Aspects of Noncommutativity and Holography in Field Theory and String Theory

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To my parents

Abstract

This thesis addresses two topics: noncommutative Yang-Mills theories and the AdS/CFT correspondence.

In the first part we study a partial summation of the θ-expanded perturbation theory. The latter allows one to define noncommutative Yang-Mills theories with arbitrary gauge groups G as a perturbation expansion in the noncommutativity parameter θ. We show that for $G \subset U(N)$, $G \neq U(M)$, $M < N$, one does not find a finite set of θ-summed Feynman rules.

In the second part we study quantities which are important for the realization of the holographic principle in the AdS/CFT correspondence: boundaries, geodesics and the propagators of scalar fields. They should play a role in the holographic setup in the BMN limit as well. We observe how these quantities behave in the limiting process from $\text{AdS}_5 \times \text{S}^5$ to the 10-dimensional plane wave which is the spacetime in the BMN limit.

Keywords:
Noncommutative Yang-Mills theory
Feynman rules
AdS/CFT and BMN correspondence
Holographic principle

Zusammenfassung

Die Arbeit beschäftigt sich mit zwei Themen: den nichtkommutativen Yang-Mills-Theorien und der AdS/CFT-Korrespondenz.

Im ersten Teil wird eine teilweise Aufsummation der θ-entwickelten Störungstheorie untersucht. Letztere stellt einen Weg dar, nichtkommutative Yang-Mills-Theorien mit beliebigen Eichgruppen G als Störungsentwicklung im Nichtkommutativitätsparameter θ zu definieren. Es wird gezeigt, daß man im Fall $G \subset U(N)$, $G \neq U(M)$, $M < N$ keinen endlichen Satz von θ-summierten Feynman-Regeln finden kann.

Im zweiten Teil werden Bausteine untersucht, die für eine Realisierung des holographischen Prinzips in der AdS/CFT-Korrespondenz von Bedeutung sind: Ränder, Geodäten und die Propagatoren skalarer Felder. Sie sollten auch für eine holographische Formulierung im BMN Limes wichtig sein. Das Verhalten dieser Größen im Limesprozeß von $AdS_5 \times S^5$ zu der 10-dimensionalen ebenen Gravitationswelle, welche die Raumzeit im BMN-Limes ist, wird studiert.

Schlagwörter:

Nichtkommutative Yang-Mills-Theorie

Feynman-Regeln

AdS/CFT- und BMN-Korrespondenz

Holgraphisches Prinzip

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Part I

Introduction

Chapter 1

General Introduction

In the energy range accessible with present experiments, three of the four fundamental interactions are successfully described by quantum field theories. The latter provide a theoretical description of the strong, weak and electromagnetic interactions which are collectively denoted as the Standard Model (of particle physics). The remaining interaction is gravity and its classical description is given by the theory of General Relativity.

The quantum field theories in the Standard Model are all examples of gauge theories, where spin one particles are responsible for transmitting the interaction. Gauge theories contain more degrees of freedom than necessary for the description of the physical (measurable) quantities. The gauge transformations relate physically equivalent field configurations, and thus form a group such that a sequential application of two gauge transformations is itself a gauge transformation. For example, the gauge group of the Standard Model is given by $SU(3) \times SU(2) \times U(1)$, where the first factor refers to the theory of the strong interaction which is called quantum chromodynamics (QCD), and the second and third factors describe the electroweak interaction that includes the weak and the electromagnetic parts. In addition to the gauge fields, a gauge theory may contain additional fields the gauge fields interact with. In the case of the Standard Model these are the fermionic quark and lepton fields and the scalar Higgs field.

Naively, one would expect that a direct observation of the particles that are associated with the Standard Model fields is possible in their elementary form. Depending on the interaction, one furthermore should find several compound systems of the elementary particles, bound together by the fundamental forces. But this naive expectation is not quite true. For instance, one does not observe free quarks. The hadrons, which are bound

states of the the strong interaction, behave differently compared to electromagnetic bound states. Consider for instance positronium which is a state of an electron and a positron bound via the electromagnetic force. If one separates the electron from the positron, the force of electromagnetism decreases, and it is possible to break up a positronium into its elementary constituents. In contrast to this, trying to separate two quarks inside the hadron leads to an effectively constant force, corresponding to an effective potential which is linearly increasing with the distance. Hence at a certain point, where the potential energy is high enough to create a quark anti-quark pair, the hadron breaks up into two hadrons, such that one does not find free quarks. A necessary (but not sufficient) ingredient for this behaviour is the interaction of the gauge bosons with each other. This is one of the main differences between QCD and quantum electrodynamics (QED) and is due to the gauge group of QCD being non-Abelian, whereas QED has an Abelian gauge group.

The absence of unbound quarks in nature makes it difficult to show their existence. Hence, it becomes understandable that before the formulation of QCD, much work was spent in trying to explain the spectrum of hadrons without taking into account that they might be non-fundamental composite objects. One of the surprising issues of the hadronic spectrum is that the hadrons can be sorted into groups in such a way that, within every group, one finds a linear relation between the mass squares m^2 of the hadrons and their spins J. In the linear relation $\alpha' m^2 = \alpha_0 + J$ the slope $\alpha' \sim 1\,\mathrm{GeV}^{-2}$ is universal for all hadrons, only the intercept α_0 is different for each group of hadrons. If one plots m^2 versus J one obtains the famous Regge trajectories [146, 147]. They were realized within dual models[1] [159, 181, 183] which successfully describe the behaviour of hadron scattering amplitudes in the so called Regge regime. For example, a $2 \to 2$ scattering process in four dimensions is specified by the three Mandelstam variables s, t and u as kinematical invariants. Only two of them, choose for instance s and t being related to the center of mass energy and the scattering angle in the process, are independent. In the Regge regime, which is given by $s \to \infty$ with $t = $ fixed, the scaling behaviour of the scattering amplitude proportional to $s^{\alpha(t)}$ with $\alpha(t) = \alpha_0 + \alpha' t$ is successfully reproduced by the dual models. It was discovered [126, 128, 167] that the dual models describe the dynamics of relativistic strings and thus we will henceforth denote them as (hadronic) string theories.

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[1]For an introduction with a detailed summary of the historical developments see [154].

Besides the above outlined successes these string theories contain some issues that are unwanted in a theory of hadrons. They predict the existence of a variety of massless particles not detected in the hadronic spectrum. Depending on the concrete model, different values for the intercept α_0 are theoretically favoured[2] that are in disagreement with the phenomenologically preferred value $\alpha_0 = \frac{1}{2}$. Furthermore, the behaviour of the scattering amplitudes in the hard fixed angle scattering regime, where s and t are large with fixed ratio $\frac{s}{t}$, is too soft compared to the experimental results. Here the string theories predict an exponential falloff, while experimentally one finds a powerlike behaviour. Experimental data indicate that the probed structure in this regime is not an extended object but instead a pointlike particle. This means that the probe particles no longer interact with the hadron as a whole but with pointlike constituents from which the latter is built.

Hence, a theory of hadrons is not a theory of fundamental objects. It turned out that QCD is the appropriate description of the constituents from which hadrons are built. But the strong interactions are far from being understood completely. Perturbative QCD is very successful in describing the phenomenology of strong interactions at high energies but the effects at low energy, where the coupling constant is large and hence perturbation theory is not applicable, are still hard to analyze. The absence of free quarks described above is an effect of this property called confinement, that until now cannot be successfully explained. The hadronic string theories can be interpreted as effective descriptions of confinement where the open string collects the effects of the flux tubes transmitting the forces between the quarks that are situated at the endpoints of this QCD string (see e. g. [39, 141, 143, 144, 157] and references therein). To be able to analyze quantum field theories non-perturbatively and thereby understand the mechanism of confinement beyond such an effective description is a challenge for theoretical physics.

Besides this fact there is another lack of understanding. General Relativity is the relevant theory of gravity at length scales that are large compared to the fundamental length of gravity, the Planck length $l_{\mathrm{P}} = M_{\mathrm{P}}^{-1}$ ($M_{\mathrm{P}} = 1.22 \times 10^{19}\,\mathrm{GeV}$ is the Planck mass). But a description of gravity at very short distances comparable to the Planck length requires a quantum theory of gravity. The naive attempt to quantize general relativity fails. To be more precise, quantum field theories suffer from divergencies that have to be regularized and absorbed into the physical parameters of the theory. This procedure is known as renormalization. In contrast to QCD and the electroweak interaction which

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[2]An enhancement of symmetries [184] and the decoupling of negative norm states [38, 80] is found.

are renormalizable quantum field theories[3], naive quantum gravity is non-renormalizable. The fields with highest spin for which a quantum field theoretical description is known are the spin one gauge fields, but the graviton (the quantum of gravitation) carries spin two.

One fundamental difference between gravity and gauge theories is that spacetime itself is affected by gravity and hence is dynamical. A good way to understand this is to think about what happens if one wants to observe smaller and smaller structures in spacetime. The probe wavelength has to be comparable or smaller than the minimum length one wants to resolve. This means one has to put higher and higher energy into the system. Energy is a source of gravitation and hence the spacetime is deformed in the measurement process. This is a good reason why the spacetime at small scales of the order of magnitude of the Planck length would look differently from what one observes at large scales. In particular, if one increases the energy above a critical value, the gravitational collapse of the region would be inevitable and the desired information would be absorbed by the black hole which would form. The density of quantum states of a black hole turns out to depend on the area of the horizon and it suggests that only one bit of information can be stored in a surface element of the order of the squared Planck length [168, 175, 176]. Hence, a formulation of quantum gravity should include a mechanism that makes it impossible to resolve structures that are smaller than the Planck length. This clearly introduces an upper bound on momenta and hence an ultraviolet cutoff in the quantum theory.

One way to realize such a cutoff is to introduce extended objects to replace the pointlike particles as fundamental objects. Heuristically speaking, to smear out interactions and make them non-local prohibits one from probing the spacetime at arbitrarily small scales. The string theories, that originally appeared as a proposal to describe the hadrons, are theories of 1-dimensional extended objects. If they are formulated as a theory of gravity [155] where now $\alpha' \sim M_\mathrm{P}^{-2}$, the disadvantage that occurred in the form of the exponential falloff of the amplitudes then turns into an advantage to guarantee a nice high energy behaviour. Moreover, the appearance of the unwanted massless modes is important. These modes contain the desired graviton. Quantization leads to constraints on the dimension and the shape of the spacetime in which the string theories live. The so called critical dimension which is necessary for a consistent quantum theory of strings is much higher than four, in particular it is given by $D = 26$ and $D = 10$ for the bosonic and the

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[3]See [172, 173, 177] and further references given in [47, 124].

supersymmetric string theories, respectively.[4] The additional dimensions do not rule out string theories as theories of gravity. Since we do not know how spacetime looks like at short distances comparable to the Planck length, there is the possibility that additional compact dimensions exist. They should simply be highly curved and thus so tiny that it is impossible to detect them at energy scales that are accessible today. Much better, the additional dimensions naturally lead to a unified description of gauge theories and gravity in our four-dimensional perspective.

Besides fixing the dimension, the consistency of the quantum theory further restricts the choice of a classical background on which the string theories can be formulated. By classical background we mean that one specifies a spacetime plus values for the fields of the theory. A consistent classical background has to fulfill certain differential equations that at the leading order turn out to be the Einstein equations for the metric and the Yang-Mills field equations for gauge fields. This means that at energies small compared to the Planck scale one finds string theories encompass general relativity [155, 188, 189] and gauge theories [127].

Putting together all the above given observations, string theories are promising candidates for a unified formulation of the four fundamental interactions [155]. In the following will not review string theories further and refer the reader to the literature [82, 83, 110, 139, 140].

The next two Sections contain a short introduction and motivation of the two topics we are dealing with in this thesis. A short summary of the structure of the corresponding part will be given at the end of each of these sections.

1.1   Noncommutative Yang-Mills theories

We have seen that the understanding of quantum field theories and of quantum gravity is highly relevant for a successful description of all fundamental interactions. Quantum field theories and in particular Yang-Mills (YM) theories are far from being understood completely. One way to learn more about them is to analyze modified YM theories which

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[4]It is not strictly necessary to work in the critical dimension. The price to pay is that the worldsheet metric becomes dynamical even in conformal gauge and introduces one new degree of freedom. Another possibility is to work with a non-constant dilaton such that the critical dimension is modified. This clearly breaks Poincaré invariance in the target space.

do not necessarily play a direct role in the description of nature.

For instance, one can deform the spacetime on which YM theories are formulated. The case of a noncommutative spacetime is of particular interest. In the canonical case that will be of importance here, the commutator of two spacetime coordinates x^μ and x^ν is given by

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} \ , \tag{1.1}$$

where $\theta^{\mu\nu}$ is a constant tensor that necessarily has length dimension two. In this way one has introduced a new parameter $\sqrt{\|\theta\|}$ into the theory that can be used as an expansion parameter for a perturbative analysis.[5]

Noncommutativity gives rise to a topological classification of Feynman diagrams [23, 174]. One replaces each line of a graph by a double line such that one obtains the so called ribbon graphs. The genus h of a Feynman diagram is then defined as the minimal genus of all surfaces on which its ribbon graph can be drawn without the crossing of lines. It then turns out that planar diagrams in the noncommutative theory are given by essentially the same expressions that one finds in the ordinary (commutative) theory. The only difference is that an overall phase factor multiplies each planar graph, being the same for each graph with fixed external momenta [72].

The situation is different for non-planar diagrams. Inside the loop integrals of the corresponding expressions, phase factors occur in the noncommutative case that depend on the loop momentum k. In the limit of larger and larger $\theta^{\mu\nu}$ these phase factors oscillate with smaller and smaller period and hence they increasingly suppress the non-planar contributions. Particularly, in the perturbation expansion for maximal noncommutativity, i. e. $\theta^{\mu\nu} \to \infty$ or alternatively all momenta being large at fixed $\theta^{\mu\nu}$, only planar diagrams survive. Hence, $\frac{1}{\|\theta\|}$ plays the role of a topological expansion parameter, i. e. $\|\theta\|$ is the analog of the rank N of the gauge group in the large N genus expansion [174] that will be described in Section 1.2 below.

Making spacetime coordinates noncommutative is not only useful for learning more about gauge theories. In addition it enables one to study some aspects of gravity without using gravity itself. The noncommutativity of the coordinates gives rise to the non-locality of interactions. Hence, one can study the influence of non-locality on renormalization properties in the framework of gauge theories and one avoids to work with theories of

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[5]$\|\theta\|$ denotes the maximum of the absolute values of all entries of $\theta^{\mu\nu}$ in its canonical skew-diagonal form, see [171].

gravity. Noncommutativity and hence non-locality of the interactions do not improve the ultraviolet behaviour of planar diagrams since only a multiplicative phase factor arises. Non-planar diagrams, however, behave differently. The oscillatoric behaviour of the phase factors inside the loop integrals renders all one-loop non-planar diagrams finite[6]. This gives rise to the remarkable issue of UV/IR mixing [121]. The effective UV cutoff depends on the external momenta p in such a way that the UV divergencies reappear whenever $p_\mu \theta^{\mu\nu} \to 0$. This means that at small momenta the noncommutative phase factors inside the loop integrals are irrelevant and hence turning on noncommutativity replaces the standard ultraviolet divergencies with a singular infrared behaviour.

As we have seen before, non-locality and improved high energy behaviour of the amplitudes should be important ingredients in a theory of quantum gravity. The facts that noncommutative YM theories capture some aspects of gravity and that gravity naturally occurs in string theories, leads to the question of how deeply gauge and string theories are related. Could it be possible that some gauge and string theories describe the same physics? Before describing examples of this kind in section 1.2, we motivate and summarize the analysis in Part II of this thesis.

During investigating noncommutative field theories one could have the idea of simplifying the problem by studying a truncated version of the full noncommutative theory. One expands in powers of $\theta^{\mu\nu}$ and only considers some of the leading terms. However, in such an expansion effects like UV/IR mixing are lost. They require the full $\theta^{\mu\nu}$-dependence. On the other hand, a frequently addressed task in the context of noncommutative gauge theories is their consistent formulation for gauge groups different from $U(N)$. In contrast to the case of ordinary gauge theories this question is highly non-trivial since a priori noncommutative theories appear to be consistent only for $U(N)$ gauge groups. In some approaches that deal with such a modification, an explicit expansion in powers of $\theta^{\mu\nu}$ is required and this prevents one from studying effects like UV/IR mixing for these theories.

Part II deals with the question of how compatible the extension of noncommutative YM theories to arbitrary gauge groups is with keeping the exact dependence on the noncommutativity parameter $\theta^{\mu\nu}$.

- In Chapter 2 we start with a review of how noncommutative YM theories arise from string theories. This connection implies the existence of a map between the

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[6]This does not hold for all non-planar diagrams at higher loops.

- noncommutative and the ordinary description. The map is of particular importance in a formulation of noncommutative YM theories with arbitrary gauge groups.

- In Chapter 3 we introduce noncommutative YM theories. We review the effects of noncommutativity on the choice of the gauge group. Then we present the Faddeev-Popov gauge fixing procedure and use it to derive the map between the noncommutative and ordinary ghost fields. We will first extract the well-known Feynman rules for noncommutative $U(N)$ YM theories. We summarize the aforementioned approaches of how to implement other gauge groups in noncommutative gauge theories. Based on our work [61], we will then discuss what happens in a construction of $\theta^{\mu\nu}$ exact Feynman rules if the gauge group is a subgroup of $U(N)$.

- A short introduction to the used formalism as well as some detailed calculations that are useful for fixing notations and support the analysis of Part II are collected in Appendix A.

1.2   Dualities of gauge and string theories

In 1974 'tHooft [174] found a property of $SU(N)$ gauge theories that suggests a relation to string theory. He observed that the perturbation expansion can be regarded as an expansion in powers of $\lambda = g^2 N$ instead of an expansion in powers of the YM coupling constant g. In addition to the expansion in λ, one can classify Feynman graphs in powers of N^{-2}. A perturbative expansion in N^{-2} is possible in the case of large N with λ kept fixed. It can be interpreted with the help of the double line notation for Feynman graphs [23, 174], that was already introduced in section 1.1. Here, each line now represents one fundamental index of the $N \times N$ representation matrices of the gauge group. The large N expansion then turns out to be a topological expansion in which a Feynman diagram with genus h is of order N^{-2h}. If one now takes the so called 'tHooft limit, where $N \to \infty$ and λ is kept fixed, only the planar graphs which can be drawn on a sphere ($h = 0$) survive. The genus expansion of Feynman diagrams in a gauge theory resembles the genus expansion of string theory, such that a relation between these two different types of theories was conjectured.

1.2.1 The AdS/CFT correspondence

The first concrete example of a gauge/string duality was proposed by Maldacena in 1997 [113] and is known as AdS/CFT correspondence in the literature. One case of particular importance is the conjecture that the 4-dimensional supersymmetric ($\mathcal{N} = 4$) YM theory should describe the same physics as type II B string theory on an $\mathrm{AdS}_5 \times \mathrm{S}^5$ background with some additional Ramond-Ramond flux switched on.

Up to now the conjecture could not be analyzed in full generality, but it has passed several non-trivial checks. One regime in which concrete computations can be done on the string theory side is where the string theory can be replaced by type II B supergravity. On the gauge theory side this regime corresponds to first taking $N \to \infty$ at fixed λ and then taking $\lambda \to \infty$. It is therefore perfectly inaccessible with perturbation theory. But turning the argument around, if one assumes that the AdS/CFT correspondence is correct, then there is a nice tool to analyze this concrete gauge theory in the large N limit at strong 'tHooft coupling λ by working in the dual supergravity description.

How can it happen that a lower dimensional gauge theory contains the same amount of information as the higher dimensional string theory? The physical picture behind this is that the AdS/CFT correspondence is holographic. Since this issue is one of the main motivations for Part III of this thesis, let us explain it in more detail.

Originally if we were to talk about holography we refer to a particular technique in photography. The use of a coherent light source like a laser enables one to store not only brightness and colour information of a three dimensional object on the two dimensional film. One splits the laser beam into two parts such that one hits the film directly and the other hits the object on the side that is directed towards the film. The direct part of the beam and the light that is reflected from the object generate an interference pattern that is stored on the film. The result is that one has encoded information about the varying distance between the surface of the object and the film. Viewing a holographic film, the object then appears three dimensional and the part that was directed towards the film could in principle be completely reproduced without information loss, at least if the resolution of the film were arbitrarily high. This is in sharp contrast to the ordinary photography which is simply a (orthogonal) projection, such that all the information about length scales perpendicular to the film is lost. In the real case of a finite resolution of the film there is an information loss in both cases, both images appear blurred at

sufficiently small distances. But the essential difference between the holographic and the ordinary image, that one does or does not have any information about the third dimension, remains.

In quantum gravity, holography [168, 175, 176] appears in connection with a black hole which information is stored on its horizon. The holographic screen is the horizon and a single grain of the film corresponds to an area element of Planck size. It is the minimum area on which information can be stored. One could now argue [176] that the emergence of holography in this context is only a special case of a holographic principle that is of universal validity in theories of gravity. Indeed, the holographic principle plays an essential role in connection with the AdS/CFT correspondence.

Let us now draw the analogy to the AdS/CFT correspondence. The complete three dimensional space in the photography example becomes the 5-dimensional anti-de Sitter spacetime AdS_5.[7] The lower dimensional holographic screen which corresponds to the two dimensional film in the above example is given by the 4-dimensional boundary of AdS_5. The easiest way to understand this is to represent AdS_5 as a full cylinder. The boundary of the cylinder then is the holographic screen and the radial coordinate is the coordinate perpendicular to the film. It is called the holographic direction. Strings live in the full cylinder, and the four dimensional theory lives on its boundary. The interference pattern that encodes the position perpendicular to the screen in holographic photography corresponds to the energy scale in the boundary theory. Like in the case of the black hole, the Planck length corresponds to the finite resolution of the film [168].

Although one finds many similarities between holographic photography and the AdS/CFT correspondence one should not drive the analogy too far. In contrast to photography where it should in principle not matter where the screen is put, in the AdS/CFT correspondence this choice can have non-trivial effects. The reason is that the geometry is not only given by AdS_5 but instead by the 10-dimensional product space $AdS_5 \times S^5$. For instance, the choice to take the boundary of AdS_5 as a holographic screen implies that it is four dimensional, because there the S^5 is shrunk to a point. Any slice at a constant holographic coordinate value somewhere in the interior of $AdS_5 \times S^5$ is 9-dimensional because there the S^5 has finite size. Hence, such a slice would define a 9-dimensional holographic screen. Indeed, this difference is one main reason why in a certain limit of

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[7]The additional 5-dimensional sphere to form a 10-dimensional spacetime simply produces the Kaluza-Klein tower of states.

the AdS/CFT correspondence a unique holographic description has not yet been found.

Before we describe this limit it is worth to recapitulate what happened with string theories until now. They were originally introduced to describe the strong interactions. Then they were replaced by QCD that has many advantages but is little under control beyond perturbation theory. String theories were proposed as the theories of gravity. But they reentered the regime of gauge theories as possible dual descriptions that might be a key tool to study non-perturbative effects in gauge theories.

This seems to explain why in their original formulation as theories of hadrons string theories covered some aspects of hadron physics. In particular it becomes more plausible why strings are an effective description of the QCD flux tubes. Furthermore, it sheds new light on the unwanted issues of string theories in this direct formulation describing hadrons. They could be regarded as a hint that one was working in the wrong setup (one should work in the spirit of the AdS/CFT correspondence instead of trying to formulate them directly as theories of hadrons) including the wrong choice of scales (one should use $\alpha' \sim M_{\mathrm{P}}^{-2}$ instead of $\alpha' \sim 1\,\mathrm{GeV}^{-2}$). For instance, it was indeed found that the too soft behaviour of string amplitudes in the hard fixed angle regime can be avoided in the framework of the AdS/CFT correspondence [142].

1.2.2 The BMN limit of the AdS/CFT correspondence

At the present time, a proof of the AdS/CFT correspondence is still out of reach. Even an analysis of string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ beyond the supergravity approximation lies outside present capabilities. Berenstein, Maldacena and Nastase [21] formulated a new limit of the AdS/CFT correspondence that goes beyond the supergravity approximation. The proposal is based on the observation [28] that the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background can be transformed with the so called Penrose-Güven limit [88, 135] to a plane wave background [27] on which string theory is quantizable [118, 119]. Berenstein, Maldacena and Nastase translated the limit to the gauge theory side and proposed that a certain subsector of operators in the gauge theory should then be related to type II B string theory on the plane wave. This limit reveals further aspects of the presumably deep connection between gauge and string theories. A very nice picture is that the operators which are compound objects made from the fields of the gauge theory can be interpreted as discretized strings where each field operator represents a single string bit.

Later it was found [75, 86] that the BMN limit can be embedded into the more general framework of expanding the string theory about a classical string solution on $\text{AdS}_5 \times \text{S}^5$ and that explicit checks for several classical solutions can be performed because the relevant gauge theory parts are integrable.

But what is still missing is a satisfactory description of how holography in the AdS/CFT correspondence appears in its BMN limit. Connected with this, it is not clear what the dual gauge theory is and where it lives. Is it a one dimensional theory on the one dimensional boundary of the plane wave spacetime, or does it live on a screen in the interior that could have any dimension between one and nine?

In Part III of this thesis we discuss the behaviour of some geometrical and field theoretical quantities in the limiting process from the AdS/CFT correspondence to its BMN limit. The idea is that one should observe how quantities that are relevant in the holographic description are transformed in the limit and hence learn more about the fate of holography.

- In Chapter 4 we review the ingredients that are essential for an understanding of the backgrounds in the AdS/CFT correspondence and in its BMN limit. Furthermore, we will review the limiting processes with which some of the backgrounds can be related and which play an important role in understanding the idea of the AdS/CFT correspondence and how its BMN limit arises.

- In Chapter 5 we review how holography is understood in the AdS/CFT correspondence in more detail. We then turn our attention to its plane wave limit and give a brief summary of some work to define a holographic setup in the BMN limit. This gives an impression that the picture is less unique than in the AdS/CFT correspondence itself.

- In Chapter 6 we analyze how the boundary structure behaves in the limiting process. Furthermore, to get some information on the causal structure, we derive and classify all geodesics in the original background and discuss their fate in the limit. This Chapter is completely based on our analysis in [62].

- In Chapter 7 we come back to the arguments given in Chapter 5, that the propagators are ingredients of particular importance in a holographic setup. We focus on the bulk-to-bulk propagator in generic $\text{AdS}_{d+1} \times \text{S}^{d'+1}$ backgrounds. We compute

it in particular cases and analyze some of its properties in detail. In particular, we show how the well-known result in the 10-dimensional plane wave spacetime is obtained by taking the Penrose limit. This chapter is mainly based on our analysis in [60].

- Some detailed calculations and useful formulas that refer to Part III are collected in Appendix B.

Part II

Noncommutative geometry

Chapter 2

Noncommutative geometry from string theory

The intention of this Chapter is to review how noncommutative gauge theories arise from open strings in the presence of a constant B-field. There exist several approaches [1, 7–9, 43, 45, 46, 63, 156] to this problem but we will mainly follow the argumentation of [158].[1] First we will discuss the σ-model description and then later on we will focus on the low energy effective action. Besides giving insights into the mechanism of how field theories can be derived from string theories, the discussion leads us to an essential ingredient for our own analysis: the Seiberg-Witten (SW) map, which provides a translation between ordinary and noncommutative gauge theory quantities.

2.1 The low energy limit of string theory with constant background B-field

2.1.1 σ-model description

The action of a bosonic string which propagates in a background consisting of a spacetime metric g_{MN}, an antisymmetric field B_{MN} and a dilaton ϕ can be written as

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \left[\left(\delta^{ab} g_{MN}(X) - 2\pi i\alpha' \epsilon^{ab} B_{MN}(X) \right) \partial_a X^M \partial_b X^N + \alpha' \mathcal{R}\phi(X) \right]. \quad (2.1)$$

Here the string sweeps out a worldsheet Σ with scalar curvature $\mathcal{R}$ and we have chosen conformal gauge and work with Euclidean signature such that the worldsheet metric is δ_{ab}.

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[1]See also the introduction in [179].

Furthermore, we assume a $(D = 26)$-dimensional target space which leads to a critical string theory where the worldsheet metric remains non dynamical in conformal gauge after quantization.[2] If the theory describes open strings we may add additional terms such as

$$-i \int_{\partial\Sigma} \mathrm{d}t \, A_M(X)\partial_\| X^M + \frac{1}{2\pi} \int_{\partial\Sigma} \mathrm{d}t \, k \, \phi(X) \qquad (2.2)$$

to the action (2.1), which couple a background gauge field A_M and the dilaton ϕ to the boundary $\partial\Sigma$ of the worldsheet Σ. Here $\partial_\|$ denotes the tangential derivative along the worldsheet boundary and k is the geodesic curvature of the boundary. If the 2-form B fulfills $\mathrm{d}B = 0$, which includes the case of a constant B_{MN}, and if the dilaton ϕ is constant, the action for open strings can be cast into the form

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \, g_{MN}\partial_a X^M \partial^a X^N - \frac{i}{2} \int_{\partial\Sigma} \mathrm{d}t \left(B_{MN} X^M + 2A_N \right)\partial_\| X^N + \phi\chi \,, \qquad (2.3)$$

where χ denotes the Euler number of the worldsheet Σ. The second integral shows that one can alternatively describe the constant B-field by a gauge field $A_M = -\frac{1}{2}B_{MN}X^N$. The field strength derived from it is $F_{MN} = B_{MN}$ with the usual definition

$$F_{MN} = \partial_M A_N - \partial_N A_M \,. \qquad (2.4)$$

Up to now the endpoints of the open strings can move unconstrained in the spacetime, the string obeys Neumann boundary conditions. In the framework of Dp-branes this setup corresponds to the case of a spacetime filling D$(D-1)$-brane. If instead we impose Neumann boundary conditions in $p+1$ dimensions and Dirichlet boundary conditions in the remaining $D-p-1$ dimensions, this defines the string endpoints to lie on a Dp-brane. This means that they can move freely in $p+1$ spacetime directions and are stuck at fixed positions in the remaining $D - p - 1$ spatial dimensions. We split the coordinate indices in the following way

$$M, N = 0, \ldots, D \,, \qquad \mu, \nu = 0, \ldots, p \,, \qquad m, n = p+1, \ldots, D \,, \qquad (2.5)$$

such that capital Latin indices $M, N, \ldots$ run over all spacetime directions whereas lower case Greek $(\mu, \nu, \ldots)$ and Latin $(m, n, \ldots)$ indices denote directions respectively parallel and perpendicular to the Dp-brane. The boundary of the string worldsheet lies completely on the brane. This means that the coordinates $X^m|_{\partial\Sigma}$ are constant and thus $\partial_\| X^m|_{\partial\Sigma} = 0$.

---

[2]We of course choose string backgrounds which preserve the conformal invariance.

Then only the components $B_{\mu\nu}$ and A_ν along the brane contribute to the boundary term of (2.3) and we can set all other components to zero without loss of generality. To simplify the analysis we will now in addition assume that the field $B_{\mu\nu}$ on the Dp-brane has maximum rank and that the metric g_{MN} is independent of X^M and of block-diagonal form with respect to the coordinate split (2.5). The background fields then read

$$g_{MN} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & g_{mn} \end{pmatrix} , \qquad B_{MN} = \begin{pmatrix} B_{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix} . \tag{2.6}$$

In this setup the boundary terms in (2.3) (with $A_N = 0$) modify the boundary conditions of the open strings. One finds

$$X^m\big|_{\partial\Sigma} = 0 ,$$
$$\left(g_{\mu\nu}\partial_\perp + 2\pi i\alpha' B_{\mu\nu}\partial_\parallel\right) X^\nu\big|_{\partial\Sigma} = 0 \tag{2.7}$$

for the directions perpendicular and parallel to the Dp-brane respectively. $\partial_\perp$ denotes the derivative normal to the boundary of the worldsheet. It is important to mention that the equations of motion are not affected by the addition of the boundary terms in (2.3). To analyze the theory with the boundary conditions (2.7) it is advantageous to choose coordinates (τ, σ) in which the boundary of the open string worldsheet is parameterized by τ and is located at constant $\sigma = 0, \pi$. The string worldsheet then describes a strip in the complex plane of $w = \sigma - i\tau$ which can be conformally mapped to the upper half plane via the holomorphic transformation $z = e^{iw} = e^{\tau + i\sigma}$. The origin of the half plane corresponds to $\tau \to -\infty$ and half circles around the origin refer to constant τ, see Fig. 2.1.

In these coordinates the boundary of the worldsheet is the real axis where $z = \bar{z}$ and the derivatives ∂_τ and ∂_σ are given by

$$\partial_\tau = z\partial + \bar{z}\bar{\partial} , \qquad \partial_\sigma = iz\partial - i\bar{z}\bar{\partial} , \qquad \partial = \frac{\partial}{\partial z} , \qquad \bar{\partial} = \frac{\partial}{\partial \bar{z}} . \tag{2.8}$$

Using (2.6), the boundary conditions (2.7) then read in $(z, \bar{z})$ coordinates

$$X^m\big|_{z=\bar{z}} = 0 ,$$
$$\left(g_{\mu\nu}(\partial - \bar{\partial}) + 2\pi\alpha' B_{\mu\nu}(\partial + \bar{\partial})\right) X^\mu\big|_{z=\bar{z}} = 0 . \tag{2.9}$$

Since the boundary terms in the action (2.3) do not modify the equations of motion, the propagator is standard for the directions perpendicular to the Dp-brane. The complete
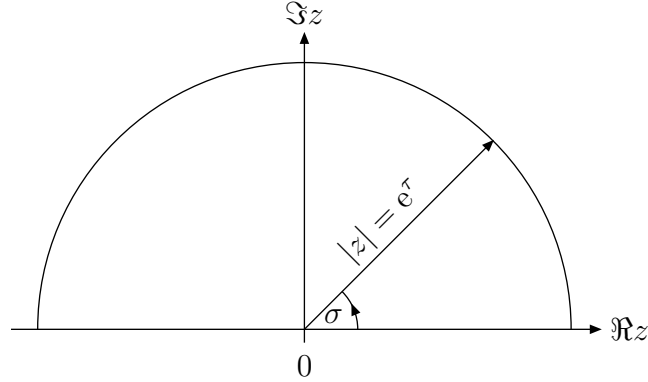
Figure 2.1: Description of the string worldsheet with the complex coordinate z. The worldsheet of the open string is given by the upper half plane with $\Im z \geq 0$ and its boundary is the real axis. A slice of constant worldsheet 'time' τ is given by a half circle with radius $|z| = e^{\tau}$. The infinite past and future are given by the origin and the circle with infinite radius respectively.

solution with the boundary conditions (2.9) in the constant background (2.6) is given by [1, 158]

$$
\begin{aligned}
\langle X^{\mu}(z)X^{\nu}(z')\rangle &= -\alpha'\Big[g^{\mu\nu}\ln\frac{|z-z'|}{|z-\bar{z}'|} + G^{\mu\nu}\ln|z-\bar{z}'|^2 + \frac{1}{2\pi\alpha'}\theta^{\mu\nu}\ln\frac{z-\bar{z}'}{\bar{z}-z'} + D^{\mu\nu}\Big] \,, \\
\langle X^{\mu}(z)X^{n}(z')\rangle &= 0 \,, \\
\langle X^{m}(z)X^{n}(z')\rangle &= -\alpha' g^{mn}\ln|z-z'| \,,
\end{aligned}
$$

(2.10)

where $D^{\mu\nu}$ is a constant and we have used the abbreviations

$$
\begin{aligned}
G_{MN} &= g_{MN} - (2\pi\alpha')^2\big(B\frac{1}{g}B\big)_{MN} \,, \\
G^{MN} &= \Big(\frac{1}{g+2\pi\alpha'B}\Big)^{MN}_{\mathrm{S}} = \Big(\frac{1}{g+2\pi\alpha'B}g\frac{1}{g-2\pi\alpha'B}\Big)^{MN} \,, \\
\theta^{MN} &= 2\pi\alpha'\Big(\frac{1}{g+2\pi\alpha'B}\Big)^{MN}_{\mathrm{A}} = -(2\pi\alpha')^2\Big(\frac{1}{g+2\pi\alpha'B}B\frac{1}{g-2\pi\alpha'B}\Big)^{MN} \,.
\end{aligned}
$$

(2.11)

They are related via

$$
G^{MN} + \frac{\theta^{MN}}{2\pi\alpha'} = \Big(\frac{1}{g+2\pi\alpha'B}\Big)^{MN} \,,
$$

(2.12)

which is trivially fulfilled for the directions $(M,N) = \{(m,\nu),(\mu,n),(m,n)\}$, where $B_{MN} = 0$, $\theta^{MN} = 0$.

Due to the third term in the first line of (2.10), the propagator is single-valued if the branch cut of the logarithm lies in the lower half plane. As a consistency check for $(M, N) = (\mu, \nu)$ one recovers the propagator on the disk with Neumann boundary conditions [139] for $B_{\mu\nu} = 0$.

The quantities given in (2.10) and (2.11) have a simple interpretation. In conformal field theories there exists a map between asymptotic states of incoming and outgoing fields and operators which are called vertex operators. If one wants to compute string theoretical scattering amplitudes one has to insert the vertex operators into the two dimensional surface. The topology of the latter determines the order in the string coupling constant and the type of vertex operators that can couple. Closed string vertex operators couple to all surfaces, inserting them at points in the interior. However, open string vertex require that the surface possesses a boundary where they have to be inserted. The short distance singularity of two vertex operators that approach each other can either be read off from the propagator or from their operator product expansion. The anomalous dimensions of the vertex operators determine the short distance behaviour in the operator product expansion. Hence, one concludes that the singularity of the propagator (2.10) if two interior points coincide determines the anomalous dimensions of closed string vertex operators. From (2.10) one finds in this case that for $(M, N) = (\mu, \nu)$ the only singular term is the numerator of the first logarithmic term which is similar to the term for $(M, N) = (m, n)$. The short-distance behaviour is

$$\left\langle X^M(z)X^N(z') \right\rangle \sim -\alpha' g^{MN} \ln |z - z'| \tag{2.13}$$

and its coefficient enters the expressions for the anomalous dimensions of the closed string vertex operators. Thus, g_{MN} is the metric seen by closed strings.

On the other hand since open strings couple to the disk by inserting the corresponding vertex operators into the boundary of the disk, their anomalous dimensions are determined by the short distance singularity of (2.10) for both points at the boundary, where $z = \bar{z} = s$. One finds

$$\left\langle X^\mu(s)X^\nu(s') \right\rangle = -\alpha' G^{\mu\nu} \ln(s - s')^2 + \frac{i}{2}\theta^{\mu\nu}\epsilon(s - s') \,, \tag{2.14}$$

with an appropriately chosen $D^{\mu\nu} = -\frac{i}{2\alpha'}\theta^{\mu\nu}$ and with

$$\epsilon(s) = \begin{cases} -1 & s < 0 \\ 1 & s > 0 \end{cases} \,. \tag{2.15}$$

Open string vertex operators see the metric $G_{\mu\nu}$. We will therefore denote $G_{\mu\nu}$ as the open string metric.

We want to show that the sign function $\epsilon(s)$ in (2.14) is responsible for the non vanishing of the commutator of the two fields X^μ, X^ν at the same boundary point. Remember first that according to the previous discussion around fig. 2.1, the radial coordinate of the half plane refers to the worldsheet 'time'. 'Time' ordering therefore translates to radial ordering on the complex z plane. The equal time commutator of two fields is defined as the difference of two limits of the time ordered product of these fields. The first [second] limit is to let the time coordinate of the second field approach the time coordinate of the first one from below [above]. The translation to radial ordering (denoted by R) is obvious and one obtains using (2.14)

$$\left\langle \left[X^\mu(s), X^\nu(s)\right]\right\rangle = \left\langle \lim_{\varepsilon\to 0} \mathrm{R}\left(X^\mu(s)X^\nu(s-\varepsilon) - X^\mu(s)X^\nu(s+\varepsilon)\right)\right\rangle = \frac{i}{2}\left(\theta^{\mu\nu} - \theta^{\nu\mu}\right) = i\theta^{\mu\nu}\ . \tag{2.16}$$

The parameter $\theta^{\mu\nu}$ can be interpreted as the noncommutativity parameter in a space where the embedding coordinates on the Dp-brane describe the noncommutative coordinates.

We will now determine the effect of the additional terms in (2.3) on string scattering amplitudes. As we have mentioned above, an element of the string S-matrix is a correlator of vertex operators that describe the asymptotic states. The correlators (at fixed order in the string coupling) are defined as a path integral over all fields X^M and metrics of the 2-dimensional surface of particular topology with vertex operators inserted and with the action (2.3) (with $A_N = 0$). An appropriate gauge fixing procedure is also understood. Consider first the simplest vertex operator for an open string tachyon with momentum p which is given by $:\mathrm{e}^{ip_\mu X^\mu}:$. Here $:$ denotes normal ordering and indices are raised and lowered with the metric in (2.11). Using (2.14), the operator product of two open string tachyon vertex operators for $s > s'$ is given by

$$:\mathrm{e}^{ip\cdot X}(s)::\mathrm{e}^{iq\cdot X}(s'): \sim (s-s')^{2\alpha' p\cdot q}\,\mathrm{e}^{-\frac{i}{2}\theta^{\mu\nu}p_\mu q_\nu}:\mathrm{e}^{i(p+q)\cdot X}(s'):\ , \tag{2.17}$$

where '$\sim$' denotes that we have only kept the most singular terms and we have defined $p \cdot q = G^{\mu\nu}p_\mu q_\nu$. One can capture the complete θ-dependence on the R. H. S. with the Moyal-Weyl $*$-product [122] which is defined as

$$f(x)*g(x) = \mathrm{e}^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial\xi^\mu}\frac{\partial}{\partial\eta^\nu}}\,f(x+\xi)g(x+\eta)\Big|_{\xi=\eta=0} \tag{2.18}$$

and which is a special example of a noncommutative $*$-product, see Appendix (A.4) for some details. One especially finds for this product

$$x^\mu * x^\nu - x^\nu * x^\mu = i\theta^{\mu\nu} \tag{2.19}$$

in accord with (2.16). The product of two tachyon vertex operators (2.17) can then be written as

$$:\mathrm{e}^{ip\cdot X}(s)\!: *\!:\mathrm{e}^{iq\cdot X}(s')\!:\Big|_{\langle X^\mu(s)X^\nu(s')\rangle_{\theta=0}} = :\mathrm{e}^{ip\cdot X}(s)\!::\mathrm{e}^{iq\cdot X}(s')\!: . \tag{2.20}$$

The above result means that the normal ordering of two tachyon vertex operators can either be performed by using (2.14) for contractions or alternatively by replacing the ordinary product with the $*$-product and contracting with the two point function (2.14) without the θ-dependent term. Both procedures capture the entire θ-dependence.

This discussion can be generalized to products of arbitrary open string vertex operators. A generic open string vertex operator is given by a polynomial P that depends on derivatives of the X and an exponential function to ensure the right behaviour under translations

$$V(p, s) = :P[\partial X, \partial^2 X, \dots]\,\mathrm{e}^{ip\cdot X}(s)\!: . \tag{2.21}$$

If one now normal orders products of these vertex operators, one finds that only the exponential factors generate a dependence on $\theta^{\mu\nu}$ in contrast to the contractions which include at least one field of the polynomial prefactors. The reason for this is that in the two point function (2.14) the θ-dependent term can be disregarded if derivatives are taken and an appropriate regularization is used (like point splitting regularization in Subsection 2.1.3). Therefore, one obtains the same θ-dependent exponential factor as if one had used tachyon vertex operators. It is now simple to see the θ-dependence of string amplitudes. One has to insert the vertex operators into the boundary of the string worldsheet, and one has to integrate over the insertion points.[3] In case of an n-point amplitude with vertex operators V_k, $k = 1, \dots, n$ one obtains

$$\left\langle \prod_{k=1}^n V_k(p_k, s_k) \right\rangle_{G,\theta} = \exp\left\{ -\frac{i}{2} \sum_{k>l} \theta^{\mu\nu}(p_k)_\mu (p_l)_\nu \epsilon(s_k - s_l) \right\} \left\langle \prod_{k=1}^n V_k(p_k, s_k) \right\rangle_{G,\theta=0} . \tag{2.22}$$

The subscript of a correlator denotes which parameters the correlator has to be computed with. The complete θ-dependence is thus described by the exponential prefactor if the

---

[3]Note that the gauge fixing procedure does not fix the worldsheet metric completely. Depending on the topology of the worldsheet, a remnant, the conformal Killing group (CKG), has to be fixed by inserting some vertex operators at fixed points without performing an integration. For the disk, three vertex operator positions have to be fixed.

theory is formulated in terms of the open string metric G_{MN}. Due to momentum conservation the prefactor is invariant under cyclic permutations of the p_k. Performing the integrations of the above expression over (some of) the insertion points s_k then produces contributions with different phase factors if the vertex operators exchange their positions in a non cyclic way. The appearance of the prefactor can be exactly described by the $*$-product (2.18) of the corresponding n fields, and one then rewrites (2.22) as

$$\Big\langle V_1(p_1, s_1) \cdots V_n(p_n, s_n) \Big\rangle_{G,\theta} = \Big\langle V_1(p_1, s_1) * \cdots * V_n(p_n, s_n) \Big\rangle_{G,\theta=0} . \qquad (2.23)$$

The above expression is an important result for the discussion of the low energy effective theory.

2.1.2   The Seiberg-Witten limit

In order to find the low energy description of the theory with action (2.3) one has to get rid of stringy effects in the correlation functions (2.23). It is clear that in an appropriate limit one should send to zero the parameter α', which is proportional to the square of the string length. In this case, where one wants to keep the effects of a constant B-field, one cannot simply keep constant the other parameters. As can be seen from (2.11), the B-dependence and especially $\theta^{\mu\nu}$ vanishes and one finds the same theory without initial B-field, if one keeps constant the other parameters in the limit. One should instead keep the open sting metric $G_{\mu\nu}$ and $\theta^{\mu\nu}$ finite and different from zero. This is natural because the correlators (2.23) which will be discussed in the limit depend on these quantities and on α'. In the Seiberg-Witten limit some components of the closed string metric $g_{\mu\nu}$ and α' are sent to zero at constant B-field in the following way

$$\alpha' \sim \sqrt{\epsilon} \to 0 \,, \qquad g_{\mu\nu} \sim \epsilon \to 0 \,, \qquad g_{mn} = \text{const.} \,, \qquad B_{\mu\nu} = \text{const.} \qquad (2.24)$$

This then ensures that G_{MN} and θ^{MN} have reasonable limits. The expressions (2.11) become

$$G_{MN} = \begin{cases} -(2\pi\alpha')^2 \big(B g^{-1} B\big)_{\mu\nu} \\ g_{mn} \end{cases} \,,$$

$$G^{MN} = \begin{cases} -\frac{1}{(2\pi\alpha')^2} \big(B^{-1} g B^{-1}\big)^{\mu\nu} \\ g^{mn} \end{cases} \,, \qquad (2.25)$$

$$\theta^{MN} = \begin{cases} \big(B^{-1}\big)^{\mu\nu} \\ 0 \end{cases} \,,$$

The propagator (2.14) at the boundary reads

$$\left\langle X^\mu(s)X^\nu(s')\right\rangle = \frac{i}{2}\theta^{\mu\nu}\epsilon(s-s') \tag{2.26}$$

in the Seiberg-Witten limit. Thus, from (2.17) and (2.20) it can be seen that the only remnant of the θ-dependence is the exponential prefactor that is described by the $*$-product (2.18). Instead of taking the Seiberg-Witten limit one can equivalently send $B_{\mu\nu} \to \infty$ without scaling the metric $g_{\mu\nu}$ and taking the $\alpha \to 0$ limit.

The interpretation of the Seiberg-Witten limit is as follows. Sending $\alpha' \to 0$ is an infinite tension limit. However, instead of leading to a point particle, the rescaling of the metric $g_{\mu\nu}$ leads to the improvement of the resolution such that a remnant of the 1-dimensionality of the string survives. The different positions of the two endpoints of the string remain observable. Simultaneously, as the tension runs to infinity, the string is made rigid (excitations in the form of massive oscillator modes are removed from the spectrum). It can be seen as a rigid rod, possessing two distinct endpoints but no further internal structure. In [160] it was shown that the two endpoints of the string on the Dp-brane are separated if a B-field is present. They can be described by a dipole [160, 169]. In the Seiberg-Witten limit, the boundary degrees of freedom (the endpoints of the dipole) are governed by the action

$$-\frac{i}{2}\int_{\partial\Sigma} \mathrm{d}t\, B_{\mu\nu}X^\mu\partial_\parallel X^\nu \ . \tag{2.27}$$

This can be regarded as the action of charged particles with the world lines X^i which move in a strong magnetic field $B_{\mu\nu}$ [112].

2.1.3 The low energy effective action

The next step is to determine the effective low energy description of the open strings moving in the constant B-field.

At first let us neglect that there is a B-field present and discuss the general procedure. The part of the low energy effective action (LEEA) for the open strings is obtained in the following way: compute the correlation functions of the vertex operators for the massless open string states and expand in powers of α'. The coefficient at a certain order in α' of the one-particle irreducible piece of one of the amplitudes then enters the effective action at this particular order in α'. It describes the coupling of the fields that correspond to the vertex operators in this string amplitude. In this way one finds that the leading non

constant term in the effective action describes a $U(1)$ gauge theory. Furthermore, it turns out [73] that the LEEA with full dependence on α' is the Dirac-Born-Infeld (DBI) action as long as the field strength (2.4) fulfills $\sqrt{2\pi\alpha'}\left|\frac{\partial F}{F}\right| \ll 1$. Its Lagrangian in presence of a background B-field with $B_{\mu n} = B_{mn} = 0$ and with a gauge field a_μ on the Dp-brane is given by

$$\mathcal{L}[g, B, a] = T_p\sqrt{\det\left[g_{\mu\nu}^{\mathrm{pb}} + 2\pi\alpha'(B_{\mu\nu} + f_{\mu\nu})\right]} \ , \qquad f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \ , \qquad (2.28)$$

where we have used lower case notation for the gauge field and the field strength for reasons that will become clear later. T_p denotes the Dp-brane tension at zero B-field

$$T_p = \frac{1}{g_{\mathrm{s}}(2\pi)^p(\alpha')^{\frac{p+1}{2}}} \ , \qquad (2.29)$$

where g_{s} denotes the closed string coupling constant. The pullback metric is defined in terms of the $D - p - 1$ scalars $2\pi\alpha'\phi^m = X^m$ as

$$g_{\mu\nu}^{\mathrm{pb}} = g_{MN}\partial_\mu X^M \partial_\nu X^N = g_{\mu\nu} + (2\pi\alpha')^2 g_{mn}\partial_\mu\phi^m\partial_\nu\phi^m \qquad (2.30)$$

and the last equality follows in static gauge where $X^\mu = \xi^\mu$ and for a block diagonal metric, see (2.6). For our purpose the scalars play no role and so we will neglect them in the following.

Although we have started the discussion without B-field on the Dp-brane, we have introduced it in (2.28). That this is a consistent generalization can be seen from the symmetries of the underlying σ-model. Since derivatives of $B_{\mu\nu}$ and $f_{\mu\nu}$ have been neglected in the DBI action, we can regard them as constant and thus the DBI action should possess the same symmetries as the underlying σ-model description (2.3). The latter is invariant under the transformation

$$B \to B' = B + \mathrm{d}\omega \ , \qquad a \to a' = a - \omega \ , \qquad \omega \in \Omega^1 \ , \qquad (2.31)$$

where ω is a one-form. This symmetry is respected by $(B + f)_{\mu\nu} = (B' + f')_{\mu\nu}$ where $f'_{\mu\nu}$ has to be computed with a'_μ. We have therefore found one low energy description of the σ-model (2.3) with constant B-field. However, one has to be careful if one wants to expand (2.28) in powers of α' to get the Seiberg-Witten limit (2.24) because the open string metric is not kept constant. As we will now see, there exists a second description in terms of the open string metric and different fields where it is easier to take the Seiberg-Witten limit.

We remember that the correlation functions (2.23) which are formulated with the open string metric $G_{\mu\nu}$ have a simple dependence on $\theta^{\mu\nu}$ and there is no explicit B-dependence left. The Seiberg-Witten limit (2.24) in this formulation is the ordinary $\alpha' \to 0$ limit. For $\theta^{\mu\nu} = 0$ the DBI Lagrangian would then simply be given by (2.28) but with g_{MN} replaced by G_{MN} and with $B_{\mu\nu} = 0$. In addition one should rename the gauge fieldto stress that they are different from the ones in the other formulation. Let us first give a more intuitive argument of how a nontrivial $\theta^{\mu\nu}$ modifies the description. We have already seen that the product between the vertex operators is simply replaced by the $*$-product (2.18) to describe the dependence on $\theta^{\mu\nu}$. This dependence has to be reproduced by the LEEA. Moreover one can check that the field theory limit of the correlator of three gauge fields[4] does not vanish anymore if $*$-products occur between them. We find self interactions of the gauge fields and therefore the $U(1)$ gauge theory becomes non-Abelian. This motivates that the recipe to construct the DBI Lagrangian for non vanishing $\theta^{\mu\nu}$ is to replace the field strength $f_{\mu\nu}$ by a non-Abelian version and to replace all products of fields by the $*$-product. The DBI Lagrangian should thus be given by[5]

$$\mathcal{L}[G, B, A] = \frac{g_{\mathrm{s}}}{G_{\mathrm{o}}^2} T_p \sqrt{\det\left[G_{\mu\nu}^{\mathrm{pb}} + 2\pi\alpha' F_{\mu\nu}\right]} \ , \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - iA_\mu * A_\nu + iA_\nu * A_\mu \ ,$$
$$(2.32)$$

where the first fraction replaces the string coupling g_{s} inside the tension by G_{o}, the open string coupling, which has to be determined later and which should be appropriately rescaled in the $\alpha' \to 0$ limit. In the infinite tension limit $\alpha' \to 0$ one finds with the relation (A.38) that (2.32) describes a noncommutative $U(1)$ gauge theory with action[6]

$$\frac{(\alpha')^{\frac{3-p}{2}}}{4(2\pi)^{p-2}G_{\mathrm{o}}^2} \int \mathrm{d}^{p+1}\xi \ \sqrt{G} G^{\mu\mu'} G^{\nu\nu'} F_{\mu\nu} * F_{\mu'\nu'} \ . \qquad (2.33)$$

The above result has been verified by a direct computation of scattering amplitudes of massless open strings [161]. A complete derivation of the noncommutative DBI action (2.32) along the lines of [73] can be found in [105].

Here we will give another argument based on the σ-model picture that motivates the

---

[4]See e. g. [139].

[5]In principle one should evaluate the determinant using $*$-products. Effectively, however, this does not make a difference here because in the DBI action terms of order $\mathcal{O}(\partial F)$ are neglected. The difference caused by choosing the ordinary product instead of the $*$-product is exactly of this form. Of course one must not neglect the $*$-product between the gauge fields.

[6]Here we have written the $*$-product between the $F_{\mu\nu}$. The expression is exactly the same as if we had taken the ordinary product because of the property (A.69). But even if both actions were not exactly the same they would only differ by terms $\mathcal{O}(\partial F)$ that have been neglected anyway.

appearance of the $*$-product in (2.32) and thus in (2.33). In the σ-model action (2.3) we already included a background gauge field and it is easy to see that a constant $B_{\mu\nu}$ on the Dp-brane can alternatively be described by switching on a background gauge field $A_\nu = \frac{1}{2} B_{\mu\nu} X^\mu$ with field strength $F_{\mu\nu} = B_{\mu\nu}$. This is a consequence of the symmetry under (2.31). Absorbing the B-field completely into the gauge field A_μ, the part of the action that survives the Seiberg-Witten limit (2.27) therefore becomes

$$S_b = -i \int_{\partial\Sigma} \mathrm{d}t \, A_\mu \partial_\| X^\mu = -i \int \mathrm{d}s \, A_\mu \partial_s X^\mu \ . \tag{2.34}$$

Naively this action is invariant under the gauge transformation

$$A_\mu \to A'_\mu = A_\mu + \delta_\lambda A_\mu \ , \qquad \delta_\lambda A_\mu = \partial_\mu \lambda \ . \tag{2.35}$$

However, this is no longer true if we quantize the fields X^μ. In this case the product of operators has to be regularized and this can change the naive gauge invariance. Under the gauge transformation (2.35) the integrand of the path integral transforms as follows

$$\delta \, \mathrm{e}^{-S_b} = \mathrm{e}^{-S_b} \, i \int \mathrm{d}s \, {:}\partial_\mu \lambda \partial_s X^\mu{:} = \sum_{n=1}^{\infty} \frac{i^{n+1}}{n!} \Big(\int \mathrm{d}s \, {:}A_\mu \partial_s X^\mu{:} \Big)^n \int \mathrm{d}s \, {:}\partial_s \lambda{:} \ . \tag{2.36}$$

So for instance the $n = 1$ term in the above sum is the product

$$- \int_{\partial\Sigma} \mathrm{d}s \, {:}A_\mu \partial_s X^\mu{:} \int \mathrm{d}s' \, {:}\partial_{s'} \lambda{:} \tag{2.37}$$

which has to be regularized because the integrations along the boundary lead to divergencies at $s = s'$. Using point splitting regularization, we cut out the region $|s - s'| < \varepsilon$ and we obtain

$$\begin{aligned}
\int_{\partial\Sigma} \mathrm{d}s \, {:}A_\mu \partial_s X^\mu{:} \int \mathrm{d}s' \, {:}\partial_{s'} \lambda{:} &= \int_{\partial\Sigma} \mathrm{d}s \, {:}A_\mu \partial_s X^\mu{:} \Big(\int_{-\infty}^{s-\varepsilon} \mathrm{d}s' + \int_{s+\varepsilon}^{\infty} \mathrm{d}s' \Big) {:}\partial_{s'} \lambda{:} \\
&= \int_{\partial\Sigma} \mathrm{d}s \, {:}A_\mu \partial_s X^\mu{:} {:}\big(\lambda(X(s+\varepsilon)) - \lambda(X(s-\varepsilon)) \big){:}
\end{aligned} \tag{2.38}$$

for λ sufficiently fast vanishing at $s \to \pm\infty$. If we now take the limit $\varepsilon \to 0$ and use the propagator (2.26) for contractions and the definition (2.18) for the $*$-product we find

$$\int \mathrm{d}s \, {:}\big(A_\mu * \lambda - \lambda * A_\mu \big) \partial_s X^\mu{:} \ . \tag{2.39}$$

Applying the point splitting regularization method, one therefore discovers that the gauge transformation has to be modified to remain a symmetry of the quantized theory. The new gauge transformation is

$$\hat{\delta}_\lambda A_\mu = \partial_\mu \lambda + i\lambda * A_\mu - iA_\mu * \lambda \ . \tag{2.40}$$

This expression is complete, even if the other terms in (2.36) with $n > 1$ are considered [158].

The consequences for the LEEA are now as follows. A background gauge field can be seen as a coherent state of open massless strings. The LEEA should therefore be invariant under the gauge transformations which transform the background fields. The gauge transformation (2.40) is a non-Abelian one and the DBI Lagrangian (2.32) respects this symmetry with the non-Abelian field strength $F_{\mu\nu}$. The same gauge invariance is obviously present in the noncommutative $U(1)$ gauge theory (2.33) as it is a limit of the DBI action (2.32).

In [158] the above change in the description of the theory by either keeping the explicit dependence on $B_{\mu\nu}$, or by interpreting it as a boundary condition and thus absorbing it into the definition of G_{MN} and $\theta^{\mu\nu}$, is denoted as background independence. In fact one can choose arbitrary steps in between and absorb only parts of the $B_{\mu\nu}$ dependence. This leads to different values for G_{MN} and $\theta^{\mu\nu}$, but the low energy descriptions are equivalent.

