_i\delta_{j(n)}(t))^{r-1} \neq 0$.

The same arguments show that $_{j(n)-j(1)+i}\tilde{\delta}_\zeta|_{F(t)} = c_1\delta + \ldots + c_{r-1}\delta^{r-1}$ for $\zeta < rj(n) + i$.

Therefore we have

$$_{j(n)-j(1)+i}\delta_{(p-1)j(n)+i} + {}_{j(n)-j(1)+i}\tilde{\delta}_{(p-1)j(n)+i} +$$

$$\sum_{w=1}^{(p-1)j(n)+i-1}{}_{j(n)-j(1)+i}\delta_w \cdot {}_{j(n)-j(1)+i}\tilde{\delta}_{(p-1)j(n)+i-w} = 0,$$

where $_{j(n)-j(1)+i}\delta_w \cdot {}_{j(n)-j(1)+i}\tilde{\delta}_{(p-1)j(n)+i-w}|_{F(t)} = c_1\delta + \ldots + c_{p-1}\delta^{p-1}$ only if $w = rj(n)$. In this case $c_{p-1} = \frac{1}{p-1-r}{}_i\tilde{\delta}_{j(n)+i}(t)({}_i\delta_{j(n)}(t))^{p-2}$.

Since $_{j(n)-j(1)+i}\tilde{\delta}_{j(n)}(t) \in Z(D)$, we have $_{j(n)-j(1)+i}\delta_i({}_{j(n)-j(1)+i}\tilde{\delta}_{j(n)})^{p-1} = 0$. So using induction, we get $_{j(n)-j(1)+i}\delta_{(p-1)j(n)+i}|_{F(t)} = c_1\delta + \ldots + c_{p-1}\delta^{p-1}$ with

$$c_{p-1} = -(1 + \ldots + \frac{1}{p-1})_i\tilde{\delta}_{j(n)+i}(t)({}_i\delta_{j(n)}(t))^{p-2} -$$

$$(1+\ldots+\frac{1}{p-2})_i\tilde{\delta}_{j(n)+i}(t)({}_i\delta_{j(n)}(t))^{p-2}-\ldots-{}_i\tilde{\delta}_{j(n)+i}(t)({}_i\delta_{j(n)}(t))^{p-2}=$$

$$-{}_i\tilde{\delta}_{j(n)+i}(t)({}_i\delta_{j(n)}(t))^{p-2}\neq 0.$$

Therefore, ${}_{j(1)}\tilde{\delta}_{j(n+1)+i}(t^p)=-{}_{j(1)}\tilde{\delta}_{j(n)+i}(t)c_{p-1}\in Z(\bar{D})$. This proves ii) and iii). The lemma is proved.
□

The theorem is proved.
□

Lemma 0.50 *Let D be a splittable division algebra. Let $n=|Gal(Z(\bar{D})/\bar{F})|$.*

There exists a parameter z such that

$$zaz^{-1}=\alpha(a)+\delta_n(a)z^n+\delta_{2n}(a)z^{2n}+\ldots,\quad a\in\bar{D}$$

So, $\delta_j=0$ if $n\not| j$.

One can repeat the proof of lemma 0.29 to prove the lemma.

Proposition 0.51 *Let D be a good splittable division algebra. Suppose $Z(\bar{D})/\bar{F}$ is not a separable extension.*

Then p does not divide $|Gal(Z(\bar{D})/\bar{F})|$.

Proof. Suppose p divides $|Gal(Z(\bar{D})/\bar{F})|$. By lemma 0.50 there exists a parameter z such that

$$zaz^{-1}=\alpha(a)+\delta_n(a)z^n+\delta_{2n}(a)z^{2n}+\ldots,\quad a\in\bar{D},$$

where $n=|Gal(Z(\bar{D})/\bar{F})|$.

Since $Z(\bar{D})/\bar{F}$ is a compositum of a purely inseparable extension and Abelian Galois extension, there exists an element $u\in Z(\bar{D})$ such that $\alpha(u)=u$, i.e. u is a purely inseparable element; so by theorem 0.43 $u^p\in Z(D)$.

In this case lemma 0.44 holds for $l=0$ and we can repeat the arguments in the proof of lemma 0.45 to show that $\delta_{pi}(u^p)\neq 0$, which is a contradiction.
□

Proposition 0.52 *Let D be a good splittable division algebra. Then we have $D\cong D_1\otimes_F D_2$, where D_1, D_2 are division algebras such that D_1 is an inertially split algebra, $Z(\bar{D_2})/\bar{F}$ is a purely inseparable extension and D_2 is a good splittable algebra (D_1 or D_2 may be trivial).*

So, $D\sim A\otimes_F B\otimes_F D_2$, where A is a cyclic division algebra and B is an unramified division algebra.

Proof. By [25], p.261, $D \cong D_1 \otimes_F \ldots \otimes_F D_k$, where $[D : F] = p_1^{r_1} \ldots p_k^{r_k}$ and $[D_i : F] = p_i^{r_i}$. Let $p_2 = p$. By proposition 0.51, $Z(\bar{D_2})/\bar{F}$ is a purely inseparable extension. Since D_i are defectless over F, $D_1, D_3, \ldots D_k$ are inertially split. Therefore, by theorem 0.37 the algebra $D_1 \otimes D_3 \otimes \ldots \otimes D_k$ is good splittable.

Let L be an inertial lift of a Galois part of the extension $Z(\bar{D})/\overline{Z(D)}$. Consider the centraliser $D' = C_D(L)$. It's clear that $D' \cong D_2 \otimes_L B$, where B is a division algebra similar to the algebra $D_1 \otimes D_3 \otimes \ldots \otimes D_k \otimes L$. The algebra B is inertial, because $Z(\bar{B})/\overline{Z(B)}$ is trivial and B is inertially split. Since $\bar{D'} \cong \bar{D_2} \otimes \bar{B}$ and $\bar{D'} \hookrightarrow D'$ is a good embedding, $\bar{D'}$ contain a subalgebra $\bar{B} \subset \bar{B} \otimes_L L \cong B \subset D'$. Now the centraliser $C_{D'}(B) \cong D_2 \otimes_F L$ and it is good splittable, so D_2 is good splittable.

Decomposition theorems [9], Thm. 5.6-5.15 complete the proof.
□

Proposition 0.53 *Let D_2 be a good splittable division algebra such that $Z(\bar{D_2})/\overline{Z(D_2)}$ is a purely inseparable extension. Then $D_2 \cong D_3 \otimes_{Z(D_2)} D_4$, where D_3 is an unramified division algebra and D_4 is a good splittable division algebra such that $\bar{D_4}$ is a field, $\bar{D_4}/\overline{Z(D_2)}$ is a purely inseparable extension, $[\bar{D_4} : \overline{Z(D_2)}] = [\Gamma_{D_4} : \Gamma_{Z(D_2)}]$.*

Proof. For a good embedding there exists a subfield $\overline{Z(D_2)} \subset K \subset Z(\bar{D_2})$ such that the extension $Z(\bar{D_2})/K$ has degree p. Then by theorem 0.37 and 0.43 there exists a lift $\tilde{K}$ of K in D_2, i.e. $\bar{\tilde{K}} = K$, $\Gamma_{\tilde{K}} = \Gamma_{Z(D_2)}$, $K \subset \tilde{K}$.

Consider the centraliser $C_1 = C_{D_2}(\tilde{K})$. We have $\bar{C_1} = \bar{D_2}$, $Z(\bar{C_1})/\overline{Z(C_1)}$ is a purely inseparable extension of degree p, say $Z(\bar{C_1}) = \overline{Z(C_1)}(u)$. Using similar arguments as in the proof of theorem 0.43 one can show that there exists a parameter z such that $C_1 \cong \bar{C_1}((z))$ as a vector space with the relation

$$zaz^{-1} = a + \delta_i(a)z^i + c_{2i}\delta_i^2(a)z^{2i} + \ldots, \quad c_{ki} \in \mathbb{F}_p$$

and $zuz^{-1} = u + xz^i$, where $x \in Z(C_1)$. Therefore, δ_i^p is a derivation trivial on the centre $Z(\bar{C_1})$, hence by Scolem-Noether theorem it is an inner derivation.

We claim that $z^p \in Z(C_1)$. To prove it, consider a subalgebra $W = \bar{C_1}((z^i)) \subset C_1$ (note that $Z(W) \neq Z(C_1)$). It exists because of the type of the relation in C_1.

We have

$$z^{-i}az^i = a - i\delta_i(a)z^i, \quad a \in \bar{C_1}$$

in W. Therefore,

$$z^{-pi}az^{pi} = a - i^p\delta_i^p(a)z^{pi}, \quad a \in \bar{C_1}$$

and

$$z^{pi}az^{-pi} = a + \delta_1'(a)z^{pi} + {\delta_1'}^2(a)z^{2pi} + \ldots,$$

where $\delta_1' = i^p\delta_i^p$. So,

$$z^p az^{-p} = a + \frac{1}{i}\delta_1'(a)z^{pi} + c_2\frac{1}{i^2}{\delta_1'}^2(a)z^{2pi} + \ldots,$$

where c_k are given by (1) in theorem 0.43. So, $z^p \in Z(C_1)$ iff $\delta_i^p = 0$. Suppose $\delta_i^p \neq 0$. Consider an element $Y \in Z(C_1)$, $w(Y) > 0$. Let

$$Y = a_1 z^p + \dots$$

First note that

$$Y = a_1 z^p + a_2 z^{2p} + a_3 z^{3p} + \dots, \quad a_i \in \bar{C}_1$$

Indeed, Y must satisfy $[Y, u] = 0$. Since $u \in Z(\bar{C}_1)$, we then have $[z^{i_k}, u] = 0$ for every k, where

$$Y = \sum_{k=1}^{\infty} a_k z^{i_k}$$

Therefore, $p | i_k$.

Then, Y must satisfy $Ya = aY$ for any $a \in \bar{C}_1$. Therefore, $a_1, \dots a_i \in Z(\bar{C}_1)$ and we must have

$$a a_{i+1} - a_{i+1} a = a_1 \delta_1'(a)/i$$

and

$$a a_{2i+1} - a_{2i+1} a = a_i \delta_1'(a) + a_1 c_2 {\delta_1'}^2(a).$$

Since $\Delta(a) = a a_{2i+1} - a_{2i+1} a$ is an inner derivation, we get ${\delta_1'}^2 = \delta$, where δ is a derivation, which is a contradiction. Therefore, ${\delta_1'}^2 = \delta = 0$ and $\delta_1' = 0$, and $z^p \in Z(C_1)$.

Consider the centraliser $C_2 = C_{C_1}(\tilde{K}(z))$. It's clear that $[\bar{C}_2 : Z(\bar{C}_2)] = [\bar{C}_2 : \overline{Z(C_2)}] = ind\bar{C}_1$ and there exists a subalgebra $\bar{C}_2 \subset C_2$, $\bar{C}_2 \subset \bar{D}_2$. Consider now the centraliser $C_3 = C_{D_2}(Z(D_2)(z))$. We have $\bar{C}_2 \subset C_3$, $\bar{C}_3 \cong \bar{C}_2$, because $[Z(\bar{C}_3) : \overline{Z(D_2)}] = [\Gamma_{C_3} : \Gamma_{Z(D_2)}] = [Z(\bar{C}_2) : \overline{Z(D_2)}] = [K : \overline{Z(D_2)}]$. By induction on dimension of $\bar{D}_2$ we get the existence of a subalgebra $\bar{C}_4 \subset \bar{D}_2$ such that $[\bar{C}_4 : Z(\bar{C}_4)] = [\bar{C}_2 : Z(\bar{C}_2)]$, $Z(\bar{C}_4) = \overline{Z(D_2)}$. Therefore there exists an unramified subalgebra $C_4 \subset D_2$ such that $[C_4 : Z(D_2)] = [C_4 : Z(D_2)] = [C_4 : Z(C_4)] = [\bar{D}_2 : Z(\bar{D}_2)]$. By Double Centraliser Theorem, $D_2 \cong C_4 \otimes_{Z(D_2)} D_4$, where D_4 is a division algebra with $\bar{D}_4 = Z(\bar{D}_2)$. Since $\bar{D}_2 \hookrightarrow D_2$ is a good embedding, $[\bar{D}_4 : \overline{Z(D_2)}]$ must be equal to $[\Gamma_{D_4} : \Gamma_{Z(D_2)}]$. It is easy to see that D_4 is also a good splittable division algebra.

The proposition is proved.

□

Proposition 0.54 *Let D_2 be a good splittable division algebra such that $\bar{D}_2$ is a field, $\bar{D}_2/\overline{Z(D_2)}$ is a purely inseparable extension and $d_{D_2}(u_k) \leq 2i(u_k)$ or $d_{D_2}(u_k) = \infty$ for all generators u_k of the extension $\bar{D}_2/\overline{Z(D_2)}$.*

Then $D_2 \cong A_1 \otimes_{Z(D_2)} \dots \otimes_{Z(D_2)} A_m$, where A_i are cyclic division algebras of degree p, $[\bar{A}_i : \overline{Z(D_2)}] = [\Gamma_{A_i} : \Gamma_{Z(D_2)}]$.

Proof. The proof immediately follows from theorem 0.43 and [26], Thm.3, §2.8.(see [1] for the proof of this theorem).
□

So, we get the following decomposition theorem.

Theorem 0.55 *Let D be a finite dimensional good splittable central division algebra over a field F with a discrete complete rank 1 valuation, $char(F) = p > 2$, such that $d_{D_2}(u_k) \leq 2i(u_k)$ or $d_{D_2}(u_k) = \infty$ for all generators u_k of the extension $Z(\bar{D})/\overline{Z(D)}$.*

Then $D \cong D_1 \otimes_F D_2 \otimes_F A_1 \otimes_F \ldots \otimes_F A_m$, where A_i are cyclic division algebras of degree p, $[\bar{A}_i : \overline{Z(D)}] = [\Gamma_{A_i} : \Gamma_{Z(D)}]$, D_1 is an inertially split division algebra, $(ind(D_2), p) = 1$, D_2 is an unramified division algebra (D_1, D_2, A_i may be trivial).

Recall that a field F is called a *C_i-field* if any homogeneous form $f(x_1, \ldots, x_n)$ of degree d in $n > d^i$ variables with coefficients in F has a non-trivial zero.

Corollary 8 *The following conjecture: the exponent of A is equal to its index for any division algebra A (here we don't demand that A is splittable) over a C_2-field F (see for example [26], 3.4.5.) has the positive answer for $F = F_1((t))$, where F_1 is a C_1-field.*

Proof. 1) Let's prove that A is splittable. Since $\bar{F}$ is a C_1-field, $\bar{A}$ is a field. We can assume $\bar{A}/\bar{F}$ is a purely inseparable extension. We claim that $\bar{A} = \bar{F}(u)$ for some $u \in \bar{A}$, so by classical Cohen's theorem, A is splittable.

Indeed, suppose $\bar{A} = \bar{F}(u_1, \ldots, u_r)$. Consider the field $K = \bar{F}(u_1^p, \ldots, u_r^p)$. By Tsen's theorem, K and $\bar{A}$ are C_1-fields. So, the form $x_1^p + x_2^p u_1 + \ldots + x_p^p u_1^{p-1} + x_{p+1}^p u_2$ has a non-trivial zero in $\bar{A}$. But $x_i^p \in K$ and elements $1, u_1, \ldots, u_1^{p-1}, u_2$ are linearly independent over K, a contradiction.

2) Assume the corollary is known in the prime exponent case. We deduce the corollary by ascending induction on $e = expA$. If e is not a prime number, then write $e = lm$. By assumption $A^{\otimes m}$ can be split by a field extension $F \subset F'$ of degree l. This implies that $A_{F'}$ has exponent dividing m. Note that F' is also a Laurent series field. By the induction hypothesis applied to the pair $(F', A_{F'})$, there exists a field extension $F' \subset L$ of degree dividing m splitting $A_{F'}$. Therefore A is split by the extension $F \subset L$ of degree dividing lm and we conclude the corollary.

3) So, let $expA = l$ be a prime number. By the basic properties of the exponent and the index (see, e.g. [26]) we have then $indA = l^k$ for some natural k.

Suppose $(l, p = charF) = 1$.

It is known that the conjecture is true for all division algebras of index $indA = 2^a 3^b$ (see, e.g. [26]), so we can assume $charF \neq 2, 3$. Then we can assume F contains the group μ_l of l-roots of unity, because $[F(\mu_l) : F] < l$ and we can reduce the problem to the algebra $A \otimes_F F(\mu_l)$. Then by the Merkuriev-Suslin theorem A is similar to the tensor product of symbol-algebras of index l.

Every symbol-algebra of index l over F is good splittable and cyclic and its residue field is a cyclic Kummer extension of $\bar{F}$. To conclude the statement of the corollary it is sufficient to prove that every two symbol algebras A_1, A_2 contain F-isomorphic maximal subfields.

Since A_i, $i = 1, 2$ is cyclic, it contains an element z_i, $z_i^l \in F$. Since A_i is a good splittable algebra and by lemma 0.50 (which is true also if $charF = 0$), we can assume $v(z_i^l) = 1$ (v is the valuation on F).

To prove it we show that A_1 contains any l-root of elements u in F with $v(u) \neq 0$. Since for any element $1 + b$, $v(b) > 0$ there exists an element $(1 + b)^{1/l} \in F$, it is sufficient to prove that A_1 contains any l-root of elements ct, $c \in \bar{F}$, where we fix some embedding $i : \bar{F} \hookrightarrow F$.

Indeed, since A_1 is a good splittable algebra and by lemma 0.50 (which is true also if $charF = 0$) we can assume there exists an element z such that $v(z^l) = 1$, $z^l = ct$, $c \in \bar{F}$, $ad(z)$ acts on $\bar{A}_1$, where $\bar{A}_1$ is embedded in A_1 by a good embedding with respect to i. Note that for any element $b \in \bar{A}_1$ we have $(bz)^l = N_{\bar{A}_1/\bar{F}}(b)z^l$. But the norm map $N_{\bar{A}_1/\bar{F}}$ is surjective because $\bar{F}$ is a C_1-field (see, e.g. [26], 3.4.2), so for any c there exists b such that $(bz)^l = ct$.

4) Suppose now $expA = p$. Then $indA = p^k$.

By Albert's theorem (in [1]) there exists a field $F' = F(u_1^{1/p}, \dots, u_k^{1/p})$ which splits A. Using the same arguments as in 1) one can show that every such a field has maximum two generators, say $F' = F(u_1^{1/p}, u_2^{1/p})$. Therefore, $indA \leq p^2$. If $indA = p$, there is nothing to prove, so we assume $indA = p^2$ and F' is a maximal subfield in A.

5) Suppose F_1 is a perfect field.

By Albert's theorem, $A \cong A_1 \otimes_F A_2$, where A_1, A_2 are cyclic algebras of degree p, $A_1 = (L_1/F, \sigma_1, u_1)$, $A_2 = (L_2/F, \sigma_2, u_2)$. Since F_1 is perfect, $\bar{A}_1/\bar{F}$, $\bar{A}_2/\bar{F}$ are Galois extensions. So, A_1, A_2 are good splittable. Let us show that A_1, A_2 have common splitting field of degree p over F. This leads to a contradiction.

By lemma 0.50 there exist parameters $z_1 \in A_1$, $z_2 \in A_2$ such that they act on $\bar{A}_1$, $\bar{A}_2$ as Galois automorphisms. Note that then $z_1^p, z_2^p \in F$. Let us show that $F(z_1)$ splits A_2.

Consider the centralizer $D = C_A(F(z_1))$. Consider the element $t_1 = z_2 z_1^{-1}$. We have $t_1^p \in F$, $w(t_1) = 0$, where w denote the unique extension of the valuation v on F. Since $\bar{D}/Z(\bar{D})$ is a Galois extension, there exists an element $b_1 \in F$ such that $w(t_1 - b_1) > 0$. Since $(t_1 - b_1)^p \in F$, there exists natural k_1 such that $w((t_1 - b_1)z_1^{-k_1}) = 0$. Denote $t_2 = (t_1 - b_1)z_1^{-k_1}$. We have again $t_2^p \in F$. Repeating this arguments and using the completeness of $D \subset A$ we get
$z_2 = t_1 z_1 = (t_2 z_1^{k_1} + b_1)z_1 = \dots = b_1 z_1 + b_2 z_1^{k_1+1} + \dots$,
so, $z_2 \in F(z_1) = Z(D)$.

6) Suppose F_1 is not perfect.

Since F' is generated by two elements over F, it contains all p-roots of F. Then, every two elements $u, z \in F$ such that $z^{1/p} \notin F(u^{1/p})$, where $z^{1/p}, u^{1/p} \in F'$, also generate F' over F. This follows from the same arguments as in 1), 4).

Now take $u \in F_1 \backslash F_1^p$, $z = u + t$. It's clear that p-roots of these elements generate F' over F. Moreover, the fields $F(u^{1/p}), F(z^{1/p})$ are *"unramified"* over F, i.e. $[\overline{F(u^{1/p})} : \bar{F}] = p = [F(u^{1/p}) : F]$, $[\overline{F(z^{1/p})} : \bar{F}] = p$. Denote $u_1 = u^{1/p}$, $u_2 = z^{1/p}$ in F'. Then by Albert's theorem, $A \cong A_1 \otimes_F A_2$, where A_1, A_2 are cyclic algebras of degree p, $A_1 = (L_1/F, \sigma_1, u)$, $A_2 = (L_2/F, \sigma_2, z)$.

Since the fields $F(u^{1/p}) \subset A_1$, $F(z^{1/p}) \subset A_2$ are "unramified" and purely inseparable of degree p over F, the algebras A_1, A_2 are good splittable. Moreover, there exist embeddings $\bar{A}_1 \hookrightarrow A_1$, $\bar{A}_2 \hookrightarrow A_2$ such that $u_1 \in \bar{A}_1$, $u_2 \in \bar{A}_2$. Then by theorem 0.43 there exist parameters $z_1 \in A_1$, $z_2 \in A_2$ such that $z_1^p, z_2^p \in F$ and

$$z_2 u_2 z_2^{-1} = u_2 + c z_2^i,$$

where $c \in F$, $v(c) = 0$. So, for the element $u_2' = c^{-1} u_2$ we have

$$z_2 u_2' z_2^{-1} = u_2' + z_2^i,$$

and $u_2' \notin F$, $u_2'^p \in F$.

Since $\bar{F}$ is a C_1-field, we have $\bar{A}_1 = \bar{A}_2$ and therefore there exist an element $b \in F(u_1) \subset A_1 \subset A$ such that $w(u_2' - b) > 0$, where w is the unique extension of v on A. Since b commutes with u_2', we have $(u_2' - b)^p \in F$. Therefore $w(u_2' - b) \in 1/p\mathbb{Z}$ (we assume the value groups of w and v lie in a common divisible hull $\Gamma_v \otimes_{\mathbb{Z}}$ "). Hence $w((u_2' - b) z_2^{-pw(u_2' - b)}) = 0$. Put $u_3 = (u_2' - b) z_2^{-pw(u_2' - b)}$.

Note that

$$z_2 (u_2' - b) z_2^{-1} = (u_2' - b) + z_2^i,$$

$$z_2 u_3 z_2^{-1} = u_3 + z_2^{i_1}, \quad i_1 < i$$

So, the elements $(u_2' - b), z_2$ generate a division algebra C of degree p over F and $u_3 \in C$. Then, u_3^p commutes with z_2 if $i_1 > 0$. Therefore, in this case $u_3^p \in F$ and C is a good splittable division algebra. Note that $u_1 \in C_A(C)$, so $A \cong A_1' \otimes_F C$ with $u_1 \in A_1'$. Using the same arguments we get that there exists an element u_4 with $w(u_4) = 0$ and

$$z_2 u_4 z_2^{-1} = u_4 + z_2^{i_2}, \quad i_2 \leq 0$$

So, i_2 must be equal to 0 and therefore u_4, z_2 generate a division algebra C' of degree p over F such that $\bar{C}'/\bar{F}$ is a Galois extension and $u_1 \in C_A(C')$. So, $A \cong D \otimes_F C'$ with $u_1 \in D$.

Therefore, A contains the maximal subfield $F(u_1)F(u_4)$, which is a compositum of a purely inseparable and Galois extension. Moreover, this field is "unramified" over F, so it is good splittable field and A is a good splittable algebra with p dividing

$|Gal(\bar{A}/\bar{F})|$. But this is a contradiction with proposition 0.51.

□

Corollary 9 *Let A be a central division p-algebra over a C_2-field $F = F_1((t))$, F_1 is a C_1-field. Then A contains a maximal purely inseparable over F subfield, i.e. A is a cyclic algebra.*

Moreover, A is a good splittable algebra.

Proof. The proof of the first statement is by induction on degree of A. If $indA = p$, then by Tignol's theorem in [32] A is cyclic, so it contains such a maximal subfield.

If $indA = p^k$, $k > 1$, then by assumption a division algebra similar to $A^{\otimes p}$ has the exponent and index p^{k-1} and so can be split by a field extension $F \subset F'$ of degree p^{k-1}. By corollary 8, the exponent and the index of $A_{F'}$ is p, so there exists an extension L/F of degree p^k such that L splits A.

To prove the second statement note that it is sufficient to prove it only for algebras A with $\bar{A}/\bar{F}$ — purely inseparable. Now to prove the assertion we use lemma 0.24. Note that, using a similar induction, it is sufficient to prove the statement for algebras A of degree p.

Let z be a purely inseparable element in A, $indA = p$. If $F(z)$ is an "unramified" over F, there is nothing to prove. So, we may assume $F(z)$ is totally ramified over F and z is a parameter of A.

Choose an element $a \in A$ such that $\bar{a}$ generates $\bar{A}$ over $\bar{F}$. Suppose $a \in \bar{A}$ for some embedding $\bar{A} \hookrightarrow A$. Suppose

$$zaz^{-1} = a + \delta_i(a)z^i + \delta_{i+1}(a)z^{i+1} + \dots$$

Then we have

$$z^p a z^{-p} = a + \sum_{k=pi}^{\infty} \sum_{(i_1,\dots i_p)} \delta_{i_1} \dots \delta_{i_p}(a) z^k = a, \tag{2}$$

where $\sum i_j = k$ and the second sum is taken over all such nonrepeating sets $(i_1, \dots, i_p)$. Therefore, $\delta_i^p(a)$ must be equal to zero. Since δ_i is a derivation, it is trivial on $\bar{F}(a^p)$.

Every element in $\bar{F}(a)$ can be written as a polynomial $c_1 + c_2 a + \dots c_p a^{p-1}$, where $c_i \in \bar{F}(a^p)$. Therefore, we can write $\delta_i = \delta_i(a)\partial/\partial(a)$. So, $\delta_i^p(a) = \delta_i(a)\partial/\partial(a)(\delta_i^{p-1}(a))$. Hence $\partial/\partial(a)(\delta_i^{p-1}(a)) = 0$ and $\delta_i^{p-1}(a) \in \bar{F}(a^p)$.

