Adv. Geom. **7** (2007), 475–528
DOI 10.1515 / ADVGEOM.2007.030

Advances in Geometry

Transitive projective planes

Nick Gill*

(Communicated by W. M. Kantor)

Abstract. A long-standing conjecture is that any transitive finite projective plane is Desarguesian. We make a contribution towards a proof of this conjecture by showing that a group acting transitively on the points of a non-Desarguesian projective plane must not contain any components.

1 Background definitions and main results

We say that a projective plane is *transitive* (respectively *primitive*) if it admits an automorphism group which is transitive (respectively primitive) on points. Kantor [22] has proved that a projective plane $\mathcal{P}$ of order x admitting a point-primitive automorphism group G is Desarguesian and $G \geq \mathrm{PSL}(3, x)$, or else $x^2 + x + 1$ is a prime and G is a regular or Frobenius group of order dividing $(x^2 + x + 1)(x + 1)$ or $(x^2 + x + 1)x$.

Kantor's result, which depends upon the Classification of Finite Simple Groups, represents the strongest success in the pursuit of a proof to the conjecture mentioned in the abstract. A corollary of Kantor's result is that a group acts primitively on the points of a projective plane $\mathcal{P}$ if and only if it acts primitively on the lines of $\mathcal{P}$. We also know, by a combinatorial argument of Block, that a group acts transitively on the points of a projective plane $\mathcal{P}$ if and only if it acts transitively on the lines of $\mathcal{P}$ [5].

Our primary result is the following:

Theorem A. *Suppose that G acts transitively on a projective plane $\mathcal{P}$ of order x. Then one of the following cases holds:*

- *$\mathcal{P}$ is Desarguesian, $G \geq \mathrm{PSL}(3, x)$ and the action is 2-transitive on points;*
- *G does not contain a component. In particular all minimal normal subgroups of G are elementary abelian.*

*This paper contains results from the author's PhD thesis. I would like to thank my supervisor, Professor Jan Saxl. Professor Bill Kantor has also given much helpful advice for which I am very grateful.

Here a *component* C of a group G is defined to be a subnormal quasi-simple subgroup of G. We note that Theorem A implies that if an almost simple group (or almost quasi-simple group) G acts transitively on the lines of a projective plane $\mathcal{P}$ of order x then $\mathcal{P}$ is Desarguesian and G has socle $\mathrm{PSL}(3, x)$. Note that definitions for group theory terms used here are provided in Section 4.

Theorem A also relates to two other results that already exist in the literature. The first is Kantor's result on primitive projective planes [22] which has already been mentioned and which is used in the proof of Theorem A; Theorem A can be thought of as a generalization of Kantor's result. The second is Ho's result that a finite projective plane admitting more than one abelian Singer group is Desarguesian [20, Theorem 1]; this result is implied by Theorem A and [20, Lemma 4.3 and Theorem 2] – details are given in [16]. In fact [16] outlines a number of results about line-transitive projective planes that follow from Theorem A.

Finally we note that all groups and sets that we consider in this paper are finite.

2 Overview of proof

To prove Theorem A we need to analyse many different possible transitive group actions on finite projective planes. The framework for our analysis of the transitive projective planes will be given by results in [9] and [7]. The key theorem is the following:

Theorem 1. [7, Theorem 2] *Let G act transitively on a projective plane $\mathcal{P}$ and let M be a minimal normal subgroup of G. Then M is either abelian or simple.*

In fact we are able to state our results more strongly by rewriting this result in terms of *components*. Hence the theorem which will provide the framework for our analysis is the following:

Theorem 2. *Suppose that G acts transitively on a projective plane $\mathcal{P}$. Then G contains at most one component.*

The proof of this theorem, which involves rewriting proofs of similar theorems from [9] and [7], is given in Section 3. In Section 4 we give the basic lemmas and notation which will be used throughout the remainder of the paper.

In the remaining sections we use Theorem 2 to examine the possible unique components of a group G acting transitively on a projective plane. Existing results in the literature are generally limited to the case where the component is simple and G is almost simple.

3 Framework results

We prove Theorem 2 which states that if a group G acts transitively upon a projective plane then G contains at most one component. Our proof of Theorem 2 starts with some preliminary results.

Note first that if C is a component of G then $C^\circ := \langle C^g : g \in G\rangle \cong C \circ C^{g_1} \circ \cdots \circ C^{g_m}$ is a normal subgroup of G where $g_1, \ldots, g_m \in G$; furthermore, if C and D are components of G with C not G-conjugate to D then $[C, D] = 1$ and so $[C^\circ, D^\circ] = 1$.

We need some information about the fixed points of automorphisms of a projective plane $\mathcal{P}$ of order x: If an automorphism g fixes at least x points then g is called *quasicentral* and g fixes $x+1$, $x+2$ or $x+\sqrt{x}+1$ points [14, 4.1.7]. In the first two cases g fixes a *fan*, namely a line $\mathfrak{L}$ and a point α and all the points on $\mathfrak{L}$ and all the lines incident with α. The distinction between the two cases depends on whether or not α lies on $\mathfrak{L}$. In the third case the set of fixed points and fixed lines of g forms a subplane of $\mathcal{P}$ of order $\sqrt{x}$.

In addition we have the following lemma:

Lemma 3. [14, 3.1.2 and 4.1.6] *Let $\mathcal{P}$ be a projective plane of order x. If H is a group of automorphisms of $\mathcal{P}$ which does not fix* (*point-wise*) *a subplane of $\mathcal{P}$ then the fixed set of H lies inside a fan. If, on the other hand, H point-wise fixes a subplane of order y then either $y^2 = x$ or $y(y+1) \leq x-2$.*

We are now ready to prove our first result which is very similar to [9, Theorem 3]:

Proposition 4. *Let G be a transitive automorphism group of a projective plane $\mathcal{P}$ of order greater than 4. Let G have normal subgroups M and N such that $M_\alpha \neq 1$ and $N_\alpha \neq 1$ for some point α. Then $[N, M] \neq 1$.*

Proof. Let M and N be two normal subgroups of G such that there is a point α so that $M_\alpha \neq 1$ and $N_\alpha \neq 1$ and $[M, N] = 1$.

Consider the point $\beta \in \alpha N$ and let $n \in N$ be such that $\beta = \alpha n$. If $m \in M_\alpha$, then $\beta m = \alpha n m = \alpha m n = \beta$. Thus αN is fixed point-wise by M_α. If $\beta \in \alpha N\backslash\{\alpha\}$ and $\mathfrak{L}$ is the line through α and β, then M_α fixes $\mathfrak{L}$ set-wise. Thus there is a line $\mathfrak{L}$ through α which is fixed by M_α and M_α fixes at least two points. A similar result applies with N replacing M.

Next we show that every line through α is fixed either by M_α or N_α. Assume that this is false and let $\mathfrak{L}$ be a line through α which is fixed by neither. Since G is line-transitive, there is some point β such that M_β fixes $\mathfrak{L}$. Now, since $[M, N] = 1$, N_α acts on the set of fixed lines of M_β. Thus each image of $\mathfrak{L}$ under the action of N_α is a line through α fixed by M_β. Since N_α does not fix $\mathfrak{L}$, it follows that M_β fixes α. However, this means that $M_\beta = M_\alpha$ and hence M_α fixes $\mathfrak{L}$ which is a contradiction to our assumption.

Thus, for one of M_α and N_α, the number of lines through α which are fixed must be at least $k/2$. Without loss of generality, this is true for N_α. We now show that the set of fixed points of N_α forms a subplane of $\mathcal{P}$. By the lemma above it is sufficient to prove that $N_G(N_\alpha)$ acts transitively on the set of lines fixed by N_α; to show this we demonstrate that $N_{\mathfrak{L}} = N_\alpha$ for any line $\mathfrak{L}$ fixed by N_α.

Let $\mathfrak{L}$ be any line through α which is fixed by N_α. Let $m \in M$ such that $\mathfrak{L}m \neq \mathfrak{L}$. Then, since $[M, N] = 1$, it follows that $\mathfrak{L}mN_{\mathfrak{L}} = \mathfrak{L}N_{\mathfrak{L}}m = \mathfrak{L}m$, that is $N_{\mathfrak{L}}$ fixes $\mathfrak{L}m$ and so $N_{\mathfrak{L}}$ fixes $\mathfrak{L}m \cap \mathfrak{L} = \{\beta\}$, say. Then $N_\alpha \subseteq N_{\mathfrak{L}} \subseteq N_\beta$, and since N_α is conjugate to N_β, we obtain $N_\alpha = N_{\mathfrak{L}}$.

Since N is normal in G, $N_G(N_{\mathfrak{L}})$ is transitive on the lines fixed by $N_{\mathfrak{L}} = N_\alpha$. Thus the fixed set of N_α is a subplane of $\mathcal{P}$ with line size at least $k/2$. This is a contradiction of the lemma above. □

Corollary 5. *Suppose that G acts transitively on a projective plane $\mathcal{P}$. Then all components of G are conjugate in G.*

Proof. If $\mathcal{P}$ is Desarguesian then G contains at most one component and the statement holds.

By [14, 3.2.15] a non-Desarguesian projective plane has order at least 9. Thus by the previous theorem any two normal subgroups M and N of G with $M_\alpha \neq 1$ and $N_\alpha \neq 1$ for some point α satisfy $[N, M] \neq 1$.

Now suppose that C and D are components of G which are not conjugate in G. Then C° and D° are distinct normal subgroups of G. Note that any component contains an involution and, since the number of points in $\mathcal{P}$ is odd, each involution must fix a point. The theorem implies that $[C^\circ, D^\circ] \neq 1$. This is a contradiction. □

We can now prove Theorem 2. Our method of proof is very similar to that of Camina [7, Theorem 1]. First we state some preliminary results:

Lemma 6. [9, Theorem 1] *Let $\mathcal{P}$ be a finite linear space and let G be a line-transitive automorphism group of $\mathcal{P}$. Let N be a normal subgroup of G. Then N acts faithfully on each of its point orbits.*

Lemma 7. [21, XIII.13.1] *Let A be an abelian automorphism group of a projective plane of order x. Then $|A| \leq x^2 + x + 1$.*

Theorem 2. *Suppose that G acts transitively on a projective plane $\mathcal{P}$. Then G contains at most one component.*

Proof. By Corollary 5, $\mathcal{P}$ is non-Desarguesian of order x and all components are conjugate in G. Let C be a component of G and let C° be the normal closure of C in G. Write $C^\circ = C_1 \circ \cdots \circ C_m$ with each C_i isomorphic to C and suppose that $m \geq 2$.