Besides discussing this issue one can ask the question in which sense the above procedure depends on the chosen regularization scheme. Assume in the following that the entire B-dependence is dealt with in the boundary conditions for the CFT and therefore it modifies the propagator like in (2.14). If we had chosen another method (like e. g. Pauli-Villars regularization) to regularize the operator products in in (2.36) then the standard Abelian $U(1)$ gauge transformation (2.35) would have been preserved and this would have to be respected by the LEEA. We already know how the corresponding LEEA with all required symmetries looks like. It is the DBI Lagrangian given in (2.28). In [158] the authors formulate a smooth interpolation between the two descriptions by introducing a two-form Φ such that the relation (2.12) is replaced by

$$\left(\frac{1}{G + 2\pi\alpha'\Phi}\right)^{MN} + \frac{\theta^{MN}}{2\pi\alpha'} = \left(\frac{1}{g + 2\pi\alpha'B}\right)^{MN} \tag{2.41}$$

and the corresponding DBI Lagrangian with the parameters that fulfill this relation reads

$$\mathcal{L}[G, \Phi, A^\Phi] = \frac{g_{\mathrm{s}}}{(G_o^\Phi)^2} T_p \sqrt{\det\left[G_{\mu\nu}^{\mathrm{pb}} + 2\pi\alpha'(\Phi_{\mu\nu} + F_{\mu\nu})\right]} \; . \tag{2.42}$$

In particular, the $*$-product above has to be evaluated with $\theta^{\mu\nu}$ that obeys (2.41) and the open string coupling G_o^Φ carries an index Φ to indicate that its value depends on Φ. For the same reason the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu^\Phi - \partial_\nu A_\mu^\Phi - i A_\mu^\Phi * A_\nu^\Phi + i A_\nu^\Phi * A_\mu^\Phi \tag{2.43}$$

is constructed with gauge fields A_μ^Φ to indicate that they are different quantities for different Φ. The two LEEAs (2.28) and (2.32) follow from the above generalized description evaluated at $\Phi_{\mu\nu} = B_{\mu\nu}$ and $\Phi_{\mu\nu} = 0$ with the identification $A_\mu^B = a_\mu$, $A_\mu^0 = A_\mu$ respectively. Remember that different values for Φ describe the same theory but regularized in different ways. The value of Φ is connected to the parameters of a regularization scheme that interpolates smoothly between Pauli-Villars ($\Phi_{\mu\nu} = B_{\mu\nu}$) and the point splitting regularization ($\Phi_{\mu\nu} = 0$). Schemes that show the existence of Φ were found in [6]. There should thus exist a map that relates the gauge fields A_μ^Φ for different Φ. In particular for the extreme cases ($\Phi_{\mu\nu} = 0, B_{\mu\nu}$) one should find

$$a_\mu \to A_\mu[a] = a_\mu + A'_\mu[a] \tag{2.44}$$

that relates the field a_μ in the first ordinary effective description to the field A_μ in the second noncommutative one. This map is essential for our analysis and we will deal with it in the next Section. It is very important to stress here that the relation between the two descriptions is essentially different from the aforementioned background field transformations. There, the regularization scheme was always point splitting and the low energy description depends on the separation of $B_{\mu\nu}$ into background part and contribution to the boundary conditions of the CFT. Here, however, we have chosen a fixed separation and varied the regularization of the underlying CFT. For further investigations concerning the relation between the ordinary and the noncommutative descriptions see [50, 92].

At the end of this Section the open string coupling constant G_o^Φ in (2.42) will be related to the closed string coupling constant g_s of (2.29) by using the equivalence

$$\mathcal{L}[g, B, a] = \mathcal{L}[G, \Phi, A^\Phi] + \mathcal{O}(\partial F) + \text{total derivatives} , \tag{2.45}$$

which is the statement that the effective descriptions (2.42) are the same for all $\Phi_{\mu\nu}$ and are in particular related to $\Phi_{\mu\nu} = B_{\mu\nu}$. The additional total derivatives arise in the exact equality because the derivation of the effective actions is insensitive to them. The corrections by derivative are caused by the fact that in the approximation by the DBI actions they are neglected. We determine the open string coupling constant G_o^Φ from this equality by setting all dynamical fields to zero. One then finds

$$(G_o^\Phi)^2 = g_s \sqrt{\frac{\det(G + 2\pi\alpha'\Phi)}{\det(g + 2\pi\alpha'B)}} . \tag{2.46}$$

For $\Phi_{\mu\nu} = 0$ this gives G_{o}, see (2.32) and (2.33)

$$G_{\mathrm{o}}^2 = g_{\mathrm{s}} \sqrt{\frac{\det G}{\det(g + 2\pi\alpha'B)}} = \sqrt{\frac{\det(g - 2\pi\alpha'B)}{\det g}} \; , \qquad (2.47)$$

where the last equality follows with the help of (2.11). In the Seiberg-Witten limit it becomes

$$G_{\mathrm{o}}^2 = g_{\mathrm{s}} \sqrt{\det\left(2\pi\alpha'B\frac{1}{g}\right)} \; . \qquad (2.48)$$

The effective Yang-Mills coupling is given by the prefactor of the F^2 term in the expansion of (2.32) which is (2.33) and it reads

$$\frac{1}{g^2} = \frac{(\alpha')^{\frac{3-p}{2}}}{(2\pi)^{p-2}G_{\mathrm{o}}^2} = \frac{(\alpha')^{\frac{3-p}{2}}}{(2\pi)^{p-2}g_{\mathrm{s}}} \sqrt{\frac{\det(g - 2\pi\alpha'B)}{\det g}} \; . \qquad (2.49)$$

As already mentioned in the discussion of (2.33), to keep g^2 finite in the limit $\alpha' \to 0$ we should scale the string coupling constants like

$$G_{\mathrm{o}} \sim \epsilon^{\frac{3-p}{4}} \; , \qquad g_{\mathrm{s}} \sim \epsilon^2 \; . \qquad (2.50)$$

The corresponding expression for arbitrary rank of $B_{\mu\nu}$ can be found in [158].

The above presented analysis can be generalized in the presence of N coincident Dp-branes. The gauge symmetry is then enhanced and again, depending on the chosen regularization, one finds an ordinary or a noncommutative Yang-Mills theory in the $\alpha' \to 0$ limit. The DBI action for N parallel Dp-branes [125], however, is more complicated and only confirmed to coincide with the direct computation of amplitudes up to fourth order in α'.[7] For a discussion of the non-Abelian DBI action in connection with noncommutativity see [51].

2.2   Construction of the Seiberg-Witten map

In Subsection 2.1.3 we have seen that, depending on the chosen regularization, the Seiberg-Witten limit of string theory in the presence of a constant background B-field leads to equivalent effective description on the worldvolume of the Dp-brane. In particular one finds an ordinary and a noncommutative formulation. The unique origin of these two

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[7]I am grateful to S. Stieberger for this comment.

descriptions predicts the existence of a map between the gauge fields A_μ and a_μ of the noncommutative and the ordinary YM theories. In this Section we will now analyze this mapping in more detail. All fields and gauge parameters will refer to the YM case. The first naive assumption is that the map could have the form $A_\mu = A_\mu[a, \partial a, \partial \partial a, \dots]$, $\Lambda = \Lambda[\lambda, \partial \lambda, \partial \partial \lambda, \dots]$. However, it is easy to see that this cannot be true. Remember that the ordinary and the noncommutative gauge transformation for a $U(1)$ gauge field are given by

$$\delta_\lambda a_\mu = \partial_\mu \lambda , \qquad \hat{\delta}_\Lambda A_\mu = \partial \Lambda_\mu + i \left[\Lambda \overset{*}{,} A_\mu \right] , \qquad (2.51)$$

where we define the $*$-commutator and $*$-anticommutator as follows

$$\left[A \overset{*}{,} B \right] = A * B - B * A , \qquad \left\{ A \overset{*}{,} B \right\} = B * A + B * A . \qquad (2.52)$$

In the case of a $U(1)$ gauge group, the $*$-product is simply given by the expression in (2.18). In the non-Abelian case we will denote the tensor product of (2.18) with the matrix multiplication as $*$-product. The ordinary gauge transformation in (2.51) is Abelian whereas the noncommutative gauge transformation is a non-Abelian one. A simple redefinition of the gauge parameter $\lambda \to \Lambda = \Lambda[\lambda, \partial \lambda, \partial \partial \lambda, \dots]$ can never change the Abelian to a non-Abelian gauge group. The only requirement on the map is that gauge equivalent configurations in the ordinary theory are mapped to gauge equivalent configurations in the noncommutative theory. This led Seiberg and Witten to the ansatz that the gauge parameter of one theory depends on both, the gauge parameter and the gauge field of the other theory [158]. In short we will write

$$A = A[a, \partial a, \partial \partial a, \dots] = A[a] , \qquad \Lambda = \Lambda[\lambda, \partial \lambda, \partial \partial \lambda, \dots, a, \partial a, \partial \partial a, \dots] = \Lambda[\lambda, a] . \qquad (2.53)$$

The requirement that gauge orbits are mapped to gauge orbits reads for infinitesimal transformations

$$A_\mu[a] + \hat{\delta}_\Lambda A_\mu[a] = A_\mu[a + \delta_\lambda a] . \qquad (2.54)$$

We will now evaluate this expression to first order in the noncommutativity parameter $\theta^{\mu\nu}$. One inserts the definitions for infinitesimal ordinary and noncommutative gauge transformations of the non-Abelian gauge fields

$$\delta a_\mu = \partial \lambda + i \left[\lambda, a_\mu \right] , \qquad \hat{\delta}_\Lambda A_\mu = \partial_\mu \Lambda + i \left[\Lambda \overset{*}{,} A_\mu \right] \qquad (2.55)$$

into (2.54). Then one expands the $*$-product (2.18) and the maps in powers of $\theta^{\mu\nu}$

$$A[a] = a + A'[a] , \qquad \Lambda[\lambda, a] = \lambda + \Lambda'[\lambda, a] , \qquad (2.56)$$

where $A'[a]$ and $\Lambda'[\lambda, a]$ depend linearly on $\theta^{\mu\nu}$. In this way one finds from (2.54) the expression

$$A'_\mu[a + \delta_\lambda a] - A'_\mu[a] - i\left[\lambda, A'_\mu[a]\right] - i\left[\Lambda'[\lambda, a], a_\mu\right] = -\frac{1}{2}\theta^{\alpha\beta}\left(\partial_\alpha\lambda\partial_\beta a_\mu + \partial_\alpha a_\mu\partial_\beta\lambda\right) , \quad (2.57)$$

where $[\ ,\ ]$ denotes the commutator with ordinary matrix product. The above equation is solved by

$$A'_\mu(a) = -\frac{1}{4}\theta^{\alpha\beta}\left\{a_\alpha, \partial_\beta a_\mu + f_{\beta\mu}\right\} , \qquad \Lambda'(\lambda, a) = \frac{1}{4}\theta^{\alpha\beta}\left\{\partial_\beta\lambda, a_\alpha\right\} \qquad (2.58)$$

and hence the Seiberg-Witten map at order $\mathcal{O}(\theta)$ reads [158]

$$A_\mu(a) = a_\mu - \frac{1}{4}\theta^{\alpha\beta}\left\{a_\alpha, \partial_\beta a_\mu + f_{\beta\mu}\right\} ,$$
$$\Lambda(\lambda, a) = \lambda + \frac{1}{4}\theta^{\alpha\beta}\left\{\partial_\beta\lambda, a_\alpha\right\} , \qquad (2.59)$$

where $\{\ ,\ \}$ denotes the ordinary matrix anticommutator and $f_{\mu\nu}$ the ordinary YM field strength

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu - i\left[a_\mu, a_\nu\right] . \qquad (2.60)$$

The result (2.59) is of central importance for our later analysis of noncommutative YM theories with gauge groups which are not $U(N)$. It is easy to derive from the above transformation formula for the gauge field the mapping of the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\left[A_\mu \overset{*}{,} A_\nu\right] , \qquad (2.61)$$

for which one finds

$$F_{\mu\nu} = f_{\mu\nu} - \frac{1}{4}\theta^{\alpha\beta}\left(2\left\{f_{\mu\alpha}, f_{\beta\nu}\right\} - \left\{a_\alpha, D_\beta f_{\mu\nu} + \partial_\beta f_{\mu\nu}\right\}\right) , \qquad D_\beta = \partial_\beta + i\left[\ , a_\beta\right] . \quad (2.62)$$

In [158] the authors derive differential equations which determine the Seiberg-Witten map. One can evaluate (2.59) for an infinitesimal variation of $\theta^{\mu\nu}$. The differential equations then read

$$\delta A_\mu = \delta\theta^{\alpha\beta}\frac{\partial}{\partial\theta^{\alpha\beta}}A_\mu = -\frac{1}{4}\delta\theta^{\alpha\beta}\left\{A_\alpha \overset{*}{,} \partial_\beta A_\mu + F_{\beta\mu}\right\} ,$$
$$\delta\Lambda = \delta\theta^{\alpha\beta}\frac{\partial}{\partial\theta^{\alpha\beta}}\Lambda = \frac{1}{4}\delta\theta^{\alpha\beta}\left\{\partial_\alpha\Lambda \overset{*}{,} A_\beta\right\} , \qquad (2.63)$$
$$\delta F_{\mu\nu} = \delta\theta^{\alpha\beta}\frac{\partial}{\partial\theta^{\alpha\beta}}F_{\mu\nu} = \frac{1}{4}\delta\theta^{\alpha\beta}\left(2\left\{F_{\mu\alpha} \overset{*}{,} F_{\nu\beta}\right\} - \left\{A_\alpha \overset{*}{,} D_\beta F_{\mu\nu} + \partial_\beta F_{\mu\nu}\right\}\right) .$$

In Appendix A.2 we present a check that these differential equations lead to the invariance (2.45) of the DBI action (2.42) under variations of $\Phi_{\mu\nu}$. The invariance under finite changes of $\Phi_{\mu\nu}$ is proven in [108]. There the author uses an expression for the (inverse) Seiberg-Witten map in the $U(1)$ case that is exact in $\theta^{\mu\nu}$, the validity of which is proven in [109, 123, 129]. The differential equation (2.63) is solved exactly in $\theta^{\mu\nu}$ but as an expansion in powers of the noncommutative $U(1)$ gauge field A_μ in [117]. An investigation of the Seiberg-Witten map order by order in $\theta^{\mu\nu}$ can be found in [70] and up to quadratic order in [94].

At the end of this Section we note that the Seiberg-Witten map is by far not unique but it possesses enormous freedom. This has been observed by [11], see also [25, 94]. For our analysis, it will be sufficient to work with the expression (2.59), because in linear order in $\theta^{\mu\nu}$ the ambiguity of A_μ has the form of a gauge transformation [11].

Chapter 3

Noncommutative Yang-Mills theories

In this Chapter we will deal with noncommutative YM theories. They are the generalizations of the noncommutative $U(1)$ gauge theory, that appeared in Chapter 2, to different gauge groups. The noncommutative YM theory with gauge group $U(N)$ is the generalization of the $U(1)$ gauge theory (see (2.33) with the YM coupling constant given in (2.49)), if one considers open strings on a stack of N Dp-branes instead of a single Dp-brane with a constant B-field on the branes. The noncommutative $U(N)$ YM theory then is the effective description of open strings in this background with the boundary conditions (2.7) in the Seiberg-Witten limit (2.24) if one uses point splitting regularization. A d-dimensional gauge theory corresponds to the choice of D$(d-1)$-branes which world volume coordinates we denote with x^μ. For simplicity we assume that the open string metric introduced in Chapter 2 obeys $G = |\det G| = 1$. Worldvolume indices are understood to be lowered and raised with respectively $G_{\mu\nu}$ and $G^{\mu\nu}$, which signature we will fix to 'mostly minus' in this Chapter. The noncommutative YM action with gauge group $U(N)$ then reads

$$S_{\text{YM}} = -\frac{1}{2g^2} \int \mathrm{d}^d x \, \text{tr} \left(F_{\mu\nu} * F^{\mu\nu} \right) = -\frac{1}{2g^2} \int \mathrm{d}^d x \, \text{tr} \left(F_{\mu\nu} F^{\mu\nu} \right) . \tag{3.1}$$

Here 'tr' denotes the trace w. r. t. the gauge group. The $*$-product in the second equality has been removed because of the following reason: the spacetime integration corresponds to the operator trace in the space of Weyl operators which can be used to describe the noncommutativity of the coordinates (see Appendices A.3,(A.4) and A.5 for a short introduction of this formalism). Here we will work in ordinary spacetime where noncommutativity is described by the $*$-product and the operator trace becomes a spacetime integral. The cyclicity of the operator trace of two Weyl operators then translates into the rule that one can remove the $*$-product between two functions under a spacetime integral, see

(A.69). The noncommutative field strength is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\left[A_\mu \overset{*}{,} A_\nu\right] \ , \tag{3.2}$$

where the $*$-product in the $*$-commutator above is defined as the tensor product of $*$-multiplication of functions with ordinary matrix multiplication of the representation matrices of the gauge Lie algebra (see (2.52)).

In the next Section we will analyze the $*$-commutator in more detail and discuss which influence noncommutativity exerts on the choice of the gauge group. The Faddeev-Popov gauge fixing procedure will be applied in Section 3.2 to noncommutative YM theories. This leads to a relation between the noncommutative and ordinary sets of ghost fields. In Section 3.3 the Feynman rules for noncommutative $U(N)$ YM theories will be worked out. Two proposals that allow one to choose other gauge groups than $U(N)$ for noncommutative YM theories will be summarized in Section 3.4. In Section 3.5 we will then analyze for these theories which kind of Feynman rules can be defined.

3.1 The gauge groups in noncommutative Yang-Mills theories

Ordinary YM theories can be realized with different gauge groups and for some of them string theory setups are known. This is different for noncommutative YM theories. This Section briefly reviews the complications which arise in the definition of noncommutative gauge theories with non-Abelian gauge groups in general and shows that $U(N)$ groups play a special role. The arguments are based on [111].

Let T^A, $A = 1, \ldots, N^2$ denote the $N \times N$ matrix representation of the generators of the $U(N)$ Lie algebra $\mathfrak{u}(N)$ which fulfill[1]

$$\left[T^A, T^B\right] = if^{AB}{}_C T^C \ , \qquad \{T^A, T^B\} = d^{AB}{}_C T^C \ , \qquad \mathrm{tr}(T^A T^B) = \frac{1}{2}\delta^{AB} \ . \tag{3.3}$$

Here $f^{AB}{}_C$ denote the antisymmetric structure constants and $d^{AB}{}_C$ are symmetric in all indices. Obviously, the anticommutation relation does not hold in the Lie algebra $\mathfrak{u}(N)$, but in its $N \times N$ matrix-representation. That it closes on the representation matrices is a specialty of $\mathfrak{u}(N)$ because it respects the Hermiticity condition on the elements. It does,

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[1] In contrast to the usual definition in most textbooks we have included the trace part into $d^{AB}{}_C$. The reason is to keep expressions simple.

however, not respect further conditions (i. e. tracelessness) which may enter the definition of subalgebras of $\mathfrak{u}(N)$.

Let $\mathfrak{t}^A$, $A = 1, \ldots, N^2$ be the generators of $\mathfrak{u}(N)$ and let $\{\mathfrak{t}^a\} \subset \{\mathfrak{t}^A\}$, $\mathfrak{t}^{a'} \in \{\mathfrak{t}^A\} \setminus \{\mathfrak{t}^a\}$, $\{\mathfrak{t}^A\} = \{\mathfrak{t}^a\} \cup \{\mathfrak{t}^{a'}\}$. A subalgebra $\mathfrak{g}$ of $\mathfrak{u}(N)$ is defined exactly as a subset of generators $\mathfrak{g} = \{\mathfrak{t}^a\}$ that closes under the commutator, i. e. $[\mathfrak{t}^a, \mathfrak{t}^b] = f^{ab}{}_c \mathfrak{t}^c$ and thus $f^{ab}{}_{c'} = 0$. For a corresponding matrix representation one therefore has

$$\left[T^a, T^b \right] = i f^{ab}{}_c T^c \, , \qquad \left\{ T^a, T^b \right\} = d^{ab}{}_C T^C = d^{ab}{}_c T^c + d^{ab}{}_{c'} T^{c'} \, . \tag{3.4}$$

As is obvious from this expression, the anticommutator does not necessarily close on the representation of the Lie subalgebra. It is easy to check that the tracelessness $\operatorname{tr} T^a = 0$ is not preserved by the anticommutator and that therefore it does not close onto the representation of the $\mathfrak{su}$, $\mathfrak{so}$ and $\mathfrak{sp}$ subalgebras of $\mathfrak{u}(N)$.

The importance of the previous discussion becomes evident if one computes the $*$-commutator (as defined in (2.52)) of two Lie algebra valued functions $f(x) = f_a(x)\mathfrak{t}^a$, $g(x) = g_a(x)\mathfrak{t}^a$. It can be decomposed as

$$\begin{aligned} [f \overset{*}{,} g] &= \frac{1}{2}(f_a * g_b + g_b * f_a)\left[\mathfrak{t}^a, \mathfrak{t}^b\right] + \frac{1}{2}(f_a * g_b - g_b * f_a)\left\{\mathfrak{t}^a, \mathfrak{t}^b\right\} \\ &= \frac{1}{2}\left\{f_a \overset{*}{,} g_b\right\}\left[\mathfrak{t}^a, \mathfrak{t}^b\right] + \frac{1}{2}\left[f_a \overset{*}{,} g_b\right]\left\{\mathfrak{t}^a, \mathfrak{t}^b\right\} \\ &= f_a \cos\left(\overleftarrow{\partial}_\rho \tfrac{1}{2}\theta^{\rho\sigma}\overrightarrow{\partial}_\sigma\right)g_b\left[\mathfrak{t}^a, \mathfrak{t}^b\right] + i f_a \sin\left(\overleftarrow{\partial}_\rho \tfrac{1}{2}\theta^{\rho\sigma}\overrightarrow{\partial}_\sigma\right)g_b\left\{\mathfrak{t}^a, \mathfrak{t}^b\right\} \end{aligned} \tag{3.5}$$

and it is immediately clear that the noncommutative gauge transformations defined in (A.83) are not Lie algebra valued because they exactly contain the $*$-commutator which depends on the anticommutator of two Lie algebra generators. One is therefore in general lead to define the noncommutative gauge theories to take values in the enveloping algebra which has as basis vectors all symmetric products of the original Lie algebra generators and which is therefore infinite dimensional. The generators read [94]

$$t^{A_1} \cdots t^{A_n} = \frac{1}{n!} \sum_{\pi \in S_n} \mathfrak{t}^{A_{\pi(1)}} \cdots \mathfrak{t}^{A_{\pi(n)}} \, , \tag{3.6}$$

where the sum runs over all $n!$ permutations of $1, \ldots, n$. The Lie algebra generators are simply the ones with $n = 1$. The infinite tower of basis elements requires an infinite number of coefficients, given by infinitely many field components. This seems to prevent one from formulating a reasonable theory. In Section 3.4 we will discuss proposals for a solution of this problem that works for generic gauge groups.

For a $U(N)$ gauge group, however, the problem solves itself. As we already mentioned, the corresponding $\mathfrak{u}(N)$ algebra is special because the Hermiticity condition on the representation matrices is respected by the anticommutator. This means that even the anticommutator of two representation matrices closes and can be expressed as a linear combination of the matrices, see (3.3). The symmetric products of generators defined in (3.6) therefore collapse in a representation (they become linear combinations of the $n = 1$ elements), and one finds for the components of the $*$-anticommutator with $f(x) = f_A(x)T^A$, $g(x) = g_A(x)T^A$,

$$
\begin{aligned}
2\,\mathrm{tr}\left(\left[f\,{}^*_,g\right]T_C\right) &= \frac{i}{2}(f_A * g_B + g_B * f_A)f^{AB}{}_C + \frac{1}{2}(f_A * g_B - g_B * f_A)d^{AB}{}_C \\
&= \frac{i}{2}\left\{f_A\,{}^*_,g_B\right\}f^{AB}{}_C + \frac{1}{2}\left[f_A\,{}^*_,g_B\right]d^{AB}{}_C \\
&= if_A \cos\left(\overleftarrow{\partial}_\rho \tfrac{1}{2}\theta^{\rho\sigma}\overrightarrow{\partial}_\sigma\right)g_B f^{AB}{}_C + if_A \sin\left(\overleftarrow{\partial}_\rho \tfrac{1}{2}\theta^{\rho\sigma}\overrightarrow{\partial}_\sigma\right)g_B d^{AB}{}_C \ .
\end{aligned}
\tag{3.7}
$$

The above given discussion shows that YM theories with gauge groups $U(N)$ are special. Their noncommutative counterparts can directly be formulated. More effort is needed if one wants to construct noncommutative YM theories with different gauge groups. We will summarize two different approaches in Section 3.4.

3.2   Gauge fixing in noncommutative Yang-Mills theories

In the following we will review the Faddeev-Popov gauge fixing procedure [67] for YM theories[2]. After a general description we will apply the formalism to the case of noncommutative YM theories and discuss its behaviour under the field redefinition defined by the Seiberg-Witten map. In the general description we always use capital variables to describe the gauge quantities. Most of the formalism applies similarly to the ordinary and the noncommutative YM theories. We will explicitly point out where differences occur.

Due to gauge invariance, path integration in YM theory should run over gauge inequivalent configurations only. Alternatively one can integrate over all configurations and divide by the volume V_g of the gauge orbits such that the partition function is given by

$$
Z[0] = \int \frac{\mathcal{D}A}{V_\mathrm{g}}\,\mathrm{e}^{iS[A]}
\tag{3.8}
$$

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[2]See e. g. [93, 124, 151] for more details.

with the integration measure

$$\mathcal{D}A = \prod_{a,\mu} \mathcal{D}A_\mu^a \ . \tag{3.9}$$

Here the indices a and μ run over all components w. r. t. the gauge group and spacetime respectively. The Faddeev-Popov trick allows one to split the integration over all configurations into gauge inequivalent (gauge fixed) ones and integrations along the gauge orbits. This then enables one to cancel the volume factor V_{g} and to formulate a gauge fixed version of (3.8). Consider a gauge field configuration $A_{0\mu}^a$. The gauge orbit that includes $A_{0\mu}^a$ is spanned by acting with the gauge transformation (A.80) or (A.74) with all possible choices of the gauge parameter on the gauge field configuration $A_{0\mu}^a$. A configuration that is gauge equivalent to $A_{0\mu}^a$ can be seen as a functional $A_\mu^a[A_0, \Lambda]$.

We now single out the configuration $A_{0\mu}^a$ by imposing a condition that is fulfilled by exactly one element[3] $A_{0\mu}^a$

$$\mathcal{F}_a[A_0] = 0 \ , \tag{3.10}$$

where $\mathcal{F}_a$ is a gauge fixing functional for each component w. r. t. the gauge group. This then allows for a separation between $A_{0\mu}^a$ and the gauge equivalent configurations. The trick is to manipulate the measure in the path integral (3.16) by introducing an identity

$$1 = \int \mathcal{D}\Lambda \, \delta[\Lambda] = \int \mathcal{D}\Lambda \, \det \frac{\delta \mathcal{F}\big[A[A_0, \Lambda]\big]}{\delta \Lambda} \, \delta\big[\mathcal{F}\big[A[A_0, \Lambda]\big]\big] \ . \tag{3.11}$$

Here $\delta[\ ]$ is the δ-functional which can be seen as an infinite product of δ-functions at each point in spacetime and the second equality is based on the functional analog of the well known identity

$$\delta\big(f(x)\big) = \frac{1}{|\det \partial_i f_j|} \delta(x) \tag{3.12}$$

for the ordinary δ-function depending on a vector valued function f_i of arguments x_i, $i = 1, \ldots, d$ with only $f_i(0) = 0$. The determinant in (3.11) refers to group and spacetime 'indices' which means more explicitly that one has to see its argument as a matrix carrying two independent pairs of 'indices' (a, b) and (x, y)

$$\frac{\delta \mathcal{F}_a\big[A[A_0(x), \Lambda(x)]\big]}{\delta \Lambda_b(y)} \ . \tag{3.13}$$

The gauge fixing (3.10) and the obvious identity $A_\mu^a[A_0, 0] = A_{0\mu}^a$ now allows one to write

$$1 = \Delta[A_0] \int \mathcal{D}\Lambda \, \delta\big[\mathcal{F}\big[A[A_0, \Lambda]\big]\big] \ , \qquad \Delta[A_0] = \det \frac{\delta \mathcal{F}\big[A[A_0, \Lambda]\big]}{\delta \Lambda}\bigg|_{\Lambda=0} \ . \tag{3.14}$$

---

[3]Strictly speaking this is only true in a perturbative analysis.

This is an obvious consequence of the functional form of the identity $f(x)\delta(x) = f(0)\delta(x)$. It is important to notice that $\Delta[A_0]$ is invariant under gauge transformations. Let $A' = A[A_0, \Lambda']$ then one finds

$$1 = \Delta[A'] \int \mathcal{D}\Lambda \, \delta\big[\mathcal{F}\big[A[A[A_0, \Lambda'], \Lambda]\big]\big] = \Delta[A'] \int \mathcal{D}\tilde{\Lambda} \, \delta\big[\mathcal{F}\big[A[A_0, \tilde{\Lambda}]\big]\big] = \frac{\Delta[A']}{\Delta[A_0]} \ , \quad (3.15)$$

where one has to use that two sequential gauge transformations are again a gauge transformation and that the measure fulfills $\mathcal{D}\Lambda = \mathcal{D}(\Lambda\Lambda') = \mathcal{D}(\tilde{\Lambda})$. In particular $u(\Lambda)u(\Lambda') = u(\tilde{\Lambda})$ or $U(\Lambda) * U(\Lambda') = U(\tilde{\Lambda})$ enter the ordinary and noncommutative gauge transformations (A.80) and (A.74) respectively. One now inserts (3.14) into the path integral (3.8) to obtain

$$Z[0] = \int \frac{\mathcal{D}A' \, \mathcal{D}\Lambda}{V_\mathrm{g}} \Delta[A_0] \, \delta\big[\mathcal{F}\big[A[A_0, \Lambda]\big]\big] \, \mathrm{e}^{iS[A']} = \int \frac{\mathcal{D}A' \, \mathcal{D}\Lambda}{V_\mathrm{g}} \Delta[A'] \, \delta\big[\mathcal{F}\big[A[A', \Lambda]\big]\big] \, \mathrm{e}^{iS[A']} \ ,$$
$$(3.16)$$

where we have renamed the integration measure and the argument of the action and then used the invariance (3.15). Due to the invariance of $\Delta[A_0]$, of the action and of the path integral measure under gauge transformations we can now perform a gauge transformation that transforms A' to A. The integrand, especially the argument of $\mathcal{F}$, then becomes independent of Λ such that the integration can be performed to cancel precisely V_g in the denominator. One thus obtains for the gauge fixed path integral

$$Z[0] = \int \mathcal{D}A \, \Delta[A] \, \delta\big[\mathcal{F}[A]\big] \, \mathrm{e}^{iS[A]} \ . \tag{3.17}$$

The determinant Δ as defined in (3.14) can be formulated as a fermionic Gaussian path integral

$$\Delta[A] = \int \mathcal{D}C \, \mathcal{D}\bar{C} \exp\left\{ -i \int \mathrm{d}^d x \, \mathrm{d}^d y \, \bar{C}^a(x) \mathcal{M}_{ab}(x, y) C^b(y) \right\} \ , \tag{3.18}$$

with the integration measures $\mathcal{D}C = \prod_a \mathcal{D}C^a$, $\mathcal{D}\bar{C} = \prod_a \mathcal{D}\bar{C}^a$ and the definition

$$\mathcal{M}_{ab}(x, y) = \left. \frac{\delta \mathcal{F}_a\big[A[A_0(x), \Lambda(x)]\big]}{\delta \Lambda_b(y)} \right|_{\Lambda=0} \tag{3.19}$$

Finally, the δ-functional in (3.17) can be removed by replacing its argument with $\mathcal{F}_a[A] - f_a$, where f_a are functions of spacetime and compute the average of all f_a with a Gaussian weight. This means integrate the functional with

$$\int \mathcal{D}f \, \mathrm{e}^{-\frac{i}{2\kappa} \int \mathrm{d}^d x f_a f^a} \ , \tag{3.20}$$

where κ is a real parameter. The δ functional then effectively cancels against the integration and it replaces f_a in the exponential by $\mathcal{F}_a[A]$. The final form of the gauge fixed functional therefore is (up to an unimportant constant N' in front)

$$Z[0] = N' \int \mathcal{D}A \, \mathcal{D}C \, \mathcal{D}\bar{C} \, \exp\left\{ iS[A] - i \int \mathrm{d}^d x \left(\frac{1}{2\kappa} \mathcal{F}_a[A] \mathcal{F}^a[A] + \bar{C}^a M_{ab} C^b \right) \right\} , \quad (3.21)$$

where M_{ab} is an operator that acts on C_b. It is defined as an operator that has to act on the δ-function to give the matrix $\mathcal{M}_{ab}(x, y)$ which is given in (3.19)

$$M_{ab}\delta(x - y) = \mathcal{M}_{ab}(x, y) . \quad (3.22)$$

It is not difficult to evaluate (3.19), since (Λ is set to zero at the end) it is sufficient to replace $A[A_0, \Lambda]$ by the infinitesimal version (A.84) or (A.83) of the respectively ordinary or noncommutative gauge transformation. Furthermore, we take the gauge fixing functional $\mathcal{F}$ to be linear in its argument such that it can be described by

$$\mathcal{F}_a[A] = \mathcal{O}_{ab}^\mu A_\mu^b , \quad (3.23)$$

where an operator $\mathcal{O}_{ab}^\mu$ acts on A_μ^a. One then finds for M_{ab}

$$
\begin{aligned}
M_{ab}\delta(x - y) = \mathcal{O}_{ac}^\mu \Big(&\delta_b^c \partial_\mu \delta(x - y) - \frac{1}{2}(\delta(x - y) \bullet A_\mu^d + A_\mu^d \bullet \delta(x - y)) f_{bd}{}^c \\
&+ \frac{i}{2}(\delta(x - y) \bullet A_\mu^d - A_\mu^d \bullet \delta(x - y)) d_{bd}{}^c \Big) .
\end{aligned}
\quad (3.24)
$$

Here the $\bullet$-product either denotes the $*$-product or the ordinary one.

At the end of this general discussion we compare gauge fixing in the ordinary and in the noncommutative case and specifically work out how the ghost fields are related in both cases. Here we will now explicitly use lower case and capital letters for variables of the ordinary and noncommutative case respectively. We begin with the gauge fixing functional and use the Seiberg-Witten map (2.53) to translate this condition between the ordinary and noncommutative case

$$\mathcal{F}[A] = \mathcal{F}[A[a]] \quad (3.25)$$

One can then write the expressions (3.19) in both cases as

$$
\begin{aligned}
\mathcal{M}_{ab}^{\mathrm{NC}}(x, y) &= \frac{\delta \mathcal{F}_a[A(x)]}{\delta A} \cdot \frac{\delta A[A_0, \Lambda]}{\delta \Lambda_b(y)} \bigg|_{\Lambda=0} , \\
\mathcal{M}_{ab}^{\mathrm{ord}}(x, y) &= \frac{\delta \mathcal{F}_a[A(x)]}{\delta A} \cdot \frac{\delta A[a]}{\delta a} \cdot \frac{\delta a[a_0, \lambda]}{\delta \lambda_b(y)} \bigg|_{\lambda=0} ,
\end{aligned}
\quad (3.26)
$$

where $\cdot$ abbreviates the summation and integration over all omitted indices and spacetime coordinates. A relation between $\mathcal{M}^{\mathrm{NC}}$ and $\mathcal{M}^{\mathrm{ord}}$ follows from a comparison of the R. H. S. in the above given expressions. It reads

$$\mathcal{M}^{\mathrm{NC}} \cdot \frac{\delta A[a]}{\delta a} \cdot \frac{\delta a[a_0, \lambda]}{\delta \lambda_b} \bigg|_{\lambda=0} = \mathcal{M}^{\mathrm{ord}} \cdot \frac{\delta A[A_0, \Lambda]}{\delta \Lambda_b} \bigg|_{\Lambda=0} . \tag{3.27}$$

Since $\Delta = \det \mathcal{M}$, taking the determinant on both sides then relates Δ^{NC} and Δ^{ord}. Both can be expressed as in (3.18) using $\mathcal{M}^{\mathrm{NC}}$ and the ghost fields $\bar{C}$, C in the noncommutative and $\mathcal{M}^{\mathrm{ord}}$, $\bar{c}$, c in the ordinary formulation. According to (3.18), the exponents are then given by

$$\bar{C}^a \cdot \mathcal{M}^{\mathrm{NC}}_{ab} \cdot C^b , \qquad \bar{c}^a \cdot \mathcal{M}^{\mathrm{ord}}_{ab} \cdot c^b \tag{3.28}$$

in the noncommutative and the ordinary formulation. A variable transformation between the two sets of ghost fields in the path integrals should produce the required determinants that are necessary to fulfill (3.27). If both exponents given in (3.28) are set equal, the transformation properties of fermionic path integrals guarantee that the required determinant is generated.[4] Using the relation (3.27) the equality of both expressions in (3.28) can be achieved with the choice

$$\bar{C} = \bar{c} , \qquad \frac{\delta A[A_0, \Lambda]}{\delta \Lambda} \bigg|_{\Lambda=0} C = \frac{\delta A[a]}{\delta a} \cdot \frac{\delta a[a_0, \lambda]}{\delta \lambda} \bigg|_{\lambda=0} c , \tag{3.29}$$

where again indices and spacetime dependence have been omitted. The solution to the above equation is already known. It is precisely the Seiberg-Witten map for the gauge parameter that maps the ghosts. The basic relation from which the map has been derived is (2.54) which can be cast into the form

$$A[a] + \hat{\delta}_\Lambda A[a] = A[a + \delta_\lambda a] = A[a] + \frac{\delta A[a]}{\delta a} \delta_\lambda a \tag{3.30}$$

for infinitesimal gauge transformations. For these one has the relation

$$\hat{\delta}_\Lambda A[a] = \frac{\delta A[A_0, \Lambda]}{\delta \Lambda} \bigg|_{\Lambda=0} \Lambda , \qquad \delta_\lambda a = \frac{\delta a[a_0, \lambda]}{\delta \lambda} \bigg|_{\lambda=0} \lambda , \tag{3.31}$$

such that the relation for the gauge parameters read

$$\frac{\delta A[A_0, \Lambda]}{\delta \Lambda} \bigg|_{\Lambda=0} \Lambda = \frac{\delta A[a]}{\delta a} \cdot \frac{\delta a[a_0, \lambda]}{\delta \lambda} \bigg|_{\lambda=0} \lambda . \tag{3.32}$$

---

[4]It is important to remember that the fermionic path integral measure transforms with the reciprocal determinant, see e. g. [151].

This is exactly the same as (3.29) for the ghosts. Therefore, the solution for C in (3.29) is simply given by the expression for the gauge parameter $\Lambda = \Lambda[\lambda, a]$ with λ replaced by c. At order $\mathcal{O}(\theta)$ one thus finds from (2.59)

$$C[c, a] = c + \frac{1}{4}\theta^{\alpha\beta}\left\{\partial_\beta c, a_\alpha\right\} \ . \tag{3.33}$$

This relation is found in [26] for the Abelian case via a discussion of the Becchi-Rouet-Stora-Tyutin (BRST) transformation [19]. The above connection between the transformation of the gauge parameter and the ghosts is not surprising if one remembers that the BRST transformation of the gauge field is given by a gauge transformation with the anticommuting ghost as a parameter.

3.3 Feynman rules for noncommutative Yang-Mills theories with gauge groups $U(N)$

In this Section we will show how to extract the known Feynman rules [10, 32, 171] for noncommutative YM theories with $U(N)$ gauge groups, which can be consistently formulated as discussed in Section 3.1. We will extract them from the path integral formulation that was presented in Appendix A.1.2. The action is given by (3.1) and we fix the gauge with the Lorentz condition

$$\mathcal{F}_a[A] = \mathcal{O}^\mu_{ab}A^b_\mu = \frac{1}{g}\delta_{ab}\partial^\mu A^b_\mu = 0 \ . \tag{3.34}$$

One can now write down the complete action that enters the gauge fixed functional (3.21) with M_{ab} given in (3.24) for the above given $\mathcal{O}^\mu_{ab}$ and with $*$-products. The complete Lagrangian is

$$\mathcal{L} = \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP}} \ . \tag{3.35}$$

With the noncommutative field strength (3.2), each piece in the above expression reads

$$\begin{aligned}
\mathcal{L}_{\text{YM}} &= -\frac{1}{2g^2}\operatorname{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) , \\
\mathcal{L}_{\text{GF}} &= -\frac{1}{2g^2\kappa}\operatorname{tr}\left((\partial^\mu A_\mu)^2\right) , \\
\mathcal{L}_{\text{FP}} &= -2\operatorname{tr}\left(\bar{C}\partial^\mu(\partial_\mu C + i\left[C \stackrel{*}{,} A_\mu\right])\right) ,
\end{aligned} \tag{3.36}$$

or after evaluating the trace over the gauge group

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4g^2} F^A_{\mu\nu} F^{\mu\nu}_A \ ,$$

$$\mathcal{L}_{\text{GF}} = -\frac{1}{4g^2\kappa} (\partial^\mu A^A_\mu)(\partial_\nu A^\nu_A) \ ,$$

$$\mathcal{L}_{\text{FP}} = -\bar{C}^A \partial^\mu \left(\partial_\mu C_A - \frac{1}{2} \left\{ C^B {}^*_, A^C_\mu \right\} f_{BCA} + \frac{i}{2} \left[C^B {}^*_, A^C_\mu \right] d_{BCA} \right) \ ,$$

$$= -\bar{C}^A \partial^\mu \left(\partial_\mu C_A - C^B \cos \left(\overleftarrow{\partial}_\rho \tfrac{1}{2} \theta^{\rho\sigma} \overrightarrow{\partial}_\sigma \right) A^C_\mu f_{BCA} - C^B \sin \left(\overleftarrow{\partial}_\rho \tfrac{1}{2} \theta^{\rho\sigma} \overrightarrow{\partial}_\sigma \right) A^C_\mu d_{BCA} \right) \ .$$
$$(3.37)$$

To find the above results we have used (3.3), (3.5) and (A.66). To make the expressions more readable we will use the abbreviation

$$\mathcal{O}_\rho \tfrac{1}{2} \theta^{\rho\sigma} \mathcal{P}_\sigma = \mathcal{O} \wedge \mathcal{P} \ , \tag{3.38}$$

where $\mathcal{O}$ and $\mathcal{P}$ denote any operators that carry one spacetime index. The components of the field strength tensor are then given by

$$F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + \frac{1}{2} \left\{ A^B_\mu {}^*_, A^C_\nu \right\} f_{BC}{}^A - \frac{i}{2} \left[A^B_\mu {}^*_, A^C_\nu \right] d_{BC}{}^A$$
$$= \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + A^B_\mu \cos \left(\overleftarrow{\partial} \wedge \overrightarrow{\partial} \right) A^C_\nu f_{BC}{}^A + A^B_\mu \sin \left(\overleftarrow{\partial} \wedge \overrightarrow{\partial} \right) A^C_\nu d_{BC}{}^A \ . \tag{3.39}$$

One finds for $\mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{GF}}$ after adding appropriate total derivative terms

$$\mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{GF}} \cong -\frac{1}{g^2} \text{tr} \left(- A_\mu \big(G^{\mu\nu} \Box - (1-\kappa^{-1}) \partial^\mu \partial^\nu \big) A_\nu \right.$$

$$\left. - 2i \partial_\mu A_\nu \left[A^\mu {}^*_, A^\nu \right] - \frac{1}{2} \left[A_\mu {}^*_, A_\nu \right] \left[A^\mu {}^*_, A^\nu \right] \right)$$

$$= \frac{1}{2g^2} A^A_\alpha \delta_{AB} \big(G^{\alpha\beta} \Box - (1-\kappa^{-1}) \partial^\alpha \partial^\beta \big) A^B_\beta$$

$$- \frac{1}{g^2} G^{\mu\beta} G^{\alpha\gamma} (\partial_\mu)_1 \big(\cos \left((\partial)_2 \wedge (\partial)_3 \right) f_{BCA} + \sin \left((\partial)_2 \wedge (\partial)_3 \right) d_{BCA} \big) A^A_\alpha A^B_\beta A^C_\gamma$$

$$- \frac{1}{4g^2} G^{\alpha\gamma} G^{\beta\delta} \big(\cos \left((\partial)_1 \wedge (\partial)_2 \right) f_{ABE} + \sin \left((\partial)_1 \wedge (\partial)_2 \right) d_{ABE} \big)$$

$$\times \big(\cos \left((\partial)_3 \wedge (\partial)_4 \right) f_{CD}{}^E + \sin \left((\partial)_3 \wedge (\partial)_4 \right) d_{CD}{}^E \big) A^A_\alpha A^B_\beta A^C_\gamma A^D_\delta \ . \tag{3.40}$$

We have used the notation $(\partial)_i$ to indicate that the derivative acts on the gauge field at the ith position in the product on the right hand side of the operator. To extract Feynman rules for this field theory means to find all connected tree-level Green functions. Comparing the above expression with (A.15) and (A.24), one obtains

$$K^{\alpha\beta}_{AB} = -\frac{1}{g^2} \delta_{AB} \big(G^{\alpha\beta} \Box - (1-\kappa^{-1}) \partial^\alpha \partial^\beta \big) \ . \tag{3.41}$$

Going through the procedure of (A.28), the 2-point function in momentum space is found to be

$$\tilde{G}^{AB}_{\alpha\beta}(p,q) = -i\tilde{\Delta}^{AB}_{\alpha\beta}(p)(2\pi)^4\delta(p+q) , \qquad \tilde{\Delta}^{AB}_{\alpha\beta}(p) = \frac{g^2}{p^2}\delta^{AB}\left(G_{\alpha\beta} - (1-\kappa)\frac{p_\alpha p_\beta}{p^2}\right) . \quad (3.42)$$

The proper 3- and 4-point vertices are found by comparing the interaction parts of (3.40) with (A.29) and then using (A.36) to get the momentum space expressions where the momenta leave the interaction point. One obtains for the 3-point vertex

$$\tilde{G}^{\alpha\beta\gamma}_{ABC}(p,q,r)_\mathrm{c} = \frac{1}{g^2}G^{\alpha\gamma}p^\beta\big(\cos(q\wedge r)f_{BCA} - \sin(q\wedge r)d_{BCA}\big)(2\pi)^d\delta(p+q+r)$$
$$+ 5 \text{ perm} , \qquad\qquad\qquad (3.43)$$

where 'perm' denotes the remaining permutations of the three momenta, Lorentz and group indices. The 4-point vertex reads

$$\tilde{G}^{\alpha\beta\gamma\delta}_{ABCD}(p,q,r,s)_\mathrm{c} = -\frac{i}{g^2}G^{\alpha\gamma}G^{\beta\delta}\big(\cos(p\wedge q)f_{ABE} - \sin(p\wedge q)d_{ABE}\big)$$
$$\times\big(\cos(r\wedge s)f_{CD}{}^E - \sin(r\wedge s)d_{CD}{}^E\big) \qquad (3.44)$$
$$\times (2\pi)^4\delta(p+q+r+s) + 5 \text{ perm} .$$

Here 'perm' denotes the remaining permutations after dividing out the 4-dimensional symmetry group S of the vertex which includes the permutations

$$S = \big\{(\pi(1),\pi(2),\pi(3),\pi(4))\big\} = \big\{(1,2,3,4),(2,1,4,3),(3,4,1,2),(4,3,2,1)\big\} \quad (3.45)$$

of the indices at the four legs 1, 2, 3, 4 (this removes the prefactor $\frac{1}{4}$, see (A.36)).

The ghost Lagrangian $\mathcal{L}_\mathrm{FP}$ reads, after adding a total derivative that modifies the interaction term

$$\mathcal{L}_\mathrm{FP} \cong -\bar{C}^A\delta_{AB}\partial^\mu\partial_\mu C^B$$
$$- (\partial^\beta)_1\Big(\cos\big((\partial)_3\wedge(\partial)_2\big)f_{CBA} + \sin\big((\partial)_3\wedge(\partial)_2\big)d_{CBA}\Big)\bar{C}^A A^B_\beta C^C . \qquad (3.46)$$

From this one finds for the 2-point function of the ghost C and antighost $\bar{C}$ respectively

$$\tilde{G}^{AB}_{(C)}(p,q) = \tilde{G}^{AB}_{(\bar{C})}(p,q) = \frac{i}{p^2}\delta^{AB}(2\pi)^d\delta(p+q) . \qquad (3.47)$$

Again using (A.36), the 3-point vertex that describes the antighost-gauge-ghost interaction can easily be read-off from (3.46) and is found to be

$$\tilde{G}^\beta_{(\bar{C}AC),ABC}(p,q,r) = -p^\beta\Big(\cos(q\wedge r)f_{CBA} - \sin(q\wedge r)d_{CBA}\Big)(2\pi)^d\delta(p+q+r) , \quad (3.48)$$

$$p,\alpha,A \ \text{———} \ p,\beta,B \qquad = -g^2 \frac{i}{p^2} \delta^{AB} \Big(G_{\alpha\beta} - (1-\kappa)\frac{p_\alpha p_\beta}{p^2} \Big)$$

$$
\begin{aligned}
&= -\frac{1}{g^2} \big(\cos(p \wedge q) f_{ABC} - \sin(p \wedge q) d_{ABC} \big)\\
&\quad \big(G^{\alpha\beta}(q-p)^\gamma + G^{\beta\gamma}(r-q)^\alpha + G^{\gamma\alpha}(p-r)^\beta \big)
\end{aligned}
$$

with legs p,α,A ; q,β,B ; r,γ,C

$$
\begin{aligned}
&\quad -\frac{i}{g^2} \big(\cos(p \wedge q) f_{ABE} - \sin(p \wedge q) d_{ABE} \big)\\
&= \quad\ \ \big(\cos(r \wedge s) f_{CD}{}^{E} - \sin(r \wedge s) d_{CD}{}^{E} \big)\\
&\quad \big(G^{\alpha\gamma} G^{\beta\delta} - G^{\alpha\delta} G^{\beta\gamma} \big) + 2 \text{ perm}
\end{aligned}
$$

with legs q,β,B ; p,α,A ; r,γ,C ; s,δ,D

$$p,A \ \cdots\!\blacktriangleleft\!\cdots \ p,B \quad = \quad p,A \ \cdots\!\blacktriangleright\!\cdots \ p_2,B \qquad = \frac{i}{p^2} \delta^{AB}$$

$$= p^\beta \big(\cos(p \wedge q) f_{ABC} + \sin(p \wedge q) d_{ABC} \big)$$

with legs p,A ; q,β,B ; r,C

Figure 3.1: Feynman rules [10, 32, 171] for noncommutative $U(N)$ YM theory, $p_\rho \frac{1}{2}\theta^{\rho\sigma} q_\sigma = p \wedge q$. The straight lines denote gauge bosons and the dashed arrow lines pointing to a vertex or away from it denote ghosts and antighosts respectively. Momentum conservation is understood and all momenta point to the vertices.

where $(\bar{C}AC)$ indicates that the momenta (leaving the interaction point) and indices (p, A), (q, β, B) and (r, C) refer to the antighost, gauge field and ghost respectively. Clearly the spacetime index β is assigned to the gauge field. Due to the fact that the three fields are different, no summation over permutations appears here.

The rules look very similar to that of ordinary YM theory. Modifications are due to the presence of the $*$-commutator in (3.2) instead of the ordinary commutator. This introduces the symmetric d_{ABC} and generates additional momentum dependent trigonometric factors in the vertices which are responsible for the UV/IR effect [121]. We already described this effect in the Introduction. It received a lot of attention in particular with respect to its stringy origin and its implications for the renormalization program. However, the UV/IR effect is not manifest in $\theta^{\mu\nu}$-expanded perturbation theory for $U(N)$. For $G \neq U(N)$, besides a conjecture for $SO(N)$ in [31], Feynman rules in terms of the full noncommutative A_μ are not known. Therefore, our goal in Section 3.5 will be to get information on these rules by studying some issues of partial summing the known θ-expanded rules. Such rules would allow one to study UV/IR mixing similar to the $U(N)$ case.

3.4 Construction of noncommutative Yang-Mills theories with gauge groups $G \neq U(N)$

In Section 3.1 we have seen that noncommutativity requires a careful choice of the underlying gauge group and that one has to give up the gauge field and the gauge transformation parameters being Lie algebra valued. Instead they take values in the enveloping algebra. For $U(N)$ gauge groups, however, the choice of a matrix representation enables one to introduce matrix multiplication and the anticommutator in addition to the commutator, and therefore one can avoid to work with the enveloping algebra. In contrast to the $U(N)$ case the formulation of gauge theories with other gauge groups is less unique. One has to work with the full infinite dimensional enveloping algebra and thus to introduce infinitely many degrees of freedom. In the following we will discuss two different kinds of constructions which avoid these problems. The one we describe first is perturbative in the noncommutativity parameter $\theta^{\mu\nu}$ and it is based on the enveloping algebra description. In the second approach, which is exact $\theta^{\mu\nu}$, one starts with a $U(N)$ theory and imposes additional constraints which are respected by the $*$-commutator. The constraints define a subgroup of $U(N)$ (endowed with the $*$-commutator).

3.4.1   The enveloping algebra approach

In a series of papers noncommutative gauge theories were discussed performing a perturbation expansion in $\theta^{\mu\nu}$. In [111] the authors formulate a general setup for defining noncommutative gauge theories. Besides the canonical structure (2.18) they analyzed the Lie algebra structure and the quantum space structure. The guiding principle of this discussion is to reduce the infinite number of fields that arises as coefficients in the enveloping algebra to a finite number [95]. The authors show that the enveloping algebra valued components (which have basis vectors (3.6) with $n > 1$) can be expressed in terms of the finite number of Lie algebra valued ones ($n = 1$ in (3.6)) and their derivatives. It turns out that there exists a solution in which the coefficient in front of the basis element (3.6) formed from n Lie algebra generators is of order $\mathcal{O}(\theta^{n-1})$.[5] The coefficient of (3.6) is then given by the corresponding $\mathcal{O}(\theta^{n-1})$ term in the expansion of the Seiberg-Witten map. That means the ordinary gauge field a_μ is the leading $n = 1$ coefficient. Some more details of this derivation are presented in Appendix A.6.

In [94, 96] the authors construct non-Abelian gauge theories in the enveloping algebra approach of [95] and determine the coefficients of the fields and the gauge transformation up to $\mathcal{O}(\theta^2)$. As already mentioned, this corresponds to an expansion of the Seiberg-Witten map up to the same order. See [15] for a related analysis which includes a discussion of ambiguities in the Seiberg-Witten map.

From the above summarized references one can now extract the recipe to construct non-Abelian gauge theories with arbitrary gauge groups G perturbatively to a certain order $\mathcal{O}(\theta^n)$. One should take the fields and gauge parameter of an ordinary gauge theory with gauge group G and insert them into the Seiberg-Witten map expanded up to $\mathcal{O}(\theta^n)$. This gives the noncommutative fields and gauge parameter which then enter the action of the noncommutative gauge theory.

3.4.2   Subgroups of $U(N)$ via additional constraints

The authors of [33] define gauge transformations in a subgroup of the noncommutative $U(N)$ gauge group (which is endowed with the $*$-commutator). The corresponding infinitesimal gauge transformations and the gauge field then do not belong to a subalgebra of

---

[5]In [94] the authors remark that one can change this behaviour by using the freedom in the Seiberg-Witten map.

the $\mathfrak{u}(N)$ Lie algebra (both endowed with the standard commutator). This is not inconsistent because in Section 3.1 we have seen that the noncommutative gauge field and gauge transformations are not Lie algebra valued anyway. The subgroup is defined by setting up constraints on the gauge field and gauge transformation parameter. Their construction works for $SO(N)$ and $Sp(N)$ subgroups, because in these cases an anti-automorphism of the noncommutative algebra of functions can be used to formulate the required constraints. This approach is exact in the noncommutativity parameter $\theta^{\mu\nu}$. For vanishing $\theta^{\mu\nu}$ one recovers the ordinary $SO(N)$ and $Sp(N)$ gauge theories. It is a disadvantage of this approach that the formulation of the anti-automorphism requires one to interpret the elements of the algebra not only as spacetime dependent but also as functions of the noncommutativity parameter $\theta^{\mu\nu}$, which then is treated as a variable and not as a (constant) parameter. But if one allows for an expansion in $\theta^{\mu\nu}$ then the dependence on $\theta^{\mu\nu}$ has an interpretation in the context of the Seiberg-Witten map (2.53) where the noncommutative gauge fields are indeed given by a power series in $\theta^{\mu\nu}$. The authors find that their constraint translates to the condition that the ordinary gauge field a_μ, which is mapped to the noncommutative gauge field A_μ, takes values in the corresponding ordinary Lie subalgebra. The $\theta^{\mu\nu}$-expanded version of this constraint thus is in perfect agreement with the enveloping algebra approach. Some more details about setting up the constraint and about its relation to the enveloping algebra approach can be found in Appendix A.7. For the $SU(N)$ case an alternative constraint has been proposed in [44].

For completeness let us remark that the authors of [33] furthermore construct a string theory setup from which their theories follow in the Seiberg-Witten limit (2.24). The conditions on the gauge fields and gauge transformations which are formulated with the anti-automorphism correspond to an orientifold projection. The string theory background is given by D_p-branes on top of an orientifold plane with a constant B-field which is parallel to and possesses opposite signs on both sides of the orientifold plane.

The above analysis was refined in [16] where the authors use a modified version of the anti-automorphism of [33] that allows for a relaxation of the condition that the elements of the algebra depend on $\theta^{\mu\nu}$. On the string side the authors discuss stable (supersymmetric) background configurations of Dp-branes and orientifold planes that correspond to the constructed gauge theories.

In [31] the authors addressed the problem of how to compute 1-loop amplitudes in noncommutative $SO(N)$ gauge theories from string theory. The setup they use corre-

sponds to the setup in [33] that we explained above. After a review of the techniques to calculate the annulus and Möbius contribution in the ordinary case the authors extend their analysis to the noncommutative case. They find that the θ-dependence is given by a simple factor multiplying the 1-loop amplitude (at $\theta^{\mu\nu} = 0$) which is the same behavior already found at tree level [33, 158], see (2.22). The authors conjecture Feynman rules for the noncommutative $SO(N)$ theory from the tree level amplitudes. These rules are *exact* in θ and therefore should be suitable for the study of effects that require the full θ-dependence (like UV/IR mixing [121], see the Introduction and the end of Section 3.3). They observe, however, that there is a mismatch between the field theory limit of the string 1-loop amplitudes and the field theory computation using these Feynman rules.

3.5 Feynman rules for noncommutative Yang-Mills theories with gauge groups $G \neq U(N)$

In the previous Section we have described in brief two proposals for a formulation of noncommutative YM theories with gauge groups $G \neq U(N)$. In the enveloping algebra approach described in Subsection 3.4.1, it was found that the coefficients of the enveloping algebra valued basis elements are related to the Lie algebra valued terms via the Seiberg-Witten map (2.53). The noncommutative gauge field and gauge transformation parameter of a gauge theory with group G are the Seiberg-Witten map of the ordinary gauge field and parameter that take values in $\mathfrak{g}$, the Lie algebra of G. On the other hand the approach described in Subsection 3.4.2 imposes constraints on the noncommutative $U(N)$ gauge field and parameter to define gauge theories with groups $G \subset U(N)$. Using the Seiberg-Witten map to translate these constraints to the ordinary fields and parameter leads exactly to the same result as found in the enveloping algebra approach: the ordinary fields and parameter take values in $\mathfrak{g}$ which then is a subalgebra of $\mathfrak{u}(N)$.

Since the first approach is *perturbative* in $\theta^{\mu\nu}$ and defines a theory in terms of the ordinary fields, whereas the second is *exact* in $\theta^{\mu\nu}$ and uses the noncommutative fields, it is an interesting question to ask how both quantum field theoretical formulations are related. We want to see what happens if one tries to resum the $\theta^{\mu\nu}$-expansion in the enveloping algebra approach. In particular it is important to clarify if and how the Feynman rules for such a theory can be constructed and if one can confirm the rules of [31]. If they are consistent one should be able to reproduce them because they were taken

from the framework of [33] which is compatible with the enveloping algebra approach.