If $\delta_i^{p-1}(a) = 0$, then let j be the maximal natural such that $\delta_i^j(a) \neq 0$, $\delta_i^j(a) \in \bar{F}(a^p)$. Now put $a_1 = \delta_i^{j-1}(a)(\delta_i^j(a))^{-1}$. Note that a_1 generates $\bar{A}$ over $\bar{F}$ Since $\delta_i^j(a) = \partial/\partial(a)(\delta_i^{j-1}(a))\delta_i(a)$, we have $\delta_i(a_1) = 1$.

So we can put $a := a_1$ and assume $\delta_i = \partial/\partial(a)$. Now the proof is by induction on k in the formula 2. For $k = ip + 1$ we have

$$\sum_{(i_1,\dots,i_p)} \delta_{i_1} \dots \delta_{i_p}(a) = 0$$

By lemma 0.44, $\delta_{i+1} = \delta_i^l + c\delta_i$, so $\delta_{i_1} \dots \delta_{i_p}(a) = 0$ if $i_p = i$. Therefore, we have

$$\delta_i \dots \delta_i \delta_{i+1}(a) = 0$$

Therefore, there exists an element $b \in \bar{A}$ such that $\delta_i(b) = \delta_{i+1}(a)$ and by lemma 0.24 there exists an element $a_2 = a + b_2 z$ such that

$$z a_2 z^{-1} = a_2 + z^i + \delta'_{i+2} z^{i+2} + \dots$$

Note that here the coeffitients on the right hand side belong to another embedding of $\bar{A}$ given by element a_2. Since $\bar{A}$ is a C_1-field, $\bar{A}$ is generated by $\bar{a_2}$ over $\bar{F}$. So, the p-basis of $\bar{A}$ consists of 1 element. So, by classical Cohen's theorem, any lifting of this element gives an embedding of $\bar{A}$. Now using induction and completeness of A we get that there exists an element a_3 such that

$$z a_3 z^{-1} = a_3 + z^i$$

and $\bar{a_3}$ generates $\bar{A}$ over $\bar{F}$. Therefore, a_3^p commutes with z, from here follows that a_3 is a purely inseparable element and $F(a_3)$ is an "unramified" extension.
The corollary is proved.
□

This corollary concludes the proof of theorem 0.36.

0.5 Classes of conjugate elements

Let K be a splittable local skew field of characteristic 0 whose first residue skew field is commutative and whose last residue skew field k is contained in its centre. We have classified these skew fields in the preceding section. In this section we give necessary and sufficient conditions for two elements of K to be conjugate.

We fix a representation of K in the form $k((u))((z))$.

Definition 0.56 *Let $\alpha = Id$. A residue $res_{i,r}$ on K is defined to be a map $res_{i,r} : k((u))((z)) \mapsto k$*

$$res_{i,r}(X) = res \frac{x_i}{u^{\delta_i}} du$$

where $X = \sum_l x_l z^l$.

Proposition 0.57 *Let $\alpha = Id$. Let $L, M \in K$, $\nu(L) = \nu(M) = -1$,*
$M = b_{-1} z^{-1} + b_0 + b_1 z + \dots$,
$L = a_{-1} z^{-1} + a_0 + a_1 z + \dots$.
The following assumptions are equivalent:

(i) there is an $S \in K$, $\nu(S) = 0$, *such that* $M = S^{-1}LS$

(ii) $a_{-1} = b_{-1}$, $a_0 = b_0$, $\ldots$, $a_{i-2} = b_{i-2}$;

$$res \frac{a_{i-1} - b_{i-1}}{u^{\delta_i} a_{-1}} du \in \mathbb{Z} \quad \textit{and} \quad u \frac{a_{i-1} - b_{i-1}}{u^{\delta_i} a_{-1}} \in k[[u]]$$

$res_{i,r}(M^j) = res_{i,r}(L_j^j)$ *for all* $j \geq 1$, *where* $L_j = \tilde{S}_j^{-1} L_{j-1} \tilde{S}_j$, $L_0 := L$, $\tilde{S}_j = \tilde{S}_j(M, L_{j-1})$.

Proof K has the form $k((u))((z))$ with the relation $zuz^{-1} = u + u^{\delta_i} z^i + \ldots$. Thus we have:

$$SM = s_0 b_{-1} z^{-1} + (s_0 b_0 + s_1 b_{-1}) + \ldots + (\sum_{j=-1}^{i-2} b_j s_{i-2-j}) z^{i-2} + (\sum_{j=-1}^{i-1} b_j s_{i-1-j}) z^{i-1} + \ldots$$

$$LS = s_0 a_{-1} z^{-1} + (s_0 a_0 + s_1 a_{-1}) + \ldots + (\sum_{j=-1}^{i-2} a_j s_{i-2-j}) z^{i-2} + (-a_{-1} s_0^{\delta_i} + \sum_{j=-1}^{i-1} a_j s_{i-1-j}) z^{i-1} + \ldots$$

It follows that the condition $a_{-1} = b_{-1}$, $a_0 = b_0$, $\ldots$, $a_{i-2} = b_{i-2}$ is necessary for M and L to be conjugate. Another necessary condition is given by the following equation for s_0:

$$\frac{s_0^{\delta_i}}{s_0} = \frac{a_{i-1} - b_{i-1}}{a_{-1}}$$

Since δ_i is a differentiation, we have

$$\frac{\frac{\partial}{\partial u} s_0}{s_0} = \frac{a_{i-1} - b_{i-1}}{u^{\delta_i} a_{-1}}$$

Thus we obtain the second necessary condition:

$$res \frac{a_{i-1} - b_{i-1}}{u^{\delta_i} a_{-1}} du \in \mathbb{Z} \quad \text{and} \quad u \frac{a_{i-1} - b_{i-1}}{u^{\delta_i} a_{-1}} \in k[[u]]$$

Conversely, if these two conditions hold, then there is an $s_0 \in k((u))$ such that the first $i+1$ summands in $L_1 = s_0^{-1} L s_0$ are the same as those in M. It is clear that L and M are conjugate if and only if L_1 and M are conjugate. The conjugating element $\tilde{S}$ has the form $1 + \cdots$ ($\tilde{S}$ can be written as $(1 + s_1 z)(1 + s_2 z^2) \ldots$.) Note that for every $x_{-1} z^{-1} + x_0 + x_1 z + \ldots \in K$ holds:

$$(1 + s_j z^j)^{-1} (x_{-1} z^{-1} + x_0 + x_1 z + \ldots)(1 + s_j z^j) = x_{-1} z^{-1} + x_0 + x_1 z + \ldots + x_{i+j-2} z^{i+j-2} +$$

$$(x_{i+j-1} + j x_{-1}^{\delta_i} s_j + x_{-1} s_j^{\delta_i}) z^{i+j-1} + \ldots$$

since the proof of lemma 0.11, (ii) implies that

$$(1+s_jz^j)^{-1}(x_{-1}+x_0z+x_1z^2+\ldots)(1+s_jz^j)=x_{-1}+x_0z+\ldots+x_{i+j-2}z^{i+j-1}+$$
$$(x_{i+j-1}+jx_{-1}^{\delta_i}s_j)z^{i+j}+\ldots),$$

and

$$(1+s_jz^j)^{-1}z^{-1}(1+s_jz^j)=(1+s_jz^j)^{-1}(z^{-1}+s_jz^{j-1}-s_j^{\delta_i}z^{i+j-1}+\ldots)=z^{-1}-s_j^{\delta_i}z^{i+j-1}+\ldots$$

It follows that

$$(s_1a_{-1})^{\delta_i}=b_i-a_i \quad (j=1),$$

if $M=\tilde{S}^{-1}L_1\tilde{S}$, where a_i is the coefficient of L_1. This equation is soluble if and only if

$$res\frac{b_i-a_i}{u^{\delta_i}}du=0,$$

that is, $res_{i,r}(M)=res_{i,r}(L_1)$.

Conversely, if the residues are equal then there is an $s_1\in k((u))$ such that the first $i+2$ summands in $L_2=(1+s_1z)^{-1}L_1(1+s_1z)$ are the same as those in M.

Proceeding by induction, we obtain at the kth step that if $M=\bar{S}^{-1}L_k\bar{S}$, then

$$ks_ka_{-1}^{\delta_i}+a_{-1}s_k^{\delta_i}=b_{i+k-1}-a_{i+k-1}.$$

To solve this equation, we substitute $s_k=a_{-1}^{-k}s$ into it and obtain the equation

$$s'=a_{-1}^{k-1}\frac{b_{i+k-1}-a_{i+k-1}}{u^{\delta_i}},$$

which is solvable if and only if $res\frac{a_{-1}^{k-1}a_{i+k-1}}{u^{\delta_i}}=res\frac{u^{-r}a_{-1}^{k-1}b_{i+k-1}}{u^{\delta_i}}$. On the other hand, the coefficient of z^i in M^k has the form

$$ka_{-1}^{k-1}b_{i+k-1}+f_M$$

where f_M is a polynomial in $b_{i+k-2},\ldots,b_{-1}$ and the values of δ_j at these points. The corresponding coefficient in L_k^k has the form

$$ka_{-1}^{k-1}a_{i+k-1}+f_{L_k}$$

and $f_{L_k}=f_M$, since $a_j=b_j$ for $j\le i+k-2$. It follows that $res_{i,r}L_k^k=res_{i,r}M^k$ if and only if $res\frac{a_{-1}^{k-1}a_{i+k-1}}{u^{\delta_i}}=res\frac{a_{-1}^{k-1}b_{i+k-1}}{u^{\delta_i}}$. which completes the proof of the proposition.
□

Definition 0.58 *Let $\alpha \neq Id$. We say that the residue res_α of$X = \sum_l x_l z^l$ is equal to zero, if*

$$x_0 \in im(\alpha - Id)$$

We say that two elements have the same residue if the residue of their difference is equal to zero.

We define $\varphi : k((u)) \mapsto k((u))$, $\varphi(x) = x^{\alpha^{-1}}/x$.

Proposition 0.59 *Let $\alpha \neq Id$. Let $L, M \in K$, $\nu(L) = \nu(M) = -1$,*
$M = b_{-1}z^{-1} + b_0 + b_1 z + \ldots$,
$L = a_{-1}z^{-1} + a_0 + a_1 z + \ldots$.
The following conditions are equivalent:

(i) there exists an $S \in K$, $\nu(S) = 0$, such that $M = S^{-1}LS$

(ii) $b_{-1}/a_{-1} \in im\varphi$;

$res_\alpha(M^j) = res_\alpha(L_j^j)$ for all $j \geq 1$, where $L_j = \tilde{S}_j^{-1} L_{j-1}\tilde{S}_j$, $L_0 := L$, $\tilde{S}_j = \tilde{S}_j(M, L_{j-1})$.

Proof is similar to that of the preceding proposition. We have

$$SM = s_0 b_{-1} z^{-1} + (s_0 b_0 + s_1 b_{-1}^{\alpha}) + \ldots$$

$$LS = a_{-1} s_0^{\alpha^{-1}} z^{-1} + (a_0 s_0 + a_{-1} s_1^{\alpha^{-1}}) + \ldots$$

Therefore, $s_0 b_{-1} = a_{-1} s_0^{\alpha^{-1}}$, that is $b_{-1}/a_{-1} \in \mathrm{im}\varphi$. If this condition holds, then we put $L_1 = s_0^{-1} L s_0$. The first coefficients in L_1 and M are equal.

Now we observe that

$$(1 + s_j)^{-1}(x_{-1}z^{-1} + x_0 + x_1 z + \ldots)(1 + s_j z^j) = x_{-1}z^{-1} + \ldots$$

$$+ x_{j-2}z^{j-2} + (x_{j-1} + s_j x_{-1}^{\alpha^j} - x_{-1} s_j^{\alpha^{-1}})z^{j-1} + \ldots$$

for any $x_{-1}z^{-1} + x_0 + x_1 z + \ldots \in K$, which follows from the calculation in the proof of Lemma 0.11, (i).

The arguments used in the proof of the preceding proposition yield at the first step the following condition that is necessary for conjugacy:

$$s_1 a_{-1}^{\alpha} - a_{-1} s_1^{\alpha^{-1}} = \alpha(s_1^{\alpha^{-1}} a_{-1}) - (s_1^{\alpha^{-1}} a_{-1}) = b_0 - a_0$$

This equation is soluble if and only if $(b_0 - a_0) \in \mathrm{im}(\alpha - Id)$. which is equivalent to the equality $res_\alpha M = res_\alpha L_1$.

At the jth step we have the condition

$$s_j a_{-1}^{\alpha^j} - a_{-1} s_j^{\alpha^{-1}} = a_{j-1} - b_{j-1}$$

Hence,

$$(a_{-1}^{\alpha} a_{-1}^{\alpha^2} \dots a_{-1}^{\alpha^{j-1}})(a_{j-1} - b_{j-1}) = (a_{-1}^{\alpha} a_{-1}^{\alpha^2} \dots a_{-1}^{\alpha^j}) s_j - (a_{-1} \dots a_{-1}^{\alpha^{j-1}}) s_j^{\alpha^{-1}} =$$

$$\alpha((a_{-1} \dots a_{-1}^{\alpha^{j-1}}) s_j^{\alpha^{-1}}) - (a_{-1} \dots a_{-1}^{\alpha^{j-1}}) s_j^{\alpha^{-1}}$$

This equation is soluble if and only if $(a_{-1} \dots a_{-1}^{\alpha^{j-1}})(a_{j-1} - b_{j-1}) \in \mathrm{im}(\alpha - Id)$, which is equivalent to the equality $res_{\alpha}(M^j) = res_{\alpha}(L_j^j)$, since the first $(j-1)$ coefficients in L_j are equal to the corresponding coefficients in M, and the coefficient of the 0th power of z in M^j is

$$a_{-1} \dots a_{-1}^{\alpha^{-j+2}} b_{j-1}^{\alpha^{-j+1}} + b_{j-1} a_{-1}^{\alpha} \dots a_{-1}^{\alpha^{j-1}} +$$

$$\text{a sum of monomials with indices} < j-1$$

The corresponding coefficient in L_j^j is

$$a_{-1} \dots a_{-1}^{\alpha^{-j+2}} a_{j-1}^{\alpha^{-j+1}} + a_{j-1} a_{-1}^{\alpha} \dots a_{-1}^{\alpha^{j-1}} +$$

$$\text{a sum of monomials with indices} < j-1$$

Hence,

$$(a_{-1} \dots a_{-1}^{\alpha^{-j+2}} b_{j-1}^{\alpha^{-j+1}} - a_{-1} \dots a_{-1}^{\alpha^{-j+2}} a_{j-1}^{\alpha^{-j+1}} + b_{j-1} a_{-1}^{\alpha} \dots a_{-1}^{\alpha^{j-1}} - a_{j-1} a_{-1}^{\alpha} \dots a_{-1}^{\alpha^{j-1}}) =$$

$$([a_{-1} \dots a_{-1}^{\alpha^{-j+2}} b_{j-1}^{\alpha^{-j+1}} - a_{-1} + \dots a_{-1}^{\alpha^{-j+2}} a_{j-1}^{\alpha^{-j+1}}] -$$

$$\alpha[\dots] + \alpha[\dots] - \alpha^2[\dots] + \alpha^2[\dots] \dots + \alpha^{j-1}[\dots] +$$

$$b_{j-1} a_{-1}^{\alpha} \dots a_{-1}^{\alpha^{j-1}} - a_{j-1} a_{-1}^{\alpha} \dots a_{-1}^{\alpha^{j-1}}) =$$

$$(2[a_{-1}^{\alpha} \dots a_{-1}^{\alpha^{j-1}} (a_{j-1} - b_{j-1})])$$

□

Remark It was shown in [18], that for the residue $res_{1,0}$ in the skew field of pseudodifferential operators holds $res_{1,0}[X, Y] = 0$, where $[X, Y]$ is the commutator of two pseudodifferential operators. The next statements provide other examples of skew fields with this property.

Lemma 0.60 *Let K be a skew field such that $\alpha^n \neq Id$ or $\alpha^n = Id$, $i_n = \infty$. Let $X, Y \in K$. Then $res_{\alpha}[X, Y] = 0$.*

Proof It is sufficient to prove the assertion for $X = u^l z^k$, $Y = u^m z^q$.
If $k+q \neq 0$, then $res_\alpha(XY) = res_\alpha(YX) = 0$. In the case $k+q=0$ we have:

$$XY - YX = u^l (u^m)^{\alpha^k} - u^m (u^l)^{\alpha^{-k}} = \alpha^k (u^m (u^l)^{\alpha^{-k}}) - u^m (u^l)^{\alpha^{-k}} \in \mathrm{im}(\alpha - Id)$$

□

In this case our propositions can be stated as follows:

Corollary 10 *Let K be a skew field such that $\alpha = Id$, $i=1$, $r=0$, $a=0$ ((In this case K is the ring $k((u))((\partial^{-1}))$ of pseudodifferential operators.) Let $L, M \in K$, $\nu(L) = \nu(M) = -1$,*
$M = b_{-1}z^{-1} + b_0 + b_1 z + \ldots$,
$L = a_{-1}z^{-1} + a_0 + a_1 z + \ldots$.
The following conditions are equivalent:

(i) there is an $S \in K$, $\nu(S) = 0$, such that $M = S^{-1}LS$

(ii) $a_{-1} = b_{-1}$;

$$res \frac{a_0 - b_0}{a_{-1}} du \in \mathbb{Z} \quad \textit{and} \quad \frac{u(a_0 - b_0)}{a_{-1}} \in k[[u]]$$

$res_{1,0}(M^j) = res_{1,0}(L^j)$ for all $j \geq 1$.

Corollary 11 *Assume that $\alpha^n \neq Id$ for all $n \in \mathbb{N}$. Let $L, M \in K$, $\nu(L) = \nu(M) = -1$,*
$M = b_{-1}z^{-1} + b_0 + b_1 z + \ldots$,
$L = a_{-1}z^{-1} + a_0 + a_1 z + \ldots$.
The following conditions are equivalent:

(i) there is an $S \in K$, $\nu(S) = 0$, such that $M = S^{-1}LS$

(ii) $b_{-1}/a_{-1} \in im\varphi$;

$res_\alpha(M^j) = res_\alpha(L^j)$ for all $j \geq 1$.

The following examples show that the identity $res_{\ldots}([X,Y]) = 0$ does not hold in other cases.

Example (i) Let K be a skew field with $\alpha = 1$, $a(0,\ldots,0) \neq 0$, $r \neq 1$. We assume that K has the form specified in Theorem 0.35. Let $M = z^{-1}$, $L = z^{-1} + z^i \in k((z)) \subset K$. If $res_{i,r}([X,Y]) = 0$ holds, then M and L are conjugate by Proposition 0.57. Let $S = 1 + s_1 z + \ldots$. We have

$$SM = z^{-1} + s_1 + s_2 z + \ldots = LS = (z^{-1} + z^i)(1 + s_1 z + \ldots) =$$

$(z^{-1}+s_1+s_2z+\ldots)+(z^i-s_1^{\delta_i}z^i)+(s_1z^{i+1}-s_2^{\delta_i}z^{i+1})+\ldots+(s_iz^{2i}+s_1^{\delta_i^2-\delta_{2i}}z^{2i}-s_{i+1}^{\delta_i}z^{2i})+\ldots$

Hence, $1-s_1^{\delta_i}=0$. Since $r\neq 1$, this equation is soluble, and $s_1=(1-r)^{-1}c^{-1}u^{1-r}$. Solving the next equations, we obtain $s_2, s_3, \ldots$. Each of these elements consists of a single monomial whose valuation is different from $r-1$.

Further, we have $s_iz^{2i}+s_1^{\delta_i^2-\delta_{2i}}z^{2i}-s_{i+1}^{\delta_i}z^{2i}=0$. By Theorem 0.35, if $a(0,\ldots,0)\neq 0$, then $s_1^{\delta_i^2-\delta_{2i}}$ contains a monomial whose valuation is equal to $r-1$. Therefore, the equation is insoluble with respect to s_{i+1}, and M is not conjugate to L. This contradiction completes the proof of the assertion.

(ii) Let K be a skew field with $\alpha=1$, $a(0,\ldots,0)=0$. In this case $i>1$, since $r=0$ for $i=1$, and we obtain the ring of pseudodifferential operators. We assume that K has the form specified in Theorem 0.35. Then $zuz^{-1}=u+cu^rz^i+r(i+1)/2c^2u^{2r-1}z^{2i}$. Therefore, $\delta_{2i}=\delta_i^2$. Then for any $x\in k((u))$ holds:

$$z^{-1}xz=x-x^{\delta_i}z^i+.\ \ z^{>2i}$$

We put $X=u^{-r-1}z^{-i}$, $Y=u^2$. Then

$$XY=u^{1-r}z^{-i}+\ldots+Cu^{r-1}z^i+\ldots,\quad C\in{}'',C\neq 0$$

Hence $res_{i,r}([X,Y])\neq 0$.

An example with $a(0,\ldots,0)\neq 0$, $r=1$ can be obtained likewise. (iii) Let K be a skew field with $\alpha^n=1$, $i_n\neq\infty$. We put $X=u^{-r_n}z^{-i_n}$ and $Y=u$. Then

$$XY=\xi^{-i_n}u^{1-r_n}z^{-i_n}+C+\ldots$$

where $C=-i_n\xi^{-i_n+1}c\neq 0$. Hence, $res_\alpha([X,Y])\neq 0$.

Remark These examples show that the Scolem-Noether theorem does not hold for skew fields defined here.

Let K be the ring $k((u))((\partial_u^{-1}))$ of pseudodifferential operators. We have shown that this is the only slew field such that $res_{1,0}([X,Y])=0$. Let us deduce a criterion for two elements of this skew field to be conjugate.

Let $n\in\mathbb{N}$ be a certain number. Consider the skew field $K'=k((t))((\partial_t^{-1}))$, where $t^n=u$. Then $\partial_t=nt^{n-1}\partial_u$, and $K\subset K'$.

Lemma 0.61 *Let $L=l_{-m}\partial_t^m+\ldots+l_0+l_1\partial_t^{-1}+\ldots\in K'$ -be an arbitrary element of K'.*

$L\in K$ if and only if $l_i\in t^ik((t^n))$.

Proof Assume that $L\in K$. Then $L=b_{-m}\partial_u^m+\ldots$, where $b_i\in k((u))=k((t^n))$. Let $j\in\mathbb{N}$. We have:
$\partial_u^j=(n^{-1}t^{1-n}\partial_t)^j$, $\partial_u^{-j}=(\partial_t^{-1}nt^{n-1})^j$.

We prove first the assertion of the lemma for l_{-i} $(i > 0)$. For $i = 1$ we have $\partial_u^i = n^{-1}t^{1-n}\partial_t$ and $b_{-1}\partial_u = l_{-1}n^{-1}t^{1-n}\partial_t$. The assertion of the lemma holds, since $t^{1-n} \in tk((t^n))$.

For an arbitrary i we have

$$\partial_u^i = \frac{\partial_t}{\partial_t t}(n^{-1}t^{1-n})(n^{-1}t^{1-n}\partial_t)^{i-1} + (n^{-1}t^{1-n})^2\partial_t^2(n^{-1}t^{1-n}\partial_t)^{i-2} =$$

$$(1-n)(n^{-1}t^{-n})(n^{-1}t^{1-n}\partial_t)^{i-1} + (n^{-1}t^{1-n})^2\partial_t^2(n^{-1}t^{1-n}\partial_t)^{i-2}$$

Since the coefficients in the expression for L in K belong to $k((t^n))$, it is sufficient to show that the lemma holds for ∂_u^i.

We prove by induction that the assumption of the lemma holds for all the coefficients in $(n^{-1}t^{1-n}\partial_t)^{i-1}$. The same is true for $(n^{-1}t^{1-n}\partial_t)^{i-2}$. Let $(n^{-1}t^{1-n}\partial_t)^{i-2} = \sum_{k=0}^{i-2}\tilde{l}_k\partial_t^k$ (Let us note that there are no negative powers of ∂_t in the expansion of ∂_u^i, $i > 0$, and the minimal power of∂_t is equal to 1). We have:

$$(n^{-1}t^{1-n})^2\partial_t^2(\sum_{k=0}^{i-2}\tilde{l}_k\partial_t^k) = (n^{-1}t^{1-n})^2(\sum_{k=0}^{i-2}\tilde{l}_k\partial_t^{k+2} + \sum_{k=0}^{i-2}\tilde{l}'_k\partial_t^{k+1} + \sum_{k=0}^{i-2}\tilde{l}''_k\partial_t^k)$$

Therefore, $(n^{-1}t^{1-n})^2\tilde{l}_k \in t^{k+2}k((t^n))$, $(n^{-1}t^{1-n})^2\tilde{l}'_k \in t^{k+1}k((t^n))$, $(n^{-1}t^{1-n})^2\tilde{l}''_k \in t^k k((t^n))$.

For $i = 0$ we have $l_0 = b_0 \in k((t^n))$.

Let us prove that the assertion of the lemma holds for ∂^{-i}, $i > 0$. For $i = 1$ we have:
$\partial_u^{-1} = n\sum_{k=0}^{n-1}(t^{n-1})^{(k)}\partial_t^{-1-k}C_k^{-1}$.

Assume that for $k < i$ it is proved $\partial_u^{-k} = \sum_{j=0}^{\infty}\tilde{l}_j\partial_t^{-k-j}$, $\tilde{l}_j \in t^{-k-j}k((t^n))$.

$$\partial_u^{-i} = (\partial_t^{-1}nt^{n-1})^i = (n\sum_{k=0}^{n-1}C_k^{-1}(t^{n-1})^{(k)}\partial_t^{-1-k})(\partial_t^{-1}nt^{n-1})^{i-1} =$$

$$(n\sum_{k=0}^{n-1}C_k^{-1}(t^{n-1})^{(k)}\partial_t^{-1-k})(\sum_{j=0}^{\infty}\tilde{l}_j\partial_t^{-i+1-j})$$

For every $k \in \{0, \ldots, n-1\}$ $\partial_t^{-1-k}\tilde{l}_j = \sum_{p=0}^{\infty}C_p^{-1-k}\tilde{l}_j^{(p)}\partial_t^{-1-k-p}$. This yields the following conditions on the coefficients for fixed k and j:
at $\partial_t^{-1-k-p-i+1-j}$, $p \geq 0$, the coefficient belongs to $t^{-1-k-i+1-j-p}k((t^n))$.

Conversely, assume that the assumptions of the lemma on the coefficients hold. We have obtained that
$\partial_u^i = \sum_{j\geq 0}c_j\partial_t^{i-j}$, and $c_j \in t^{i-j}k((t^n))$ for any $i \in \mathbb{Z}$.