Let D be a Sylow 2-subgroup of C°. Since $\mathcal{P}$ has an odd number of points there is a point α so that D fixes α. Thus $(C_i)_\alpha \neq 1$ for $1 \leq i \leq m$. Since G acts transitively on $\mathcal{P}$ this is true for all points α. Choose α so that $(C_1)_\alpha$ has maximal order. Observe that $[C_2, (C_1)_\alpha] = 1$ so αC_2 consists of points fixed by $(C_1)_\alpha$.

Now C° is faithful on all its point orbits by Lemma 6. This implies that αC_2 contains at least 5 points as C_2 is quasisimple and normal in C°. The fixed set of $(C_1)_\alpha$ is either a subplane or lies inside a fan. But, since C_2 does not fix any point, we conclude that $(C_1)_\alpha$ fixes a subplane whose order is at most $\sqrt{x}$.

We now show that for any line $\mathfrak{L}$ incident with α there is a j so that $(C_j)_\alpha$ fixes $\mathfrak{L}$. Choose a line $\mathfrak{L}$ incident with α. If $(C_1)_\alpha$ fixes $\mathfrak{L}$ there is nothing to prove. We know that there exists a line, $\mathfrak{L}_1$, which is incident with α and is fixed by $(C_1)_\alpha$. But G is transitive on lines so there is $g \in G$ with $\mathfrak{L}_1 g = \mathfrak{L}$. Then $\beta = \alpha g$ is incident with $\mathfrak{L}$ and $((C_1)_\alpha)^g$ fixes $\mathfrak{L}$. But there exists j so that $((C_1)_\alpha)^g = (C_j)_\beta$ since g permutes the factors C_i. Let

$i \neq j$. Then $(C_i)_\alpha$ commutes with $(C_j)_\beta$ and so acts on the set of lines fixed by $(C_j)_\beta$. If $(C_i)_\alpha$ fixes $\mathfrak{L}$ then we have proved our claim. If not we see that $(C_j)_\beta$ fixes at least two lines through α and so fixes α. However $((C_1)_\alpha)^g = (C_j)_\beta$ so by the maximality of $(C_1)_\alpha$ we have $(C_j)_\alpha = (C_j)_\beta$ and the claim is proved.

Let y be the order of the subplane fixed by $(C_i)_\alpha$. Then $m(y+1) \geq x+1$. If $y = \sqrt{x}$ then this implies that $m \geq \sqrt{x}$. If $y \neq \sqrt{x}$ then Lemma 3 implies that $y(y+1) \leq x-2$. Thus $m \geq y+1$ and so $m \geq \sqrt{x+1} > \sqrt{x}$.

Since C° has an abelian subgroup of order at least 5^m it follows from Lemma 7 that $x^2+x+1 \geq 5^m \geq 5^{\sqrt{x}}$. This has no solutions. □

4 Basic results and notation

The notation outlined in this section will hold throughout the rest of the paper. We also state here a number of basic results which will be used repeatedly throughout the paper.

4.1 Projective plane results. Consider a projective plane $\mathcal{P}$ of order x with $v = x^2 + x + 1$ points and lines.

Lemma 8. [22, p. 33] *Let G act transitively on a projective plane with G_α a point-stabilizer. If p_1 is a prime $\equiv 2(3)$ then G_α contains some Sylow p_1-subgroup of G. Moreover, G_α contains a subgroup of index at most 3 in a Sylow 3-subgroup of G.*

For g an element of G we write n_g for the size of the G-conjugacy class of g in G and r_g for the number of these conjugates lying in a point-stabilizer G_α, for some fixed point α in $\mathcal{P}$. Furthermore, d_g is the number of fixed points of g. We will sometimes also write $r_g(B)$ for the number of G-conjugates of g lying in a subgroup B of G, so $r_g = r_g(G_\alpha)$.

We know already that if an automorphism g fixes at least x points then g is called *quasicentral* and g fixes $x+1$, $x+2$ or $x+\sqrt{x}+1$ points [14, 4.1.7]. Furthermore, if an automorphism has $x+1$ or $x+2$ fixed points then it is known as a *perspectivity* and Wagner has proved that if G contains a nontrivial perspectivity and G acts transitively on $\mathcal{P}$ then $\mathcal{P}$ is Desarguesian and $G \geq \mathrm{PSL}(3,x)$ [34].

Now any involution is quasicentral ([14, 3.1.6]) and so all the groups G that we consider contain quasicentral automorphisms. By Wagner's result we will be interested in the situation when x is a square, say $x = u^2$, and all quasicentral automorphisms, in particular all involutions, have u^2+u+1 fixed points.

We will be particularly interested in properties of integers of the form u^2+u+1 where u is an integer.

Lemma 9. *If $x = u^2$ then $x^2+x+1 = (u^2+u+1)(u^2-u+1)$, where $(u^2+u+1, u^2-u+1) = 1$.*

Lemma 10. [27, p. 11] *If $u^2+u+1 = p_1^a$ where p_1 is a prime, then either $p_1^a = p_1$ or $p_1^a = 7^3$.*

Lemma 11. [22, p. 33] *If $x = u^2$ and $x^2 + x + 1 = p^a m$ for a prime p with $a > 1$, then either $m > 8p^a$ or $p^a = u^2 \pm u + 1 = 7^3$.*

Lemma 12. *Let $x = u^2$ and let g be an involution acting on projective plane $\mathcal{P}$ with $u^2 + u + 1$ fixed points. Then*

- $\frac{n_g}{r_g} = u^2 - u + 1$*;*
- $d_g = u^2 + u + 1$*;*
- $v = \frac{n_g}{r_g} d_g$ *and* $(\frac{n_g}{r_g}, d_g) = 1$.

Proof. Count pairs of the form (α, g), where α is a point and g is an involution fixing α, in two different ways. Then $|\{(\alpha, g) : \alpha g = \alpha\}| = v r_g = n_g d_g$. We know already that $d_g = u^2 + u + 1$ thus we must have $\frac{n_g}{r_g} = u^2 - u + 1$ and the result follows. □

Lemma 13. *Suppose that g is an involution acting on projective plane $\mathcal{P}$ with $u^2 + u + 1$ fixed points. If $n_g = 2^c p^a m$ where $(m, 2p) = 1$ then the largest power of p in v is less than or equal to* $\max(p^a, m + 2\sqrt{m} + 2)$.

Proof. If $p | \frac{n_g}{r_g}$ then clearly the highest power of p dividing v divides p^a. If not, then $u^2 - u + 1 = \frac{n_g}{r_g}$ divides m. Then the highest power of p dividing v divides $d_g = u^2 + u + 1 < (u^2 - u + 1) + 2\sqrt{u^2 - u + 1} + 2$. □

It is in our exploitation of the last two results that our treatment will differ substantially from that of Kantor in the primitive case. We will make use of the equalities outlined in Lemma 12, taking g to be a member of a small conjugacy class of involutions.

4.2 Group theory results and notation. We begin with a general lemma which will be useful throughout the chapter.

Lemma 14. *Let $C < A \times B$. Suppose $|A| < |B : N|$ where N is the largest proper normal subgroup of B. Then either:*

- $C \leq A \times B_1$ *for* $B_1 < B$*; or*
- $C = A_1 \times B$ *for* $A_1 \leq A$.

Proof. Suppose $C \not\leq A \times B_1$ for $B_1 < B$. Then define $B_1 = \{(1, b) : (a, b) \in C\} \cong B$ and observe that the projection $C \to A, (a, b) \mapsto a$ has kernel $K = \{(1, b) \in C\} \lhd B_1$. But $|B_1 : K| \leq |A| < |B : N|$ where N is the largest proper normal subgroup of B. Thus $K = B_1$ and $C = A_1 \times B$ for some $A_1 \leq A$ as required. □

Now we want to show that a group G with unique component L cannot act transitively on a projective plane $\mathcal{P}$ unless it contains a non-trivial perspectivity.

Recall that L is a component of G provided L is a subnormal quasi-simple subgroup of G; a *quasi-simple* group C is one such that $C = C'$ (C is equal to its commutator subgroup) and $C/Z(C)$ is simple. We also define an *almost simple* group to be a group G such that $N \trianglelefteq G \leq \mathrm{Aut}(N)$ where N is a non-abelian simple group; an almost simple group can also be thought of as a group with *non-abelian simple socle*, the *socle* of a group G being the product of the minimal normal subgroups of G. For a fuller discussion see [3].

We write $H.G$ for an extension of a group H by a group G and $H : G$ for a split extension. An integer n denotes a cyclic group of order n, while $[n]$ (respectively $[q^n]$) denotes an arbitrary soluble group of order n (respectively q^n) and p^n denotes an elementary abelian group of order p^n where p is a prime. We write $|H|_p$ for the highest divisor of $|H|$ which is a power of a prime p.

Put $L_\alpha = G_\alpha \cap L$, the stabilizer of a point α in the action of L on $\mathcal{P}$. In general, we will set M to be a maximal subgroup of the component L which contains L_α. Define $L^\dagger := L/Z(L)$ and $M^\dagger := M/(Z(L) \cap M)$.

Write $G = (L \circ C_G(L)).N$ where N is a subgroup of $\mathrm{Out}\, L$. Then $G/C_G(L)$ is an almost simple group and we use results about the maximal subgroups of such groups:

When $L^\dagger$ is a classical simple group we use the results of Aschbacher [1] as described in Kleidman and Liebeck [23]. These results give information about the maximal subgroups of a group $L^\dagger.N$ where the simple socle $L^\dagger$ is a classical group.