The aim of this Section, which is based on our work [61], is to get information on Feynman rules for noncommutative gauge theories with gauge groups $G \neq U(N)$ that include the exact dependence on $\theta^{\mu\nu}$ and especially to decide if the rules for $SO(N)$ given in [31] are consistent. In principle one should work with the approach of Subsection 3.4.2 that uses a constraint. We have seen that this constraint has the interpretation that under the Seiberg-Witten map the ordinary fields and parameter are restricted to a subalgebra of $\mathfrak{g}$. One can therefore resolve the constraint by using the formulation of the theory in the ordinary fields which is obtained by use of the full Seiberg-Witten map (exact in $\theta^{\mu\nu}$). Since the latter is not known in the non-Abelian case one is forced to work with its θ-expansion, i. e. the constraint is perturbative in θ. We will show that one can nevertheless extract statements from such a setup that are universal, i. e. that do not depend on the order to which the constraint has been expanded. For this one has to analyze a resummation of the θ-expansion. To be more precise the formulation is as follows: first express the noncommutative $U(N)$ gauge field A_μ and gauge transformation Λ via the Seiberg-Witten map (2.59) in terms of the ordinary $U(N)$ gauge field a_μ and gauge transformation λ, respectively. After that both a_μ and λ are constrained to take values in the Lie subalgebra $\mathfrak{g} \subset \mathfrak{u}(N)$ of $G \subset U(N)$. The gauge theory with gauge group G is now defined by the noncommutative Yang-Mills action (3.1) together with the corresponding gauge transformations (A.74), (A.83) *and* the constraint that (2.53) is valid with $a_\mu, \lambda \in \mathfrak{g}$. After choosing the Lorenz gauge condition and the Feynman gauge ($\kappa = 1$) we have from (3.36) the complete setup

$$S[A, C, \bar{C}] = -\frac{1}{2g^2} \int \mathrm{d}^d x \, \mathrm{tr} \left(F_{\mu\nu} F^{\mu\nu} + (\partial^\mu A_\mu)^2 \right) - 2 \int \mathrm{d}^d x \, \mathrm{tr} \left(\bar{C} \partial^\mu (\partial_\mu C + i \, [C \overset{*}{,} A_\mu]) \right) , \tag{3.49}$$

where according to (2.59) and (3.33) one has

$$\begin{aligned}
A_\mu[a] &= a_\mu - \frac{1}{4}\theta^{\alpha\beta} \left\{ a_\alpha, \partial_\beta a_\mu + f_{\beta\mu} \right\} + \mathcal{O}(\theta^2) , \\
\Lambda[\lambda, a] &= \lambda + \frac{1}{4}\theta^{\alpha\beta} \left\{ \partial_\beta \lambda, a_\alpha \right\} + \mathcal{O}(\theta^2) , \\
C[c, a] &= c + \frac{1}{4}\theta^{\alpha\beta} \left\{ \partial_\beta c, a_\alpha \right\} + \mathcal{O}(\theta^2) , \\
\bar{C} &= c .
\end{aligned} \tag{3.50}$$

Inserting these transformation formulas into (3.49) then leads to an action for the ordinary gauge fields a_μ and ghosts c, $\bar{c}$ which is a power series in $\theta^{\mu\nu}$ and one can determine the

Feynman rules. Besides the standard propagators and vertices like in the $U(N)$ theory in Fig. 3.1 one has in general an infinite set of additional vertices with an increasing number of legs, derivatives, and powers of $\theta^{\mu\nu}$. For our further discussion it is useful to stress that all these vertices are generated by the $\theta^{\mu\nu}$-expansion of the whole action, i. e. both the noncommutative kinetic and interaction term. In the following we call this kind of perturbation theory the *θ-expanded perturbation theory* for the noncommutative G gauge theory. It is extensively studied in the references of Subsection 3.4.1. On the other side for the $U(N)$ case it is straightforward to get directly from (3.49) Feynman rules in terms of A_μ, C and $\bar{C}$ as already discussed in Section 3.3. They are *exact* in $\theta^{\mu\nu}$. We will now follow the strategy:

1. In Subsection 3.5.1 we will introduce the path integral formulation for the constrained theory and show how the constraint is resolved by a variable transformation inside the path integral.

2. In Subsection 3.5.2 we will then extract the θ-expanded Feynman rules for this formulation and discuss the behaviour of the theory under resummation of the θ-expansion for general gauge groups G and compare with the case of $U(N)$.

3. In Subsection 3.5.3 we will prove that for $G \neq U(N)$ one generates infinitely many connected n-point Green functions and one therefore cannot define a consistent set (a finite number) of Feynman rules.

4. In Subsection 3.5.4 we will discuss a counterexample that the Feynman rules for $SO(N)$ given in [31] cannot be derived from the theory.

3.5.1   Path integral quantization of the constrained theory

We will now formulate the θ-expanded perturbation theory and define our task in precise technical terms. The original noncommutative interactions are kept as some of the vertices of our wanted Feynman rules. The theory is summed with respect to the vertices generated by the expansion of the kinetic term only. We start with (3.49), the noncommutative $U(N)$ YM-theory in Feynman gauge described with the noncommutative gauge field A_μ and Faddeev-Popov ghosts C and $\bar{C}$ and separate

$$S[A, C, \bar{C}] = S_{\text{kin}}[A, C, \bar{C}] + S_I[A, C, \bar{C}] \, , \tag{3.51}$$

with

$$S_{\text{kin}}[A, C, \bar{C}] = -\frac{1}{g^2} \int \mathrm{d}^d x \, \text{tr} \, \partial_\mu A_\nu \partial^\mu A^\nu - \int \mathrm{d}^d x \, \partial_\mu \bar{C} \partial^\mu C . \tag{3.52}$$

According to (A.7) the generating functional for noncommutative G Green functions is given by

$$Z_G[J, \bar{\eta}, \eta] = \int_{a,c,\bar{c} \in \mathfrak{g}} \mathcal{D}A \, \mathcal{D}\bar{C} \, \mathcal{D}C \, \mathrm{e}^{i(S[A,C,\bar{C}] + A \cdot J + \bar{\eta} \cdot C + \bar{C} \cdot \eta)} , \tag{3.53}$$

where we have introduced sources J, η and $\bar{\eta}$ for the gauge field, ghost and anti-ghost respectively. By the notation $\int_{a,c,\bar{c} \in \mathfrak{g}}$ we indicate the integration over A, C, $\bar{C}$ with the constraint that their image under the inverse Seiberg-Witten map is in $\mathfrak{g}$, i. e. $a, c, \bar{c} \in \mathfrak{g}$. From now on we will work with the $N \times N$ matrix representation of $\mathfrak{u}(N)$ that fulfills (3.3). For $U(N)$ the constraint is trivially solved by $A_\mu = A_\mu^A T_A$ and free integration over A_μ^A, C^A, $\bar{C}^A$.

To explore the possibility of noncommutative G Feynman rules, which after some possible projection work with the $U(N)$ vertices, we write (3.53) using (3.51) as (see (A.16))

$$Z_G[J, \bar{\eta}, \eta] = \mathrm{e}^{iS_I[\frac{\delta}{i\delta J}, \frac{\delta}{i\delta\bar{\eta}}, \frac{\delta}{i\delta\eta}]} Z_G^{\text{kin}}[J, \bar{\eta}, \eta] , \tag{3.54}$$

with

$$Z_G^{\text{kin}}[J, \bar{\eta}, \eta] = \int_{a,c,\bar{c} \in \mathfrak{g}} \mathcal{D}A \, \mathcal{D}\bar{C} \, \mathcal{D}C \, \mathrm{e}^{i(S_{\text{kin}}[A,C,\bar{C}] + A \cdot J + \bar{\eta} \cdot C + \bar{C} \cdot \eta)} . \tag{3.55}$$

Denoting by $\mathcal{J}$ the functional determinant for changing the integration variables from A, C, $\bar{C}$ to a, c, $\bar{c}$ we get

$$Z_G^{\text{kin}}[J, \bar{\eta}, \eta] = \int_{a,c,\bar{c} \in \mathfrak{g}} \mathcal{D}a \, \mathcal{D}\bar{c} \, \mathcal{D}c \, \mathcal{J}[a, c, \bar{c}] \, \mathrm{e}^{i(S_{\text{kin}}[a,c,\bar{c}] + s_1[a,c,\bar{c}] + A[a] \cdot J + \bar{\eta} \cdot C[c,a] + \bar{c} \cdot \eta)} . \tag{3.56}$$

The new quantity $s_1[a, c, \bar{c}]$ appearing above is defined via (3.50) and (3.52) by

$$S_{\text{kin}}[A[a], C[c,a], \bar{c}] = S_{\text{kin}}[a, c, \bar{c}] + s_1[a, c, \bar{c}] . \tag{3.57}$$

Applying (A.10) and (A.11) here, the logarithm of (3.56) divided by $Z_G^{\text{kin}}[0, 0, 0]$ is the generating functional for the connected Green functions of the composites A, C, $\bar{C}$ in the field theory with elementary fields $a, c, \bar{c}$ interacting via $s_1 - i \ln \mathcal{J}$. Therefore, it can be represented by (see (A.12))

$$\ln \frac{Z_G^{\text{kin}}[J, \bar{\eta}, \eta]}{Z_G^{\text{kin}}[0]} = \sum_n \frac{i^n}{n!} \int \mathrm{d}^d x_1 \ldots \mathrm{d}^d x_n \, \langle A(x_1) \ldots A(x_n) \rangle_{\text{c}}^{\text{kin}} J(x_1) \ldots J(x_n) + \ldots , \tag{3.58}$$

where $\langle A(x_1) \ldots A(x_n) \rangle_c^{\text{kin}}$ stands for the n-point connected Green function of A_μ in this field theory. The dots at the end represent the corresponding ghost and mixed ghost and gauge field terms.

Neglecting $\mathcal{J}$ (a justification will follow in Subsection 3.5.2) these are just the Green functions for the composites A, C, $\bar{C}$ obtained within θ-expanded perturbation theory by partial summation of all diagrams built with vertices generated by the θ-expansion of the noncommutative kinetic term only.

In the $U(N)$-case $Z_{U(N)}^{\text{kin}}[J, \bar{\eta}, \eta]$ as given by (3.55) is a trivial Gaussian integral and one obtains an expression similar to (A.19). It is the generating functional of Green functions for A, C, $\bar{C}$ treated as free fields. Then in (3.58) only the two point functions $\langle AA \rangle_c^{\text{kin}}$ and $\langle C\bar{C} \rangle_c^{\text{kin}}$ are different from zero. In addition they are equal to the free propagators as discussed in Section 3.3 (see (3.42), (3.47)).

Starting with free fields and imposing a constraint in the generic case generates an interacting theory. We want to decide what happens in our case (3.55) for $G \subset U(N)$. By some special circumstance it could be that only the connected two-point functions are modified. Another less restrictive possibility would be that connected n-point functions beyond some finite $n_0 > 2$ vanish. In both cases from (3.54) we would get Feynman rules with a finite number of building blocks.

For $U(N)$ the equivalent representation (3.56) is due to a simple field redefinition of a free theory. Therefore, looking at the n-point functions of the (in terms of $a, c, \bar{c}$, see (3.50)) composite operators A, C, $\bar{C}$ the summation of the perturbation theory with respect to $s_1[a, c, \bar{c}] - i \ln \mathcal{J}$ must yield the free field result guaranteed by (3.55).[6]

On the other side for $G \subset U(N)$ we cannot directly evaluate (3.55) and are forced to work with (3.56). It will turn out to be useful to study both $U(N)$ and $G \subset U(N)$ in parallel. Since the result for $U(N)$ is a priori known, one has some checks for the calculations within the s_1-perturbation theory.

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[6]This is a manifestation of the equivalence theorem [98].

3.5.2 s_1-perturbation theory for $U(N)$ and $G \subset U(N)$

In both cases our gauge fields take values in the $N \times N$ matrix representation of the Lie algebra $\mathfrak{u}(N)$. We write

$$A_\mu = A_\mu^A T_A \; , \tag{3.59}$$

where we have used (3.3) for the generators T^A of $\mathfrak{u}(N)$. Then (3.50) implies

$$A_\mu^M = a_\mu^M - \frac{1}{2}\theta^{\alpha\beta} a_\alpha^P \partial_\beta a_\mu^Q d_{PQ}{}^M + \frac{1}{4}\theta^{\alpha\beta} a_\alpha^P \partial_\mu a_\beta^Q d_{PQ}{}^M - \frac{1}{4}\theta^{\alpha\beta} a_\alpha^P a_\beta^Q a_\mu^R d^{MS}{}_P f_{SQR} + O(\theta^2) \; ,$$

$$C^M = c^M + \frac{1}{4}\theta^{\alpha\beta} \partial_\alpha c^P a_\beta^Q d_{PQ}{}^M + O(\theta^2)$$

$$\bar{C}^M = \bar{c}^M \; .$$

$$\tag{3.60}$$

In the case $G \subset U(N)$, $G \neq U(M)$, $M < N$ [7] we indicate the *generators spanning the Lie algebra* $\mathfrak{g}$ *of* G with a *lower case* Latin index and the *remaining* ones with a *primed lower case* Latin index. Upper case Latin indices run over all $U(N)$ generators, see (3.3) and (3.4). Since $\mathfrak{g}$ is the Lie subalgebra of $\mathfrak{u}(N)$ that corresponds to the subgroup $G \subset U(N)$ we have according to (3.4)

$$f_{abc'} = 0 \; , \quad \forall a, b, c' \qquad \text{and} \qquad d_{abc'} \neq 0 \quad \text{for some } a, b, c' \; . \tag{3.61}$$

As discussed in the previous Section, the noncommutative G gauge field theory is then defined by unconstrained functional integration over $a_\mu^b, c^b, \bar{c}^b$ and by the condition

$$a_\mu^{b'} = c^{b'} = \bar{c}^{b'} = 0 \; . \tag{3.62}$$

In spite of (3.62) via (3.60) with (3.61) one has non-vanishing $A_\mu^{b'}$ and $C^{b'}$.

We are interested in (3.56), i. e. the Green functions of A, C, $\bar{C}$, which are composites in terms of $a, c, \bar{c}$. For the diagrammatic evaluation one gets from (3.60) the *external* vertices where all momenta are directed to the interaction point, and a slash denotes a derivative of the field at the corresponding leg. (We write down the $\propto \theta^0$ and $\propto \theta^1$ contributions only. Momentum conservation at all vertices is understood.)

$$p,\mu,M \; \rule[0.5ex]{2cm}{0.4pt} \; k,\alpha,A \quad = \begin{cases} \delta_\mu^\alpha \delta_A^M & \text{for } M = m \\ 0 & \text{for } M = m' \end{cases} \; , \tag{3.63}$$

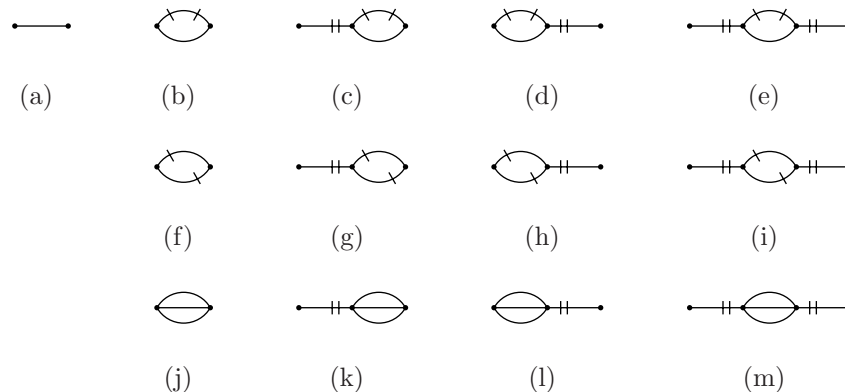[7]In the following we sometimes implicitly understand that $G \subset U(N)$ excludes $U(M)$ subgroups.

$$\begin{matrix} k_2,\beta,B \\ \\ p,\mu,M \\ \\ k_1,\alpha,A \end{matrix} \qquad = i \left(\frac{1}{4}\theta^{\beta\alpha}\delta_\mu^\nu - \frac{1}{2}\theta^{\beta\nu}\delta_\mu^\alpha \right) d_{AB}{}^M(k_1)_\nu \ , \qquad (3.64)$$

$$\begin{matrix} k_3,\gamma,C \\ \\ p,\mu,M \qquad k_2,\beta,B \\ \\ k_1,\alpha,A \end{matrix} \qquad = -\frac{1}{4}\theta^{\alpha\beta}d^{ME}{}_A f_{EBC}\delta_\mu^\gamma \ , \qquad (3.65)$$

and

$$p,M \ \cdots\!\!\blacktriangleleft\!\!\cdots \ k,\alpha,A \qquad = \qquad p,M \ \cdots\!\!\blacktriangleright\!\!\cdots \ k,A \qquad = \begin{cases} \delta_A^M & \text{for } M = m \\ 0 & \text{for } M = m' \end{cases} \ , \qquad (3.66)$$

$$\begin{matrix} k_2,\beta,B \\ \\ p,M \\ \\ k_1,A \end{matrix} \qquad = \frac{i}{4}\theta^{\nu\beta}d_{AB}{}^M(k_1)_\nu \ . \qquad (3.67)$$

The insertion of (3.60) into (3.57) yields $s_1[a,c,\bar{c}]$ generating the *internal* vertices

$$\begin{matrix} k_3,\gamma,C \\ \\ k_1,\alpha,A \ \longrightarrow\!\!+\!\!+\!\!< \\ \\ k_2,\beta,B \end{matrix} \qquad = \frac{1}{g^2} \left(\frac{1}{4}\theta^{\gamma\beta}G^{\nu\alpha} - \frac{1}{2}\theta^{\gamma\nu}G^{\alpha\beta} \right) d_{ABC}k_1^2(k_2)_\nu \ , \qquad (3.68)$$

$$\begin{matrix} k_4,\delta,D \\ \\ k_1,\alpha,A \ \longrightarrow\!\!+\!\!+\!\!< \ k_3,\gamma,C \\ \\ k_2,\beta,B \end{matrix} \qquad = \frac{i}{4g^2}\theta^{\delta\gamma}d_{AD}{}^E f_{ECB}G^{\alpha\beta}k_1^2 \ , \qquad (3.69)$$

$$\begin{matrix} k_3,C \\ \\ k_1,A \ \cdots\!\!\blacktriangleleft\!\!+\!\!+\!\cdots\!< \\ \\ k_2,\beta,B \end{matrix} \qquad = -\frac{1}{4}\theta^{\beta\nu}d_{ABC}k_1^2(k_3)_\nu \ . \qquad (3.70)$$

Figure 3.2: Contributions to $\langle A_\mu^M A_\nu^N \rangle_\mathrm{c}^\mathrm{kin}$ up to order θ^2

Figure 3.3: Some additional vanishing contributions to $\langle A_\mu^M A_\nu^N \rangle_\mathrm{c}^\mathrm{kin}$

The double slash stems from the derivatives in (3.52) after partial integration and denotes the action of $\Box = \partial_\mu \partial^\mu$ at the corresponding leg.

The propagators were given in Fig. 3.1 and simplify in Feynman gauge ($\kappa = 1$) to

$$- ig^2 G_{\alpha\beta} \delta^{AB} \frac{1}{k^2} \, , \qquad i\delta^{AB} \frac{1}{k^2} \tag{3.71}$$

for the commuting gauge field and ghosts, respectively.

Up to now we have not taken into account the functional determinant $\mathcal{J}$ in (3.56). To simplify the analysis we use dimensional regularization. Then this determinant is equal to one, and all diagrams containing momentum integrals not depending on any external momentum or mass parameter (tadpole type) are zero, see e. g. [191]. For other regularizations these tadpole type diagrams just cancel the determinant contributions, at least in the $U(N)$ case.

After these preparations we consider the 2-point function $\langle A_\mu^M A_\nu^N \rangle_\mathrm{c}^\mathrm{kin}$ within the perturbation theory with respect to s_1, see (3.56). Fig. 3.2 shows all diagrams up to order θ^2 which do not vanish in dimensional regularization. To give an impression which diagrams are absent, Fig. 3.3 presents some of them.
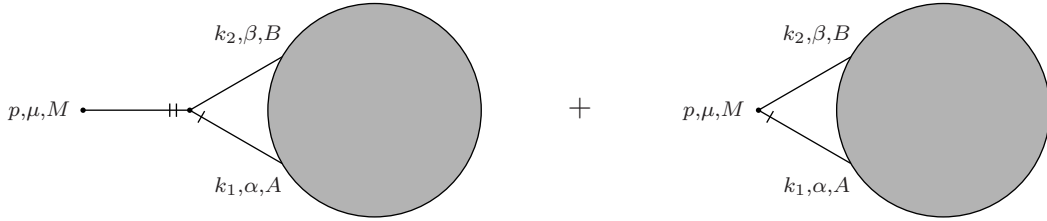
Figure 3.4: Canceling graphs from the $\propto a^2$ terms of the $\propto \theta$ terms of the SW map
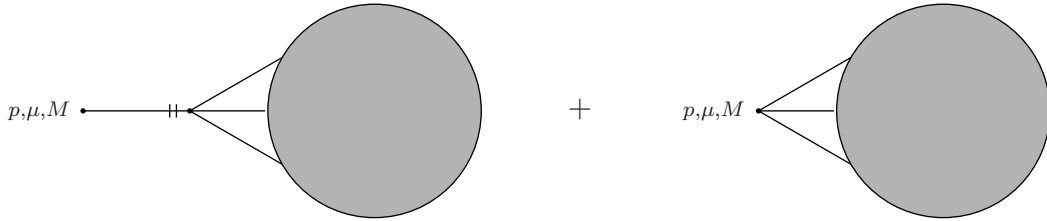
Figure 3.5: Canceling graphs from the $\propto a^3$ terms of the $\propto \theta$ terms of the SW map

Let us first continue with the $U(N)$ case. Then a straightforward analysis shows that the diagrams in Fig. 3.2(b) - 3.2(e) cancel among each other. The same is true for Fig. 3.2(f) - 3.2(i) and for Fig. 3.2(j) - 3.2(m).

The cancellation mechanism is quite general. Let us denote by $M(k_1, \alpha, A|k_2, \beta, B)$ an arbitrary subdiagram with two marked legs, denoted by a shaded bubble in Fig. 3.4. Then the sum of the two diagrams in Fig. 3.4 is equal to

$$
\begin{aligned}
&-\frac{ig^2}{p^2}G_{\mu\nu}\delta^{MN}\frac{1}{g^2}\left(\frac{1}{4}\theta^{\beta\alpha}G^{\lambda\nu}-\frac{1}{2}\theta^{\beta\lambda}G^{\nu\alpha}\right)d_{NAB}p^2(k_1)_\lambda M(k_1, \alpha, A|k_2, \beta, B) \\
&+i\left(\frac{1}{4}\theta^{\beta\alpha}\delta^\lambda_\mu-\frac{1}{2}\theta^{\beta\lambda}\delta^\alpha_\mu\right)d_{AB}{}^M(k_1)_\lambda M(k_1, \alpha, A|k_2, \beta, B) = 0 \ .
\end{aligned}
\tag{3.72}
$$

A similar general cancellation mechanism holds for diagrams of the type shown in Fig. 3.5.

Considering the s_1-perturbation theory we have convinced ourselves that for $U(N)$ the propagator of the composite fields assumes the form

$$
\langle A^M_\mu A^N_\nu\rangle^{\text{kin}}_{\text{c}} = -ig^2 G_{\mu\nu}\delta^{MN}\frac{1}{p^2} + O(\theta^3) \ .
\tag{3.73}
$$

Of course, from the representation (3.55) we know a priori that there are in all orders of θ no corrections to the free propagator. Nevertheless the above exercise was useful, since it unmasked the cancellation mechanism for Fig. 3.4 and Fig. 3.5 as being essential

for establishing the already known result purely within s_1-perturbation theory. It is straightforward to check also the vanishing of connected n-point functions for $n > 2$.

What changes if we switch from $U(N)$ to $G \subset U(N)$? First of all, then we do not know the answer in advance and we have to rely only on s_1-perturbation theory. Secondly, in this perturbation theory the above cancellation mechanism is no longer present for external points that carry a primed index and are thus related to elements of $\mathfrak{u}(N)$ that are not in $\mathfrak{g}$. Then according to (3.63) the external vertex to start with in the first diagrams of Fig. 3.4 and Fig. 3.5 is zero, i. e. the partners to cancel the second diagrams disappear. This observation is a strong hint that for $G \subset U(N)$ there remain non-vanishing connected Green functions $\langle A(x_1) \ldots A(x_n) \rangle_c^{\text{kin}}$ for all integer n. An explicit proof will be given in the next Section.

3.5.3 Non-vanishing n-point Green functions generated by $\ln Z_G^{\text{kin}}$

The connected Green functions

$$G_c^{\text{kin}, m_1' \ldots m_n'}{}_{\mu_1 \ldots \mu_n}(p_1, \ldots, p_n) = \left\langle A_{\mu_1}^{M_1}[a(x_1)] \ldots A_{\mu_n}^{M_n}[a(x_n)] \right\rangle_c^{\text{kin}}$$

are power series in θ and g. To prove their non-vanishing for generic θ and g it is sufficient to extract at least one non-zero contribution to G_c^{kin} of some fixed order in θ and g.

To find for our purpose the simplest tractable component of the Green function it turns out to be advantageous to restrict all of the group indices M_i to primed indices that do not correspond to generators of the Lie algebra of G. Then the Green function simplifies in first nontrivial order of the Seiberg-Witten map to:

$$\left\langle \prod_{i=1}^{n} \left(A^{(2)m_i'}{}_{\mu_i}[a(x_i)] + A^{(3)m_i'}{}_{\mu_i}[a(x_i)] \right) \right\rangle_c^{\text{kin}}. \tag{3.74}$$

Here $A^{(2)}$, $[A^{(3)}]$ denote the $\propto \theta$ part of the Seiberg-Witten map (3.60) with quadratic, [cubic] dependence on the ordinary field a_μ. Thus, the above function is $\mathcal{O}(\theta^n)$. Focusing now on the special contribution which is exactly $\propto \theta^n$, it is clear that in addition to the external vertices further θ-dependence (e. g. higher order corrections to the Seiberg-Witten map) is not allowed. That means this special part of the connected Green function is universal with respect to the θ-expansion of the constraint (3.50) where $a_\mu \in \mathfrak{g}$.
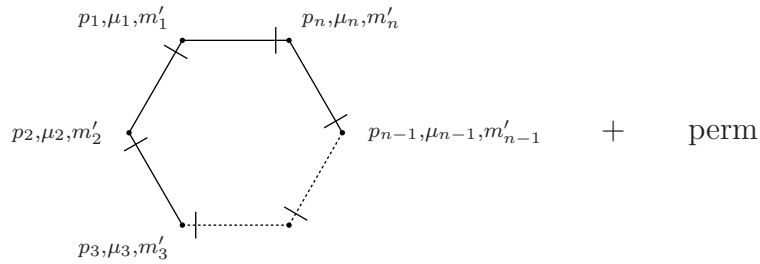
Figure 3.6: graphs $\propto \theta^n g^{2n}$ of the connected n-point Green function

The special contribution to the Green function $\propto \theta^n$ then consists of n to $\frac{3}{2}n$, $[\frac{3}{2}(n-1)+1]$ internal lines for n even, [odd]. Two or three of these originate from each of the n points (external vertices). There are no further internal vertices present stemming from the interaction term $s_1[a, c, \bar{c}]$ in (3.56) since this would increase the power in θ.

In our normalization where the coupling constant g is absorbed into the fields, each propagator enlarges the power of the diagram in g by g^2. Thus, for general coupling g it is sufficient to check the non-vanishing of all connected diagrams with the same number of propagators. Here we choose the minimum case of n propagators where we can neglect all contributions from $A^{(3)}$ in (3.74). Then it follows that the connected $\propto \theta^n g^{2n}$ contributions to the Green function are given by the type of diagrams shown in Fig. 3.6.

These diagrams are 1PI in terms of the ordinary fields a_μ, but we will further on denote them only as connected. One could imagine that with another field redefinition it might be possible to transform all these diagrams to connected, but not 1PI diagrams. It could then happen that one can construct them from a finite set of building blocks. A redefined field in which the type of diagrams in Fig. 3.6 could appear as connected 1PI diagrams would be given by the remnant of the Seiberg-Witten map without the leading contribution. Then, each external vertex in Fig. 3.6 would be the origin of a single line and the diagram could possibly be 1PI. Therefore, we are careful with our notation, but we have some arguments why such a field transformation is not relevant or possible here. Firstly, such a redefinition would not affect our analysis, because such a finite set would describe a perturbation expansion in orders of $\theta^{\mu\nu}$, and this is what we do not want. Secondly, the equivalence theorem [98, 191] is formulated for the field redefinitions that have to start with a term linearly in the fields. In the remnant of the Seiberg-Witten map

which is the candidate for the required field redefinition, the linear term is absent.

We will now determine the total number of the diagrams in Fig. 3.6. The two lines starting at each point are distinguishable due to the derivative at one leg. To construct all connected contributions we connect the first leg of the first external vertex to one of the $2n - 2$ other legs that do not start at the same external point. The next one is connected to one of the remaining $2n - 4$ allowed legs, such that no disconnected subdiagram is produced and so on. We thus have to add-up $(2n - 2)!! = (n - 1)!2^{n-1}$ diagrams. All of them can be drawn like the one shown in Fig. 3.6 by permuting the external momenta, Lorentz and group indices and the internal legs.

To sum-up all diagrams it is convenient to define two classes of permutations: the first includes all permutations that interchange the two distinct legs at one or more external vertices with the distribution of the external momenta, Lorentz and group indices held fixed. The second contains all permutations which interchange the external quantities such that this cannot be traced back to a permutation of the distinct lines at the external vertices. We call its elements proper permutations in the following.

In total 2^n combinations exist, generated by interchanging the distinct legs when the external points are fixed. The proper permutations are the ones which cannot be mapped to each other by acting with the Dihedral group D_n. D_n is the symmetry group of an n-sided regular polygon. It is $2n$ dimensional and it is represented by

$$D_n = \left\{ x, y \,|\, x^2 = y^2 = \mathbb{1}, \, (xy)^n = \mathbb{1} \right\} = \left\{ S^i R^j \,|\, S^n = R^2 = \mathbb{1}, \, RSR^{-1} = S^{-1} \right\} , \quad (3.75)$$

where the $2n$ elements are given by reflections along n of the symmetry axes through the origin, $n - 1$ rotations with angle $2\pi \frac{k}{n}$, $k = 1, \ldots, n - 1$ and the identity. There are $n!$ configurations of the external points and with each one $2n - 1$ others are identified by acting with D_n, i. e. there are $\frac{n!}{2n} = \frac{(n-1)!}{2}$ proper permutations. This is consistent with the total number of diagrams.

The connected $\propto \theta^n g^{2n}$ contributions to the momentum space Green function can thus

be cast into the following form:

$$G_{\mathrm{c}}^{\mathrm{kin},m_1'...m_n'}{}_{\mu_1...\mu_n}(p_1,\ldots,p_n)\big|_{\propto\theta^n g^{2n}} = \sum_{\{i_1,\ldots,i_n\}\in\frac{S_n}{D_n}} \quad (3.76)$$

Here the brackets around the external vertices denote a sum over both configurations where the two legs are interchanged. These n sums are then multiplied, and describe exactly the 2^n permutations of the distinct two legs at each vertex.

The sum of the two permutations at one external vertex occurring n times in (3.76) reads

$$=-\frac{i}{4}d_{AB}{}^M\left[2(\theta^{\beta\gamma}q_\gamma\delta_\mu^\alpha+\theta^{\alpha\gamma}r_\gamma\delta_\mu^\beta)+\theta^{\alpha\beta}(q_\mu-r_\mu)\right]\,.$$

Using this, the analytic expression for the $\propto\theta^n g^{2n}$ part of the connected Green function is given by

$$G_{\mathrm{c}}^{\mathrm{kin},m_1'...m_n'}{}_{\mu_1...\mu_n}(p_1,\ldots,p_n)\big|_{\propto\theta^n g^{2n}}$$

$$= \sum_{\{i_1,\ldots,i_n\}\in\frac{S_n}{D_n}} \frac{g^{2n}}{4^n}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\prod_{r=1}^n d^{a_r m_{i_r}'}{}_{a_{r+1}}\Big[-2\theta^{\alpha_r\gamma_r}(q_{r-1})_{\gamma_r}G_{\mu_{i_r}\alpha_{r+1}}+2\theta_{\alpha_{r+1}}{}^{\gamma_r}(q_r)_{\gamma_r}\delta_{\mu_{i_r}}^{\alpha_r}$$

$$-\theta^{\alpha_r}{}_{\alpha_{r+1}}(q_{r-1}+q_r)_{\mu_{i_r}}\Big]\frac{1}{q_{r-1}^2}\,,$$

$$(3.77)$$

where summation over α_r appearing twice in the sequence of multiplied square brackets is understood. Thereby one has to identify $a_{n+1}=a_1$, $\alpha_{n+1}=\alpha_1$, $p_{i_n}=-\sum_{r=1}^{n-1}p_{i_r}$. The q_r are defined by

$$q_r = q_r(k,p_{i_1},\ldots,p_{i_r}) = k+\sum_{s=1}^r p_{i_s}\,. \qquad (3.78)$$

In Appendix A.8 we prove that this expression is indeed non-zero at least for even n and the most symmetric non-trivial configuration of the external momenta, Lorentz and group

indices. This means that non-vanishing connected n-point functions for arbitrary high n exist in the kinetic perturbation theory, leading to infinitely many building blocks in the θ-summed case. In other words one needs infinitely many elements to formulate Feynman rules for the noncommutative G gauge theory if one insists on keeping the noncommutative $U(N)$ vertices as components.

Due to the fact that the expressions discussed above cannot be affected by higher order corrections of (3.60) this statement is universal, i. e. independent of the power in θ up to which the constraint $a_\mu \in \mathfrak{g}$ is implemented.

3.5.4   The case with sources restricted to the Lie algebra of G

Up to now we have looked for Feynman rules working with the original $U(N)$ vertices and sources J^M taking values in the full $\mathfrak{u}(N)$ Lie algebra. This seemed to be natural since in the enveloping algebra approach for $G \subset U(N)$ the noncommutative gauge A^M-field, although constrained, carries indices M running over all generators of $\mathfrak{u}(N)$.

There is still another option to explore. First one can restrict the sources $J, \eta, \bar{\eta}$ in (3.53) by hand to take values in $\mathfrak{g}$ only. Then instead of pulling out in (3.54) the complete interaction S_I one separates only those parts of S_I, which yield vertices whose external legs carry lower case Latin indices referring to $\mathfrak{g}$ exclusively. The remaining parts of S_I, generating vertices with at least one leg owning a primed index, are kept under the functional integral. The functional integration and the constraint remain unchanged. We denote this splitting of S_I by

$$S_I[A, C, \bar{C}] = S_i[A, C, \bar{C}] + S_i'[A, C, \bar{C}] \tag{3.79}$$

and the sources by hatted quantities

$$\hat{J}^{a'} = \hat{\bar{\eta}}^{a'} = \hat{\eta}^{a'} = 0 . \tag{3.80}$$

Then

$$Z_G[\hat{J}, \hat{\bar{\eta}}, \hat{\eta}] = \mathrm{e}^{iS_i[\frac{\delta}{i\delta \hat{J}}, \frac{\delta}{i\delta \hat{\bar{\eta}}}, \frac{\delta}{i\delta \hat{\eta}}]} \hat{Z}_G[\hat{J}, \hat{\bar{\eta}}, \hat{\eta}] \tag{3.81}$$

and

$$
\begin{aligned}
\hat{Z}_G[\hat{J}, \hat{\bar{\eta}}, \hat{\eta}] &= \int_{a,c,\bar{c} \in \mathfrak{g}} \mathcal{D}A \, \mathcal{D}\bar{C} \, \mathcal{D}C \ \mathrm{e}^{i(S_{\mathrm{kin}}[A,C,\bar{C}] + S_i'[A,C,\bar{C}] + A\hat{J} + \hat{\bar{\eta}}C + \bar{C}\hat{\eta})} \\
&= \int_{a,c,\bar{c} \in \mathfrak{g}} \mathcal{D}a \, \mathcal{D}\bar{c} \, \mathcal{D}c \, \mathcal{J} \ \mathrm{e}^{i(S_{\mathrm{kin}}[a,c,\bar{c}] + \hat{s}_1[a,c,\bar{c}] + A[a]\hat{J} + \hat{\bar{\eta}}C[c,a] + \bar{c}\hat{\eta})} \ ,
\end{aligned}
\tag{3.82}
$$

where $\hat{s}_1[a, c, \bar{c}]$ is defined by

$$S_{\text{kin}}[A[a], C[c, a], \bar{c}] + S'_i[A[a], C[c, a], \bar{c}] = S_{\text{kin}}[a, c, \bar{c}] + \hat{s}_1[a, c, \bar{c}] \ . \tag{3.83}$$

If now the generating functional of connected Green functions

$$\ln \frac{\hat{Z}_G[\hat{J}, \hat{\bar{\eta}}, \hat{\eta}]}{\hat{Z}_G[0]} = \sum_n \frac{i^n}{n!} \int \mathrm{d}^d x_1 \ldots \mathrm{d}^d x_n \, \langle A(x_1) \ldots A(x_n) \rangle_{\text{c}}^{\text{kin}+S'_i} \hat{J}(x_1) \ldots \hat{J}(x_n) + \ldots \ ,$$

$$\tag{3.84}$$

e. g. for $G = SO(N)$, would generate only the free propagators (like (3.58) does for $G = U(N)$), the $SO(N)$ Feynman rules conjectured in ref. [31] would have been derived via partial summation of the θ-expanded perturbation theory in the enveloping algebra approach. As was argued in [33] and explained before, the constraint they use is equivalent to requiring that the image under the inverse SW map is in $\mathfrak{SO}(N)$. Thus, a generation of additional vertices would disprove the rules of [31].

In the remaining part of this Section we show that there are additional vertices. For this purpose we consider $\langle A(x_1) \ldots A(x_n) \rangle_{\text{c}}^{\text{kin}+S'_i}$ and look at it as a power series in g^2 and θ. To prove that a contribution with $n > 2$ to (3.84) it is not identically zero, it is sufficient to find a particular non-vanishing order in g^2 and θ.

New vertices arise from expressing the noncommutative fields A_μ either in the original noncommutative kinetic term or 3-point or 4-point interactions in S'_i (see (3.79) via (3.60)) in terms of the ordinary field a_μ. To take into consideration only the vertices (3.68), (3.69) and (3.70) at $\mathcal{O}(\theta)$ generated by an expansion of the kinetic term is not sufficient if one wants to avoid to work with higher orders of the Seiberg-Witten map. The reason is as follows. Because of (3.80) it is clear that only the components $\langle A^{m_1}(x_1) \ldots A^{m_n}(x_n) \rangle_{\text{c}}^{\text{kin}+S'_i}$ (with unprimed lower case Latin indices) contribute to (3.84). For $SO(N)$, where $d_{abc} = 0$ [53, 76], it follows immediately that all $\mathcal{O}(\theta)$ contributions from the kinetic term vanish in this case and contributions can only start at $\mathcal{O}(\theta^2)$. At this order, however, a mixing with the $\mathcal{O}(\theta^2)$ terms in the Seiberg-Witten map occurs.

If one does not want to work with the higher order terms in the Seiberg-Witten map, one has to look at diagrams that include the remnants of the 3- and 4-point vertices in S'_i of (3.79). They are given by the diagrams of Fig. 3.1, but carry at least one primed index. It is clear that these primed indices have to be converted to unprimed lower case indices via θ-dependent terms in the expansion of the noncommutative fields. Each primed index increases the order in θ by at least one. We will now search for interactions

of only gauge fields of lowest possible order in θ.[8] The number of primed indices then has to be minimized. As can be seen from Fig. 3.1 both pure gauge vertices include factors with the structure constants. From (3.61) one can immediately see that the contributions with only one primed index vanish for both vertices so that two primed indices is the minimal number. The next step is to ensure that these primed indices are only internal, i. e. one has to look at the θ-expansion of the noncommutative gauge fields and choose the θ-dependent terms for the fields that are attached to the two legs that carry primed indices. It is easy to see that the linear order terms of (3.50) are sufficient to transform each primed index into two or three unprimed indices. Although the lowest order in θ is now $\mathcal{O}(\theta^2)$, no mixing with higher order contributions to the Seiberg-Witten map can occur in this case because one is forced to put θ-dependent terms on two legs. With the help of a single 3-vertex S_i' one can now construct a 5-, 6- of 7-point function that carries only unprimed lower case indices. A single 4-vertex of S_i' leads to an 6-, 7- or 8-point function of this kind. A tree level n-point function with k vertices possesses $k-1$ internal lines that connect the vertices. If l of these vertices come from the $\propto \theta$ terms of the Seiberg-Witten map and $k-l$ vertices are taken from S_i' and thus of order g^{-2} the total order of the tree level n-point function is $g^{2(n+l-1)}\theta^l$. Even for generic g and θ the 6- and 7-point functions constructed from the 3- and 4-point vertices mix because the Seiberg-Witten map at linear order generates two and three unprimed legs from a primed one. This could in principle lead to a cancellation between the two contributions even if both are separately non zero. The situation is different for the 5- and the 8-point functions. There is no other tree level diagram of the same order that leads to a 5-point function and thus to show that it does not vanish is a proof that there are new interactions. For the 8-point function there is a mixing but it cannot lead to serious cancellations because its the standard mixing in YM theory. The underlying 4-point vertex mixes with the 4-point amplitude made of two 3-point vertices. This mechanism ensures gauge invariance of the full amplitude but does not annihilate it. Hence, it is sufficient to check the non-vanishing of the 8-point amplitude that is built with the 4-point vertex alone. In Appendix A.9 for $SO(3)$ $\langle A^{m_1}(x_1)\ldots A^{m_8}(x_8)\rangle_c^{\text{kin}+S_i'}$ is explicitly computed in lowest order in g and θ and shown not to vanish. This serves as a counterexample that disproves the $SO(N)$ Feynman rules of [31].

---

[8]Note that the original 3-point or 4-point interactions Fig. 3.1 by themselves are θ-dependent via the $*$-product. But since we are searching for the lowest order in θ this further θ-dependence can be disregarded.

The more ambitious program to exclude rules based on the vertices in S_i and an arbitrary but finite number of additional building blocks would require to show, similar to the previous Section, that there is no n_0 assuring vanishing connected n-point functions for $n > n_0$. Although we have practically no doubt concerning this conjecture, a rigorous proof is beyond our capabilities since for increasing n higher and higher orders of the SW-map contribute. This happens because in contrast to the proof in Subsection 3.5.3 one is forced to look at Green functions with all external group indices referring to generators of the Lie algebra of G since no primed indices of the remaining generators spanning $U(N)$ are probed.

Part III

The BMN limit of the AdS/CFT correspondence

Chapter 4

Relations between string backgrounds

In this chapter we will briefly discuss some p-brane solutions of supergravity. Then we will introduce anti-de Sitter (AdS) spacetimes, the product spaces AdS × S and the pp wave spacetimes. We will show how AdS × S arises as a near horizon limit of certain p-brane solutions and we will discuss the Penrose-Güven limit in more detail since they play an essential role in the AdS/CFT correspondence and in its BMN limit. We will mostly work in general dimensions. This will enable us to encompass certain types of solutions without repeating similar calculations and it will shed some light on the general structure. Special emphasis is put on the $(D = 10)$-dimensional solutions as concrete setups in the formulation of the AdS/CFT correspondence and its BMN limit. We will also present the corresponding solutions in $D = 11$.

4.1 The backgrounds

4.1.1 Some p-brane solutions of supergravity

That part of a generic supergravity action which is sufficient for our purpose to present some p-brane solutions in various dimensions D can be written down in closed form. First it is advantageous to define

$$d = p + 1 , \qquad d' = D - p - 3 , \tag{4.1}$$

such that $D = d + d' + 2$. One should keep in mind that a supergravity theory in D dimensions restricts the allowed d and hence the possible $(d-1)$-brane solutions. The

relevant (bosonic) part of a supergravity action in the Einstein-frame reads [64]

$$ S_D = \frac{1}{2\kappa^2} \int \mathrm{d}^D x \, \sqrt{-g} \Big[\mathcal{R} - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2} e^{-a(d)\phi} |F_{d+1}|^2 \Big] \tag{4.2} $$

for the metric g_{MN} (from now on we choose the signature 'mostly plus'), the dilaton[1] ϕ and a $(d+1)$-form field strength F_{d+1} which is defined as $F_{d+1} = \mathrm{d}A_d$. We have used the abbreviations

$$ |F_{d+1}|^2 = \frac{1}{(d+1)!} g^{M_1 N_1} \dots g^{M_{d+1} N_{d+1}} F_{N_1 \dots N_{d+1}} F_{M_1 \dots M_{d+1}} \; . \tag{4.3} $$

and

$$ a(d)^2 = 4 - 2\frac{dd'}{d+d'} \; . \tag{4.4} $$

The equations of motions which derive from the above given action read

$$
\begin{aligned}
\mathcal{R}^{MN} - \frac{1}{2} g^{MN} \mathcal{R} &= \frac{1}{2}\Big(\partial^M \phi \partial^N \phi - \frac{1}{2} g^{MN} (\partial\phi)^2 \Big) \\
&\quad + \frac{1}{2}\Big(\frac{1}{d!} F^M{}_{P_1 \dots P_d} F^{NP_1 \dots P_d} - \frac{1}{2} g^{MN} |F_{d+1}|^2 \Big) e^{-a(d)\phi} \; ,
\end{aligned}
$$
$$ \partial_M \big(\sqrt{-g}\, e^{-a(d)\phi} F^{MP_1 \dots P_d} \big) = 0 \; , $$
$$ \partial_M \big(\sqrt{-g}\, g^{MN} \partial_N \phi \big) = -\frac{a(d)}{2} \sqrt{-g}\, e^{-a(d)\phi} |F_{d+1}|^2 \; , \tag{4.5} $$

and they admit the so called extremal solutions of the form [64, 136]

$$
\begin{aligned}
\mathrm{d}s^2_{d-1} &= H(y)^{-\frac{d'}{d+d'}} \, \mathrm{d}x^\mu \, \mathrm{d}x_\mu + H(y)^{\frac{d}{d+d'}} \, \mathrm{d}y^i \, \mathrm{d}y_i \; , \\
e^{2\phi} &= H(y)^{a(d)} \; , \qquad H(y) = 1 + \Big(\frac{R_2}{y} \Big)^{d'} \; , \\
F_{d+1} &= \mathrm{vol}(\mathbb{R}^{1,d-1}) \wedge \mathrm{d}H^{-1}(y) \; ,
\end{aligned}
\tag{4.6}
$$

where $\mu = 0, \dots, d-1$ and $i = d, \dots, d+d'+1$. The coordinate $y = \sqrt{y^i y_i}$ measures the transverse separation from the brane that resides at $y = 0$, and $\mathrm{vol}(\mathbb{R}^{1,d-1})$ is the volume form of $R^{1,p}$. The parameter R_2 is related to κ and the D$(d-1)$-brane tension T_{d-1} as [64]

$$ R_2^{d'} = \frac{2\kappa^2 T_{d-1}}{d' \Omega_{d'+1}} = g_s (4\pi)^{\frac{d'}{2}-1} \alpha'^{\frac{d'}{2}} \Gamma\big(\tfrac{d'}{2}\big) \; , \tag{4.7} $$

where $\Omega_{d'+1}$ is the volume of the unit $\mathrm{S}^{d'+1}$ (B.59). Furthermore, we have used that κ and T_p can be expressed in terms of the squared string length α' and the string coupling g_s

---

[1]There is no dilaton in 11-dimensional supergravity. One simply has to ignore it if one wants to deal with this case. Only the solutions with $a(d) = 0$ are then relevant.

as[2] [136]

$$2\kappa^2 = (2\pi)^{d+d'-1}\alpha'^{\frac{d+d'}{2}}g_s^2 \ ,$$

$$T_{d-1} = \frac{1}{g_s(2\pi)^{d-1}\alpha'^{\frac{d}{2}}} \ . \tag{4.8}$$

In $D = d + d' + 2$ dimensions with g denoting the determinant of the metric, we now use the conventions

$$\varepsilon_{0...D-1} = 1 \ , \qquad \varepsilon^{0...D-1} = \frac{1}{g} \tag{4.9}$$

for the total antisymmetric tensor density and define the Hodge-duality operator as

$$\star \left(\mathrm{d}x^{M_1} \wedge \cdots \wedge \mathrm{d}x^{M_d}\right) = -\frac{\sqrt{-g}}{(D-d)!}\varepsilon^{M_1...M_d}{}_{M_{d+1}...M_D} \, \mathrm{d}x^{M_{d+1}} \wedge \cdots \wedge \mathrm{d}x^{M_D} \ . \tag{4.10}$$

After rewriting the second term in the metric (4.6) in polar coordinates

$$\mathrm{d}y^i \, \mathrm{d}y_i = \mathrm{d}y^2 + y^2 \, \mathrm{d}\Omega^2_{d'+1} \ , \tag{4.11}$$

one then finds for the differential form that occurs in F_{d+1} of (4.6)

$$\star \left(\mathrm{vol}(\mathbb{R}^{1,d-1}) \wedge \mathrm{d}y\right) = H(y)^{2-\frac{a^2(d)}{2}}y^{d'+1} \, \mathrm{vol}(\Omega_{d'+1}) \tag{4.12}$$

with $\mathrm{vol}(\Omega_{d'+1})$ denoting the volume form of the unit $\mathrm{S}^{d'+1}$, see (B.58). The Hodge-dual field strength then reads

$$\star F_{d+1} = -H(y)^{-\frac{a^2(d)}{2}}y^{d'+1}\partial_y H(y) \, \mathrm{vol}(\Omega_{d'+1}) \ . \tag{4.13}$$

The above given solutions are $(d-1)$-branes embedded in $D = d + d' + 2$ dimensions. They have an isometry group $SO(1, d-1) \times SO(d'+2)$. For $D = 10$ and $d = 4$ one obtains the D3-brane solution of type IIB supergravity with the self-dual 5-form flux, i. e. the flux has to fulfill the relation

$$F_5 = \star F_5 \ . \tag{4.14}$$

This can be achieved by replacing

$$F_5 \to \frac{1}{2}(F_5 + \star F_5) \tag{4.15}$$

if one remembers that in case of Minkowski signature the action of the Hodge-star on a p-form ω_p fulfills

$$\star \star \omega_p = -(-1)^{p(D-p)}\omega_p \ . \tag{4.16}$$

---

[2]In $D = 11$ where the dilaton is absent one has to set $g_s = 1$.

For $D = 11$ and $d = 3$ or $d = 4$ one obtains the M2- and M5-branes where the 4-form field strength of the theory is given by the last line in (4.6) for the M2- brane and by (4.13) for the M5-brane respectively.

At the end let us discuss the measurement of energy in presence of a gravitational source like the $(d-1)$-brane (4.6). The warp factor that multiplies the time differential $-\,\mathrm{d}x_0^2$ in the metric (4.6) determines the gravitational redshift that is caused by the $(d-1)$-brane. An observer infinitely far away from the p-branes measures energy with the operator $E_\infty = -i\partial_0$. At finite distance y he uses $E_y = -iH(y)^{\frac{d'}{2(d+d')}}\partial_0$. The energy of a particle at y measured by an observer at infinity is thus redshifted by

$$E_\infty = H(y)^{-\frac{d'}{2(d+d')}}E_y \ . \tag{4.17}$$

4.1.2 Anti-de Sitter spacetime

$(d+1)$-dimensional Anti de-Sitter spacetime AdS_{d+1} with radius R_1 is a solution to the Einstein equations (B.2) with cosmological constant

$$\Lambda = -\frac{1}{2R_1^2}d(d-1) \ . \tag{4.18}$$

The Riemann and Ricci tensors and the scalar curvature read

$$\mathcal{R}_{\mu\nu\rho\sigma} = -\frac{1}{R_1^2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \ , \qquad \mathcal{R}_{\mu\nu} = -\frac{d}{R_1^2}g_{\mu\nu} \ , \qquad \mathcal{R} = -\frac{d(d+1)}{R_1^2} \ . \tag{4.19}$$

It is a maximally symmetric spacetime. That means its metric possesses as many symmetries (Killing vectors) as $(d+1)$-dimensional flat space, namely $\frac{(d+1)(d+2)}{2}$. The space can be realized as a coset

$$\mathrm{AdS}_{d+1} = \frac{SO(2,d)}{SO(1,d)} \ , \tag{4.20}$$

where $SO(2,d)$ and $SO(1,d)$ are the isometry group and the stabilizer respectively. Furthermore, it can be constructed as an embedding of the hyperboloid

$$-X_0^2 - X_{d+1}^2 + \sum_{i=1}^{d} X_i^2 = -R_1^2 \tag{4.21}$$

into the flat $(d+2)$-dimensional $\mathbb{R}^{2,d}$ with metric

$$\mathrm{d}s_{\mathbb{R}^{2,d}}^2 = -\,\mathrm{d}X_0^2 - \mathrm{d}X_{d+1}^2 + \sum_{i=1}^{2} \mathrm{d}X_i^2 \ . \tag{4.22}$$

With the coordinates

$$0 \le t \le 2\pi \ , \qquad 0 \le \rho \ , \qquad 0 \le \bar\rho < \frac{\pi}{2} \tag{4.23}$$

one can realize the embedding as follows

$$\begin{aligned}
X_0 &= R_1 \cosh\rho \cos t = R_1 \sec\bar\rho \cos t \ , \\
X_{d+1} &= R_1 \cosh\rho \sin t = R_1 \sec\bar\rho \sin t \ , \\
X_i &= R_1 \sinh\rho \,\omega_i = R_1 \tan\bar\rho \,\omega_i \ , \qquad i = 1,\ldots,d \ , \qquad \sum_i \omega_i^2 = 1 \ ,
\end{aligned} \tag{4.24}$$

where we have presented two alternative choices ρ and $\bar\rho$. These coordinates with the above given ranges cover the hyperboloid exactly once and are therefore denoted as global coordinates. But what we mean with AdS_{d+1} (if not otherwise stated) is the universal covering of the hyperboloid with the coordinate ranges

$$0 \le t \ , \qquad 0 \le \rho \ , \qquad 0 \le \bar\rho < \frac{\pi}{2} \ , \tag{4.25}$$

i. e. one allows a multiple wrapping of the hyperboloid in the global time direction. The induced metric on the hyperboloid becomes in the two sets of global coordinates

$$\begin{aligned}
\mathrm{d}s^2_{\mathrm{AdS}} &= R_1^2\big(-\cosh^2\rho \,\mathrm{d}t^2 + \mathrm{d}\rho^2 + \sinh^2\rho \,\mathrm{d}\Omega^2_{d-1}\big) \\
&= R_1^2 \sec^2\bar\rho \big(-\mathrm{d}t^2 + \mathrm{d}\bar\rho^2 + \sin^2\bar\rho \,\mathrm{d}\Omega^2_{d-1}\big) \ ,
\end{aligned} \tag{4.26}$$

where $\mathrm{d}\Omega^2_{d-1}$ is the metric of the unit S^{d-1}. In the second line the term in parenthesis describes a space with topology $\mathbb{R} \times \mathrm{S}^d$. It is the $(d+1)$-dimensional Einstein static universe (ESU). From the second line it is then easy to read off the conformal boundary of AdS_{d+1}. One simply has to check where the conformal factor diverges. This happens at $\bar\rho = \frac{\pi}{2}$ or equivalently $\rho \to \infty$. The structure of the boundary is found from the terms in parenthesis by computing it at these values of the coordinates. In the second line of (4.26), the factor in front of the unit S^{d-1} is one and hence the boundary itself is given by $\mathbb{R} \times \mathrm{S}^{d-1}$. In addition, two points i^- and i^+ of timelike past and future infinity belong to the boundary structure of AdS_{d+1} [89]. The reason why one has to include these points is that one cannot make the time coordinate finite without pinching off the spatial directions.

The metric of AdS_{d+1} can be cast into a very simple form if one introduces a patchwise coordinate system that covers only one half of AdS_{d+1}. The embedding reads in these

coordinates $(s, x_0, \vec{x})$, where $s > 0$

$$
\begin{aligned}
X_0 &= \frac{1}{2s}\big(1 + s^2(R_1^2 + \vec{x}^2 - x_0^2)\big) \ , \\
X_{d+1} &= R_1\, s\, x_0 \ , \\
X_i &= R_1\, s\, x_i \ , \qquad i = 1, \ldots, d-1 \ , \\
X_d &= \frac{1}{2s}\big(1 - s^2(R_1^2 - \vec{x}^2 + x_0^2)\big) \ ,
\end{aligned}
\tag{4.27}
$$

and the metric is given by

$$
\mathrm{d}s_{\mathrm{AdS}}^2 = R_1^2\Big(\frac{\mathrm{d}s^2}{s^2} + s^2(-\,\mathrm{d}x_0^2 + \mathrm{d}\vec{x}^2)\Big) \ .
\tag{4.28}
$$

One can then perform a coordinate change $x_\perp = s^{-1}$, $0 < x_\perp$ to obtain the metric in the so called Poincaré coordinates

$$
\mathrm{d}s_{\mathrm{AdS}}^2 = \frac{R_1^2}{x_\perp^2}(\mathrm{d}x_\perp^2 - \mathrm{d}x_0^2 + \mathrm{d}\vec{x}^2) \ .
\tag{4.29}
$$

It is easy to identify the region of the hyperboloid that is covered by these coordinates. Subtract the embedding coordinates X_0 and X_d in global as well as in the Poincaré coordinates

$$
X_0 - X_d = R_1(\sec\bar{\rho}\cos t - \tan\bar{\rho}\,\omega_d) = R_1^2\frac{1}{x_\perp}
\tag{4.30}
$$

and use the condition $0 < x_\perp$ which then leads to the restriction

$$
\cos t > \sin\bar{\rho}\,\omega_d \ .
\tag{4.31}
$$

With the coordinate ranges $0 \le \rho < \frac{\pi}{2}$ and $-1 \le \omega_d \le 1$ one finds that the Poincaré patch looks different for the two hemispheres with $\omega_d < 0$ and $\omega_d \ge 0$. For the case of AdS$_2$, where $\omega_1 \in S^0 = \{-1, 1\}$, the patch is shown in Fig. 4.1.