Consider the highest monomial in L:

$$l_{-m}\partial_t^m = l_{-m}c_0^{-1}\partial_u^m - l_{-m}(\sum_{j\geq 1} c_j c_0^{-1}\partial_t^{m-j})$$

We have $l_{-m}c_0^{-1} \in k((t^n))$, $l_{-m}c_jc_0^{-1} \in t^{m-j}k((t^n))$. Hence, $L = l_{-m}c_0^{-1}\partial_u^m + L_1$, where $\nu(L_1) > \nu(L)$, and the the assumptions of the lemma hold for the coefficients in L_1. We complete the proof by induction.
□

Lemma 0.62 *Let $L, M \in K \subset K'$ and $\nu(L) = \nu(M) = -n$. Let $M = SLS^{-1}$, where $S \in K'$. Then $S \in K$ if and only if*

$$res\frac{l_{\nu(L)+1} - m_{\nu(M)+1}}{l_{\nu(L)}} = 0 \quad and \quad t\frac{l_{\nu(L)+1} - m_{\nu(M)+1}}{l_{\nu(L)}} \in k[[t]]$$

Proof is similar to that of Proposition 0.57.
□

Theorem 0.63 *Let $L, M \in K = k((u))((\partial_u^{-1}))$, $\nu(L) = \nu(M) < 0$,*
$M = m_{\nu(M)}\partial_t^{-\nu(M)} + \ldots$,
$L = l_{\nu(L)}\partial_t^{-\nu(L)} + \ldots$.
The following assumptions are equivalent:

(i) there is an $S \in K$, $\nu(S) = 0$, such that $M = S^{-1}LS$

(ii) $\nu(L) = \nu(M)$, $m_{\nu(M)} = l_{\nu(L)}$,

$$res\frac{l_{\nu(L)+1} - m_{\nu(M)+1}}{l_{\nu(L)}} = 0 \quad and \quad t\frac{l_{\nu(L)+1} - m_{\nu(M)+1}}{l_{\nu(L)}} \in k[[t]]$$

$res(M^{j/(-\nu(M))}) = res(L^{j/(-\nu(L))})$ for all $j \geq 1$ in K'.

Proof follows immediately from Corollary 10, Lemmas 0.61, 0.62 and the fact that L (and M) has precisely one nth root in K'.
□

Theorem 0.64 *Assume that $L, M \in K = k((u))((\partial_u^{-1}))$ and $\nu(L) = \nu(M) = 0$. Then*
(i) If $l_0 = m_0 \neq const$ and $l_1 = m_1$, then $M = SLS^{-1}$.
(ii) If $l_0 = m_0 = const$, then $M = SLS^{-1}$ if and only if $(M - m_0)^{-1} = S(L - l_0)^{-1}S^{-1}$ (see Theorem 0.63)

Proof is obvious.
□

0.6 New equations of KP-type on skew fields

In this section we give an answer on a question given in [22]. Namely, the classical KP-hierarchy is constructed by means of the ring of pseudo-differential operators $P = k((x))((\partial^{-1}))$. This ring is a skew field. The point is to consider other skew fields instead of this one. We will study if there exist some new non-trivial generalisations of the KP-hierarchy for a list of two-dimensional skew fields. In particular, we give a number of new partial differential equations of the KP-type.

For every two-dimensional skew field from the list of theorem 1.5 we can write down a decomposition $K = K_+ + K_-$, where $K_- = \{L \in K : ord(L) < 0\}$ and K_+ consists of the operators containing only ≥ 0 powers of z, and a "KP-hierarchy" in the Lax form:

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L],$$

where $L \in z^{-1} + K_- \otimes k[[\dots, t_m, \dots]]$. Let $L = z^{-1} + u_1 z + u_2 z^2 + \dots$, where $u_m = u_m(u, t_1, t_2, \dots)$. Further we will denote $\partial/\partial t_n$ as ∂_n.

One can check that if the canonical automorphism α in the classification theorem 1.5 is not trivial, then our "KP-hierarchy" became trivial, i.e. it can be easily linearised and solvable. We omit calculations here. So, it can be assumed that $\alpha = id$. The same is true if $i > 1$, because $[(L^n)_+, L] = -[(L^n)_-, L] = 0 \mod \wp^i$ in this case, where $\wp$ is a maximal ideal of the first valuation in K. So, our "KP-hierarchy" again is linear and easily solvable in this case.

So, we assume $i = 1$, hence, $r = 0$ and $c = 1$, and there is only only one non-trivial parameter a. If $a = 0$, K is isomorphic to the ring P of pseudo-differential operators. Denote by $u', u'', \dots$ the subsequent derivatives by x.

First for $n = 1$, we get

$$\partial_1 u_1 = u_1'$$

This means that we can take $t_1 = x$ for u_1.

Now we write down the first two equations for $n = 2$ and the first equation for $n = 3$.

$$\partial_2 u_1 = u''_1 + 2u_2' \tag{3}$$

$$\partial_2 u_2 = 2u_3' + 2u_1 u_1' + u''_2 + 2ax^{-1}u_2' \tag{4}$$

$$\partial_3 u_1 = u'''_1 + 3u''_2 + 3u_3' + 6u_1 u_1' + 3a(x^{-1}u''_1 - x^{-2}u_1') \tag{5}$$

Let us introduce the new notation: $u = u_1(x, y, t)$ with $y = t_2$, $t = t_3$. Also we use the standart notation $u_t, u_y, u_{yy}, \dots$ for derivatives.

We can eliminate u_3' from equations 4 and 5 and then we get

$$3u_{2y} - 2u_t = -6uu' - 3u''_2 - 2u''' + 6ax^{-1}u_2' - 6ax^{-1}u'' + 6ax^{-2}u' \tag{6}$$

From 3 we find

$$u'''_2 = 1/2(u''_y - u'''), u'_{2y} = 1/2(u_{yy} - u''_y)$$

Differentiating equation 6 by x and inserting these expressions we finally get new KP-equation

$$(4u_t - u''' - 12uu')' = 3u_{yy} + 6a(5x^{-2}u'' - x^{-2}u_y - 3x^{-1}u''' + x^{-1}u'_y - 4x^{-3}u')$$

One can see that if $a = 0$, we get the usual KP-equation (see also explicite calculations in [21]).

Chapter 1

Classification of automorphisms of a two-dimensional local field.

1.1 Basic results.

In this chapter let K be a two-dimensional local field, $K \cong k((u))((z))$; $Aut_k(K)$ be a group of continuous k-automorphisms of a field K with respect to the topology given by fixed parametrisation, i.e. by the parameters u and z (see [35] concerning the connection between a topology and a parametrisation).

Introduce the following notation. By Greece letters α, β, γ we will denote automorphisms of a field K. An overline will denote the residue homomorphism. As before, ν denote a valuation on the field K, $\bar{\nu}$ denote a valuation on the field $\bar{K}$, $\wp, \bar{\wp}$ are valuation ideals of the valuations $\nu, \bar{\nu}$, $\mu(k)$ is the group of roots of the unity, $Aut_k(\bar{K})$ is a group of continuous k-automorphisms of the field $\bar{K}$.

Recall some results from chapter 1, section 3.

Definition 1.1 *Let $\bar{K}$ be a one-dimensional local field with the residue field k, $char\bar{K} = chark$, $\bar{\alpha} \in Aut_k(\bar{K})$. Put*
$\xi(\bar{\alpha}) = \bar{\alpha}(u)u^{-1} \quad mod \quad \in k$ *and define* $i(\bar{\alpha}) \in \mathbb{N} \cup \infty$ *by the following:*
$i(\bar{\alpha}) = 1$ *if* $\xi(\bar{\alpha}) \notin \mu(k)$, *else*
$i(\bar{\alpha}) = \bar{\nu}((\bar{\alpha}^n - Id)(u))$, *where* $n \geq 1$, $\xi(\bar{\alpha})$ *is a primitive root of degree* n, $ord(\xi(\bar{\alpha})) = n$.

Proposition 1.2 *Let k be an arbitrary field, $chark = 0$. Any automorphism $\bar{\alpha} \in Aut_k(k((u)))$ with $\bar{\alpha}(u) = \xi(\bar{\alpha})u + a_2u^2 + \ldots$ is conjugate with the automorphism $\bar{\beta}$: $\bar{\beta}(u) = \xi(\bar{\alpha})u + xu^{i(\bar{\alpha})} + x^2yu^{2i(\bar{\alpha})-1}$, where $x \in k^*/k^{*(i(\bar{\alpha})-1)}$, $y \in k$.*

Two automorphisms $\bar{\beta}$, $\bar{\beta}'$ are conjugate iff
$(\xi(\bar{\beta}), i(\bar{\beta}), x(\bar{\beta}), y(\bar{\beta})) = (\xi(\bar{\beta}'), i(\bar{\beta}'), x(\bar{\beta}'), y(\bar{\beta}'))$.

Corollary 12 *1) $i(\bar{\alpha}) = 1$ iff $\bar{\alpha}$ is an automorphism of infinite order and $\xi(\bar{\alpha})$ has infinite order;*

2) $1 < i(\bar{\alpha}) < \infty$ *iff* $\bar{\alpha}$ *has infinite order and* $\xi(\bar{\alpha})$ *has finite order;*
3) $i(\bar{\alpha}) = \infty$ *iff* $\bar{\alpha}$ *has finite order.*

Remark. i) In the notation of proposition we have $n|(i(\bar{\alpha}) - 1)$.
ii) if k is a field of characteristic $p > 0$, then the following fact remains true: the automorphism $\bar{\alpha} \in Aut_k(k((u)))$ with $\bar{\alpha}(u) = \xi(\bar{\alpha})u + a_2u^2 + \ldots$ is conjugate with the automorphism $\bar{\beta}$: $\bar{\beta}(u) = \xi(\bar{\alpha})u + xu^{i(\bar{\alpha})} + \ldots$, where $x \in k^*/k^{*(i(\bar{\alpha})-1)}$.

Lemma 1.3 *Let* $a_0 \in \bar{K}$, $\bar{\alpha} \in Aut_k(\bar{K})$. *The linear map* $T = \bar{\alpha} - a_0 : \bar{K} \to \bar{K}$ *has the following property:*
if $\bar{\alpha}^n = Id$ *for some* n, *then* $\dim(kerT) = \dim(cokerT) = d$, *where* $d = 0$ *or* ∞;
if $\bar{\alpha}^n \neq Id$ *and* $chark = 0$, *then* $\dim(kerT) = \dim(cokerT) = d$, *where* $d = 0$ *or* 1;
if $\bar{\alpha}^n \neq Id$ *and* $chark = p$, *then one of the following cases holds:*
1) $\dim(kerT) = \dim(cokerT) = 0$ *or*
2) $\dim(kerT) = 0$, $\dim(cokerT) = \infty$ *or*
3) $\dim(kerT) = 1$, $\dim(cokerT) = \infty$.

Proof. By proposition 1.2 we can assume $\bar{\alpha}(u) = \xi u + xu^{i(\bar{\alpha})} + \ldots$, where ξ is a primitive n-th root of unity.

If $\bar{\alpha}(u) = \xi u$, $\xi^n = 1$, then the first claim of lemma is clear, so from now on $\bar{\alpha}(u) \neq \xi u$ (note that we have proved the first claim in the case $chark = 0$, because, by corollary 12, any automorphism of a finite order looks like this).

Suppose the element a_0 satisfy one of the following properties:
$\bar{\nu}(a_0) \neq 0$ or
$\bar{\nu}(a_0) = 0$ but $\overline{a_0} \neq \xi^j$ for all $j \in \mathbb{Z}$. Let's study values of the valuation $\bar{\nu}$ on elements $T(u^l)$ for different l. We have:

$$\bar{\alpha}(u^l) - a_0u^l = (\xi u + xu^{i(\bar{\alpha})} + \ldots)^l - a_0u^l = \xi^l u^l(1 + \xi^{-1}xu^{i(\bar{\alpha})-1} + \ldots)^l - a_0u^l$$

Therefore:
$\bar{\nu}(T(u^l)) = l$ or $l + \bar{\nu}(a_0)$ if $\bar{\nu}(a_0) < 0$.

So, we can solve any equation $\bar{\alpha}(y) - a_0y = Y$, and the map T is surjective. It is injective, because the values $\bar{\nu}(T(u^l))$ are finite and $\bar{\nu}(T(u^l)) \neq \bar{\nu}(T(u^{l_1}))$ if $l \neq l_1$.

Suppose now $\overline{a_0} = \xi^j$. Since the injectivity and the projectivity of the map $\bar{\alpha} - a_0$ are defined by the existence and the uniqueness of a solution of the equation $\bar{\alpha}(y) - a_0y = Y$ for any $Y \in \bar{K}$, we can replace y by yu^j and assume that $\overline{a_0} = 1$. Then a_0 can be written as the product $a_0 = \prod_{j=1}^{\infty}(1 + a_{0j}u^j)$, $a_{0j} \in k$.

Put $q = \bar{\nu}(a_0 - 1)$. There are two possible case: $q < i(\bar{\alpha}) - 1$ and $q \geq i(\bar{\alpha}) - 1$.

Let $q < i(\bar{\alpha}) - 1$. Then we can assume $n|q$. To prove it we have to prove that a_0 can be written as the product $a_0^1\frac{\bar{\alpha}(x)}{x}$ for some x, where $\bar{\nu}(a_0^1) = q_1 > q$, $n|q_1$.

Note that $\frac{\bar{\alpha}(1+cu^l)}{1+cu^l} = \frac{1+c\bar{\alpha}(u^l)}{1+cu^l} = 1 + c(\bar{\alpha}(u^l) - u^l)(1 + cu^l)^{-1}$, where c is a constant.

$$\bar{\alpha}(u^l) - u^l = (\xi u + xu^{i(\bar{\alpha})} + \ldots)^l - u^l = \xi^l u^l(1 + \xi^{-1}xu^{i(\bar{\alpha})-1} + \ldots)^l - u^l \qquad (1.1)$$

From this formula we get the following property:
if $n \not| l$, then $\bar{\nu}(\bar{\alpha}(u^l) - u^l) = l$ and $\frac{\bar{\alpha}(1+cu^l)}{1+cu^l} = 1 + (\xi^l - 1)cu^l + \ldots$. Hence, a_0 can be represented as the product above, because there exists a constant c such that the value $\bar{\nu}(a_0\frac{\bar{\alpha}(1+cu^l)}{1+cu^l})$ increases.

So, let we now have: $q < i(\bar{\alpha}) - 1$ and $n|q$. Let's study values of the valuation $\bar{\nu}$ on elements $T(u^l)$ for different l. By formula (1.1) we have:
$\bar{\nu}(T(u^l)) = l$ if $n \not| l$
$\bar{\nu}(T(u^l)) = l + q$ if $(l, n) \neq 1$.

Therefore, we can solve any equation $\bar{\alpha}(y) - a_0 y = Y$, and the map T is surjective. It is injective, because all the values $\bar{\nu}(T(u^l))$ are finite and $\bar{\nu}(T(u^l)) \neq \bar{\nu}(T(u^{l_1}))$ if $l \neq l_1$.

Consider now the case $q \geq i(\bar{\alpha}) - 1$. As in the first case we can assume that $n|q$. We divide this case into three cases:
$q = i(\bar{\alpha}) - 1$,
$q > i(\bar{\alpha}) - 1$ and q is finite
q is infinite, i.e. $a_0 = 1$.

Let $q = i(\bar{\alpha}) - 1$. Let $a_0 = 1 + wu^q + \ldots$. Note that

$$\frac{\bar{\alpha}(cu^{ln})}{cu^{ln}} = \frac{(\xi u + xu^{i(\bar{\alpha})} + \ldots)^{ln}}{u^{ln}} = 1 + nl\xi^{-1}xu^{i(\bar{\alpha})-1} + \ldots$$

Hence, if $w \neq nl\xi^{-1}x$ for all l, we can apply the same arguments as in the first case and get that T is injective and surjective, i.e. $d = 0$. Otherwise, we can write $a_0 = a_0^1\bar{\alpha}(u^{ln})/u^{ln}$, where $\bar{\nu}(a_0^1 - 1) > q$, and reduce this case to the case $q > i(\bar{\alpha}) - 1$.

Let $chark = 0$. We claim that the case $q > i(\bar{\alpha}) - 1$ can be reduced to the case $q = \infty$. In this connection it is necessary to show that $a_0 = \frac{\bar{\alpha}(A)}{A}$. We know, that

$$\frac{\bar{\alpha}(1 + cu^l)}{1 + cu^l} = 1 + c(\bar{\alpha}(u^l) - u^l)(1 + cu^l)^{-1}$$

and $\bar{\alpha}(u^l) - u^l$ has the valuation equal to l if $(l, n) = 1$, and to $(i(\bar{\alpha}) - 1) + l$ if $n|l$, $(l, chark) = 1$ and $l \neq 0$.

From here we get the necessary result, because we can multiply a_0 sequentially by suitable elements of the form $1 + cu^j + \ldots$, each of which can be got from a certain element of the form $1 + c_j u^j$ or $1 + c_j u^{j-(i(\bar{\alpha})-1)}$. It is clear that the product $A = \prod_{j=q}^{\infty}(1 + c_j u^{j-(i(\bar{\alpha})-1)})$ converges.

Let now $chark = 0$ and $q = \infty$, i.e. $a_0 = 1$. Then we claim that $d = 1$. Let us first find the dimension of the kernel of the map T. To do that we investigate the values of the valuation $\bar{\nu}$ of the elements $T(u^l)$ by different l. We have:
$\bar{\nu}(T(u^l)) = l$ if $n \not| l$
$\bar{\nu}(T(u^l)) = l + (i(\bar{\alpha}) - 1)$ if $n|l$ and $l \neq 0$
$\bar{\nu}(T(1)) = \infty$ if $l = 0$, i.e. $T(1) = 0$.

From this follows that the kernel is one-dimensional and consists of the elements of the field k, because all $\bar{\nu}(T(u^l))$ are finite if $l \neq 0$ and $\bar{\nu}(T(u^l)) \neq \bar{\nu}(T(u^{l_1}))$ if $l \neq l_1$.

On the other hand, we get also that the cokernel is one-dimensional, because we can get an element with any value of valuation except an element with the value $(i(\bar{\alpha})-1)$, and there exists a pullback of any convergent (to zero) sequence, which is also converge to zero.

Now we must examine the cases, when $char k = p$. Let us first consider the case $char k = p$ and $q = \infty$.

We prove that $\bar{\alpha}^n = 1$ if and only if $dim_k(ker(\bar{\alpha}-1)) = \infty$.

Let $\bar{\alpha}^n = 1$, $n = p^k m$, $(p, m) = 1$. It is obvious that if exists an element $x \in \bar{K}$, $x \notin k$ such that $(\bar{\alpha}-1)(x) = 0$, then $dim_k(ker(\bar{\alpha}-1)) = \infty$. Suppose, that there is no such an element. Therefore:
$\bar{\alpha}^m(u) = u + a_1$, $a_1 \in \bar{K}$, $\bar{\nu}(a_1) > 1$, $a_1 \neq 0$,
$\bar{\alpha}^{2m}(u) = u + 2a_1 + a_2$, $a_2 \in \bar{K}$, $\bar{\nu}(a_2) > \bar{\nu}(a_1)$, $a_2 \neq 0$,
$\ldots$,
$\bar{\alpha}^{p^k m}(u) = u + \ldots + a_{p^k}$, $a_{p^k} \in \bar{K}$, $\bar{\nu}(a_{p^k}) > \bar{\nu}(a_{p^k-1})$, $a_{p^k} \neq 0$,
and we get a contradiction.

Conversely, let $dim_k(ker(\bar{\alpha}-1)) = infty$. Assume $F = ker(\bar{\alpha}^m - 1)$, $m = ord(\xi(\bar{\alpha}))$.

It's clear that F is a field.

Let $n \in \mathbb{N}$ be a minimal positive value of the valuation $\bar{\nu}$ on this field.

Then $n = p^k$, $k \in \mathbb{Z}$. For, if $n = p^k l$, $(l, p) = 1$, then there exists an element $x \in F$ with such a value and, moreover, $x = d^l$, $d \in \bar{K}$. But then $d \in F$, because $\xi(\bar{\alpha}^m) = 1$, a contradiction.

So, $\bar{K}/F$ is a finite algebraic extension of degree p^k, therefore $\bar{\alpha}^m$ is an automorphism of a finite order. It is easy to see that the order is equal to n, i.e. $\bar{\alpha}$ is a generator of the cyclic Galois group $Gal(\bar{K}/(kerT))$.

Remark. In particular, we have got a description of a subgroup of elements of finite order in the so-called "Nottingham" group. See [3], [12], [5] for further details about this group (i.e. the group $Aut_k(\bar{K})$, $char\bar{K} = p$).

Let $\bar{\alpha}$ be an automorphism of infinite order. Then $kerT = k$, $dim_k(kerT) = 1$. Let $(i(\bar{\alpha})-1, p) = 1$. We claim that for any integer $N > 0$ there exist numbers $h(N) \in \mathbb{N}$, $h(N) > h(N-1)$ and x, $h(N-1) < x \leq h(N)$ such that the maximal value of the valuation on a preimage of arbitrary element with the value x less than $-N$ (or the preimage is empty). From this follows that one can construct infinitely many elements, which are not in the image of the map T.

For $N = 1$ it's clear — $h(1) = x = i(\bar{\alpha}) - 1$. For arbitrary N consider the vector space $< T(u^l), -N \leq l \leq s(k) >$, where $s(k) = (i(\bar{\alpha})-1) + p(i(\bar{\alpha})-1) + \ldots + p^k(i(\bar{\alpha})-1)$. It's clear that $dim_k < T(u^l), -N \leq l \leq s(k) > \leq (s(k)+N)$. From the other hand side,
$\bar{\nu}(T(u^{p^k(i(\bar{\alpha})-1)})) = 2p^k(i(\bar{\alpha})-1) > s(k)$,
$\bar{\nu}(T(u^{p^k(i(\bar{\alpha})-1)+p^{k-1}(i(\bar{\alpha})-1)})) = 2p^{k-1}(i(\bar{\alpha})-1) + p^k(i(\bar{\alpha})-1) > s(k)$,

...
$\bar{\nu}(T(u^{p^k(i(\bar{\alpha})-1)+\ldots+p(i(\bar{\alpha})-1)})) > s(k)$.

So, our property holds for $k \gg N$. Indeed, assume the converse. Then $< u^l, h(N-1) \le l \le s(k) > \subset < T(u^l), -N \le l \le s(k) >$ for all k, and $s(k) - h(N-1) \le s(k) + N - k$ for all k, a contradiction.

In the case $(i(\bar{\alpha}) - 1, p) \neq 1$ we have: $\bar{\nu}(T(u^l)) \neq \bar{\nu}(T(u^{l_1}))$ if $l \neq l_1$. Therefore, the cokernel of the map T has infinite dimension.

Let now $char k = p$, $q > (i(\bar{\alpha}) - 1)$ and q is finite. Note that T is not injective if and only if $a_0 = \bar{\alpha}(A)/A$. Therefore, if T is not injective, this case is equivalent to the case $q = \infty$.

Let T be injective.

Here two cases are possible:

1) there exists an integer $i \ge \bar{\nu}(a_0 - 1)$ such that for some $s_1, s_2 \in \bar{K}$ $a_0 = s_2\bar{\alpha}(s_1)/s_1$, where $\bar{\nu}(s_2 - 1) = i$ and there are no elements s_1', s_2' such that $a_0 = s_2'\bar{\alpha}(s_1')/s_1'$, where $\bar{\nu}(s_2' - 1) > i$.

2) for any integer i such that $i \ge \bar{\nu}(a_0 - 1)$ there exist $s_1, s_2 \in \bar{K}$ such that $a_0 = s_2\bar{\alpha}(s_1)/s_1$, where $\bar{\nu}(s_2 - 1) \ge i$.

For example, the first case takes the place when $\bar{\nu}(T(a_0 - 1)) < (i(\bar{\alpha}) - 1)$, the second — when $\bar{\nu}(T(a_0 - 1)) > (i(\bar{\alpha}) - 1)$. If $\bar{\nu}(T(a_0 - 1)) = (i(\bar{\alpha}) - 1)$, then may take place either the first or the second case.

Indeed, we have seen that

$$\frac{\bar{\alpha}(cu^{ln})}{cu^{ln}} = 1 + nl\xi^{-1}xu^{i(\bar{\alpha})-1} + \ldots \tag{1.2}$$

$$\frac{\bar{\alpha}(1 + cu^l)}{1 + cu^l} = 1 + c(\bar{\alpha}(u^l) - u^l)(1 + cu^l)^{-1} \tag{1.3}$$

Hence, if $\bar{\nu}(T(a_0-1)) < (i(\bar{\alpha})-1)$ or $\bar{\nu}(T(a_0-1)) = (i(\bar{\alpha})-1)$, but $a_0 = 1 + wu^{i(\bar{\alpha})-1} + \ldots$ and $w \neq nl\xi^{-1}x$ for all l, then 1) holds. If the rest inequalities hold, then one can see that 2) may take place.

Let the case 1) holds. Let's show that T is surjective.

Indeed, this case is equivalent to the property $\bar{\nu}(\bar{\alpha}(y) - a_0 y) \le i + \bar{\nu}(y)$, $y \in \bar{K}$. But this means that
$< T(u^l), N \le l \le N_1 > \subset < u^l, N + min\{i(\bar{\alpha}) - 1, i\} \le l \le N_1 + max\{i(\bar{\alpha}) - 1, i\} >$ for all integers N, N_1, $N < N_1$.

From this follows that the cokernel of the map T cannot have infinite dimension. Suppose it has finite dimension, i.e. it is not equal to zero. Choose an element of the minimal value κ of the valuation in the cokernel and choose a number N_1: $N_1 + i < \kappa$. Complete a basis of the vector space $< T(u^l), N \le l \le N_1 >$ with respect to the basis of the vector space $< u^l, N + min\{i(\bar{\alpha}) - 1, i\} \le l \le N_1 + max\{i(\bar{\alpha}) - 1, i\} >$. Denote

these elements by e_j, $j \in \{1, \dots, |i - i(\bar\alpha) + 1|\}$. Then for any integer $\tilde N_1 > N_1$ we will have
$< T(u^l), e_j, N \le l \le \tilde N_1, j \in \{1, \dots, |i - i(\bar\alpha) + 1|\} >=< u^l, N + min\{i(\bar\alpha) - 1, i\} \le l \le \tilde N_1 + max\{i(\bar\alpha) - 1, i\} >$, but this contradicts to the existence of elements of the cokernel.