We will sometimes precede the structure of a subgroup of a projective group with $\hat{}$ which means that we are giving the structure of the pre-image in the corresponding universal group (we call this *hat notation*). For a given element $g \in L$ we will often write g^* for an element in the corresponding universal group which projects onto g. The symbol * will also be used in a different way, with groups, e.g. P_1^*, to signal that a group is a subgroup of a section of L or $L^\dagger$. Write $\mathrm{GF}(q)$ for the finite field of size q.

We now prove a small result which will be very useful:

Lemma 15. *Suppose that G has a unique component L and G acts transitively on the set of points of a projective plane $\mathcal{P}$. Then, except when $L = P\Omega^+(8, q)$, there exists $L \leq H \leq G$ such that $H/C_H(L) \leq \Gamma L$ and H acts transitively on the set of points of $\mathcal{P}$. Here ΓL is the full semilinear classical group associated with L.*

Proof. The result is trivial except when $L^\dagger = \mathrm{PSL}(n, q)$ while $G/C_G(L)$ contains an inverse-transpose automorphism of L and when $L = \mathrm{Sp}(4, 2^f)$ while $G/C_G(L)$ contains a graph automorphism of L. In both cases G contains a normal subgroup H of index 2 such that $H/C_H(L) \leq \Gamma L$. Since we are acting on a set of odd order, any transitive action of G induces a transitive action of H as required. $\square$

Lemma 15 implies that, to prove Theorem A, it is enough to show that the subgroup H cannot act transitively upon a non-Desarguesian projective plane as this implies that the same must hold for G. Thus, except when $L^\dagger = P\Omega^+(8, q)$, we assume that $G/C_G(L) \leq \Gamma L$.

We will write $M \in \mathcal{C}_i$ to mean that $M^\dagger$ is in the i-th family of natural maximal subgroups of $L^\dagger$ given by Kleidman and Liebeck [23]. When M is parabolic we will write $M = P_m$ to mean that M is a maximal parabolic subgroup fixing a totally singular subspace W of dimension m inside the natural classical geometry V of dimension n.

When $L^\dagger$ is an exceptional simple group we use different sources to find information about maximal subgroups M of L. When M is parabolic we refer to [10, 19, 18]. In some other cases, the maximal subgroups are completely enumerated; in particular for

$L^\dagger = {}^2B_2(q)$ [32], for $L^\dagger = {}^2G_2(q)$ [24, 35], for $L^\dagger = G_2(q)$ [24, 13], for $L^\dagger = {}^2F_4'(q)$ [28, 12] and for $L^\dagger = {}^3D_4(q)$ [25].

In both classical and exceptional cases, we appeal to a result of Liebeck and Saxl [26] and Kantor [22] which gives the maximal subgroups of odd index in an almost simple group. In particular, when the socle is a finite simple classical group acting on a classical geometry V, such a maximal subgroup either lies in $\mathcal{C}_1$ (stabilizers of totally singular or non-singular subspaces) for characteristic 2 or, when the characteristic is odd, lies in $\mathcal{C}_1$, $\mathcal{C}_2$ (stabilizers of decompositions into subspaces of fixed dimension, $V = \oplus_{i=1}^t V_i$) or $\mathcal{C}_5$ (stabilizers of subfields) or is in a small set of listed exceptions.

Finally, when $L^\dagger$ is a sporadic simple group we refer to [2] which, amongst many other things, lists the maximal subgroups of odd index.

Our analysis becomes slightly simpler by using the following result of Camina and Praeger which is a corollary of Lemma 6:

Lemma 16. [9, Corollary 1] *Let N be an abelian normal subgroup of a group G. Suppose that G acts line-transitively on a finite linear space $\mathcal{P}$. Then N acts semiregularly on the points of $\mathcal{P}$.*

In the case where $\mathcal{P}$ is a projective plane we can apply Lemma 8. Thus if L is a unique component of G then $Z(L)$ is normal in G and must have order only divisible by primes congruent to $1(3)$ or by 3 to the first power. In the case where L is a group of Lie type, for instance, this implies that L is simple unless it is isomorphic to $E_6(q)$, ${}^2E_6(q)$, $\mathrm{U}(n,q)$ or $\mathrm{PSL}(n,q)$ for certain n.

4.3 Hypothesis. Finally we state our hypothesis for the rest of the paper:

Hypothesis. 1. Suppose that G is a group with a unique component L;
2. Suppose that G acts transitively on a set of points of order $v = x^2 + x + 1$ where $x = u^2, u \in \mathbb{Z}, u \geq 2$;
3. Suppose that all involutions fix $u^2 + u + 1$ points;
4. Suppose that $L_\alpha \leq M$ where M is a maximal subgroup of L of odd index and that $v > |L : M|$;
5. Except when $L^\dagger = P\Omega^+(8,q)$, suppose that $G/C_G(L) \leq \Gamma L$;
6. Finally suppose that $Z(L)$ has order only divisible by primes congruent to $1(3)$ or by 3 to the first power.

Throughout the rest of the paper we will set $L^\dagger$ to be in a particular family of simple groups and will prove the following result (which, in turn, implies Theorem A):

Result. *If $L \neq \mathrm{PSL}(2,q)$, then our hypothesis leads to a contradiction. If $L = \mathrm{PSL}(2,q)$, then our hypothesis along with two extra suppositions* (*described in Section* 7) *leads to a contradiction.*

This result is entirely group theoretic and makes no reference to the geometry of projective planes. Note also that Lemmas 8 to 13 all apply under our hypothesis since they depend only on the number of points $x^2 + x + 1$.

5 $L^\dagger$ is alternating or sporadic

In this section we prove that, if $L^\dagger$ is alternating or sporadic, then the hypothesis in Section 4.3 leads to a contradiction. This implies the following proposition:

Proposition 17. *Suppose G has a unique component L such that $L^\dagger$ is isomorphic to an alternating group, A_n with $n \geq 5$, or a sporadic simple group. Then G does not act transitively on a projective plane.*

When $L^\dagger$ is a sporadic simple group, the maximal subgroups of $L^\dagger$ of odd index are given by Aschbacher [2]. Aschbacher's list implies that any maximal subgroup M of odd index in L has index divisible by 9 or by a prime congruent to $2(3)$. Since L_α must lie in such a maximal subgroup this contradicts Lemma 8.

Suppose that $L^\dagger \cong A_n$, the alternating group on n letters. If $n \neq 6, 7$ then $Z(L) \leq 2$ [30]; thus, by Lemma 16, $L = L^\dagger = A_n$. If $n = 6, 7$ then $Z(L) \leq 6$ and so, by Lemma 16, $L = A_n$ or $L = 3.A_n$.

Assume for the moment that $n > 7$ and so $L = A_n$. Let $g \in L = A_n$ be a double transposition. Then $n_g = n(n-1)(n-2)(n-3)/8$. Now A_n contains an abelian subgroup, H, of size $2^{\lfloor \frac{n}{2} \rfloor - 1}$ which contains at least $\lfloor \frac{n}{2} \rfloor (\lfloor \frac{n}{2} \rfloor - 1)$ L-conjugates of g.

Since H lies inside a Sylow 2-subgroup of L, we know that H lies in L_α for some point α. We conclude that

$$\frac{n_g}{r_g} \leq \frac{n(n-1)(n-2)(n-3)}{8\lfloor \frac{n}{2} \rfloor (\lfloor \frac{n}{2} \rfloor - 1)}.$$

Next we refer to Lemma 7 and observe that $|H| \leq v$. Furthermore, for $u > 2$, we have $v < 2\left(\frac{n_g}{r_g}\right)^2$. Hence

$$2^{\lfloor \frac{n}{2} \rfloor - 1} \leq 2\frac{n^2(n-1)^2(n-2)^2(n-3)^2}{2^6\lfloor \frac{n}{2} \rfloor^2 (\lfloor \frac{n}{2} \rfloor - 1)^2}.$$

Thus $2^{\lfloor \frac{n}{2} \rfloor} < n^4$ and $n \leq 43$. If $u = 2$ then $v = 21$ and again we can conclude that $n \leq 43$. Now to examine the cases where $7 < n \leq 43$ we use a method similar to that in [8, Section 5].

Consider the usual permutation action of $L = A_n$ as $\mathrm{Alt}(\Omega)$, acting on a set Ω of size n. Then L_α contains a Sylow p-subgroup of L for every prime $p \equiv 2(3)$ and a subgroup of index 3 in a Sylow 3-subgroup of L.

Let Γ be the longest orbit of L_α in Ω. If $8 \leq n \leq 10$ then, since L_α contains a Sylow 2-group and a Sylow 5-group of L, L_α^Γ must be primitive; if $11 \leq n \leq 21$ then the same conclusion comes from the primes 2 and 11; if $22 \leq n \leq 33$ then the same conclusion comes from the primes 2 and 17; and if $34 \leq n \leq 43$ then the same conclusion comes from the primes 2 and 29. Now L_α^Γ has odd index in $\mathrm{Alt}(\Gamma)$ and 5 does not divide the index. By [26] this means that L_α^Γ contains $\mathrm{Alt}(\Gamma)$.

For $n \geq 11, n \neq 39$, we claim that $|\Gamma| \geq n - 2$. This is proved using Lemma 8 for each individual value of n. We do not reproduce this here but consider, for instance, when

$n = 16$: Then L_α contains elements with cycle type (11) and $(8, 8)$ and so $|\Gamma| = 16 \geq n - 2$.

Let us examine this case, where $n \geq 11, n \neq 39$. Consider again, g, a double transposition with $n_g = n(n-1)(n-2)(n-3)/8$. Then $r_g \geq (n-2)(n-3)(n-4)(n-5)/8$ and so $\frac{n_g}{r_g} \leq \frac{n(n-1)}{(n-4)(n-5)} < 3$ for $n \geq 11$. This is impossible.

For $n = 39$ it turns out, using Lemma 8, that $|\Gamma| \geq 34$. Then $\frac{n_g}{r_g} < 3$ and this case is excluded.

For $n = 8$ or 10, the same argument gives $|\Gamma| = n$ and no action exists. For $n = 9$, $|\Gamma| \geq 5$ and, referring to [26], L_α lies in an intransitive subgroup of L and this contradicts Lemma 8.