With the help of the embedding one can define the so called chordal distance, defined as the square length of the straight line that connects two points on the submanifold. Using the metric (4.22) and the embedding (4.21) one finds in global as well as in Poincaré coordinates

$$
\begin{aligned}
u(x, x') = (X(x) - X(x'))^2 &= 2R_1^2\Big[-1 + \cosh\rho\cosh\rho'\cos(t - t') - \sinh\rho\sinh\rho'\,\omega^i\omega_i'\Big] \\
&= \frac{R_1^2}{x_\perp x_\perp'}\Big[(x_\perp - x_\perp')^2 - (x_0 - x_0')^2 + (\vec{x} - \vec{x}')^2\Big] \ .
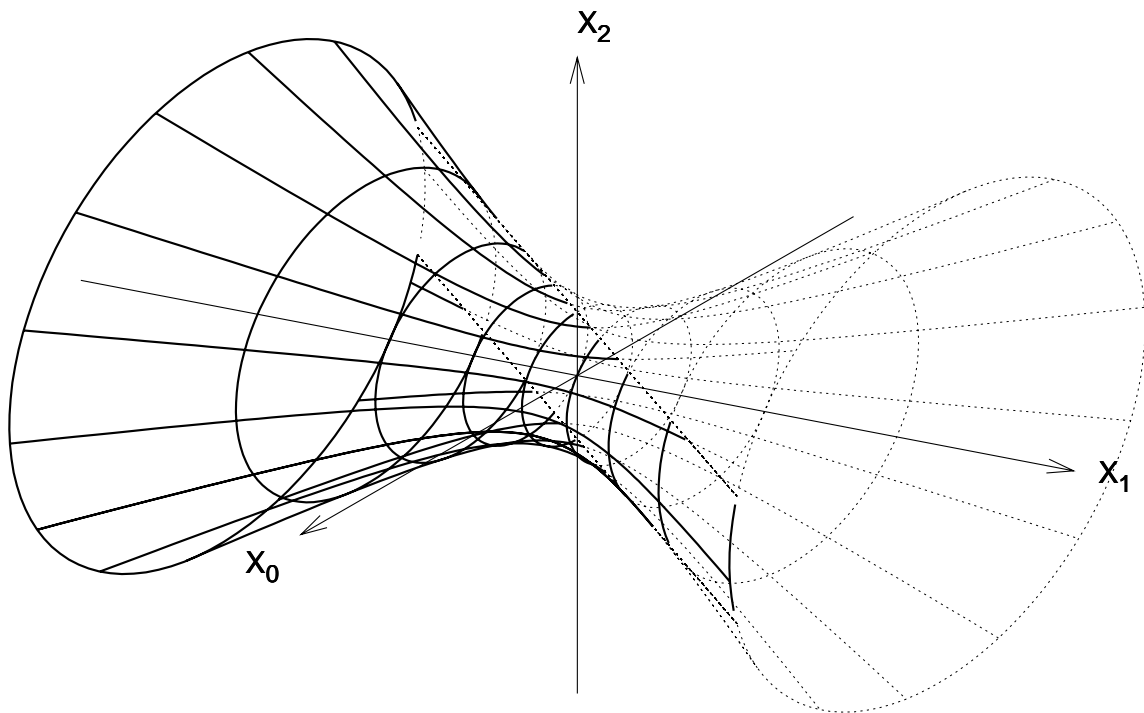\end{aligned}
\tag{4.32}
$$

Figure 4.1: AdS$_2$ embedded as a hyperboloid into $\mathbb{R}^{2,1}$. In this picture constant global coordinates ρ, $\bar{\rho}$ are represented as circles and constant t are given by the remaining curves. For generic AdS$_{d+1}$ with $d > 1$ only one half of the hyperboloid can be drawn because X_1 has to be replaced by $|\vec{X}|$ and every point then represents a $(d-1)$-dimensional sphere. The part drawn with full lines denotes the Poincaré patch that fulfills (4.31).

An important point is the conformal mapping of AdS_{d+1} to other spacetimes. The metric without the conformal factor in the second line of (4.26) describes one half of the Einstein static universe (ESU) which has topology $\mathbb{R} \times \text{S}^d$. AdS_{d+1} can therefore be conformally mapped to one half of the $(d+1)$-dimensional ESU. The fact that only one half is covered is caused by the coordinate range $0 \leq \bar{\rho} < \frac{\pi}{2}$ instead of $0 \leq \bar{\rho} \leq \pi$ that covers the full ESU. The spatial part of the boundary at $\bar{\rho} = \frac{\pi}{2}$ is mapped to a $(d-1)$-dimensional subsphere within the S^d of the ESU. From (4.29) it follows immediately that AdS_{d+1} is conformally flat.

4.1.3 The product spaces AdS × S

The direct product space of of $(d+1)$-dimensional anti-de Sitter space with radius R_1 and of a $(d'+1)$-dimensional sphere with radius R_2 is *not* a solution to the Einstein equations (B.2) with cosmological constant. This can be easily seen from the components of the Ricci tensor and the scalar curvature written down in (4.33) below. But geometries of this type arise as near horizon limits of p-brane solutions in supergravity theories. This type of spacetime is of interest for us because it is an essential ingredient in the formulation of the AdS/CFT correspondence. The non-vanishing components of the Riemann and Ricci tensors and of the scalar curvature read

$$
\begin{aligned}
\mathcal{R}_{\mu\nu\rho\sigma} &= -\frac{1}{R_1^2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \,, & \mathcal{R}_{\mu\nu} &= -\frac{d}{R_1^2}g_{\mu\nu} \,, & \mathcal{R} &= -\frac{d(d+1)}{R_1^2} + \frac{d'(d'+1)}{R_2^2} \,. \\
\mathcal{R}_{mnrs} &= \frac{1}{R_2^2}(g_{mr}g_{ns} - g_{ms}g_{nr}) \,, & \mathcal{R}_{mn} &= \frac{d'}{R_2^2}g_{mn} \,,
\end{aligned}
\tag{4.33}
$$

It is not a maximally symmetric spacetime because the number of Killing vectors is given by $\frac{(d+1)(d+2)}{2} + \frac{(d'+1)(d'+2)}{2}$ instead of $\frac{(d+d'+2)(d+d'+3)}{2}$ for the corresponding flat $(d+d'+2)$-dimensional space. As a group manifold it reads

$$
\text{AdS}_{d+1} \times \text{S}^{d'+1} = \frac{SO(2,d)}{SO(1,d)} \times \frac{SO(d'+2)}{SO(d'+1)} \,.
\tag{4.34}
$$

The sphere itself can be described by the embedding

$$
\sum_{i=1}^{d'+2} Y_i^2 = R_2^2 \,,
\tag{4.35}
$$

in the flat $(d' + 2)$-dimensional $\mathbb{R}^{d'+2}$ with the standard Euclidean metric. One choice of coordinates that is very useful is given by

$$
\begin{aligned}
Y_1 &= R_2 \cos \vartheta \cos \psi \ , \\
Y_2 &= R_2 \cos \vartheta \sin \psi \ , \\
Y_i &= R_2 \sin \vartheta \, \hat{\omega}_i \ , \qquad i = 3, \ldots, d+2 \ , \qquad \sum_i \hat{\omega}_i^2 = 1 \ .
\end{aligned}
\tag{4.36}
$$

Some more details about these coordinates can be found in Appendix B.7 . The induced metric then assumes the form

$$
\mathrm{d}s_{\mathrm{S}}^2 = R_2^2 \, \mathrm{d}\Omega_{d'+1}^2 = R_2^2 \big(\cos^2 \vartheta \, \mathrm{d}\psi^2 + \mathrm{d}\vartheta^2 + \sin^2 \vartheta \, \mathrm{d}\hat{\Omega}_{d'-1}^2 \big) \ ,
\tag{4.37}
$$

and the chordal distance reads

$$
v(y, y') = (Y(y) - Y(y'))^2 = 2R_2^2 \Big[1 - \cos \vartheta \cos \vartheta' \cos(\psi - \psi') - \sin \vartheta \sin \vartheta' \, \hat{\omega}^i \hat{\omega}_i' \Big] \ .
\tag{4.38}
$$

The chordal distance of a direct product space like $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ is then given with (4.32) as the direct sum $u + v$ of the distances in both spaces (remember that the chordal distances are squared distances).

The metric of the complete product space $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ reads in the global or Poincaré coordinates for AdS_{d+1} respectively

$$
\begin{aligned}
\mathrm{d}s_{\mathrm{AdS}\times\mathrm{S}}^2 &= R_1^2 \sec^2 \bar{\rho} \Big(-\mathrm{d}t^2 + \mathrm{d}\bar{\rho}^2 + \sin^2 \bar{\rho} \, \mathrm{d}\Omega_{d-1}^2 + \frac{R_2^2}{R_1^2} \cos^2 \bar{\rho} \, \mathrm{d}\Omega_{d'+1}^2 \Big) \\
&= R_1^2 \frac{1}{x_\perp^2} \Big(-\mathrm{d}x_0^2 + \mathrm{d}x_\perp^2 + \mathrm{d}\vec{x}^2 + \frac{R_2^2}{R_1^2} x_\perp^2 \, \mathrm{d}\Omega_{d'+1}^2 \Big) \ .
\end{aligned}
\tag{4.39}
$$

Both expressions indicate that $R_1 = R_2$ is a special value. At $R_1 = R_2$ the metric in the first line is up to the conformal factor the $(d + d' + 2)$-dimensional Einstein static universe (ESU) with the topology $\mathbb{R} \times \mathrm{S}^{d+d'+1}$. In contrast to pure AdS that is conformal to only one half of an ESU, $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ is conformal to a complete ESU. In the Poincaré coordinates the $(d' + 1)$-dimensional sphere and the coordinate $x_\perp$ together form a flat $(d' + 2)$-dimensional space with $x_\perp$ as radial coordinate. The expression in parenthesis then is the metric of $\mathbb{R}^{1,d+d'+1}$ and in this case $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ is conformally flat as a whole, and this is not only valid for its factors AdS_{d+1} and S^{d+1} separately. A computation of the Weyl tensors in Appendix B.2 shows that $R_1 = R_2$ is the necessary and sufficient condition for the above statements for $d, d' > 0$.

The conformal boundary of $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ is the same as for pure AdS_{d+1}, see the discussion around (4.26). This can be easily seen by the fact that at the boundary, where $\bar{\rho} = \frac{\pi}{2}$, the factor in front of the part that refers to $\mathrm{S}^{d'+1}$ in the metric becomes zero. Hence, the sphere shrinks to a point at the boundary and has no influence on the boundary structure. Like in the case of pure AdS_{d+1} one has to add two points i^- and i^+ representing timelike past and future infinity to obtain the complete boundary structure.

4.1.4   pp wave and plane wave spacetimes

The 'plane fronted waves with parallel rays' (pp waves in short) contain the plane waves as a subclass. The latter were originally discussed as solutions of the vacuum Einstein equations that describe gravitational waves far away from their sources, see for instance [66]. Brinkmann [36, 37] introduced the 'plane fronted wave' and Rosen assumed that spacetime filling plane waves do not exist due to singularities. He rejected them as unphysical. Robinson [148], however, found that the singularities are caused by the choice of the coordinate system and are not physical singularities. In [30] the authors discussed plane waves in general and defined them as solutions of the vacuum Einstein equations with as many symmetries as found for an electromagnetic plane wave. Penrose [134] showed that a plane wave admits no spacelike hypersurface which would be adequate for the global specification of Cauchy data and that it is therefore impossible to embed the plane wave globally into a pseudo-Euclidean space. Later Cahen and Wallach [41] classified all Lorentzian symmetric spaces. The plane wave metric is of that type [71].

The pp wave / plane wave spacetimes attract new attention as solutions of the 11-dimensional [91, 102] and of type II B supergravity [27]. The most general D-dimensional pp wave metric is given by [29]

$$\mathrm{d}s^2 = -4\,\mathrm{d}z^+\,\mathrm{d}z^- + H(z^+, z)(\mathrm{d}z^+)^2 + 2A_i(z^+, z)\,\mathrm{d}z^i\,\mathrm{d}z^+ + \mathrm{d}\vec{z}^2 \ . \tag{4.40}$$

It has flat transverse $(i = 1, \ldots, D - 2)$ $(D - 2)$-dimensional space ('plane fronted') and a covariantly constant null killing vector ∂_{z^-} ('parallel rays'). In the following we will consider the case without $A_i(z^+, z)$. In [29] it was shown that one can remove $A_i(z^+, z)$ by a field redefinition if it is of the form $A_i(z^+, z) = A_{ij}(z^+)z^j$. The metric then has the form

$$\mathrm{d}s^2 = -4\,\mathrm{d}z^+\,\mathrm{d}z^- + H(z^+, z)(\mathrm{d}z^+)^2 + \mathrm{d}\vec{z}^2 \ . \tag{4.41}$$

and the non-vanishing components of the Christoffel connection (B.5) read

$$\Gamma^-{}_{++} = -\frac{1}{4}\partial_+ H \;, \qquad \Gamma^-{}_{+i} = \Gamma^-{}_{i+} = -\frac{1}{4}\partial_i H \;, \qquad \Gamma^i{}_{++} = -\frac{1}{2}\partial_i H \;. \tag{4.42}$$

Using (B.3) and (B.4) one finds

$$\mathcal{R}^-{}_{i+j} = \frac{1}{4}\partial_i \partial_j H \;, \qquad \mathcal{R}_{++} = -\frac{1}{2}\Delta H \;, \qquad \mathcal{R} = 0 \tag{4.43}$$

for the only independent components of the Riemannian curvature tensor and for the complete Ricci tensor and the curvature scalar. The Laplacian in the $(D-2)$-dimensional transverse space is denoted with Δ. The pp wave thus obeys the Einstein equations (B.2) in the vacuum ($\Lambda = 0$) if H is a harmonic function in the $(D-2)$-dimensional transverse space.

For the subclass of the plane waves the function H in (4.41) is quadratic in the transverse coordinates z^i [90, 166]

$$H(z^+, z) = H_{ij}(z^+) z^i z^j \;. \tag{4.44}$$

Hence, the plane wave is Ricci flat if the matrix H_{ij} is traceless

$$\delta^{ij} H_{ij} = 0 \;. \tag{4.45}$$

Another special case is given by

$$H_{ij}(z^+, z) = H(z^+)\delta_{ij} z^i z^j \tag{4.46}$$

which is relevant in what follows. It is easy to see that in this case the pp wave is conformally flat. The Ricci-flat solution is denoted as purely gravitational and the conformally flat one as purely electromagnetic respectively [134].

With the metric (4.41) one can construct ($D = 10$)-dimensional string and ($D = 11$)-dimensional supergravity backgrounds by switching on some flux such that (4.5) is obeyed after inserting (4.43). For type II B one finds solutions of the form

$$\begin{aligned} \mathrm{d}s^2 &= -4\,\mathrm{d}z^+\,\mathrm{d}z^- + H(z^+, z)(\mathrm{d}z^+)^2 + \mathrm{d}\vec{z}^2 \;, \\ F_5 &= \mathrm{d}z^+ \wedge \varphi(z^+, z) \;, \end{aligned} \tag{4.47}$$

where H has to fulfill

$$\Delta H = -|\varphi|^2 \;, \qquad |\varphi|^2 = \frac{1}{4!}\varphi_{ijkl}\varphi^{ijkl} \tag{4.48}$$

and Δ is the Laplacian in the transverse $(D - 2 = 8)$-dimensional space. For 11-dimensional supergravity these solutions look rather similar with

$$
\begin{aligned}
\mathrm{d}s^2 &= -4\,\mathrm{d}z^+\,\mathrm{d}z^- + H(z^+, z)(\mathrm{d}z^+)^2 + \mathrm{d}\vec{z}^2 \ , \\
F_4 &= \mathrm{d}z^+ \wedge \varphi(z^+, z) \ ,
\end{aligned}
\tag{4.49}
$$

where H has to fulfill

$$
\Delta H = -\varphi^2 \ , \qquad \varphi^2 = \frac{1}{3!}\varphi_{ijk}\varphi^{ijk}
\tag{4.50}
$$

and Δ is the Laplacian in the transverse $(D - 2 = 9)$-dimensional space.

An important subclass of the above given solutions in both cases is again of the plane wave type with an H of the form

$$
H(z^+, z) = H(z) = H_{ij}z^i z^j \ ,
\tag{4.51}
$$

where H_{ij} is a constant symmetric matrix and $i, j = 1, \ldots, D - 2$ are indices in the 8- and 9-dimensional transverse subspaces for type IIB and 11-dimensional supergravity respectively. The corresponding spacetimes are called homogeneous plane waves. They are Lorentzian symmetric spaces generically discussed in [41] and are denoted as Cahen-Wallach (CW) spaces in the literature [27, 28, 71]. The 5- and 4-form fluxes are homogeneous in these cases and null, i. e. $|F_5|^2 = 0$ and $|F_4|^2 = 0$ respectively and the complete solutions is denoted as homogeneous pp (Hpp) wave [71].

The above solutions have at least 16 Killing spinors and therefore preserve at least one half of supersymmetry. For a special chosen matrix H_{ij} one finds that in both cases, type IIB and 11-dimensional supergravity, there exists one solution that admits 32 Killing spinors and thus preserves maximal supersymmetry. For type IIB the solution becomes [27][3]

$$
H_{ij} = -\mu^2\delta_{ij} \ , \qquad \varphi = 2\mu(\mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3 \wedge \mathrm{d}z^4 + \mathrm{d}z^5 \wedge \mathrm{d}z^6 \wedge \mathrm{d}z^7 \wedge \mathrm{d}z^8) \ .
\tag{4.52}
$$

Since H is of the form (4.46) the spacetime is conformally flat. In $D = 11$ dimensions one finds [102]

$$
H_{ij} = \begin{cases} -\frac{1}{9}\mu\delta_{ij} & i, j = 1, \ldots, 3 \\ -\frac{1}{36}\mu\delta_{ij} & i, j = 4, \ldots, 9 \end{cases} \ , \qquad \varphi = \mu\,\mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3 \ ,
\tag{4.53}
$$

---

[3]Note that in [27] there is an additional factor of 2 in φ. The result presented here coincides with [150].

which is denoted as Kowalski-Glikman (KG) solution in the literature [27, 28, 71]. It is important to remark that a common factor in H_{ij} can be absorbed in a simultaneous rescaling of z^+ and z^-. In particular this means that the parameter μ can be set to an arbitrary non-zero value.

It has been shown by Penrose [134] that it is impossible to globally embed the plane wave spacetimes into a pseudo-Euclidean spacetime. However, an isometric embedding of the D-dimensional CW spaces with metric

$$\mathrm{d}s^2 = -4\,\mathrm{d}z^+\,\mathrm{d}z^- + H_{ij}z^i z^j (\mathrm{d}z^+)^2 + \mathrm{d}\vec{z}^2 \tag{4.54}$$

in $\mathbb{R}^{2,D}$ is possible [28]. The metric of $\mathbb{R}^{2,D}$ (4.22) via the coordinate transformations

$$Z^1_+ = \frac{1}{2}(Z_0 + Z_d)\ , \quad Z^1_- = \frac{1}{2}(Z_0 - Z_d)\ , \quad Z^2_+ = \frac{1}{2}(Z_{d+1} + Z_{d-1})\ , \quad Z^2_- = \frac{1}{2}(Z_{d+1} - Z_{d-1}) \tag{4.55}$$

can be transformed to

$$\mathrm{d}s^2 = -4\sum_{k=1}^{2}\mathrm{d}Z^k_+\,\mathrm{d}Z^k_- + \sum_{i=1}^{D}\mathrm{d}Z^2_i\ . \tag{4.56}$$

If the hypersurface is defined as

$$\sum_{k=1}^{2} Z^k_+ Z^k_+ = 1\ , \qquad H_{ij}Z^i Z^j + 4\sum_{k=1}^{2} Z^k_+ Z^k_- = 0 \tag{4.57}$$

and parameterized as follows

$$
\begin{aligned}
Z^1_+ &= \cos z^+\ , \\
Z^1_- &= -z^- \sin z^+ - \frac{1}{4}H_{ij}z^i z^j \cos z^+\ , \\
Z^2_+ &= \sin z^+\ , \\
Z^2_- &= z^- \cos z^+ - \frac{1}{4}H_{ij}z^i z^j \sin z^+\ , \\
Z_i &= z_i\ ,
\end{aligned}
\tag{4.58}
$$

one finds that the induced metric is given by (4.54). The chordal distance in the plane wave reads

$$
\begin{aligned}
\Phi(z,z') &= -4\sum_{k=1}^{2}(Z^k_+(z) - Z^k_+(z'))(Z^k_-(z) - Z^k_-(z')) + \sum_{i=1}^{D}(Z_i(z) - Z_i(z'))^2 \\
&= -4(z^- - z'^-)\sin(z^+ - z'^+) + 2H_{ij}(z^i z^j + z'^i z'^j)\sin^2 \tfrac{z^+ - z'^+}{2} + (\vec{z} - \vec{z}')^2\ .
\end{aligned}
\tag{4.59}
$$

4.2   Limits of spacetimes

In this Section we discuss two limiting processes that relate the previously discussed spacetimes with each other and that are of relevance in the AdS/CFT correspondence and in its BMN limit. The first so called near horizon limit applies to the p-brane solutions of supergravity and, for special choices of the spacetime dimensions, produces a background of the form $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$. The second so called Penrose-Güven limit applies to any supergravity solution, transforming the geometry into a plane wave and additionally acting on the other fields of the background.

4.2.1   The near horizon limit

In the AdS/CFT correspondence the near horizon limit [79] of the p-brane backgrounds plays an important role. In the setup of type II B string theory in the presence of a stack of D3-branes, one has closed strings in the bulk and open strings that describe the excitation of the branes ending on the branes. In the near horizon limit these two different string sectors decouple, and two alternative descriptions of the near horizon regions are related via the AdS/CFT correspondence. The near horizon limit can also be applied to other geometries e. g. given by a configuration of different branes or to black holes [42, 68, 97, 113]. In the former case it extracts information about the shape of the background close to the branes. In particular, the metric in the limit describes how the spacetime looks like in this region. For some special brane configurations one obtains geometries of the form $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1} \times M^{D-(d+d'+2)}$, where M is a $(D - (d + d' + 2))$-dimensional Euclidean manifold. Here we will restrict ourselves to the case a single (stack of) $(d-1)$-branes where the limiting spacetime is a direct product of only two factors and show under which conditions the limiting geometry becomes $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$.

The starting point is the $(d-1)$-brane solution of supergravity given in (4.6). We remember that the coordinate $y = \sqrt{y^i y_i}$ describes the distance from the $(d-1)$-brane with 'extension' R_2. To focus into the region close to the brane therefore means to discuss the solution for $R_2 \gg y$. For general d and d', however, one does not find an $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ geometry close to the brane. The condition on the dimensions turns out to be

$$dd' = 2(d + d') \Leftrightarrow a(d)^2 = 0 \;, \tag{4.60}$$

where $a(d)$ is defined in (4.4) and from (4.6) one immediately finds that all these solutions

have a constant dilaton ϕ. The metric in this case reads for $R_2 \gg y$

$$
\begin{aligned}
\mathrm{d}s^2 &= \left(\frac{R_2}{y}\right)^{-\frac{d'^2}{d+d'}} \mathrm{d}x^\mu \, \mathrm{d}x_\mu + \left(\frac{R_2}{y}\right)^{\frac{dd'}{d+d'}} (\mathrm{d}y + y^2 \, \mathrm{d}\Omega^2_{d'+1}) \\
&= \left(\frac{R_2}{y}\right)^{-\frac{2d'}{d}} \mathrm{d}x^\mu \, \mathrm{d}x_\mu + \left(\frac{R_2}{y}\right)^2 \mathrm{d}y^2 + L^2 \, \mathrm{d}\Omega^2_{d'+1} \; .
\end{aligned}
\tag{4.61}
$$

One now has to replace y by a new variable such that the first two terms combine to the metric of AdS_{d+1} (in the Poincaré coordinates). The redefinition reads

$$
\frac{y}{R_2} = \left(\frac{\gamma R_2}{x_\perp}\right)^{\frac{d}{d'}} , \qquad \frac{\mathrm{d}y}{y} = -\frac{d}{d'} \frac{\mathrm{d}x_\perp}{x_\perp} ,
\tag{4.62}
$$

where γ is a real parameter that has to be determined. The metric now becomes

$$
\mathrm{d}s^2 = \left(\frac{\gamma R_2}{x_\perp}\right)^2 \left[\left(\frac{d}{d'\gamma}\right)^2 \mathrm{d}x_\perp^2 + \mathrm{d}x^\mu \, \mathrm{d}x_\mu\right] + L^2 \, \mathrm{d}\Omega^2_{d'+1} \; .
\tag{4.63}
$$

A comparison with (4.29) shows that γ has to fulfill

$$
\gamma = \frac{R_1}{R_2} = \frac{d}{d'} ,
\tag{4.64}
$$

in order for the first part of the metric to describe AdS_{d+1} ((in the Poincaré coordinates). Then it is clear that R_1 and R_2 are the radii of AdS_{d+1} and of $\mathrm{S}^{d'+1}$ respectively.

What remains to be done is to determine the near horizon limit of the fluxes. In polar coordinates one has $\mathrm{d}H^{-1} = \partial_y H^{-1} \, \mathrm{d}y$, and one finds in the limit $R_2 \gg y$ using the variable substitution (4.62) and $\gamma R_2 = R_1$

$$
\mathrm{d}H^{-1} = -d\left(\frac{R_1}{x_\perp}\right)^d \frac{\mathrm{d}x_\perp}{x_\perp} \; .
\tag{4.65}
$$

With the volume form of AdS_{d+1} which in the Poincaré coordinates reads

$$
\mathrm{vol}(\mathrm{AdS}_{d+1}) = \left(\frac{R_1}{x_\perp}\right)^{d+1} \mathrm{d}x_\perp \wedge \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{d-1}
\tag{4.66}
$$

one then finds for the flux

$$
F_{d+1} = \frac{d}{R_1} \mathrm{vol}(\mathrm{AdS}_{d+1})
\tag{4.67}
$$

and for its Hodge-dual (4.13) with $a(d) = 0$

$$
\star F_{d+1} = \frac{d'}{R_2} \mathrm{vol}(\mathrm{S}^{d'+1}) ,
\tag{4.68}
$$

where $\mathrm{S}^{d'+1}$ denotes the sphere with radius R_2.

At the end we discuss the measurements of energies in the near horizon geometry. The exponential factor in the variable redefinition (4.62) is strictly positive and therefore $R_2 \gg y$ corresponds to $R_2 \ll x_\perp$. According to the discussion around (4.17) the deeper a particle is located in the interior of AdS the more redshifted its wavelength appears to an observer at some fixed position further outside (with smaller $x_\perp$). The energy of a particle at $x_\perp$ in AdS measured by an observer at infinity reads

$$E_\infty = \frac{R_1}{x_\perp} E_{x_\perp} \ . \tag{4.69}$$

4.2.2   The Penrose-Güven limit

One of the main ingredients that lead to the formulation of the BMN limit of the AdS/CFT correspondence is the observation that the 10-dimensional plane wave background of type II B string theory arises as a Penrose-Güven limit of the $AdS_5 \times S^5$ background. In [135] Penrose describes that the expansion of the metric around a null geodesic in a congruence of null geodesics without conjugate (focal) points leads to a plane wave metric. The absence of conjugate (focal) points (that are defined as points where infinitesimally neighboured null geodesics intersect) is important because otherwise the coordinate system breaks down. Intuitively this expansion is the one-dimensional analog of the expansion of a spacetime around a single point which results in the flat tangent space at that point. Before the work of Penrose appeared it was already observed by Pirani [137] that the gravitational field of a fast moving particle the more approaches a plane wave the higher its velocity.

The work of Penrose was later extended by Güven [88], who included all massless fields into the limiting process. This enables one to apply the limiting procedure to a complete string background and then to arrive at a new consistent background.[4]

Let us in brief review the Penrose-Güven limit. A general string background in D dimensions consists of all bosonic fields including the metric g_{MN}, the NS-NS fields B_{MN} and ϕ, a background gauge field A_M and the $(p+1)$-form R-R potentials A_{p+1}. One first chooses a coordinate system so that the metric takes the form

$$\mathrm{d}s^2 = 2\,\mathrm{d}z^+(-2\,\mathrm{d}z^- + \alpha\,\mathrm{d}z^+ + \beta_i\,\mathrm{d}z^i) - C_{ij}\,\mathrm{d}z^i\,\mathrm{d}z^j \ , \tag{4.70}$$

---

[4]The homogeneity of the actions of the theories under the required rescaling guarantees that the limit of a consistent background again is itself a consistent background.

where C_{ij} is a symmetric positive definite $(D-2) \times (D-2)$ matrix. This coordinate system breaks down at the nearest conjugate point where $\det C = 0$. The above choice of coordinates singles out a congruence of null geodesics with $z^+, z^i = \text{const.}$ and z^- the affine parameter along the geodesics. The next step is to fix a particular gauge. We will collectively deal with all fields B_{MN}, A_M and A_{p+1} and denote them as A_{p+1}. The formulas for B and the gauge field A follow from the expressions for A_{p+1} by simply setting $p = 1$ and $p = 0$ respectively. The $(p+1)$-form possesses a gauge freedom of the form

$$A_{p+1} \rightarrow A'_{p+1} = A_{p+1} + \mathrm{d}\Lambda_p , \qquad \Lambda_p \in \Omega^p , \tag{4.71}$$

where Λ_p is a p-form. It is used to set

$$(A_{p+1})_{-i_1 \dots i_{p-1}} = 0 . \tag{4.72}$$

The next step is to rescale the coordinates with a positive real number Ω in the following way

$$z^- = \bar{z}^- , \qquad z^+ = \Omega^2 \bar{z}^+ , \qquad z^i = \Omega \bar{z}^i \tag{4.73}$$

and to define new fields according to

$$\bar{g}_{\mu\nu}(\Omega) = \Omega^2 g_{\mu\nu}(\Omega) , \qquad \bar{\phi}(\Omega) = \phi(\Omega) ,$$
$$\bar{A}_{p+1}(\Omega) = \Omega^{-p-1} A_{p+1}(\Omega) , \qquad \bar{F}_{p+2}(\Omega) = \mathrm{d}\bar{A}_{p+1}(\Omega) = \Omega^{-p-1} F_{p+2}(\Omega) . \tag{4.74}$$

One then finds in the $\Omega \rightarrow 0$ limit

$$\mathrm{d}\bar{s}^2 = -4 \, \mathrm{d}\bar{z}^+ \, \mathrm{d}\bar{z}^- - C_{ij} \, \mathrm{d}\bar{z}^i \, \mathrm{d}\bar{z}^j ,$$
$$\bar{\phi} = \bar{\phi}(\bar{z}^-)$$
$$\bar{A}_{p+1} = \frac{1}{(p+1)!} \bar{A}(\bar{z}^-)_{i_1 \dots i_{p+1}} \, \mathrm{d}\bar{z}^{i_1} \wedge \dots \wedge \mathrm{d}\bar{z}^{i_{p+1}} + \text{gauge} \tag{4.75}$$
$$\bar{F}_{p+2} = \frac{1}{(p+1)!} \bar{F}(\bar{z}^-)_{-i_1 \dots i_{p+1}} \, \mathrm{d}\bar{z}^- \wedge \mathrm{d}\bar{z}^{i_1} \wedge \dots \wedge \mathrm{d}\bar{z}^{i_{p+1}} ,$$

which is a representation of the plane wave metric and of the fields in Rosen coordinates. One can transform to Brinkmann (or harmonic) coordinates (u, v, y^i) with the relations

$$\bar{z}^- = u , \qquad \bar{z}^+ = v - \frac{1}{4} C'_{ij}(\bar{z}^-) Q^i{}_k(\bar{z}^-) Q^j{}_l(\bar{z}^-) y^k y^l , \qquad \bar{z}^i = Q^i{}_j(\bar{z}^-) y^j , \tag{4.76}$$

where a prime denotes a derivative w. r. t. the argument and the matrix $Q^i{}_j$ fulfills

$$C_{ij} Q^i{}_k Q^j{}_l = \delta_{jl} , \qquad C_{ij} \left((Q')^i{}_k Q^j{}_l - Q^i{}_k (Q')^j{}_l \right) = 0 . \tag{4.77}$$

If one then defines

$$h_{kl} = -\left(C'_{ij}Q^j{}_l + C_{ij}(Q'')^j{}_l\right)Q^i{}_k \tag{4.78}$$

one obtains the final expression for the plane wave background

$$\begin{aligned}
\mathrm{d}\bar{s}^2 &= 2\,\mathrm{d}u\,\mathrm{d}v - h_{kl}(u)y^k y^l\,\mathrm{d}u^2 - \delta_{kl}\,\mathrm{d}y^k\,\mathrm{d}y^l\ , \\
\bar{\phi} &= \bar{\phi}(u) \\
\bar{F}_{p+2} &= \frac{1}{(p+1)!}\bar{F}(u)_{ui_1\ldots i_{p+1}}\,\mathrm{d}u \wedge \mathrm{d}y^{i_1} \wedge \cdots \wedge \mathrm{d}y^{i_{p+1}}\ .
\end{aligned} \tag{4.79}$$

Note that the field strengths but not the potentials retain their form under the coordinate transformation.

Before we discuss special cases let us collect what are called the hereditary issues in the literature [28, 78]. Due to Geroch [78] a property is called hereditary if, whenever a family of spacetimes has this property, all limits of this family have this property, too. Here we will follow the definition of [28] where a property is called hereditary if it is valid for an initial consistent background (not necessarily for a complete family of spacetimes) and remains valid in the Penrose-Güven limit (not necessarily in all limits) and not only for the metric but for all fields of the background. The hereditary properties are [28]

1. The Penrose limit of a Ricci flat spacetime is Ricci flat.

2. The Penrose limit of a conformally flat spacetime is conformally flat.

3. The Penrose limit of an Einstein space is always Ricci flat.

4. The Penrose limit of a locally symmetric space (e. g. the CW spaces) is locally symmetric.

5. The Penrose-Güven limit of a solution of the supergravity equations of motion remains a solution.

6. The dimension of the symmetry algebra does not decrease in the Penrose-Güven limit.

7. The number of conserved supersymmetries does not decrease in the Penrose-Güven limit.

We will now review the Penrose limits of AdS_{d+1} and $\text{AdS}_{d+1} \times \text{S}^{d'+1}$.

Since AdS_{d+1} is a conformally flat locally symmetric Einstein space it follows from the above given list of hereditary properties that its Penrose limit must be a conformally flat locally symmetric Ricci flat space which is of course isometric to $\mathbb{R}^{1,d}$.

For $\text{AdS}_{d+1} \times \text{S}^{d'+1}$ the situation is more subtle but it can nevertheless be discussed for general dimensions [28]. Remember that $\text{AdS}_{d+1} \times \text{S}^{d'+1}$ can be represented as the group manifold (4.34). Since the space is homogeneous we can go to an arbitrary point p_0 of this spacetime and obtain the different Penrose limits by taking all possible different null geodesics that cross p_0. It is clear that two null geodesics that are now completely specified by their directions can only lead to different Penrose limits if one cannot map the tangent vectors (which are null) of these geodesics at p_0 into each other by using the isometries that keep p_0 as a fixed point. These isometries are precisely given by the elements of the stabilizer which is $H = SO(1,d) \times SO(d'+1)$. The number of different Penrose limits is therfore given by the number of different orbits of null directions under the action of H. Since $\text{S}^{d'+1}$ has only spacelike directions it is clear that the component of the tangential vector in the AdS_{d+1} part can either be null or timelike. If it is null it has to lie on an S^{d-1} and the geodesic is stationary in the sphere. If, however, the AdS part of the tangential vector is timelike then it can point in any timelike direction and is not confined to the lightcone. Its tip lies on the ball B^d of future pointing timelike directions. If one has chosen the component in AdS one can only choose the direction in $\text{S}^{d'+1}$, i. e. the component of the tangent vector in $\text{S}^{d'+1}$ is then confined to end on an $\text{S}^{d'}$. The AdS part of the stabilizer is $SO(1,d)$. One can therefore map all vectors starting at p_0 with fixed length to each other by a spatial rotation and then identify timelike vectors with different lengths via a Lorentz boost. In $\text{S}^{d'+1}$ the second factor of the stabilizer $SO(d'+1)$ allows one to identify with each other all vectors with equal length that start at p_0 . That means there are only two distinct orbits and therefore two different Penrose limits in $\text{AdS}_{d+1} \times \text{S}^{d'+1}$. One is obtained by expanding around a null geodesic which has no movement in the sphere $\text{S}^{d'+1}$. The other corresponds to a geodesic that has a non-vanishing movement in $\text{S}^{d'+1}$. The first choice is not very interesting. The Penrose limit is the same as in case of pure AdS_{d+1}, that means flat Minkowski space. The limit for the second choice of geodesic will be worked out in the following.

The simplest way to obtain the plane wave spacetimes from $\text{AdS}_{d+1} \times \text{S}^{d'+1}$ is to cast

(4.26) into the form

$$\mathrm{d}s^2 = R_1^2 \Big(-\cosh^2 \rho \, \mathrm{d}t^2 + \mathrm{d}\rho^2 + \sinh^2 \rho \, \mathrm{d}\Omega_{d-1}^2 + \frac{R_2^2}{R_1^2} \big(\cos^2 \vartheta \, \mathrm{d}\psi^2 + \mathrm{d}\vartheta^2 + \sin^2 \vartheta \, \mathrm{d}\hat{\Omega}_{d'-1}^2 \big) \Big) \,.$$
$$(4.80)$$

Going to lightcone coordinates

$$z^+ = \frac{1}{2}\Big(t + \frac{R_2}{R_1}\psi \Big) \,, \qquad z^- = \frac{R_1^2}{2}\Big(t - \frac{R_2}{R_1}\psi \Big) \,, \tag{4.81}$$

one replaces all coordinates as follows

$$t = z^+ + \frac{z^-}{R_1^2} \,, \qquad \psi = \frac{R_1}{R_2}\Big(z^+ - \frac{z^-}{R_1^2} \Big) \,, \qquad \rho = \frac{r}{R_1} \,, \qquad \vartheta = \frac{y}{R_2} \,, \tag{4.82}$$

and expands in powers of R_1^{-1} and R_2^{-1}. Neglecting terms $\mathcal{O}(R_1^{-1})$ and $\mathcal{O}(R_2^{-1})$ the metric reads

$$\begin{aligned}
\mathrm{d}s^2 = R_1^2 \Big(&- \Big[(\mathrm{d}z^+)^2 + \frac{r^2}{R_1^2}(\mathrm{d}z^+)^2 + \frac{2}{R_1^2}\,\mathrm{d}z^+\,\mathrm{d}z^- \Big] + \frac{\mathrm{d}r^2}{R_1^2} + \frac{r^2}{R_1^2}\,\mathrm{d}\Omega_{d-1}^2 \\
&+ \Big[(\mathrm{d}z^+)^2 - \frac{y^2}{R_2^2}(\mathrm{d}z^+)^2 - \frac{2}{R_1^2}\,\mathrm{d}z^+\,\mathrm{d}z^- \Big] + \frac{\mathrm{d}y^2}{R_1^2} + \frac{y^2}{R_1^2}\,\mathrm{d}\hat{\Omega}_{d'-1}^2 \Big) \\
= &-4\,\mathrm{d}z^+\,\mathrm{d}z^- - \Big(r^2 + \frac{R_1^2}{R_2^2}y^2 \Big)(\mathrm{d}z^+)^2 + \mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_{d-1}^2 + \mathrm{d}y^2 + y^2\,\mathrm{d}\hat{\Omega}_{d'-1}^2 \,.
\end{aligned}
\tag{4.83}$$

This result is exact if we take the limit $R_1, R_2 \to \infty$, $\frac{R_1}{R_2} = $ fixed, which clearly corresponds to the plane wave limit. Indeed, the above result has the form of a plane wave spacetime if we take the embedding space coordinates ω and $\hat{\omega}$ respectively for the subspheres of AdS_{d+1} and $\mathrm{S}^{d'+1}$ and define Cartesian coordinates

$$\begin{aligned}
x_i &= r\,\omega_i \,, & i &= 1, \dots, d \,, & \sum_i \omega_i^2 &= 1 \,, \\
y_{i'} &= y\,\hat{\omega}_{i'} \,, & i' &= 1, \dots, d' \,, & \sum_{i'} \hat{\omega}_{i'}^2 &= 1 \,.
\end{aligned}
\tag{4.84}$$

The metric then becomes with $\vec{z} = (\vec{x}, \vec{y})$

$$\mathrm{d}s^2 = -4\,\mathrm{d}z^+\,\mathrm{d}z^- + H_{ij}z^i z^j (\mathrm{d}z^+)^2 + \delta_{ij}\,\mathrm{d}z^i\,\mathrm{d}z^j \,,$$
$$H_{ij} = \begin{cases} -\delta_{ij} & i,j = 1, \dots, d \\ -\frac{R_1^2}{R_2^2}\delta_{ij} & i,j = d+1, \dots, d+d' \end{cases} \,.
\tag{4.85}$$

This metric includes precisely the ones of the 10-dimensional plane wave solutions of type II B (4.52), where $R_1 = R_2$, $d = d' = 4$ and of the 11-dimensional KG solution (4.53) with

$R_2 = 2R_1$ and $d = 3$, $d' = 6$ or $R_2 = \frac{1}{2}R_1$ and $d = 3$, $d' = 6$ after changing a common factor of H_{ij} by a redefinition of z^+ and z^-.

At the end we have to compute the flux in the the Penrose-Güven limit. From (4.67) and (4.68) one can see that the rescaling (4.74) is necessary because the $(d+1)$-form flux is of order $\mathcal{O}(R_2^d)$. With our coordinate choice (4.82) where the scale factor is included, the radii dependence should cancel out up to finite ratios $\frac{R_1}{R_2}$. The volume forms in global coordinates read

$$
\begin{aligned}
\mathrm{vol}(\mathrm{AdS}_{d+1}) &= R_1^{d+1} \cosh \rho (\sinh \rho)^{d-1} \, \mathrm{d}t \wedge \mathrm{d}\rho \wedge \mathrm{vol}(\Omega_{d-1}) \,, \\
\mathrm{vol}(\mathrm{S}^{d'+1}) &= R_2^{d'+1} \cos \theta (\sin \theta)^{d'-1} \, \mathrm{d}\psi \wedge \mathrm{d}\theta \wedge \mathrm{vol}(\hat{\Omega}_{d'-1}) \,,
\end{aligned}
\tag{4.86}
$$

where $\mathrm{vol}(\Omega_{d-1})$, $\mathrm{vol}(\hat{\Omega}_{d'-1})$ are volume forms on the corresponding unit spheres. Changing the coordinates according to (4.82) and only keeping the leading contribution then results in

$$
\mathrm{vol}(\mathrm{AdS}_{d+1}) = R_1 \, r^{d-1} \, \mathrm{d}z^+ \wedge \mathrm{d}r \wedge \mathrm{vol}(\Omega_{d-1}) \,, \quad \mathrm{vol}(\mathrm{S}^{d'+1}) = R_1 \, y^{d'-1} \, \mathrm{d}z^+ \wedge \mathrm{d}y \wedge \mathrm{vol}(\hat{\Omega}_{d'-1}) \,.
\tag{4.87}
$$

Expressed in the variables (4.84), one then finds for the $(d+1)$-form flux

$$
F_{d+1} = d \, \mathrm{d}z^+ \wedge \mathrm{vol}(\mathbb{R}^d)
\tag{4.88}
$$

and the dual flux becomes with $D = d + d' + 2$

$$
\star F_{d+1} = d' \frac{R_1}{R_2} \, \mathrm{d}z^+ \wedge \mathrm{vol}(\mathbb{R}^{d'}) = d \, \mathrm{d}z^+ \wedge \mathrm{vol}(\mathbb{R}^{d'}) \,,
\tag{4.89}
$$

where the last equality follows with (4.64). According to (4.15) the selfdual 5-form field strength now reads

$$
F_5 = 2 \, \mathrm{d}z^+ \wedge (\mathrm{vol}_x(\mathbb{R}^4) + \mathrm{vol}_y(\mathbb{R}^4)) \,,
\tag{4.90}
$$

where the subscript denotes which set of coordinates one has to take from (4.84). We have found the flux of the 10-dimensional plane wave background (4.52).

Chapter 5

Holography

In this Chapter we will first briefly describe the AdS/CFT correspondence. Special emphasis will be put on the concrete realization of holography in which the bulk-to-boundary propagator plays an important role. As a crucial point for the motivation of the analysis in Chapter 7 we will show how this propagator is determined from the bulk-to-boundary propagator. Secondly, we will give a short description of the BMN limit of the AdS/CFT correspondence. Then we will summarize the proposals about the realization of holography in this limit. On that basis we will then fix the direction of the analysis in the Chapters 6 and 7.

5.1   The AdS/CFT correspondence and holography

In the AdS/CFT correspondence [113] string theory on an AdS space is related to a CFT on the boundary of this space. Here we will mainly describe the correspondence where the string theory is type II B on $\mathrm{AdS}_5 \times \mathrm{S}^5$. The boundary CFT then is the superconformal $\mathcal{N} = 4$ super Yang-Mills (SYM) theory in 4 dimensions. The following review will be rather short. For more details we refer the reader to the reviews [3, 55, 58, 101, 136].

Let us first describe the string theory setup in brief. We have seen in Subsection 4.1.1 that 10-dimensional type II B supergravity admits a 3-brane solution with non-vanishing self dual 5-form flux F_5. This 3-brane corresponds to a D3-brane in the full string theory and thus type II B supergravity in its 3-brane background should be a low energy description of type II B string theory with a D3-brane. On the other hand [106] the effective action can be formulated including type II B supergravity in the bulk and

the DBI action on the branes. Instead of a single D3-brane one then takes a stack of N D3-branes where in general the parameter (4.7) of the $(d-1)$-brane solution (4.6) is replaced by

$$R_2^{d'} \rightarrow N R_2^{d'} \ , \tag{5.1}$$

where $D = d + d' + 2$. The gauge group on the branes is extended to the non-Abelian $U(N)$. In the near horizon limit described in Subsection 4.2.1 the following now happens [3, 4, 57]. The modes in the throat region, which is the regime close to the brane, become trapped and decouple from the dynamics outside this region.[1] The action of type II B supergravity in its 3-brane background thus splits into two decoupled parts, the near horizon part which is type II B supergravity in AdS$_5 \times$ S^5 and a part which lives outside the horizon. In the second description the interaction of the D3-brane with the bulk vanishes in the limit and one finds two decoupled systems similar to the situation above. One part is $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ on the worldvolume of the brane and the other part is exactly the same already found in the near horizon limit of the first description. Since both descriptions should be equivalent Maldacena [113] conjectured that supergravity in AdS$_5 \times$ S^5 should be a dual description of the 4-dimensional $\mathcal{N} = 4$ $SU(N)$ SYM gauge theory and he extended this conjecture to full string theory. The parameters of both theories are related in the following way, see (4.7) and (4.64)

$$R_1 = R_2 = R \ , \qquad R^4 = \lambda \alpha'^2 \ , \qquad \lambda = g^2 N \ , \qquad g^2 = 4\pi g_s \ , \tag{5.2}$$

where g is the YM coupling and λ the 'tHooft coupling constant. For the time being, however, it seems hopeless to work with the full string theory on AdS$_5 \times$ S^5. The reason is that the background includes the non-vanishing Ramond-Ramond flux F_5 and one is thus forced to work with the Green-Schwarz superstring action which up to now could not be quantized successfully in this case. The approximation one now uses is type II B supergravity instead of the full string theory. It is faithful as long as the curvature radius of the background is large compared to the string length. To avoid string loop corrections one should furthermore take $g_s \rightarrow 0$. This can be realized by the following sequence of limits

$$N \rightarrow \infty \ , \quad \text{with} \quad \lambda = \text{fixed} \ , \quad \text{then} \quad \lambda \rightarrow \infty \ . \tag{5.3}$$

In this limit the correspondence then relates classical supergravity in AdS$_5 \times$ S^5 to the large N limit of strongly coupled $\mathcal{N} = 4$ SYM.

---

[1]This is not completely true. The modes which describe the center of mass position of the stack of branes correspond to the $U(1)$ degrees of freedom and do not decouple [3].

If we regard the sphere as simply generating the complete Kaluza-Klein (KK) tower of massive modes, the conjecture then states that a 5-dimensional supergravity theory should be equivalent to a 4-dimensional CFT. Since equivalent means that both include the same information, this correspondence is an example of the holographic principle: higher dimensional information can be stored in a lower dimensional description. Thereby the lower dimensional theory should not contain more than one degree of freedom per Planck area [170, 176]. In the following we will explain more precisely how holography is understood in the AdS/CFT correspondence.

The AdS/CFT correspondence relates the correlation functions of operators $\mathcal{O}_\Delta$ with conformal dimension Δ in the d-dimensional CFT to the classical value of the action S_{IIB} of (dimensionally reduced) type IIB supergravity in AdS_{d+1} evaluated with boundary condition $\bar{\phi}_\Delta(\bar{x})$, $\bar{x} = (x_0, \vec{x})$ as follows [85, 186]

$$\left\langle \exp \int \mathrm{d}^d \bar{x} \, \bar{\phi}_\Delta(\bar{x}) \mathcal{O}_\Delta(\bar{x}) \right\rangle_{\text{CFT}} = \mathrm{e}^{-S_{\text{IIB}}[\phi_\Delta[\bar{\phi}_\Delta]]} \; . \tag{5.4}$$

We remark that the above formulation is given for Euclidean AdS_{d+1} and that we will keep dealing with the Euclidean case to the end of this Section. The Lorentzian formulation can be found in [13, 14]. The metric with Euclidean signature in Poincaré coordinates is given by (4.29) but with the minus sign in front of $\mathrm{d}x_0^2$ converted to plus. The boundary of AdS_{d+1} is situated at $x_\perp = 0$. The supergravity modes $\phi_\Delta(x)$, $x = (x_\perp, \bar{x})$ that correspond to the operators in the CFT are given by the non-normalizable modes[2] which scale as [55, 74]

$$\bar{\phi}_\Delta(\bar{x}) = \lim_{x_\perp \to 0} \phi_\Delta(x) x_\perp^{\Delta - d} \; . \tag{5.5}$$

One can regard the bulk supergravity fields ϕ_Δ as functionals of the corresponding boundary values by computing the convolutions

$$\phi_\Delta[\bar{\phi}_\Delta](x) = \int \mathrm{d}^d \bar{x}' \, K_\Delta(x, \bar{x}') \bar{\phi}_\Delta(\bar{x}') \; , \tag{5.6}$$

where $K_\Delta(x, \bar{x}')$ denotes the corresponding bulk-to-boundary propagator which for a scalar field is defined as [55, 74]

$$K_\Delta(x, \bar{x}') = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}} \Gamma(\Delta - \frac{d}{2})} \left(\frac{x_\perp}{x_\perp^2 + (\bar{x} - \bar{x}')^2} \right)^\Delta \; . \tag{5.7}$$

---

[2]The normalizable modes determine the vacuum structure (VEVs) of the CFT. See [186] and [13] for differences between the Euclidean and Lorentzian case.

It behaves near the boundary like

$$\lim_{x_\perp \to 0} K_\Delta \left(x, \bar{x}' \right) x_\perp^{\Delta - d} = \delta^d (\bar{x} - \bar{x}') \ . \tag{5.8}$$

The given prescription can now be interpreted as follows [186]. The d-dimensional CFT lives on the d-dimensional boundary of AdS_{d+1}. Sources that couple to the operators of the CFT are identified with the boundary values for the corresponding non-normalizable supergravity modes which solve the supergravity equations of motion. The bulk supergravity modes are constructed via a convolution of the boundary value with the corresponding bulk-to-boundary propagator. According to (5.4) an n-point correlation function of CFT operators can now be evaluated by taking the corresponding n functional derivatives w. r. t. the boundary values $\bar{\phi}_\Delta$ (and after that setting $\bar{\phi}_\Delta = 0$).

One can represent the different contributions to the n-point function by so called Witten diagrams [186], which are Feynman diagrams for the AdS/CFT correspondence. AdS_{d+1} is represented by a disk and its d-dimensional boundary becomes the boundary circle of the disk. The n points are now distributed along the circle and from each point originates a line into the interior of the disk. These lines represent the bulk-to-boundary propagators. The n lines in the interior are now combined into vertices found in the type II B supergravity action. If a diagram contains several vertices in the interior they have to be connected with lines that correspond to the bulk-to-bulk propagator in AdS_{d+1}.[3]

The bulk-to-bulk propagator $G_\Delta(x, x')$ is defined as a solution of the differential equation

$$(\Box_x - m^2) G_\Delta(x, x') = -\frac{1}{\sqrt{g_{\text{AdS}}}} \delta(x, x') \ , \tag{5.9}$$

where $\Box_x$ is the Laplace operator on Euclidean AdS_{d+1} acting on the first argument of $G_\Delta(x, x')$, and g_{AdS} is the determinant of the metric. The solution reads [40, 55]

$$G_\Delta(x, x') = \frac{\Gamma(\Delta)}{R_1^{d-1} 2\pi^{\frac{d}{2}} \Gamma(\Delta - \frac{d}{2} + 1)} \left(\frac{\xi}{2} \right)^\Delta F\left(\frac{\Delta}{2}, \frac{\Delta}{2} + \frac{1}{2}; \Delta - \frac{d}{2} + 1; \xi^2 \right) \ , \quad \xi = \frac{2R_1^2}{u + 2R_1^2} \ , \tag{5.10}$$

where $u = u(x, x')$ is the chordal distance given in (4.32) and hence in Poincaré coordinates ξ has the form

$$\xi = \frac{2x_\perp x'_\perp}{x_\perp^2 + x'^2_\perp + (\bar{x} - \bar{x}')^2} \ . \tag{5.11}$$

---

[3]Such a diagram is generated if the exponential function on the R. H. S. of (5.4) is expanded to at least quadratic order in the action.

It becomes zero for $x'_\perp \to 0$.

As is shown in Appendix B.3 the bulk-to-boundary propagator $K_\Delta(x, \bar{x}')$ can be obtained from the bulk-to-bulk propagator $G_\Delta(x, x')$ as follows (see (B.20))

$$K_\Delta(x, \bar{x}') = -R_1^{d-1} \big[(d - \Delta) x'^{-\Delta}_\perp - x'^{1-\Delta}_\perp \partial'_\perp \big] G_\Delta(x, x') \big|_{x'_\perp = 0} . \qquad (5.12)$$

Thus, using $G_\Delta(x, x')$, the boundary value problem can be solved.

We have seen that the AdS/CFT correspondence is holographic in the sense that a theory of gravity in AdS_{d+1} can be described by a conformal field theory on the d-dimensional boundary of that space. It was shown [170] that the boundary theory respects the restriction that only one bit of information per Planck area is stored and hence it provides a true holographic description. This result is related to the fact that infrared effects in the bulk theory correspond to ultraviolet effects in the boundary theory, known as UV/IR connection.

The role of the coordinate perpendicular to the boundary, which is called holographic direction, is of central importance. In the Poincaré coordinates, $x_\perp$ is the holographic coordinate and $x_\perp = 0$ is the position of the boundary where the metric (4.29) diverges. Bulk computations thus require a cutoff $\delta > 0$ [170, 186], where now $x_\perp \geq \delta$, to regularize the infinite size of the volume. From the perspective of the bulk theory, δ is an infrared cutoff. For its interpretation in the boundary theory we remember that due to (4.69) an observer on the boundary measures the lower energy the more the source of this energy lies in the interior of AdS. The infrared region of the boundary theory therefore corresponds to the deep interior of the AdS, whereas the ultraviolet regime is dominated by the region close to the boundary. Hence, the infrared cutoff δ in the bulk theory becomes an ultraviolet cutoff in the boundary theory [170].

5.2 The BMN correspondence and holography

In Section 4.2.2 we have seen that the 10-dimensional maximally supersymmetric plane wave background arises as a Penrose-Güven limit of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background of type II B string theory. Berenstein, Maldacena and Nastase [21] then translated the limit to the gauge theory side and formulated a new limit of the full AdS/CFT correspondence, which is technically tractable even beyond the supergravity approximation. In the following we

will denote the correspondence in this limit as BMN correspondence and the limit that has to be taken on both sides of the correspondence collectively as the BMN limit.

As we have seen in Section 5.1, the AdS/CFT correspondence obeys the holographic principle and hence it is natural to investigate how holography can be established in its BMN limit. Before we start to present a summary of some work that appeared in this context, we will in brief describe the BMN correspondence. More detailed reviews are [138, 150, 152].

The proposal of Berenstein, Maldacena and Nastase [21] starts with the discussion of how the Penrose-Güven limit translates to the gauge theory side. We have seen in Subsection 4.2.2 that the Penrose-Güven limit of $AdS_5 \times S^5$ can be realized by sending the common embedding radius R to infinity. The result is given in (4.52) and the parameter μ will be kept in the following. The requirement to have finite lightcone momenta

$$p^- = \frac{\mu}{2}(E - J) \,, \qquad p^+ = \frac{E + J}{2\mu R^2} \tag{5.13}$$

in the limit leads to the condition that the energy E and the angular momentum J of a supergravity mode both scale with R^2 but that $E - J =$ finite. E and J are identified with the conformal dimension Δ and with the charge of a $U(1)$ subgroup of the $SO(6)$ R-symmetry group respectively. The $U(1)$ subgroup is the one singled out in the Penrose-Güven limit in the S^5. According to (5.2) the limit on the gauge theory side becomes

$$N \to \infty \,, \qquad \frac{J}{\sqrt{N}} = \text{fixed} \,, \qquad g = \text{fixed} \,. \tag{5.14}$$

The operators that survive the limit have to obey

$$\Delta - J = \text{finite} \geq 0 \,. \tag{5.15}$$

A chiral primary operator (CPO) $\mathcal{O}_k$ in the scalar sector of $\mathcal{N} = 4$ SYM that has conformal dimension $\Delta = k$ reads

$$\mathcal{O}_k = C^{a_1 \dots a_k} \, \text{tr}[\phi_{a_1} \dots \phi_{a_k}] \,, \tag{5.16}$$

where $C^{a_1 \dots a_k}$ are traceless symmetric tensors (that correspond to the spherical harmonics on S^5) and ϕ_a, $a = 1, \dots, 6$ denote the scalars of $\mathcal{N} = 4$ SYM. The field combination $Z = \frac{1}{\sqrt{2}}(\phi_5 + i\phi_6)$ carries definite scaling dimension $\Delta = 1$ and charge $J = 1$. The remaining four scalars ϕ_a and covariant derivatives $D_\mu Z$, $\mu = 1, \dots, 4$ carry $\Delta - J = 1$. They are denoted as 'impurities' that are inserted into the composite operators of the

string state operator mapping. A sample of the dictionary reads [21, 138]

$$|0, p^+\rangle \leftrightarrow \frac{1}{\sqrt{JN^J}} \, \mathrm{tr}[Z^J] \, ,$$

$$\alpha_0^{\dagger i} |0, p^+\rangle \leftrightarrow \frac{1}{\sqrt{N^J}} \, \mathrm{tr}[\Phi_i Z^J] \, , \tag{5.17}$$

$$\alpha_n^{\dagger i} \alpha_{-n}^{\dagger j} |0, p^+\rangle \leftrightarrow \frac{1}{\sqrt{JN^J}} \sum_{l=0}^{J} \mathrm{tr}[\Phi_i Z^l \Phi_j Z^{J-l}] \, \mathrm{e}^{\frac{2\pi i n l}{J}} \, .$$

Here $|0, p^+\rangle$ denotes the string groundstate and we have listed only the bosonic operators with up to two impurities

$$\Phi_i = (D_1 Z, \dots, D_4 Z, \phi_1, \dots \phi_4) \, , \qquad i = 1, \dots, 8 \, . \tag{5.18}$$

The phase factor in (5.17) is chosen such that, according to (5.13), the anomalous 'twist' $\Delta - J$ of the composite operators match with the energy eigenvalues of the plane wave string Hamiltonian in light cone gauge [21, 119]

$$H_{\mathrm{lc}} = 2p^- = \sum_{n=-\infty}^{\infty} N_n \mu \sqrt{1 + \frac{n^2}{(\mu p^+ \alpha')^2}} \, . \tag{5.19}$$

Here N_n is the occupation number at level n. From (5.13) one finds with (5.2) that the parameters in the limit are related as follows

$$\frac{1}{(\mu p^+ \alpha')^2} = \frac{g^2 N}{J^2} = \lambda' \tag{5.20}$$

and we have defined the expansion parameter λ' that is the effective coupling constant in the BMN limit. One now immediately sees that string states only built with zero-mode oscillators $\alpha_0^{\dagger i}$ (see (5.17)) are protected against corrections in powers of λ'. The non-protected states contain a minimum number of two oscillators with $n \neq 0$. They correspond to the two-impurity operators with a non-trivial phase factor that depends on their separation within the trace. The phase factor leads to the matching between the anomalous twist $\Delta - J$ and the light cone energy in the planar limit

$$\Delta - J = 2\sqrt{1 + \lambda' n^2} = 2 + \lambda' n^2 + \mathcal{O}(\lambda'^2) \, . \tag{5.21}$$

At one loop this is demonstrated in [21]. The two loop result is presented in [84] and a complete reproduction to all orders was obtained in [153].