Let the case 2) holds. This case is a negation of the case 1); so, for any natural i there exists $N_1 \in \mathbb{N}$ such that $< T(u^l), N \le l \le N_1 > \not\subset < u^l, N + min\{i(\bar\alpha) - 1, i\} \le l \le N_1 + max\{i(\bar\alpha) - 1, i\} >$. Repeating the arguments of the case $a_0 = 1$ we get that there exists a converge sequence in $\bar K$ such that the maximal values of the valuation on preimages of elements of this sequence tend to $-\infty$. From this we get that the cokernel has infinite dimension.

The lemma is proved.

□

Corollary 13 *In the notation of lemma let* $chark = 0$*. Then* $d = 1$ *if and only if* $\bar\alpha$ *has infinite order and* $a_0 = \bar\alpha(x)/x$ *for some* $x \in \bar K$*;* $d = \infty$ *iff* $\bar\alpha$ *has finite order and* $a_0 = \xi^j$*,* $j \in \mathbb{Z}$*;* $d = 0$ *in the rest cases.*

Let $\alpha \in Aut_k(K)$. Then the automorphism $\bar\alpha \in Aut_k(\bar K)$ and its invariants $\xi(\bar\alpha) \in k^*$, $i(\bar\alpha)$, $x(\bar\alpha) \in k^*/(k^*)^{i(\bar\alpha)-1}$, $y(\bar\alpha) \in k$ are defined (see def. and prop. 1.2). Put $a_0 = \overline{\alpha(z)z^{-1}} \in k((u))$. Note that the number $\bar\nu(a_0)$ does not depend on the choice of the parameter z.

Theorem 1.4 *(***Theorem I)** *Let* $chark = 0$*. Let* $\bar\nu(a_0) \ne 0$ *or* $\bar\nu(a_0) = 0$*, but* $\bar a_0 \notin \{\xi(\bar\alpha)^m, m \in \mathbb{Z}\}$*. Then*
1) The automorphism α *is conjugate with an automorphism* β *given by the formula*

$$\beta(u) = \xi u + x u^{i(\bar\alpha)} + x^2 y u^{2i(\bar\alpha)-1}$$

$$\beta(z) = u^{\bar\nu(a_0)} \bar a_0 (1 + a_n u^n + a_{2n} u^{2n} + \dots a_{i(\bar\alpha)-1} u^{i(\bar\alpha)-1}) z$$

where $\xi = \xi(\bar\alpha)$*,* $x = x(\bar\alpha)$*,* $y = y(\bar\alpha)$*,* $a_{nq} \in k$*,* $q \in \{1, \dots, (i(\bar\alpha) - 1)/n\}$*,* $a_{i(\bar\alpha)-1} \notin n\xi(\bar\alpha)^{-1} x(\bar\alpha)\mathbb{Z}'$*,* $\mathbb{Z}' = \mathbb{Z}\backslash\{0\}$*,* $n = ord(\xi(\bar\alpha))$*.*
2) $\bar\nu(a_0)$*,* $\bar a_0$*,* $a_j^{(i(\bar\alpha)-1)}/x(\bar\alpha)^j$*,* $\xi(\bar\alpha)$*,* $x(\bar\alpha)$*,* $y(\bar\alpha)$*,* $i(\bar\alpha)$ *is the complete system of invariants with respect to the conjugation.*

Assume

$$\alpha(u) = c_0 + c_1 z + c_2 z^2 + \dots, \quad c_i \in k((u))$$

$$\alpha(z) = a_0 z + a_1 z^2 + \dots, \quad a_i \in k((u))$$

Let us denote the additional notation:
$i \in \mathbb{N} \cup \{\infty\}$ — such a minimal positive integer that $a_0^i = \bar\alpha(Y)/Y$ for some $Y \in k((u))$,
$j = min_q\{iq : c_{iq} \ne 0\}$, $q \ge 0$,

$i(\alpha) = min_q\{iq+1 : a_{iq} \neq 0\}$,
$\tilde{a}_0 = 1 + a_n u^n + a_{2n} u^{2n} + \ldots + a_{i(\bar{\alpha})-1} u^{i(\bar{\alpha})-1}$,

$\bar{f} \in Aut_k(\bar{K})$ — such an automorphism that $\bar{f}^{-1}\bar{\alpha}\bar{f}(u) = \xi(\bar{\alpha})u + xu^{i(\bar{\alpha})} + yu^{2i(\bar{\alpha})-1}$,
$\tilde{y}_2$ — such a solution of the equation $\bar{\alpha}(Y)/Y = \xi(\bar{\alpha}) + i(\bar{\alpha})xx_0^{i(\bar{\alpha})-1} + (2i(\bar{\alpha})-1)yx_0^{2i(\bar{\alpha})-2}$, $x_0 = \bar{f}(u)$ that $\overline{(\tilde{y}_2 u^{-\bar{\nu}(\tilde{y}_2)})} = 1$,
$\tilde{y}_1$ — such a solution of the equation $\bar{\alpha}(Y)/Y = \tilde{a}_0^i$ that $\overline{(\tilde{y}_1 u^{-\bar{\nu}(\tilde{y}_1)})} = 1$,
$B_1 = \tilde{y}_1^{-(i(\alpha)-1)/i}\tilde{a}_0$,
$B_2 = \tilde{y}_1^{-2(i(\alpha)-1)/i}\tilde{a}_0$,
$A_q = \tilde{y}_1^{-(j+q-1)/i}\bar{f}^{-1}(\bar{\alpha}(\tilde{y}_2))$.
Note that B_1, B_2, A_q are defined uniquely.

Theorem 1.5 (**Theorem II**) *Assume chark $= 0$ and let $\bar{\nu}(a_0) = 0$ and $\bar{a}_0 \in \{\xi(\bar{\alpha})^m, m \in \mathbb{Z}\}$ and $\bar{\alpha}$ be of infinite order.*

Then α is conjugate to β, that is defined according to the next four possible cases:
a) $i(\alpha) - 1 = j$, $i(\alpha) < \infty$, so

$$\beta(u) = \xi(\bar{\alpha})u + xu^{i(\bar{\alpha})} + x^2yu^{2i(\bar{\alpha})-1} + r_1A_1u^{i(\bar{\alpha})-1}z^j$$

$$\beta(z) = \tilde{a}_0z + s_1B_1u^{i(\bar{\alpha})-1}z^{i(\alpha)} + s_2B_2u^{i(\bar{\alpha})-1}z^{2i(\alpha)-1}$$

where $r_1 \in k^*/(k^*)^j$, $s_2 \in k$, $s_1^{j(i(\bar{\alpha})-1)}x^{(i(\bar{\alpha})-2)(i(\alpha)-1)}/r_1^{(i(\ bar\alpha)-1)(i(\alpha)-1)} \in k$.
b) $i(\alpha) - 1 < j$, $i(\alpha) < \infty$

$$\beta(u) = \xi(\bar{\alpha})u + xu^{i(\bar{\alpha})} + x^2yu^{2i(\bar{\alpha})-1}$$

$$\beta(z) = \tilde{a}_0z + s_1B_1u^{i(\bar{\alpha})-1}z^{i(\alpha)} + s_2B_2u^{i(\bar{\alpha})-1}z^{2i(\alpha)-1}$$

where $s_1 \in k^*/k^{*(i(\alpha)-1, i(\bar{\alpha})-1)}$, $s_2 \in k$.
c) $i(\alpha) - 1 > j$, $i(\alpha) < \infty$

$$\beta(u) = \xi(\bar{\alpha})u + xu^{i(\bar{\alpha})} + x^2yu^{2i(\bar{\alpha})-1} + r_1A_1u^{i(\bar{\alpha})-1}z^j + \ldots + r_{i(\alpha)-1-j}A_{i(\alpha)-1-j}u^{i(\bar{\alpha})-1}z^{i(\alpha)-1}$$

$$\beta(z) = \tilde{a}_0z + s_1B_1u^{i(\bar{\alpha})-1}z^{i(\alpha)} + s_2B_2u^{i(\bar{\alpha})-1}z^{2i(\alpha)-1}$$

where $r_1 \in k^*/k^{*j}$, $s_1^{j(i(\bar{\alpha})-1)}x^{(i(\bar{\alpha})-2)(i(\alpha)-1)}/r_1^{(i(\bar{\alpha})-1)(i(\alpha)-1)} \in k$, $r_q, s_2 \in k$, $q \neq 1$.
d) $i(\alpha) = \infty$ ($j \leq \infty$)

$$\beta(u) = \xi(\bar{\alpha})u + xu^{i(\bar{\alpha})} + x^2yu^{2i(\bar{\alpha})-1} + r_1A_1u^{i(\bar{\alpha})-1}z^j$$

$$\beta(z) = \tilde{a}_0z$$

where $r_1 \in k^*/k^{*j}$.

We denote $j(\alpha) := j$ in the cases a), c), d), and $j(\alpha) := \infty$ in the case b). Then $\bar{\nu}(a_0)$, $a_j^{(i(\bar{\alpha})-1)/j}/x(\bar{\alpha})$, $\xi(\bar{\alpha})$, $x(\bar{\alpha})$, $y(\bar{\alpha})$, $i(\bar{\alpha})$, $i(\alpha)$, $j(\alpha)$, i, and the elements r_q, s_1, s_2 with the relations defined in the items a)-d) are complete system of invariants with respect to the conjugation.

Moreover, $i|j(\alpha)$, $i|(i(\alpha)-1)$ (we accept that in the case $i=\infty$ $i|j$ means, that $j=\infty$, $i(\alpha)=\infty$, i.e. there are no elements with z^j, $z^{i(\alpha)}$). $i \neq \infty$ if and only if $\tilde{a}_0 = 1 + a_{i(\bar{\alpha})-1}u^{i(\bar{\alpha})-1}$, where $a_{i(\bar{\alpha})-1} = q\xi^{-1}x$, $q \in$ ".

Let us introduce the additional notation:
$q_a := -\bar{\nu}(c_j) \mod j$, i.e. $0 \geq q_a > -j$;
$q_b := q_a + min\{\bar{\nu}(a_j) - q_a; -1\}$;
in the case of $q_b - q_a = -1$ we denote
$c_b/c_a := \{res_u(c_j/a_j)$ if $a_j \neq 0$,
$c_b/c_a := 1$ otherwise $\}$;

in the case of $c_b/c_a \in$ " we denote
$q_1 := 1$ if $c_b/c_a \in \mathbb{Z}$,
q_1 is a denominator of the fraction $c_b/c_a = p_1/q_1$, where $(p_1, q_1) = 1$, $q_1 > 0$ otherwise;

in the last case let us denote by $p_1 \in \mathbb{Z}'$ the numerator of this fraction;
in the case $q_1 < j(\alpha)$, $q_1 \not| j(\alpha)$ we denote by
$n_1 \in \mathbb{N}$ a number that satisfies the properties $n_1 < q_1$, $q_1|(j(\alpha) - n_1)$;
in the case, when the equation $(x+1)/j = p_1/q_1$ is solvable, we denote by
$i_b \in \mathbb{N}$ such a number that $i_b - q_a + 1$ is a solution of this equation.

Consider the equations

$$0 = -(1+w)n_1^2p_1 + n_1jw - (q-1)(2+w)q_1n_1 + q_1(-2+(1+w)q_a)+$$

$$q_1[p_1(j(q-1)(w+2)+j-(q-1)^2q_1+(q-1)q_1+2q_1)+2j(q_a-1)+(q-1)q_1((1+w)q_a-2)] \quad (1.4)$$

$$(p_1^2(-1+q-q_1(-1+q-2q^2+qq_1))-qq_1^2(q_a-1)+p_1q_1(1-3q-(q-1)qq_1+qq_a)) = 0 \quad (1.5)$$

Theorem 1.6 *(***Theorem III)** *Let $char k = 0$ and let $\bar{\nu}(a_0) = 0$ and $\bar{a}_0 \in \{\xi(\bar{\alpha})^m, m \in \mathbb{Z}\}$ and $\bar{\alpha}^n = Id$.*

Then α is conjugate to β defined in one of the following ways, depending on the possible cases:

O) $i = \infty$. Then

$$\beta(u) = \xi u$$

$$\beta(z) = B_0 z$$

where $B_0 \in k((u^n))$*, that is,* β *has a form of a canonical automorphism of one-dimensional local field* $F((z))$*, where* $F = k((u^n))$ *in the appropriate case.*

O') $i < \infty$. Then $i = 1$ and this case is divided into two ones:

I) $j > i(\alpha) - 1$, i.e. $j(\alpha) = \infty$

$$\beta(u) = \xi u$$

$$\beta(z) = z + B_1 z^{i(\alpha)} + B_1^2 B_2 z^{2i(\alpha)-1}$$

where $B_1, B_2 \in k((u^n))$*,* $B \in k((u^n))^*/k((u^n))^{*(i(\alpha)-1)}$*, i.e.* β *has the form of a canonical automorphism of one-dimensional local field* $F((z))$*, where* $F = k((u^n))$*.*

II) $j \le i(\alpha) - 1$ (and in this case $j = i(\alpha) - 1$). This case has two subsections:

A) $q_b - q_a < -1$. Then

$$\beta(u) = \xi u + r u^{q_a} z^j$$

$$\beta(z) = z + s_1 u^{q_b} z^{j+1} + s_2 B_2 z^{2j+1}$$

where $B_2 = u^{q_b - 1}$ *if* $q_a \neq 0$*, and 0 otherwise,*
the numbers $r = s_1 (c_b/c_a)^{-1} \in k^*/(k^*)^{(j,q_a)}$*,* $s_2, s_1^{j(q_a-1)}/r^{(i(\alpha)-1)q_b} \in k$*.*

B) $q_b - q_a = -1$. This case has two possibilities:

1) $c_b/c_a \notin$ ”. Then

$$\beta(u) = \xi u + r u^{q_a} z^j$$

$$\beta(z) = z + s_1 u^{q_a - 1} z^{j+1}$$

where numbers $r \in k^*/(k^*)^{(j,q_a)}$*,* $s_1^{j(q_a-1)}/r^{(i(\alpha)-1)q_b} \in k$*.*

2) $c_b/c_a \in$ ”. This case has three possibilities:

a) $q_1 = j$. Then

$$\beta(u) = \xi u + r u^{q_a} z^j$$

$$\beta(z) = z + s_1 B_1 z^{j+1} + s_2 u^{-p_1 - 1 + q_a} z^{2j+1} + s_3 u^{-p_1 + 2q_a - 2} z^{3j+1}$$

where $B_1 = u^{q_a - 1}$ *if* $(x + 1 - q_a)/j \neq p_1/q_1$ *for all* $x \in \mathbb{N}$*,*
and $B_1 = u^{q_a-1} + r_{i_b} u^{i_b}$*,* $i_b > q_a - 1$ *otherwise,*
$s_2, s_3, r_{i_b} \in k$*,* $r \in k^*/(k^*)^{(j,q_a)}$*,* $s_1^{j(q_a-1)}/r^{(i(\alpha)-1)q_b} \in k$*.*

b) $q_1 < j$. *Here we have two sub-cases:*

i) $q_1 | j$. *Then*

$$\beta(u) = \xi u + r u^{q_a} z^j$$

$$\beta(z) = z + s_1 B_1 z^{j+1} + (s_{2,1} u^{-p_1 q_1^{-1}(q_1 - j) + 2q_a - 2} + s_{2,2} u^{-p_1 - 1 + q_a}) z^{j+1+q_1} + s_3 u^{-2p_1 - 1 + q_a} z^{j+1+2q_1} + s_{j/q_1} u^{-p_1 q_1^{-1}((j/q_1+1)q_1 - j) + 2q_a - 2} z^{j+1+(1+j/q_1)q_1} + s_{q_2} B_{q_2} z^{j+1+q_1(1+q_2)}$$

where $B_1 = u^{q_a - 1}$ *if* $(x + 1 - q_a)/j \neq p_1/q_1$ *for all* $x \in \mathbb{N}$ *and*
$B_1 = u^{q_a-1} + r_{i_b} u^{i_b}$, $i_b > q_a - 1$ *otherwise,*
$B_{q_2} = u^{-p_1(1+q_2)-1+q_a}$ *if* $-q_a + 1 - (q_2 - 1)p_1 = 0$, *and 0 otherwise,*
$s_2, s_3, s_{j/q_1}, s_{q_2} \in k$, $r \in k^*/(k^*)^{(j,q_a)}$, $s_1^{j(q_a-1)}/r^{(i(\alpha)-1)q_b} \in k$.

ii) $q_1 \not| j$. *Then we again have two cases:*

i') $n_1 | q_1$ *and* $-q_1 n_1^{-1}(p_1 q_1^{-1}(n_1 - j) + q_a) = -p_1 + 1$. *Then*

$$\beta(u) = \xi u + r u^{q_a} z^j$$

$$\beta(z) = z + s_1 u^{q_a-1} z^{j+1} + s_2 u^{-p_1 q_1^{-1}(n_1 - j) + 2q_a - 2} z^{j+1+n_1} + s_q B_q z^{j+1+lq_1},$$

where $B_q = u^{-p_1 q - 1 + q_a}$ *if (1.4) is fulfilled and 0 otherwise,*
l *is a solution of an equation (1.4);*
$s_2, s_q \in k$, $r \in k^*/(k^*)^{(j,q_a)}$, $s_1^{j(q_a-1)}/r^{(i(\alpha)-1)q_b} \in k$.

ii') $n_1 \not| q_1$ *or* $-q_1 n_1^{-1}(p_1 q_1^{-1}(n_1 - j) + q_a) \neq -p_1 + 1$. *Then*

$$\beta(u) = \xi u + r u^{q_a} z^j$$

$$\beta(z) = z + s_1 u^{q_a-1} z^{j+1} + s_2 u^{-p_1 q_1^{-1}(n_1 - j) + 2q_a - 2} z^{j+1+n_1} + s_3 u^{-p_1 - 1 + q_a} z^{j+1+q_1} + s_4 u^{-p_1 - 1 + q_a} z^{j+1+2q_1} + s_q B_q z^{j+1+lq_1}$$

where $B_q = u^{-p_1 q - 1 + q_a}$ *if* $-q_a + 1 - (q - 1)p_1 = 0$ *and 0 otherwise,*
l *is a solution of the equation* $-q_a + 1 - (q - 1)p_1 = 0$,
$s_2, s_q, s_3, s_4 \in k$, $r \in k^*/(k^*)^{(j,q_a)}$, $s_1^{j(q_a-1)}/r^{(i(\alpha)-1)q_b} \in k$.

c) $q_1 > j$. *There are two possibilities:*

i) $j \not| q_1$. *Then*

$$\beta(u) = \xi u + r u^{q_a} z^j$$

$$\beta(z) = z + s_1 u^{q_a-1} z^{j+1} + s_2 u^{-p_1 - 1 + q_a} z^{j+1+q_1} + s_3 u^{-p_1 - 2 + 2q_a} z^{2j+1+q_1} + s_4 u^{-2p_1 - 1 + q_a} z^{j+1+2q_1} + s_q B_q z^{j+1+lq_1}$$

where $B_q = u^{-p_1 q-1+q_a}$ *if* $-q_a+1-(q-1)p_1 = 0$, *and 0 otherwise,*
l *is a solution of the equation* $-q_a+1-(q-1)p_1 = 0$,
$s_2, s_q, s_3, s_4 \in k$, $r \in k^*/(k^*)^{(j,q_a)}$, $s_1^{j(q_a-1)}/r^{(i(\alpha)-1)q_b} \in k$.

ii) $j|q_1$. *It also has two possible cases:*

i') $-p_1+q_a-2 \neq q_1(q_a-1)/j$. *Then*

$$\beta(u) = \xi u + r u^{q_a} z^j$$

$$\beta(z) = z + s_1 u^{q_a-1} z^{j+1} + s_2 u^{-p_1-1+q_a} z^{j+1+q_1} + s_3 u^{-p_1-2+2q_a} z^{2j+1+q_1} + s_4 u^{-2p_1-1+q_a} z^{j+1+2q_1} + s_q B_q z^{j+1+lq_1}$$

where $B_q = u^{-p_1 q-1+q_a}$ *if* $-q_a+1-(q-1)p_1 = 0$, *and 0 otherwise,*
l *is a solution of an equation* $-q_a+1-(q-1)p_1 = 0$,
$s_2, s_q, s_3, s_4 \in k$, $r \in k^*/(k^*)^{(j,q_a)}$, $s_1^{j(q_a-1)}/r^{(i(\alpha)-1)q_b} \in k$.

ii') $-p_1+q_a-2 = q_1(q_a-1)/j$. *Then*

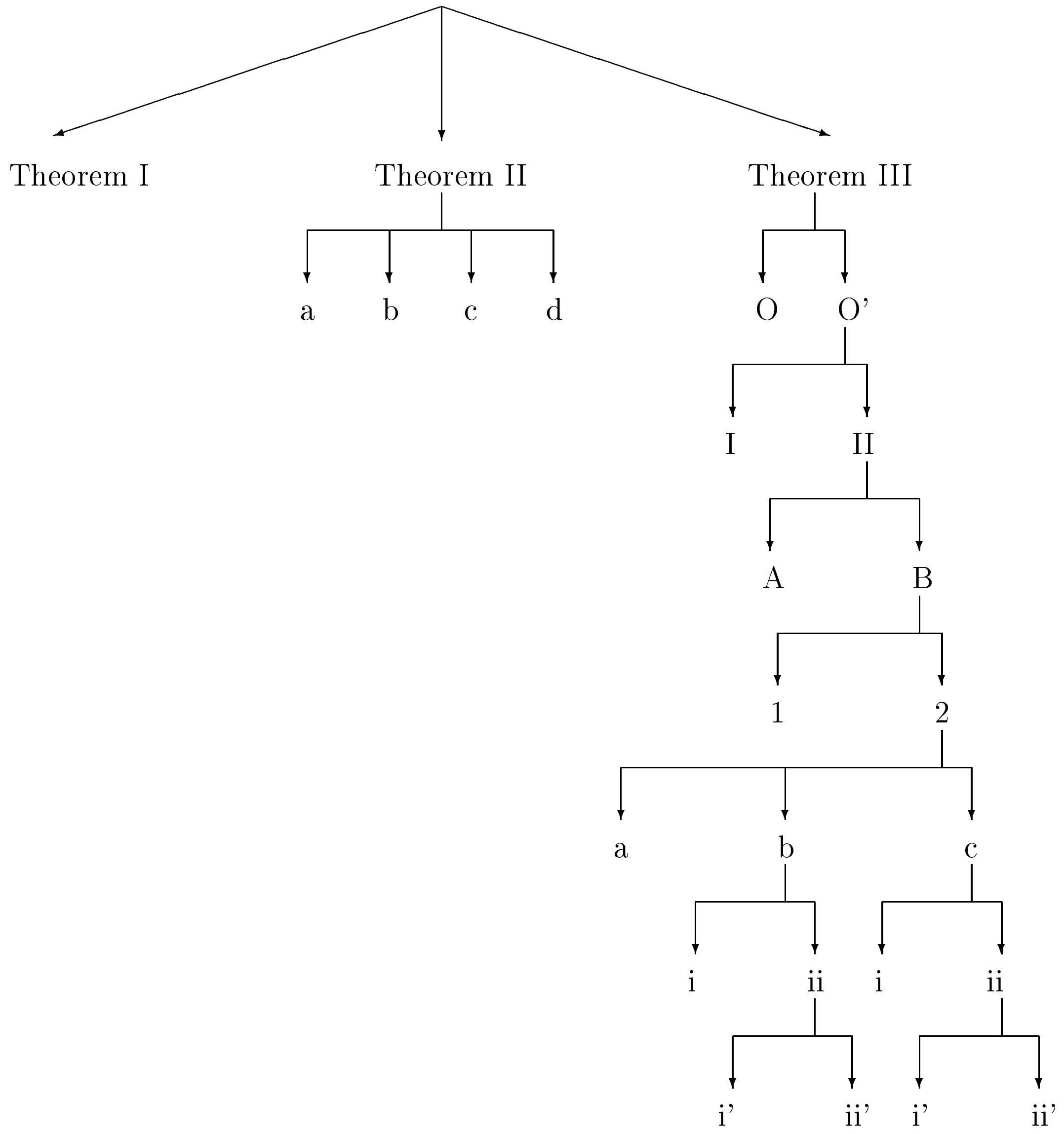
$$\beta(u) = \xi u + r u^{q_a} z^j$$

$$\beta(z) = z + s_1 u^{q_a-1} z^{j+1} + s_2 u^{-p_1-1+q_a} z^{j+1+q_1} + s_{q_n,1} B_{q_n,1} z^{2j+1+l_1 q_1} + s_{q_n,2} B_{q_n,2} z^{2j+1+l_2 q_1} + s_{q_m,1} B_{q_m,1} z^{j+1+l'_1 q_1} + \ldots + s_{q_m,w} B_{q_m,w} z^{j+1+l'_w q_1}$$

where $B_{q_n,i} = u^{-p_1 l_i-2+2q_a}$ *if (1.5) is satisfied, and 0 otherwise,*
$B_{q_m,j} = u^{-p_1 l'_j-1+q_a}$ *if* l'_q *are defined, and 0 otherwise,*
l_1, l_2 *are the solutions of equation (1.5),* $l'_1, \ldots, l'_w$ *are solutions of some equation of degree* $w = q_1/j$,
$s_2, s_{q_n,i}, s_{q_m,j} \in k$, $r \in k^*/(k^*)^{(j,q_a)}$, $s_1^{j(q_a-1)}/r^{(i(\alpha)-1)q_b} \in k$.

$\bar{\nu}(a_0)$, $\xi(\bar{\alpha})$, $i(\bar{\alpha})$, $i(\alpha)$, $j(\alpha)$, i, q_a, q_b, q_1, n_1, i_b, *and those elements with relations that were defined in all items are the complete system of invariants with respect to the conjugation.*

Classification of conjugacy classes



Corollary 14 *For given $\bar{\nu}(a_0)$, n, $i(\bar{\alpha})$, $i(\alpha)$, $j(\alpha)$, i, q_a, q_b, q_1, n_1, i_b, the set of conjugacy classes of the automorphism α is parametrised by only finite number of parameters, except the cases Th.III O), O')(I), Th.I $i(\bar{\alpha}) = \infty$.*

Proof of theorems (and of corollary) Recall that $\bar{\alpha}$ is an automorphism on the field $\bar{K}$, $\bar{\alpha} = \alpha \mod \wp$.

It is clear that if two automorphisms α, β are conjugate, then the automorphisms $\bar{\alpha}, \bar{\beta}$ are conjugate in the group $Aut_k(\bar{K})$. To prove the theorem we must prove the existence of an automorphism f such that $\alpha = f\beta f^{-1}$ and β is an automorphism, as defined in the formulation of the theorem. Thereto it would be also proved, that the automorphism β can be uniquely reconstructed by the automorphism α and any β-like automorphism gives its own conjugacy class.