Now suppose $n \leq 7$. If $n = 5$ or 6 then Lemma 8 implies that $|L : L_\alpha| \leq 3$. This is impossible since no subgroup of such small index exists in L. We are left with $n = 7$.

When $n = 7$ we know that L_α contains an element of order 5. Examining [12] this means that $M^\dagger = S_5$ or A_6. In fact we must have $L_\alpha = S_5$ or A_6. In both cases $\frac{n_g}{r_g}$ is not an integer. Thus all cases are excluded.

Remark. It is worth noting that we could prove Proposition 17 directly by appealing to [17, Theorem 1] and then dealing with the cases where $n < 21$.

6 $L^\dagger = \mathrm{PSL}(n, q), n > 3$

In this section we assume that $n > 3$ and prove that, if $L^\dagger = \mathrm{PSL}(n, q)$, then the hypothesis in Section 4.3 leads to a contradiction. This implies the following proposition:

Proposition 18. *If G has a unique component such that $L^\dagger$ is isomorphic to $\mathrm{PSL}(n, q)$ with $n > 3$, then G does not act transitively on a projective plane.*

Consider $\mathrm{SL}(n, q)$ acting naturally on a vector space V. Recall that a *transvection*, g^* say, in $\mathrm{SL}(n, q)$ is an automorphism of V such that $g^* - I$ has rank 1 and square 0. We now state the following preliminary result:

Lemma 19. *Let C be a conjugacy classes of involutions in L corresponding to either*

- *diagonalizable involutions in the natural modular representation of $\mathrm{SL}(n, q)$ with q odd; or to*
- *the projective image of transvections in $\mathrm{SL}(n, q)$, where $q = 2^a$ for some integer a.*

Then C is invariant under ΓL.

Proof. Consider the diagonalizable case first. We need to consider the actions by conjugation of automorphisms of $\mathrm{SL}(n,q)$ on a diagonal matrix,

$$g^* = \begin{pmatrix} -1 & & & & & \\ & \ddots & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}.$$

Clearly a field automorphism will preserve g^*. Similarly an automorphism lying in $\mathrm{GL}(n,q)$ of form,

$$\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & a \end{pmatrix}$$

where $a \in \mathrm{GF}(q)^*$, also preserves g^*. These generate the full outer automorphism group of $\mathrm{SL}(n,q)$ in $\Gamma L(n,q)$ and we are done. In the case where we have a transvection we consider the actions by conjugation of automorphisms of $\mathrm{SL}(n,q)$ on a matrix,

$$g^* = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ & 1 & & \ddots & \vdots \\ & & & \ddots & 0 \\ & & & & 1 \end{pmatrix}.$$

Clearly both field automorphisms and the automorphism in $\mathrm{GL}(n,q)$ exhibited above preserve g^* and we are done. □

Much of the ensuing treatment will involve counting involutions g. We will take care to ensure that g is always of one of the two types in this lemma thus ensuring that $n_g = r_g(L) = |L : C_L(g)|$ and $r_g = r_g(L_\alpha)$. Also, observe that we may exclude $\mathrm{PSL}(4,2) \cong A_8$. We begin by restricting the family within which M, a maximal subgroup of L containing L_α, may lie:

6.1 L_α must lie in a parabolic subgroup. By Liebeck and Saxl [26], we know that L_α lies inside a maximal subgroup M where

- for q odd, $M \in \mathcal{C}_1, \mathcal{C}_2$ or $\mathcal{C}_5$; or $n = 4$;
- for q even, $M \in \mathcal{C}_1$.

Lemma 20. *L_α cannot lie inside a maximal subgroup from families $\mathcal{C}_i, i > 1$.*

Proof. We may assume that q is odd. In $\mathrm{SL}(n,q)$, define

$$g^* = \begin{pmatrix} -1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

Then g^* is centralized in $\mathrm{SL}(n,q)$ by $(\mathrm{SL}(2,q)\times\mathrm{SL}(n-2,q)).(q-1)$. Then the projective image, g, of g^* is an involution in L and n_g divides

$$q^{2(n-2)}(q^{n-1}+\cdots+q+1)(q^{n-2}+\cdots+q+1)/(q+1).$$

Examining the order of subgroups M in $\mathcal{C}_2$ of $\mathcal{C}_5$ we find that $|M|_p \leq q^{\frac{1}{4}(n-1)n}$ and hence $|L:M|_p \geq q^{\frac{1}{4}(n-1)n}$. Since $n>3$, we know that q^2 divides the index of any maximal subgroup in $\mathcal{C}_2$ or $\mathcal{C}_5$. In the case where $n=4$, the only maximal subgroups of odd index which do not lie in families $\mathcal{C}_1$, $\mathcal{C}_2$ or $\mathcal{C}_5$ also have index divisible by q^2. Hence $p\geq 7$ by Lemma 8. Then, by Lemma 13, the largest power of p in v is $q^{2(n-2)}$.

Thus, for $n>4$, $q^{\frac{1}{2}n(n-1)-2(n-2)} = q^{\frac{1}{2}(n^2-5n+8)}$ divides the order of L_α. We therefore need to have $\frac{1}{2}(n^2-5n+8)\leq\frac{1}{4}(n-1)n$ and so $n<7$.

If n is 5 or 6, the only possibility that fits this inequality is when $M=N_L(L(n,q_0))$ for $q=q_0^2$. But then $|L:M|$ is even and so this case can be excluded. This possibility can also be excluded when $n=4$. However when $n=4$ we also need to consider the following further possibilities (note that when $n=4$ we can assume that $L=\mathrm{PSL}(4,q)$):

- $M = \hat{}(\mathrm{SL}(2,q)\times\mathrm{SL}(2,q)).(q-1).2$. (Recall that we use hat notation $\hat{}$ to indicate that we are giving the structure of the pre-image of M in $\mathrm{SL}(4,q)$.) In this case $|L:M| = n_g = \frac{1}{2}q^4(q^2+1)(q^2+q+1)$. Then we know that the maximum power of p in v is q^4 hence L_α contains Sylow p-subgroups of M. However the index of a parabolic subgroup in $\mathrm{SL}(2,q)$ is even, hence we must have $\hat{}(\mathrm{SL}(2,q)\times\mathrm{SL}(2,q)).2 < L_\alpha$. Then we know that for some α, $L_\alpha > \hat{}\begin{pmatrix}\mathrm{SL}(2,q) & \\ & \mathrm{SL}(2,q)\end{pmatrix}$. Since L_α also contains a Sylow 2-subgroup of $\mathrm{PSL}(4,q)$, this implies that L_α must contain the projective image of $\begin{pmatrix}1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1\end{pmatrix}$ which is L-conjugate to g and so $r_g\geq q^2(q+1)^2$. Thus $\frac{n_g}{r_g}\leq\frac{1}{2}q^2(q^2+1)$ and $v\leq q^4(q^2+1)(q^2+q+1)$ and so $v=\frac{1}{2}q^4(q^2+1)(q^2+q+1)$ contradicting Lemma 11.
- $M = L(4,q_0).[\frac{c}{(q-1,4)}(q_0-1,4)]$ where $c=(q-1)/(q_0-1,\frac{q-1}{(q-1,4)})$ and $q=q_0^3$. Then $|L:M| = (q_0^{12}(q_0^8+q_0^4+1)(q_0^6+q_0^3+1)(q_0^4+q_0^2+1))/(\frac{c}{(q-1,4)}(q_0-1,4))$. Now we know that $p\equiv 1(3)$ and so the highest power of 3 in c is 3. Then we have $9\big||L:M|$ which is impossible.
- M is of odd index but does not lie in families $\mathcal{C}_1,\mathcal{C}_2$ or $\mathcal{C}_5$. Examining [23, 26] we find that there are two possibilities: Either $M\in\mathcal{C}_6$ and $M\cong 2^4.A_6$ or $M\in\mathcal{C}_8$ and $M\cong\mathrm{PGSp}(4,q)$. In the former case, q^6 divides $|L:M|$ which is a contradiction. In the latter case, since $p\equiv 1(3)$, we find that 9 divides $|L:M|$ which, again, is a contradiction. □

Thus we assume from here on that L_α lies inside $M \in \mathcal{C}_1$. This means that L_α must always lie inside a parabolic subgroup, P_m, which stabilizes a subspace W of dimension m in the natural vector space for G. We now seek to bound m.

6.2 L_α lies in P_m, m small. We begin by noting some preliminary facts which we will use to establish which parabolic groups P_m are possible candidates to contain L_α. In particular we will show that m is small.

Lemma 21. *Suppose L_α lies inside P_m. For $r|\binom{n}{m}$, r prime, there exists an integer a such that $(1+q^a+\cdots+q^{a(r-1)})$ divides $|L:P_m|$ which, in turn, divides v.*

Corollary 22. *Suppose L_α lies inside P_m.*

- *If $p \equiv 1(3)$ then for each prime r dividing $\binom{n}{m}$, we have $r \equiv 1(3)$ or $r = 3$ and $9 \nmid \binom{n}{m}$.*
- *If p is odd then $\binom{n}{m}$ is odd, and so either n is odd, or n is even and m is even.*
- *If $p = 2$ then $\binom{n}{m} \not\equiv 0(4)$.*

Proof. We need only prove the final statement. Suppose $4|\binom{n}{m}$. Then either $(q^2+1)|v$ or $(q+1)^2|v$. This means that either v is divisible by a prime congruent to $2(3)$ or that $9\big|v$. Both of these are impossible. □

Note that, since $(n,q) \neq (4,2)$, the smallest index of a parabolic subgroup in $\mathrm{PSL}(n,q)$, $n \geq 4$ is 31 ([23, table 5.2A]). Since x is a square we know that $v \geq 91$ and so $d_g < 2\frac{n_g}{r_g}$.

6.2.1 Case n odd, p odd. In this case L contains the projective image, g, of

$$g^* = \begin{pmatrix} -1 & & & \\ & \ddots & & \\ & & -1 & \\ & & & 1 \end{pmatrix}.$$

Then $n_g = q^{n-1}(q^{n-1}+\cdots+q+1)$. Furthermore, since $n \geq 4$, g is conjugate in G to the projective image, h, of at least one other diagonal matrix. Then g and h commute and lie in an elementary abelian 2-group. Since L_α contains a Sylow 2-subgroup of L, we must have $r_g \geq 2$.