This is of course not the end of the story. Up to now we have only discussed that the planar part of the BMN gauge theory matches with the free string spectrum. The correspondence should also work if we include string interactions. In the gauge theory they are argued to correspond to non-planar contributions. It was shown in [48, 103] that the naively expected suppression of non-planar diagrams does not hold. The reason is that the number of elementary fields in the BMN operators grows with J and hence the number of diagrams at each order compensate the $\frac{1}{N}$ suppression. In this way a certain class of non-planar contributions survives the limit [103], and their non-planarity is controlled by the effective genus counting parameter

$$g_2 = \frac{J^2}{N} \ .$$
(5.22)

A genus h contribution then is of order $(g_2)^{2h}$. To describe interactions on the string theory side, light cone string field theory is required [131, 138]. There exist two different proposals for the cubic interaction vertex. The first one is worked out in [130, 132, 133, 164, 165]. For this vertex a relation like $H_{\mathrm{lc}} = \mu(\Delta - J)$ holds even for non-planar diagrams if one interpretes it as an operator relation, where both sides act in different Hilbert spaces. The picture behind this is that multiple string interactions correspond to the two point functions of multi-trace operators on the gauge theory side [20, 182]. The situation is different for the second vertex which is constructed in [59]. It is proposed to correspond to the gauge theory 3-point function of BMN operators in the original basis[4], because a relation like in the case of the first vertex does not hold. Compared to the first proposal, this proposal becomes problematic beyond tree level, because the gauge theory 3-point function in the original basis of operators then is no longer of the form dictated by conformal invariance Furthermore, it is unclear how this proposal should be extended to more than 3-point interactions [20]. A clear decision between the two proposals should be possible as soon as it has become clear if or in which sense the 3-point function enters the duality.

After this short summary of some aspects of the BMN correspondence we will now in brief present some proposals of how holography could be realized in this limit.

In [52] the authors come to the conclusion that the gauge theory dual is Euclidean and lives in a 4-dimensional subspace that is formed by four of the eight transversal coordinates z_i in (4.52). One of their arguments for the Euclidean signature in this case

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[4]This is the basis before one has taken into account operator mixing at non-planar level.

is that the duality relates the operators of the lower dimensional theory to the transverse oscillators (in light cone gauge). In their setup the holographic coordinate is given by z^+.

In [100] the authors argue that the holographic direction in the plane wave background (4.52) is z^+ by analyzing the behaviour of particles in the background. In particular they identify possible non-compact directions and analyze which of these directions effectively do not confine particles. They find that z^+ is effectively non-compact in this sense. Furthermore, via a coordinate transformation the conformal flatness of the plane wave (4.52) becomes manifest. In this new coordinates the metric is given by flat Minkowski space with a conformal factor that depends only on $x^+ = -\cot \frac{z^+}{2}$. This coordinate system is seen as analogously to the Poincaré patch for AdS. The gauge theory dual then should live on a slice of constant $x^+ \to -\infty$. Since the wave equation in these coordinates is only of first order in the light cone coordinates x^+ and in x^-, only the non-normalizable modes are found. The absence of the normalizable modes lead the authors to the proposal that one has to impose boundary conditions on both slices (constant $x^+ \to -\infty$ and $x^- \to -\infty$) and that the data on the slice of constant x^- should then determine the vacuum structure. This situation is different compared to the full Lorentzian AdS/CFT correspondence, where normalizable and non-normalizable modes exist [13, 14]. The authors give a concrete proposal of how to compute correlation functions with the bulk-to-boundary propagator in their setup. They determine the bulk-to-bulk and bulk-to-boundary propagator directly by using the eigenmodes of the wave equation in their specific coordinate system. The concrete case of bosonic strings in $\text{AdS}_3 \times \text{S}^3 \times M^{20}$ supported by NS-NS 3-form flux is dealt with in [24] and is in agreement with the proposal of [100].

A different proposal is worked out in [107]. The authors argue that the holographic direction is given by the first radial coordinate that corresponds to r in (4.83) and originates from the AdS part. The holographic screen is identified as the corresponding sphere plus the lightcone direction z^+. The boundary theory should be the original $\mathcal{N} = 4$ SYM one, living at $r \to \infty$. The setup is very similar to the one in the AdS/CFT correspondence. In their approach both normalizable and non-normalizable modes are present and the allowed modes carry positive lightcone momentum p^+.

The approach of [22] starts with working out the structure of the conformal boundary of the plane wave which turns out to be a 1-dimensional null line parameterized by the coordinate z^+ in (4.52). Since this coordinate contains the time direction of the AdS

part, the authors propose that the boundary theory should be a one-dimensional quantum mechanical system that can be realized as a matrix model. It should describe the lowest KK modes of the 4-dimensional $\mathcal{N} = 4$ SYM theory after compactification on S^3. The authors furthermore state that the observables of the boundary theory should be finite time transition matrix elements between states that describe multiple strings. However a comparison with the string calculation then requires care because of the following reason. A naive interpretation of z^+ as a time variable is not justified if one defines the lightcone coordinates as in (4.82). The periodicity of ψ then implies a periodicity of z^+. Instead [149] one can define $z^+ = t$ and keep z^- as in (4.82). Then only z^- becomes periodic with a large period of $2\pi R_1 R_2$. The authors argue that the effect of this compactification should be considered in the normalization of the wavefunctions as follows: bulk modes should be normalized considering z^- as non-compact because one uses the plane wave metric (4.52) which is the exact limit $R_1 = R_2 \to \infty$ of $\text{AdS}_5 \times S^5$. But in the boundary theory one should normalize the states considering z^- as compact. The authors of [22] furthermore comment on the proposals of [52, 100, 107]. They remark that [52] does not consider that the BMN correspondence is a limit of the AdS/CFT correspondence. Their argument is that to put the gauge theory on a Euclidean space does not follow from the original AdS/CFT correspondence where the Euclidean version only occurs after Wick rotating to Euclidean AdS. They criticize [100] with the argument that it is not allowed to work in a patch along the lines of the Poincaré patch in AdS. They state that the translation to an Euclidean version is not possible in the plane wave case, however that it is essential in the full AdS/CFT correspondence to allow working with the Poincaré patch. Their comment on the proposal [107] is that one should not forget that the null geodesic around which one expands is in the center of AdS and that ,blowing up its neighbourhood, the old boundary of AdS, where the original $\mathcal{N} = 4$ SYM lives, lies outside the plane wave.

This result of [22] that the conformal boundary is a one-dimensional null line was confirmed by [115] where the authors used the construction of Geroch, Kronheimer and Penrose [77, 89] to attach a causal boundary to a spacetime, see also [90]. The construction uses the indecomposable past (IP) and future (IF) sets. Proper indecomposable past (PIP) and future (PIF) sets are the pasts and futures of points in the spacetime, whereas terminal indecomposable past (TIP) and future (TIF) sets are not the pasts and futures of points in the spacetime. The idea is to regard TIPs and TIFs as representing points of the causal boundary of the spacetime. We do not want to give all exact definitions

here but physically motivate what is happening. TIPs can be seen as the pasts of future inextendible timelike curves. A future inextendible timelike curve is one without a future endpoint. Precisely, a future endpoint p_0 to a future directed (parameter γ increases in the future direction) non spacelike curve is defined by taking a neighbourhood around p_0 and demanding that by increasing the parameter γ along that curve one always enters the neighbourhood at $\gamma = \gamma_0$ but never leaves it again for arbitrary $\gamma > \gamma_0$. It is then clear that a future inextendible timelike curve is one on which for every point p on the curve one can find a point q that lies in the future of p. The PIP of p is thus included in the PIP of q but there is no PIP that includes all other PIPs of points on the curve. Hence, it is a TIP that includes all PIPs of points on the curve. This TIP now represents the future endpoint of the curve that is an element of the causal boundary of the spacetime. This motivates why one regards TIPs and TIFs to represent points of the causal boundary of the spacetime. It is interesting to note that for AdS and the plane wave spacetimes, points of the causal boundary are represented by TIPs *and* TIFs (see [89] for AdS and [115] for the plane wave).[5] In less technical words, the boundaries of these spacetimes allow for an *exchange* of information with the interior. This is in contrast i. e. to Minkowski-space which boundary points are *either* represented by TIPs *or* TIFs and therefore can *either* be influenced by *or* influence the bulk but not both simultaneously.

The problem, where the lower dimensional holographic partner should reside, was addressed in [162] from a different perspective. The idea is to introduce a holographic screen that is not necessarily connected with the boundary structure of the space. The author discusses the Penrose limit of pure AdS which is flat Minkowski space. He shows that a projection on scale invariant states fixes a de Sitter (dS) hypersurface of codimension one in AdS which in the flat space limit becomes Minkowski space and the isometry group $SO(2, d)$ then becomes the conformal group on it. The analysis has been extended in [163] to $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ where the Penrose limit is taken along a null geodesic with movement in the sphere.

In [5] the author argues that the dual gauge theory in the BMN limit is not an effective one-dimensional theory but the full 4-dimensional SYM theory. The underlying picture is that already in the full AdS/CFT correspondence the SYM theory should be seen regarded as living on any hypersurface with constant holographic coordinate (ρ in global and $x_\perp$ in Poincaré coordinates), where its constant value is related to the energy scale

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[5]We disregard some special points here and in the following.

of the SYM theory. According to (4.83), in the plane wave background each hypersurface has constant r and is given by the 3-dimensional sphere from the AdS part with z^+ as time direction.

In [190] the author proposes that one should interpret the bulk-boundary connection in the BMN correspondence as a tunneling phenomenon. The proposal starts from the observation that particles that move in the S^5 of the original $\mathrm{AdS}_5 \times \mathrm{S}^5$ background never reach the conformal boundary. They are confined in a region in the interior of AdS. Hence, a theory on the boundary of $\mathrm{AdS}_5 \times \mathrm{S}^5$ seems to have no influence on or be influenced by this region. As we have seen in Section 4.2.2, the Penrose limit that leads to the plane wave spacetime zooms into the neighbourhood of a null geodesic with movement in S^5 and therefore, applying the above considerations, its bulk theory seems to be disconnected from the AdS boundary. The author observes that with a purely imaginary action one finds trajectories that connect two boundary points by entering the bulk of AdS. He argues that the tunneling picture follows with a double Wick rotation that leads to Euclidean time in AdS and transforms the angle coordinate ψ of (4.82), which parameterizes the null geodesic in the S^5, the role of a time coordinate. The author proposes to relate the Euclidean S-matrix to the operator product expansion of BMN operators on the boundary.

The authors of [114] analyze the 3-point correlation function of two BMN and one non-BMN chiral primary operator (CPO) in the AdS/CFT correspondence. They compare the CFT and supergravity calculations before and after the BMN limit. They find the expected agreement [104] in the full AdS/CFT context, even using the coordinates of (4.82) and taking the $R \to \infty$ limit. They argue, however, that in a holographic setup where one wants to relate the amplitude to a correlation function of local operators in the dual theory, the amplitude should be truncated. In their proposal one should remove the part that describes the propagation from the AdS boundary to the geodesic along which the Penrose limit is taken and replace it by a δ-function. To be more precise the time (or in the variables of (4.82) z^+-dependence of the amplitude which has poles with a period of π should be replaced by a δ-function with a single pole to guarantee locality in time. This leads to a mismatch between the string and the gauge theory result. The authors give possible explanations for the mismatch such that this does not necessarily lead to an exclusion of a holographic principle in the BMN limit.

A completely different point of view is taken in [86]. There, the authors find that the BMN limit is a concrete realization of general considerations in [145] about operators

with large spin. It can be embedded into the more general framework of finding classical solutions for single strings on $AdS_5 \times S^5$ and computing the quantum fluctuations around them. The BMN limit then has the following interpretation [75, 86]: the null geodesic around which the Penrose limit is taken on the string side is a classical pointlike string solution.[6] The quantum fluctuations around this solution that enter the action in quadratic order can then be interpreted as the embedding coordinates of a string in the 10-dimensional plane wave background. In [180] the author now argues that to try to find a holographic setup in the BMN limit might be misleading because the limit simply represents the 1-loop approximation of the σ-model.

The above given brief descriptions show that holography in the BMN limit of the AdS/CFT correspondence is by far less understood than in the AdS/CFT correspondence itself. Although the above proposals are different, they may not necessarily exclude each other, and they might be equivalent descriptions [100]. In principle one can follow two ways to find a holographic setup in the BMN correspondence. One can either work directly in the BMN limit and disregard the fact that it follows from the AdS/CFT correspondence, or one can try to get information on the holographic principle by following the limiting process from the full AdS/CFT to the BMN correspondence, and thereby observe what happens to the ingredients in a holographic setup. The advantage of the second approach should be that it excludes holographic formulations which are not related to holography in the full AdS/CFT correspondence.

In the following we will work in the spirit of this second proposal and first discuss geometrical quantities in Chapter 6. In Section 6.1 we will investigate the behaviour of the boundary structure of $AdS_5 \times S^5$ in the Penrose limit to the plane wave. This extends the analysis of [22] where it is discussed *after* the limit has been taken. To analyze the causal structure in connection with a holographic picture, geodesics, and in particular null geodesics, reaching the holographic screen out of the bulk, play a central role. In Section 6.2 we will determine all possible geodesics in $AdS_5 \times S^5$ and in the 10-dimensional plane wave. Section 6.3 then deals with the question of how the boundary reaching null geodesics in $AdS_5 \times S^5$ are translated to the plane wave geodesics in the Penrose limit.

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[6]In [75] it is shown that there exist two pointlike string solutions in $AdS_5 \times S^5$. Both move along an equator of S^5. One does, whilst the other does not move in spatial directions of AdS_5. Both are equivalent by a coordinate transformation in $AdS_5 \times S^5$. This fits perfectly with the considerations in Subsection 4.2.2 that all configurations with different velocity components in AdS_5 and in S^5 are equivalent as long as the velocity in S^5 is different from zero.

After having discussed the geometrical quantities, we refer to the observations in Section 5.1 and argue that propagators should play an essential role in the realization of holography. Hence, in Chapter 7 we will focus on the scalar propagator in a generic $AdS_{d+1} \times S^{d'+1}$ background and analyze for $d = d' = 4$ the behaviour when taking the Penrose limit. In Section 7.1 we will use the differential equation for the propagator to work out under which circumstances simple powerlike solutions can be found. The generalization to arbitrary dimensions and curvature radii is useful to identify the general mechanism that leads to these kind of solutions. In Section 7.2 they will then be rederived from the solutions in flat space, using a conformal map. An interpretation of the result including global properties will be given in Section 7.3. In Section 7.4 the propagator will be explicitly constructed by summing up the Kaluza-Klein modes. For the particular case of $AdS_5 \times S^5$ the Penrose limit will then be discussed in Section 7.5.

Chapter 6

Boundaries and geodesics in AdS × S and in the plane wave

We have mentioned in Section 5.2 that the boundaries of $AdS_5 \times S^5$ and of the 10-dimensional plane wave exchange information with the interior. This is equivalent to saying that each point on these boundaries is represented by a TIP as well as TIF.[1] A light ray travels from the interior to the boundary, is reflected and travels back into the interior in finite time. It is interesting to investigate how this picture is translated in the Penrose limiting process from $AdS_5 \times S^5$ to the plane wave background. Therefore, in Section 6.1 we will first analyze how the boundary of $AdS_5 \times S^5$ approaches the plane wave boundary in the limiting process. The analysis will be carried out in the two sets of coordinates introduced in [21] and [22] which from now on we denote as BMN and BN coordinates respectively. In Section 6.2 we will then determine all geodesics in $AdS_5 \times S^5$ and in the plane wave. The boundary reaching null geodesics in both spacetimes will then be of particular interest for us because they are the light rays traveling between bulk and boundary. Locally, $AdS_5 \times S^5$ geodesics converge to plane wave geodesics because the Penrose limit which connects both spacetimes is realized by sending a parameter (that the metric depends on) to infinity. But our results will be useful for analyzing global aspects. In particular, in Section 6.3 we will work out how differently some null geodesics approach in both spaces their corresponding conformal boundary and in which sense these $AdS_5 \times S^5$ null geodesics approach in the Penrose limit the null geodesics running to the plane wave conformal boundary. This Chapter is based on our work [62].

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[1]From now on we will neglect that the two separate points i^- and i^+ of timelike past and future infinity are part of the conformal boundary.

6.1 Common description of the conformal boundaries

In this Section we will relate the conformal boundaries of $\text{AdS}_5 \times \text{S}^5$ and of the plane wave in a suitable coordinate system. Since the angular coordinate ψ in the coordinates (4.80) for $\text{AdS}_5 \times \text{S}^5$ is constrained by $-\pi \leq \psi \leq \pi$, one finds from (4.82) that the coordinates z^+ and z^- are restricted to

$$R^2 z^+ - \pi R^2 \leq z^- \leq R^2 z^+ + \pi R^2 \ , \tag{6.1}$$

where we have set $R = R_1 = R_2$ for the radii of AdS_5 and of S^5 respectively. This is a strip in the (z^+, z^-)-plane bounded by the two parallel straight lines with slope R^2 and crossing the z^+-axis at $-\pi$ and π, respectively. For $R \to \infty$ this strip becomes the coordinate range $-\infty < z^- < \infty$, $-\pi \leq z^+ \leq \pi$. Taking the limit for the metric, the identification of the two boundaries of the strip is given up, and it makes sense to extend to the whole (z^+, z^-)-plane. If one wants to avoid the restriction to the strip already for finite R, one has to puncture S^5 at its poles and to go then to the universal covering obtained by allowing ψ to take any real value.

The sequence of coordinate transformations, done in [22] to analyze the conformal boundary of the plane wave geometry (6.17), can be summarized as follows. Writing $\mathrm{d}\vec{z}^2 = z^2 \, \mathrm{d}\Omega_7^2$, the Ω_7 coordinates remain untouched. Then in a first step one transforms[2] in the patch $z^+ \in (-\frac{\pi}{2}, \frac{\pi}{2})$ the coordinates z^+, z^-, z to θ, φ, ζ

$$
\begin{aligned}
\cot \theta &= \frac{\left((1 - z^2) \tan z^+ - 4 z^-\right) \cos z^+}{2z} \ , \\
\tan \tfrac{\varphi \pm \zeta}{2} &= \frac{1}{2}(1 + z^2) \tan z^+ + 2 z^- \pm \frac{z}{\sin \theta \cos z^+} \ .
\end{aligned}
\tag{6.2}
$$

The new coordinates are constrained by

$$0 \leq \theta \leq \pi \ , \qquad 0 \leq \zeta \leq \pi \ , \qquad |\varphi \pm \zeta| \leq \pi \ . \tag{6.3}$$

The second step uses the periodicity properties of the trigonometric functions to glue the other z^+-strips, resulting in the final coordinate range

$$0 \leq \theta \leq \pi \ , \qquad 0 \leq \zeta \leq \pi \ , \qquad -\infty < \varphi < \infty \ . \tag{6.4}$$

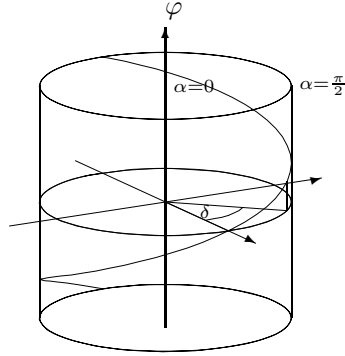[2] Note that our definitions for $z^\pm$ follow [21] and thus slightly differ from [22].

Figure 6.1: Part of the boundary of $AdS_5 \times S^5$ and of the plane wave in BN coordinates $(\delta = \beta + \pi)$

Then the plane wave metric, up to a conformal factor, turns out to be that of the Einstein static universe $\mathbb{R} \times S^9$. The analysis of singularities of the conformal factor, determining the conformal boundary of the plane wave, becomes most transparent after a change of parameterization of S^9. Let denote $(z_1, z_2, \vec{z})$ Cartesian coordinates in an embedding $\mathbb{R}^{10}$, then the parameterization by θ, ζ is related to that by α, β via[3]

$$z_1 = \cos\zeta = \sin\alpha\cos\beta \ ,$$
$$z_2 = \cos\theta\sin\zeta = \sin\alpha\sin\beta \ ,$$
$$|\vec{z}| = \sin\theta\sin\zeta = \cos\alpha \ .$$

The range for α, β is

$$0 \leq \alpha \leq \frac{\pi}{2} \ , \qquad 0 \leq \beta \leq 2\pi \ . \tag{6.5}$$

Now the plane wave metric in these BN coordinates takes the form [22]

$$ds_{\mathrm{pw}}^2 = \frac{1}{|\, e^{i\varphi} + \sin\alpha\, e^{i\beta}\,|^2} \left(-d\varphi^2 + d\alpha^2 + \sin^2\alpha\, d\beta^2 + \cos^2\alpha\, d\Omega_7^2 \right) \ . \tag{6.6}$$

The conformal factor is singular if and only if $\alpha = \frac{\pi}{2}$ and $\varphi = \beta + (2k+1)\pi$, $k \in \mathbb{Z}$. Since at $\alpha = \frac{\pi}{2}$ the S^7 part due the $\cos^2\alpha$ factor in front of $d\Omega_7^2$ shrinks to a point, the conformal boundary[4] of the plane wave is one-dimensional, see also Fig. 6.1

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[3]We shift α to $\alpha - \frac{\pi}{2}$ and β to $\beta - \pi$ relative to [22].

[4]Here and for the $AdS_5 \times S^5$ case below, while speaking about the conformal boundary, we omit the two isolated points $i^{\pm}$ for timelike infinity.

To avoid confusion in comparing Fig. 6.1 with similar looking pictures for $\mathrm{AdS}_5 \times \mathrm{S}^5$, where the half of some Einstein static universe is depicted, it is appropriate to stress that Fig. 6.1 represents the whole Einstein static universe $R \times S^9$ although the radius variable of the cylinder runs from zero to $\frac{\pi}{2}$ only. This range for α is due to its special role in the parameterization of S^9 in (6.5).

The coordinate transformations just discussed for the identification of the conformal boundary of the plane wave of course can also be applied to the $\mathrm{AdS}_5 \times \mathrm{S}^5$ metric. A priori these new coordinates are not a favourite choice to give any special insight into the $\mathrm{AdS}_5 \times \mathrm{S}^5$ geometry. In particular they are not well suited to find the conformal boundary.

But we can turn the argument around. Since we know already the conformal boundary of $\mathrm{AdS}_5 \times \mathrm{S}^5$, we can look where this boundary is situated in the new coordinates and hope to find some illuminating picture for its degeneration in the $R \to \infty$ limit which produces the plane wave metric.

As we have discussed in connection with the metrics (4.26) and (4.80), the conformal boundary of respectively AdS_{d+1} and $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ is at $\rho \to \infty$ with all other coordinates kept fixed at arbitrary finite values.

Translating this into the coordinates (4.82), (4.84) it is at $r \to \infty$ and z^+, z^-, y, ω_i, $\tilde{\omega}_{i'}$, $i, i' = 1, \ldots 3$ fixed at arbitrary finite values. Before applying (6.2) we define z for the $\mathrm{AdS}_5 \times \mathrm{S}^5$ case by

$$r = z \cos \chi \, , \qquad y = z \sin \chi \, , \tag{6.7}$$

i. e. $z^2 = r^2 + y^2$. In the following coordinate transformation according to (6.2) χ, ω_i, $\tilde{\omega}_{i'}$ remain untouched. The conformal $\mathrm{AdS}_5 \times \mathrm{S}^5$ boundary is now at $z \to \infty$, $\chi \to 0$, z^+, z^-, ω_i, $\tilde{\omega}_{i'}$ fixed at arbitrary finite values. The expansion of the first line of (6.2) yields

$$\cot \theta = -\frac{z}{2} \sin z^+ + \mathcal{O}(z^{-1}) \, . \tag{6.8}$$

From this one finds (as above we again start with $|z^+| \leq \frac{\pi}{2}$ and glue the other z^+ patches afterwards)

$$\lim_{\substack{z \to \infty \\ z^+ > 0}} \theta = \pi \, , \qquad \lim_{\substack{z \to \infty \\ z^+ < 0}} \theta = 0 \, , \qquad \lim_{\substack{z \to \infty \\ z^+ = 0}} \theta = \frac{\pi}{2} \, . \tag{6.9}$$

Furthermore, by coupling $z^+ \to 0$ in a suitable way with $z \to \infty$ one can reach any $\theta \in (0, \pi)$

$$\lim_{\substack{z \to \infty \\ z^+ = c/z}} \theta = \arctan\left(-\frac{2}{c}\right) \, . \tag{6.10}$$

In the second equation of (6.2) one has to insert

$$\frac{1}{\sin\theta} = \frac{z}{2}|\sin z^+|\sqrt{1 + \frac{4}{z^2\sin^2 z^+}} \ . \tag{6.11}$$

At fixed z^+ the expansion of the second equation of (6.2) then reads

$$\tan\tfrac{\varphi\pm\zeta}{2} = \left(1 \pm \epsilon(\sin z^+)\right)\frac{z^2}{2}\tan z^+ \pm \frac{1}{|\sin z^+|\cos z^+} + \mathcal{O}(z^{-1}) \ , \tag{6.12}$$

where ϵ is the sign function defined in (2.15). From the above result one finds (again for $|z^+| \leq \frac{\pi}{2}$)

$$\left\{\begin{array}{l} z^+ < 0 \quad (\text{i. e. } \theta \to 0) \\ z^+ > 0 \quad (\text{i. e. } \theta \to \pi) \end{array}\right\} \implies \lim_{z\to\infty}\tan\tfrac{\varphi+\zeta}{2} = \left\{\begin{array}{c} \text{finite} \\ \infty \end{array}\right\}, \quad \lim_{z\to\infty}\tan\tfrac{\varphi-\zeta}{2} = \left\{\begin{array}{c} \infty \\ \text{finite} \end{array}\right\}. \tag{6.13}$$

In addition one gets for $z \to \infty$ coupled as in (6.10) with $z^+ \to 0$ that (6.11) becomes

$$\frac{1}{\sin\theta} = \sqrt{\frac{c^2}{4} + 1} + \mathcal{O}(z^{-1}) \ . \tag{6.14}$$

The second equation of (6.2) then reads

$$\tan\tfrac{\varphi\pm\zeta}{2} = z\left(\pm\sqrt{\frac{c^2}{4} + 1} + \frac{c}{2}\right) + \mathcal{O}(z^0) \ , \tag{6.15}$$

such that one gets in this case

$$0 < \theta < \pi \implies \lim_{\substack{z\to\infty \\ z^+=c/z}}\tan\tfrac{\varphi\pm\zeta}{2} = \pm\infty \ . \tag{6.16}$$

Putting together (6.9)-(6.16), we see that in the projection onto the three coordinates φ, θ, ζ the conformal boundary of the $(|z^+| < \frac{\pi}{2})$-patch of $\mathrm{AdS}_5 \times \mathrm{S}^5$ is mapped to the one-dimensional line starting at $(\varphi, \zeta, \theta) = (-\pi, 0, 0)$, running first with $\theta = 0$ and $\zeta - \varphi = \pi$ to $(\varphi, \zeta, \theta) = (0, \pi, 0)$, then with $\varphi = 0$ and $\zeta = \pi$ to $(\varphi, \zeta, \theta) = (0, \pi, \pi)$ and finally with $\theta = \pi$ and $\varphi + \zeta = \pi$ to $(\varphi, \zeta, \theta) = (\pi, 0, \pi)$, see also Fig. 6.2

Translating this via (6.5) into the coordinates (φ, α, β) we find the line[5] $\alpha = \frac{\pi}{2}$, $\beta = \pi + \varphi$, $-\pi < \varphi < +\pi$. After gluing the other z^+-patches we can conclude:

---

[5]Note that the piece from $(\varphi, \zeta, \theta) = (0, \pi, 0)$ to $(\varphi, \zeta, \theta) = (0, \pi, \pi)$ with $\varphi = 0$ and $\zeta = \pi$ is mapped to one point $(\varphi, \alpha, \beta) = (0, \frac{\pi}{2}, \pi)$.
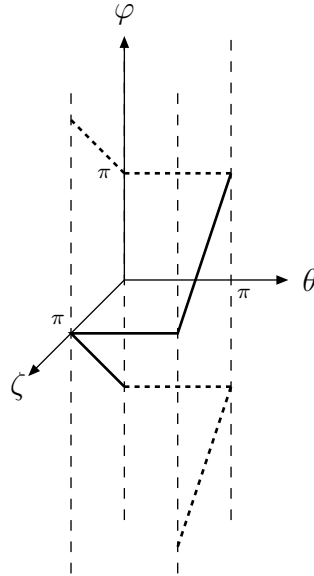
Figure 6.2: Part of the boundary of $AdS_5 \times S^5$ and of the plane wave in (φ, ζ, θ) coordinates

The projection onto the coordinates (φ, α, β) of the conformal boundary of $AdS_5 \times S^5$ coincides with that of a part of the conformal boundary of the 10-dimensional plane wave (see (4.41) with $H(z^+, z) = -\delta_{ij} z^i z^j$)

$$ds^2_{pw} = -4 \, dz^+ \, dz^- - \vec{z}^2 (dz^+)^2 + d\vec{z}^2 \ . \tag{6.17}$$

That in this projection only a part of the plane wave boundary line appears as the $AdS_5 \times S^5$ boundary is due to the restriction to the $AdS_5 \times S^5$-strip (6.1). Note that this restriction can be circumvented as discussed at the beginning of this Section.

Taking into account the other seven coordinates, the $AdS_5 \times S^5$ boundary is of course not one-dimensional. But by using the same coordinates both for $AdS_5 \times S^5$ *and* the plane wave, we now have visualized the degeneration of the conformal boundary in the process of approaching the plane wave limit. In the projection to three of the BN coordinates (φ, α, β) the boundary stays throughout this process at the same location. The extension in the remaining coordinates degenerates to a point in the limit.

The picture is more involved if one compares the two boundaries in the BMN coordinates (z^+, z^-, z). As noted in [22], due to the singularity of the coordinate transformation on the boundary line in (φ, α, β), the limits in (z^+, z^-, z) which map to this boundary

line are not unique. Besides

$$\text{Limit (i):} \quad z \to \infty \, , \qquad z^+, z^- = \text{finite} \, , \tag{6.18}$$

just discussed above, the second limit is

$$\text{Limit (ii):} \quad z^- \to \pm\infty \, , \qquad z^+, z = \text{finite} \, , \tag{6.19}$$

as trivially seen from (6.2).

Now the situation looks a little bit cumbersome. The conformal boundary of AdS$_5 \times$ S^5 is realized via limit (i), that of the plane wave via both limits (i) *and* (ii), although the the plane wave itself is a limit of AdS$_5 \times$ S^5.

To get a better understanding of this situation, we are now asking what happens in the $R \to \infty$ limit with geodesics, in particular null geodesics, which reach the conformal AdS$_5 \times$ S^5 boundary. After an explicit construction of all the geodesics both for AdS$_5 \times$ S^5 and the plane wave (6.17) in the next Section, we come back to this question.

6.2   Geodesics in AdS$_5 \times$ S^5 and in the plane wave

6.2.1   Geodesics in AdS$_5 \times$ S^5

We start with the AdS$_5 \times$ S^5 metric, where the AdS$_5$ part is given in the global by coordinates in the first line of (4.26)

$$\mathrm{d}s^2 = R^2(-\,\mathrm{d}t^2 \cosh^2 \rho + \mathrm{d}\rho^2 + \sinh^2 \rho \, \mathrm{d}\Omega_3^2 + \mathrm{d}\Omega_5^2) \, . \tag{6.20}$$

The geodesic equations for the AdS$_5$ and S^5 coordinates decouple. Geodesics on S^5 are great circles. Whether the geodesic in the total manifold AdS$_5 \times$ S^5 moves in S^5 or stays at a fixed S^5 position has consequences for the overall causal property (spacelike, timelike, null) only. There is no effect on the AdS$_5$ coordinates. Therefore, we can concentrate on the AdS$_5$ part. It can be regarded as a warped product of a two-dimensional space with coordinates t and ρ and a 3-dimensional sphere. In Appendix B.4 we present the details how the geodesic equations can be simplified in this case. The warp factor in front of the spherical part has only influence on the parameterization of the geodesics, but not on their shape. They are still given by great circles on the subsphere. The simplified

geodesic equations for t, ρ and f, where f captures the effects of the movement in the 3-dimensional unit sphere, are given by (B.32) and read

$$\ddot{t} + 2\dot{\rho}\dot{t}\tanh\rho = 0 \ , \tag{6.21}$$

$$\ddot{\rho} + (\dot{t}^2 - \dot{f}^2)\sinh\rho\cosh\rho = 0 \ , \tag{6.22}$$

$$\ddot{f} + 2\dot{\rho}\dot{f}\coth\rho = 0 \ . \tag{6.23}$$

To be more precise, $\dot{f}$ is given in terms of the metric $g(\Omega_3)_{mn}$ and the angle velocities $\dot{y}^m$ of the unit S^3 as

$$\dot{f}^2 = g(\Omega_3)_{mn}\dot{y}^m\dot{y}^n \ . \tag{6.24}$$

Hence, $f(\tau)$ itself is the angle between the position vectors along the geodesic at $\tau = 0$ and at τ and $\dot{f}$ has the interpretation as velocity in the 3-dimensional unit sphere. Remark that there is some freedom in the definition of f such that the above result corresponds to a particular choice, see Appendix B.4 for more details. Summarizing the discussion so far, the coordinates in the S^3 either remain constant ($\dot{f} = 0$) or describe a movement on a great circle ($\dot{f} \neq 0$).

We will now solve the geodesic equations for t, ρ and f. Straightforward integration of (6.21) and (6.23) yields

$$\dot{t} = \frac{b}{\cosh^2\rho} \ , \qquad \dot{f} = \frac{\tilde{b}}{\sinh^2\rho} \ , \qquad b, \tilde{b} \quad \text{constant} \ . \tag{6.25}$$

With (6.22) one derives an equation for ρ alone

$$\ddot{\rho} + \frac{b^2}{\cosh^3\rho}\sinh\rho - \frac{\tilde{b}^2}{\sinh^3\rho}\cosh\rho = 0 \ . \tag{6.26}$$

This equation can be integrated after multiplication with $\dot{\rho}$. If one introduces the integration constant c one finds

$$\dot{\rho}^2 = \frac{c}{R^2} - \frac{\tilde{b}^2}{\sinh^2\rho} + \frac{b^2}{\cosh^2\rho} \ . \tag{6.27}$$

This is precisely the condition for the parameter to be an affine one for the full metric (6.20). In fact, if we denote the coordinates on AdS$_5$ with x, the condition that τ is an affine parameter reads

$$G_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = R^2(-\dot{t}^2\cosh^2\rho + \dot{\rho}^2 + \dot{f}^2\sinh^2\rho) = c \ , \tag{6.28}$$

and one finds (6.27) with the help of (6.25).

Are all solutions of (6.27) also solutions of (6.26)? At first the constancy of the scalar product of the tangential vector with itself is of course a much weaker condition than the geodesic equations. But in writing down (6.27) we already have implied the geodesic equations for all coordinates, except for ρ. Under these circumstances, at least as long as $\dot{\rho} \neq 0$, the constant scalar product condition is equivalent to the geodesic equation for the last coordinate ρ. If $\rho = \text{const.} \neq 0$ it follows immediately from (6.22) that $\dot{t} = \dot{f} = \text{const.}$ If $\rho = 0$ the only possibilities are geodesics that move only in the time direction of AdS_5 or stay at a point.

Since $\dot{\rho}^2$ is a non-negative quantity, from (6.27) and

$$\frac{c}{R^2} - \frac{\tilde{b}^2}{\sinh^2 \rho} + \frac{b^2}{\cosh^2 \rho} \leq \frac{c}{R^2} + b^2 \ , \qquad \forall \rho \ ,$$

as a byproduct, we find a constraint on c and the integration constant b

$$\frac{c}{R^2} + b^2 \geq 0 \ . \tag{6.29}$$

For further analyzing the consequences of the positiveness of both sides of (6.27) we introduce the abbreviations

$$A = \frac{c}{R^2} \ , \qquad B = b^2 + \frac{c}{R^2} - \tilde{b}^2 \ , \qquad C = -\tilde{b}^2 \ . \tag{6.30}$$

Then first of all, by these definitions and the inequality (6.29) the constants A, B, C are universally constrained by

$$C \leq 0 \ , \qquad B \geq A + C \ , \qquad B \geq C \ . \tag{6.31}$$

In addition, checking whether there are real ρ-values for which the R. H. S. of (6.27) is non-negative, it turns out that only four classes of ranges[6] of the constants A, B, C are allowed. Integrating case by case first (6.27) and then (6.25) for the four classes one finds:

type I

$$A > 0 \ ,$$
$$0 \leq \frac{\sqrt{B^2 - 4AC} - B}{2A} \leq \sinh^2 \rho \ , \tag{6.32}$$

---

[6]The special case $A = B = C = 0$ corresponds to a point, not to a curve.

$$\rho = \mathrm{arsh}\,\sqrt{\frac{1}{4A}\left(\mathrm{e}^{\pm 2\sqrt{A}(\tau+\tau_0)} + (B^2 - 4AC)\,\mathrm{e}^{\mp 2\sqrt{A}(\tau+\tau_0)}\right) - \frac{B}{2A}}\,,$$

$$t = \pm\arctan\left(\frac{\mathrm{e}^{\pm 2\sqrt{A}(\tau+\tau_0)} + 2A - B}{2b\sqrt{A}}\right) + t_0\,, \qquad (6.33)$$

$$f = \pm\arctan\left(\frac{\mathrm{e}^{\pm 2\sqrt{A}(\tau+\tau_0)} - B}{2\tilde{b}\sqrt{A}}\right) + f_0\,,$$

type II

$$A < 0\,, \qquad B^2 - 4AC > 0\,, \qquad B > 0\,,$$

$$0 \le \frac{B - \sqrt{B^2 - 4AC}}{-2A} \le \sinh^2\rho \le \frac{B + \sqrt{B^2 - 4AC}}{-2A}\,, \qquad (6.34)$$

$$\rho = \mathrm{arsh}\,\sqrt{\frac{1}{-2A}\left(B \pm \sqrt{B^2 - 4AC}\,\sin(2\sqrt{-A}(\tau+\tau_0))\right)}\,,$$

$$t = \pm\frac{1}{2}\,\mathrm{arccot}\left(\frac{2\sqrt{-A}\,b\cos(2\sqrt{-A}(\tau+\tau_0))}{\sqrt{B^2 - 4AC} \pm (B - 2A)\sin(2\sqrt{-A}(\tau+\tau_0))}\right) + t_0\,, \qquad (6.35)$$

$$f = \pm\frac{1}{2}\,\mathrm{arccot}\left(\frac{2\sqrt{-A}\,\tilde{b}\cos(2\sqrt{-A}(\tau+\tau_0))}{\sqrt{B^2 - 4AC} \pm B\sin(2\sqrt{-A}(\tau+\tau_0))}\right) + f_0\,,$$

type III

$$A = 0\,, \qquad B > 0\,,$$

$$0 \le \frac{-C}{B} \le \sinh^2\rho\,, \qquad (6.36)$$

$$\rho = \mathrm{arsh}\,\sqrt{B(\tau+\tau_0)^2 - \frac{C}{B}}\,,$$

$$t = \arctan\left(\frac{B(\tau+\tau_0)}{b}\right) + t_0\,, \qquad (6.37)$$

$$f = \arctan\left(\frac{B(\tau+\tau_0)}{\tilde{b}}\right) + f_0\,,$$

type IV

$$A < 0\,, \qquad B^2 - 4AC = 0\,, \qquad B \ge 0\,,$$

$$\sinh^2\rho = \frac{B}{-2A}\,, \qquad (6.38)$$

$$\rho = \text{arsh} \sqrt{\frac{B}{-2A}} \;,$$
$$t = \sqrt{-A}\tau + t_0 \;,$$
$$f = \pm\sqrt{-A}\tau + f_0 \;.$$

$$(6.39)$$

Perhaps it is useful to stress, that in the absence of any movement in the S^3, i. e. for $C = 0$, the formulas (6.33), (6.35) and (6.37) for ρ simplify to

type I with C=0

$$\rho = \text{arsh}\left(\sqrt{\left|\frac{B}{A}\right|}\,|\sinh\left(\sqrt{A}(\tau + \tau_0')\right)|\right) \;,$$

$$(6.40)$$

type II with C=0

$$\rho = \text{arsh}\left(\sqrt{\left|\frac{B}{-A}\right|}\,|\sin\left(\sqrt{-A}(\tau + \tau_0')\right)|\right) \;,$$

$$(6.41)$$

type III with C=0

$$\rho = \text{arsh}\left(\sqrt{B}|\tau + \tau_0|\right) \;.$$

$$(6.42)$$

The $\pm$ alternative in (6.33) and (6.35) has been absorbed into the shift of the integration constant τ_0 to τ_0'.

The causal properties of the geodesics and their relation to the conformal boundary (note footnote 5) can be summarized in the following table.

type	causal properties w. r. t. AdS_5	causal properties w. r. t. $AdS_5 \times S^5$	reaches conf. bound. of $AdS_5 \times S^5$
I	space-like	space-like	yes
II	time-like	all	no
III	null	null or space-like	yes
IV	time-like	all	no

For later use it is important to stress, that null geodesics in the sense of full $AdS_5 \times S^5$ reaching the boundary have to be of *type III*. For them no movement in S^5 is allowed while a movement in S^3 is possible as long as $b^2 > \tilde{b}^2$.

6.2.2 Geodesics in the plane wave

Here the metric is given by the 10-dimensional version of (4.41) with $H(z^+, z) = -\delta_{ij} z^i z^j$

$$ds^2 = -4 \, dz^+ \, dz^- - \vec{z}^2 (dz^+)^2 + d\vec{z}^2 \ , \tag{6.43}$$

such that one finds for the geodesic equations (B.23) with the Christoffel connection (4.42)

$$\ddot{z}^+ = 0 \ , \tag{6.44}$$

$$\ddot{z}^- + \frac{1}{2} \dot{z}^+ \frac{d}{d\tau} \vec{z}^2 = 0 \ , \tag{6.45}$$

$$\ddot{z}^i + (\dot{z}^+)^2 z^i = 0 \ . \tag{6.46}$$

Then (6.44) implies linear dependence of z^+ on the the affine parameter τ

$$z^+ = \alpha \tau + z_0^+ \ . \tag{6.47}$$

Obviously now the geodesics fall into two classes, *type A* with $\alpha = 0$ and *type B* with $\alpha \neq 0$.

type A

$$z^+ = \text{const.} \ , \qquad z^- = \beta \tau + z_0^- \ , \qquad z^i = \gamma^i \tau + z_0^i \ . \tag{6.48}$$

The scalar product of their tangential vector with itself is given by $(\gamma^i)^2$ (see Appendix (B.5) for a relation between the geodesic and the chordal distances). This implies:

All *type A* geodesics are null or space-like. Space-like *type A* geodesics reach infinity in the transversal coordinates $\vec{z}$. *Type A* null geodesics are given by constant z^+ and z^i as well as z^- running between $\pm \infty$.

If we choose $\alpha \neq 0$ in (6.47) we find after the integration of (6.45) and of (6.46) the second type of geodesics given by

type B

$$z^+ = \alpha \tau + z_0^+ \ ,$$
$$z^- = \frac{1}{8} \sum_i (\beta^i)^2 \sin\left(2\alpha(\tau + \tau_0^i)\right) + \gamma \tau + z_0^- \ , \tag{6.49}$$
$$z^i = \beta^i \sin\left(\alpha(\tau + \tau_0^i)\right) \ .$$

The scalar product of the tangential vector with itself is now equal to $-4\alpha\gamma$ (see Appendix (B.5) for a relation between the geodesic and the chordal distances), and we conclude:

All *type B* geodesics either stay at $z^i = 0$ (for $\beta^i = 0$) or oscillate in the transversal coordinates z^i (for $\beta^i \neq 0$). All space or time-like *type B* geodesics ($\gamma \neq 0$) reach $\pm\infty$ both in z^+ and z^-. *Type B* null geodesics ($\gamma = 0$) reach $\pm\infty$ *only* with respect to z^+. Furthermore, they stay at fixed $\vec{z}$ and z^- ($\vec{\beta} = 0$) or oscillate both in z^i and z^- ($\beta^i \neq 0$).

In conclusion null geodesics reaching the conformal boundary of the plane wave, see (6.18), (6.19), are necessarily of *type A*. There are no null geodesics reaching the conformal boundary within the asymptotic regime of limit (i).

Closing this Section we comment on a simple discussion of the plane wave null geodesics in using the BN coordinates of (6.6). In general null geodesics are invariant under a Weyl transformation. Such a transformation only effects the choice of affine parameters along the null geodesics. Null geodesics with respect to (6.6) without the Weyl factor are given by great circles in S^9 accompanied by a compensating movement along the time-like direction φ. If we discuss S^9 as an embedding in $\mathbb{R}^{10}$, reaching $\alpha = \frac{\pi}{2}$ is equivalent to reaching the (z_1, z_2)-plane. There are of course great circles within this plane. They correspond to null geodesics either winding at $\alpha = \frac{\pi}{2}$ in constant distance to the conformal plane wave boundary around the cylinder in Fig. 6.1 up to $\varphi \to \pm\infty$ or they wind in the orthogonal direction crossing the conformal plane wave boundary. In the sense of $\mathbb{R} \times S^9$ there is nothing special with such a crossing. But going back to the metric including the Weyl factor, starting from an inside point, the boundary is reached at infinite affine parameter. Furthermore, there are of course great circles staying completely away from the (z_1, z_2)-plane (i. e. $\alpha = \frac{\pi}{2}$). They correspond to null geodesics generically oscillating in $0 < \alpha < \frac{\pi}{2}$ and running up to $\varphi \to \pm\infty$. Finally, great circles can also intersect the (z_1, z_2)-plane. Then they correspond to null geodesics oscillating in α and touching $\alpha = \frac{\pi}{2}$. Obviously some of them reach the conformal boundary line of the plane wave. According to the above analysis in BMN coordinates they are of *type A*, too.

6.3 Conformal boundaries and geodesics

As discussed in Section 6.2.1, only null-geodesics of *type III* reach the conformal boundary of $AdS_5 \times S^5$. They necessarily stay at fixed S^5-position. Translating (6.37) into the

coordinates of (4.82) we get with $R = R_1 = R_2$

$$
\begin{aligned}
z^+ &= \frac{1}{2}\Big(\arctan\Big(\frac{B(\tau + \tau_0)}{b}\Big) + t_0 + \psi \Big) \,, \\
z^- &= \frac{R^2}{2}\Big(\arctan\Big(\frac{B(\tau + \tau_0)}{b}\Big) + t_0 - \psi \Big) \,, \\
r &= R \operatorname{arsh} \sqrt{B(\tau + \tau_0)^2 - \frac{C}{B}} \,, \\
f &= \arctan\Big(\frac{B(\tau + \tau_0)}{\tilde{b}}\Big) + f_0 \,, \\
y &= R\vartheta \,.
\end{aligned}
\tag{6.50}
$$

Our goal is to find in the $R \to \infty$ limit a correspondence to null geodesics of the plane wave. Therefore, our $AdS_5 \times S^5$ geodesics have to stay at least partially within the range of finite z^+, z^-, r, y. Taking $R \to \infty$ at fixed τ would send all z^- to infinity. But of course the affine parameter itself is determined only up to a constant rescaling. Therefore, the best procedure is to eliminate the affine parameter completely.

First from (6.50) we conclude, that along the full range of a *type III* null geodesic, i. e. for $(-\infty < \tau < \infty)$, the coordinate z^+ runs within an interval of length $\frac{\pi}{2}$: $z^+ \in \big(\frac{1}{2}(t_0 + \psi) - \frac{\pi}{4}, \frac{1}{2}(t_0 + \psi) + \frac{\pi}{4}\big)$ and z^- runs within an interval of length $\frac{\pi}{2}R^2$: $z^- \in \big(R^2(\frac{t_0 - \psi}{2} - \frac{\pi}{4}), R^2(\frac{t_0 - \psi}{2} + \frac{\pi}{4})\big)$. To ensure that the z^- interval for $R \to \infty$ stays at least partially within the range of finite values both endpoints of the interval have to have the opposite sign. Thus, we have to restrict t_0 and ψ by

$$
-\frac{\pi}{2} < t_0 - \psi < \frac{\pi}{2} \,.
\tag{6.51}
$$

In addition one has universally

$$
|f(\tau = +\infty) - f(\tau = -\infty)| = \pi \,.
\tag{6.52}
$$

As already mentioned around (6.24), f can be understood as the angle along the great circle in S^3 on which our null geodesics is running. Therefore, for *type III* geodesics the positions for $\tau = -\infty$ and $\tau = +\infty$ within the S^3 are always antipodal to each another.[7]

After these preparations we now eliminate the affine parameter and express z^+, r and

---

[7] In the limiting case, where the null geodesics goes through $r = 0$, f becomes a step function.

f in terms of z^- (note that for *type III* we have $A = 0$ and $B = b^2 - \tilde{b}^2$)

$$z^+ = \frac{z^-}{R^2} + \psi \ ,$$

$$r = R \operatorname{arsh} \sqrt{\frac{\tan^2(\frac{2z^-}{R^2} - t_0 + \psi) + \frac{\tilde{b}^2}{b^2}}{1 - \frac{\tilde{b}^2}{b^2}}} \ , \tag{6.53}$$

$$f = f_0 + \arctan\left(\frac{b}{\tilde{b}} \tan\left(\frac{2z^-}{R^2} - t_0 + \psi\right)\right) \ ,$$

$$y = R\vartheta \ .$$

The minimal value for r is

$$r_{\min} = R \operatorname{arsh}\left(\frac{|\frac{\tilde{b}}{b}|}{\sqrt{1 - \frac{\tilde{b}^2}{b^2}}}\right) \ . \tag{6.54}$$

Since we insist on finite $r_{\min}$ for $R \to \infty$ we have to rescale ($r_0 = \lim_{R\to\infty} r_{\min}$)

$$\frac{|\tilde{b}|}{|b|} = \frac{r_0}{R} \ . \tag{6.55}$$

Although we have now realized finite $r_{\min}$, the z^- value where $r_{\min}$ is reached stays finite for $R \to \infty$ only if (6.51) is replaced by the stronger rescaling condition

$$t_0 - \psi = \frac{a}{R^2} \ . \tag{6.56}$$

Altogether, to stay at least with part of the *type III* null geodesics within the range of finite BMN coordinates, it is mandatory to perform the rescalings (6.55), (6.56) and to keep y fixed. The remaining parameters replacing $t_0, \psi, b, \tilde{b}, f_0, \vartheta$ are ψ, a, b, r_0, f_0, y.

Considering now at fixed z^- the $R \to \infty$ limit of (6.53) one arrives at

$$z^+ = \psi + \mathcal{O}(R^{-2}) \ ,$$

$$r = r_0 + \mathcal{O}(R^{-2}) \ ,$$

$$f = f_0 + \mathcal{O}(R^{-2}) \ , \tag{6.57}$$

$$y = \mathrm{const.}$$

Constant r, y via (6.7) give constant z. In addition, constant f, i. e. no movement in the S^3, and the a priori absence of any movement in S^5 leads to constant $\vec{z}$. This together with the constancy of z^+ implies:

An $\text{AdS}_5\times\text{S}^5$ null geodesics, reaching the conformal boundary, for any finite z^--interval at $R \to \infty$ converges uniformly to a *type A* null geodesics of the plane wave.

However, the approach of the $\text{AdS}_5 \times \text{S}^5$ null geodesics to the conformal boundary of $\text{AdS}_5 \times \text{S}^5$ is realized within the asymptotic regime of limit (i), see (6.18), (6.50), but that of the plane wave null geodesics within the regime of limit (ii), see (6.19) and text after (6.48). That means even for large R, after a region of convergence, on their way to the boundary they diverge from one another at the very end (in the z^+, z^-, z coordinates under discussion).

In a global setting the situation is most simply illustrated for *type III* null geodesics crossing the origin of the transverse BMN coordinates $\vec{z}$, i. e. $r_0 = y = 0$. We also put $a = 0$, the case $a \neq 0$ can be simply recovered by the replacement $z^- \to z^- - \frac{a}{2}$. Then first of all z^- runs between $\pm\frac{\pi}{4}R^2$. Furthermore, (6.53) implies

$$
\begin{aligned}
\frac{r}{R} &= F\left(\tfrac{z^-}{R^2}\right) , \quad \text{with} \quad F(z) = \text{arsh}\,|\tan(2z)| , \\
f &= f_0 \pm \frac{\pi}{2}\epsilon(z^-) ,
\end{aligned}
\tag{6.58}
$$

where ϵ is the sign function (2.15). The plane wave geodesic is at $z = y = r = 0, -\infty < z^- < \infty$. It is the uniform limit for $R \to \infty$ in the region $|z^-| < R^{1-\varepsilon}$, $\varepsilon > 0$. This convergence is due to the different powers of R on the l.h.s. and in the argument of the function F on the R. H. S. of (6.58), see also Fig. 6.3.

The picture in Fig. 6.3 has to be completed by the freedom to choose a point on S^3 to fix the direction in the space of the $\vec{z}$ coordinates. This completely specifies the *type III* null geodesics under discussion. Then the conformal boundary reaching null geodesics of $\text{AdS} \times \text{S}^5$ crossing the origin of the transversal BMN coordinates $\vec{z} = 0$ form a cone with base S^3. The three parameters to specify the S^3 position together with ψ nicely correspond to the four-dimensionality of the $\text{AdS} \times \text{S}^5$ boundary. In the $R \to \infty$ limit this cone degenerates.
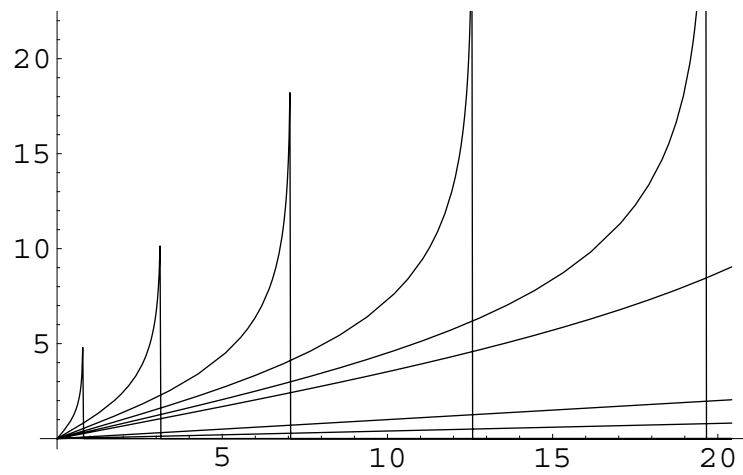
Figure 6.3: Approach of boundary reaching $AdS_5 \times S^5$ null geodesics to a boundary reaching null geodesics of the plane wave. The plane wave null geodesics runs along the horizontal axis up to infinity. The plot shows r versus z^- for $AdS_5 \times S^5$ null geodesics (6.58) in the cases $R = 1, 2, 3, 4, 5, 6, 20, 50$.

Chapter 7

The scalar bulk-to-bulk propagator in AdS × S and in the plane wave

In this Chapter we will focus on the scalar bulk-to-bulk propagator $G(z, z')$ in $\text{AdS}_{d+1} \times \text{S}^{d'+1}$ defined as solution of a differential equation similar to (5.9) in pure AdS_{d+1}. From the first sight it might be less obvious why we deal especially with this propagator. It appears to be more natural to analyze the bulk-to-boundary propagator instead, because it determines the bulk supergravity fields from their boundary values via (5.6) and thus enter directly the holographic description. Furthermore, it was shown [56] that the bulk-to-bulk propagator is no longer needed in computing any correlation functions in the AdS/CFT setup.

However, the bulk-to-boundary propagator cannot be directly used to get information about Penrose limiting process from $\text{AdS}_5 \times \text{S}^5$ to the 10-dimensional plane wave. As we have learned in Section 6.3, only a limited region around the particular null geodesics, at which the Penrose limit is taken, converges to the plane wave spacetime. The $\text{AdS}_5 \times \text{S}^5$ boundary is not part of the plane wave spacetime. Hence, a reasonable limit (where both points are part of the plane wave spacetime) of the bulk-to-boundary propagator in AdS_5 does not exist.

The situation is different for the bulk-to-bulk propagator in the full $\text{AdS}_5 \times \text{S}^5$ spacetime. Here one can choose both points within the region of convergence such that the limit of the propagator is well defined. The discussion in Section 5.1 has shown that its study can be useful to get information about a holographic setup. There we have seen that the bulk-to-boundary propagator in AdS_{d+1} (see (5.7)) is related to the bulk-to-bulk

propagator (see (5.10)) if the boundary has codimension one compared to the bulk dimension. It was sufficient to work with the AdS_{d+1} propagators and deal with the additional dimensions from the sphere in a Kaluza-Klein decomposition. In this case the relation (5.12) holds between the bulk-to-bulk and the bulk-to-boundary propagator.

The situation is somewhat different in case of the BMN correspondence, where the direct product structure of the underlying spacetime breaks down and one has to deal with all dimensions on equal footing. The hope is that, depending on how holography is effectively realized in the plane wave, one can nevertheless find some relation that allows one to compute the appropriate bulk-to boundary from the bulk-to-bulk propagator in the plane wave similarly to (5.12) in the case of AdS_{d+1}. Furthermore, the scalar bulk-to-bulk propagator in the BMN plane wave has been constructed in [116] by a direct approach leaving the issue of its derivation via a limiting process as an open problem.

Motivated by this perspective, and because it is an interesting problem in its own right, we will present in this Chapter the construction of the scalar propagator on $\text{AdS}_{d+1} \times \text{S}^{d'+1}$ spaces with the respective embedding radii R_1 and R_2, and then discuss for $\text{AdS}_5 \times \text{S}^5$ and $R_1 = R_2$ its behaviour in the Penrose limiting process to the 10-dimensional plane wave. Allowing for generic dimensions d and d' as well as generic curvature radii R_1 and R_2 is very helpful to understand the general mechanism for the construction of the propagator. Of course only some of these spacetimes are parts of consistent supergravity backgrounds, see Sections 4.1.1 and 4.2.1. This Chapter is mainly based on our work [60].

In Section 7.1 we will focus on the differential equation defining the scalar propagator in generic $\text{AdS}_{d+1} \times \text{S}^{d'+1}$ backgrounds. Within this Section we will be able to find the propagator in conformally flat situations, i. e. for equal embedding radii of AdS_{d+1} and $\text{S}^{d'+1}$ and for masses corresponding to Weyl invariant actions. For a comparison we will also discuss the case of pure AdS_{d+1}. In the Weyl invariant case, by using the conformal transformation that relates the Poincaré patch to flat space, the solutions of the differential equations can alternatively be obtained from the well-known propagators in flat space. We will demonstrate this in Section 7.2. However, since only patches of AdS_{d+1} and $\text{AdS}_{d+1} \times \text{S}^{d'+1}$ are conformal to flat space, it is not appropriate for a discussion of global aspects of the solutions. But as we have seen in Sections 4.1.2 and 4.1.3, AdS_{d+1} and $\text{AdS}_{d+1} \times \text{S}^{d'+1}$ can be globally conformally mapped respectively to one half and to the complete ESU of corresponding dimension. We will use this to discuss some global issues of the propagators in Section 7.3. With the hope to get the propagator for generic masses,

in Section 7.4 we study its KK mode sum. We will be able to perform the sum, whenever a linear relation holds between the conformal dimension of the KK mode and the quantum number parameterizing the eigenvalue of the Laplacian on the sphere. Beyond the cases treated in the previous Sections this applies to certain additional mass values, but fails to solve the full generic problem. As a byproduct, the comparison with the result of Section 7.1 yields a theorem on the summation of certain products of Gegenbauer and Legendre functions.

In Section 7.5 we will apply the plane wave limit to $\mathrm{AdS}_5 \times \mathrm{S}^5$. We will explicitly show that the massless propagator indeed reduces to the expression of [116]. Furthermore, we will present the limit of the full differential equation which is fulfilled by the propagator of massive scalar fields given in [116].

7.1 The differential equation for the propagator and its solution

7.1.1 The scalar propagator on $\mathbf{AdS}_{d+1} \times \mathbf{S}^{d'+1}$

The scalar propagator is defined as the solution of

$$(\Box_z - M^2)G(z, z') = \frac{i}{\sqrt{-g}}\delta(z, z') \, , \tag{7.1}$$

with suitable boundary conditions at infinity. $\Box_z$ denotes the d'Alembert operator on $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$, acting on the first argument of the propagator $G(z, z')$. Again, we denote the coordinates referring to the AdS_{d+1} factor by x and those referring to the $\mathrm{S}^{d'+1}$ factor by y, i. e. $z = (x, y)$. In a continuation to Euclidean space the R. H. S. of (7.1) is changed to $-\frac{1}{\sqrt{g}}\delta(z, z')$ such that it is consistent with (5.9). The change is due to the procedure to fix the normalization by integrating the L. H. S. over space(time). In the Lorentzian case the additional factor $-i$ is generated by the required Wick-rotation.