Assume

$$f(u) = x_0 + x_1 z + x_2 z^2 + \ldots$$

$$f(z) = y_0 z + y_1 z^2 + \ldots$$

We choose the parameter $x_0 \in \bar{K}$ in such a way that $\bar{\alpha}(x_0)$ has a canonical form, that is $\bar{\alpha}(x_0) = \xi x_0 + x x_0^{i(\bar{\alpha})} + y x_0^{2i(\bar{\alpha})-1}$. Recall that x is a representative of a class $k^*/k^{*(i(\bar{\alpha})-1)}$.

1.2 Proof of the theorems I and II

Let a_0 fulfil the assumptions of the theorem. We prove, that there exists an automorphism f such that $\alpha f(u) = f\beta(u)$; $\alpha f(z) = f\beta(z)$, where β is an automorphism, as defined in the theorem. To do that, we prove by induction that $\alpha f(u) = f\beta(u) \mod \wp^m$ and $\alpha f(z) = f\beta(z) \mod \wp^{m+1}$ for all $m \in \mathbb{N}$.

From (1.2), (1.1), (1.3) (which remain true also in the case of finite order automorphism $\bar{\alpha}$) follows, that the set of representatives of classes of the elements $\bar{f}^{-1}(a_0\bar{a_0}^{-1}\bar{\alpha}(y_0)/y_0)$ can be described as the set of the elements $\{u^{\bar{\nu}(a_0)}(1 + a_n u^n + a_{2n}u^{2n} + \ldots a_{i(\bar{\alpha})-1}u^{i(\bar{\alpha})-1}), a_{nq} \in k, a_{i(\bar{\alpha})-1} \neq nl\xi^{-1}x$, where ξ is a primitive root from 1 of a degree n, $l \in \mathbb{Z}\backslash\{0\}\}$. From the definition of the element a_0 follows that the elements a_{nq} are uniquely defined by automorphism α, that is, they don't depend on the choice of parameter z, and $\bar{a_0}$ is defined up to multiplication by an element ξ^m, $m \in \mathbb{Z}$.

Assume $\tilde{\bar{a}}_0 = \bar{a_0}u^{\bar{\nu}(a_0)}(1+a_n u^n + a_{2n}u^{2n} + \ldots a_{i(\bar{\alpha})-1}u^{i(\bar{\alpha})-1})$. Then we have for $m = 1$ that

$\overline{\alpha f(u)} = \overline{\alpha(x_0)} = \overline{\alpha}(x_0) = \xi x_0 + x x_0^{i(\bar{\alpha})} + \ldots = \overline{f\beta(u)}$; $\alpha f(z) = \alpha(y_0)\alpha(z) = \bar{\alpha}(y_0)a_0 z = f(\tilde{\bar{a}}_0)y_0 z = f(\tilde{\bar{a}}_0)f(z) = f\beta(z) \mod \wp^2$.

For an arbitrary m we replace α by $f_{m-2}^{-1}\alpha f_{m-2}$ for a suitable automorphism f_{m-2} (that is, for any automorphism with suitable coefficients $x_0, \ldots, x_{m-2}$, $y_0, \ldots, y_{m-2}$), and now can consider that $c_0 = \xi u + x u^{i(\bar{\alpha})} + \ldots$, $c_1 = \ldots = c_{m-2} = 0$, $a_0 = \tilde{\bar{a}}_0$, $a_1 = \ldots = a_{m-2} = 0$, $x_0 = u$, $x_1 = \ldots = x_{m-2} = 0$, $y_0 = 1$, $y_1 = \ldots = y_{m-2} = 0$. Then

$$\alpha f(u) = \alpha(u) + \alpha(x_{m-1})\alpha(z^{m-1}) = \xi u + x u^{i(\bar{\alpha})} + \ldots + c_{m-1}z^{m-1} + \bar{\alpha}(x_{m-1})\tilde{a}_0^{m-1}z^{m-1} \mod \wp^m$$

$$f\beta(u) = \xi(u + x_{m-1}z^{m-1}) + x(u + x_{m-1}z^{m-1})^{i(\bar{\alpha})} + \ldots = \xi u + x^{i(\bar{\alpha})}u + \ldots + \xi x_{m-1}z^{m-1} +$$

$$i(\bar{\alpha})x x_{m-1}u^{i(\bar{\alpha})-1}z^{m-1} + \ldots =$$

$$\xi u + x^{i(\bar{\alpha})}u + \ldots + x_{m-1}(\frac{\partial}{\partial(u)}(\bar{\alpha}(u)))z^{m-1} \mod \wp^m$$

Hence,

$$\bar{\alpha}(x_{m-1})\tilde{a}_0^{m-1}+c_{m-1}=x_{m-1}(\frac{\partial}{\partial(u)}(\bar{\alpha}(u))). \tag{1.6}$$

And in the same way,

$$\alpha f(z)=\alpha(z)+\alpha(y_{m-1})\alpha(z^m)=\tilde{a}_0z+a_{m-1}z^m+(\bar{\alpha}(y_{m-1})+\ldots)(\tilde{a}_0z+\ldots)^m \mod \wp^{m+1}$$

$$f\beta(z)=f(\tilde{a}_0)f(z)=(\tilde{a}_0+\frac{\partial}{\partial(u)}(\tilde{a}_0)x_{m-1}z^{m-1})(z+y_{m-1}z^m)=$$

$$\tilde{a}_0z+\frac{\partial}{\partial(u)}(\tilde{a}_0)x_{m-1}z^m+\tilde{a}_0y_{m-1}z^m \mod \wp^{m+1}$$

Hence,

$$\bar{\alpha}(y_{m-1})\tilde{a}_0^m+a_{m-1}=\tilde{a}_0y_{m-1}+\frac{\partial}{\partial(u)}(\tilde{a}_0)x_{m-1}. \tag{1.7}$$

By Corollary 12, if the conditions of the theorem are fulfilled, the equations (1.6), (1.7) have the unique solution with any c_{m-1}, a_{m-1} and with any m, whence follows the proof of the case 1). By Proposition 1.2 the proof of the case 2) is evident.

Proof of the Theorem II

If $i=\infty$, we can apply entirely the same arguments as in the Theorem I, and get that α is conjugate to the automorphism β, where β has the same form as in the Theorem I (i.e. this case corresponds to the case d), when $j=\infty$). In order that these arguments remain true, we must only show that the element $a:=\xi+i(\alpha)xx_0^{i_{\bar{\alpha}}-1}+\ldots$ in (1.6) can be represented in the form $\bar{\alpha}(y)/y$. But it follows directly from the relations (1.2), (1.3), (1.1).

Let $i<\infty$. We prove that there exists such an automorphism f that $\alpha f(u)=f\beta(u)$, $\alpha f(z)=f\beta(z)$, where automorphism β is as defined in the theorem. The proof is the same as in the Theorem I.

The case $m=1$ coincides with the case $m=1$ from the Theorem I. Applying the same arguments as there, we get equations of the form (1.6) and (1.7). By Corollary 2, these equations are solvable if $i \not| (m-1)$. They may be unsolvable if $i|(m-1)$. Since $char k=0$, the kernel and the cokernel of the maps

$$T_{m-1,1}=\bar{\alpha}(x_{m-1})\tilde{a}_0^{m-1}-(\xi+xu^{i(\bar{\alpha})-1}+\ldots)x_{m-1}, \quad T_{m-1,2}=\bar{\alpha}(y_{m-1})\tilde{a}_0^m-\tilde{a}_0y_{m-1}$$

are one-dimensional if $i|(m-1)$.

We put $x'_k=\tilde{y}_1^{k/i}\tilde{y}_2^{-1}x_k$, $y'_k=\tilde{y}_1^{k/i}y_k$ for $k=iq$, $q\in\mathbb{N}$. Then

$$\bar{\alpha}(x_k)\tilde{a}_0^k-(\xi+xu^{i(\bar{\alpha})-1}+\ldots)x_k=\bar{\alpha}(\tilde{y}_1^{-k/i}\tilde{y}_2x'_k)\frac{\bar{\alpha}(\tilde{y}_1^{k/i})}{\tilde{y}_1^{k/i}}-\frac{\bar{\alpha}(\tilde{y}_2)}{\tilde{y}_2}x'_k\tilde{y}_1^{-k/i}=\bar{\alpha}(\tilde{y}_2)\tilde{y}_1^{-k/i}(\bar{\alpha}(x'_k)-x'_k),$$

$$\bar\alpha(y_k)\tilde a_0^{k+1}-\tilde a_0 y_k=\bar\alpha(\tilde y_1^{-k/i}y'_k)\frac{\bar\alpha(\tilde y_1^{k/i})}{\tilde y_1^{k/i}}\tilde a_0-\tilde a_0 y'_k\tilde y_1^{-k/i}=\tilde a_0\tilde y_1^{-k/i}(\bar\alpha(y'_k)-y'_k)$$

Now we can write down the kernel and cokernel of these maps in the explicit form. For $T_{k,1}$ the kernel is $\tilde y_1^{-k/i}\tilde y_2(x'_k)_0$, where $(x'_k)_0\in k$, cokernel — $cu^{i(\bar\alpha)-1}\tilde y_1^{-k/i}\bar\alpha(\tilde y_2)$, $c\in k$; in the same way, for $T_{k,2}$ the kernel is $\tilde y_1^{-k/i}(y'_k)_0$, where $(y'_k)_0\in k$, cokernel — $c_1u^{i(\bar\alpha)-1}\tilde y_1^{-k/i}\tilde a_0\bar\alpha$.

Step 1 We show that α is conjugate to an automorphism α', which has all the coefficients c'_q and a'_q, $q\geq 1$, satisfying the property:

$$\text{if } i\not| q,\text{ then } a'_q=c'_q=0;\text{ if } i|q,\text{ then } \bar\nu(c'_q\tilde y_1^{q/i}\bar\alpha(\tilde y_2^{-1}))\geq i(\bar\alpha)-1,\ \bar\nu(a'_q\tilde y_1^{q/i}\tilde a_0^{-1})\geq i(\bar\alpha)-1 \tag{1.8}$$

We even show that in (1.8) we have either equalities or $c_q=0$ $(a_q=0)$.

In fact, let α' be such an automorphism that $c'_0=c_0=\xi u+xu^{i(\bar\alpha)}+\ldots$, $a'_0=\tilde a_0$. Let us find the rest coefficients satisfying these properties. Applying induction on m, we have for arbitrary m that

$$\alpha f(u)=\alpha(u)+\alpha(x_{m-1})\alpha(z^{m-1})=$$

$$\bar\alpha(u)+c'_1z+\ldots c'_{m-1}z^{m-2}+c_{m-1}z^{m-1}+(\bar\alpha(x_{m-1})+\ldots)(\tilde a_0z+\ldots)^{m-1}\mod \wp^m$$

$$f\alpha'(u)=f(\bar\alpha(u))+f(c'_1)f(z)+\ldots+f(c'_{m-2})f(z^{m-2})+f(c'_{m-1})f(z^{m-1})=$$

$$\bar\alpha(u)+\frac{\partial}{\partial u}(\bar\alpha(u))x_{m-1}z^{m-1}+c'_1z+\ldots+c'_{m-1}z^{m-1}\mod \wp^m$$

Hence,

$$\bar\alpha(x_{m-1})\tilde a_0^{m-1}+c_{m-1}=x_{m-1}(\frac{\partial}{\partial u}(\bar\alpha(u)))+c'_{m-1} \tag{1.9}$$

If $i\not|(m-1)$, then $c'_{m-1}=0$ and by Corollary 2 the solution of this equation exists and is unique. If $i|(m-1)$, then for solvability of this equation it is enough to select c'_{m-1} in the form $cu^{i(\bar\alpha)-1}\tilde y_1^{-(m-1)/i}\bar\alpha(\tilde y_2)$, i.e. $\bar\nu(c'_{m-1}\tilde y_1^{(m-1)/i}\bar\alpha(\tilde y_2^{-1}))\geq i(\bar\alpha)-1$.

Further,

$$\alpha f(z)=\alpha(z)+\alpha(y_{m-1})\alpha(z^m)=$$

$$\tilde a_0z+a'_1z^2+\ldots+a'_{m-2}z^{m-1}+a_{m-1}z^m+(\bar\alpha(y_{m-1})+\ldots)(\tilde a_0^mz^m+\ldots)\mod \wp^{m+1}$$

$$f\alpha'(z)=f(\tilde a_0)f(z)+f(a'_1)f(z^2)+\ldots+f(a'_{m-1})f(z^m)=$$

$$\tilde a_0z+\frac{\partial}{\partial u}(\tilde a_0)x_{m-1}z^m+\tilde a_0y_{m-1}z^m+(a'_1+\frac{\partial}{\partial u}(a'_1)x_{m-1}z^{m-1})(z+y_{m-1}z^m)^2+\ldots$$

$$+(a'_{m-1}+\frac{\partial}{\partial u}(a'_{m-1})x_{m-1}z^{m-1})(z+y_{m-1}z^m)^m=\tilde a_0z+\frac{\partial}{\partial u}(\tilde a_0)x_{m-1}z^m+\tilde a_0y_{m-1}z^m+a'_1z^2+\ldots$$

$$+a'_{m-1}z^m\mod \wp^{m+1}$$

Hence

$$\bar{\alpha}(y_{m-1})\tilde{a}_0^m + a_{m-1} = \tilde{a}_0 y_{m-1} + \frac{\partial}{\partial u}(\tilde{a}_0)x_{m-1} + a'_{m-1}, \tag{1.10}$$

and in the same to the previous case way we get the desired result.

Step 2 Here two cases are possible:
1) $j \geq i(\alpha)$;
2) $j < i(\alpha)$.

Case 1). We show that $\alpha = f^{-1}\beta' f$, where $\beta'(u) = \bar{\alpha}(u)$.

To do that we find the sequential conjugations $\alpha \mapsto \alpha' = f_{mi}^{-1}\alpha f_{mi}$, where $f_{mi}(u) = u + x_{mi}z^{mi}$, $f_{mi}(z) = z$, $m \geq 1$, $x_{mi} = \tilde{y}_1^{-m}\tilde{y}_2(x'_{mi})_0$. If $m = (j - i(\alpha) + 1)/i$, we have for the coefficients c'_q that:

$$\alpha f(u) = \alpha(u) + \alpha(x_{im})\alpha(z^{im}) = \bar{\alpha}(u) + c_j z^j + \ldots +$$

$$(\bar{\alpha}(x_{im}) + \bar{\alpha}(\frac{\partial}{\partial u}x_{im})z^j + \ldots)(\tilde{a}_0 z + a_{i(\alpha)-1}z^{i(\alpha)} + \ldots)^{im} =$$

$$\bar{\alpha}(u) + c_j z^j + \bar{\alpha}(x_{im})\tilde{a}_0^{im}z^{im} + \bar{\alpha}(x_{im})\tilde{a}_0^{im-1}a_{i(\alpha)-1}z^j \mod \wp^{j+1}$$

$$f\alpha'(u) = f(\bar{\alpha}(u)) + f(c'_{(m+1)i}z^{(m+1)i}) + \ldots = \bar{\alpha}(u) + x_{im}\frac{\partial}{\partial u}(\bar{\alpha}(u))z^{im} + \ldots + f(c'_{(m+1)i}z^{(m+1)i}) + \ldots$$

$$+ f(c'_j z^j) \mod \wp^{j+1} \tag{1.11}$$

Since $x_{mi} = \tilde{y}_1^{-m}\tilde{y}_2(x'_{mi})_0$, the equation at z^{mi} has the form

$$\bar{\alpha}(x_{mi})\tilde{a}_0^{im} - x_{mi}(\frac{\partial}{\partial u}(\bar{\alpha}(u))) = 0 \tag{1.12}$$

We show that all the coefficients c'_q in (1.11) can be chosen so that $\bar{\nu}(\tilde{y}_1^{q/i}\bar{\alpha}(\tilde{y}_2^{-1})c'_q) > i(\bar{\alpha}) - 1$.

In order to do that if $q < j$, we prove, applying induction on q/i, that all the coefficients at z in degrees higher than im in $f(\bar{\alpha}(u))$, $f(c'_{li}z^{li})$ satisfy this property, supposing that c'_{li} satisfies this property at $l < q/i$.

For $f(u^l)$, $l > 1$ we have by Newton's binomial formula that

$$f(u^l) = u^l + \sum_{k=1}^{l} u^{l-k}x_{im}^k z^{imk}C_l^k,$$

whence
$\bar{\nu}(u^{l-k}x_{im}^k\tilde{y}_1^{mk}\bar{\alpha}(\tilde{y}_2^{-1})) = l-k+(k+1)\bar{\nu}(\tilde{y}_2) = l-k+(k-1)i(\bar{\alpha}) = l-k+(k-1)(i(\bar{\alpha})-1) > (i(\bar{\alpha}) - 1)$ for $k > 1$,
what proves our assertion for $f(\bar{\alpha}(u))$. For $f(c'_{li}z^{li})$ we have $f(c'_{li}z^{li}) = f(c'_{li})z^{li}$, and, using Newton's binomial formula again, we get

$l > i(\bar\alpha)-1-\bar\nu(\tilde y_1^l)+\bar\nu(\tilde y_2) = i(\bar\alpha)-1-\bar\nu(\tilde y_1^l)+i(\bar\alpha)$, where from $l-1+(k-1)(i(\bar\alpha)-1) > i(\bar\alpha)-1-\bar\nu(\tilde y_1^l)$ for all k, what proves our assertion in this case also.

At z^j we have the equation

$$\bar\alpha(x_{im})\tilde a_0^{im-1}a_{i(\alpha)-1} + c_j = c'_j, \tag{1.13}$$

and we must only solve the equation

$$\tilde y_1^{j/i}\bar\alpha(\tilde y_2^{-1}c_j + \tilde y_1^{j/i}\bar\alpha(\tilde y_2^{-1})a_{i(\alpha)-1}\tilde a_0^{im-1}\bar\alpha(x_{im}) = 0 \mod \wp^{i(\bar\alpha)} \tag{1.14}$$

in order to finish the induction step for the coefficients c'_q. We have:

$$\tilde y_1^{j/i}\bar\alpha(\tilde y_2^{-1})c_j + \tilde y_1^{j/i}\bar\alpha(\tilde y_2^{-1})a_{i(\alpha)-1}\tilde a_0^{im-1}\bar\alpha(x_{im}) =$$

$$\tilde y_1^{j/i}\bar\alpha(\tilde y_2^{-1}c_j + \tilde y_1^{j/i}\bar\alpha(\tilde y_2^{-1})\bar\alpha(\tilde y_1^{-m})\bar\alpha(\tilde y_2)(x'_{mi})_0 a_{i(\alpha)-1} =$$
$$\tilde y_1^{j/i}\bar\alpha(\tilde y_2^{-1})c_j + \tilde y_1^{(i(\alpha)-1)/i}a_{i(\alpha)-1}(x'_{mi})_0 \mod \wp^{i(\bar\alpha)}$$

Since $\bar\nu(\tilde y_1^{(i(\alpha)-1)/i}a_{i(\alpha)-1}) = i(\bar\alpha)-1 = \bar\nu(\tilde y_1^{j/i}\bar\alpha(\tilde y_2^{-1}c_j)$, there exists a unique constant $(x'_{mi})_0$, with which the equation (1.14) is solvable.

Let us show that the coefficients c'_q, $q > j$, a'_q, $q \geq 1$ satisfy the properties (1.8).

As for coefficients c'_q, it is remained to prove, that the coefficients at z^d in $\alpha(x_{im})\alpha(z^{im})$ for $d > j$ satisfy (1.8) . It's clear that (1.8) remains true if $i \not| q$. But if $i|q$, then $\alpha(z^{im}) = z^{im}D$, where D is a series with coefficients of the same behaviour as a_q. It follows from the Newton's binomial formula. Applying the same arguments as for $f(c'_{li})z^{li}$, we get that (1.8) holds for $\alpha(x_{im})z^{im}$, where from (1.8) also holds for the product $\alpha(x_{im})z^{im}D$, because (1.8) holds for each series $\alpha(x_{im})z^{im}d_qz^q$, where $D = \sum_{q\geq 0, i|q} d_qz^q$.

For the coefficients a'_q we have

$$\alpha f(z) = \alpha(z)$$

$$f\alpha'(z) = f(\tilde a_0)z + f(a'_{i(\alpha)-1})z^{i(\alpha)} + \ldots \tag{1.15}$$

where from, using calculations for $f(u^l)$, we get that (1.8) holds for a'_q. Therefore, since $\tilde a_0 = 1 \mod \wp^{i(\bar\alpha)-1}$, we have $\bar\nu(a'_q\tilde y_1^{q/i}\tilde a_0^{-1}) > i(\bar\alpha)-1$ for all $q < i(\alpha)-1$.

To complete the induction, let's show that $\alpha' = f^{-1}\alpha'' f$, where the coefficients c''_q, a''_q of the automorphism α'' satisfy (1.8) and $c''_q = 0$, $1 \leq q \leq j$, $a''_q = 0$, $1 \leq q < i(\alpha)-1$. The proof is again by induction on m ($\wp^m$). Let's use the calculations before the formulas (1.9) and (1.10). The equation (1.9) is solvable with $c'_{m-1} = 0$, $\bar\nu(c_{m-1}\tilde y_1^{(m-1)/i}\bar\alpha(\tilde y_2^{-1})) > i(\bar\alpha)-1$, $m-1 < i(\alpha)-1$, and to complete the proof we only have to check that the properties of the coefficients c'_q, a'_q remain true by the conjugation $f_{m-1}^{-1}\alpha' f_{m-1}$. But this follows from the same arguments as in the case, when $f_{m-1}(u) = u + x_{m-1}z^{m-1}$, $f_{m-1}(z) = z$, $\bar{\nu}(x_{m-1}\tilde{y}_1^{(m-1)/i}\tilde{y}_2^{-1}) = 0$, because we used only the inequation $\bar{\nu}(x_{m-1}\tilde{y}_1^{(m-1)/i}\tilde{y}_2^{-1}) \geq 0$, which is true also in our case, because x_{m-1} is a solution of an equation of the type (1.9). In the same way one can prove this fact in the case $f_{m-1}(u) = u$, $f_{m-1}(z) = z + y_{m-1}z^m$, $\bar{\nu}(y_{m-1}\tilde{y}_1^{(m-1)/i}) \geq 0$. From the other hand side, one can see from the equation (1.9) that the conjugation $f_{m-1}(u) = u$, $f_{m-1}(z) = z+y_{m-1}z^m$ does not change the coefficient c'_{m-1}, so any conjugation f_{m-1} can be decomposed into composition of two conjugations f'_{m-1}, f''_{m-1} such that $f'_{m-1}(z) = z$, $f''_{m-1}(u) = u$.

Thus, we have proved that α is conjugate to α'', where $j'' > j$, $i''_\alpha = i(\alpha)$. Since our arguments do not depend on j, we get the required assertion by induction.

In the same way with Proposition 1.2 it is proved now that $\alpha = f^{-1}\beta f$, where $\beta(u) = \bar{\alpha}(u)$, $\beta(z) = \tilde{a}_0 z + a_{i(\alpha)-1}z^{i(\alpha)} + a_{2(i(\alpha)-1)}z^{2i(\alpha)-1}$, where $\bar{\nu}(a_{i(\alpha)-1}\tilde{y}_1^{(i(\alpha)-1)/i}) = i(\bar{\alpha}) - 1$, $\bar{\nu}(a_{2(i(\alpha)-1)}\tilde{y}_1^{2(i(\alpha)-1)/i}) = i(\bar{\alpha}) - 1$ (i.e it is the case b) of the theorem).

Case 2). This case is divided into two ones:
a) $j = i(\alpha) - 1$,
b) $j < i(\alpha) - 1$.

Let us look first at the case a). We show that $\alpha = f\beta f^{-1}$, where β is defined in the case a) of the theorem.

To do that we make sequential substitutions $\alpha \mapsto \alpha' = f_{mi}^{-1}\alpha f_{mi}$, $f_{mi}(u) = u + x_{mi}z^{mi}$, $f_{mi}(z) = z + y_{mi}z^{mi+1}$, where $x_{mi} = \tilde{y}_1^{-m}\tilde{y}_2(x'_{mi})_0$, $y_{mi} = \tilde{y}_1^{-m}(y'_{mi})_0$. It is enough to show that for every m a corresponding automorphism α' has coefficients c'_q, a'_q, which satisfy the property:
$\bar{\nu}(\tilde{y}_1^{q/i}\bar{\alpha}(\tilde{y}_2^{-1})c_q) > i_{\bar{\alpha}} - 1$, $\bar{\nu}(\tilde{y}_1^{q/i}a'_q) > i(\bar{\alpha}) - 1$, $i|q$, $im \leq q \leq mi + j$, $q \neq j, 2j$, $c'_q = a'_q = 0$, $q < im$,
because then α' can be reduced to the case, when the appropriate coefficients c'_q, a'_q are equal to zero. That is done using the same substitutions, as by deriving equations (1.9), (1.10), and with the help of result from the case 1). Since for every m the number of necessary conjugations is finite, the desired automorphism f: $\alpha = f\beta f^{-1}$ exists.