Thus $\frac{n_g}{r_g} \leq \frac{1}{2}q^{n-1}(q^{n-1}+\cdots+q+1)$, $d_g \leq q^{n-1}(q^{n-1}+\cdots+q+1)$ and $v \leq \frac{1}{2}q^{2n-2}(q^{n-1}+\cdots+q+1)^2$. Now observe that

$$\frac{1}{2}(q^{n-1}+\cdots+q+1)^2 \geq q^{2n-1} \text{ implies } (q^n-1)^2 \geq 2q^{2n-1}(q-1)^2,$$

hence $q^{2n} \geq 2q^{2n-1}(q-1)^2$, which gives $q \geq 2(q-1)^2$ and $q < 3$. We know that $q \geq 3$ hence $\frac{1}{2}(q^{n-1}+\cdots+q+1)^2 < q^{2n-1}$ and $v < q^{4n-3}$. But $|L:P_m| > q^{m(n-m)}$ hence, for $n \geq 23$, we have $m \leq 4$. We use Corollary 22 to narrow down the possibilities:

1. For $p \equiv 1(3)$ we find, by explicit calculation using Corollary 22, that $m \leq 4$ for all n. In fact, checking small n we find that if $m = 1, 2$ then $n \geq 7$; if $m = 3$ then $n \geq 39$; if $m = 4$ then $n > 70$.
2. For $p \not\equiv 1(3)$ then $\frac{n_g}{r_g} | 3(q^{n-1} + \cdots + q + 1)$. Hence $d_g < 3.q^n$ and so $v < 9q^{2n}$. For $n \geq 11$ this implies that $m \leq 2$.
   Checking the cases where $n < 11$ we find that $m \leq 2$ or $(n, m) = (7, 3)$. This final case will be dealt with along with other exceptional cases at the end of Section 6.3.9.

6.2.2 Case n even, p odd. Note that in this case we must have m even and L contains the projective image, g, of

$$g^* = \begin{pmatrix} -1 & & & & \\ & \ddots & & & \\ & & -1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}.$$

Now $n_g = q^{2(n-2)}(q^{n-2} + \cdots + q^2 + 1)(q^{n-2} + \cdots + q + 1)$. Again $r_g \geq 2$ and so $\frac{n_g}{r_g} \leq \frac{1}{2}q^{2(n-2)}(q^{n-2} + \cdots + q^2 + 1)(q^{n-2} + \cdots + q + 1)$. This gives $d_g \leq q^{2(n-2)}(q^{n-2} + \cdots + q^2 + 1)(q^{n-2} + \cdots + q + 1)$ and so $v \leq \frac{1}{2}q^{4(n-2)}(q^{n-2} + \cdots + q^2 + 1)^2(q^{n-2} + \cdots + q + 1)^2$.

In a similar fashion to before we know that, for $q \geq 3$ and $n \geq 4$,

$$\frac{1}{2}(q^{n-2} + \cdots + q^2 + 1)^2(q^{n-2} + \cdots + q + 1)^2 < q^{4n-7}$$

and so $v < q^{8n-15}$. But $|\operatorname{PSL}(n, q) : P_m| > q^{m(n-m)}$ hence, for $n \geq 70$, we have $m \leq 8$. Once again we use Corollary 22 to narrow down the possibilities:

(1) For $p \equiv 1(3)$, we find that $n < 70$ implies that $m = 2$. In fact $(n, m) = (14, 2)$, $(38, 2)$ or $(62, 2)$.
(2) For $p \not\equiv 1(3)$, $\frac{n_g}{r_g} | 3(q^{n-2} + \cdots + q^2 + 1)(q^{n-2} + \cdots + q + 1) < 3q^{2n-3}$. Thus $v < 9q^{4n-5}$. But $|G : P_m| > q^{m(n-m)}$. Thus for $n \geq 18$ we must have $m \leq 4$. For $n < 18$, $m \leq 4$ or $(n, m) = (14, 6)$. This final case will be dealt with along with other exceptional cases in Section 6.3.9.

6.2.3 Case $p = 2$. In this case G contains the projective image, g, of

$$g^* = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}.$$

Here g^* is a transvection and $n_g = (q^{n-1} - 1)(q^{n-1} + \cdots + q + 1)$. Examining a Sylow-2 subgroup of $\operatorname{PSL}(n, q)$ we see that it contains at least $2(q^{n-1} - 1)$ L-conjugates of g.

Since L_α must contain one such Sylow 2-subgroup, we conclude that $r_g \geq 2(q^{n-1}-1)$ and so $\frac{n_g}{r_g} < \frac{1}{2}(q^{n-1}+\cdots+q+1)$. Since $d_g < 2\frac{n_g}{r_g}$, $v < \frac{1}{2}(q^{n-1}+\cdots+q+1)^2$. Also, since $L_\alpha < P_m$ and $|\operatorname{PSL}(n,q) : P_m| > q^{m(n-m)}$, we conclude that, for $n \geq 10$, $m \leq 2$.

For $n < 10$, the fact that $4 \not| \binom{n}{m}$ implies that $(n,m) = (7,3), (8,4)$ or $(9,4)$ if $m > 2$. We rule out these three possibilities in turn:

- $(9,4)$: This gives $q^{4(9-4)} > q^{2n}$ which is a contradiction.
- $(8,4)$: In this case, $(q^4+1)\big||G : P_4|$ which is impossible.
- $(7,3)$: In this case, $|G : P_3| = (q^2-q+1)(q^4+\cdots+q+1)(q^6+\cdots+q+1) > \frac{1}{2}(q^6+\cdots+q+1)^2 > v$ which is a contradiction.

Note that if $m = 2$ and $n \equiv 0,1(4)$ then $(q^2+1)\big|v$ which is impossible. Hence when $m = 2$ we assume that $n \equiv 2,3(4)$.

6.2.4 Cases to be examined. We now state those values of m for which $L_\alpha < P_m$ gives a potential transitive action of G:

1. $p = 2$: $m = 1$ ($n \geq 5$) or 2 ($n \geq 6$);
2. $p \not\equiv 1(3)$, p odd:
   - n odd: $m = 1$ ($n \geq 5$), $m = 2$ ($n \geq 7$) or $(n,m) = (7,3)$;
   - n even: $m = 2$ ($n \geq 6$), $m = 4$($n \geq 12$) or $(n,m) = (14,6)$;
3. $p \equiv 1(3)$:
   - n even: $m = 2$ ($n = 14$ or $n \geq 38$), $m = 4,6,8$ ($n > 70$);
   - n odd: $m = 1,2$ ($n \geq 7$), $m = 3$ ($n \geq 39$), $m = 4$ ($n > 70$).

Remark. Note that $n = 4$ is now done. We will assume that $n \geq 5$ from now on.

All that remains is to go through the listed cases one at a time assuming that L_α lies inside the given P_m and so $|L : P_m|$ divides v. We seek a contradiction. We begin with a preliminary lemma and corollary which will be useful for counting the number of involutions in L_α:

Lemma 23. *Suppose that q is an odd prime power. Assume that the following two matrices are involutions in $\mathrm{SL}(n,q)$, then they are conjugate in $\mathrm{SL}(n,q)$:*

$$\begin{pmatrix} V & X_1 \\ 0 & W \end{pmatrix}, \begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix}$$

where $V \in \mathrm{GL}(m,q)$, $W \in \mathrm{GL}(n-m,q)$ and $X_1 \in M(m \times (n-m), q)$, the set of m by $n-m$ matrices over the field of q elements.

Proof. Since these matrices are involutions we must have $VX_1 + X_1W = 0$. Take X such that $2X = -X_1W$. Then $AX = X_1 + XW$ and we find that:

$$\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}\begin{pmatrix} V & X_1 \\ 0 & W \end{pmatrix} = \begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix}\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}. \qquad \square$$

Corollary 24. *Let q be odd and suppose that L_α lies inside a parabolic subgroup, P_m, of L where $P_m = \hat{}A : (B : C)$ with $C = q-1$ and*

$$A = \begin{pmatrix} I & M(m \times (n-m), q) \\ & I \end{pmatrix}, \quad B = \begin{pmatrix} \mathrm{SL}(m,q) & \\ & \mathrm{SL}(n-m,q) \end{pmatrix}.$$

Define $\pi(L_\alpha)$ to be equal to the following set:

$$\left\{ \begin{pmatrix} Y_1 & \\ & Y_2 \end{pmatrix} : \begin{pmatrix} Y_1 & Z \\ & Y_2 \end{pmatrix} \in A : (B : C), \text{ for some } Z \in M(m \times (n-m), q) \right\},$$

the projection of P_m onto the Levi quotient restricted to L_α. Now assume that L_α contains an involution g which is the projective image of an involution in $\mathrm{SL}(n,q)$, $g^* = \left(\begin{smallmatrix} X_1 & Y \\ & X_2 \end{smallmatrix}\right)$.

Then r_g is greater than or equal to the number of $\pi(L_\alpha)$-conjugates of the block diagonal matrix $\left(\begin{smallmatrix} X_1 & \\ & X_2 \end{smallmatrix}\right)$ in $\pi(L_\alpha)$.

Recall that, in our statement of the corollary, we use hat notation $\hat{}$ to indicate that we are giving the structure of the pre-image of P_m in $\mathrm{SL}(n,q)$. Note that in what follows we will assume that L_α lies in a parabolic subgroup which is L-conjugate to one of the above form. In fact, in $\mathrm{PSL}(n,q)$ where $n \geq 3$, there are two conjugacy classes of parabolic subgroups. However, since these two classes are fused by a graph automorphism, our method extends trivially to cover the other class.

6.3 Remaining cases.

6.3.1 Case $p = 2$, $m = 1$.

Take g^* a transvection as before, with $n_g = (q^{n-1}-1)(q^{n-1}+\cdots+q+1)$. Recall that $r_g \geq 2(q^{n-1}-1)$ and so $\frac{n_g}{r_g} \leq \frac{1}{2}(q^{n-1}+\cdots+q+1)$ and so $v < \frac{1}{2}(q^{n-1}+\cdots+q+1)^2$.