We first look for solutions at $z \neq z'$ and discuss the behaviour at $z = z'$ afterwards. The constructions of AdS_{d+1} and $\mathrm{S}^{d'+1}$ via the embeddings in $\mathbb{R}^{2,d}$ and in $\mathbb{R}^{d'+2}$ respectively were presented in Sections 4.1.2 and 4.1.3.[1] The embeddings endow us with the chordal distances that can be used to measure distances between two points on the manifolds. For AdS_{d+1} and $\mathrm{S}^{d'+1}$ they are defined in 4.32 and 4.38 and denoted as $u(x, x')$ and $v(y, y')$,

---

[1] Remember that with AdS_{d+1} we always mean the universal covering, if not otherwise indicated.

respectively. The chordal distance u is a unique function of x and x' if one restricts oneself to the hyperboloid. On the universal covering it is continued as a periodic function. For later use we note that on the hyperboloid and on the sphere the antipodal points $\tilde{x}$ and $\tilde{y}$ to given points x and y are defined by changing the sign of the embedding coordinates X and Y respectively. From (4.32) and (4.38) one then finds with $\tilde{u} = u(x, \tilde{x}')$, $\tilde{v} = v(y, \tilde{y}')$

$$u + \tilde{u} = -4R_1^2 \ , \qquad v + \tilde{v} = 4R_2^2 \ . \tag{7.2}$$

Using the homogeneity and isotropy of both AdS_{d+1} and $\text{S}^{d'+1}$ it is clear that the propagator can depend on z, z' only via the chordal distances $u(x, x')$ and $v(y, y')$. Strictly speaking this at first applies only if AdS_{d+1} is restricted to the hyperboloid. Up to subtleties due to time ordering (see the end of Section 7.3) this remains true also on the universal covering. The d'Alembert operator then simplifies to

$$\begin{aligned}
\Box_z &= \Box_x + \Box_y \ , \\
\Box_x &= 2(d+1)\Big(1 + \frac{u}{2R_1^2}\Big)\frac{\partial}{\partial u} + \Big(\frac{u^2}{R_1^2} + 4u\Big)\frac{\partial^2}{\partial u^2} \ , \\
\Box_y &= 2(d'+1)\Big(1 - \frac{v}{2R_2^2}\Big)\frac{\partial}{\partial v} - \Big(\frac{v^2}{R_2^2} - 4v\Big)\frac{\partial^2}{\partial v^2} \ .
\end{aligned} \tag{7.3}$$

One can now ask for a solution of (7.1) at $z \neq z'$ that only depends on the total chordal distance $u + v$. Indeed, using (7.3), it is easy to derive that such a solution exists if and only if

$$R_1 = R_2 = R \ , \qquad M^2 = \frac{d'^2 - d^2}{4R^2} \ . \tag{7.4}$$

Furthermore, it is necessarily powerlike and given by

$$G(z, z') \propto (u + v)^{-\frac{d+d'}{2}} \ . \tag{7.5}$$

Extending this to $z = z'$ we find just the right power for the short distance singularity to generate the δ-function on the R. H. S. of (7.1). Hence, after fixing the normalization we end up with

$$G(z, z') = \frac{\Gamma(\frac{d+d'}{2})}{4\pi^{\frac{d+d'}{2}+1}} \frac{1}{(u + v + i\varepsilon(t, t'))^{\frac{d+d'}{2}}} \ . \tag{7.6}$$

Note that due to (7.2) besides the singularity at $z = z'$ there is another one at the total antipodal point where $z = \tilde{z}' = (\tilde{x}', \tilde{y}')$. We have introduced an $i\varepsilon$-prescription by replacing $u \to u + i\varepsilon$, where ε depends explicitly on time. We will comment on this in Section 7.3. In particular, we will see that on the universal covering of the hyperboloid

the singularity at the total antipodal point does not lead to an additional δ-source on the R. H. S. of (7.1).

Scalar fields with mass m^2 in AdS_{d+1} via the AdS/CFT correspondence are related to CFT fields with conformal dimension

$$\Delta_\pm(d, m^2) = \frac{1}{2}\left(d \pm \sqrt{d^2 + 4m^2 R_1^2}\right) . \tag{7.7}$$

Note that the exponent of $(u+v)$ in the denominator of the propagator (7.6) is just equal to $\Delta_+(d, M^2)$. From the AdS_{d+1} point of view the $(d+d'+2)$-dimensional mass M^2 is the mass of the KK zero mode of the sphere. We will come back to these issues in Section 7.4.

For completeness let us add another observation. Disregarding for a moment the source structure, under the conditions (7.4) there is a solution of (7.1), that depends only on $(u-v)$. The explicit form is

$$\tilde{G}(z, z') \propto \frac{1}{(u - v + 4R^2 + i\varepsilon(t, t'))^{\frac{d+d'}{2}}} . \tag{7.8}$$

It has the same asymptotic falloff as (7.6). But due to (7.2) it has singularities only at the semi-antipodal points where $z = z'_\mathrm{s} = (x', \tilde{y}')$ and $z = \tilde{z}'_\mathrm{s} = (\tilde{x}', y')$. We will say more on $\tilde{G}(z, z')$ in Sections 7.2 and 7.3.

At the end of this Subsection we give a simple interpretation of the conditions (7.4). We have already seen from the discussion referring to the $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ metric (4.39) that $R_1 = R_2$ is exactly the condition for conformal flatness of the complete product space $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ as a whole and that it is then conformal to the complete Einstein static universe. Furthermore, the mass condition just singles out the case of a scalar field coupled in Weyl invariant manner to the gravitational background. The corresponding D-dimensional action is

$$S = -\frac{1}{2}\int \mathrm{d}^D z \sqrt{-g}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{D-2}{4(D-1)}\mathcal{R}\phi^2\right] . \tag{7.9}$$

Inserting the constant curvature scalar $\mathcal{R}$ for $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ (4.33) with equal radii one gets for the mass just the value in (7.4).

Altogether in this Subsection we have constructed the scalar $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ propagator for the case of Weyl invariant coupling to the metric in conformally flat situations. The Weyl invariant coupled field is the natural generalization of the massless field in flat space.

7.1.2 A remark on the propagator on pure AdS$_{d+1}$

Having found for AdS$_{d+1}$ × S$^{d'+1}$ such a simple expression for the scalar propagator, one is wondering whether the well known AdS propagators can also be related to simple powers of the chordal distance.

In (5.10) we have given the general massive scalar propagator on pure AdS$_{d+1}$ space corresponding to the two distinct conformal dimensions $\Delta_\pm$ defined in (7.7).

Again, here a powerlike solution of the differential equation (5.9) (continued to Minkowski signature) on pure AdS$_{d+1}$ with the d'Alembert operator given in (7.3) exists for the Weyl invariant coupled mass value

$$m^2 = \frac{1 - d^2}{4R_1^2} \; . \tag{7.10}$$

There is no condition on R_1 because in Subsection 4.1.2 we have seen that pure AdS spaces are always conformally flat. The related value for the conformal dimension from (7.7) is then $\Delta_\pm = \frac{d \pm 1}{2}$. The powerlike solution is given by

$$G(x, x') = \frac{\Gamma(\frac{d-1}{2})}{4\pi^{\frac{d+1}{2}}} \frac{1}{(u + i\varepsilon(t, t'))^{\frac{d-1}{2}}} \; . \tag{7.11}$$

In contrast to the AdS$_{d+1}$ × S$^{d'+1}$ case here the exponent of u is given by $\Delta_-(d, m^2)$. We have again kept the option of a time dependent $i\varepsilon(t, t')$ and will comment on it in Section 7.3.

The above solution can indeed be obtained from (5.10) by taking the sum of the expressions for Δ_+ and Δ_-. In addition one finds another simple structure by taking the difference. They are given by

$$
\begin{aligned}
\frac{1}{2}(G_{\Delta_-} + G_{\Delta_+}) &= \frac{\Gamma(\frac{d-1}{2})}{4\pi^{\frac{d+1}{2}}} \frac{1}{(u + i\varepsilon(t, t'))^{\frac{d-1}{2}}} \; , \\
\frac{1}{2}(G_{\Delta_-} - G_{\Delta_+}) &= \frac{\Gamma(\frac{d-1}{2})}{4\pi^{\frac{d+1}{2}}} \frac{1}{(u + 4R^2 + i\varepsilon(t, t'))^{\frac{d-1}{2}}} \; .
\end{aligned}
\tag{7.12}
$$

Both expressions are derived by using (5.10) and (B.48) of Appendix B.6. The first combination has the right short distance singularity to be a solution of (7.1). The second combination resembles (7.8). We will say more on these linear combinations in Sections 7.2 and 7.3.

7.1.3   Comment on masses and conformal dimensions on AdS$_{d+1}$

On AdS spaces one has to respect the Breitenlohner-Freedman bounds [34, 35]. The first bound is derived from the condition that there is no energy flux through the boundary of AdS$_{d+1}$. This condition with the requirement to have real energies effectively is a restriction on $\Delta_\pm$ to be real. One finds that the masses have to obey

$$m^2 \geq -\frac{d^2}{4R_1^2} \; . \tag{7.13}$$

Furthermore, the so called unitarity bound requires

$$\Delta > \frac{d-2}{2} \; . \tag{7.14}$$

This implies that for $-\frac{d^2}{4R_1^2} \leq m^2 < \frac{4-d^2}{4R_1^2}$ both, Δ_+ and Δ_- are allowed, since they are both normalizable. On the other side for $\frac{4-d^2}{4R_1^2} \leq m^2$ only Δ_+ is allowed.

The masses for Weyl invariant coupling are $\frac{1-d^2}{4R_1^2}$ and $\frac{d'^2-d^2}{4R_1^2}$ for AdS$_{d+1}$ and AdS$_{d+1} \times$ S$^{d'+1}$, respectively. Hence, in our Weyl invariant cases for pure AdS Δ_+ and Δ_- are allowed while for AdS$_{d+1} \times$ S$^{d'+1}$ with $d' > 1$ only Δ_+ is allowed.

7.2   Derivation of the propagator from the flat space one

In the previous Section we have shown that a simple powerlike solution of (7.1) can be found if the underlying spacetime is AdS$_{d+1}$ or a conformally flat product space AdS$_{d+1} \times$ S$^{d'+1}$ and if the corresponding scalar field is Weyl invariant coupled to the curvature of the background. Both properties allow for a mapping of the differential equation, the scalar field and the propagator to flat space. The other way around, one can use Weyl invariance in this special case to construct the propagator of Weyl invariant coupled fields on conformally flat backgrounds from the flat space massless propagator.

We will use this standard construction to rederive the AdS$_{d+1} \times$ S$^{d'+1}$ expressions (7.6) and (7.8) from the flat space solutions.

The relevant Weyl transformation in a D-dimensional manifold is

$$g_{\mu\nu} \to \varrho \, g_{\mu\nu} \; , \qquad \phi \to \phi' = \varrho^{\frac{2-D}{4}} \phi \; . \tag{7.15}$$

If then the metric is of the form $g_{\mu\nu}(z) = \varrho(z)\,\eta_{\mu\nu}$ one finds the following relation between the propagator in curved and flat space

$$G(z,z') = \big(\varrho(z)\,\varrho(z')\big)^{\frac{2-D}{4}}\,G_{\text{flat}}(z,z')\;,\qquad G_{\text{flat}}(z,z') = \frac{\Gamma(\frac{D-2}{2})}{4\pi^{\frac{D}{2}}}\frac{1}{\big((z-z')^2+i\epsilon\big)^{\frac{D-2}{2}}}\;. \tag{7.16}$$

It can be derived either by formal manipulations with the corresponding functional integral or by using the covariance properties of the defining differential equation.[2]

Applying the formula first to pure AdS one gets with the metric (4.29), that the propagator in Poincaré coordinates reads

$$G(x,x') = \frac{\Gamma(\frac{d-1}{2})}{R_1^{d-1}4\pi^{\frac{d+1}{2}}}\left(\frac{1}{x_\perp x_\perp'}\big[(x_\perp-x_\perp')^2 - (x^0-x'^0)^2 + (\vec{x}-\vec{x}')^2 + i\epsilon\big]\right)^{\frac{1-d}{2}}\;. \tag{7.17}$$

With the help of (4.32) it is easy to see that this expression is equal to (7.11).

The Poincaré patch of pure AdS$_{d+1}$, shown in Fig. 4.1 for $d=1$, is conformal to a flat half space with $x_\perp \geq 0$. $x_\perp = 0$ corresponds to the conformal boundary of AdS. Let us first disregard that the flat half space represents only one half of AdS$_{d+1}$ and discuss global issues later. We can then implement either Dirichlet or Neumann boundary conditions by the standard mirror charge method. To $x = (x_\perp, x^0, x^1, \ldots, x^{d-1})$ we relate the mirror point[3]

$$\tilde{x} = (-x_\perp, x^0, x^1, \ldots, x^{d-1}) \tag{7.18}$$

and the mirror propagator by

$$\tilde{G}_{\text{flat}}(x,x') = G_{\text{flat}}(x,\tilde{x}')\;. \tag{7.19}$$

Then $\frac{1}{2}(G_{\Delta_-} - G_{\Delta_+})$ in the second line of (7.12) turns out to be just the Weyl transformed version of $\tilde{G}_{\text{flat}}(x,x')$. Equivalently we can state, that G_{Δ_+} and G_{Δ_-} are the Weyl transformed versions respectively of the Dirichlet and Neumann propagator in the flat halfspace.

The situation is different for AdS$_{d+1} \times S^{d'+1}$ spacetimes. According to (4.39), $x_\perp \geq 0$ becomes a radial coordinate of a full $(d'+2)$-dimensional flat subspace of a total space with coordinates

$$z = \big(x_0, \vec{x}, x_\perp \tfrac{\vec{Y}}{R}\big)\;, \tag{7.20}$$

---

[2]Of course, the discussion has to be completed by considering also the boundary conditions.

[3]Using $x_\perp < 0$ for parameterizing the second Poincaré patch the mirror point is at the antipodal position on the hyperboloid (see (4.27) and Fig. 4.1).

where $\vec{Y}^2 = R^2$ are the embedding coordinates of $S^{d'+1}$. The boundary of the AdS part is mapped to the origin of the $(d'+2)$-dimensional subspace. Similarly to the pure AdS case, $G(z, z')$ from (7.6) is the Weyl transform of $G_{\text{flat}}(z, z')$. To see this one has to cast the length square on the $(d'+2)$-dimensional subspace, which appears in the denominator of the propagator, into the form

$$\frac{1}{R^2}(x_\perp \vec{Y} - x'_\perp \vec{Y}')^2 = x_\perp^2 + x_\perp'^2 - 2\frac{x_\perp x'_\perp}{R^2}\vec{Y}\vec{Y}' = (x_\perp - x'_\perp)^2 + \frac{x_\perp x'_\perp}{R^2}v \ , \qquad (7.21)$$

where we have used (4.38) and remember that $u(x, x')$ is given by (4.32). In addition, with

$$z_{\text{s}} = \left(x_0, \vec{x}, -x_\perp \frac{\vec{Y}}{R}\right) \ , \qquad \tilde{G}_{\text{flat}}(z, z') = G_{\text{flat}}(z, z'_{\text{s}}) \qquad (7.22)$$

we find that the second simple solution (7.8) is the Weyl transformed version of $\tilde{G}_{\text{flat}}(z, z')$.

The coordinates (7.20) and (7.22) are related by replacing $\vec{Y}$ by $-\vec{Y}$, i. e. z_{s} is related to z by going to the antipodal point in the sphere, according to the definition of z_{s} after (7.8). The two points z, z_{s} are elements of $\mathbb{R}^{d+d'+2}$ lying in the first Poincaré patch where $x_\perp \geq 0$.

As we mentioned before, one has to be careful with global issues. We work in the Poincaré patch that only covers points with $x_\perp \geq 0$. It is easy to see that the coordinates (7.20) of z and (7.22) of z_{s} remain unchanged if one simultaneously replaces $x_\perp$ by $-x_\perp$ and $\vec{Y}$ by $-\vec{Y}$. This operation switches from z and z_{s} respectively to the total antipodal positions $\tilde{z}$ and $\tilde{z}_{\text{s}}$, that are covered by a second Poincaré patch with $x_\perp < 0$. Thus, the latter points, being elements of the complete manifold, are not covered by the first Poincaré patch. In the context of pure AdS$_{d+1}$, the mirror point $\tilde{x}$ in (7.18) related to x is outside of the first Poincaré patch but it is still a point in AdS$_{d+1}$ covered by the second Poincaré patch. Hence, $\tilde{x}$ is not an element of the flat half space that is conformal to the first Poincaré patch. We will now analyze the global issues more carefully by working with the corresponding ESU.

7.3 Relation to the ESU

As discussed in Subsections 4.1.2 and 4.1.3, AdS$_{d+1}$ and AdS$_{d+1} \times$ S$^{d'+1}$ with $R_1 = R_2$ are conformal to respectively one half and to the full ESU of the corresponding dimension. This conformal relation has been used in [12] at $d = 3$ to find consistent quantization

schemes on AdS_4. In case of the Weyl invariant mass value (7.10) the quantization prescription on the ESU leads to two different descriptions for pure AdS. One can either choose transparent boundary conditions or reflective boundary conditions at the image of the AdS boundary. The reflectivity of the boundary is guaranteed for either Dirichlet or Neumann boundary conditions. This is realized by choosing a subset of modes with definite symmetry properties, whereas in the transparent case all modes are used. Quantization in the reflective case leads one to the solutions $G_{\Delta_\pm}$. These results motivate why we will work on the ESU in the following. We will find the antipodal points and see how the mirror charge construction works. Then we will discuss what this implies for the well known propagators in AdS_{d+1} and our solutions for $\text{AdS}_{d+1} \times \text{S}^{d'+1}$ in the Weyl invariant cases.

A convenient global coordinate system with coordinate $\bar{\rho}$, where the conformal equivalence between AdS_{d+1} or $\text{AdS}_{d+1} \times \text{S}^{d'+1}$ and the corresponding ESU is obvious, was defined in (4.24). In these coordinates a point $\tilde{x}$ antipodal to the point $x = (t, \bar{\rho}, x_\Omega)$ in AdS_{d+1} is given by

$$\tilde{x} = (t + \pi, \bar{\rho}, \tilde{x}_\Omega) \ , \tag{7.23}$$

where x_Ω denotes the angles of the $(d-1)$-dimensional subsphere of AdS_{d+1} with embedding coordinates ω_i (see(4.24)), such that one finds[4]

$$\omega_i(\tilde{x}_\Omega) = -\omega_i(x_\Omega) \ . \tag{7.24}$$

The above relation (7.23) must not be confused with the relation between two points that are antipodal to each other on the sphere of the ESU at fixed time.

We now want to visualize the above relation on the sphere of the ESU. For convenience we choose AdS_2 such that the ESU has topology $\mathbb{R} \times \text{S}^1$. The subsphere of AdS_2 is given by $\text{S}^0 = \{-1, 1\}$ such that we have $\omega = \pm 1$. Hence, the transformation of x_Ω as prescribed in (7.23) becomes a flip between the two points of the S^0. The information contained in S^0 can be traded for an additional sign information of $\bar{\rho}$, and therefore the transformation from x_Ω to $\tilde{x}_\Omega$ simply corresponds to an reflection at $\bar{\rho} = 0$. We will now describe the time shift. After the transformation of the spatial coordinates is performed, one has found the antipodal event at time $t + \pi$. To relate it to an event at the original time t one simply travels back in time along any null geodesics that crosses the spatial position of the antipodal event. On the ESU these null geodesics are clearly great circles. They meet at

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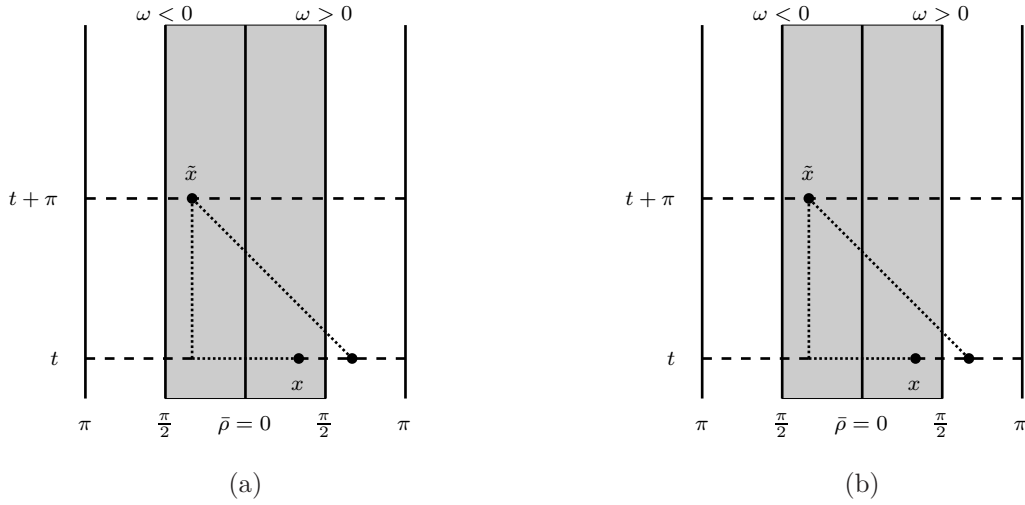
[4]See (B.74) for an explicit relation between the angles.

Figure 7.1: AdS_2 (Fig. 7.1(a)) and $\mathrm{AdS}_2 \times \mathrm{S}^0$ (Fig. 7.1(b)) conformally mapped to the corresponding ESU. The regions that are covered are displayed as gray-filled regions. The ESU is given by a cylinder such that one has to identify the two boundaries of the strip where $\rho = \pi$. The two points of the S^0 within AdS_2 and of the extra factor S^0 in the product space are $\omega = \pm 1$ and $\frac{Y}{R} = \pm 1$, respectively. $\tilde{x}$ and $\tilde{z}$, z_s, $\tilde{z}_\mathrm{s}$ are the antipodal points to x and z in respectively AdS_2 and $\mathrm{AdS}_2 \times \mathrm{S}^0$. They are constructed by following the lines with small dashsize. The horizontal direction corresponds to the transformation in the space coordinates and the vertical one is associated to the time shift. The diagonal lines then point to the source at the corresponding conjugate point where null geodesics intersect. The conjugate points can be regarded as effective time shifted sources with the same time coordinate as the original event x or z.

two points on the sphere. One is at the spatial position of the event and the other point is the antipodal point on the sphere of the ESU. The time it takes for a massless particle to travel between these two points is given by π, see Fig. 7.1. In this way one now arrives at an event that can have caused the event at later time $t + \pi$, and that has the same time coordinate as x, and its coordinate value $\bar{\rho}$ is given by a reflection at $\bar{\rho} = \frac{\pi}{2}$ on S^1. As $\bar{\rho} = \frac{\pi}{2}$ is the position of the AdS boundary, the mirror image to x is situated outside of the region that corresponds to AdS. The effect of the original source at x in combination with the mirror source either at $\tilde{x}$ as given in (7.23) or at equal times mirrored at the boundary is that a light ray that travels to the boundary of AdS is reflected back into the interior.

Let us now discuss what happens in the case of $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$. The point $z = (t, \bar{\rho}, x_\Omega, y)$ possesses the total antipodal point $\tilde{z}$ and the two semi-antipodal points z_s and

$\tilde{z}_{\mathrm{s}}$ given by

$$\tilde{z} = (t + \pi, \bar{\rho}, \tilde{x}_{\Omega}, \tilde{y}) \ , \qquad z_{\mathrm{s}} = (t, \bar{\rho}, x_{\Omega}, \tilde{y}) \ , \qquad \tilde{z}_{\mathrm{s}} = (t + \pi, \bar{\rho}, \tilde{x}_{\Omega}, y) \ , \qquad (7.25)$$

where x_{Ω} is as in the pure AdS_{d+1} case and fulfills (7.24) and y are all angle coordinates of $\mathrm{S}^{d'+1}$.

In Fig. 7.1 the case of $\mathrm{AdS}_2 \times \mathrm{S}^0$, is shown. The effect of the factor S^0 can be alternatively described by adding to the range $0 \le \bar{\rho} \le \frac{\pi}{2}$ the interval $\frac{\pi}{2} \le \bar{\rho} \le \pi$. This is possible because in the ESU at $\bar{\rho} = \frac{\pi}{2}$ the S^0 shrinks to a point. The complete ESU is now covered by the image of $\mathrm{AdS}_2 \times \mathrm{S}^0$. The map to an antipodal position within the AdS_2 factor is as before, one finds the spatial coordinates by reflecting at $\bar{\rho} = 0$. Within the S^0 factor, the antipodal position is found by reflecting at $\bar{\rho} = \frac{\pi}{2}$. Using this, it can be seen that w. r. t. the point z, the point $\tilde{z}$ is at the antipodal position on the S^1 of the ESU. Traveling back in time from $t + \pi$ to t along a null geodesic, one arrives at z from where one started. In the same way, the two semi-antipodal points z_{s}, $\tilde{z}_{\mathrm{s}}$ are connected with each other by light rays. On the sphere of the ESU the z and z_{s} are related by a reflection at $\bar{\rho} = \frac{\pi}{2}$. Here, in contrast to the case of AdS_2, even the mirror events at equal times are situated within the image of $\mathrm{AdS}_2 \times \mathrm{S}^0$. The above results are straightforwardly generalized to arbitrary dimensions.

Coming back to the discussion in Section 7.2, we can now make more precise statements about the mirror charge method to impose definite boundary conditions at $\bar{\rho} = \frac{\pi}{2}$. A linear combination of the two solutions like in (7.12) does not necessarily generate additional δ-sources on the R. H. S. of the differential equation (5.9), although both powerlike solutions in (7.12) have singularities within AdS_{d+1}, the expression in the first line has one at $x = x'$ and the expression in the second line has one at $x = \tilde{x}'$. The singularity of the second expression only appears at $t = t' + \pi$, and its contribution to the R. H. S. of the differential equation (5.9) depends on the time ordering prescription. In the cases where the θ-function used for time ordering has an additional step at $t = t' + \pi$, a second δ-function is generated (see [12] for a discussion of AdS_4). With the standard time ordering one finds that $G_{\Delta_{\pm}}$ are solutions with a source at $x = x'$ only. For AdS_4 this was obtained in [65].

The situation is different for $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$, where the propagator (7.6) has singularities at $z = z'$, $z = \tilde{z}'$ and the second solution (7.8) has singularities at $z = z'_{\mathrm{s}}$, $z = \tilde{z}'_{\mathrm{s}}$. Again, whether the singularities at $z = \tilde{z}'$ and $z = \tilde{z}'_{\mathrm{s}}$ appear as δ-sources on the R. H. S. of the differential equation (7.1), depends on the chosen time ordering. However in

contrast to the pure AdS_{d+1} case, the singularity of the second solution (7.8) at $z = z'_s$ always leads to a δ-source on the R. H. S. of (7.1) but at the wrong position. This result corresponds to the above observation on the ESU that the mirror sources at equal times are not part of the image of AdS_{d+1} but of $AdS_{d+1} \times S^{d'+1}$.

At the end let us give some comments on the $i\varepsilon(t, t')$-prescription. First of all, one has to introduce it in all expressions (7.6), (7.8) and (7.12), since all of them have singularities at coincident or antipodal positions. Secondly, as worked out for AdS_4, a time independent $\varepsilon(t, t') = \epsilon$ refers to taking the step function $\theta(\sin(t - t'))$ for time ordering [12] which is appropriate if one restricts oneself to the hyperboloid. Standard time ordering with $\theta(t - t')$, being appropriate on the universal covering, yields a time dependent $\varepsilon(t, t') = \epsilon \, \text{sgn}((t - t') \sin(t - t'))$ [65]. As mentioned in Section 7.1, due to the time dependence of $\varepsilon(t, t')$, the coordinate dependence of the solutions is not entirely included in u and v.

7.4   Mode summation on $\mathbf{AdS}_{d+1} \times \mathbf{S}^{d'+1}$

In this Section we will use the propagator on pure AdS_{d+1} given by (5.10) and the spherical harmonics on $S^{d'+1}$ to construct the propagator on $AdS_{d+1} \times S^{d'+1}$ via its mode expansion, summing up all the KK modes. We will be able to perform the sum only for special mass values where the conformal dimensions $\Delta_\pm$ of the scalar modes are linear functions of l, with l denoting the lth mode in the KK tower. Even a mixing of several scalar modes of this kind is allowed. The mixing case is interesting because it occurs in supergravity theories on $AdS_{d+1} \times S^{d+1}$ backgrounds [49, 54, 87, 99, 120]. For example in type II B supergravity in $AdS_5 \times S^5$ the mass eigenstates of the mixing matrix for scalar modes [87, 99] correspond to the bosonic chiral primary and descendant operators in the AdS/CFT dictionary [104]. For these modes $\Delta_\pm$ depend linearly on l.

The main motivation for investigating the mode summation was the hope to find the propagator for generic mass values. But forced to stay in a regime of a linear $\Delta_\pm$ versus l relation we can give up the condition of conformal flatness, but remain restricted to special mass values. We nevertheless present this study since several interesting aspects are found along the way. Furthermore, in the literature it is believed that an explicit computation of the KK mode summation is too cumbersome [116]. We will show how to deal with the mode summation by discussing the $AdS_3 \times S^3$ case first, allowing for unequal radii but necessarily a special mass value. The result will then be compared to

the expressions in the previous Sections by specializing to equal embedding radii.

Having discussed this special case we will comment on the modifications which are necessary to deal with generic $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ spacetimes.

The results of the previous Sections in connection with the expression for the mode summation in the conformally flat and Weyl invariant coupled case lead to the formulation of a summation rule for a product of Legendre functions and Gegenbauer polynomials. An independent proof of this rule is given in Appendix B.8. With this it is possible to discuss the results in generic dimensions without doing all the computations explicitly. Furthermore, the sum rule might be useful for other applications, too.

For the solution of (7.1) we make the following ansatz[5]

$$G(z, z') = \frac{1}{R_2^{d'+1}} \sum_I G_I(x, x') Y^I(y) Y^{*I}(y') , \qquad (7.26)$$

where we sum over the multiindex $I = (l, m_1, \ldots, m_{d'})$ such that $l \geq m_1 \geq \cdots \geq m_{d'-1} \geq |m_{d'}| \geq 0$, Y^I denote the spherical harmonics on $\mathrm{S}^{d'+1}$, and '$*$' means complex conjugation. Some useful relations for the spherical harmonics can be found in Appendix B.7.

The mode dependent Green function on AdS_{d+1} then fulfills

$$\left(\Box_x - M^2 - \frac{l(l+d')}{R_2^2} \right) G_I(x, x') = \frac{i}{\sqrt{-g_{\mathrm{AdS}}}} \delta(x, x') , \qquad (7.27)$$

which follows when decomposing the d'Alembert operator like in (7.3) and using (B.61). The solution of this equation[6] was already given in (5.10), into which the (now KK mode dependent) conformal dimensions enter. They were already defined in (7.7), and the AdS mass is a function of the mode label l

$$m^2 = M^2 + m_{\mathrm{KK}}^2 = M^2 + \frac{l(l+d')}{R_2^2} . \qquad (7.28)$$

In the following as a simple example we will present the derivation of the propagator on $\mathrm{AdS}_3 \times \mathrm{S}^3$ via the KK mode summation. Compared to the physically more interesting $\mathrm{AdS}_5 \times \mathrm{S}^5$ background the expressions are easier and the general formalism becomes clear.

---

[5]This ansatz is designed to generate a solution that corresponds to (7.6). If one wants to generate a solution corresponding to (7.8) one has to replace either y or y' by the corresponding antipodal coordinates $\tilde{y}$ or $\tilde{y}'$.

[6]As explained around (7.1), the R. H. S. of the equation deviates from the one in (5.9) due to the continuation procedure between the Lorentzian and the Euclidean case.

Evaluating (5.10) for $d = d' = 2$ the AdS$_3$ propagator for the lth KK mode is given by

$$G_\Delta(x, x') = \frac{1}{R_1 2^{\Delta+1}\pi} \xi^\Delta F\left(\tfrac{\Delta}{2}, \tfrac{\Delta}{2} + \tfrac{1}{2}; \Delta; \xi^2\right) = \frac{1}{R_1 4\pi} \frac{1 + \sqrt{1-\xi^2}}{\sqrt{1-\xi^2}} \left[\frac{\xi}{1 + \sqrt{1-\xi^2}}\right]^\Delta .$$
(7.29)

From (B.50), (7.7) and (7.28) one finds that the mode dependent positive branch of the conformal dimension reads

$$\Delta = \Delta_+ = 1 + \frac{R_1}{R_2}\sqrt{\frac{R_2^2}{R_1^2} + l(l+2) + M^2 R_2^2} .$$
(7.30)

The spherical part follows from (B.64) of Appendix B.7 where we discuss it in more detail and is given by

$$\sum_{m_1 \geq |m_2| \geq 0}^{l} Y^I(y) Y^{*I}(y') = \frac{(l+1)}{2\pi^2} C_l^{(1)}(\cos\Theta) , \qquad \cos\Theta = \frac{Y \cdot Y'}{R_2^2} = 1 - \frac{v}{2R_2^2} .$$
(7.31)

Remember that the $C_l^{(\beta)}$ denote the Gegenbauer polynomials and Y, Y' in the formula for Θ are the embedding space coordinates of the sphere, compare with (4.36) and (4.38). One thus obtains from (7.26)

$$G(z, z') = \frac{1}{8\pi^3 R_1 R_2^3} \frac{1 + \sqrt{1-\xi^2}}{\sqrt{1-\xi^2}} \sum_{l=0}^{\infty} (l+1)\left[\frac{\xi}{1 + \sqrt{1-\xi^2}}\right]^\Delta C_l^{(1)}(\cos\Theta) .$$
(7.32)

In this formula Δ is a function of the mode parameter l and we can explicitly perform the sum only for special conformal dimensions which are linear functions of l

$$\Delta = \Delta_+ = \frac{R_1}{R_2}l + \frac{R_1 + R_2}{R_2} ,$$
(7.33)

following from (7.30) after choosing the special mass value

$$M^2 = \frac{1}{R_2^2} - \frac{1}{R_1^2} .$$
(7.34)

The sum then simplifies and can explicitly be evaluated by a reformulation of the l-dependent prefactor as a derivative and by using (B.68)

$$\sum_{l=0}^{\infty}(l+1)q^l C_l^{(1)}(\eta) = \left(q\frac{\partial}{\partial q} + 1\right)\sum_{l=0}^{\infty} q^l C_l^{(1)}(\eta) = \frac{1 - q^2}{(1 - 2q\eta + q^2)^2} .$$
(7.35)

With the replacements

$$q = \left[\frac{\xi}{1 + \sqrt{1-\xi^2}}\right]^{\frac{R_1}{R_2}} , \qquad \eta = \cos\Theta$$
(7.36)

one now finds after some simplifications

$$G(z, z') = \frac{1}{8\pi^3 R_1 R_2^3} \frac{1}{\sqrt{1-\xi^2}} \xi^{1+\frac{R_1}{R_2}} \frac{(1+\sqrt{1-\xi^2})^{\frac{R_1}{R_2}} - (1-\sqrt{1-\xi^2})^{\frac{R_1}{R_2}}}{\left[(1+\sqrt{1-\xi^2})^{\frac{R_1}{R_2}} - 2\xi^{\frac{R_1}{R_2}}\cos\Theta + (1-\sqrt{1-\xi^2})^{\frac{R_1}{R_2}}\right]^2} .$$
(7.37)

For the conformally flat case $R_1 = R_2 = R$, where (7.34) becomes the mass generated by the Weyl invariant coupling to the background, the above expression simplifies to

$$G(z, z') = \frac{1}{4\pi^3 R^4} \frac{\xi^2}{(2 - 2\xi\cos\Theta)^2} = \frac{1}{4\pi^3} \frac{1}{(u + v + i\varepsilon(t, t'))^2} ,$$
(7.38)

where we have restored the $i\varepsilon(t, t')$-prescription. This result exactly matches (7.6).

The way to perform the KK mode summation on generic $\text{AdS}_{d+1} \times \text{S}^{d'+1}$ backgrounds is very similar to the one presented above. One finds a linear relation between l and Δ

$$\Delta_\pm = \pm\frac{R_1}{R_2}l + \frac{dR_2 \pm d'R_1}{2R_2}$$
(7.39)

at the $(d + d' + 2)$-dimensional mass value

$$M^2 = \frac{d'^2 R_1^2 - d^2 R_2^2}{4R_1^2 R_2^2} .$$
(7.40)

This expression is a generalization of (7.34) and it reduces to (7.4) in the conformally flat case. For generic dimension the way of computing the propagator is very similar to the one presented for the $\text{AdS}_3 \times \text{S}^3$ background. However the steps (7.29) to express the hypergeometric function in the AdS propagator and (7.35) to compute the sum become more tedious. For dealing with the hypergeometric functions see the remarks in Appendix B.6. The sum generalizes in the way, that higher derivatives and more terms enter the expression (7.35).

Next we discuss the mode summation in the conformally flat case $R_1 = R_2$ at the Weyl invariant mass value but for generic d and d'. In this case with the corresponding conformal dimensions

$$\Delta = \Delta_+ = l + \frac{d + d'}{2} ,$$
(7.41)

using (5.10) and (B.64), the propagator is expressed as

$$G(z, z') = \frac{\Gamma(\frac{d'}{2})}{4\pi} \left(\frac{\xi}{2\pi R^2}\right)^{\frac{d+d'}{2}}$$
$$\times \sum_{l=0}^{\infty} \frac{\Gamma(l + \frac{d+d'}{2})}{\Gamma(l + \frac{d'}{2})} \left(\frac{\xi}{2}\right)^l F\left(\frac{l}{2} + \frac{d+d'}{4}, \frac{l}{2} + \frac{d+d'}{4} + \frac{1}{2}; l + \frac{d'}{2} + 1; \xi^2\right) C_l^{(\frac{d'}{2})}\left(1 - \frac{v}{2R^2}\right) .$$
(7.42)

This equality together with the solution (7.6) has lead us to formulate a sum rule for the above given functions at generic d and d'. The above series should exactly reproduce (7.6). In Appendix B.8 we give an independent direct proof of the sum rule.

Considering the mode summation one finds an interpretation of the asymptotic behaviour of (7.6) observed in Subsection 7.1.1. The asymptotic regime $u \to \infty$ corresponds to $\xi \to 0$. As the contribution of the lth mode is proportional to $\xi^{\Delta_+} \sim \xi^l$, the conformal dimension of the zero mode determines the asymptotic behaviour.

Note also that the additional singularity of (7.6) at the total antipodal position $z = \tilde{z}'$ can be seen already in (7.26). Under antipodal reflection in AdS_{d+1} the pure AdS propagator fulfills $G_{\Delta_\pm}(x, \tilde{x}') = (-1)^{\Delta_\pm} G_{\Delta_\pm}(x, x')$. On the sphere the spherical harmonics at antipodal points are related via $Y^I(y) = (-1)^l Y^I(\tilde{y})$. Hence, in case that $\Delta_\pm$ is given by (7.41), replacing z' by the total antipodal point $\tilde{z}'$ leads to the same expression for the mode sum up to an l-independent phase factor.

One final remark to the choice of Δ_+. What happens if one performs the mode expansion with AdS propagators based on Δ_-? First in any case for high enough KK modes Δ_- violates the unitarity bound (7.14). But ignoring this condition from physics one can nevertheless study the mathematical issue of summing with Δ_-. The corresponding series is given by (7.35) after replacing q by q^{-1}. It is divergent since for real u the variable q in (7.36) obeys $|q| \le 1$ (case $R_1 = R_2$). One can give meaning to the sum by the following procedure. q as a function of ξ has a cut between $\xi = \pm 1$. If $|q| \le 1$ on the upper side of the cut, then $|q| \ge 1$ on the lower side. Hence, it is natural to define the sum with Δ_- as the analytic continuation from the lower side. By this procedure we found both for $\mathrm{AdS}_3 \times \mathrm{S}^3$ and $\mathrm{AdS}_5 \times \mathrm{S}^5$ up to an overall factor -1 the same result as using Δ_+. The sign factor can be understood as a consequence of the continuation procedure.

7.5 The plane wave limit

The propagator in the 10-dimensional plane wave was constructed in [116]. We will now demonstrate how this propagator in the massless case arises as a limit of our $\mathrm{AdS}_5 \times \mathrm{S}^5$ propagator (7.6) by following the limiting process. As an additional consistency check we will take the $R \to \infty$ limit of the differential equation (7.1) using (7.3) to obtain the equation on the plane wave background and find that it is fulfilled by the massive propagator given in [116].

We first take the chordal distances u and v introduced in (4.32) and (4.38)

$$
\begin{aligned}
u &= 2R^2\Big[-1 + \cosh\rho\cosh\rho'\cos(t-t') - \sinh\rho\sinh\rho'\,\omega^i\omega_i'\Big] , \\
v &= 2R^2\Big[+1 - \cos\vartheta\cos\vartheta'\cos(\psi-\psi') - \sin\vartheta\sin\vartheta'\,\hat\omega^i\hat\omega_i'\Big] ,
\end{aligned}
\tag{7.43}
$$

then we apply the variable transformations (4.82) and (4.84) such that one gets at large $R = R_1 = R_2$ up to terms vanishing for $R \to \infty$

$$
\begin{aligned}
u &= 2R^2\Big[-1 + \cos\Delta z^+ + \tfrac{1}{R^2}\Big(-(\vec{x}^2+\vec{x}'^2)\sin^2\tfrac{\Delta z^+}{2} + \tfrac{(\Delta\vec{x})^2}{2} - \Delta z^-\sin\Delta z^+\Big)\Big] , \\
v &= 2R^2\Big[+1 - \cos\Delta z^+ + \tfrac{1}{R^2}\Big(-(\vec{y}^2+\vec{y}'^2)\sin^2\tfrac{\Delta z^+}{2} + \tfrac{(\Delta\vec{y})^2}{2} - \Delta z^-\sin\Delta z^+\Big)\Big] ,
\end{aligned}
\tag{7.44}
$$

where $\Delta z^\pm = z^\pm - z'^\pm$, $\Delta\vec{x} = \vec{x} - \vec{x}'$, $\Delta\vec{y} = \vec{y} - \vec{y}'$ and $\vec{x} = r\,\vec{\omega}$, $\vec{y} = y\,\hat{\vec{\omega}}$. In the $R \to \infty$ limit the sum of both chordal distances is thus given by

$$
\Phi = \lim_{R\to\infty}(u+v) = -2(\vec{z}^2+\vec{z}'^2)\sin^2\tfrac{\Delta z^+}{2} + (\vec{z}-\vec{z}')^2 - 4\Delta z^-\sin\Delta z^+ ,
\tag{7.45}
$$

where $\vec{z} = (\vec{x},\vec{y})$, $\vec{z}' = (\vec{x}',\vec{y}')$ and Φ refers to the notation of [116]. Φ is precisely the $R \to \infty$ limit of the total chordal distance on $\mathrm{AdS}_5 \times \mathrm{S}^5$, which is the chordal distance in the plane wave, compare with (4.59) using $H_{ij} = -\delta_{ij}$. It remains finite as both $\sim R^2$ terms in (7.44) cancel. This happens due to the expansion around a *null* geodesic.

The massless propagator in the plane wave background in the $R \to \infty$ limit of (7.6) with $d = d' = 4$ thus becomes

$$
G_{\mathrm{pw}}(z,z') = \frac{3}{2\pi^5}\frac{1}{(\Phi + i\varepsilon(z^+,z'^+))^4} ,
\tag{7.46}
$$

which agrees with [116].

In addition we checked the massive propagator of [116] which fulfills the differential equation on the plane wave background. This equation can be obtained from (7.1) and (7.3) by taking the $R \to \infty$ limit. In the limit the sum of both chordal distances is given in (7.45). The difference is given by

$$
\lim_{R\to\infty}\frac{u-v}{R^2} = 4(\cos\Delta z^+ - 1)
\tag{7.47}
$$

this has to be substituted into (7.3). Finally, one obtains the differential equation

$$
\Big[4\cos\Delta z^+\Big(5\frac{\partial}{\partial\Phi} + \Phi\frac{\partial^2}{\partial\Phi^2}\Big) + 4\sin\Delta z^+\frac{\partial}{\partial\Phi}\frac{\partial}{\partial\Delta z^+} - M^2\Big]G_{\mathrm{pw}}(z,z') = \frac{i}{\sqrt{-g_{\mathrm{pw}}}}\delta(z,z') ,
\tag{7.48}
$$

which is fulfilled by the expression given in [116]. As already noticed in Section 7.1, in contrast to the massless propagator the massive one depends not only on the total chordal distance Φ but in addition on (7.47).

Part IV

Summary and conclusions

This thesis addressed two topics. The first topic dealt with the formulation of Feynman rules for noncommutative YM theories with general gauge groups. In the second part, ingredients for the formulation of the holographic principle in the AdS/CFT correspondence and in its BMN limit were analyzed.

In part II we have discussed some issues of the noncommutative version of pure YM theory. We have first reviewed how the noncommutative $U(1)$ theory arises from string theory in a background with constant B-field and how this led to the formulation of the Seiberg-Witten map. The latter played an essential role in the formulation of noncommutative gauge theories with arbitrary gauge groups G.

We have focused on the task of obtaining information about the Feynman rules for noncommutative YM theories with gauge groups G. We have shown how the Seiberg-Witten map between the sets of ghost fields can be extracted from the Faddeev-Popov gauge fixing procedure. Before we went to the crucial problem of analyzing the noncommutative YM theories with gauge groups $G \neq U(N)$, we rederived the well known Feynman rules for $G = U(N)$.

To get information about the Feynman rules in the case of general $G \subset U(N)$, we then started from the path integral formulation, imposing a constraint on the integral over the noncommutative fields. In terms of the Seiberg-Witten map the latter was interpreted as the restriction that the noncommutative fields are mapped to ordinary fields of an ordinary YM theory with gauge group G. This constraint was then resolved by using the power series of the Seiberg Witten map to replace the constrained integration over the noncommutative fields by an unconstrained one over the ordinary fields. In this way we arrived at the enveloping algebra approach, where additional interaction vertices were generated from the θ-expansion of the Seiberg-Witten map.

To get information about the Feynman rules without θ-expansion, we studied the issue of partial summation of the above described θ-expanded perturbation theory. In this analysis we kept the noncommutative YM vertices already found for $U(N)$ and focused on the remaining kinetic part of the perturbation theory. For $G = U(N)$, we found agreement with the expected result for an unconstrained integration, that only the connected 2-point Green functions of the noncommutative fields should be generated. We found that this was guaranteed by a cancellation mechanism between two types of diagrams that are different w. r. t. one of their interaction vertices. In one type of diagrams the latter is generated by the expansion of the kinetic term in the action, in the second type of

diagrams the vertex has its origin in an expansion of the corresponding source term.

For arbitrary $G \subset U(N)$, $G \neq U(M)$, $M < N$ this cancellation mechanism breaks down because the leading contributions in the expansion of the source terms vanishes and hence the first type of diagrams is absent. The number of legs of non-vanishing connected Green functions generated by the remaining uncanceled parts is not bounded from above. Hence, there are no Feynman rules based on the noncommutative $U(N)$ YM vertices and, besides perhaps suitably modified propagators, at most a finite number of additional building blocks with gauge field or ghost legs.

As usual in the case of no go theorems one has to be very carefully in stressing the input made. Our negative statement is bound to the a priori decision to work with the noncommutative $U(N)$ YM vertices. Of course, at this stage we cannot exclude the existence of rules that contain the exact θ-dependence and that are based on some clever modification of these vertices. We also cannot exclude that the infinite set of building blocks with gauge field and ghost legs by means of some additional auxiliary field could be resolved into rules with only a finite number of building blocks.

To make contact with the conjectured rules for $SO(N)$, we have then modified our setup extracting only the pure $SO(N)$ components of the noncommutative $U(N)$ YM vertices. If the rules hold, the remaining parts must generate nothing beyond a connected two point function. Taking the $SO(3)$ case as a counterexample, we were able to show explicitly that there is a non-vanishing connected 8-point function. Hence, the conjectured Feynman rules are inconsistent with the framework in which they were defined.

Our analysis has shown that a lot of effort is required to obtain some information about the Feynman rules with full θ-dependence for arbitrary gauge groups G. It has furthermore shown that the cases $G = U(N)$ and $G \subset U(N)$ are essentially different: the first leads to an unconstrained path integral formulation where it is straightforward to extract the Feynman rules, while the latter requires a constrained path integral leading to a more involved analysis. Many questions remain unanswered or arise new from our analysis. For instance, the task to find a general formalism to derive Feynman rules in case of arbitrary G remains unsolved. Of course, this is closely connected to the question if one can formulate a more general no go theorem that excludes any clever attempt to formulate Feynman rules, including the possibility to modify the original vertices and to introduce additional auxiliary fields.

In Part III we have analyzed some ingredients that are important for a formulation of the holographic principle. We have reviewed the AdS/CFT correspondence and its BMN limit. Thereby, we discussed the connection between the underlying $\text{AdS}_{d+1} \times \text{S}^{d'+1}$ and plane wave backgrounds in detail. We especially focused on the realization of holography in the AdS/CFT correspondence and summarized proposals for the less understood BMN case. From this discussion it turned out that the boundary structure and the geodesics are important geometrical ingredients in a holographic setup. We have furthermore shown that the bulk-to-boundary propagator plays an essential role in a holographic formulation, and we have derived its relation to the bulk-to-bulk propagator. We have motivated that, to get information about holography in the BMN limit, one should observe how the aforementioned quantities behave in the limiting procedure from the AdS/CFT to the BMN correspondence. In particular, this meant that we had to analyze the the boundaries, geodesics and propagators in $\text{AdS}_5 \times \text{S}^5$ and observe their behaviour in the 10-dimensional plane wave which arises in a Penrose limit.

For the discussion of the boundaries we have worked in two different coordinate systems: the first set that we denoted as the BMN coordinates were introduced to obtain the plane wave spacetime directly in the limit of infinite embedding radii from the $\text{AdS}_{d+1} \times \text{S}^{d'+1}$ geometry. They are the Brinkmann coordinates in the plane wave case. The second set which we called the BN coordinates are convenient for finding the conformal boundary of the plane wave.

We have shown that in the BN coordinate system the coordinates of the boundary of $\text{AdS}_5 \times \text{S}^5$, in the projection to three coordinates appears to be located at the same spiraling line as the plane wave boundary. Of course for $\text{AdS}_5 \times \text{S}^5$ on this line the extension with respect to the other 7 coordinates is not degenerated to a point. But we have generated a perhaps useful intuitive picture: The boundary is always at the same line, taking the limit $R \to \infty$ the extension in the remaining 7 coordinates shrinks to a point. This then implies also the degeneration of the 3 remaining dimensions of the conformal boundary of $\text{AdS}_5 \times \text{S}^5$. In the BMN coordinates it turned out that, due to the singularity of the coordinate transformation at the boundary line, the approach to this line is realized within two different asymptotic regimes that we called (i) and (ii). Only limit (i) corresponds to the conformal boundary of $\text{AdS}_5 \times \text{S}^5$.

We have then given a complete classification for geodesics, both for the original full $\text{AdS}_5 \times \text{S}^5$ and the plane wave and we identified the boundary reaching null geodesics. In

$AdS_5 \times S^5$ we found four different types of geodesics, but only one type of null geodesics reaches the conformal boundary. They stay at constant position in the S^5 and have to approach the boundary in the limit (i). In contrast to the $AdS_5 \times S^5$ case, the boundary reaching null geodesics of the plane wave approach it within limit (ii). This implied that for $R \to \infty$ the convergence of $AdS_5 \times S^5$ geodesics to plane wave geodesics is not uniform outside the region $|z^-| < R^{1-\varepsilon}$, $\varepsilon > 0$. Hence, the naive picture is supported that in BMN coordinates the $AdS_5 \times S^5$ space up to the order of magnitude of R looks like a plane wave. Furthermore, we have found that at each point with finite BMN coordinates, the null geodesics of $AdS_5 \times S^5$ reaching the conformal boundary form a cone with base S^3. For $R \to \infty$, in the range where the BMN coordinates are fixed or grow slower than R, this cone degenerates to the single plane wave null geodesic crossing the point under consideration and reaching the plane wave conformal boundary. Therefore, all points in this range effectively notice a degeneration of the boundary.

We have then studied the the bulk-to-bulk propagator of a scalar field in $AdS_{d+1} \times S^{d'+1}$ backgrounds. With the help of the previously presented results we have explained that an analysis of the bulk-to-boundary propagator itself is of little use if one wants to observe the behaviour in the Penrose limit. The reason is that the point on the boundary lies outside of the region of convergence. However, we explained why the behaviour of the bulk-to-bulk propagator can be studied in the plane wave limit. At least in the case where the holographic screen has codimension one compared to the bulk it is related to the corresponding propagator with one point in the bulk and the other on the holographic screen.

We have analyzed the propagator in $AdS_{d+1} \times S^{d'+1}$ backgrounds with generic dimensions d, d' and generic embedding radii R_1 and R_2 for both factors, such that we could extract general statements about its construction. First, we have discussed the defining wave equation for the propator with δ-source in this background. On conformally flat backgrounds for Weyl invariant coupled fields the propagator turned out to be simply powerlike in the sum of both chordal distances. In this case a further powerlike solution to the wave equation was found to exist. It depends only on the difference of the chordal distances and has singularities if the points are antipodal to each other either in the AdS_{d+1} or in the $S^{d'+1}$ part. To make contact with the Weyl invariant coupled case in pure AdS space, we have in brief presented a simple powerlike solution and a solution with a singularity at the antipodal point that are linear combinations of the well known

AdS propagators with the corresponding $\Delta_{\pm}$ values.

An alternative construction from the well known propagator in flat space was given by using the Weyl invariance that admits the required conformal mapping. Only the Poincaré patch that covers one half of AdS could be dealt with in this way. This has prevented us from studying global issues of the solutions in flat space.

To analyze some global properties, we have then used the fact that AdS_{d+1} and $\text{AdS}_{d+1} \times \text{S}^{d'+1}$ can be conformally mapped to respectively one half and to the full corresponding ESU. We focussed on the source structure of the previously found solutions of the corresponding differential equation. It turned out that in AdS_{d+1} one of the two solutions has a singularity if the two points are antipodal to each other. We found that it depends on the time ordering prescription whether this leads to a contribution to the δ-sources on the R. H. S. of the equation. In $\text{AdS}_{d+1} \times \text{S}^{d'+1}$, one finds that one of the solutions has singularities if both points coincide or if one point is at the total antipodal position. The other solution has singularities if one of the points is at the antipodal position either in AdS_{d+1} or in $\text{S}^{d'+1}$. In this case both solutions contribute to the δ-sources on the R. H. S. of the differential equation. The chosen time ordering can only influence if the other singularities in both solutions lead to additional δ-sources. Hence, in contrast to the $\text{AdS}_{d+1} \times \text{S}^{d'+1}$ case, in AdS_{d+1} one has the option to construct two independent solutions for the propagator without changing the canonical source structure.

In addition for $\text{AdS}_{d+1} \times \text{S}^{d'+1}$ we have investigated the KK decomposition of the propagator using spherical harmonics. We have noted that the summation can be performed even in non conformally flat backgrounds, but only for special mass values. The relevant condition is that the conformal dimension of the field mode is a linear function of the KK mode parameter. In the conformally flat case for a Weyl invariant coupled field the uniqueness of the solution of the differential equation in combination with the KK decomposition led to the formulation of a theorem that sums up a product of Legendre functions and Gegenbauer polynomials. We presented an independent proof for this theorem.

For $\text{AdS}_5 \times \text{S}^5$ we explicitly performed the Penrose limit on our expression for the propagator to find the result on the plane wave background. We found agreement with the result obtained by an explicit construction in the plane wave and got an interpretation for the spacetime dependence of this result. It simply depends on the $R \to \infty$ limit of the sum of both chordal distances on the original $\text{AdS}_5 \times \text{S}^5$, that is the chordal distance on the 10-dimensional plane wave. In the general massive case there is an additional

dependence on the suitable rescaled difference of both chordal distances. We formulated the differential equation in the limit and checked that the well known massive propagator on the plane wave background is a solution.

From the above summarized observations one could draw the following rough picture of what might happen in the limit from the AdS/CFT correspondence to its BMN limit. The fact that only a limited region in the interior of $AdS_5 \times S^5$ converges to the 10-dimensional plane wave, together with the fact that only a subset of nearly protected operators survives on the boundary theory seems to indicate that the limit is accompanied by a projection. One could speculate that the bulk region between the part that converges to the plane wave and the boundary of $AdS_5 \times S^5$ is responsible for this projection. In this limit the old boundary is blown apart. This could lead to a selection process that an observer in the geometry converging to the plane wave can only measure and influence some of the degrees of freedom of the boundary theory. On the level of the corresponding sources the selection process could be realized as follows: only those sources are present which correspond to modes decaying sufficiently slow for increasing R when travelling through the region between the $AdS_5 \times S^5$ boundary and the region that converges to the plane wave.

Furthermore, one of the most important points is to find the right holographic screen in the BMN correspondence. We have seen that several proposals with holographic screens of various dimensions exist. In particular if one regards the boundary as the holographic screen the dual theory should be 1-dimensional. The type of holographic screen then determines the further steps of how the propagator with one point in the bulk and one point on the screen can be derived from our result for the bulk-to-bulk propagator.

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Appendix A

Appendix to Part II

A.1 Path integral quantization of quantum field theories

In this Appendix we will shortly review the path integral approach [69] that allows us to quantize a given field theory. It is convenient for a discussion of symmetries and especially for understanding the covariant gauge fixing procedure in non-Abelian theories. This Appendix should be regarded as a brief review of some of the tools that we will need for our own analysis. A more detailed introduction to the method can for instance be found in [124].

A.1.1 The path integral approach

In quantum field theories one wants to compute correlation functions of the fields. The theory is usually defined by an action S. In the path integral approach the extraction of an n-point function of the fields which we collectively denote with ϕ is given by

$$\langle 0 | \, \mathrm{T} \left[\hat{\phi}(x_1) \ldots \hat{\phi}(x_n) \right] | 0 \rangle = N \int \mathcal{D}\phi \; \mathrm{e}^{iS[\phi]} \, \phi(x_1) \ldots \phi(x_n) \; , \qquad N = \left[\int \mathcal{D}\phi \; \mathrm{e}^{iS[\phi]} \right]^{-1} . \tag{A.1}$$

The L. H. S. of this expression shows the n-point Green function in the canonical operator formalism, where $|0\rangle$ is the vacuum, T denotes time ordering and $\hat{\phi}(x)$ is a field operator at position x. The above integral on the R. H. S. is given for a theory in Minkowski space and it is related to the version in Euclidean space by performing a Wick rotation which replaces i in front of the action in the exponent by -1. The fields ϕ under the integral are

classical functions. In the following we will omit the time ordering symbol, the operator-hat and the zeros in writing vacuum matrix elements. It is worth remarking that the calculations below can be intuitively understood if one interpretes the path integral as the continuum limit of a number of integrals over $\phi(x_i)$ at discrete lattice points x_i.

The functional derivative in d spacetime dimensions is defined as

$$\frac{\delta}{\delta\phi(x')}\mathcal{F}[\phi(x)] = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big(\mathcal{F}[\phi(x)+\varepsilon\delta^d(x-x')] - \mathcal{F}[\phi(x)]\Big) . \tag{A.2}$$

In particular one finds

$$\frac{\delta}{\delta\phi(x')}\phi(x) = \delta^d(x-x') \tag{A.3}$$

and the chain rule

$$\frac{\delta}{\delta\phi(x')}\mathcal{F}\big[\mathcal{G}[\phi(x)]\big] = \int \mathrm{d}^d y \, \frac{\delta\mathcal{F}\big[\mathcal{G}[\phi(x)]\big]}{\delta\mathcal{G}[\phi(y)]}\frac{\delta\mathcal{G}[\phi(y)]}{\delta\phi(x')} = \frac{\delta\mathcal{F}\big[\mathcal{G}[\phi(x)]\big]}{\delta\mathcal{G}} \cdot \frac{\delta\mathcal{G}}{\delta\phi(x')} , \tag{A.4}$$

where we have defined the abbreviation $\cdot$ to indicate that possible indices are contracted and that the spacetime dependence is integrated over. In particular this means

$$f \cdot g = \int \mathrm{d}^d x \, f_I(x)g^I(x) , \tag{A.5}$$

and

$$f \cdot \mathcal{M} \cdot g = \int \mathrm{d}^d x \, \mathrm{d}^d y \, f^I(x)\mathcal{M}_{IJ}(x,y)g^J(y) , \tag{A.6}$$

for indices I, J that collectively denote spacetime and group indices.