Let us write down the calculations for an arbitrary m:

$$\alpha f(z) = \alpha(z)+\alpha(y_{mi})\alpha(z^{mi+1}) = \tilde{a}_0 z+B_1u^{i(\bar{\alpha})-1}z^{i(\alpha)}+B_2u^{i(\bar{\alpha})-1}z^{2i(\alpha)-1}+a_{mi+i(\alpha)-1}z^{mi+i(\alpha)}+$$

$$\bar{\alpha}(y_{mi})\tilde{a}_0^{mi+1}z^{mi+1} + \bar{\alpha}(\frac{\partial}{\partial u}y_{mi})Au^{i(\bar{\alpha})-1}z^{im+1+j}+$$

$$(mi+1)\tilde{a}_0^{mi}B_1u^{i(\bar{\alpha})-1}\bar{\alpha}(y_{mi})z^{mi+i(\alpha)} \mod \wp^{mi+i(\alpha)+1}$$

$$f\alpha'(z) = f(\tilde{a}_0)f(z) + f(a'_{im})f(z^{im+1}) + \ldots + f(a'_{im+i(\alpha)-1}f(z^{im+i(\alpha)}) =$$

$$\tilde{a}_0 z+\frac{\partial}{\partial u}(\tilde{a}_0)x_{mi}z^{mi+1}+\tilde{a}_0 y_{mi}z^{mi+1}+a'_{im}z^{im+1}+\ldots+a'_{im+i(\alpha)-1}z^{im+i(\alpha)}+i(\alpha)y_{mi}a'_{i(\alpha)-1}z^{im+i(\alpha)}+$$

$$\frac{\partial}{\partial u}(a'_{i(\alpha)-1})x_{mi}z^{im+i(\alpha)} \mod \wp^{mi+i(\alpha)+1} \tag{1.16}$$

Because of the special form of y_{im}, $\bar{\alpha}(y_{mi})\tilde{a}_0^{im+1} = \tilde{a}_0 y_{mi}$. The coefficients a'_q, $q < im + i(\alpha) - 1$ can be chosen so that they have the pointed properties, in the same way, as in the case 1). For $q = im + i(\alpha) - 1$ it is necessary to show, that there exists $(y'_{mi})_0$:

$$\tilde{y}_1^{m+(i(\alpha)-1)/i}(a_{mi+i(\alpha)-1}+\bar{\alpha}(\frac{\partial}{\partial u}(y_{mi}))Au_{i(\bar{\alpha})-1}+(mi+1)\tilde{a}_0^{mi}B_1u^{i_{\bar{\alpha}}-1}\bar{\alpha}(y_{mi})-i(\alpha)y_{mi}a'_{i(\alpha)-1}-$$

$$\frac{\partial}{\partial u}(a'_{i(\alpha)-1})x_{mi}) = 0 \mod \wp^{i(\bar{\alpha})} \tag{1.17}$$

Since
$\bar{\nu}(\tilde{y}_1^{m+(i(\alpha)-1)/i}A\bar{\alpha}(\frac{\partial}{\partial u}y_{mi})u^{i(\bar{\alpha})-1}) > i(\bar{\alpha}) - 1$, $\bar{\nu}(\tilde{y}_1^{m+(i(\alpha)-1)/i}a_{mi+i(\alpha)-1}) \geq i(\bar{\alpha}) - 1$,
$\bar{\nu}(\tilde{y}_1^{m+(i(\alpha)-1)/i}B_1\bar{\alpha}(y_{mi})u^{i(\bar{\alpha})-1}) = i(\bar{\alpha}) - 1$, $\bar{\nu}(\frac{\partial}{\partial u}(a'_{i(\alpha)-1})x_{mi}\tilde{y}_1^{m+(i(\alpha)-1)/i}) \geq i(\bar{\alpha}) - 1 - 1 + i(\bar{\alpha}) > i(\bar{\alpha}) - 1$,
the element $(y'_{mi})_0$ exists and is defined uniquely if $(im+1) \neq i(\alpha)$, i.e. $q \neq 2j$. Further,

$$\alpha f(u) = \alpha(u) + \alpha(x_{mi})\alpha(z^{mi}) = \bar{\alpha}(u) + Au^{i(\bar{\alpha})-1}z^j + c_{mi+i(\alpha)-1}z^{mi+j} + \bar{\alpha}(x_{mi})\tilde{a}_0^{mi}z^{mi}+$$

$$\bar{\alpha}(\frac{\partial}{\partial u}x_{mi})Au^{i(\bar{\alpha})-1}z^{mi+j} + mi\tilde{a}_0^{mi-1}B_1u^{i_{\bar{\alpha}}-1}\bar{\alpha}(x_{mi})z^{mi+i(\alpha)-1} \mod \wp^{mi+i(\alpha)}$$

$$f\alpha'(u) = f(\bar{\alpha}(u))+f(c'_{im})f(z^{im})+\ldots+f(c'_{im+j})f(z^{im+j}) = \bar{\alpha}(u)+\frac{\partial}{\partial u}(\bar{\alpha}(u))x_{im}z^{im}+c'_{im}z^{im}+\ldots$$

$$+ c'_{im+j}z^{im+j} + \frac{\partial}{\partial u}(c'_j)x_{mi}z^{im+j} + jy_{mi}c'_jz^{mi+j}, \tag{1.18}$$

whence we get similarly that we must solve an equation over $(x'_{mi})_0$:

$$\tilde{y}_1^{m+j/i}\tilde{y}_2^{-1}(c_{mi+j} + mi\tilde{a}_0^{mi-1}B_1u^{i_{\bar{\alpha}}-1}\bar{\alpha}(x_{mi})-$$

$$\frac{\partial}{\partial u}(c'_j)x_{mi} - jy_{mi}c'_j + \bar{\alpha}(\frac{\partial}{\partial u}x_{mi})Au^{i(\bar{\alpha})-1}) = 0 \mod \wp^{i(\bar{\alpha})} \tag{1.19}$$

Since $(y'_{mi})_0$ was already defined (if $mi = j$, we can take $(y'_{mi})_0$ equal to a constant),
$\bar{\nu}(\tilde{y}_1^{m+j/i}\tilde{y}_2^{-1}\bar{\alpha}(\frac{\partial}{\partial u}x_{mi})Au^{i(\bar{\alpha})-1} > i(\bar{\alpha}) - 1$,
$\bar{\nu}(\tilde{y}_1^{m+j/i}\tilde{y}_2^{-1}\frac{\partial}{\partial u}(c'_j)x_{mi}) > i(\bar{\alpha}) - 1$, $\bar{\nu}(\tilde{y}_1^{m+j/i}\tilde{y}_2B_1u^{i(\bar{\alpha})-1}\bar{\alpha}(x_{mi})) = i(\bar{\alpha}) - 1$,
so an element $(x'_{mi})_0$ does exist.

Let us now examine the case b). Now by the similar arguments as in a), we get that α is conjugate to β,
$\beta(u) = \bar{\alpha}(u) + A_1z^j + A_2z^{j+1} + \ldots + A_{i(\alpha)-1-j}z^{i(\alpha)-1}$,
$\beta(z) = \tilde{a}_0z + B_1z^{i(\alpha)} + b_2z^{2i(\alpha)-1}$, $\bar{\nu}(A_q\tilde{y}_1^{q/i}\bar{\alpha}(\tilde{y}_2^{-1})) = i(\bar{\alpha}) - 1$ or $A_q = 0$, $\bar{\nu}(B_q\tilde{a}_0^{-1}\tilde{y}_1^{q/i}) = i(\bar{\alpha}) - 1$ if $i(\alpha)$ is finite, and

$\beta(u) = \bar{\alpha}(u) + Az^j$,
$\beta(z) = \tilde{a}_0 z$ if $i(\alpha) = \infty$ (see cases c) and d) correspondingly).

In fact, let us use formulas 1.16 and (1.18). Since $\bar{\nu}(A_q\bar{\alpha}(\frac{\partial}{\partial u}y_{mi})\tilde{y}_1^{m+j/i+q-1}) > i(\bar{\alpha}) - 1$, the arguments from deriving the formula (1.17) remain true for all coefficients a_q, $q \geq i + i(\alpha) - 1$, $q \neq 2i(\alpha) - 1$, and the property from the case a) is realized. Similarly, since $\bar{\nu}(\bar{\alpha}(\frac{\partial}{\partial u}x_{mi})A_q\tilde{y}_1^{m+j/i+q-1}\tilde{y}_2^{-1}) > i(\bar{\alpha}) - 1$, we can apply formula (1.19) for the coefficients c_q, $q \geq i + i(\alpha) - 1$ and get the desired result.

Remark. In the case b) of theorem, if $\tilde{a}_0 \neq 1$ or $\tilde{a}_0 = 1$ but $y \neq 0$, where y is a second parameter of the canonical representation of $\bar{\alpha}$, one can show by direct calculations that α is conjugate with β: $\beta(u) = \bar{\alpha}(u) + Az^j$, $\beta(z) = \tilde{a}_0 z + Bz^{i(\alpha)}$, where A satisfies (1.8) and B does not. But, if $\tilde{a}_0 = 1$ and $y = 0$, then for any $k \geq 1$ $\frac{\partial}{\partial u}(B)\tilde{y}_2 \in Im(\bar{\alpha} - Id)$, where $B = c\tilde{y}_2^{-1}u^{1+k(1-i(\bar{\alpha}))}$, whence, by formulas (1.17) and (1.19), one can derive that β does not exist and the number of parameters can not be decreased.

Remark. 1. In the case of characteristic $p > 0$ we have in general $dim(kerT) \neq dim(cokerT)$, as it was shown in lemma 1.3. From this follows that automorphisms can not be parameterised by finite number of parameters in more cases than in the case of $chark = 0$. For example, α can not be always redused to β, where $\beta(u) = \bar{\alpha}(u) + A_1z^j + \ldots + A_k u^{j+k}$: k may be equal to the infinity.

2. The classification can be easily generalised to the case of n-dimensional local field, because we used only the property $dim(kerT) = dim(cokerT)$ and arguments with valuations. In the case of multidimensional equal characteristics local fields of characteristic 0 all our arguments can be carried over to the case of higher dimension if we assume that the value group of $\bar{\nu}$ is $\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$.

Now we only have to prove that the automorphisms β, β' are conjugate if and only if $\beta = \beta'$, where β, β' are automorphisms from the formulation of theorem. It's clear that if β is conjugate with β', then $\tilde{a}_0 = \tilde{a'}_0$ and $\bar{\beta}(u) = \bar{\beta}'(u) = \bar{\alpha}(u)$ is a nesessary condition, whence β is defined up to the change $u \mapsto x_0 : \bar{\alpha}(x_0)$ has the canonical view and $z \mapsto cz$, $c \in k^*$.

Then, β and β' must have the same numbers j, j' and $i(\alpha), i'_\alpha$. Indeed, if β and β' are conjugate, then $\beta = f^{-1}\beta' f$, and f can be decomposed in a composition of automorphisms $f = f_1 f_2 \ldots f_m$, where $f_q(u) = u + x_q z^q$, $f_{q_1}(z) = z + y_{q_1}z^{q+1}$. Then from (1.9), (1.10) follows that for $q < min\{j, j'\}$ we have $x_q \in kerT_{q,1}$, for $q_1 < min\{i(\alpha) - 1, i'_\alpha - 1\}$ we have $y_{q_1} \in kerT_{q,2}$. From the proof of the case a) follows that the conjugations f_q with this numbers preserve properties (1.8) of the coefficients c_q, a_{q_1} for $q \leq min\{j, j'\}$, $q_1 \leq min\{i(\alpha) - 1, i'_\alpha - 1\}$. Therefore, if $j \neq j'$ or $i(\alpha) \neq i'_\alpha$, then the first nonzero coefficient of $\beta(u)$ or $\beta'(u)$ or $\beta(z)$ or $\beta'(z)$ must lie in the kernel of the map $T_{j(j'),1}$ ($T_{i(\alpha)(i'_\alpha),2}$), but this contradicts to the choice of these coefficients. Therefore, $j = j'$ and $i(\alpha) = i'_\alpha$. So, β and β' are in the same class defined by the pair $(j, i(\alpha))$. In this case the equality follows from the special choice of coefficients of $z^j, z^{i(\alpha)}$ and the proves of the corresponding cases.

1.3 Proof of the theorem III

Let $\bar{\alpha}^n = 1$. Then, by Proposition 1.2, there exists x_0 such that $\bar{\alpha}(x_0) = \xi x_0$, where ξ is a primitive root of 1. As in the theorem II, we consider that $\alpha(u) = \xi u + c_j z^j + \ldots$, $\alpha(z) = a_0 z + a_{i(\alpha)-1} z^{i(\alpha)} + \ldots$.

At first we note that $i = 1$ or ∞. Indeed, if $a_0^i = \frac{\bar{\alpha}(y)}{y}$, then, as it was shown in the theorem II, we may suppose $a_0 = 1 + cu^{qn} + \ldots$. But in this case $a_0^i = 1 + icu^{qn} + \ldots$. Further,
$a_0^i \bar{\alpha}(a_0^i) \ldots \bar{\alpha}^{n-1}(a_0^i) = \frac{\bar{\alpha}(y)}{y} \frac{\bar{\alpha}^2(y)}{\bar{\alpha}(y)} \ldots \frac{\bar{\alpha}^n(y)}{\bar{\alpha}^{n-1}(y)} = 1$. So we have that:
$a_0^i \bar{\alpha}(a_0^i) \ldots \bar{\alpha}^{n-1}(a_0^i) = (1 + icu^{qn} + \ldots) \ldots (1 + icu^{qn} + \ldots) = 1 + nciu^{qn} + \ldots \neq 1$, where from $a_0 = 1$ or $i = \infty$. Thus, $\alpha(u) = \xi u + c_j u^j + \ldots$, $\alpha(z) = z + a_{i(\alpha)-1} z^{i(\alpha)} + \ldots$ (in this case $dim(kerT_{k,1}) = \infty = dim(kerT_{k,2}) = dim(cokerT_{k,1,2})$).

Further let us consider that $c_q \in uk((u^n))$, $a_q \in k((u^n))$, because by going over to conjugations as in (1.9) and (1.10), we can solve all the equations
$\bar{\alpha}(y) - \xi y = c_q \mod uk((u^n))$, $\bar{\alpha}(y) - y = a_q \mod k((u^n))$
(in theorem 2 we have reduced general case to a case $a_q, c_q \in cokerT_{q,1}, T_{q,2}$ in the same way).

As in Proposition 1, it is proved that if $i = \infty$, then takes place the case O) of the theorem.

Let $i = 1$. The case $j \geq i(\alpha)$ coincide with the case $j \geq i(\alpha)$ of the theorem 2: by writing over the formula (1.11), we get that holds (1.12) and there from holds $c'_q = 0$, $q < j$, and the equation (1.13) always has a solution $x_m \in uk((u^n))$, when $a_q \in k((u^n))$, $c_q, c'_q \in uk((u^n))$. It holds from (1.15) that by conjugation the coefficients $a'_q = 0$, $q < i(\alpha) - 1$. All other arguments from the theorem 2 should be applied here also, and in the same way as with the case O), we get the case O') I).

Let now $j + 1 = i(\alpha)$ and let $f(z) = y_0 z$, $y_0 \in k((u^n))$. Then we have:

$$\alpha f(z) = \alpha(y_0)\alpha(z) = (y_0 + \frac{\partial}{\partial u}(y_0) c_j z^j + \ldots)(z + a_{i(\alpha)-1} z^{i(\alpha)} + \ldots)$$

$$f\alpha'(z) = f(z) + f(a'_1 z^2) + \ldots = y_0 z + f(a'_1) y_0^2 z^2 + \ldots \tag{1.20}$$

So we get from here that $a'_q = 0$, $q < j$, $\bar{f}(a'_j) y_0^{j+1} = \frac{\partial}{\partial u}(y_0) c_j + y_0 a_{i(\alpha)-1}$. If the equation $\frac{\partial}{\partial u}(y_0) c_j + y_0 a_{i(\alpha)-1} = 0$ isn't solvable, then $dlog(y_0) \neq -a_{i(\alpha)-1}/c_j$, where from $\bar{\nu}(a_{i(\alpha)-1}/c_j) < -1$ or $\bar{\nu}(a_{i(\alpha)-1}/c_j) = -1$, but $res(-a_{i(\alpha)-1}/c_j) \notin \mathbb{Z}$. If it is solvable, then we can consider that $\bar{\nu}(a'_{i(\alpha)-1}/c'_j) = -1$, $res(a'_{i(\alpha)-1}/c'_j) \in$ ”, setting $y_0 = u$.

The case $j+1 < i(\alpha)$ is reduced to a case $j+1 = i(\alpha)$: indeed, by setting $y_0 = u$, we get that $a'_j \neq 0$, and it's the first non equal to zero coefficient in the decomposition $\alpha'(z)$, while $\bar{\nu}(a'_j/c'_j) = -1$, $res(a'_j/c'_j) \in$ ”.

The next part of the proof is following the proof of the theorem 2: in order to prove the rest items of the theorem, we shall go over to conjugations, and as a result get formulas of the kind (1.17) and (1.19). However, we cannot completely repeat the arguments from the previous case, because the kernel and cokernel of corresponding maps $T_{k,1}$ and $T_{k,2}$ are infinite-dimensional in our case. The formulas (1.17) and (1.19) are now written down as

$$\frac{\partial}{\partial u}(y_m)A + (m+1-i(\alpha))By_m - \frac{\partial}{\partial u}(B)x_m = a \tag{1.21}$$

$$mBx_m - \frac{\partial}{\partial u}(A)x_m - jy_mA + \frac{\partial}{\partial u}(x_m)A = b \tag{1.22}$$

where $A = c_j$, $B = a_{i(\alpha)-1}$, a and b — arbitrary elements from $uk((u^n))$ and $k((u^n))$ correspondingly, $x_m \in uk((u^n))$, $y_m \in k((u^n))$. It turns out that the solutions of this system strongly depend on the properties of the numbers, defined before the formulation of the theorem 3. From now on in the proof we are going to investigate the solvability of this system in dependence from the behaviour of these numbers.

First of all we note that we can put $A = c_1u^k$, $k \leq 0$, $|k| < j$, $c_1 \in k$. Indeed, we write down the conjugation $f^{-1}\alpha f$, $f(z) = y_0z$, $y_0 = u^q$, where $q \geq 0$ is a minimal positive integer such that $qi \geq \bar{\nu}(A)$, $f(u) = x_0$, $\bar{\nu}(x_0) = 1$. Then

$$\alpha f(u) = \alpha(x_0) = \xi x_0 + \frac{\partial}{\partial u}(x_0)c_jz^j + \ldots$$

$$f\alpha'(u) = f(\xi u) + f(c'_jz^j) + \ldots = \xi x_0 + \bar{f}(c'_j)y_0^jz^j + \ldots,$$

where from we get $y_0^j\frac{\bar{f}(c'_j)}{c_j} = \frac{\partial}{\partial u}(x_0)$. We consider $c'_j = A = c_1u^k$, $k = -\bar{\nu}(y_0^j/c_j)$. Then $(-k+1)c_1y_0^j/c_j = \frac{\partial}{\partial u}(x_0^{-k+1})$. We can choose $c_1 \in k$ so, that the equation would be solvable. And here also $a'_j = \bar{f}^{-1}(\frac{\partial}{\partial u}(y_0)y_0^{-j-1}c_j + y_0^{-j}a_j)$.

We show that in all the cases (except the cases 2) a), 2) b) i) of the theorem) such a conjugation could be found, that it holds $A = c_1u^k$, $B = c_2u^{k_1}$. By that it appears, that the coefficients A_1, B_1 in all the cases of the theorem have the form, as mentioned above.

Let $\bar{\nu}(B/A) = -1$, $res(B/A) = p_1/q_1 \in$ ”, $(p_1, q_1) = 1$.

We note that q_1 doesn't depend on conjugation. Indeed, A and B change only by conjugation $f^{-1}\alpha f$, $f(u) = x_0$, $\bar{\nu}(x_0) = 1$, $f(z) = y_0 z$. But then from the (1.20) follows, that $res(B'/A') - res(B/A) \in \mathbb{Z}$, hence we get that q_1 doesn't depend on conjugations.

Let us now show that there is such a conjugation, that $B/A = res(B/A)u^{-1}$, if $q_1 \not| j$ or $res(B/A) < 0$. Therefore we look for a conjugation f, $f(u) = x_0$, $f(z) = y_0 z$, so that the automorphism $\alpha' = f^{-1}\alpha f$ would have $A' = c_1 u^k$, $k < 0$, $B' = c_2 u^{k-1}$. For that, considering (1.20), we must solve a system

$$\bar{f}(A')y_0^j = \frac{\partial}{\partial u}(x_0)c_j, \quad \bar{f}(B')y_0^{j+1} = \frac{\partial}{\partial u}(y_0)c_j + y_0 a_j$$

Dividing the first equation by the second, we get: $\bar{f}(B'/A')y_0 = \frac{\frac{\partial}{\partial u}(y_0) + y_0 B/A}{\frac{\partial}{\partial u}(x_0)}$, where from

$$\frac{c_2}{c_1}x_0^{-1}\frac{\partial}{\partial u}(x_0) = \frac{\frac{\partial}{\partial u}(y_0)}{y_0} + B/A, \quad c_1 y_0^j A^{-1} = \frac{\partial}{\partial u}(x_0)x_0^{-k} = (1-k)^{-1}\frac{\partial}{\partial u}(x_0^{1-k}) \quad (1.23)$$

We look for x_0, y_0 in a form of $x_0 = u(1 + \varepsilon_1 u + \varepsilon_2 u^2 + \ldots)$, $y_0 = u^\lambda(1 + \omega_1 u + \ldots)$. Let $c = c_2/c_1$, $B/A = c_{ba}u^{-1} + \gamma_0 + \gamma_1 u + \ldots$. Then from the first equation (1.23) we get:

$$cu^{-1} + c(\varepsilon_1 + 2\varepsilon_2 u + 3\varepsilon_3 u^2 + \ldots)(1 + \varepsilon_1 u + \varepsilon_2 u^2 + \ldots)^{-1} =$$

$$\lambda u^{-1} + (\omega_1 + 2\omega_2 u + \ldots)(1 + \omega_1 u + \ldots)^{-1} + (c_{ba}u^{-1} + \gamma_0 + \gamma_1 u + \ldots)$$

Suppose $c = \lambda + c_{ba} \neq 0$ (we can always find such $\lambda \geq 0$). By comparing the coefficients in the left and right sides, we get linear equations of the form

$$c\varepsilon_i = \omega_i + i^{-1}\gamma_{i-1} + i^{-1}\psi_i,$$

where ψ_i — certain polynomial from ε_q, ω_q, $q < i$ (they are determined from the previous equations). From the second equation we get:

$$c_1 u^{\lambda j}(1 + \omega_1 u + \ldots)^j c_a^{-1} u^{-k_a} = u^{-k} + (2-k)\varepsilon_1 u^{1-k} + (3-k)(\varepsilon_2 + \ldots)u^{2-k} + \ldots$$

(where $A = c_a u^{k_a}$). Suppose $c_1 = c_a$, $-k = \lambda j - k_a$. Then $k \leq 0$. Because of $c_{ba} \neq 0$, we can put $\lambda = 0$. Hence $k = k_a > -j$. Comparing the coefficients, we get linear equations of the form

$$j\omega_i = (i + 1 - k)\varepsilon_i + \tilde{\psi}_i = (i + 1 + \lambda j - k_a)\varepsilon_i + \tilde{\psi}_i \quad (1.24)$$

For every i the system has a solution, if $(i + 1 - k_a)/j \neq c_{ba} = p_1/q_1$, what holds true always under the condition that $q_1 \not| j$ or $c_{ba} = res(B/A) < 0$. If these conditions are not fulfilled, then B' can have the form $B' = c_2 u^{k-1} + c_{i_b}u^{i_b}$, what is evident from the arguments, mentioned above.

If $res(B/A) \notin$ ", then applying the same thoughts, we also get the same result.

Let $\bar{\nu}(B/A) < -1$.
Then we look for B' in the form $B' = c_2 u^{k+\bar{\nu}(B/A)}$. System (1.23) will now have the form

$$cx_0^{\bar{\nu}(B/A)}\frac{\partial}{\partial u}(x_0) = \frac{\frac{\partial}{\partial u}(y_0)}{y_0} + B/A, \quad c_1 y_0^j A^{-1} = (1-k)^{-1}\frac{\partial}{\partial u}(x_0^{1-k})$$

Hence

$$cu^{\bar{\nu}(B/A)} + cu^{\bar{\nu}(B/A)+1}(\varepsilon_1 + 2\varepsilon_2 u + \ldots)(1+\varepsilon_1 u + \ldots)^{-1} =$$
$$\lambda u^{-1} + (\omega_1 + 2\omega_2 u + \ldots)(1+\omega_1 u+\ldots)^{-1} + (c_{ba}u^{\bar{\nu}(B/A)} + \gamma_{\bar{\nu}(B/A)+1}u^{\bar{\nu}(B/A)+1} + \ldots)$$

whence $c = c_{ba}$ and equation i looks like following:

$$ci\varepsilon_i = \omega_{i+\bar{\nu}(B/A)+1}(i + \bar{\nu}(B/A) + 1) + \psi_i$$

where $\omega_{i+\bar{\nu}(B/A)+1} = 0$ if $(i + \bar{\nu}(B/A) + 1) \leq 0$. Equations (1.24) are written over without changes, where from we get that every system i is solvable, and our proposition is proved.

Let us now go back to a system (1.21), (1.22). We show, that system of the equations (1.21), (1.22) is solvable, if $m \neq j$. It holds:

$$y_m = (mB/(jA) - \frac{\partial}{\partial u}(A)/(jA))x_m + \frac{\partial}{\partial u}(x_m)/j - b/(jA) \tag{1.25}$$

Hence

$$\frac{\partial^2}{\partial u^2}(x_m) + \frac{\partial}{\partial u}(x_m)((2m-j)B/A - \frac{\partial}{\partial u}(A)/A) + x_m((m-j)\frac{\partial}{\partial u}(B)B/(BA) -$$
$$(2m-j)\frac{\partial}{\partial u}(A)B/A^2 - (\frac{\partial^2}{\partial u^2}(A)\frac{\partial}{\partial u}(A))/(\frac{\partial}{\partial u}(A)A) +$$
$$(\frac{\partial}{\partial u}(A))^2/A^2 + (m-j)mB^2/A^2) - \frac{\partial}{\partial u}(b/A) - (m-j)Bb/A^2 - ja/A = 0 \tag{1.26}$$

We set $q = \bar{\nu}(B/A)$. From this: $q \leq -1$, if $q = -1$, then $res(B/A) \notin \mathbb{Z}$.
Let us show that the equation

$$\frac{\partial^2}{\partial u^2}(x_m) + \frac{\partial}{\partial u}(x_m)((2m-j)B/A - \frac{\partial}{\partial u}(A)/A) + x_m((m-j)\frac{\partial}{\partial u}(B)B/(BA) -$$
$$(2m-j)\frac{\partial}{\partial u}(A)B/A^2 - (\frac{\partial^2}{\partial u^2}(A)\frac{\partial}{\partial u}(A))/(\frac{\partial}{\partial u}(A)A) +$$
$$(\frac{\partial}{\partial u}(A))^2/A^2 + (m-j)mB^2/A^2) = cu^k \mod \wp^{k+1} \tag{1.27}$$

is solvable, if $q < -1$ or $q = -1$, but $res(B/A) \notin$ ", for all $k \in \mathbb{Z}$ and every constant $c \in k$. From here we immediately get the solvability of the equation (1.26) for all b and a, and also of a system (1.21), (1.22). From here will follow the proof of the items A), B) 1).