Then we suppose that $L_\alpha = \hat{}A.B.C \leq P_1 = \hat{}[q^{n-1}] : (\mathrm{SL}(n-1,q).(q-1))$. Since L_α contains a Sylow 2-subgroup of L, $A = [q^{n-1}]$ with $B \leq \mathrm{SL}(n-1,q)$, $C \leq (q-1)$. Now $|L : P_1| = q^{n-1}+\cdots+q+1$ and thus $|\mathrm{SL}(n-1,q) : B| < \frac{1}{2}(q^{n-1}+\cdots+q+1)$. We know that B contains a Sylow 2-subgroup of $\mathrm{SL}(n-1,q)$ and so we are in one of the following situations:

- $B \leq P^*_{m_1}$, a parabolic subgroup of $\mathrm{SL}(n-1,q)$. For $n \geq 5$ and $m_1 \geq 2$ observe that $|\mathrm{SL}(n-1,q) : P^*_{m_1}| > q^{2(n-3)} > \frac{1}{2}(q^{n-1}+\cdots+q+1)$ which is impossible. Thus $m_1 = 1$ and $B < [q^{n-2}] : \mathrm{GL}(n-2,q)$. In this case $(q^{n-1}+\cdots+q+1)\cdot(q^{n-2}+\cdots+q+1)$ divides v and $B = [q^{n-2}] : B_1^*$ where $|\mathrm{GL}(n-2,q) : B_1^*| < q$. Thus $B > B_1^* > \mathrm{SL}(n-2,q)$.
- $B = \mathrm{SL}(n-1,q)$.

Consider the second situation first. We know that $\pi(L_\alpha)$ contains $\left(\begin{smallmatrix} 1 & \\ & \mathrm{SL}(n-1,q) \end{smallmatrix}\right)$ for some α, and we also know that the projective images of the following matrices are conjugate in the group L:

$$g^* = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}, \quad h^* = \begin{pmatrix} 1 & & & & & \\ & 1 & 0 & \cdots & 0 & 1 \\ & & 1 & & & 0 \\ & & & \ddots & & \vdots \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}.$$

Thus, by Corollary 24, $r_g \geq r_g(\hat{\ }\mathrm{SL}(n-1,q)) \geq (q^{n-2}-1)(q^{n-2}+\cdots+q+1)$. This implies that $\frac{n_g}{r_g} < q(q+1)$ and $v \leq q^4+q^2+1$. This is a contradiction for $n \geq 5$.

Thus we assume that we are in the first situation. The same argument though implies that $r_g \geq r_g(\hat{\ }\mathrm{SL}(n-2,q)) \geq (q^{n-3}-1)(q^{n-3}+\cdots+q+1)$. This implies that $\frac{n_g}{r_g} < (q^2+1)^2$ and so $\frac{n_g}{r_g} \leq q^4+q^2+1$. This means that $v \leq q^8+4q^6+7q^4+6q^2+3$. We know that $(q^{n-1}+\cdots+q+1)(q^{n-2}+\cdots+q+1)|v$ which gives a contradiction for $n \geq 6$.

For $n=5$ we find that $(q^3+q^2+q+1)|v$ hence $(q^2+1)|v$ which implies that a prime $p_1 \equiv 2(3)$ divides v which is a contradiction.

6.3.2 Case $p = 2, m = 2$. We assume here that $n \geq 6$ and $L_\alpha \leq P_2 \cong \hat{\ }[q^{2(n-2)}]:(\mathrm{SL}(2,q)\times\mathrm{SL}(n-2,q)).(q-1)$. Now P_2 has index $(q^{n-1}+\cdots+q+1)(q^{n-2}+\cdots+q+1)/(q+1)$. We know, as before, that $v < \frac{1}{2}(q^{n-1}+\cdots+q+1)^2$ hence $|P_2 : L_\alpha| < q(q+1)$. Now observe that $\mathrm{SL}(n-2,q)$ does not have a subgroup of index less than $q(q+1)$ hence $L_\alpha > \mathrm{SL}(n-2,q)$. As for $m=1$, this implies that $v \leq q^8+4q^6+7q^4+6q^2+3$. This must be greater than the index of P_2 and so we must have $n=6$.

In fact when we examine $n=6$ we find that, to satisfy the bound, we must have $q=2$. Explicit calculation of n_g, r_g and $|L:P_2|$ excludes this possibility.

Remark. From here on we assume that p is odd and $n \geq 5$.

6.3.3 Case p odd, $p \not\equiv 1(3)$, n odd, m=1. For the next two cases take g as before for p odd and n odd with $n_g = q^{n-1}(q^{n-1}+\cdots+q+1)$. We suppose that $L_\alpha = \hat{\ }A.B.C < P_1 = \hat{\ }[q^{n-1}]:(\mathrm{SL}(n-1,q).(q-1))$. Here $A \leq [q^{n-1}]$, $B \leq \mathrm{SL}(n-1,q)$ and $C \leq q-1$. Note that $|L:P_1| = q^{n-1}+\cdots+q+1$.

Suppose first that $p \neq 3$. Then $\frac{n_g}{r_g}|q^{n-1}+\cdots+q+1$ and so $v < 2(q^{n-1}+\cdots+q+1)^2$. Then $|P_1 : L_\alpha| < 2(q^{n-1}+\cdots+q+1)$. Now L_α contains a Sylow-p subgroup of L since $p \equiv 2(3)$. Hence B either lies in a parabolic subgroup, $P^*_{m_1}$, of $\mathrm{SL}(n-1,q)$ or $B = \mathrm{SL}(n-1,q)$.

Observe that if m_1 is odd then $|\,\mathrm{SL}(n-1,q):P^*_{m_1}|$ is even. Thus we must assume that m_1 is even, in which case $|\,\mathrm{SL}(n-1,q):P^*_{m_1}| > q^{2(n-3)} > 2(q^{n-1}+\cdots+q+1)$ for $n \geq 6$. This is a contradiction. For $n=5$, P^*_2 also has even index in $\mathrm{SL}(4,q)$ so can be excluded. Hence we assume that $B = \mathrm{SL}(n-1,q)$ and $|C|$ is even. We know that, for some α, $\pi(L_\alpha)$ contains $\begin{pmatrix} \pm 1 & \\ & \mathrm{SL}(n-1,q).2 \end{pmatrix}$. Thus, appealing to Corollary 24, we conclude that $r_g \geq r_g(\hat{}\operatorname{SL}(n-1,q).2) \geq q^{n-2}(q^{n-2}+\cdots+q+1)$ and so $\frac{n_g}{r_g} < q(q+1)$. This means that $v \leq q^4+q^2+1$ which is a contradiction for $n \geq 5$.

We are left with the case where $p=3$. Now L_α contains a group of index 3 in a Sylow-3 subgroup of L and $|L:L_\alpha|$ is odd. Hence B either lies in a parabolic subgroup, $P^*_{m_1}$ of $\operatorname{SL}(n-1,q)$ or $B=\operatorname{SL}(n-1,q)$. The case where $B=\operatorname{SL}(n-1,q)$ is ruled out exactly as for $p \neq 3$.

Let $B \leq P^*_{m_1} < \operatorname{SL}(n-1,q)$ and suppose that $n \geq 8$. Then $v > q^7+\cdots+q+1 > 1333$ and $\frac{n_g}{r_g} > 31$. This, combined with the fact that $\frac{n_g}{r_g} \leq 3(q^{n-1}+\cdots+q+1)$, means that $v < 12(q^{n-1}+\cdots+q+1)^2$.

Now B lies in $P^*_{m_1}$ and so m_1 must be even. Then $|\operatorname{SL}(n-1,q):P^*_{m_1}| > q^{2(n-3)} > 12(q^{n-1}+\cdots+q+1)$ for $n \geq 8$ which is a contradiction. We are left with $n=5$ or 7. If $n=5$ then we exclude it as for $p \neq 3$.

For $n=7$, we know that $d_g < 2\frac{n_g}{r_g} \leq 6(q^6+\cdots+q+1)$ and so $v < 18(q^6+\cdots+q+1)^2$. Thus we require that $q^{2(7-3)} < |\operatorname{SL}(n-1,q):P^*_{m_1}| < 18(q^6+\cdots+q+1)$. This is impossible for $q \geq 9$.

When $q=3$ we find that $\frac{n_g}{r_g}|3(q^6+\cdots+q+1)=3279$. Now $\frac{n_g}{r_g}=u^2-u+1$ for some integer u and so $\frac{n_g}{r_g} \leq q^6+\cdots+q+1$ and we refer to the case where $p \neq 3$.

Remark. Note that we have now covered all possible cases where $n=5$ and we assume that $n \geq 6$ from here on.

6.3.4 Case p odd, $p \not\equiv 1(3)$, n odd, $m=2$. In this case $L_\alpha = \hat{}A.B.C \leq P_2 \cong \hat{}[q^{2(n-2)}]:(\operatorname{SL}(2,q)\times\operatorname{SL}(n-2,q)).(q-1)$ where $A \leq [q^{n-1}]$, $B \leq \operatorname{SL}(2,q)\times\operatorname{SL}(n-2,q)$ and $C \leq q-1$. Now $|L:P_2| = (q^{n-3}+\cdots+q^2+1)(q^{n-1}+\cdots+q+1)$.

Now we know that $\frac{n_g}{r_g}|3(q^{n-1}+\cdots+q+1)$. Thus $v < 12(q^{n-1}+\cdots+q+1)^2$ and hence $|P_2:L_\alpha| < 12(q+1)^2$. If $(n,q) \neq (7,3)$ then no subgroup of $\operatorname{SL}(n-2,q)$ has index less than $12(q+1)^2$ unless $(n,q)=(7,3)$. If $(n,q)=(7,3)$ then the only subgroups of $\operatorname{SL}(5,q)$ with indices less than $12(3+1)^2$ are the parabolic subgroups. These have indices in $\operatorname{SL}(5,q)$ divisible by 11 and so can be excluded. This implies that in all cases $B=B^*\times\operatorname{SL}(n-2,q)$ for B^* some subgroup of $\operatorname{SL}(2,q)$.