The functional derivative allows one to define a generating functional $Z[J]$ from which all Green functions can be obtained. One introduces external sources $J(x)$ for the fields and defines

$$Z[J] = \int \mathcal{D}\phi \, \mathrm{e}^{iS[\phi]+i\phi\cdot J} , \tag{A.7}$$

such that $Z[0] = N^{-1}$ gives the normalization factor N in (A.1). The n-point function can now be obtained by taking n functional derivatives of the generating functional and setting $J = 0$ afterwards

$$G(x_1,\ldots,x_n) = \big\langle\phi(x_1)\ldots\phi(x_n)\big\rangle = \frac{1}{Z[0]}\frac{\delta}{i\delta J(x_1)}\cdots\frac{\delta}{i\delta J(x_n)}Z[J]\bigg|_{J=0} . \tag{A.8}$$

The generating functional can therefore be written as a series expansion in J

$$\frac{Z[J]}{Z[0]} = \sum_n \frac{i^n}{n!}\int \mathrm{d}^d x_1\ldots\mathrm{d}^d x_n \, \big\langle\phi(x_1)\ldots\phi(x_n)\big\rangle J(x_1)\ldots J(x_n) . \tag{A.9}$$

In principle a theory can now be specified by giving the action, the generating functional, or all possible Green functions.

There is, however, some redundancy because the above Green functions contain pieces which factorize in independent parts. An n-point function in general includes terms which are already known from correlators with less field insertions. The removal of these contributions leads to the definition of the connected n-point Green function which consists only of the non-factorizable part. The generating functional of the connected Green functions $W[J]$ is related to the functional $Z[J]$ via

$$Z[J] = \mathrm{e}^{iW[J]} \ . \tag{A.10}$$

We thus have

$$G(x_1, \ldots, x_n)_{\mathrm{c}} = \left\langle \phi(x_1) \ldots \phi(x_n) \right\rangle_{\mathrm{c}} = \frac{\delta}{i\delta J(x_1)} \cdots \frac{\delta}{i\delta J(x_n)} \ln \frac{Z[J]}{Z[0]}\bigg|_{J=0} \ , \tag{A.11}$$

where the subscript c indicates the connected part. The series expansion is given by

$$\ln \frac{Z[J]}{Z[0]} = \sum_n \frac{i^n}{n!} \int \mathrm{d}^d x_1 \ldots \mathrm{d}^d x_n \left\langle \phi(x_1) \ldots \phi(x_n) \right\rangle_{\mathrm{c}} J(x_1) \ldots J(x_n) \ . \tag{A.12}$$

That (A.10) in terms of Feynman graphs describes the connected part can be easily checked by computing some examples. An argument based on the cluster property of $W[J]$ can be found in [191].

But even the connected Green functions contain redundant information. They are all given by tree diagrams of so called proper Green functions. The intuitive definition of a tree diagram in terms of Feynman graphs is that it is decomposed into two parts by cutting one internal line. In contrast to this, the proper Green functions do not decay into two parts by cutting only one internal line. They are one particle irreducible (1PI). The generating functional of the proper Green functions $\Gamma[\phi]$ is defined as the Legendre transform of $W[J]$

$$\Gamma[\phi] = W[J] - \int \mathrm{d}^d x \, \phi(x)J(x) \ , \tag{A.13}$$

where $\phi(x)$ is the field in presence of the source $J(x)$, i. e.

$$\phi(x) = \frac{\delta}{\delta J(x)} W[J] \ , \qquad J(x) = -\frac{\delta}{\delta \phi(x)} \Gamma[J] \ . \tag{A.14}$$

One can obtain $\Gamma[\phi]$ from the above equations by first inserting a power series of $W[J]$ in the first equation, then inverting the first equation to obtain J as a functional of ϕ

and then equating this with the second equation. This procedure requires the inverse of the connected 2-point function which is the proper 2-point function. One can convince oneself in this way that $\Gamma[\phi]$ generates the proper Green functions. A proof along the lines that $\Gamma[\phi]$ is the effective action can be found in [185]. An alternative proof that adds a disconnected piece with an infinitesimal parameter ε to the propagator and then shows that the $\mathcal{O}(\varepsilon)$ contribution to $\Gamma[\phi]$ is connected is presented in [191].

The knowledge of all proper Green functions up to a given order in the perturbation expansion is sufficient for a computation of all Green functions up to the same order as tree diagrams. This is the statement that $\Gamma[\phi]$ is the effective action of the underlying quantum field theory.

To be more precise, the proper Green functions are amputated, i. e. their external lines are removed. In a tree diagram the proper n-point functions with $n > 2$ are joined with the connected 2-point function. External legs are restored by using the connected 2-point function, too.

We can now define two different sets of building blocks from which all Green functions can be computed. One set is extracted from the action $S[\phi]$ itself. The second set is extracted from the effective action $\Gamma[\phi]$ and it contains the effective building blocks that already include the quantum corrections. With this set only tree diagrams have to be computed.

At the end of this introduction let us discuss the physical meaning of the different kinds of Green functions. The amputated n-point Green function with all external momenta being on-shell is the n-point contribution to the S-matrix. It is the amplitude that describes the scattering of n particles with each other. One assumes that the interaction takes place at a finite region in space and time and the incoming and outgoing particles are produced and respectively measured infinitely far away from the interaction region. This is implemented by the amputation and the choice of on-shell external momenta of the n-point Green function. The amputated Green function contains contributions where not all particles really interact with each other. One particle could enter the interaction region and leave it without interacting with all the other particles. On the level of Green functions this means that the contribution to the n-point Green function describing such a scattering process factorizes into a 2- and an $n-2$-point function. As the n-point Green function includes contributions from all possible scattering scenarios of n particles it encompasses contributions that factorize into two and more Green functions with less

legs. However, besides these contributions there is one piece where all n particles interact and which therefore cannot be split into independent factors. This part contains new information. The other parts are already known as products of Green functions with less than n legs.

A.1.2 Feynman graphs from path integrals

In this Subsection we will describe how a path integral can be evaluated in perturbation theory. The discussion will be rather general and it will lead us to a prescription how to find the momentum space Feynman rules.

A theory of several fields (or field components) ϕ^I is given by an action

$$S[\phi] = -\frac{1}{2} \int \mathrm{d}^d x \, \mathrm{d}^d y \, \phi(x)^I \mathcal{K}_{IJ}(x,y) \phi^J(y) - \int \mathrm{d}^d x \, V[\phi(x)] \,, \qquad (A.15)$$

where $\mathcal{K}$ describes an operator acting to the right, I, J are multi-indices and V is a potential term. We will now show how to deal with the path integral (see (A.7)) perturbatively in powers of the potential, assuming that the potential describes a small perturbation to the quadratic part of the action. First, remove the potential term inside the path integral by rewriting it in terms of functional derivatives

$$Z[J] = \mathrm{e}^{-i \int \mathrm{d}^d x \, V[\frac{\delta}{i \delta J}]} \int \mathcal{D}\phi \, \mathrm{e}^{-\frac{i}{2}\phi \cdot \mathcal{K} \cdot \phi + i\phi \cdot J} \,. \qquad (A.16)$$

The path integral with the remaining integrand can now be evaluated. The field redefinition

$$\phi'^I(x) = \phi^I(x) - \int \mathrm{d}^d y \, \Delta^{IJ}(x,y) J_J(y) \,, \qquad (A.17)$$

where Δ is the classical two point function which is the inverse of the operator $\mathcal{K}_{IJ}$

$$\int \mathrm{d}^d y \, \mathcal{K}_{IJ}(x,y) \Delta^{JI'}(y,x') = \delta_I^{I'} \delta^d(x-x') \qquad (A.18)$$

transforms the integral into one of Gaussian type. The integral is independent of J and therefore contributes only to the normalization. The dependence on the sources J is completely included in the term that remaines from completing the square in the exponent. We denote the free generating functional with Z^{kin} and it is given by

$$Z_{\mathrm{kin}}[J] = \int \mathcal{D}\phi \, \mathrm{e}^{-\frac{i}{2}\phi \cdot \mathcal{K} \cdot \phi + i\phi \cdot J} = \mathrm{e}^{\frac{i}{2} J \cdot \Delta \cdot J} \,. \qquad (A.19)$$

The complete generating functional then reads

$$Z[J] = \mathrm{e}^{-i \int \mathrm{d}^d x \, V[\frac{\delta}{i\delta J}]} \, \mathrm{e}^{\frac{i}{2} J \cdot \Delta \cdot J} \ . \tag{A.20}$$

One can now determine the n-point Green functions to arbitrary order in the perturbation expansion as follows. First one expands the exponential function up to order n in V. Then one acts with the derivatives inside the V on the second exponential factor. The last step is to project out from this result, all terms which are of order n in J. The projection is performed by acting with n functional derivatives, like in (A.8), and setting $J = 0$ afterwards.

The free theory

In the case of a free theory, where the first factor in (A.20) is absent, the generating functional is given by

$$Z[J] = Z_{\mathrm{kin}}[J] = \mathrm{e}^{\frac{i}{2} J \cdot \Delta \cdot J} \ . \tag{A.21}$$

From the definition (A.10) the generating functional for the connected Green functions is as follows

$$W_{\mathrm{kin}}[J] = \frac{1}{2} \int \mathrm{d}^d x \, \mathrm{d}^d y \, J_I(x) \Delta^{IJ}(x, y) J_J(y) \ . \tag{A.22}$$

The consequence of this result is that (using (A.11)) in the free case the only connected Green function is the 2-point function

$$G^{I_1 I_2}(x_1, x_2) = \left\langle \phi(x_1) \phi(x_2) \right\rangle_{\mathrm{c}} = -i\Delta^{I_1 I_2}(x_1, x_2) \ . \tag{A.23}$$

A generic n-point Green function is therefore either 0 if n is odd or it is a sum over all possibilities to factorize the n-point function into a product of $\frac{n}{2}$ 2-point functions if n is even[1].

From now on assume that the operator $\mathcal{K}$ of (A.15) has the following form

$$\mathcal{K}_{IJ}(x, y) = K_{IJ}\delta^d(x - y) \ , \tag{A.24}$$

where K_{IJ} is a matrix-valued differential operator that acts on x to the right. The relation (A.18) then becomes

$$K_{IJ}\Delta^{JI'}(x, x') = \delta_I^{I'} \delta^d(x - x') \ . \tag{A.25}$$

---

[1]We exclude the possibility of non-vanishing vacuum expectation values of the fields and of a coupling to a background field

Introduce the Fourier transform of a general n-point function and its inverse as follows

$$\tilde{G}^{I_1 \dots I_n}(p_1, \dots, p_n) = \int \mathrm{d}^d x_1 \, \mathrm{e}^{-ip_1 \cdot x_1} \cdots \int \mathrm{d}^d x_n \, \mathrm{e}^{-ip_n \cdot x_n} \, G^{I_1 \dots I_n}(x_1, \dots, x_n) \, ,$$

$$G^{I_1 \dots I_n}(x_1, \dots, x_n) = \int \frac{\mathrm{d}^d p_1}{(2\pi)^d} \, \mathrm{e}^{ip_1 \cdot x_1} \cdots \int \frac{\mathrm{d}^d p_n}{(2\pi)^d} \, \mathrm{e}^{ip_n \cdot x_n} \, \tilde{G}^{I_1 \dots I_n}(p_1, \dots, p_n) \, ,$$

(A.26)

where $p \cdot x$ denotes the ordinary scalar product. This refers to the convention that all momenta are *incoming* momenta [93], i. e. momentum p_i flows to the point x_i. The δ-functions are then represented as usual

$$\delta^d(x) = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \, \mathrm{e}^{ip \cdot x} \, , \qquad \delta^d(p) = \int \frac{\mathrm{d}^d x}{(2\pi)^d} \, \mathrm{e}^{-ip \cdot x} \, .$$

(A.27)

One then finds for (A.25)

$$\tilde{\Delta}^{I_1 I_2}(p_1, p_2) = \left(\frac{1}{K_{\partial_{x_1} \to ip_1}} \right)^{I_1 I_2} (2\pi)^d \delta^d(p_1 + p_2) = \tilde{\Delta}^{I_1 I_2}(p_1)(2\pi)^d \delta^d(p_1 + p_2) \, ,$$

(A.28)

where $\left(K^{-1}_{\partial_{x_1} \to ip_1} \right)^{I_1 I_2}$ indicates the expression which one obtains from the differential operator $K_{I_1 I_2}$ via replacing the derivatives $\frac{\partial}{\partial x_1}$ with ip_1 and then taking the inverse of the matrix.

The interacting theory

Let us now discuss the interacting theory with the functional (A.20), and with a potential given by

$$V[\phi] = g\mathcal{G}\left[\phi^{I_1}, \dots, \phi^{I_N} \right] \, ,$$

(A.29)

where g is the coupling constant and $\mathcal{G}$ is a functional that is linear in all its arguments. Here we will not discuss the general perturbative expansion to all orders in g but instead focus on the extraction of the Feynman rules for the theory. The fundamental building blocks (or Feynman rules) from which all diagrams in a perturbative expansion can be built are the connected 2-point function that we have already found in (A.28), and the proper n-point tree level Green functions with $n > 2$ in momentum space. It is clear that from the potential (A.29) one only finds an N-point vertex. Its exact expression in momentum space will now be determined.

First expand (A.20) in the lowest nontrivial order in the coupling constant g. This here gives

$$Z[J] = \left\{ 1 - ig \int \mathrm{d}^d z \, \mathcal{G}\left[\tfrac{\delta}{i\delta J_{I_1}}, \dots, \tfrac{\delta}{i\delta J_{I_N}} \right] + \dots \right\} \exp \left\{ \frac{i}{2} \int \mathrm{d}^d x \, \mathrm{d}^d y \, J_I(x) \Delta^{IJ}(x, y) J_J(y) \right\} \, .$$

(A.30)

After acting with the N derivatives in $\mathcal{G}$ the term that contribute to the N-point function have to be of the order N. The only relevant term in the expansion of the second exponential factor therefore is the one proportional to $(J \cdot \Delta \cdot J)^N$. One obtains

$$G^{J_1 \ldots J_N}(x_1, \ldots, x_N)_{\mathrm{p'}} = \frac{\delta}{i\delta J_{J_1}(x_1)} \cdots \frac{\delta}{i\delta J_{J_N}(x_n)}(-i)g \int \mathrm{d}^d z \, \mathcal{G}\big[\Delta \cdot J, \ldots, \Delta \cdot J\big]$$

$$= (-i)^{N+1} g \int \mathrm{d}^d z \sum_{\pi \in S_N} \mathcal{G}\big[\Delta^{I_1 J_{\pi(1)}}(z, x_{\pi(1)}), \ldots, \Delta^{I_N J_{\pi(N)}}(z, x_{\pi(N)})\big] \, ,$$

$$\text{(A.31)}$$

where the subscript p' denotes the proper (but not truncated) part and where we have used the abbreviation

$$(\Delta \cdot J)^I(z) = \int \mathrm{d}^d y \, \Delta^{IJ}(z, y) J_J(y) \, . \tag{A.32}$$

The sum in (A.31) runs over $N!$ permutations of the permutation group S_N. It is generated because there are $N!$ possibilities for the N functional derivatives to act on the N sources J. The Green function is therefore symmetric under permutations of the indices (J_i, x_i), $i = 1, \ldots, N$ at each leg with the indices at any other leg. To find the Feynman rule for this vertex one now has to transform to momentum space and to amputate the diagram. First, insert the momentum space expressions for Δ given by

$$\Delta^{II'}(x, x') = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \, \mathrm{e}^{-ip \cdot (x-x')} \, \tilde{\Delta}^{II'}(p) \, , \tag{A.33}$$

which follows from (A.28) if p points from x to x', to obtain

$$G^{J_1 \ldots J_N}(x_1, \ldots, x_N)_{\mathrm{p'}} = (-i)^{N+1} g \int \frac{\mathrm{d}^d p_1}{(2\pi)^d} \mathrm{e}^{ip \cdot x_1} \cdots \int \frac{\mathrm{d}^d p_N}{(2\pi)^d} \mathrm{e}^{ip \cdot x_N} \int \mathrm{d}^d z \, \mathrm{e}^{-i(p_1 + \cdots + p_N)z}$$

$$\sum_{\pi \in S_N} \mathcal{G}_{(\partial)_j \to ip_{\pi(j)}}\big[\tilde{\Delta}^{I_1 J_{\pi(1)}}(p_{\pi(1)}), \ldots, \tilde{\Delta}^{I_N J_{\pi(N)}}(p_{\pi(N)})\big] \, .$$

$$\text{(A.34)}$$

Here $\mathcal{G}_{(\partial)_j \to ip_j}$ denotes the expression which is obtained from the functional $\mathcal{G}$ by replacing all derivatives that act on the jth argument by ip_j $(j = 1, \ldots, N)$. The functional then is no longer a functional. It becomes a function of the momenta p_j and reads

$$\mathcal{G}_{(\partial)_j \to ip_j}\big[\tilde{\Delta}^{I_1 J_1}(p_1), \ldots, \tilde{\Delta}^{I_N J_N}(p_N)\big] = \big(\mathcal{G}_{(\partial)_j \to ip_j}\big)_{I_1 \ldots I_N} \tilde{\Delta}^{I_1 J_1}(p_1), \ldots, \tilde{\Delta}^{I_N J_N}(p_N) \, . \tag{A.35}$$

The proper (truncated) Green function (denoted with subscript p) is now obtained by simply removing the N factors $-i\tilde{\Delta}$. Furthermore, we assume that $\mathcal{G}$ does not explicitly

$$p,I \quad \text{————} \quad p,J \qquad = -i\Big(\frac{1}{K_{\partial_x \to ip}}\Big)^{IJ}$$

$$= -ig\,\dim(S) \sum_{[\pi]\in\frac{S_N}{S}} \big(\mathcal{G}_{(\partial)_j \to -ip_{\pi(j)}}\big)^{I_{\pi(1)}\dots I_{\pi(N)}}$$

Figure A.1: General Feynman rules for the theory (A.15) with one N-point vertex (A.29). Momentum conservation is understood. The momentum p of the propagator enters the point x and all momenta of the vertex point to the vertex.

depend on the spacetime coordinates z. Then in (A.34) the z-integration can be carried out and one finds by comparing with (A.26) and using (A.27) that

$$
\begin{aligned}
\tilde{G}^{I_1\dots I_N}(p_1,\dots,p_N)_{\mathrm{p}} &= -ig \sum_{\pi\in S_N} \big(\mathcal{G}_{(\partial)_j \to ip_{\pi(j)}}\big)^{I_{\pi(1)}\dots I_{\pi(N)}}(2\pi)^d\delta^d(p_1+\dots+p_N)\\
&= -ig\,\dim(S) \sum_{[\pi]\in\frac{S_N}{S}} \big(\mathcal{G}_{(\partial)_j \to ip_{\pi(j)}}\big)^{I_{\pi(1)}\dots I_{\pi(N)}}(2\pi)^d\delta^d(p_1+\dots+p_N)\,.
\end{aligned}
$$

$$(A.36)$$

In the second line S denotes the symmetry group of $\mathcal{G}$ and the sum runs over one element of each orbit $[\pi] = \big\{\pi' \in S_N \,|\, \pi' = S\pi\big\}$. It is clear that this simplifies a concrete evaluation, for instance if $\mathcal{G}$ is symmetric in all arguments then S is the full permutation group with $\dim(S) = N!$ and the sum only consists of one element.

The above expression directly produces the momentum space Feynman rule for the N-point interaction vertex (A.29) with all momenta p_j leaving the vertex. If one wants to have all momenta to point to the vertex then one has to replace $p_j \to -p_j$, see Fig. A.1. Let us remark that if one wants to compute Feynman diagrams from the above rules one still has to deal with symmetry factors that depend on the concrete diagram.

In Chapter 3 the above given expressions are used to determine the Feynman rules for the noncommutative YM theories.

A.2 Invariance of the DBI action

Let M be a quadratic invertible matrix and δM a small variation. The inverse of $M + \delta M$ up to $\mathcal{O}(\delta M)$ is given by

$$\frac{1}{M + \delta M} = \frac{1}{M}(M - \delta M)\frac{1}{M} \ . \tag{A.37}$$

The determinant of $M + \delta M$ expanded up to $\mathcal{O}\big((\delta M)^2\big)$ reads

$$\det(M + \delta M) = \det M\Big[1 + \text{tr}(M^{-1}\delta M) + \frac{1}{2}(\text{tr}(M^{-1}\delta M))^2 - \frac{1}{2}\text{tr}(M^{-1}\delta M M^{-1}\delta M) + \dots\Big] \ . \tag{A.38}$$

With these relations we can now compute the variation of (2.42) under $\delta\theta^{\mu\nu}$. The variations

$$\Phi_{\mu\nu} \to \Phi_{\mu\nu} + \delta\Phi_{\mu\nu} \ , \qquad G_{\mu\nu} \to G_{\mu\nu} + \delta G_{\mu\nu} \ , \qquad \theta^{\mu\nu} \to \theta^{\mu\nu} + \delta\theta^{\mu\nu} \tag{A.39}$$

are not independent due to (2.41) but fulfill

$$(\delta G + \tilde{\alpha}\delta\Phi)_{\mu\nu} = \Big((G + \tilde{\alpha}\Phi)\frac{\delta\theta}{\tilde{\alpha}}(G + \tilde{\alpha}\Phi)\Big)_{\mu\nu} \ , \tag{A.40}$$

as can be seen with the help of (A.37) and $\tilde{\alpha} = 2\pi\alpha'$. Considering the part of $\mathcal{O}(\delta M)$ in (A.38) it is then easy to derive the variations

$$\delta\sqrt{\det(G + \tilde{\alpha}(\Phi + F))} = \frac{1}{2}\sqrt{\det(G + \tilde{\alpha}(\Phi + F))}\,\text{tr}\,\Big(\frac{1}{G + \tilde{\alpha}(\Phi + F)}(\delta G + \tilde{\alpha}(\delta\Phi + \delta F))\Big) \ , \tag{A.41}$$

and thus with one finds in particular from (2.46) that

$$\delta(G_0^\Phi)^2 = \frac{1}{2}(G_0^\Phi)^2\,\text{tr}\,\Big((G + \tilde{\alpha}\Phi)\frac{\delta\theta}{\tilde{\alpha}}\Big) \ . \tag{A.42}$$

The variation of the DBI Lagrangian (2.42) then reads up to a factor $g_s T_p$

$$\delta\Big(\frac{1}{(G_0^\Phi)^2}\sqrt{\det(G + \tilde{\alpha}(\Phi + F))}\Big)$$
$$= \frac{1}{(2G_0^\Phi)^2}\sqrt{\det(G + \tilde{\alpha}(\Phi + F))}\,\text{tr}\,\Big(\frac{1}{G + \tilde{\alpha}(\Phi + F)}\Big[-(G + \tilde{\alpha}\Phi)\delta\theta F + \tilde{\alpha}\delta F\Big]\Big) \ . \tag{A.43}$$

We now insert (2.63) which describes how $\delta F_{\mu\nu}$ depends on $\delta\theta^{\mu\nu}$. For the Abelian case one finds

$$\delta F_{\mu\nu} = \delta\theta^{\alpha\beta}\big(F_{\mu\alpha}F_{\nu\beta} - \frac{1}{2}A_\alpha(\partial_\beta + D_\beta)F_{\mu\nu} + \mathcal{O}(\partial F \partial F)\big) \tag{A.44}$$
$$= -(F\delta\theta F)_{\mu\nu} - \frac{1}{2}\vec{A}\delta\theta(\vec{\partial} + \vec{D})F_{\mu\nu} + \mathcal{O}(\partial F \partial F) \ ,$$

where in the second line we have used matrix notation. Furthermore, it is easy to see that

$$\partial_l \sqrt{\det(G + \tilde{\alpha}(\Phi + F))} = \frac{\tilde{\alpha}}{2} \sqrt{\det(G + \tilde{\alpha}(\Phi + F))} \, \text{tr} \left(\frac{1}{G + \tilde{\alpha}(\Phi + F)} \partial_l F \right) ,$$

$$D_l \sqrt{\det(G + \tilde{\alpha}(\Phi + F))} = \frac{\tilde{\alpha}}{2} \sqrt{\det(G + \tilde{\alpha}(\Phi + F))} \, \text{tr} \left(\frac{1}{G + \tilde{\alpha}(\Phi + F)} D_l F \right) + \mathcal{O}(\partial F D F) .$$

$$(A.45)$$

One inserts (A.44) into (A.43) and then uses the above relations to reexpress the terms where derivatives act on $F_{\mu\nu}$. After integrating by parts, one obtains

$$\delta \left(\frac{1}{(G_0^\Phi)^2} \sqrt{\det(G + \tilde{\alpha}(\Phi + F))} \right)$$
$$= \frac{1}{(2G_0^\Phi)^2} \sqrt{\det(G + \tilde{\alpha}(\Phi + F))} \left[- \text{tr}(\delta\theta F) + (\partial_\beta + D_\beta)(A\delta\theta)^\beta \right] + \mathcal{O}(\partial F) + \text{tot. der.}$$
$$= \mathcal{O}(\partial F) + \text{total derivatives} ,$$

$$(A.46)$$

where the last step follows with the identity

$$(\partial_\beta + D_\beta)(A\delta\theta)^\beta = \delta\theta^{\alpha\beta} F_{\beta\alpha} = \text{tr}(\delta\theta F) . \tag{A.47}$$

Hence, the Seiberg-Witten map translates the field strength in such a way that (2.45) holds. At the end it is important to remark that we had to take into account the second term in (A.44) although naively it is $\mathcal{O}(\partial F)$. The reason is that it contains a factor A_μ without a derivative such that it is not negligible after partial integration that shifts the derivative to this 'bare' A_μ. If one works with the known explicit solution of the differential equation (2.63) in the case of an (exactly) constant $F_{\mu\nu}$, the absence of this term is responsible for an observed mismatch [158, 187] in the relation (2.45). An extension of the above given calculation to the non-Abelian case can be found in [178].

A.3 The Weyl operator formalism

In a noncommutative spacetime the coordinates no longer commute. Instead one has the relation

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} , \tag{A.48}$$

where $\hat{x}^\mu$ are Hermitian operators and we assume $\theta^{\mu\nu}$ to be constant. One can now define the noncommutative counterpart to a function f in ordinary d-dimensional space $\mathbb{R}^d$ by taking its Fourier transform

$$\tilde{f}(k) = \int \mathrm{d}^d x \, \mathrm{e}^{-ik_\mu x^\mu} f(x) \tag{A.49}$$

and using the Weyl symbol

$$\hat{\mathcal{W}}[f] = \int \frac{\mathrm{d}^d k}{(2\pi)^D} \tilde{f}(k)\, \mathrm{e}^{ik_\mu \hat{x}^\mu} \ . \tag{A.50}$$

This procedure is described by the Hermitian operator $\hat{\Delta}(x) = \hat{\Delta}^\dagger(x)$ which is given by

$$\hat{\Delta}(x) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, \mathrm{e}^{ik_\mu \hat{x}^\mu}\, \mathrm{e}^{-ik_\mu x^\mu} \ , \tag{A.51}$$

such that

$$\hat{\mathcal{W}}[f] = \int \mathrm{d}^d x\, f(x)\, \hat{\Delta}(x) \ . \tag{A.52}$$

Derivatives can be defined as

$$\left[\hat{\partial}_\mu, \hat{x}^\nu\right] = \delta^\nu_\mu \ , \qquad \left[\hat{\partial}_\mu, \hat{\partial}_\nu\right] = 0 \ . \tag{A.53}$$

From this definition it follows immediately that

$$\left[\hat{\partial}_\mu, \hat{\Delta}(x)\right] = -\partial_\mu \hat{\Delta}(x) \ , \tag{A.54}$$

and one thus finds from (A.52) after integration by parts

$$\left[\hat{\partial}_\mu, \hat{\mathcal{W}}[f]\right] = \int \mathrm{d}^d x\, \partial_\mu f(x)\, \hat{\Delta}(x) = \hat{\mathcal{W}}[\partial_\mu f] \ . \tag{A.55}$$

The translation operator is given by $\mathrm{e}^{v^\mu \hat{\partial}_\mu}$ and it acts as follows

$$\mathrm{e}^{v^\mu \hat{\partial}_\mu}\, \hat{\Delta}(x)\, \mathrm{e}^{-v^\mu \hat{\partial}_\mu} = \hat{\Delta}(x+v) \ . \tag{A.56}$$

From this it is obvious that a trace defined for the Weyl operators is independent of $x \in \mathbb{R}^d$ because of its invariance under cyclic permutations. One then finds from (A.52) that the trace '$\hat{\mathrm{tr}}$' corresponds to an integration over spacetime

$$\hat{\mathrm{tr}}\, \hat{\mathcal{W}}[f] = \int \mathrm{d}^d x\, f(x) \ , \tag{A.57}$$

where we have normalized $\hat{\mathrm{tr}}\, \hat{\Delta}(x) = 1$. The products of operators at distinct points can be defined by using the Baker-Campbell-Hausdorff formula

$$\mathrm{e}^A\, \mathrm{e}^B = \mathrm{e}^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}([A,[A,B]]+[B,[B,A]])+\cdots} \ . \tag{A.58}$$

In the special case where the commutator is a c-number as in (A.48), one obtains

$$\mathrm{e}^{ik_\mu \hat{x}^\mu}\, \mathrm{e}^{ik'_\mu \hat{x}^\mu} = \mathrm{e}^{-\frac{i}{2}\theta^{\mu\nu}k_\mu k'_\nu}\, \mathrm{e}^{i(k+k')_\mu \hat{x}^\mu} \ . \tag{A.59}$$

Using (A.51) and assuming that θ^{-1} exists (for which an even spacetime dimension is necessary), the product of two $\hat{\Delta}$-operators reads

$$\hat{\Delta}(x)\,\hat{\Delta}(y) = \frac{1}{\pi^d|\det\theta|} \int \mathrm{d}^d z\; \hat{\Delta}(z) e^{-2i(\theta^{-1})_{\mu\nu}(z^\mu - x^\mu)(z^\nu - y^\nu)}\;. \tag{A.60}$$

With the normalization $\hat{\mathrm{tr}}\,\hat{\Delta}(x) = 1$, it follows that the operators $\hat{\Delta}(x)$ and $\hat{\Delta}(y)$ are orthonormal w. r. t. the trace operation

$$\hat{\mathrm{tr}}\big(\hat{\Delta}(x)\,\hat{\Delta}(y)\big) = \delta^d(x - y)\;. \tag{A.61}$$

Hence, the inverse of the Weyl-operator (A.52) is well defined given by

$$f(x) = \hat{\mathrm{tr}}\big(\hat{\mathcal{W}}[f]\,\hat{\Delta}(x)\big) \tag{A.62}$$

for a function f.

A.4   The ∗-product

The operation of multiplication of Weyl operators can be captured by introducing a noncommutative ∗-product in ordinary space that has to fulfill the relation

$$\hat{\mathcal{W}}[f]\,\hat{\mathcal{W}}[g] = \hat{\mathcal{W}}[f * g]\;. \tag{A.63}$$

Using the Fourier transformation (A.49), the Weyl transformation (A.52), the definition of $\hat{\Delta}$ (A.51) and the Baker-Campbell-Hausdorff relation (A.59) for the commutator (A.48), one finds for the ∗-product

$$(f * g)(x) = \iint \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d k'}{(2\pi)^d} \tilde{f}(k)\tilde{g}(k') e^{-\frac{i}{2}\theta^{\mu\nu} k_\mu k'_\nu} e^{i(k_\mu + k'_\mu)x^\mu}\;. \tag{A.64}$$

With the help of the inverse Fourier transformation the above expression can be cast into the following form

$$\begin{aligned}
(f * g)(x) &= \exp\left\{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\right\} f(x)g(y)\Big|_{y=x}\\
&= f(x)g(x) + \sum_{n=1}^\infty \left(\frac{i}{2}\right)^n \frac{1}{n!}\theta^{i_1 j_1}\cdots\theta^{i_n j_n}\partial_{i_1}\cdots\partial_{i_n}f(x)\partial_{j_1}\cdots\partial_{j_n}g(x)\;.
\end{aligned} \tag{A.65}$$

The second line shows how the exponential function has to be understood. One has obtained the Moyal-Weyl $*$-product [122]. From this result it is easy to derive expressions for the $*$-(anti)commutator

$$[f(x)\overset{*}{,}g(x)] = 2i\sin\left\{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\right\}f(x)g(y)\Big|_{y=x} = 2if(x)\sin\left(\overleftarrow{\partial}_\rho\tfrac{1}{2}\theta^{\rho\sigma}\overrightarrow{\partial}_\sigma\right)g(x) \ ,$$

$$\{f(x)\overset{*}{,}g(x)\} = 2\cos\left\{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\right\}f(x)g(y)\Big|_{y=x} = 2f(x)\cos\left(\overleftarrow{\partial}_\rho\tfrac{1}{2}\theta^{\rho\sigma}\overrightarrow{\partial}_\sigma\right)g(x) \ .$$
(A.66)

The $*$-product is associative and one finds for the product of n functions f_a, $a = 1, \ldots, n$

$$(f_1 * \cdots * f_n)(x) = \prod_{a<b}\exp\left\{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x_a^\mu}\frac{\partial}{\partial x_b^\nu}\right\}f_1(x_1)\cdots f_n(x_n)\Big|_{x_1=\cdots=x_n=x} \ . \tag{A.67}$$

The cyclicity of the operator trace (A.57) translates into the invariance of the integral

$$\hat{\text{tr}}\big(\hat{\mathcal{W}}[f_1]\cdots\hat{\mathcal{W}}[f_n]\big) = \int \mathrm{d}^d x \, f_1(x) * \cdots * f_n(x) \tag{A.68}$$

under cyclic permutations of the f_a. In particular, a trace over two Weyl operators and therefore an integral of two functions multiplied by the $*$-product reduces to the integral with the two functions being multiplied by using the ordinary product

$$\hat{\text{tr}}\big(\hat{\mathcal{W}}[f]\,\hat{\mathcal{W}}[g]\big) = \int \mathrm{d}^d x \, f(x) * g(x) = \int \mathrm{d}^d x \, f(x)g(x) \ . \tag{A.69}$$

A.5   Noncommutative Yang-Mills theories

In Yang-Mills theories the gauge fields take values in a representation of the Lie algebra of the underlying gauge group. One therefore has to generalize the formalism of appendix A.3 somewhat. The Weyl transformation (A.52) is redefined as a tensor product of the $\hat{\Delta}$-operator and the Lie algebra representation matrices to

$$\hat{\mathcal{W}}[A_\mu] = \int \mathrm{d}^d x \, \hat{\Delta}(x) \otimes A_\mu(x) \ . \tag{A.70}$$

One can then write the action of the noncommutative Yang-Mills theories in the operator space as follows

$$S_{\text{YM}} = -\frac{1}{4g^2}\hat{\text{tr}} \otimes \text{tr}\left(\hat{\mathcal{W}}[F_{\mu\nu}]\,\hat{\mathcal{W}}[F^{\mu\nu}]\right) \ , \tag{A.71}$$

where '$\hat{\text{tr}}$' and 'tr' denote the traces w. r. t. the spacetime part and the gauge group, and the field strength in operator space reads

$$\begin{aligned}\hat{\mathcal{W}}[F_{\mu\nu}] &= \left[\hat{\partial}_\mu, \hat{\mathcal{W}}[A_\nu]\right] - \left[\hat{\partial}_\nu, \hat{\mathcal{W}}[A_\mu]\right] - i\left[\hat{\mathcal{W}}[A_\mu], \hat{\mathcal{W}}[A_\nu]\right] \\ &= \hat{\mathcal{W}}\left[\partial_\mu A_\nu - \partial_\nu A_\mu - i\,[A_\mu\overset{*}{,}A_\nu]\right] \ .\end{aligned} \tag{A.72}$$

Using the definition of the operator trace (A.57) one finds

$$S_{\mathrm{YM}} = -\frac{1}{4g^2} \int \mathrm{d}^d x \, \mathrm{tr} \left(F_{\mu\nu} * F^{\mu\nu} \right) . \tag{A.73}$$

Due to the symmetry of the trace and of the integral under cyclic permutations of the argument, the above action is invariant under the gauge transformation

$$\hat{\mathcal{W}}[A_\mu] \to \hat{\mathcal{W}}[\tilde{A}_\mu] = \hat{\mathcal{W}}[U] \, \hat{\mathcal{W}}[A_\mu] \, \hat{\mathcal{W}}[U]^{-1} - i \, \hat{\mathcal{W}}[U] \left[\partial_\mu, \hat{\mathcal{W}}[U]^{-1} \right]$$
$$= \hat{\mathcal{W}} \left[U * A_\mu * U^{-1*} - iU * \partial_\mu U^{-1*} \right] , \tag{A.74}$$

because the field strength transforms according to

$$\hat{\mathcal{W}}[F_{\mu\nu}] \to \hat{\mathcal{W}}[\tilde{F}_{\mu\nu}] = \hat{\mathcal{W}}[U] \, \hat{\mathcal{W}}[F_{\mu\nu}] \, \hat{\mathcal{W}}[U]^{-1} = \hat{\mathcal{W}} \left[U * F_{\mu\nu} * U^{-1*} \right] . \tag{A.75}$$

The Weyl transformation of U^{-1*} yields the inverse in the operator space. Hence, in ordinary space it is the inverse of U w. r. t. the $*$-product. The corresponding relations are given by

$$\hat{\mathcal{W}}[U] \, \hat{\mathcal{W}}[U]^{-1} = \hat{\mathcal{W}}[U]^{-1} \, \hat{\mathcal{W}}[U] = \hat{\mathbb{1}} \otimes \mathbb{1} , \qquad U * U^{-1*} = U^{-1*} * U = \mathbb{1} . \tag{A.76}$$

They imply for a constant $\theta^{\mu\nu}$ that the following equalities hold

$$\hat{\mathcal{W}}[U] \left[\partial_\mu, \hat{\mathcal{W}}[U]^{-1} \right] = - \left[\partial_\mu, \hat{\mathcal{W}}[U] \right] \hat{\mathcal{W}}[U]^{-1} , \qquad U * \partial_\mu U^{-1*} = -(\partial_\mu U) * U^{-1*} . \tag{A.77}$$

It is important to remark that one must not identify U^{-1*} and U^{-1} because the latter is defined as the inverse of U w. r. t. the ordinary (matrix) product

$$U \, U^{-1} = \mathbb{1} , \tag{A.78}$$

and therefore it is clear that $\hat{\mathcal{W}}[U^{-1}] \neq \hat{\mathcal{W}}[U]^{-1} = \hat{\mathcal{W}}[U^{-1*}]$. A relation between U^{-1*} and U^{-1} can be worked out order by order in $\theta^{\mu\nu}$ if one expands the $*$-product in (A.76). This leads to (with an invertible $\theta^{\mu\nu}$)

$$U^{-1*} = U^{-1} + \frac{i}{2} \theta^{\mu\nu} U^{-1} (\partial_\mu U) U^{-1} (\partial_\nu U) U^{-1} + \dots . \tag{A.79}$$

In the following we will make some general comments on the gauge transformation (A.74) and especially compare with the ordinary case. In the latter we have the ordinary YM gauge field a_μ and the finite gauge transformation u which acts like

$$a_\mu \to \tilde{a}_\mu = u \, a_\mu \, u^{-1} - iu \, \partial_\mu u^{-1} . \tag{A.80}$$

If we require that the gauge field is Hermitian we have to impose the unitarity condition on u, i. e.

$$u^{-1} = u^\dagger \ . \tag{A.81}$$

The quantity u is an element of the gauge group and the gauge field a_μ takes values in the corresponding Lie algebra. The same is true for the transformed gauge field $\tilde{a}_\mu$ in (A.80).

The gauge transformation (A.74) preserves the Hermiticity of a Hermitian gauge field $A_\mu = A_\mu^\dagger$ if one chooses U as $*$-unitary, i. e.

$$U^{-1*} = U^\dagger \ . \tag{A.82}$$

One can therefore define noncommutative gauge theories with $U(N)$ gauge groups. That a naive extension to other gauge groups appears to be difficult can most easily be seen from the infinitesimal versions of the gauge transformations. With Λ as an infinitesimal gauge transformation parameter one finds from (A.74)

$$A_\mu \to A_\mu + \delta A_\mu \ , \qquad \delta A_\mu = \partial_\mu \Lambda + i \left[\Lambda \, ; A_\mu \right] \ . \tag{A.83}$$

This has to be compared with the infinitesimal gauge transformation with parameter λ in the ordinary case extracted from (A.80)

$$a_\mu \to a_\mu + \delta a_\mu \ , \qquad \delta a_\mu = \partial_\mu \lambda + i \left[\lambda , a_\mu \right] \ . \tag{A.84}$$

Since λ and a_μ are Lie algebra valued in the ordinary case, the infinitesimal gauge transformation is guaranteed to be Lie algebra valued, too. This is because the ordinary commutator of two elements of the Lie algebra is itself an element. In the noncommutative case, however, the situation is different. One has to evaluate the $*$-commutator that occurs in (A.83) and analyze in which cases it closes on the algebra, see section 3.1.

A.6 The Seiberg-Witten map from the enveloping algebra approach

In [95] the authors deal with the noncommutative coordinates $\hat{x}^\mu$ and the enveloping algebra generators on an equal footing. They replace them by ordinary quantities x^μ and t^a respectively and describe the noncommutativity with a $*$-product. A product of n

variables t^a corresponds to the symmetrized product of n Lie algebra generators $\mathfrak{t}^a$ given by (see (3.6))

$$t^{a_1} \cdots t^{a_n} = \frac{1}{n!} \sum_{\pi \in S_n} \mathfrak{t}^{a_{\pi(1)}} \cdots \mathfrak{t}^{a_{\pi(n)}} \ . \tag{A.85}$$

These generators span the corresponding enveloping algebra. The star product that describes the spacetime noncommutativity and the gauge algebra is defined by

$$(f * g)(x, t) = e^{\frac{i}{2}(\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x'^\nu} + t^a g_a(i\frac{\partial}{\partial t}, i\frac{\partial}{\partial t'}))} f(x, t) g(x', t')\big|_{x'=x, t'=t} \ . \tag{A.86}$$

Here $g_a(u, v)$ follows from the group multiplication

$$e^{u_a \mathfrak{t}^a} e^{v_b \mathfrak{t}^b} = e^{i(u_c + v_c + \frac{1}{2}g_c(u,v))\mathfrak{t}^c} \tag{A.87}$$

and with the Baker-Campbell-Hausdorff formula (A.58) one finds the expansion

$$g_c(u, v) = -u_a v_b f^{ab}{}_c + \frac{1}{6} u_a v_b (v_d - u_d) f^{ab}{}_e f^{ed}{}_c + \dots \ . \tag{A.88}$$

In [111] infinitesimal gauge transformations on noncommutative space have been defined for a field ϕ as

$$\hat{\delta}_\Lambda \, \hat{\mathcal{W}}[\phi] = i \, \hat{\mathcal{W}}[\Lambda] \, \hat{\mathcal{W}}[\phi] \ . \tag{A.89}$$

Multiplication of a field by a coordinate is not a covariant operation since

$$\hat{\delta}_\Lambda \big(\hat{\mathcal{W}}[x^\mu] \, \hat{\mathcal{W}}[\phi] \big) = i \, \hat{\mathcal{W}}[x^\mu] \, \hat{\mathcal{W}}[\Lambda] \, \hat{\mathcal{W}}[\phi] \neq i \, \hat{\mathcal{W}}[\Lambda] \, \hat{\mathcal{W}}[x^\mu] \, \hat{\mathcal{W}}[\phi] \ . \tag{A.90}$$

Therefore, one introduces covariant coordinates that commute with the gauge transformation

$$\hat{\delta}_\Lambda \big(\hat{\mathcal{W}}[X^\mu] \, \hat{\mathcal{W}}[\phi] \big) = \hat{\mathcal{W}}[X^\mu] \, \hat{\delta}_\Lambda \, \hat{\mathcal{W}}[\phi] \ . \tag{A.91}$$

With the ansatz $X^\mu = x^\mu + V^\mu(x)$ one finds for $V^\mu(x)$

$$\hat{\delta}_\Lambda \, \hat{\mathcal{W}}[V^\mu] = -i \left[\hat{x}^\mu, \hat{\mathcal{W}}[\Lambda] \right] + i \left[\hat{\mathcal{W}}[\Lambda], \hat{\mathcal{W}}[V^\mu] \right] \ , \tag{A.92}$$

where [,] denotes the commutator in the underlying noncommutative space. The corresponding expression in ordinary space is found by replacing all functions by their ordinary counterparts and all products between Weyl operators by the corresponding $*$-product. The transformation of the gauge connection then becomes

$$\hat{\delta}_\Lambda V^\mu = \theta^{\mu\nu} \partial_\nu \Lambda + i \left[\Lambda \stackrel{*}{,} V^\mu \right] \ , \tag{A.93}$$

where we have used the $*$-product (A.86) that leads one to the relation

$$- i \left[x^\mu \overset{*}{,} \Lambda \right] = \theta^{\mu\nu} \partial_\nu \Lambda . \tag{A.94}$$

The second term in (A.93) with the $*$-product (A.86) in the case of a non-vanishing g_c in (A.88) shows, that the transformation of V^μ starts either at order θ^0 or linearly in $\theta^{\mu\nu}$ if V^μ itself is of the order θ^0 or of higher order in $\theta^{\mu\nu}$. It is thus reasonable to assume that V^μ starts with a term of at most first order in $\theta^{\mu\nu}$, because with the transformation (A.93) one can always generate a term that is linear in $\theta^{\mu\nu}$. Furthermore, it is important to stress that the connection V^μ was introduced to make multiplication in a noncommutative space a covariant operation. It should vanish for $\theta^{\mu\nu} \to 0$ as the space becomes commutative. Hence, it should start at linear order in $\theta^{\mu\nu}$ and one can make the ansatz

$$V^\mu = \theta^{\mu\nu} A_\nu ,$$
$$\hat{\delta}_\Lambda A_\mu = \partial_\mu \Lambda + i \left[\Lambda \overset{*}{,} A_\mu \right] \tag{A.95}$$

for the connection. Its form becomes more clear in a comparison with the case in an ordinary space. The gauge connection V^μ itself has no counterpart in ordinary space, where the multiplication with a coordinate is a covariant operation w. r. t. the ordinary counterpart of the gauge transformation (A.89). In ordinary space, differentiation becomes covariant under gauge transformations by introducing a covariant derivative that depends on the gauge connection. In noncommutative spaces one has to start one step earlier and covariantize the coordinates themselves. This already includes the covariantization of ordinary derivatives, because in the the $*$-product (A.86) they appear at $\mathcal{O}(\theta)$. Hence, one should not wonder that a order $\theta^{\mu\nu}$, the gauge connection (A.95) coincides with the one found in the ordinary case.

We will now analyze (A.95) order by order in $\theta^{\mu\nu}$. Expansion of (A.86) gives

$$(f * g)(x,t) = \left(1 + \frac{i}{2} \theta^{\alpha\beta} \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial x'^\beta} + \dots \right) f(x,t) \circledast g(x',t') \Big|_{x'=x,t'=t} ,$$
$$f(x,t) \circledast g(x',t') = e^{\frac{i}{2} t^a g_a (i \frac{\partial}{\partial t}, i \frac{\partial}{\partial t'})} f(x,t) g(x',t') . \tag{A.96}$$

At $\mathcal{O}(\theta^0)$ one finds that the relation for $\hat{\delta}_\Lambda A_\mu$ in (A.95) can be solved with an ansatz where Λ and A_μ depend linearly on t^a

$$\Lambda = \lambda_a^1 t^a , \qquad A_\mu = a_{\mu,a}^1 t^a . \tag{A.97}$$

That this should work is clear from the previous discussion. In the $\theta^{\mu\nu} \to 0$ limit A_μ should become the well known gauge connection on ordinary space which is Lie algebra

valued and which thus depends linearly on t^a. Inserting the ansatz into (A.95) one finds

$$\hat{\delta}_\Lambda a^1_{\mu,a} t^a = \partial_\mu \lambda^1_a t^a + i\lambda^1_a a^1_{\mu,b}[t^a \circledast t'^b - t^b \circledast t'^a]\Big|_{t'=t} = \partial_\mu \lambda^1_a t^a + i\lambda^1_a a^1_{\mu,b} f^{ab}{}_c t^c , \qquad (A.98)$$

where we have used

$$t^a \circledast t'^b \Big|_{t'=t} = t^a t^b + \frac{i}{2} f^{ab}{}_c t^c . \qquad (A.99)$$

This follows from the definition of the $\circledast$-product in (A.96) with the expansion (A.88). At $\mathcal{O}(\theta^1)$ one includes terms which are quadratic in t^a in the ansatz, such that

$$\Lambda = \lambda^1_a t^a + \lambda^2_{ab} t^a t^b , \qquad A_\mu = a^1_{\mu,a} t^a + a^2_{\mu,ab} t^a t^b . \qquad (A.100)$$

The reason that this is sufficient follows from the fact that the $\mathcal{O}(\theta^0)$ terms can contribute at most in second order in t^a. This can be seen from (A.96). Using the result of $\mathcal{O}(\theta^0)$, one finds

$$\begin{aligned}
\hat{\delta}_\Lambda a^2_{\mu,ab} t^a t^b &= \partial_\mu \lambda^2_{ab} t^a t^b - \frac{1}{2}\theta^{\alpha\beta}\partial_\alpha \lambda^1_a \partial_\beta a^1_{\mu,b}(t^a \circledast t'^b + t^b \circledast t'^a)\Big|_{t'=t} \\
&\quad + i(\lambda^1_c a^2_{\mu,ab} - \lambda^2_{ab} a^1_{\mu,c})(t^c \circledast t'^a t'^b - t^a t^b \circledast t'^c)\Big|_{t'=t} \\
&= \partial_\mu \lambda^2_{ab} t^a t^b - \theta^{\alpha\beta}\partial_\alpha \lambda^1_a \partial_\beta a^1_{\mu,b} t^a t^b - 2(\lambda^1_a a^2_{\mu,cb} + \lambda^2_{ab} a^1_{\mu,c}) f^{ac}{}_d t^b t^d ,
\end{aligned} \qquad (A.101)$$

where we have used the symmetry of λ^2_{ab} and $a^2_{\mu,ab}$ under the exchange $a \leftrightarrow b$ and the relations

$$t^c \circledast t'^a t'^b \Big|_{t'=t} = t^a t^b t^c - \frac{i}{2} t^d \left[f^{ac}{}_d t^b + f^{bc}{}_d t^a - \frac{i}{6}(f^{ac}{}_e f^{eb}{}_d + f^{bc}{}_e f^{ea}{}_d) \right] , \qquad (A.102)$$

$$t^a t^b \circledast t'^c \Big|_{t'=t} = t^a t^b t^c + \frac{i}{2} t^d \left[f^{ac}{}_d t^b + f^{bc}{}_d t^a + \frac{i}{6}(f^{ac}{}_e f^{eb}{}_d + f^{bc}{}_e f^{ea}{}_d) \right] . \qquad (A.103)$$

Together, (A.98) and (A.101) read

$$\begin{aligned}
\hat{\delta}_\Lambda a^1_{\mu,a} &= \partial_\mu \lambda^1_a + i\lambda^1_b a^1_{\mu,c} f^{bc}{}_a , \\
\hat{\delta}_\Lambda a^2_{\mu,ab} &= \partial_\mu \lambda^2_{ab} - \theta^{\alpha\beta}\partial_\alpha \lambda^1_a \partial_\beta a^1_{\mu,b} - 2(\lambda^1_d a^2_{\mu,cb} + \lambda^2_{db} a^1_{\mu,c}) f^{dc}{}_a .
\end{aligned} \qquad (A.104)$$

The second equality can now be reformulated. Remember that the elements $t^{a_1}\ldots t^{a_n}$ correspond to symmetric products of the Lie algebra generators $\mathfrak{t}^a$, forming the generators of the enveloping algebra. If we define the following quantities

$$a_\mu = a^1_{\mu,a} \mathfrak{t}^a , \qquad A'_\mu = a^2_{\mu,ab} \mathfrak{t}^a \mathfrak{t}^b , \qquad \lambda = \lambda^1_a \mathfrak{t}^a , \qquad \Lambda' = \lambda^2_{ab} \mathfrak{t}^a \mathfrak{t}^b , \qquad (A.105)$$

then the second equation of (A.104) reads

$$\hat{\delta}_\Lambda A'_\mu = \partial_\mu \Lambda' - \frac{1}{2}\theta^{\alpha\beta} \{\partial_\alpha \lambda, \partial_\beta a_\mu\} + i\left[\lambda, A'_\mu\right] + i\left[\Lambda, a_\mu\right] . \qquad (A.106)$$

This is the exactly the $\mathcal{O}(\theta^1)$ expansion (2.57) of (2.54) that defines the Seiberg-Witten map. The solution is given in (2.59) and its translation to the previously used notation is given by

$$
\begin{aligned}
a_{\mu,ab}^2 t^a t^b &= -\frac{1}{2}\theta^{\alpha\beta} a_{\alpha,a}^1 (\partial_\beta a_{\mu,b}^1 + F_{\beta\mu,b}^1) t^a t^b \ , \\
\lambda_{ab}^2 t^a t^b &= \frac{1}{2}\theta^{\alpha\beta} \partial_\alpha \lambda_a^1 a_{\beta,b}^2 t^a t^b \ ,
\end{aligned}
\tag{A.107}
$$

where

$$
F_{\mu\nu,a}^1 = \partial_\mu a_{\nu,a}^1 - \partial_\nu a_{\mu,a}^1 + f^{cd}{}_a a_{\mu,c}^1 a_{\nu,d}^1 \ .
\tag{A.108}
$$

In [94] the above summarized formalism is used to work out the Seiberg-Witten map up to $\mathcal{O}(\theta^2)$. From the above construction it follows that the gauge connection (A.95) and the gauge transformation (A.89) are completely determined by the coefficients $a_{\mu,a}^1$ and λ_a^1 of the Lie algebra valued terms. One can write

$$
\hat{\delta}_\Lambda \phi(x) = i\Lambda[\lambda^1, a^1] * \phi(x) \ .
\tag{A.109}
$$

In [95] the authors check that the composition property of two gauge transformations

$$
(\hat{\delta}_{\Lambda_1} \hat{\delta}_{\Lambda_2} - \hat{\delta}_{\Lambda_2} \hat{\delta}_{\Lambda_1}) = \hat{\delta}_{i[\Lambda_1 \overset{*}{,} \Lambda_2]}
\tag{A.110}
$$

holds if the transformations are interpreted in the form (A.109).

A.7 Constraints on the gauge group via anti-automorphisms

In [16, 33] the authors present proposals of how to construct noncommutative gauge theories with some subgroups of $U(N)$. The idea is to formulate a constraint for the gauge field and the gauge parameter. Here we will present some more detail about this construction. In [33] the authors observe that the condition of (anti)-Hermiticity is preserved by the $*$-product (A.65), i. e. the relation

$$
(f * g)^\dagger = g^\dagger * f^\dagger
\tag{A.111}
$$

holds for two matrix-valued functions f and g. It follows that the finite gauge transformation of the noncommutative gauge connection A_μ then preserves (anti)-Hermiticity if the gauge parameter Λ itself is chosen to be (anti)-Hermitian (see the discussion in appendix A.5). One could have the idea to obtain $SO(N)$ or $SP(N)$ gauge group by

making the gauge field and gauge transformations real, and in addition dropping the i in the exponent of the $*$-product (A.65). However, this approach fails because then the property (A.111) for Hermitian conjugation is no longer valid. However it is essential for preserving Hermiticity. Instead, the authors of [33] formulate an additional constraint on the gauge field and the gauge parameter. They define an algebra $\mathcal{A}_\theta$ which elements are formal power series in $\theta^{\mu\nu}$. That means it is required to define the gauge field and gauge transformations as elements of this algebra, depending explicitly on $\theta^{\mu\nu}$. Then they define an anti-automorphism r of $\mathcal{A}_\theta$, which acts on $f(x,\theta) \in \mathcal{A}_\theta$ as follows

$$(\ )^{\mathrm{r}} : f(x,\theta) \mapsto f^{\mathrm{r}}(x,\theta) = f(x,-\theta) \ . \tag{A.112}$$

Acting on the coordinates x^μ themselves, this map is the identity. It reverses the order of $*$-multiplication

$$(x_1^{\mu_1} * \cdots * x_n^{\mu_n})^{\mathrm{r}} = (x_n^{\mu_n})^{\mathrm{r}} * \cdots * (x_1^{\mu_1})^{\mathrm{r}} \ . \tag{A.113}$$

The anti-automorphism r is now combined with matrix transposition t, acting on the representation matrices of the gauge Lie-algebra, in a map which is defined as $(\ )^{\mathrm{rt}} = ((\ )^{\mathrm{t}})^{\mathrm{r}}$. It has the crucial property that its action on the $*$-product of two elements $f,g \in \mathcal{A}_\theta$ is given by

$$(f * g)^{\mathrm{rt}} = g^{\mathrm{rt}} * f^{\mathrm{rt}} \ , \tag{A.114}$$

and hence it provides us with a relation similar to (A.111) for Hermitian conjugation.

In addition to the Hermiticity condition[2]

$$A_\mu^\dagger(x,\theta) = A_\mu(x,\theta) \ , \qquad \Lambda^\dagger(x,\theta) = \Lambda(x,\theta) \tag{A.115}$$

one can now impose the extra constraint

$$A_\mu^{\mathrm{rt}}(x,\theta) = -A_\mu(x,\theta) \ , \qquad \Lambda^{\mathrm{rt}}(x,\theta) = -\Lambda(x,\theta) \tag{A.116}$$

on the gauge field and gauge transformations. Using the definition of the anti-automorphism r, the constraint becomes

$$A_\mu^{\mathrm{t}}(x,-\theta) = -A_\mu(x,\theta) \ , \qquad \Lambda^{\mathrm{t}}(x,-\theta) = -\Lambda(x,\theta) \ . \tag{A.117}$$

The above relations lead to definite symmetry properties of the matrix valued expansion coefficients, if one expands the gauge field and gauge transformation parameter in power series in $\theta^{\mu\nu}$ like

$$A_\mu(x,\theta) = A_\mu^0 + \theta^{\alpha\beta} A_{\mu,\alpha\beta}^1 + \dots \ , \qquad \Lambda(x,\theta) = \Lambda^0 + \theta^{\alpha\beta} \Lambda_{\mu,\alpha\beta}^1 + \dots \ . \tag{A.118}$$

---

[2]The authors of [16, 33] work with an anti-Hermitian gauge field and gauge transformation parameter.

The matrices A^{2n} $[A^{2n+1}]$ and Λ^{2n} $[\Lambda^{2n+1}]$, $n = 0, 1 \ldots$ have to be antisymmetric [symmetric]. In combination with the Hermiticity condition (A.115) this then requires that A^{2n} $[A^{2n+1}]$ are purely imaginary [real].

One should not interpret the higher order expansion coefficients as new degrees of freedom. Instead they should be regarded as functions of A^0 and Λ^0, as explicitly realized in the Seiberg-Witten map (2.53). The latter respects the constraint (A.117). For instance, in the explicit expansion of the Seiberg-Witten map (2.59) the coefficient at linear order in θ is symmetric if the ordinary field a_μ is antisymmetric under matrix transposition. An inversion of the Seiberg-Witten map gives the constraint (A.117) formulated for A_0

$$a_\mu[A] = -a_\mu^{\mathrm{t}}[A] \ . \tag{A.119}$$

This means that the noncommutative gauge theory with the gauge group restricted by (A.117) is the image of an ordinary gauge theory with gauge group $SO(N)$ under the Seiberg-Witten map.