If $q < -1$, then
$\bar{\nu}((2m-j)B/A - \frac{\partial}{\partial u}(A)/A) = \bar{\nu}(B/A)$ (if $2m \neq j$) and ≥ -1 (if $2m = j$),
$\bar{\nu}((m-j)\frac{\partial}{\partial u}(B)B/(BA) - (2m-j)\frac{\partial}{\partial u}(A)B/A^2 - (\frac{\partial^2}{\partial u^2}(A)\frac{\partial}{\partial u}(A))/(\frac{\partial}{\partial u}(A)A) + (\frac{\partial}{\partial u}(A))^2/A^2 + (m-j)mB^2/A^2) = \bar{\nu}(B^2/A^2)$, because $m \neq j$. Thus,

$$\bar{\nu}(\frac{\partial^2}{\partial u^2}(x_m) + \frac{\partial}{\partial u}(x_m)((2m-j)B/A - \frac{\partial}{\partial u}(A)/A) + x_m((m-j)\frac{\partial}{\partial u}(B)B/(BA) -$$

$$(2m-j)\frac{\partial}{\partial u}(A)B/A^2 - (\frac{\partial^2}{\partial u^2}(A)\frac{\partial}{\partial u}(A))/(\frac{\partial}{\partial u}(A)A) + (\frac{\partial}{\partial u}(A))^2/A^2 + (m-j)mB^2/A^2) =$$

$$\bar{\nu}(\frac{\partial^2}{\partial u^2}(x_m) + \frac{\partial}{\partial u}(x_m)(2m-j)c_1u^q + x_m(m-j)mc_2u^{2q}) = \bar{\nu}(x_m(m-j)mc_2u^{2q})$$

where from immediately follows solvability of the equation (1.27).

If $q = -1$,
then we put $q_a = \bar{\nu}(A)$, $q_b = \bar{\nu}(B)$, $k = \bar{\nu}(x_m)$, $x = res(B/A)$. And now for the solvability of the equation (1.27) is necessary to show that the equation

$$k(k-1) + k(2m-j)x - kq_a + ((m-j)q_b - (2m-j)q_a)x + q_a + (m-j)mx^2 = 0 \quad (1.28)$$

doesn't have a solution.

This quadratic equation has the critical points $-\frac{k-q_a}{m-j}$, $-\frac{k-1}{m}$ (and if $m = j$, then one of the points is $-\frac{k-1}{m}$), so if $res(B/A) \notin$ ", then our assertion is proved. Moreover, in the case when $q = -1$, $res(B/A) \notin$ " we have proved the solvability of the equation (1.26), and through that also of a system (1.21), (1.22) for all m, by this proving the case B) 1).

If $m = j$, $q < -1$, $q_a = 0$, then the equation (1.27) has the form

$$\frac{\partial^2}{\partial u^2}(x_m) + \frac{\partial}{\partial u}(x_m)(mB/A - \frac{\partial}{\partial u}(A)/A) + x_m(-mB\frac{\partial}{\partial u}(A)/A^2 -$$

$$(\frac{\partial^2}{\partial u^2}(A)\frac{\partial}{\partial u}(A))/(\frac{\partial}{\partial u}(A)A) + (\frac{\partial}{\partial u}(A))^2/A^2 = cu^k \mod \wp^{k+1},$$

that is always solvable, because $\bar{\nu}(x_mB\frac{\partial}{\partial u}(A)/A^2) < \bar{\nu}(x'_mB/A) < \bar{\nu}(x''_m)$.

If $q_a \neq 0$, then this equation isn't solvable with $k = q_a - 1 + q$. Thus, if $q < -1$, α is the conjugation to automorphism β: $\beta(u) = \xi u + Au^j$, $\beta(z) = z + Bz^{i(\alpha)} + cu^{\bar{\nu}(A)-1+q}z^{2i(\alpha)-1}$ (see case A)).

The case $res(B/A) \in$ " should be studied precisely. Recall that in this case $\bar{\nu}(B/A) = -1$.

Let $res(B/A) = p_1/q_1$ $(= c_b/c_a)$, $(p_1, q_1) = 1$. The following proof of the theorem would be divided into three cases (which do not coincide with the corresponding cases from the formulation of the theorem), in order to make the proof easier:
a) $q_1 | j$, $q_1 \neq j$
b) $q_1 \not| j$
c) $q_1 = j$.

a) (see the case B) 2) b) i)).
Here $-\frac{k_1 - q_a}{m_1 - j} = -\frac{k_2 - 1}{m_1} = \frac{p_1}{q_1}$. Then there exist $c_1, c_2 \in k$ such that the equation (1.26) $+ c_1 u^{k_1 - 2} + c_2 u^{k_2 - 2} = 0$ has solutions with $m = m_1$, what follows from the solvability of the equation (1.27) for all k, except of $k = k_1 - 2$, $k = k_2 - 2$, and m_1 — is the first index, when the system (1.21), (1.22) isn't solveable in a general case. Also in this case the space of solutions of the homogeneous equation (1.26) is generated by x_1 and x_2, $\bar{\nu}(x_1) = k_1$, $\bar{\nu}(x_2) = k_2$. Thus, automorphism α is conjugate to the automorphism α',

$\alpha'(u) = \xi u + A z^j + c_{j+1+2m_1} z^{j+1+2m_1} + \ldots,$
$\alpha'(z) = z + B z^{j+1} + B_2 z^{j+1+m_1} + \ldots$, where $B_2 = c_1 u^{k_1 - 2 + q_a} + c_2 u^{k_2 - 2 + q_a}$.

Now let us investigate behaviour of the values k_{1,m_q}, k_{2,m_q} for different m_q, for which the equation (1.26) has no solutions, where k_{1,m_q}, k_{2,m_q} are solutions of the equation (1.28).

Obviously, $m_q = q q_1$, $q \in \mathbb{N}$. Note that $(k_{1,m_q} - k_{2,m_q})$ doesn't depend on m_q $(q \neq j/q_1)$. Indeed, $\frac{k_{1,m_q} - q_a}{m_q - j} = \frac{k_{2,m_q} - 1}{m_q} = -\frac{p_1}{q_1}$. Hence $k_{2,m_q} = -p_1 q + 1$, $k_{1,m_q} = -p_1(q - j/q_1) + q_a$, and $(k_{1,m_q} - k_{2,m_q}) = p_1 j / q_1 + q_a - 1$. We observe, that
$k_{2,m_q} = k_{2,m_{q-1}} + k_{2,m_1} - 1$, $k_{1,m_q} = k_{1,m_{q-1}} + k_{2,m_1} - 1 = k_{2,m_{q-1}} + k_{1,m_1} - 1$.

We write down the formula (13) for the case, when
$y_{m_1} = \omega_1 y_{m_1,1} + \omega_2 y_{m_1,2}$, $x_{m_1} = \omega_1 x_{m_1,1} + \omega_2 x_{m_1,2}$,
where $\omega_1, \omega_2 \in k$, x_{m_1}, y_{m_1} are solutions of the homogeneous system (1.21), (1.22) for $m = m_1$. Because of $\bar{\nu}(x_{m_1,1}) = k_{1,m_1}$, $\bar{\nu}(x_{m_1,2}) = k_{2,m_1}$, we have $\bar{\nu}(y_{m_1,1}(x_{m_1,1})) = k_{1,m_1} - 1$, $\bar{\nu}(y_{m_1,2}(x_{m_1,2})) = k_{2,m_1} - 1$. Indeed, from the formula (1.25), $\bar{\nu}(y_m) = \bar{\nu}(x_m) - 1$, if $m p_1/q_1 - q_a + k \neq 0$, where $k = \bar{\nu}(x_m)$. Let be $m p_1/q_1 - q_a + k = 0$. Then $p_1/q_1 = -\frac{k_{1,m_1} - q_a}{m_1}$, whence $j = 0$. It's a contradiction. Analogously for k_{2,m_1} $q_a = 1$, but $q_a \leq 0$, also a contradiction. So we have:

$$\alpha' f(z) = z + B z^{i(\alpha)} + B_2 z^{i(\alpha)+m_1} + a'_{2m_1+i(\alpha)-1} z^{2m_1+i(\alpha)} + \ldots + y_{m_1} z^{m_1+1} +$$

$$\frac{\partial}{\partial u}(y_{m_1}) A z^{m_1+i(\alpha)} + (m_1+1) B y_{m_1} z^{m_1+i(\alpha)} + (m_1+1) B_2 y_{m_1} z^{2m_1+i(\alpha)} + \ldots$$

$$f\alpha''(z) = z + y_{m_1} z^{m_1+1} + B z^{i(\alpha)} + B_2 z^{i(\alpha)+m_1} + B_3 z^{m_2+i(\alpha)} + \frac{\partial}{\partial u}(B) x_{m_1} z^{m_1+i(\alpha)} +$$

$$i(\alpha)y_{m_1}Bz^{m_1+i(\alpha)}+2^{-1}\frac{\partial^2}{\partial u^2}(B)x_{m_1}^2z^{2m_1+i(\alpha)}+C_{i(\alpha)}^2y_{m_1}^2Bz^{2m_1+i(\alpha)}+\frac{\partial}{\partial u}(B_2)x_{m_1}z^{i(\alpha)+2m_1}+$$
$$(i(\alpha)+m_1)y_{m_1}B_2z^{2m_1+i(\alpha)}+\frac{\partial}{\partial u}(B)i(\alpha)x_{m_1}y_{m_1}z^{2m_1+i(\alpha)}+a''_{2m_1+i(\alpha)-1}z^{2m_1+i(\alpha)}+\ldots$$
$$\alpha'f(u)=\xi u+Az^j+c'_{2m_1+j}z^{2m_1+j}+\ldots+\xi x_{m_1}z^{m_1}+\frac{\partial}{\partial u}(x_{m_1})Az^{m_1+j}+$$
$$m_1Bx_{m_1}z^{m_1+j}+m_1B_2x_{m_1}z^{2m_1+j}+\ldots$$
$$f\alpha''(u)=\xi u+\xi x_{m_1}z^{m_1}+Az^j+\frac{\partial}{\partial u}(A)x_{m_1}z^{m_1+j}+jy_{m_1}Az^{m_1+j}+2^{-1}\frac{\partial^2}{\partial u^2}(A)x_{m_1}^2z^{2m_1+j}+$$
$$C_j^2y_{m_1}^2Az^{2m_1+j}+\frac{\partial}{\partial u}(A)jx_{m_1}y_{m_1}z^{2m_1+j}+c''_{2m_1+j}z^{2m_1+j}+\ldots$$

As $m_1<j$, in the expression for $\alpha'f(z)$, $\alpha'f(u)$ there is the only term at $z^{2m_1+i(\alpha)}$.

For $y_{m_q}=\omega_1y_{m_q,1}+\omega_2y_{m_q,2}$, $x_{m_q}=\omega_1x_{m_q,1}+\omega_2x_{m_q,2}$ the formula 1.16 should have the form

$$\alpha'f(z)=z+Bz^{i(\alpha)}+B_2z^{i(\alpha)+m_1}+B_3z^{i(\alpha)+m_2}+a'_{m_q+m_1+i(\alpha)-1}z^{m_q+m_1+i(\alpha)}+\ldots+y_{m_Q}z^{m_q+1}+$$
$$\frac{\partial}{\partial u}(y_{m_q})Az^{m_q+i(\alpha)}+(m_q+1)By_{m_q}z^{m_q+i(\alpha)}+(m_q+1)B_2y_{m_q}z^{m_q+m_1+i(\alpha)}+$$
$$(m_q+1)B_3y_{m_q}z^{m_q+m_2+i(\alpha)}+\ldots$$
$$f\alpha''(z)=z+y_{m_q}z^{m_q+1}+Bz^{i(\alpha)}+B_2z^{i(\alpha)+m_1}+B_3z^{i(\alpha)+2m_1}+$$
$$\frac{\partial}{\partial u}(B)x_{m_q}z^{m_q+i(\alpha)}+i(\alpha)y_{m_q}Bz^{m_q+i(\alpha)}+$$
$$\frac{\partial}{\partial u}(B_2)x_{m_q}z^{i(\alpha)+m_1+m_q}+(i(\alpha)+m_1)y_{m_q}B_2z^{m_1+m_q+i(\alpha)}+a''_{m_q+m_1+i(\alpha)-1}z^{m_1+m_q+i(\alpha)}+$$
$$\frac{\partial}{\partial u}(B_3)x_{m_q}z^{i(\alpha)+m_2+m_q}+(i(\alpha)+m_2)y_{m_q}B_3z^{m_2+m_q+i(\alpha)}+\ldots \quad (1.29)$$
$$\alpha'f(u)=\xi u+Az^j+c'_{m_q+j+m_1}z^{m_q+j+m_1}+\ldots+\xi x_{m_q}z^{m_q}+\frac{\partial}{\partial u}(x_{m_q})Az^{m_q+j}+$$
$$m_qBx_{m_q}z^{m_q+j}+m_qB_2x_{m_q}z^{m_q+m_1+j}+\ldots$$
$$f\alpha''(u)=\xi u+\xi x_{m_q}z^{m_q}+Az^j+\frac{\partial}{\partial u}(A)x_{m_q}z^{m_q+j}+jy_{m_q}Az^{m_q+j}+c''_{m_q+m_1+j}z^{m_q+m_1+j}+\ldots$$

Whence follows:

$$2^{-1}(\frac{\partial^2}{\partial u^2}(B)x_{m_1}^2)=C_{q_a-1}^2(c_bu^{q_a-3}+\ldots)(\omega_1^2x_1^2+2\omega_1\omega_2x_1x_2+\omega_2^2x_2^2)$$
$$2^{-1}(\frac{\partial^2}{\partial u^2}(A)x_{m_1}^2)=C_{q_a}^2(c_au^{q_a-2}+\ldots)(\omega_1^2x_1^2+2\omega_1\omega_2x_1x_2+\omega_2^2x_2^2) \quad (1.30)$$

$$y_{m_1}^2 B = (c_b u^{q_a-1} + \ldots)((\omega_1^2 y_1^2 + 2\omega_1\omega_2 y_1 y_2 + \omega_2^2 y_2^2)$$
$$y_{m_1}^2 A = (c_a u^{q_a} + \ldots)((\omega_1^2 y_1^2 + 2\omega_1\omega_2 y_1 y_2 + \omega_2^2 y_2^2)$$
$$\frac{\partial}{\partial u}(B) i(\alpha) x_{m_1} y_{m_1} = (q_a - 1) i(\alpha)(c_b u^{q_a-2} + \ldots)(\omega_1^2 x_1 y_1 + \omega_2^2 x_2 y_2 + \omega_1\omega_2(x_1 y_2 + x_2 y_1))$$
$$\frac{\partial}{\partial u}(A) j x_{m_1} y_{m_1} = q_a j (c_a u^{q_a-1} + \ldots)(\omega_1^2 x_1 y_1 + \omega_2^2 x_2 y_2 + \omega_1\omega_2(x_1 y_2 + x_2 y_1)) \quad (1.31)$$
$$y_{m_q} B_2 = \omega_1 c_1 y_{m_q,1} u^{k_{1,m_q}-2+q_a} + \omega_1 c_2 y_{m_q,1} u^{k_{2,m_q}-2+q_a} + c_1 \omega_2 y_{m_q,2} u^{k_{1,m_q}-2+q_a} + c_2 \omega_2 y_{m_q,2} u^{k_{2,m_q}-2+q_a}$$
$$x_{m_q} B_2 = \omega_1 c_1 x_{m_q,1} u^{k_{1,m_q}-2+q_a} + \omega_1 c_2 x_{m_q,1} u^{k_{2,m_q}-2+q_a} + c_1 \omega_2 x_{m_q,2} u^{k_{1,m_q}-2+q_a} + c_2 \omega_2 x_{m_q,2} u^{k_{2,m_q}-2+q_a} \quad (1.32)$$
$$(\frac{\partial}{\partial u}(B_2)) x_{m_q} = (k_{1,m_q} - 2 + q_a)\omega_1 c_1 x_{m_q,1} u^{k_{1,m_q}-3+q_a} + (k_{2,m_q} - 2 + q_a)\omega_1 c_2 x_{m_q,1} u^{k_{2,m_q}-3+q_a} +$$
$$(k_{1,m_q} - 2 + q_a) c_1 \omega_2 x_{m_q,2} u^{k_{1,m_q}-3+q_a} + (k_{2,m_q} - 2 + q_a) c_2 \omega_2 x_{m_q,2} u^{k_{2,m_q}-3+q_a} \quad (1.33)$$

Let $(k_{1,m_q} - k_{2,m_q}) < 0$. We shall show, that in formulas (1.30)- (1.33) monomials with valuation $(k_{1,m_q} + k_{1,m_1} + q_a - 3)$, belong to the image of the map (1.27), it means that the equation (1.27) with the right side in a form of these monomials is solvable.

Indeed, in a case of $(k_{1,m_q} - k_{2,m_q}) < 0$ we have $p_1/q_1 < (-q_a+1)/j < (-q_a+1+i)/j$, if $i \geq 1$. But then $A = c_1 u^{q_a}$, $B = c_2 u^{q_b} = c_2 u^{q_a-1}$, and $y_{m_q} = \omega_1 u^{k_{1,m_q}-1} + \omega_2 u^{k_{2,m_q}-1}$, $x_{m_q} = \omega_1 u^{k_{1,m_q}} + \omega_2 u^{k_{2,m_q}}$, because all the coefficients of the homogeneous system (1.21), (1.22) have the monomial form. Since $(k_{1,m_q} + k_{1,m_1} - 1) < k_{1,m_q} + k_{2,m_q} - 1 = k_{1,m_{q+1}} < k_{2,m_{q+1}}$, the equation (1.27) has monomial solutions of a form, mentioned above.

If $(k_{1,m_q} - k_{2,m_q}) > 0$, so $(k_{1,m_q} + k_{1,m_1} - 1) > k_{1,m_q} + k_{2,m_1} - 1 = k_{1,m_{q+1}} > k_{2,m_{q+1}}$, where from follows the same result.

If $(k_{1,m_q} - k_{2,m_q}) = 0$, then y_{m_q} and x_{m_q} consist of the only monomial, i.e. $\omega_1 = 0$ and expressions in (1.30)-(1.33) are simplified to the one monomial, which is in the general case not in the image of the map (1.27).

Now we show that for all q except $q = 1$, $q = j/q_1$, $q = (1 - q_a)/p_1 + 1$ there exist the coefficients ω_1, ω_2 (coefficient ω_2, if $k_{1,m_q} = k_{2,m_q}$) are such that
$a'_{m_q+m_1+i(\alpha)-1} + \frac{\partial}{\partial u}(c''_{m_q+m_1+j})/j - c''_{m_1+m_q+j}\frac{\partial}{\partial u}(A)/(jA) + (m_{q+1} - j) p_1 u^{-1} c''_{m_q+m_1+j}/(jq_1) + (1 - i(\alpha)) B_2 y_{m_q} - \frac{\partial}{\partial u}(B_2) x_{m_q}$
belongs to the image of the map (1.27), i.e. the equation (1.27) with the right side in the form of these expressions is solvable.

According to (1.32), (1.33), we need to show that

$$(1 - i(\alpha) + m_{q-1}) c_2 \omega_2 y_{m_q,2} u^{k_2-2+q_a} - (k_{2,m_1} + q_a - 2) c_2 \omega_2 x_{m_q,2} u^{k_2-3+q_a} + \frac{\partial}{\partial u}(b)/j + b\frac{\partial}{\partial u}(A)/(jA)$$
$$+(m_{q+1} - j) p_1 u^{-1} b/(jq_1) = 0 \mod \wp^{k_{2,m_{q+1}}+q_a-1}$$

where $b = m_q c_2 \omega_2 x_{m_q,2} u^{k_{2,m_1}-2+q_a}$, only if $(q-1)q_1(1-q_a-(q-1)p_1) = 0$ and

$$(1-i(\alpha)+m_{q-1})\omega_1 c_2 y_{m_q,1} u^{k_2-2+q_a} - (k_{2,m_1}-2+q_a)c_2\omega_1 x_{m_q,1} u^{k_2-3+q_a} + \frac{\partial}{\partial u}(b)/j + b\frac{\partial}{\partial u}(A)/(jA)$$

$$+(m_{q+1}-j)p_1 u^{-1} b/(jq_1) = 0 \mod \wp^{k_{1,m_{q+1}}+q_a-1}$$

where $b = m_q c_2 \omega_1 x_{m_q,1} u^{k_{2,m_1}-2+q_a}$ only if $-(q-1)^2 q_1 p_1 + (q-1)(q_a-1)q_1 + (1-j/q_1)(jp_1 + q_1(q_a-1)) = 0$ (we remark that ω_2 does not depend on ω_1).

Since $y_{m_q,1,2} = \frac{p_1-q_a+k_{1,2,m_q}}{j} x_{m_q,1,2} u^{-1} + \ldots$, it is necessary to show that

$$-(1-i(\alpha)+m_{q-1})\frac{p_1-q_a+k_{2,m_q}}{j} - (k_{2,m_1}-2+q_a) + \frac{m_q(k_{2,m_{q+1}}-1+q_a)}{j} -$$

$$\frac{m_q q_a}{j} + \frac{(m_{q+1}-j)p_1 q}{j} \neq 0$$

if $(q-1)q_1(1-q_a-(q-1)p_1) \neq 0$,

$$-(1-i(\alpha)+m_{q-1})\frac{p_1-q_a+k_{1,m_q}}{j} - (k_{2,m_1}-2+q_a) + \frac{m_q(k_{1,m_{q+1}}-1+q_a)}{j} -$$

$$\frac{m_q q_a}{j} + \frac{(m_{q+1}-j)p_1 q}{j} \neq 0$$

if $-(q-1)^2 q_1 p_1 + (q-1)(q_a-1)q_1 + (1-j/q_1)(jp_1 + q_1(q_a-1)) \neq 0$. But

$$-(1-i(\alpha)+m_{q-1})\frac{p_1-q_a+k_{2,m_q}}{j} - (k_{2,m_1}-2+q_a) + \frac{m_q(k_{2,m_{q+1}}-1+q_a)}{j} - \frac{m_q q_a}{j} + \frac{(m_{q+1}-j)p_1 q}{j}$$

$$= \frac{(q-1)q_1(1-q_a-(q-1)p_1)}{j},$$

$$-(1-i(\alpha)+m_{q-1})\frac{p_1-q_a+k_{1,m_q}}{j} - (k_{2,m_1}-1+q_a) + \frac{m_q(k_{1,m_{q+1}}-1+q_a)}{j} - \frac{m_q q_a}{j} + \frac{(m_{q+1}-j)p_1 q}{j}$$

$$= \frac{-(q-1)^2 q_1 p_1 + (q-1)(q_a-1)q_1 + (1-j/q_1)(jp_1 + q_1(q_a-1))}{j}$$

We observe here, that $-(q-1)^2 q_1 p_1 + (q-1)(q_a-1)q_1 + (1-j/q_1)(jp_1 + q_1(q_a-1)) \neq 0$. In fact, if $-(q-1)^2 q_1 p_1 + (q-1)(q_a-1)q_1 + (1-j/q_1)(jp_1 + q_1(q_a-1))$ has solutions in integers, then its discriminant must be equal to $q_1^2 l^2$, where $l \in \mathbb{Z}$. But $D = (q_a-1)^2 q_1^2 + 4(q_a-1)q_1 p_1(q_1-j) + 4p_1^2 j(q_1-j)$, whence follows, that $j(q_1-j) = (q_1-j)^2$, what is wrong.

So, we have shown that α is conjugated to α'':
$\alpha''(u) = \xi u + Az^j$,

$\alpha''(z) = z + Bz^{i(\alpha)} + B_2 z^{i(\alpha)+q_1} + B_3 z^{i(\alpha)+2q_1} + B_{j/q_1} z^{i(\alpha)+(1+j/q_1)q_1} + B_{q_2} z^{i(\alpha)+q_1(1+q_2)}$, where $B_{j/q_1} = cu^{k_{1,m_{j/q_1+1}}-2+q_a}$, $B_{q_2} = \tilde{c}u^{k_{2,m_{1+q_2}}-2+q_a}$ if $q_1(1-q_a-(q-1)p_1) = 0$ and $B_{q_2} = 0$ otherwise.

Let's show now, that B_3 can be taken as $c_{b_3}u^{k_{2,m_2}-2+q_a}$. In order to do that, we exhibit that in formulas (1.30)-(1.33) monomials with ω_2^2 belong to the image of the map (1.27). Then the case $q = 1$ is equivalent to a general case, and since $q = 1$ is one of the solutions of $(q-1)q_1(1-q_a-(q-1)p_1) = 0$, B_3 is defined in the same way as B_q. For that, according to (1.30)-(1.33), we must show that

$$\frac{\partial}{\partial u}(b)/j - b\frac{\partial}{\partial u}(A)/(jA) + (m_{q+1}-j)p_1 u^{-1} b/(jq_1) + a = 0$$

where $b = C_j^2 \omega_2^2 y_{2,m_1}^2 c_b q_1/p_1 u^{2k_{2,m_1}-2+q_a} + C_{q_a}^2 c_b q_1/p_1 \omega_2^2 u^{2k_{2,m_1}-2+q_a} + q_a j c_b q_1/p_1 \omega_2^2 (p_1 - q_a + k_{2,m_1})/j$, $a = C_{q_a-1}^2 + C_{i(\alpha)}^2 (p_1 - q_a + k_{2,m_1})^2/j^2 + (q_a-1)i(\alpha)(p_1 - q_a + k_{2,m_1})/j$. In fact,

$$C_j^2 \frac{(p_1-q_a+k_{2,m_1})^2 q_1}{j^3 p_1}(2k_{2,m_1}-2+q_a) + C_{q_a}^2 \frac{q_1(2k_{2,m_1}-2+q_a)}{p_1 j} +$$

$$q_a j \frac{q_1(p_1-q_a+k_{2,m_1})}{p_1 j^2}(2k_{2,m_1}-2+q_a) - \frac{q_a}{j}\left(C_j^2 \frac{(p_1-q_a+k_{2,m_1})^2 q_1}{j^2 p_1} + C_{q_a}^2 \frac{q_1}{p_1} +\right.$$

$$\left. q_a j \frac{q_1(p_1-q_a+k_{2,m_1})}{p_1 j}\right) + \frac{m_2-j}{j}\left(C_j^2 \frac{(p_1-q_a+k_{2,m_1})^2}{j^2} + C_{q_a}^2 + q_a j \frac{p_1-q_a+k_{2,m_1}}{j}\right) + C_{q_a-1}^2 +$$

$$C_{i(\alpha)}^2 (p_1-q_a+k_{2,m_1})^2/j^2 + (q_a-1)i(\alpha)(p_1-q_a+k_{2,m_1})/j = 0,$$

and it proves our assumption.

The case $k_{1,m_q} = k_{2,m_q}$ is more simple, and all the arguments remain true.

b)

In this case the system $-\frac{k_1-q_a}{m-j} = \frac{p_1}{q_1}$, $-\frac{k_2-1}{m} = \frac{p_1}{q_1}$ is incompatible, that is why for all m the "cokernel" of the map (1.27) is one-dimensional, $A = c_1 u^{q_a}$, $B = c_2 u^{q_a-1}$.