Now $B=B^*\times\operatorname{SL}(n-2,q)$ implies that $\pi(L_\alpha) \geq \operatorname{SL}(n-2,q).2$ and so, by Corollary 24, $r_g > r_g(\hat{}\operatorname{SL}(n-2,q)) > q^{n-3}(q^{n-3}+\cdots+q+1)$ and $\frac{n_g}{r_g} < q^2(q^2+1)$ and so $v < q^8+q^4+1$. This gives a contradiction for $n \geq 7$.

6.3.5 Case p odd, $p \not\equiv 1(3)$, n even, $m=2$. For the next two cases, take g as earlier for p odd and n even. Then $n_g = q^{2(n-2)}(q^{n-2}+\cdots+q+1)(q^{n-2}+\cdots+q^2+1)$. As in the previous case, $L_\alpha = \hat{}A.B.C \leq P_2 \cong \hat{}[q^{2(n-2)}]:(\operatorname{SL}(2,q)\times\operatorname{SL}(n-2,q)).(q-1)$ where $A \leq [q^{2(n-2)}]$, $B \leq (\operatorname{SL}(2,q)\times\operatorname{SL}(n-2,q))$, $C \leq q-1$ and $\pi(L_\alpha)=\hat{}B.C$. Now P_2 has index in L, $(q^{n-2}+\cdots+q^2+1)(q^{n-2}+\cdots+q+1)$.

We know, by Lemma 14, that one of the following must hold:

- $B \leq (\operatorname{SL}(2,q)\times B_1)$ for some $B_1 < \operatorname{SL}(n-2,q)$;
- $B = (B_2\times\operatorname{SL}(n-2,q))$ for some $B_2 \leq \operatorname{SL}(2,q)$.

Consider the second possibility first. As previously, Corollary 24 implies that $r_g \geq r_g(\hat{}\mathrm{SL}(n-2,q)) \geq q^{2(n-4)}(q^{n-4}+\cdots+q+1)(q^{n-4}+\cdots+q^2+1)$. Then $\frac{n_g}{r_g} \leq q^4(q^2+1)^2$ and $v \leq q^{18}$ which is a contradiction for $n > 11$. We will need to consider $n = 6, 8, 10$.

We turn to the first possibility above. We know that $\frac{n_g}{r_g} | 3(q^{n-2}+\cdots+q+1)(q^{n-2}+\cdots+q^2+1)$. This implies that $v < 9(q^{n-2}+\cdots+q+1)^3(q^{n-2}+\cdots+q^2+1)$ and so $|P_2 : L_\alpha| < 9(q^{n-2}+\cdots+q+1)^2$. Thus we must have B_1 lying inside a parabolic subgroup, $P^*_{m_1}$, in $\mathrm{SL}(n-2,q)$ with $|\mathrm{SL}(n-2,q) : P^*_{m_1}| < 9(q^{n-2}+\cdots+q+1)^2$. We know that m_1 must be even. If $m_1 \geq 4$ then we know that $|\mathrm{SL}(n-2,q) : P^*_{m_1}| > q^{4(n-2-4)}$ which is a contradiction for $n \geq 12$. Thus $n-2 \leq 8$ in which case $m_1 = 4$ is not allowed and so this can also be excluded. Thus we must have $m_1 = 2$. However we know that $\binom{n}{2}$ is odd and so $n \equiv 2(4)$, hence $n-2 \equiv 0(4)$, hence $\binom{n-2}{2}$ is even and $|\mathrm{SL}(n-2,q) : P^*_2|$ is even by Lemma 21. We may exclude this possibility.

We are left with the possibility that $n = 6, 8$ or 10 and $B = B_2 \times \mathrm{SL}(n-2,q)$ for some $B_2 \leq \mathrm{SL}(2,q)$.

Observe first that $A.B.C/A$ acts on the non-identity elements of A by conjugation. Since $B = B_2 \times \mathrm{SL}(n-2,q)$, this action has orbits of size divisible by $q^{n-2}-1$. When $p = 3$, $q^{n-2}-1$ does not divide $q^{2(n-2)}/3 - 1$ hence in all cases we may assume that $A = [q^{2(n-2)}]$.

Then, for some α, $A : B$ (or its transpose) has the following form and contains the following conjugate of g^*:

$$h^* = \begin{pmatrix} I_{2\times 2} & & & & \\ & -I_{2\times 2} & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \in \begin{pmatrix} B_2 & A \\ & \mathrm{SL}(n-2,q) \end{pmatrix}.$$

Observe that $|A : C_A(h^*)| = q^4$. Thus $r_g \geq q^4 r_g(\hat{}\mathrm{SL}(n-2,q)) \geq q^{2n-4}(q^{n-4}+\cdots+q+1)(q^{n-4}+\cdots+q^2+1)$. Thus $\frac{n_g}{r_g} \leq (q^2+1)^2$. In fact we may assume that $\frac{n_g}{r_g} \leq q^4+q^2+1$ and so $d_g \leq q^4+3q^2+3$ and $v \leq (q^4+q^2+1)(q^4+3q^2+3)$.

Now $|L : P_2| = (q^{n-2}+\cdots+q^2+1)(q^{n-2}+\cdots+q+1) > (q^4+q^2+1)(q^4+3q^2+3)$ for $n \geq 6$, $q \geq 3$. This is a contradiction.

Remark. Observe that we have now completed the case where $n = 6$. We assume that $n \geq 7$ from now on.

6.3.6 Case p odd, $p \not\equiv 1(3)$, n even, $m = 4$.

Let $n \geq 12$ for this case. Similarly to the previous case, $L_\alpha = \hat{}A.B.C \leq P_4 \cong \hat{}[q^{4(n-4)}] : (\mathrm{SL}(4,q) \times \mathrm{SL}(n-4,q)).(q-1)$ where $A \leq [q^{4(n-4)}]$, $B \leq (\mathrm{SL}(4,q) \times \mathrm{SL}(n-4,q))$, $C \leq q-1$ and $\pi(L_\alpha) = \hat{}B.C$.

As before, $n_g = q^{2(n-2)}(q^{n-2}+\cdots+q+1)(q^{n-2}+\cdots+q^2+1)$ and so $\frac{n_g}{r_g} | 3(q^{n-2}+\cdots+q+1)(q^{n-2}+\cdots+q^2+1)$. This implies that $v < 9(q^{n-2}+\cdots+q+1)^3(q^{n-2}+\cdots+q^2+1)$. Then we have

$$|L : P_4||P_4 : L_\alpha| < 9(q^{n-2}+\cdots+q+1)^3(q^{n-2}+\cdots+q^2+1)$$

Since $9(q^{n-2}+\cdots+q+1)^3(q^{n-2}+\cdots+q^2+1) < q^{4n-4}$ we must have $|P_4 : L_\alpha| < q^{12}$. We know, by Lemma 14, that one of the following must hold:

- $B \leq (\mathrm{SL}(2,q) \times B_1)$ for some $B_1 < \mathrm{SL}(n-4,q)$. In this case $|\mathrm{SL}(n-4,q) : B_1| < q^{12}$. For $n \geq 12$ this implies that B_1 lies in the parabolic subgroup P_1^* of $\mathrm{SL}(n-4,q)$. But this has even index and so can be excluded.
- $B = (B_2 \times \mathrm{SL}(n-4,q))$ for some $B_2 \leq \mathrm{SL}(4,q)$.

Thus the second possibility must hold. As before Corollary 24 implies that $r_g \geq r_g(\hat{}\mathrm{SL}(n-4,q)) \geq q^{2(n-6)}(q^{n-6}+\cdots+q+1)(q^{n-6}+\cdots+q^2+1)$. Then $\frac{n_g}{r_g} < q^8(q^4+1)^2$ and

$$d_g < \frac{n_g}{r_g} + 2\sqrt{\frac{n_g}{r_g}} + 2 < (q^8+q^4+3)q^4(q^4+1)$$

giving $v \leq q^{12}(q^4+1)^3(q^8+q^4+3)$ which is a contradiction for $n \geq 12$.

6.3.7 Case p odd, $p \equiv 1(3)$, n even, $m = 2, 4, 6$ or 8. We will take g to be the projective image of,

$$g^* = \begin{pmatrix} -1 & & & & \\ & \ddots & & & \\ & & -1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}.$$

Then $n_g = q^{2(n-2)}(q^{n-2}+\cdots+q^2+1)(q^{n-2}+\cdots+q+1)$ and we know that $v < q^{8n-15}$. Recall that when $m=2$ we may assume that $n=14$ or $n \geq 38$, otherwise $n > 70$.

Let $L_\alpha = \hat{}A.B.C \leq P_m \cong \hat{}[q^{2(n-m)}] : (\mathrm{SL}(m,q) \times \mathrm{SL}(n-m,q)).(q-1)$ where $A \leq [q^{m(n-m)}]$, $B \leq (\mathrm{SL}(m,q) \times \mathrm{SL}(n-m,q))$, $C \leq q-1$ and $\pi(L_\alpha) = \hat{}B.C$. Note that $|L : P_m| > q^{m(n-m)}$ and so $|P_m : L_\alpha| < q^{8n-15-mn+m^2}$.