A.8   Proof that (3.77) does not vanish

To prove that the Green function in (3.77) is non-zero, it is sufficient to show that at least one contribution to this quantity with an *independent* tensor structure is non-vanishing at some configuration of the external momenta, Lorentz and group indices. Choosing the most symmetric non-trivial external configuration

$$p_1 = \cdots = p_{n-1} = p \ , \quad p_n = -(n-1)p \ , \quad \mu_1 = \cdots = \mu_n = \mu \ , \quad m_1' = \cdots = m_n' = m'$$
$$\tag{A.120}$$

simplifies (3.77) considerably, e. g. the summation over permutations of the external quantities simply lead to a combinatorial factor.

We first pick out all terms where – *after* performing the integral of (3.77) – the tensor structure of the μ_i is purely constructed with $G_{\mu_i\mu_j}$ such that $\theta^{\alpha\beta}$ does not carry an external Lorentz index μ_i. To minimize the number of contributing terms we choose $\theta^{\alpha\beta}(p_i)_\beta = 0$.[3] In this case the square brackets in (3.77) simplify and we use the abbreviations

$$2\Big[①_r + ②_r + ③_r\Big]k = 2\Big[-\theta^{\alpha_r\gamma_r}G_{\mu_{i_r}\alpha_{r+1}} + \theta_{\alpha_{r+1}}{}^{\gamma_r}\delta^{\alpha_r}_{\mu_{i_r}} - \theta^{\alpha_r}{}_{\alpha_{r+1}}\delta^{\gamma_r}_{\mu_{i_r}}\Big]k_{\gamma_r} \ ,$$

---

[3]This can be realized for the choice (A.120).

where Lorentz indices are not written explicitly. For the three terms inside the bracket only the following multiplications can produce a pure $G_{\mu_i\mu_j}$-structure

$$①_r②_{r+1} = \theta^{\alpha_r\gamma_r}\theta^{\gamma_{r+1}}{}_{\alpha_{r+2}}G_{\mu_{i_r}\mu_{i_{r+1}}}$$

$$②_r①_{r+1} = {}^{\gamma_r}\theta\theta^{\gamma_{r+1}}\delta^{\alpha_r}_{\mu_{i_r}}G_{\mu_{i_{r+1}}\alpha_{r+2}}$$

$$②_r③_{r+1} = {}^{\gamma_r}\theta\theta_{\alpha_{r+2}}\delta^{\alpha_r}_{\mu_{i_r}}\delta^{\gamma_{r+1}}_{\mu_{i_{r+1}}}$$

$$③_r①_{r+1} = {}^{\alpha_r}\theta\theta^{\gamma_{r+1}}\delta^{\gamma_r}_{\mu_{i_r}}G_{\mu_{i_{r+1}}\alpha_{r+2}}$$

$$③_r③_{r+1} = {}^{\alpha_r}\theta\theta_{\alpha_{r+2}}\delta^{\gamma_r}_{\mu_{i_r}}\delta^{\gamma_{r+1}}_{\mu_{i_{r+1}}} \ ,$$

where we have defined ${}^{\alpha}\theta\theta^{\gamma} = \theta^{\alpha\beta}\theta_{\beta}{}^{\gamma}$. These products are the building blocks of the complete terms with n factors, for instance like

$$\underbrace{①_1②_2\cdots①_{k-1}②_k③_{k+1}\cdots③_{j+k}\cdots}_{n} \ ,$$

where the α_1 index of the first factor is contracted with the α_{n+1} index of the last.

Further restrictions are imposed on the complete expressions: The total number of factors n has to be even because one cannot construct a pure $G_{\mu_i\mu_j}$ structure with an odd number of μ_i's. In addition the number of ③'s in the complete product of n terms has to be even as otherwise after performing the integral in (3.77) one θ would carry an index μ_i (see equations below). Then it follows that the numbers of ①'s and ②'s have to be identical.

Using the configuration (A.120) the contribution of all terms with an even number j of ③'s and an even number $n-j$ ①'s and ②'s can now be written as

$$G_{\mathrm{c}}^{\mathrm{kin},m'\ldots m'}{}_{\mu\ldots\mu}(p,\ldots,p,-(n-1)p)\big|_{\propto\theta^n g^{2n},\text{ only } G_{\mu\mu}}$$

$$= \frac{(n-1)!}{2}\frac{g^{2n}}{2^n}\Big[\prod_{r=1}^{n}d^{a_r m'}{}_{a_{r+1}}\Big]$$

$$\times \sum_{j=0,2}^{n}(\underbrace{\theta\ldots\theta}_{n})^{\gamma_{j+1}\ldots\gamma_n}\underbrace{G_{\mu\mu}\ldots G_{\mu\mu}}_{n-j}\delta^{\gamma_1}_{\mu}\ldots\delta^{\gamma_j}_{\mu}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{k_{\gamma_1}\ldots k_{\gamma_n}}{q_1^2\ldots q_n^2}\Big|_{\text{only special } G} \ ,$$

$$(A.121)$$

where the factor $\frac{(n-1)!}{2}$ stems from performing the summation over all proper permutations and $q_r = k + rp$, $r \neq n$, $q_n = k$. To make the above expression compact we have used some further abbreviations which we now explain.

The relevant part of the integral in the above expression is defined as the tensor component of the integral only made out of the metric where the metric must not pos-

sess a mixed index pair with one index from the set $\{\gamma_1, \ldots, \gamma_j\}$ and one from the set $\{\gamma_{j+1} \ldots \gamma_n\}$. It then reads

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k_{\gamma_1} \ldots k_{\gamma_n}}{q_1^2 \ldots q_n^2}\bigg|_{\text{only special } G} = I_0 \sum_{\substack{\text{perm} \\ \{i_1, \ldots, i_j\} \\ \{i_{j+1}, \ldots, i_n\}}} \prod_{r=1,3}^{n-1} G_{\gamma_{i_r} \gamma_{i_{r+1}}} \,, \qquad (\text{A.122})$$

where I_0 denotes a scalar integral which will be discussed later.

The tensor $(\theta \ldots \theta)^{\gamma_{j+1} \ldots \gamma_n}$ in (A.121) is built by summing over all possibilities to replace $\frac{n-j}{2}$ of the n summation index pairs (α_r, α_r) in the trace $\mathrm{tr}\{\theta^n\} = \theta^{\alpha_1}{}_{\alpha_2} \theta^{\alpha_2}{}_{\alpha_3} \ldots \theta^{\alpha_n}{}_{\alpha_1}$ by the index pairs $\{(\gamma_{j+1}, \gamma_{j+2}), \ldots, (\gamma_{n-1}, \gamma_n)\}$ keeping the ordering of the γ-pairs, i. e. the pair $(\gamma_{j+1}, \gamma_{j+2})$ is inserted at the positions with smallest index r of all replaced α_r and so on. All indices r of the replaced α_r either have to be odd or even, since otherwise at least two substructures ${}^{\gamma_r}\theta \ldots \theta^{\gamma_{r+1}}$ would contain an odd number of θ's vanishing when contracted with the symmetric $k_{\gamma_r} k_{\gamma_{r+1}}$ in (A.121). Some examples for illustration: If $j = n$ in (A.121) then $(\theta \ldots \theta) = \mathrm{tr}\{\theta^n\}$ and there is only one contribution. If $j = n - 2$ then there are n possibilities[4] to replace a pair α_r by the pair (γ_{n-1}, γ_n) such that $(\theta \ldots \theta)^{\gamma_{n-1} \gamma_n} = n^{\gamma_{n-1}} \theta \ldots \theta^{\gamma_n}$. For general $j \neq n$ there are $2\binom{n/2}{j/2}$ non-vanishing possibilities to replace summation indices by the γ-pairs.

The contraction of the above defined $(\theta \ldots \theta)^{\gamma_{j+1} \ldots \gamma_n}$ in (A.121) with the tensor structure of the integral (A.122) leads to a sum over products of traces of the form $\prod_i \mathrm{tr}\{\theta^{2k_i}\}$, $k_i \in \mathbb{N}$ such that $\sum_i 2k_i = n$. All these products of traces include the same sign $(\mathrm{sgn}\, \mathrm{tr}\{\theta^2\})^{\frac{n}{2}}$.[5] Thus, all summed terms in (A.121) carry the same sign such that a cancellation mechanism between different terms cannot be present. Proving the non-vanishing of (A.121) therefore only requires to show that the group structure factor and the scalar integral I_0 in (A.122) are non-zero.

For instance, the choice $m' = 0$, where the generator T^0 is given by $T^0 = \frac{1}{\sqrt{2N}}\mathbb{1}$ in an $U(N)$ theory, leads to $d_{ab0} = \sqrt{\frac{2}{N}}\delta_{ab}$. Hence, with $\dim \mathfrak{g}$ as the dimension of the Lie algebra $\mathfrak{g}$

$$\prod_{r=1}^{n} d^{a_r m'}{}_{a_{r+1}}\bigg|_{m'=0} = \left(\frac{2}{N}\right)^{\frac{n}{2}} \dim \mathfrak{g}$$

---

[4] $\frac{n}{2}$ possibilities to replace α_r with odd r and $\frac{n}{2}$ to replace the ones with even r.
[5] This can be proven by using the canonical skew-diagonal form of [171].

does not vanish.

In general the integral in (A.121) can be decomposed in scalar integrals like

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k_{\gamma_1} \dots k_{\gamma_n}}{q_1^2 \dots q_n^2} = I_0 \sum_{\substack{\text{perm} \\ \{i_1, \dots, i_n\}}} \prod_{r=1,3}^{n-1} G_{\gamma_{i_r} \gamma_{i_{r+1}}} + \text{ terms containing } p_{\gamma_i} \,,$$

where due to the choice (A.120) the q_i (3.78) now only depend on p such that the above tensor structure can only be spanned by $G_{\gamma_i \gamma_j}$ and p_{γ_i}. Notice that in (A.121) only one part of the total symmetric tensor multiplying I_0 given in (A.122) is needed. In the above expression we now choose all indices $\gamma_1 = \dots = \gamma_n = \gamma$ and the momentum p such that it has a vanishing component p_γ for the special choice of γ. Then one finds for I_0

$$I_0 = \frac{1}{n!} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{(k_\gamma)^n}{q_1^2 \dots q_n^2} \,.$$

For even n this is non-vanishing since it is positive definite after a Wick rotation.

Thus, the expression (A.121) in general does not vanish for all even n implying that at least all connected n-point Green functions with an even number of external points are therefore present in the kinetic theory such that it produces infinitely many building blocks in the θ-summed case.

A.9 A counterexample that disproves the $SO(N)$ Feynman rules of [31]

In this appendix we give an explicit proof for the non-vanishing of the lowest order contribution in g^2 and θ to $\langle A^{m_1}(x_1) \dots A^{m_8}(x_8) \rangle_{\mathrm{c}}^{\mathrm{kin}+S_i'}$ in the $SO(3)$ case.

As discussed in the main text, we focus on the 8-point vertex generated out of a 4-point interaction of the non-commutative A_μ in S_i', see Fig. 3.1. Via the definition of S_i', at least one of the A_μ has to carry a primed group index. Since we look for the lowest order in θ we can replace the $*$-product by the usual product. The interaction then has the gauge group structure $f_{N_1 N_2}{}^K f_{N_3 n_4' K}$. Due to the subgroup property of G this is zero if $N_j = n_j, j = 1, 2, 3$. Therefore, we have to start with a 4-point interaction of the A

where two of them carry a primed index. Three interaction terms contribute in this case

$$\frac{i}{g^2}\left(f_{m_1m_2}{}^af_{n'_3n'_4a}G^{\mu_1\nu_3}G^{\mu_2\nu_4} + f_{n'_4m_1}{}^{a'}f_{m_2n'_3a'}G^{\mu_1\nu_3}G^{\mu_2\nu_4} + f_{n'_4m_1}{}^{a'}f_{n'_3m_2a'}G^{\mu_1\mu_2}G^{\nu_3\nu_4}\right)$$
$$\times A^{m_1}_{\mu_1}A^{m_2}_{\mu_2}A^{m'_3}_{\nu_3}A^{m'_4}_{\nu_4}\ .$$
$$(A.123)$$

Now we replace $A^{m_i}_{\mu_i}$ by $a^{m_i}_{\mu_i}$ for $i=1,2$ and $A^{n'_i}_{\nu_i}, i=3,4$ by the term with maximum number of a within the θ^1 contribution, see (3.50), and get

$$\frac{i}{16g^2}\left(f_{m_1m_2}{}^af_{n'_3n'_4a}G^{\mu_1\mu_5}G^{\mu_2\mu_8} + f_{m_1n'_4}{}^{a'}f_{m_2n'_3a'}(G^{\mu_1\mu_2}G^{\mu_5\mu_8} - G^{\mu_1\mu_5}G^{\mu_2\mu_8})\right)$$
$$\times d_{n'_3m_3}{}^ef_{em_4m_5}d_{n'_4m_6}{}^kf_{km_7m_8}\theta^{\mu_3\mu_4}\theta^{\mu_6\mu_7}a^{m_1}_{\mu_1}a^{m_2}_{\mu_2}\ldots a^{m_8}_{\mu_8}\ .$$
$$(A.124)$$

With this interaction the $g^{18}\theta^2$ contribution to the Fourier transform of $\langle A(x_1)\ldots A(x_8)\rangle_c^{\mathrm{kin}+S'_i}$ becomes up to the momentum conservation factor equal to

$$M^{\mu_1\ldots\mu_8}_{m_1\ldots m_8} = \frac{i}{16}g^{18}\sum_{\substack{\mathrm{perm}\\ \{i_1,\ldots,i_8\}}}\theta^{\mu_{i_3}\mu_{i_4}}\theta^{\mu_{i_6}\mu_{i_7}}d_{n'_3m_{i_3}}{}^ef_{em_{i_4}m_{i_5}}d_{n'_4m_{i_6}}{}^kf_{km_{i_7}m_{i_8}}$$

$$\times\Bigg(\frac{1}{2}(G^{\mu_{i_1}\mu_{i_5}}G^{\mu_{i_2}\mu_{i_8}} - G^{\mu_{i_2}\mu_{i_5}}G^{\mu_{i_1}\mu_{i_8}})f_{m_{i_1}m_{i_2}}{}^af_{n'_3n'_4a}$$

$$+ (G^{\mu_{i_1}\mu_{i_2}}G^{\mu_{i_5}\mu_{i_8}} - G^{\mu_{i_1}\mu_{i_5}}G^{\mu_{i_2}\mu_{i_8}})f_{m_{i_1}n'_4}{}^{a'}f_{m_{i_2}n'_3a'}\Bigg)\ .$$

$$(A.125)$$

We will have reached the goal of this appendix if it can be shown that the above quantity is different from zero. Our explicit proof of $M^{\mu_1\ldots\mu_8}_{m_1\ldots m_8}\neq 0$ consists in the numerical calculation for one special choice of gauge group and Lorentz indices. To minimize the calculational effort forced by taking into account all the permutations, we looked for an index choice with a lot of symmetry with respect to the interchange of external legs. But we also had to avoid too much symmetry not to produce a zero result.

If we use the standard Gell-Mann enumeration of the nine generators of the $U(3)$ Lie algebra, see e.g. [93], the generators of the $SO(3)$ subalgebra carry the indices 2,5,7. Then

our special choice for the external legs is

	leg 1	leg 2	leg 3	leg 4	leg 5	leg 6	leg 7	leg 8
m_i	5	5	2	5	2	2	5	2
μ_i	λ	λ	μ	ν	μ	μ	ν	μ

$$\text{(A.126)}$$

The chosen Lorentz indices are all spacelike and have to fulfill

$$\mu \neq \nu \,, \quad \mu \neq \lambda \,, \quad \nu \neq \lambda \,,$$
$$\theta^{\mu\nu} \neq 0 \,, \quad \theta^{\mu\lambda} = 0 \,, \quad \theta^{\nu\lambda} = 0 \,. \tag{A.127}$$

Taking into account the list of vanishing d_{ABC} and f_{ABC} for $U(3)$ [93] we find

$$M^{\mu_1\dots\mu_8}_{m_1\dots m_8}\Big|_{\text{special}} = 6ig^{14} \left(\theta^{\mu\nu}\right)^2 f^2_{257} \left[\left(f_{345}d_{247} - f_{123}d_{157}\right)^2 + f^2_{458}d^2_{247}\right] . \tag{A.128}$$

All f and d in (A.128) are different from zero.

Appendix B

Appendix to Part III

B.1 The Einstein equations with cosmological constant

Some spacetimes that we discuss in section 4.1 are solutions of the Einstein equations or are direct products of such solutions. In the following we will in short present the equations and fix our notations and conventions. The Einstein-Hilbert action in D dimensions with a cosmological constant is given by

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^D \sqrt{-g}(\mathcal{R} - 2\Lambda) \ . \tag{B.1}$$

From it one derives the Einstein equations

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = -\Lambda g_{\mu\nu} \ . \tag{B.2}$$

The Ricci tensor $\mathcal{R}_{\mu\nu}$ and the scalar curvature $\mathcal{R}$ are computed from the Riemannian curvature tensor

$$\mathcal{R}^\alpha{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha{}_{\beta\delta} - \partial_\delta \Gamma^\alpha{}_{\beta\gamma} + \Gamma^\alpha{}_{\rho\gamma}\Gamma^\rho{}_{\beta\delta} - \Gamma^\alpha{}_{\rho\delta}\Gamma^\rho{}_{\beta\gamma} \tag{B.3}$$

as follows

$$\mathcal{R}_{\beta\delta} = \mathcal{R}^\alpha{}_{\beta\alpha\delta} \ , \qquad \mathcal{R} = g^{\beta\delta}\mathcal{R}_{\beta\delta} \ . \tag{B.4}$$

Here $\Gamma^\alpha{}_{\beta\gamma}$ denotes the Christoffel connection coefficients which are given in terms of the metric as

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2}g^{\alpha\rho}(\partial_\beta g_{\rho\gamma} + \partial_\gamma g_{\rho\beta} - \partial_\rho g_{\beta\gamma}) \ . \tag{B.5}$$

B.2   Conformal flatness

One can explicitly check conformal flatness by a computation of the Weyl tensor. The latter is defined as

$$
\mathcal{C}_{ABCD} = \mathcal{R}_{ABCD} - \frac{1}{D-2}(g_{AC}\mathcal{R}_{BD} - g_{AD}\mathcal{R}_{BC} + g_{BD}\mathcal{R}_{AC} - g_{BC}\mathcal{R}_{AD})
$$
$$
+ \frac{\mathcal{R}}{(D-1)(D-2)}(g_{AC}g_{BD} - g_{AD}g_{BC}) \tag{B.6}
$$

for a D dimensional space with coordinate indices A, B, C, D. The Weyl tensor is constructed in such a way that under a conformal transformation of the metric

$$
g_{AB} \rightarrow g'_{AB} = \varrho \, g_{AB} \tag{B.7}
$$

it transforms homogeneously[1] i. e.

$$
\mathcal{C}_{ABCD} \rightarrow \mathcal{C}'_{ABCD} = \varrho \, \mathcal{C}_{ABCD} \ . \tag{B.8}
$$

Two spaces are called conformal to each other if their Weyl tensors are related as given in the above equation. In particular a space with $\mathcal{C}_{ABCD} = 0$ can be conformally mapped to flat space and is therefore called conformally flat.

It is easy to see that all spaces with a Riemann tensor of the form

$$
\mathcal{R}_{ABCD} = \frac{\mathcal{R}}{D(D-1)}(g_{AC}g_{BD} - g_{AD}g_{BC}) \tag{B.9}
$$

in D dimensions are conformally flat. This is not necessarily true if the D-dimensional space is a direct product of conformally flat spaces. In the following we will check that $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ with embedding radii R_1 and R_2 respectively is conformally flat if and only if $R_1 = R_2$ for $d, d' > 0$. Greek indices refer to AdS_{d+1} and lower case Latin indices refer to $\mathrm{S}^{d'+1}$ in the following. Since we now that AdS_{d+1} and $\mathbb{R} \times \mathrm{S}^{d'+1}$ are solutions to the Einstein equation with cosmological constant, the Ricci tensors have to be proportional to the metrics of these spaces. As the metric of $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ is block diagonal, all contributions to the Ricci tensor with mixed indices vanish, too. To find the condition for conformally flatness of $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$, we have to check the vanishing of

$$
\mathcal{C}_{\mu\nu\rho\sigma} \ , \qquad \mathcal{C}_{mn\rho\sigma} \ , \qquad \mathcal{C}_{m\nu r\sigma} \ , \qquad \mathcal{C}_{mnrs} \ . \tag{B.10}
$$

---

[1] This is equivalent to $\mathcal{C}'^{A}{}_{BCD} = \mathcal{C}^{A}{}_{BCD}$.

If one inserts the expressions (4.33) for the non-vanishing Riemann and Ricci tensors and the scalar curvature one finds

$$
\begin{aligned}
\mathcal{C}_{\mu\nu\rho\sigma} &= \frac{d'(d'+1)}{(d+d')(d+d'+1)}\left(-\frac{1}{R_1^2}+\frac{1}{R_2^2}\right)(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho}) \,, \\
\mathcal{C}_{m\nu r\sigma} &= -\frac{dd'}{(d+d')(d+d'+1)}\left(-\frac{1}{R_1^2}+\frac{1}{R_2^2}\right)g_{mr}g_{\nu\sigma} \,, \\
\mathcal{C}_{mnrs} &= \frac{d(d+1)}{(d+d')(d+d'+1)}\left(-\frac{1}{R_1^2}+\frac{1}{R_2^2}\right)(g_{mr}g_{ns}-g_{ms}g_{nr}) \,.
\end{aligned}
\tag{B.11}
$$

It is easy to see that these components are zero and thus that $\mathrm{AdS}_{d+1}\times\mathrm{S}^{d'+1}$ is conformally flat if and only if $R_1 = R_2$ for $d, d' > 0$. Furthermore, one finds from the above given results that the product of a conformally flat higher dimensional space with $\mathbb{R}$ or S^1 is always conformally flat. Hence, the ESU is conformally flat since it is given by the direct product of a sphere with $\mathbb{R}$.

B.3   Relation of the bulk-to-bulk and the bulk-to-boundary propagator

We will show for a scalar field in an Euclidean space how the bulk-to-boundary propagator is related to the bulk-to-bulk propagator, if the boundary has codimension one w. r. t. the bulk, like in the case of the AdS/CFT correspondence. The bulk-to-bulk propagator $G(x, x')$ of a scalar field with mass m is defined as Green function that fulfills

$$
(\Box_x - m^2)G(x, x') = -\frac{1}{\sqrt{g}}\delta(x, x') \,,
\tag{B.12}
$$

with appropriate boundary conditions. Here $\Box_x$ is the Laplace operator on the $(d+1)$-dimensional Riemannian manifold M with a d-dimensional boundary which we denote with ∂M. The coordinates are x^i, the metric is g_{ij} and its determinant is g. The propagator $G(x, x')$ corresponds to a scalar field $\phi(x)$ which should obey

$$
(\Box_x - m^2)\phi(x) = J(x) \,, \qquad \lim_{x_\perp\to 0}\phi(x)x_\perp^a = \bar{\phi}(\bar{x}) \,,
\tag{B.13}
$$

where $J(x)$ are sources for the field ϕ in the interior. We have split the coordinates like $x = (x_\perp, \bar{x})$ with the boundary at $x_\perp = 0$. Boundary values $\bar{\phi}$ for the field ϕ are specified with a nontrivial scaling with $x_\perp^a$ for later convenience. The bulk-to-boundary propagator $K(x, \bar{x}')$ is defined as the solution of the equations

$$
(\Box_x - m^2)K(x, \bar{x}') = 0 \,, \qquad \lim_{x_\perp\to 0}K(x, \bar{x}')x_\perp^a = \delta(\bar{x}, \bar{x}') \,,
\tag{B.14}
$$

where the second equation implements the necessary singular behaviour at the boundary. A solution of the equations (B.13) with $J(x) = 0$ is then given by

$$\phi(x) = \int_{\partial M} \mathrm{d}^d \bar{x}' K(x, \bar{x}') \bar{\phi}(\bar{x}') . \tag{B.15}$$

Since we deal with the problem in Euclidean signature [55, 74, 186], we will denote $K(x, \bar{x}')$ as the Poisson kernel. It is not independent from the Green function defined via (B.12) as we will now show.

With (B.12) one can write an identity for the field ϕ that reads

$$\phi(x) = -\int_M \mathrm{d}^{d+1} x' \sqrt{g} \phi(x') (\Box_{x'} - m^2) G(x, x') . \tag{B.16}$$

After applying partial integration twice and using (B.13) it assumes the form

$$\phi(x) = \int_{\partial M} \mathrm{d}A'_\mu \sqrt{g} g^{\mu\nu} \big[(\partial'_\nu \phi(x')) G(x, x') - \phi(x') \partial'_\nu G(x, x') \big] - \int_M \mathrm{d}^{d+1} x' \sqrt{g} J(x') G(x, x') , \tag{B.17}$$

where $\mathrm{d}A'_\mu$ denotes the infinitesimal area element on ∂M which points into the outer normal direction and ∂'_μ denotes a derivative w. r. t. x'_μ. If one has the additional restriction that $G(x, x') = 0$ for $x' \in \partial M$ ($x'_\perp = 0$) the first term in the above boundary integral is zero. One then arrives at the 'magic rule' which for the boundary value problem in presence of a source $J(x)$ in the interior can be found in [17]. Here, however, we have to be more careful. In (B.13) we have allowed for a scaling of the boundary value with $x^a_\perp$ as written down in (B.14). For $a > 0$ the vanishing $G(x, x')$ at the boundary can be compensated and the first term in the boundary integral of (B.17) then contributes.

Considering AdS_{d+1}, this is indeed the case, because the field ϕ with conformal dimension Δ represents the non-normalizable modes ϕ_Δ which scale as given in (5.5), i. e. $a = \Delta - d$. Indicating the corresponding propagator with the suffix Δ, one finds $G(x, x') = 0$ at $x' \in \partial M$ but the vanishing is compensated by the singular behaviour of the non-normalizable modes in the limit $x'_\perp \to 0$.

We now formulate (B.17) on Euclidean AdS_{d+1} in Poincaré coordinates (4.29) (with the minus sign in front of $\mathrm{d}x_0^2$ converted to plus) with $J(x') = 0$, where one has

$$\mathrm{d}A_\mu = -\mathrm{d}^d \bar{x} \delta^\perp_\mu , \qquad \sqrt{g} = \Big(\frac{R_1}{x_\perp} \Big)^{d+1} , \qquad g^{\perp\perp} = \Big(\frac{x_\perp}{R_1} \Big)^2 . \tag{B.18}$$

The minus sign in the area element stems from the fact that the $x_\perp$-direction points into the interior of AdS_{d+1}, but one has to take the outer normal vector. Now (B.17) reads

$$\phi_\Delta(x) = -R_1^{d-1} \int \mathrm{d}^d \bar{x}' x'^{1-d}_\perp \big[(\partial'_\perp \phi(x')) G_\Delta(x, x') - \phi(x') \partial'_\perp G_\Delta(x, x') \big] . \tag{B.19}$$

Using now (5.5) and comparing with (5.6) that replaces (B.15) in the AdS$_{d+1}$-case, one finds that the relation of the bulk-to-bulk and the bulk-to-boundary propagator is given by

$$K_\Delta(x, \bar{x}') = -R_1^{d-1}\left[(d-\Delta)x_\perp'^{-\Delta} - x_\perp'^{1-\Delta}\partial_\perp'\right]G_\Delta(x, x')\big|_{x_\perp'=0} . \tag{B.20}$$

If we now insert the explicit expression (5.10) for $G_\Delta(x, x')$, we see what we already mentioned: in approaching the boundary ($x_\perp' \to 0$), $G_\Delta(x, x')$ itself goes to zero like $x_\perp'^\Delta$ but this is compensated by the singular behaviour of the prefactor in the first term of (B.20). Hence, in contrast to the situation of the 'magic rule' [17], it contributes to the bulk-to-boundary propagator. One then finds with (5.11) and with $F(a, b; c; 0) = 1$ that effectively

$$K_\Delta(x, \bar{x}') = -R_1^{d-1}(d-2\Delta)x_\perp'^{-\Delta}G_\Delta(x, x')\big|_{x_\perp'=0} , \tag{B.21}$$

which is in perfect agreement with the explicit expressions (5.7) and (5.10).

B.4 Geodesics in a warped geometry

Consider a metric of the form

$$ds_z^2 = ds_x^2 + e^{2w(x)}ds_y^2 , \tag{B.22}$$

where the subscript indicates which coordinates $z = (x, y)$ the corresponding part depends on. We use capital Latin indices for the whole space lower case Greek indices for the first part (that only depends on x) and lower case Latin indices for the second part (that depends on y and on x due to the warp factor). The components of the complete metric are G_{MN} and the two blocks are $G_{\mu\nu}$ and $G_{mn} = e^{2w(x)}g_{mn}$. The equations for the geodesics, parameterized with an affine parameter τ read

$$\ddot{z}^R + \Gamma^R{}_{MN}\dot{z}^M\dot{z}^N = 0 . \tag{B.23}$$

Derivatives w. r. t. τ are denoted with dots. It is easy to see that the only non-zero components of the connection $\Gamma^R{}_{MN}$, which has been defined in (B.5) in terms of the

metric, are then given by

$$
\begin{aligned}
\Gamma^{\rho}{}_{\mu\nu} &= \frac{1}{2}g^{\rho\kappa}(\partial_{\mu}G_{\kappa\nu} + \partial_{\nu}G_{\kappa\mu} - \partial_{\kappa}G_{\mu\nu}) \ , \\
\Gamma^{\rho}{}_{mn} &= -G^{\rho\kappa}\,\mathrm{e}^{2w}(\partial_{\kappa}w)g_{mn} \ , \\
\Gamma^{r}{}_{m\nu} &= (\partial_{\nu}w)\delta^{r}_{m} \ , \\
\Gamma^{r}{}_{\mu n} &= (\partial_{\mu}w)\delta^{r}_{n} \ , \\
\Gamma^{r}{}_{mn} &= \gamma^{r}{}_{mn} = \frac{1}{2}g^{rk}(\partial_{m}g_{kn} + \partial_{n}g_{km} - \partial_{k}g_{mn}) \ .
\end{aligned}
\tag{B.24}
$$

Here, $\gamma^{r}{}_{mn}$ is the connection that corresponds to $\mathrm{d}s_{y}^{2}$ computed with g_{mn} (without the warp factor). The geodesic equations then read

$$
\begin{aligned}
\ddot{x}^{\rho} + \Gamma^{\rho}{}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} - G^{\rho\kappa}\,\mathrm{e}^{2w}(\partial_{\kappa}w)\dot{\vec{y}}^{2} &= 0 \ , \\
\ddot{y}^{r} + \gamma^{r}{}_{mn}\dot{y}^{m}\dot{y}^{n} + 2\dot{w}\dot{y}^{r} &= 0 \ ,
\end{aligned}
\tag{B.25}
$$

where we have used $g_{mn}\dot{y}^{m}\dot{y}^{n} = \dot{\vec{y}}^{2}$ and $\dot{x}^{\mu}\partial_{\mu}w = \dot{w}$. The effect of the warp factor in (B.22) is that the scale in the part of the space with y-coordinates becomes x-dependent. Along a geodesic (where $x = x(\tau)$) the scale thus depends on τ. This means that the parameter τ is not an affine parameter for a geodesic in the space with metric $\mathrm{d}s_{y}^{2}$. However, using a reparameterization, one can make it affine w. r. t. $\mathrm{d}s_{y}^{2}$. We introduce a new parameter $\sigma = f(\tau)$ and compute the derivatives of $y(\sigma) = y(f(\tau))$ w. r. t. τ and w. r. t. σ, where we denote the latter with a prime. One finds

$$
\dot{y}^{r} = y'^{r}\dot{f} \ , \qquad \ddot{y}^{r} = y''^{r}\dot{f}^{2} + y'^{r}\ddot{f} \ .
\tag{B.26}
$$

Inserting these results into the second geodesic equations of (B.25), one obtains

$$
\dot{f}^{2}(y''^{r} + \Gamma^{r}{}_{mn}y'^{m}y'^{n}) + (\ddot{f} + 2\dot{w}\dot{f})y'^{r} = 0 \ .
\tag{B.27}
$$

If the reparameterization $\sigma = f(\tau)$ obeys the differential equation

$$
\ddot{f} + 2\dot{w}\dot{f} = 0 \ ,
\tag{B.28}
$$

then $y(\sigma)$ describes the geodesic in the geometry given by $\mathrm{d}s_{y}^{2}$ with affine parameter σ, and we find the geodesic equations

$$
\begin{aligned}
\ddot{x}^{\rho} + \Gamma^{\rho}{}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} - G^{\rho\kappa}\,\mathrm{e}^{2w}(\partial_{\kappa}w)c\dot{f}^{2} &= 0 \ , \\
y''^{r} + \gamma^{r}{}_{mn}y'^{m}y'^{n} &= 0 \ ,
\end{aligned}
\tag{B.29}
$$

where $\sigma = f(\tau)$ has to obey (B.28). We have used $\vec{y}'^2 = c$, where c is a constant because σ is the affine parameter w. r. t. $\mathrm{d}s_y^2$. It is clear that an affine parameter remains affine under constant rescalings and henceforth we can choose $\sigma = f(\tau)$ in such a way that $c = 1$ without loss of generality.

The above considerations are especially useful for finding the geodesics of AdS_{d+1} with the metric written down in the first line of (4.26). One can identify

$$\mathrm{d}s_x^2 = -\cosh^2 \rho \, \mathrm{d}t^2 + \mathrm{d}\rho^2 \ , \qquad \mathrm{d}s_y^2 = \mathrm{d}\Omega_{d-1}^2 \ , \qquad \mathrm{e}^w = \sinh \rho \ . \tag{B.30}$$

The movement of the geodesics in the $(d-1)$-dimensional spherical part is clear. They run along a great circle with speed $\dot{\vec{y}}^2 = \dot{f}^2$. We will not discuss its concrete dependence on the coordinates y, but use the function f to describe the movement in this part. The only non-vanishing connection coefficient in (t, ρ) are

$$\Gamma^t{}_{\rho t} = \Gamma^t{}_{t\rho} = \tanh \rho \ , \qquad \Gamma^\rho{}_{tt} = -\sinh \rho \cosh \rho \ . \tag{B.31}$$

The differential equations that collectively describe the movement in the $(d-1)$-dimensional subsphere of AdS_{d+1} then read

$$\ddot{t} + 2\dot{\rho}\dot{t} \tanh \rho = 0 \ ,$$
$$\ddot{\rho} + (\dot{t}^2 - \dot{f}^2) \sinh \rho \cosh \rho = 0 \ , \tag{B.32}$$
$$\ddot{f} + 2\dot{\rho}\dot{f} \coth \rho = 0 \ .$$

B.5 Relation of the chordal and the geodesic distance in the 10-dimensional plane wave

The chordal distance (4.59) in the plane wave is related to the geodesic distance. This relation will now be determined for the 10-dimensional Hpp wave solution, where $H_{ij} = -\delta_{ij}$. We start from the type B geodesics (6.49) in the plane wave and fix the free parameters in terms of two points which are connected with a geodesic segment using the parameter range $0 \leq \tau \leq 1$. We choose

$$z^+(0) = z'^+ \ , \qquad z^-(0) = z'^- \ , \qquad z^i(0) = z'^i \ ,$$
$$z^+(1) = z^+ \ , \qquad z^-(1) = z^- \ , \qquad z^i(1) = z^i \ . \tag{B.33}$$

Then from (6.47) it follows immediately

$$\alpha = z^+ - z'^+ \ , \qquad z_0^+ = z'^+ \ . \tag{B.34}$$

After that it is advantageous to use the theorems for trigonometric functions casting (6.49) into the form

$$
\begin{aligned}
z^-(\tau) &= \frac{1}{4} \sum_i z^i \beta^i \cos \alpha (\tau + \tau_0^i) + \gamma \tau + z_0^- \\
&= \frac{1}{4} \sum_i \left[\beta_1^i \beta_2^i (\cos^2 \alpha \tau - \sin^2 \alpha \tau) + \left((\beta_1^i)^2 - (\beta_2^i)^2 \right) \cos \alpha \tau \sin \alpha \tau \right] + \gamma \tau + z_0^- \ , \\
z^i(\tau) &= \beta_1^i \sin \alpha \tau + \beta_2^i \cos \alpha \tau \ ,
\end{aligned}
\tag{B.35}
$$

where β_1^i and β_2^i are given by

$$
\beta_1^i = \beta^i \cos \alpha \tau_0^i \ , \qquad \beta_2^i = \beta^i \sin \alpha \tau_0^i \ .
\tag{B.36}
$$

The initial value $z^-(0) = z'^-$ now yields

$$
z_0^- = z'^- - \frac{1}{4} \sum_i \beta_1^i \beta_2^i \ .
\tag{B.37}
$$

Inserting this result into the expression for z^- at the final value $z^-(1) = z^-$ then determines

$$
\gamma = z^- - z'^- + \frac{1}{4} \sum_i \left[2\beta_1^i \beta_2^i \sin^2 \alpha - \left((\beta_1^i)^2 - (\beta_2^i)^2 \right) \cos \alpha \sin \alpha \right] \ .
\tag{B.38}
$$

For the parameters β_1^i and β_2^i one finds from (B.36) at the initial value $z^i(0) = z'^i$

$$
\beta_2^i = z'^i \ .
\tag{B.39}
$$

Using this the condition to reach the final value $z^i(1) = z^i$ leads to

$$
\beta_1^i = \frac{1}{\sin \alpha} (z^i - z'^i \cos \alpha) \ .
\tag{B.40}
$$

From this we determine

$$
\begin{aligned}
\sum_i \beta_1^i \beta_2^i &= \frac{1}{\sin \alpha} \left(\vec{z}\vec{z}' - \vec{z}'^2 \cos \alpha \right) \ , \\
\sum_i \left((\beta_1^i)^2 - (\beta_2^i)^2 \right) &= \frac{1}{\sin^2 \alpha} \left(\vec{z}^2 - 2\vec{z}\vec{z}' \cos \alpha + \vec{z}'^2 \cos^2 \alpha - \vec{z}'^2 \sin^2 \alpha \right) \ .
\end{aligned}
\tag{B.41}
$$

Inserting this into (B.38) then yields

$$
\begin{aligned}
\gamma &= z^- - z'^- + \frac{1}{4 \sin \alpha} \left(2\vec{z}\vec{z}' - (\vec{z}^2 + \vec{z}'^2) \cos \alpha \right) \\
&= z^- - z'^- + \frac{1}{4 \sin \alpha} \left(2(\vec{z}^2 + \vec{z}'^2) \sin^2 \tfrac{\alpha}{2} - (\vec{z} - \vec{z}')^2 \right) \ .
\end{aligned}
\tag{B.42}
$$

The constant tangential vector has length $-4\alpha\gamma$. Due to our choice that τ runs over an interval of one length unit, this is directly the geodesic distance $s(z, z')$ between the two points in (B.33). From (B.34) and (B.42) one finds that it is explicitly given by

$$
\begin{aligned}
s(z, z') &= -4(z^+ - z'^+)(z^- - z'^-) - \frac{z^+ - z'^+}{\sin(z^+ - z'^+)}\Big(2(\vec{z}^2 + \vec{z}'^2)\sin^2\tfrac{(z^+ - z'^+)}{2} - (\vec{z} - \vec{z}')^2\Big) \\
&= \frac{z^+ - z'^+}{\sin(z^+ - z'^+)}\Phi(z, z') \ .
\end{aligned}
$$
(B.43)

The last line above shows the relation between the geodesic and the chordal distance (4.59) in the case $H_{ij} = -\delta_{ij}$. the chordal distance (4.59) in the case $H_{ij} = -\delta_{ij}$. It is immediately clear that the above relation holds for the type A geodesics (6.48) as well. If one takes the limit $z'^+ \to z^+$ the prefactor becomes one. For the two points (B.33) one finds from (6.48)

$$
\beta = z^- - z'^- \ , \qquad z_0^- = z'^- \ , \qquad \gamma^i = z^i - z'^i \ , \qquad z_0^i = z'^i \ .
$$
(B.44)

The length of the tangent vector is given by $(\gamma^i)^2$ and therefore the distances are related via

$$
s(z, z') = \Phi(z, z') = (\vec{z} - \vec{z}')^2 \ ,
$$
(B.45)

that coincides with (B.43).

B.6   Useful relations for hypergeometric functions

Most of the relations we present here can be found in [2, 18, 81] or are derived from there. An integral representation for the hypergeometric functions is given by

$$
F\big(a, b; c; z\big) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)}\int_0^1 \mathrm{d}u\, u^{b-1}(1-u)^{c-b-1}(1-zu)^{-a} \ , \qquad \Re c > \Re b > 0 \ . \text{ (B.46)}
$$

In particular, hypergeometric functions with parameters a, b, and c that are related to each other are important for us. One finds so called quadratic transformation formulas like

$$
\begin{aligned}
F\big(a, a + \tfrac{1}{2}; c; z^2\big) &= (1 + z)^{-2a}F\big(2a, c - \tfrac{1}{2}; 2c - 1; \tfrac{2z}{1+z}\big) \ , \\
F\big(a, b; a + b - \tfrac{1}{2}; z\big) &= \frac{1}{\sqrt{1 - z}}F\big(2a - 1, 2b - 1; a + b - \tfrac{1}{2}; \tfrac{1}{2}(1 - \sqrt{1 - z})\big) \ , \\
F\big(a, b; a + b + \tfrac{1}{2}; z\big) &= F\big(2a, 2b; a + b + \tfrac{1}{2}; \tfrac{1}{2}(1 - \sqrt{1 - z})\big) \ .
\end{aligned}
$$
(B.47)

The hypergeometric functions in the propagators (5.10) with $\Delta_{\pm} = \frac{d\pm1}{2}$ (at the mass value generated by the Weyl invariant coupling) become ordinary analytic expressions

$$
\begin{aligned}
F\big(a, a + \tfrac{1}{2}; \tfrac{1}{2}; \xi^2\big) &= \frac{1}{2}\big[(1+\xi)^{-2a} + (1-\xi)^{2a}\big] \ , \\
F\big(a + \tfrac{1}{2}, a + 1; \tfrac{3}{2}; \xi^2\big) &= -\frac{1}{4a\xi}\big[(1+\xi)^{-2a} - (1-\xi)^{-2a}\big] \ .
\end{aligned}
\tag{B.48}
$$

Setting $a = \frac{d-1}{4}$ one finds (7.12).

To find the hypergeometric functions relevant for the propagators in higher dimensional AdS spaces one can use a recurrence relation (Gauß' relation for contiguous functions)

$$
F\big(a, b; c-1; z\big) = \frac{c[c - 1 - (2c - a - b - 1)z]}{c(c-1)(1-z)}F\big(a, b; c; z\big) + \frac{(c-a)(c-b)z}{c(c-1)(1-z)}F\big(a, b; c+1; z\big) \ .
\tag{B.49}
$$

where the hypergeometric functions relevant in lower dimensional AdS spaces enter.

For odd AdS dimensions (even d) the relevant hypergeometric functions can be expressed with the above recurrence relation in terms of ordinary analytic functions. This happens because of the explicit expressions

$$
\begin{aligned}
F\big(a, a + \tfrac{1}{2}; 2a; z\big) &= \frac{2^{2a-1}}{\sqrt{1-z}}\big[1 + \sqrt{1-z}\big]^{1-2a} \ , \\
F\big(a, a + \tfrac{1}{2}; 2a + 1; z\big) &= 2^{2a}\big[1 + \sqrt{1-z}\big]^{-2a} \ .
\end{aligned}
\tag{B.50}
$$

One has to apply (B.49) n times to compute the AdS propagator at generic Δ in $d+1 = 3+2n$ dimensions. For AdS$_3$ one simply uses the first expression in (B.50). The AdS$_5$ case is of particular importance and therefore we give the explicit expression for the needed hypergeometric function

$$
F\big(\tfrac{\Delta}{2}, \tfrac{\Delta}{2} + \tfrac{1}{2}; \Delta - 1; z\big) = \frac{1}{2(1-z)^{\frac{3}{2}}}\Big[\frac{2}{1+\sqrt{1-z}}\Big]^{\Delta-1}\Big[\sqrt{1-z} + \frac{\Delta - 1}{\Delta - 2}(1-z) + \frac{1}{\Delta - 1}\Big] \ .
\tag{B.51}
$$

B.7 Spheres and spherical harmonics of arbitrary dimensions

Here we have collected some useful facts about spheres and spherical harmonics, see e. g. [18] for more details. A discussion of spherical harmonics require some results for Gegenbauer polynomials that can be found for example in [2].

The $(d'+1)$-dimensional sphere can be parameterized in several ways. One can choose the standard spherical coordinates

$$0 \leq \psi_k \leq \pi \ , \qquad k = 1, \ldots d' \ , \qquad 0 \leq \psi_{d'+1} < 2\pi \tag{B.52}$$

with the embedding

$$Y_k = R_2 \prod_{i=1}^{k-1} \sin \psi_i \cos \psi_k \ , \qquad k = 1, \ldots, d'+1 \ ,$$

$$Y_{d'+2} = R_2 \prod_{i=1}^{d'} \sin \psi_i \sin \psi_{d'+1} \tag{B.53}$$

in which the metric reads

$$\mathrm{d}s^2 = R_2^2 \big(\mathrm{d}\psi_1^2 + \sin^2 \psi_1 (\mathrm{d}\psi_2^2 + \cdots + \sin^2 \psi_{d'-1} (\mathrm{d}\psi_{d'}^2 + \sin^2 \psi_{d'} \, \mathrm{d}\psi_{d'+1}^2) \cdots) \big) \ . \tag{B.54}$$

Alternatively it is sometimes advantageous to divide the sphere into two subspheres of dimensions $\bar{d}$ and $d' - \bar{d}$ with the help of the coordinate ψ,

$$0 \leq \psi \leq \frac{\pi}{2} \ , \tag{B.55}$$

and the embedding

$$Y_k = R_2 \cos \psi \, \omega_k \ , \qquad \sum_{k=1}^{\bar{d}+1} \omega_k^2 = 1 \ , \qquad k = 1, \ldots, \bar{d}+1 \ ,$$

$$Y_l = R_2 \sin \psi \, \hat{\omega}_l \ , \qquad \sum_{l=1}^{d'-\bar{d}+1} \hat{\omega}_l^2 \ , \qquad l = 1, \ldots, d' - \bar{d}+1 \ , \tag{B.56}$$

such that the metric is then given by

$$\mathrm{d}s^2 = R_2^2 \big(\mathrm{d}\psi^2 + \sin^2 \psi \, \mathrm{d}\Omega_{\bar{d}}^2 + \cos^2 \psi \, \mathrm{d}\hat{\Omega}_{d'-\bar{d}}^2 \big) \ . \tag{B.57}$$

In the coordinates (B.53) the volume form of the unit $\mathrm{S}^{d'+1}$ reads

$$\mathrm{vol}(\Omega_{d'+1}) = \prod_{k=1}^{d'} \Big((\sin \psi_k)^{d'-k+1} \, \mathrm{d}\psi_k \Big) \wedge \mathrm{d}\psi_{d'+1} \ , \tag{B.58}$$

where the product has to be understood as the wedge product. Integrating the volume form leads to the total volume of the unit $\mathrm{S}^{d'+1}$

$$\Omega_{d'+1} = \frac{2\pi^{\frac{d'}{2}+1}}{\Gamma(\frac{d'}{2}+1)} \ . \tag{B.59}$$

Spherical harmonics $Y^I(y)$ on $S^{d'+1}$ are characterized by quantum numbers

$$I = (l, m_1, \ldots, m_{d'}) , \qquad l \geq m_1 \geq \cdots \geq m_{d'-1} \geq |m_{d'}| \geq 0 \qquad \text{(B.60)}$$

and form irreducible representations of $SO(d' + 2)$. They are eigenfunctions with respect to the Laplace operator on the sphere

$$\Box_y Y^I(y) = -\frac{l(l + d')}{R_2^2} Y^I(y) . \qquad \text{(B.61)}$$

In the coordinates (B.53) they are explicitly given by [18]

$$Y^{l, m_1, \ldots, m_{d'}}(y) = N(l, m_1, \ldots, m_{d'}) \, e^{im_{d'}\psi_{d'+1}} \prod_{k=1}^{d'} (\sin \psi_k)^{m_k} C_{m_{k-1}-m_k}^{(m_k + \frac{1}{2}(d'-k+1))}(\cos \psi_k) , \quad \text{(B.62)}$$

where $y = (\psi_1, \ldots, \psi_{d'+1})$, $m_0 = l$ and the $C_l^{(\beta)}$ are the Gegenbauer Polynomials described below. With their normalization in (B.71) the normalization factor is given by

$$(N(l, m_1, \ldots, m_{d'}))^{-2} = 2\pi \prod_{k=1}^{d'} N(m_{k-1} - m_k, m_k + \tfrac{d'-k+1}{2}) , \qquad \text{(B.63)}$$

such that the spherical harmonics are orthonormal w. r. t. integration over the unit $S^{d'+1}$. One very important relation for the spherical harmonics is

$$\sum_{m_1 \geq \cdots \geq m_{d'-1} \geq |m_{d'}| \geq 0}^{l} Y^I(y) Y^{*I}(y') = \frac{(2l + d')\Gamma(\frac{d'}{2})}{4\pi^{\frac{d'}{2}+1}} C_l^{(\frac{d'}{2})}(\cos \Theta) , \qquad \cos \Theta = \frac{\vec{Y}\vec{Y}'}{R_2^2} = 1 - \frac{v}{2R_2^2} ,$$
$$\text{(B.64)}$$

where $\vec{Y}$ and $\vec{Y}'$ are the embedding vectors which coordinates are given by (B.53). The above formula can be easily verified with the help of the homogeneity of the sphere which allows one to choose $\vec{Y}' = (1, 0, \ldots, 0)$. At this values all angles $y' = (0, \ldots, 0)$ and therefore all spherical harmonics (B.62) except of

$$Y^{l, 0, \ldots, 0}(y') = N(l, 0, \ldots, 0) C_l^{(\frac{d'}{2})}(1) \prod_{k=1}^{d'-1} C_0^{(\frac{d'-k}{2})}(1) = N(l, 0, \ldots, 0) C_l^{(\frac{d'}{2})}(1) \qquad \text{(B.65)}$$

are zero. The normalization factor reads

$$(N(l, 0, \ldots, 0))^2 = \frac{(2l + d')\Gamma(\frac{d'}{2})}{4\pi^{\frac{d'}{2}+1}} \frac{1}{C_0^{(\frac{d'}{2})}(1)} , \qquad \text{(B.66)}$$

as can be verified from (B.63) with (B.69) by using the duplication formula

$$2^{1-z}\sqrt{\pi}\Gamma(z) = \Gamma(\tfrac{z}{2})\Gamma(\tfrac{z+1}{2}) . \qquad \text{(B.67)}$$

Only the first term of the sum (B.64) survives and yields the R. H. S. with $\cos\theta = \cos\psi_1$, and ψ_1 being given by the angle between $\vec{Y}$ and $\vec{Y}'$ according to (B.53).

The Gegenbauer Polynomials $C_l^{(\beta)}$ can be defined via their generating function

$$\frac{1}{(1 - 2q\eta + q^2)^\beta} = \sum_{l=0}^{\infty} q^l C_l^{(\beta)}(\eta) \ . \tag{B.68}$$

Their standardization is given by

$$G_l^{(\beta)}(1) = \frac{\Gamma(l + 2\beta)}{\Gamma(l + 1)\Gamma(\beta)} \tag{B.69}$$

and they obey the orthogonality relation

$$\int_{-1}^{1} d\eta \, (1 - \eta^2)^{\beta - \frac{1}{2}} C_m^{(\beta)}(\eta) C_n^{(\beta)}(\eta) = N(n, \beta)\delta_{mn} \ , \tag{B.70}$$

where

$$N(n, \beta) = \frac{2^{1-2\beta}\pi\Gamma(n + 2\beta)}{\Gamma(n + 1)(n + \beta)\Gamma(\beta)^2} \ . \tag{B.71}$$

One way to represent the Gegenbauer polynomials is via Rodrigues' formula

$$C_n^{(\beta)}(\eta) = (-1)^n 2^{-n} \frac{\Gamma(\beta + \frac{1}{2})\Gamma(n + 2\beta)}{\Gamma(n + 1)\Gamma(2\beta)\Gamma(n + \beta + \frac{1}{2})} (1 - \eta^2)^{\frac{1}{2} - \beta} \frac{d^n}{d\eta^n} (1 - \eta^2)^{n + \beta - \frac{1}{2}} \ . \tag{B.72}$$

An important property is their symmetry under reflection of their arguments. One can easily read off from (B.68) that they obey

$$C_l^{(\beta)}(-\eta) = (-1)^l C_l^{(\beta)}(\eta) \ . \tag{B.73}$$

This symmetry relates the values of the spherical harmonics (B.62) at $y = (\psi_1, \ldots, \psi_{d'+1})$ to their values at the antipodal point on the sphere, which is given by

$$\tilde{y} = (\tilde{\psi}_1, \ldots, \tilde{\psi}_{d'}, \tilde{\psi}_{d'+1}) = (\pi - \psi_1, \ldots, \pi - \psi_{d'}, \psi_{d'+1} + \pi) \ . \tag{B.74}$$

Then the spherical harmonics obey

$$Y^I(\tilde{y}) = (-1)^l Y^I(y) \ . \tag{B.75}$$

B.8   Proof of the summation theorem

Using (5.10) and (B.64) for $\Delta_+ = l + \frac{d+d'}{2}$ (leading to (7.42)), one finds the solution (7.6) if the following relation holds for $\alpha \geq \beta > 0$, $2\alpha, 2\beta \in \mathbb{N}$

$$\sum_{l=0}^{\infty} \frac{\Gamma(l+\alpha)}{\Gamma(l+\beta)} \left(\frac{\xi}{2}\right)^l F\left(\tfrac{l}{2} + \tfrac{\alpha}{2}, \tfrac{l}{2} + \tfrac{\alpha}{2} + \tfrac{1}{2}; l + \beta + 1; \xi^2\right) C_l^{(\beta)}(\eta) = \frac{\Gamma(\alpha)}{\Gamma(\beta)} \frac{1}{(1 - \xi\eta)^\alpha} \ , \quad \text{(B.76)}$$

with the interpretation $\alpha = \frac{d+d'}{2}$, $\beta = \frac{d'}{2}$. We could not find the above formula in the literature. It is in fact a summation rule for a product of a special hypergeometric function for which so called quadratic transformation formulae exist and which can be expressed in terms of a Legendre function [2] and of a Gegenbauer polynomial. The identity can therefore be re-expressed in the following way

$$\left(\frac{2}{\xi}\right)^\beta (1 - \xi^2)^{\frac{\beta-\alpha}{2}} \sum_{l=0}^{\infty} \Gamma(l+\alpha)(l+\beta) P_{\alpha-\beta-1}^{-l-\beta}\left(\tfrac{1}{\sqrt{1-\xi^2}}\right) C_l^{(\beta)}(\eta) = \frac{\Gamma(\alpha)}{\Gamma(\beta)} \frac{1}{(1 - \xi\eta)^\alpha} \ . \quad \text{(B.77)}$$

The simplest way to prove[2] this relation is to use the orthogonality of the Gegenbauer polynomials (B.70) to project out a term with fixed l from the sum in (B.76). One finds

$$\frac{\Gamma(l+\alpha)}{\Gamma(l+\beta)} \left(\frac{\xi}{2}\right)^l F\left(\tfrac{l}{2} + \tfrac{\alpha}{2}, \tfrac{l}{2} + \tfrac{\alpha}{2} + \tfrac{1}{2}; l + \beta + 1; \xi^2\right) N(l,\beta) = \frac{\Gamma(\alpha)}{\Gamma(\beta)} \int_{-1}^{1} \mathrm{d}\eta \, \frac{(1 - \eta^2)^{\beta-\frac{1}{2}}}{(1 - \xi\eta)^\alpha} C_l^{(\beta)}(\eta) \ . \quad \text{(B.78)}$$

The Gegenbauer polynomials are then expressed with Rodrigues' formula (B.72). Applying partial integration l times and using

$$\frac{\mathrm{d}^l}{\mathrm{d}\eta^l} (1 - \xi\eta)^{-\alpha} = \xi^l \frac{\Gamma(l+\alpha)}{\Gamma(\alpha)} (1 - \xi\eta)^{-l-\alpha} \quad \text{(B.79)}$$

then yields

$$N(l,\beta) F\left(\tfrac{l}{2} + \tfrac{\alpha}{2}, \tfrac{l}{2} + \tfrac{\alpha}{2} + \tfrac{1}{2}; l + \beta + 1; \xi^2\right)$$
$$= \frac{\Gamma(l+\beta)}{\Gamma(\beta)} \frac{\Gamma(\beta+\tfrac{1}{2})\Gamma(l+2\beta)}{\Gamma(l+1)\Gamma(2\beta)\Gamma(l+\beta+\tfrac{1}{2})} \times \int_{-1}^{1} \mathrm{d}\eta \, \frac{(1 - \eta^2)^{l+\beta-\frac{1}{2}}}{(1 - \xi\eta)^{l+\alpha}} \ , \quad \text{(B.80)}$$

---

[2] We thank Danilo Diaz for delivering a simplification of our original recurrence proof, that allows for an extension to more general values of α and β.

where $N(l, \beta)$ is given by (B.71) With the variable substitution $u = \frac{\eta+1}{2}$ the integral on the R. H. S. can be cast into the form

$$
\begin{aligned}
\int_{-1}^{1} \mathrm{d}\eta \, \frac{(1 - \eta^2)^{l+\beta-\frac{1}{2}}}{(1 - \xi\eta)^{l+\alpha}} &= \frac{4^{l+\beta}}{(1 + \xi)^{l+\alpha}} \int_{0}^{1} \mathrm{d}u \, u^{l+\beta-\frac{1}{2}} (1 - u)^{l+\beta-\frac{1}{2}} \left(1 - \tfrac{2\xi}{1+\xi}u\right)^{-l-\alpha} \\
&= \frac{4^{l+\beta}}{(1 + \xi)^{l+\alpha}} \frac{(\Gamma(l + \beta + \frac{1}{2}))^2}{\Gamma(2l + 2\beta + 1)} F\left(l + \alpha, l + \beta + \tfrac{1}{2}; 2l + 2\beta + 1; \tfrac{2\xi}{1+\xi}\right) \\
&= 4^{l+\beta} \frac{\Gamma(l + \beta + \frac{1}{2})^2}{\Gamma(2l + 2\beta + 1)} F\left(\tfrac{l}{2} + \tfrac{\alpha}{2}, \tfrac{l}{2} + \tfrac{\alpha}{2} + \tfrac{1}{2}; l + \beta + 1; \xi^2\right) .
\end{aligned}
$$
(B.81)

In the second line we have used the integral representation for the hypergeometric functions (B.46) and the third line follows with the help of the quadratic transformation formula in the first line of (B.47). As the last step, we have to insert (B.81) into (B.80). With the duplication formula (B.67) one then finds that both sides of (B.80) match, and the proof of (B.76) is complete. The relation is valid not only for $\alpha \geq \beta > 0$, $2\alpha, 2\beta \in \mathbb{N}$, but for all $\beta > 0$.

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Hilfsmittel

Außer den angegebenen Referenzen habe ich das Computeralgebra-Programm `Mathematica` und den Fortran-Compiler `f77` verwendet. Die Feynman-Graphen und 2-dimensionalen Figuren wurden mit `Axodraw` und die 3-dimensionalen Figuren mit `Gnuplot` erstellt. Die Arbeit selbst wurde unter Verwendung von LaTeX und BibTeX verfaßt.

Selbständigkeitserklärung

Hiermit erkläre ich, die vorliegende Arbeit selbständig ohne fremde Hilfe verfaßt und nur die angegebene Literatur und Hilfsmittel verwendet zu haben.

Berlin, den 31. März 2005   Christoph Sieg