Let denote as k_{1,n_q} the solution of the equation $-\frac{k_1-q_a}{n_q-j} = \frac{p_1}{q_1}$, and as k_{2,m_q} the solution of the equation $-\frac{k_2-1}{m_q} = \frac{p_1}{q_1}$. It is clear that $m_q = qm_1 = qq_1$, as in the case a), and $n_{q+1} = n_q + m_1$ only if $n_q + m_1 \neq j$. But in this case the next value of n_{q+1} is $n_q + 2m_1$, so we consider this recurrence relation to be true always. Further, $(k_{1,n_q} - k_{2,m_q})$ doesn't depend on q, as in a), and $k_{2,m_{q+1}} = k_{2,m_q} + k_{2,m_1} - 1$, $k_{1,n_{q+1}} = k_{1,n_q} + k_{2,m_1} - 1$.

The proof, that follows, would be divided into three cases:
1) $q_1 < j$ (see case B) 2) b) ii)),
2) $q_1 > j$, $j \not| q_1$ (see case B) 2) c) i)),
3) $j | q_1$ (see case B) 2) c) ii)).

We put $B_2 = c_1 u^{k_{1,n_1}-2+q_a}$, $B_3 = c_2 u^{k_{2,m_1}-2+q_a}$, if $n_1 < m_1$, and $B_2 = c_1 u^{k_{2,m_1}-2+q_a}$, $B_3 = c_2 u^{k_{1,n_1}-2+q_a}$ otherwise.

In the case 1)
$n_1, m_1 < j$, $n_1 < m_1$. The idea of the following proof is the following: we look for the sequential conjugations f_{n_q} and f_{m_q}, where $f_{n_q}(u) = u + x_{n_q} z^{n_q}$, $f_{n_q}(z) = z + y_{n_q} z^{n_q}$, and x_{n_q}, y_{n_q} are solutions of the homogeneous system (1.21), (1.22) with $m = n_q$; $f_{m_q}(u) = u + x_{m_q} z^{m_q}$, $f_{m_q}(z) = z + y_{m_q} z^{m_q}$, where x_{m_q}, y_{m_q} are solutions of the homogeneous system (1.21), (1.22) with $m = m_q$. We choose $x_{n_q}, y_{n_q}, x_{m_q}, y_{m_q}$ so that with these conjugations the equation (1.26) become solvable with $m = n_{q+1}$, $m = m_{q+1}$.

For the automorphism f_{m_q} we can use the results from the case a), because x_{m_q} and y_{m_q} here have the form $\omega_2 x_{m_q,2}$, $\omega_2 y_{m_q,2}$, $\bar{\nu}(y_{m_q,2}) = k_{2,m_q} - 1$, $\bar{\nu}(x_{m_q,2}) = k_{2,m_q}$, and $m_1 = q_1 < j$, $j \neq m_q$. We rewrite for f_{n_q} the formula (1.29):
if $q > 1$, then

$$\alpha' f_{n_q}(u) = \xi u + Az^j + c'_{n_q+j+m_1} z^{n_q+j+m_1} + \ldots + \xi x_{n_q} z^{n_q} + \frac{\partial}{\partial u}(x_{n_q}) A z^{n_q+j} +$$
$$n_q B x_{n_q} z^{n_q+j} + n_q B_2 x_{n_q} z^{n_q+n_1+j} + n_q B_3 x_{n_q} z^{n_q+m_1+j} + \ldots$$
$$f_{n_q} \alpha''(u) = \xi u + \xi x_{n_q} z^{n_q} + Az^j + \frac{\partial}{\partial u}(A) x_{n_q} z^{n_q+j} + j y_{n_q} A z^{n_q+j} +$$
$$c''_{n_q+n_1+j} z^{n_q+n_1+j} + c''_{n_q+m_1+j} z^{n_q+m_1+j} + \ldots$$

(since $n_q > m_1$, there are no more terms with $z^{n_q+m_1+j}$),

$$\alpha' f_{n_q}(z) = z + Bz^{i(\alpha)} + B_2 z^{i(\alpha)+n_1} + B_3 z^{i(\alpha)+m_1} + a'_{n_q+m_1+i(\alpha)-1} z^{n_q+m_1+i(\alpha)} + \ldots + y_{n_q} z^{n_q+1} +$$
$$\frac{\partial}{\partial u}(y_{n_q}) A z^{n_q+i(\alpha)} + (n_q+1) B y_{n_q} z^{n_q+i(\alpha)} + (n_q+1) B_2 y_{n_q} z^{n_q+n_1+i(\alpha)} + (n_q+1) B_3 y_{n_q} z^{n_q+m_1+i(\alpha)}$$
$$f_{n_q} \alpha''(z) = z + y_{n_q} z^{n_q+1} + Bz^{i(\alpha)} + B_2 z^{i(\alpha)+m_1} + B_3 z^{i(\alpha)+n_1} + \frac{\partial}{\partial u}(B) x_{n_q} z^{n_q+i(\alpha)} + i(\alpha) y_{n_q} B z^{n_q+i(\alpha)} +$$
$$\frac{\partial}{\partial u}(B_2) x_{n_q} z^{i(\alpha)+n_1+n_q} + (i(\alpha)+n_1) y_{n_q} B_2 z^{n_1+n_q+i(\alpha)} + \frac{\partial}{\partial u}(B_3) x_{n_q} z^{i(\alpha)+m_1+n_q} +$$
$$(i(\alpha)+m_1) y_{n_q} B_3 z^{m_1+n_q+i(\alpha)} + a''_{n_q+n_1+i(\alpha)-1} z^{n_q+n_1+i(\alpha)} + a''_{n_q+m_1+i(\alpha)-1} z^{n_q+m_1+i(\alpha)} + \ldots$$

The formula remains true for $q = 1$ also, as it is seen from the calculations, similar to $q = 1$ in case a).

If $n_1 | m_1$ and $k_{1,n_q} + (m_1/n_1 - 1) k_{1,n_1} = k_{2,m_q}$, i.e. $m_1/n_1 k_{1,n_1} = k_{2,m_1}$, then the coefficient $a''_{n_q+m_1+i(\alpha)-1}$ ($c''_{n_q+m_1+i(\alpha)-1}$) depends on $a''_{n_q+n_1+i(\alpha)-1}$ ($c''_{n_q+n_1+i(\alpha)-1}$) in a general case. In this situation for almost all q the conjugation f_{n_q} can be chosen so that the equation (1.26) is solvable for $m = m_q$, and f_{m_q} so that equation (1.26) is solvable for $m = n_{q+1}$.

Thus, the arguments, similar to the case a), tell us that α is conjugated to β:
$\beta(u) = \xi u + Az^j$,
$\beta(z) = z + Bz^{i(\alpha)} + B_2 z^{i(\alpha)+n_1} + B_q z^{i(\alpha)+m_q} + B_{q_2} z^{i(\alpha)+n_{q_2}}$,

where $B_q = cu^{k_{2,m_q}-2+q_a}$ or equals to zero. It depends on that, if at least one expression from

$$\frac{\partial}{\partial u}(b)/j - b\frac{\partial}{\partial u}(A)/(jA) + (m_{q+1} - j)p_1 u^{-1} b/(jq_1) + a$$

equals to zero or not, with $b = u^{k_{1,n_q}+wk_{1,n_1}-2+q_a} n_q$, $a = (k_{1,n_1} - 2 + q_a)u^{k_{1,n_q}+wk_{1,n_1}-3+q_a} + (1 - i(\alpha) + m_{q-1})u^{k_{1,n_q}+wk_{1,n_1}-3+q_a}(p_1 - q_a + k_{1,n_q})/j$, w has the values from 1 to $q_1/n_1 - 1$, in other words, if equals to zero at least one of expressions

$$-(1+w)n_1^2 p_1 + n_1 jw - (q-1)(2+w)q_1 n_1 + q_1(-2 + (1+w)q_a) +$$

$$q_1[p_1(j(q-1)(w+2) + j - (q-1)^2 q_1 + (q-1)q_1 + 2q_1) + 2j(q_a - 1) + (q-1)q_1((1+w)q_a - 2)] \tag{1.34}$$

Further, $B_{q_2} = cu^{k_{1,n_{q_2}}-2+q_a}$ or zero, in accordance with equality to zero of the expression

$$\frac{\partial}{\partial u}(b)/j - b\frac{\partial}{\partial u}(A)/(jA) + (n_{q_2} - j)p_1 u^{-1} b/(jq_1) + a$$

with $b = n_{q_2-1} u^{k_{2,m_{q_2-1}}+k_{1,n_1}-2+q_a}$, $a = (k_{1,n_1} - 2 + q_a)u^{k_{2,m_{q_2-1}}+k_{1,n_1}-3+q_a} + (1 - i(\alpha) + m_{q_2-2})u^{k_{2,m_{q_2-1}}+k_{1,n_1}-3+q_a}(p_1 - q_a + k_{2,m_{q_2-1}})/j$, i.e.

$$-(q_2-2)^2 q_1^2 p_1 + (q_2-2)q_1 j p_1 + p_1 j(j - n_1) + (3j + n_1)q_1(q_a - 1) \tag{1.35}$$

We note, that this equation doesn't have solutions in integers, i.e. $B_{q_2} = 0$. Really, its discriminant must be equal to $q_1^2 p_1^2 l^2$, $l \in \mathbb{Z}$. But $D = q_1^2 j^2 p_1^2 + 4q_1^2 p_1^2 j(j - n_1) + 4q_1^3 p_1(q_a - 1)(3j + n_1)$, hence $0 < p_1^2 q_1^2 (j - n_1)^2 = q_1^2 p_1(q_a - 1)(3j + n_1) < 0$, a contradiction. Thus we have proved the case B) 2) b) ii) i').

If $n_1 \not| m_1$ or $(m_1/n_1)k_{1,n_1} \neq k_{2,m_1}$, then the solvability of the equation (1.26) for $m = n_{q+1}$, $a = a''_{n_q+m_1+i(\alpha)-1}$, $b = c''_{n_q+m_1+i(\alpha)-1}$ doesn't depend on coefficients $a''_{n_q+n_1+i(\alpha)-1}$ ($c''_{n_q+n_1+i(\alpha)-1}$). In this case for almost all q f_{n_q} can be chosen so that the equation (1.26) is solvable for $m = n_{q+1}$, and f_{m_q} so that equation (1.26) is solvable for $m = m_{q+1}$. Not very complicated modification of all arguments, mentioned before, leads us to the conclusion, that α is conjugated to β:
$\beta(u) = \xi u + Az^j$,
$\beta(z) = z + Bz^{i(\alpha)} + B_2 z^{i(\alpha)+n_1} + B_3 z^{i(\alpha)+m_1} + B_4 z^{i(\alpha)+2m_1} + B_q z^{i(\alpha)+m_1} + B_{q_2} z^{i(\alpha)+n_{q_2}}$,
where $B_q = cu^{k_{2,m_q}-2+q_a}$ or equals to zero in accordance with equality to zero of the expression
$q_1(1 - q_a - (q-1)p_1)$,
$B_4 = B_q$ for $q = 1$, $B_{q_2} = cu^{k_{1,n_{q_2}}-2+q_a}$ or equals to zero in accordance with equality to zero of the expression

$$\frac{\partial}{\partial u}(b)/j - b\frac{\partial}{\partial u}(A)/(jA) + (n_{q_2} - j)p_1 u^{-1} b/(jq_1) + a = 0$$

with $b = n_{q_2-1}u^{k_{1,n_{q_2-1}}+k_{2,m_1}-2+q_a}$, $a = (k_{2,m_1}-2+q_a)u^{k_{1,n_{q_2-1}}+k_{2,m_1}-3+q_a} + (1-i(\alpha)+n_{q_2-2})u^{k_{1,n_{q_2-1}}+k_{2,m_1}-3+q_a}(p_1-q_a+k_{1,n_{q_2-1}})/j$, i.e.

$$-n_1^2p_1-q_1(p_1(2j+q_1(q_2-3)^2)-(j+q_1(q_2-2))(q_a-1))+n_1q_1(-1-2p_1(q_2-3)+q_a) \quad (1.36)$$

This equation has no solutions in integers by the same reasons as (1.35), where from $B_{q_2}=0$ (see case B) 2) b) ii) ii')).

In the case 2)
$m_1 < n_1$, and since $j \not| q_1$, we can apply here the arguments from the case 1). Then the result would coincide with the result of the previous paragraph.

In the case 3)
$m_1 < n_1$, but $j|q_1$, so we rewrite the formula (1.29) for the conjugation f_{m_q}, $q \geq 1$ in the following way:

$$\alpha' f_{m_q}(u) = \xi u + Az^j + c'_{m_q+m_1+j}z^{m_q+m_1+j} + \ldots + \xi x_{m_q}z^{m_q} + \frac{\partial}{\partial u}(x_{m_q})Az^{m_q+j}+$$

$$m_qBx_{m_q}z^{m_q+j} + \frac{1}{2}\frac{\partial^2}{\partial u^2}(x_{m_q})A^2z^{m_q+2j} + C^2_{m_q}B^2x_{m_q}z^{m_q+2j} + m_qB\frac{\partial}{\partial u}(x_{m_q})Az^{m_q+2j}+$$
$$m_qB_2x_{m_q}z^{m_q+m_1+j} + m_qB_3x_{m_q}z^{m_q+n_1+j} + \ldots$$

$$f_{m_q}\alpha''(u) = \xi u + \xi x_{m_q}z^{m_q} + Az^{m_q} + Az^j + \frac{\partial}{\partial u}(A)x_{m_q}z^{m_q+j}+$$
$$jy_{m_q}Az^{m_q+j} + c''_{n_q+j}z^{n_q+j} + \ldots + c''_{m_q+m_1+j}z^{m_q+m_1+j} + \ldots$$

$$\alpha' f_{m_q}(z) = z+Bz^{i(\alpha)}+B_2z^{i(\alpha)+m_1}+B_3z^{i(\alpha)+n_1}+a'_{m_q+m_1+i(\alpha)-1}z^{m_q+m_1+i(\alpha)}+\ldots+y_{m_q}z^{m_q+1}+$$
$$\frac{\partial}{\partial u}(y_{m_q})Az^{m_q+i(\alpha)}+(m_q+1)By_{m_q}z^{m_q+i(\alpha)}+\frac{1}{2}\frac{\partial^2}{\partial u^2}(y_{m_q})A^2z^{m_q+2i(\alpha)-1}+C^2_{m_q+1}B^2y_{m_q}z^{m_q+2i(\alpha)-1}+$$
$$(m_q+1)B\frac{\partial}{\partial u}(y_{m_q})Az^{m_q+2i(\alpha)-1} + \ldots + (m_q+1)B-2y_{m_q}z^{m_q+m_1+i(\alpha)} + \ldots$$

$$f_{m_q}\alpha''(z) = z + y_{m_q}z^{m_q+1} + Bz^{i(\alpha)} + B_2z^{i(\alpha)+m_1} + B_3z^{i(\alpha)+n_1}+$$
$$\frac{\partial}{\partial u}(B)x_{m_q}z^{m_q+i(\alpha)} + i(\alpha)y_{m_q}Bz^{m_q+i(\alpha)} + \frac{\partial}{\partial u}(B_2)x_{m_q}z^{i(\alpha)+m_1+m_q}+$$
$$a''_{n_q+i(\alpha)-1}z^{n_q+i(\alpha)} + \ldots + a''_{m_q+m_1+i(\alpha)-1}z^{m_q+m_1+i(\alpha)} + (i(\alpha)+m_1)y_{m_q}B_2z^{m_1+m_q+i(\alpha)} + \ldots$$

Hence follows, that two cases are possible:

a') the solvability of the equation (1.26) for $m = m_q + m_1$ doesn't depend on coefficients $c''_{n_q+j}, \ldots, c''_{m_q+m_1}$, $a''_{n_q+j}, \ldots, a''_{n_1+m_1}$, that is it depends only on $c''_{m_q+m_1+j}$, $a''_{m_q+m_1+i(\alpha)-1}$ (and it is equal to $-p_1+q_a-2 \neq q_1(q_a-1)/j$).

b') the solvability of the equation (1.26) for $m = m_q + m_1$ depends on coefficients $c''_{n_q+j}, \ldots, c''_{m_q+m_1}, a''_{n_q+j}, \ldots, a''_{n_1+m_1}$, i.e. $-p_1 + q_a - 2 = q_1(q_a - 1)/j$.

In the case a'), repeating the proof as in the previous cases, we get the same result as in the case 2) (it corresponds to a case B) 2) c) ii) i')).

In the case b') the conjugations f_{m_q} determine the solvability of the equation (1.26) for $m = n_q$, and conjugations f_{n_q} — for $m = m_{q+1}$. Here, in this case of $m = n_q$, as seen from the formula above, can exist not more than two q — solutions of the equation

$$\frac{\partial}{\partial u}(b)/j - b\frac{\partial}{\partial u}(A)/(jA) + (n_{q_2} - j)p_1 u^{-1} b/(jq_1) + a = 0$$

with $b = 2^{-1}\frac{\partial^2}{\partial u^2}(u_{k_{2,m_q}})u^{2q_a} + C^2_{m_q}u^{2q_a-2}p_1^2/q_1^2 u^{k_{2,m_q}} + m_q p_1/q_1 u^{2q_a-1}\frac{\partial}{\partial u}(u^{k_{2,m_q}})$,
$a = 2^{-1}\frac{\partial^2}{\partial u^2}(u_{k_{2,m_q}-1})u^{2q_a}(p_1 - q_a + k_{2,m_q})/j + C^2_{m_q+1}u^{2q_a-2}p_1^2/q_1^2 u^{k_{2,m_q}-1}(p_1 - q_a + k_{2,m_q})/j + (m_q+1)p_1/q_1 u^{2q_a-1}\frac{\partial}{\partial u}(u^{k_{2,m_q}-1})(p_1 - q_a + k_{2,m_q})/j$, i.e.

$$(p_1^2(-1+q-q_1(-1+q-2q^2+qq_1)) - qq_1^2(q_a-1) + p_1q_1(1-3q-(q-1)qq_1+qq_a)) = 0 \quad (1.37)$$

and in the case $m = m_{q+1}$ not more than $(m_{q+1} - n_q)/j + 1 = q_1/j = w\ q$ — solutions of the appropriate equation

$$\frac{\partial}{\partial u}(b)/j - b\frac{\partial}{\partial u}(A)/(jA) + (n_{q_2} - j)p_1 u^{-1} b/(jq_1) + a = 0 \quad (1.38)$$

Thus, α is conjugated to β:
$\beta(u) = \xi u + Az^j$,
$\beta(z) = z + Bz^{i(\alpha)} + B_2 z^{i(\alpha)+m_1} + B_{q_n,1}z^{i(\alpha)+n_{q_1}} + B_{q_n,2}z^{i(\alpha)+n_{q_2}} + B_{q_m,1}z^{i(\alpha)+m_{q_1}} + \ldots + B_{q_m,w}z^{i(\alpha)+m_{q_w}}$,
where $B_{q_n,i} = c_i u^{k_{1,n_{q_i}}-2+q_a}$, $B_{q_m,j} = c_j u^{k_{2,m_{q_j}}-2+q_a}$ or 0 depending on solvability of corresponding equations (it is the case B) 2) c) ii) ii')).

In the case c),
when $q_1 = j$, we use arguments from both: the previous case and the case a), and get from there, that α is conjugated to β:
$\beta(u) = \xi u + Az^j$,
$\beta(z) = z + Bz^{i(\alpha)} + B_2 z^{2i(\alpha)-1} + B_3 z^{3i(\alpha)-2} + B_{q_m,1,1}z^{i(\alpha)+q_0q_1} + B_{q_m,1,2}z^{i(\alpha)+q_2q_1} + B_{q_m,1,3}z^{i(\alpha)+q_3q_1} + B_{q_m,2,1}z^{i(\alpha)+q'_1q_1} + B_{q_m,2,2}z^{i(\alpha)+q'_2q_1} + B_{q_m,2,3}z^{i(\alpha)+q'_3q_1}$,
where $B_2 = c_{b_2}u^{-p_1-1+q_a}$, $B_3 = c_{b_3}u^{-p_1+2q_a-2}$, $B_{q_m,1,i} = c_{b_{q_m,1,i}}u^{-p_1q_1^{-1}(q_iq_1-j)+2q_a-2}$, $B_{q_m,2,j} = c_{b_{q_m,2,j}}u^{-p_1q'_j+q_a-1}$ or 0 depending on solvability of corresponding equations $\frac{\partial}{\partial u}(b)/j - b\frac{\partial}{\partial u}(A)/(jA) + (n_{q_2} - j)p_1u^{-1}b/(jq_1) + a = 0$. If we denote as b_{50}, a_{50} b and a in (1.37), then the appropriate b, a, b', a' for two equations are equal to

$$b = b_{50} + c_a^{-2}m_q u^{k_{m_q,1}}u^{-p_1+1-2+q_a}$$

$$a = a_{50} + c_a^{-2}(1 - i(\alpha) + m_q - m_1)u^{-p_1+q_a-2+q_a}u^{k_{m_q,1}-1}(p_1 - q_a + k_{1,m_q})/j \qquad (1.39)$$

$$b' = b_{50} + c_a^{-2}m_q u^{k_{m_q,2}}u^{-p_1+1-2+q_a}$$

$$a = a_{50} + c_a^{-2}(1 - i(\alpha) + m_q - m_1)u^{-p_1+q_a-2+q_a}u^{k_{m_q,2}-1}(p_1 - q_a + k_{2,m_q})/j \qquad (1.40)$$

where c_a is a constant for the coefficient A, and the rest notations are taken from the case a). By direct calculations it is not difficult to show, that these equations are not solvable.

The proof of the last statement in the theorem is similar to the proof of the statement in the theorem 2.

The theorem is proved.

□

Bibliography

[1] A. A. Albert. Normal division algebras of degree p^l over f of characteristic p. *Trans. Amer. Soc.*, 40:112–126, 1936.

[2] G. Azumaya. On maximally central algebras. *Nagoya J.Math.*, 2:119–150, 1951.

[3] R. Camina. Subgroups of the nottingham group. *J.Algebra*, 196:101–113, 1997.

[4] P.M. Cohn. *Skew-fields*. Cambridge University Press, 1997.

[5] M. de Sautoy and I. Fesenko. Where the wild things are: ramification groups and the nottingham group. preprint.

[6] E.E. Demidov. *The Kadomzev-Petviashvili Hierarchy and Shottki's problem*. Independent Moscow University, Math. college, Moscow, 1995.

[7] T. Fimmel and A.N. Parshin. An introduction to the higher adelic theory. book in preparation.

[8] A. Gurevich. *Description of Galois groups of local fields with the aid of power series*. PhD thesis, Humboldt University Berlin, 1998.

[9] B. Jacob and A. Wadsworth. Division algebras over henselian fields. *J.Algebra*, 128:126–179, 1990.

[10] S.A. Jennings. Substitution groups of formal power series. *Canad. J. Math.*, 41:325–340, 1954.

[11] D. L. Johnson. The group of formal power series under substitution. *J.Austral. Math. Soc. (Series A)*, 45:298–302, 1988.

[12] B. Klopsch. *Normal subgroups and automorphisms of substitution groups of formal power series*. PhD thesis, Oxford University, 1997.

[13] F. Laubie. Extensions de lie et groups d'automorphismes de corps locaux. *Comp. Math.*, 67:165–189, 1988.

[14] F. Laubie and M. Saine. Ramification of automorphisms of $k((t))$. *J.Number theory*, 63:143–145, 1997.

[15] V. G. Lomadze. On residues in algebraic geometry. *Izv. Akad. Nauk SSSR, Ser. Mat.*, 45(6):1258–1287, 1981.

[16] M. Mulase. Category of vector bundles on algebraic curves and infinite dimensional grassmanians. *Intern. J. of Math.*, 1(3):293–342, 1990.

[17] A. N. Parshin. Infinite grassmanians and vector bundles. preprint.

[18] A. N. Parshin. Krichever correspondence for algebraic surfaces. preprint TU Braunschweig, Bericht 99/17. e-print math.AG/9911097.

[19] A. N. Parshin. On the arithmetic of two-dimensional schemes. i. distributions and residues. *Izv. Akad. Nauk SSSR, Ser. Mat.*, 40:695–729, 1976.

[20] A. N. Parshin. Abelian coverings of arithmetic schemes. *Soviet Math. Dokl.*, 19:1438–1442, 1978.

[21] A. N. Parshin. Brauer group. *Trud. mat. Inst. AN SSSR*, 165:143–170, 1984. English transl. in Proceedings of the Steclov Institute of Mathematics 1985, Issue 3.

[22] A. N. Parshin. Local class field theory. *Trudy Matem. Inst. Akad. Nauk SSSR*, 165:143–170, 1984.

[23] A. N. Parshin. On the ring of formal pseudodifferential operators. *Trudy Mat. Inst. Steklov. Akad. Nauk SSSR*, 224:291–305, 1999. English transl.

[24] A.N. Parshin. Chern classes, adeles and l-functions. *Journ. für die reine und angewandte Mathematik*, 341, 1983.

[25] R. S. Pierce. *Associative Algebras.* Springer-Verlag, New-York, 1982.

[26] V.P. Platonov and V.I. Yanchevskii. Finite-dimensional skew fields. *VINITI, Moscow*, 77:144–262, 1991. English transl. in A.I.Kostrikin and I.R.Shafarevich (eds.), Algebra IX, Encyclopaedia Math. Sci., vol. 77, Springer, Berlin, 1995, pp. 121-239.

[27] I. Quandt. On a relative version of the krichever correspondence. e-print alg-geom/9606015.

[28] D. J. Saltman. Division algebras over discrete valued fields. *Comm. Algebra*, 8:1749–1774, 1980.

[29] G. Segal and G. Wilson. Loop groups and equations of kdv type. *Publ. Math. IHES*, 61:5–65, 1985.

[30] J. P. Serre. *Corps locaux*. Hermann, Paris, 1962.

[31] O. F. G. Shilling. *The theory of Valuations*. Math. Surveys. — Amer. Math. Soc. Providence, R. I., 1956.

[32] J.-P. Tignol. Algebres a division et extensions de corps sauvagement ramifiees de degre premier. *J.Reine Angew. Math.*, 404:1–38, 1990.

[33] A. Wadsworth. *Valuation theory on finite dimensional division algebras*, volume 00. Fields institute Communications, 1997.

[34] E. Witt. Schiefkörper über diskret bewerteten körpern. *J.Reine Angew. Math.*, 176:153–156, 1936.

[35] A. Yekutieli. An explicit construction of the grothendieck residue complex. *Asterisque*, 208, 1992.

[36] I.O. York. *The group of formal power series under substitution*. PhD thesis, University of Nottingham, 1990.

[37] A.B. Zheglov. Higher local skew-fields. In I. Fesenko and M. Kurihara, editors, *Invitation to higher local fields*, volume 3, pages 281–292. Geometry and Topology Gournal, Warwick, 2000.