There are two possibilities for B, by Lemma 14:

- $B = (B_2 \times \mathrm{SL}(n-m,q))$ for some $B_2 \leq \mathrm{SL}(m,q)$. Then Corollary 24 implies that $r_g \geq r_g(\hat{}\mathrm{SL}(n-m,q)) \geq q^{2(n-m-2)}(q^{n-m-2}+\cdots+q+1)(q^{n-m-2}+\cdots+q^2+1)$. Then $\frac{n_g}{r_g} \leq q^{2m}(q^m+1)^2$ and $v \leq q^{8m+3}$ Thus we need $m(n-m) < 8m+3$ which implies that $m > \frac{n-8}{2}$ which is a contradiction.
- $B \leq (\mathrm{SL}(m,q) \times B_1)$ for some $B_1 < \mathrm{SL}(n-m,q)$. By Liebeck and Saxl [26], the projective image of B_1 in $\mathrm{PSL}(n-m,q)$ must lie in families $\mathcal{C}_1, \mathcal{C}_2$ or $\mathcal{C}_5$. The latter two possibilities imply that $n(n-1)/4 < 8n-15-mn+m^2$, hence $n^2-(33-m)n+(60-m^2) < 0$ and $n < 33-m$, which yields $n=14$ and $m=2$. We examine the remaining situation with $n=14$, $m=2$. Then one subgroup in $\mathcal{C}_2$ has index less than $q^{8n-15-mn+m^2} = q^{6n-11}$, namely the projective image of $Q_2 \cong (\mathrm{SL}(6,q) \times \mathrm{SL}(6,q)).(q-1).2$ which has even index in $\mathrm{PSL}(12,q)$. Similarly the only subgroup in $\mathcal{C}_5$ with index less than q^{6n-11} is $N_{\mathrm{PSL}(12,q)}(\mathrm{PSL}(12,q_0))$ where $q=q_0^2$. This also has even index in $\mathrm{PSL}(12,q)$ and so can be excluded.
  Thus B_1 lies in a parabolic subgroup $P_{m_1}^*$ of $\mathrm{SL}(n-m,q)$. Since $n-m$ is even, we must have m_1 even to have $i := |\mathrm{SL}(n-m,q) : P_{m_1}^*|$ odd. Observe that

$q^{m_1(n-m-m_1)} < i < q^{8n-15-mn+m^2}$. Suppose first that $m + m_1 \geq 10$. The upper and lower bounds for i imply that $(10-m)(n-10) < 8n - 15 - mn + m^2$, hence $2n < m^2 - 10m + 85$, which implies that $n < 35$ and $m = 2$. We examine the remaining situation with $n < 35, m = 2$. Referring to Corollary 22 the only value of n less than 35 for which P_2 has admissible index is $n = 14$. But in this case $m_1 = 8$ is too large to define a parabolic group in $\mathrm{SL}(12, q)$. This case is excluded. Thus we assume that $m + m_1 \leq 8$ and $m \leq 6$. We split into cases:

- Suppose that $m = 6$ and so $m_1 = 2$. Then $|L : P_6|$ odd implies that $\binom{n}{6}$ is odd and hence $n \equiv 2(4)$. However this implies that $\binom{n-6}{2}$ is even and so i is even which is impossible.
- Suppose that $m = 4$ and so $m_1 \leq 4$. Recall that, by Corollary 22, 5 does not divide $\binom{n}{4}$ hence $n \equiv 4(5)$. However this implies that 5 divides $\binom{n-4}{m_1}$ which implies, by Lemma 21, that i is divisible by a prime $p_1 \equiv 2(3)$ which is impossible.
- Suppose that $m = 2$ and so $m_1 \leq 6$. We exclude $m_1 = 2$ or 6 in the same way as we excluded $m_1 = 2$ for $m = 6$. We exclude $m_1 = 4$ in the same way as we excluded $m_1 = 4$ for $m = 4$. Hence we are done.

6.3.8 Case p odd, $p \equiv 1(3)$, n odd, $m = 1, 2, 3$ or 4. We will take g to be the projective image of,

$$g^* = \begin{pmatrix} -1 & & & \\ & \ddots & & \\ & & -1 & \\ & & & 1 \end{pmatrix}.$$

Then $n_g = q^{n-1}(q^{n-1} + \cdots + q + 1)$ and we know that $v < q^{4n-3}$. Furthermore, by Lemma 13, we know that $|v|_p \leq q^{n-1}$. Recall that, for $m = 1$ or 2, we have $n = 7$ or $n \geq 13$, for $m = 3$ we have $n \geq 39$ and for $m = 4$ we have $n > 70$.

Then, in this case, $L_\alpha = \hat{\ }A.B.C \leq P_m = \hat{\ }[q^{n-m}] : (\mathrm{SL}(n-m, q).(q-1))$ where $A \leq [q^{n-m}]$, $B \leq \mathrm{SL}(n-m, q)$, $C \leq q - 1$ and $\pi(L_\alpha) = \hat{\ }B.C$. Note that $|L : P_m| > q^{m(n-m)}$ and so $|\mathrm{SL}(n-m, q) : B| < q^{4n-3-mn+m^2}$.

There are two possibilities for B, by Lemma 14:

- $B = (B_2 \times \mathrm{SL}(n-m, q))$ for some $B_2 \leq \mathrm{SL}(m, q)$. We know that $2 \leq C$ and so, by Corollary 24, $r_g \geq r_g(\hat{\ }\mathrm{SL}(n-m, q).2) \geq q^{n-m-1}(q^{n-m-1} + \cdots + q + 1)$. Hence $\frac{n_g}{r_g} < q^m(q^m + 1)$ and $v \leq q^{4m} + q^{2m} + 1$. Thus we have $m(n-m) < 4m + 1$, hence $m^2 + (4-n)m + 1 > 0$ and $m > n - 5$. This is a contradiction.
- $B \leq (\mathrm{SL}(m, q) \times B_1)$ for some $B_1 < \mathrm{SL}(n-m, q)$. By Liebeck and Saxl [26], the projective image of B_1 in $\mathrm{PSL}(n-m, q)$ must lie in a subgroup M of $\mathrm{PSL}(m, q)$ from families $\mathcal{C}_1, \mathcal{C}_2$ or $\mathcal{C}_5$. The latter two possibilities imply that, $n(n-1)/4 < 4n - 3 - mn + m^2$, hence $n^2 - (17-4m)n + (12-4m^2) < 0$ and $n < 17 - 2m$. This implies that either $m = 2$ and $n = 7$ or $m = 1$ and $n = 7, 13$. In fact, when $m = 1$ and $n = 13$ the initial inequality is not satisfied and this possibility can be excluded. When $m = 2$ and $n = 7$, the only possibility is if $B_1 \leq M = N_{L_5(q)}(L_5(q_0))$ where $q = q_0^2$. But $|\mathrm{SL}(n-2, q) : M|$ is even here and can be excluded. When $m = 1$ and

$n = 7$ we must have M a subgroup of $\mathrm{SL}(6, q)$ in $\mathcal{C}_2$ or $\mathcal{C}_5$ and $|\,\mathrm{SL}(6, q) : M| < q^{19}$. The only such subgroups are $M = \hat{}(\mathrm{SL}(3, q))^2.(q-1).2$ and $M = N_{L(6,q)}(L(6, q_0))$ where $q = q_0^2$. Both of these subgroups have even index in $\mathrm{SL}(6, q)$ and hence B_1 does not lie inside such an M.

Thus B_1 lies in a parabolic subgroup, $P^*_{m_1}$ of $\mathrm{SL}(n - m, q)$. Write $i := |\,\mathrm{SL}(n - m, q) : P^*_{m_1}|$ and observe that $q^{m_1(n-m-m_1)} < i < q^{4n-3-mn+m^2}$. Suppose first that $m + m_1 \geq 5$. The upper and lower bounds for i imply that

$$(5 - m)(n - 5) < 4n - 3 - mn + m^2 \text{ , hence } n < m^2 - 5m + 28.$$

This implies that $n < 24$ and either $m = 1$ or $m = 2$. These cases imply that $m_1 \geq 3$. Now for i to be divisible only by primes congruent to $1(3)$ or by 3 but not 9, we must have $\binom{n-m}{m_1}$ divisible only by primes congruent to $1(3)$ or by 3 but not 9 and hence $n - m \geq 39$ which is a contradiction.

Thus $m + m_1 \leq 4$ and $m \leq 3$. Note that if m is odd then m_1 must be even since i is odd implies that $\binom{n-m}{m_1}$ is odd. This excludes $m = 3$ and ensures that, for $m = 1$, $m_1 = 2$.

Observe some facts about the remaining cases:

– Suppose that $m = 1$ and $m_1 = 2$. We must have $n \geq 39$ to ensure that n and $\binom{n-1}{2}$ are divisible only by primes congruent to $1(3)$ or by 3 but not 9. Then we have $B_1 \leq P^*_2 \cong [q^{2(n-3)}] : (\mathrm{SL}(2, q) \times \mathrm{SL}(n - 3, q)).(q - 1)$ and, since $|\,\mathrm{SL}(n - 1, q) : P^*_2| > q^{2(n-3)}$, then $|P^*_2 : B_1| < q^{n+4}$.
– Suppose that $m = 2$. If $n = 7$ then B_1 lies inside a parabolic subgroup of $\mathrm{SL}(5, q)$. But 5 divides $\binom{5}{j}$ for $j = 1, 2$ which is not allowed. Thus $n \geq 39$ as this is the next smallest number with allowable divisors of $\binom{n}{2}$. Consider $m_1 = 2$. Since $\binom{n}{2}$ is odd we must have $n \equiv 3(4)$ and so $\binom{n-2}{2}$ is even which is a contradiction. Hence $m_1 = 1$ and $B_1 \leq P^*_1 \cong [q^{n-3}] : \mathrm{SL}(n-3, q).(q-1)$. Now $|\,\mathrm{SL}(n - 2, q) : P^*_1| \geq q^{n-3}$ and so $|P^*_1 : B_1| < q^{n+4}$.

Now the only subgroup of $\mathrm{SL}(n - 3, q)$ in $\mathcal{C}_1, \mathcal{C}_2$ or $\mathcal{C}_5$ with index less than q^{n+4} is a parabolic subgroup P^*_1 which has even index. Thus, for $m = 1$ and $m = 2$, $B_1 \geq \mathrm{SL}(n - 3, q).2$ and so, by Corollary 24, $r_g \geq r_g(\hat{}\mathrm{SL}(n - 3, q).2) \geq q^{n-4}(q^{n-4} + \cdots + q + 1)$. Hence $\frac{n_g}{r_g} < q^3(q^3 + 1)$ and $v \leq q^{12} + q^6 + 1$ which is a contradiction.

6.3.9 Exceptional cases. We have deferred two cases in the process of our proof. Firstly we need to consider the possibility that $n = 7, p \not\equiv 1(3)$ is odd and $L_\alpha \leq P_3$, a parabolic subgroup stabilizing a 3-dimensional subspace in the vector space for G. We exclude this possibility as follows:

Refer to Section 6.2.1 when np is odd and suppose that $L_\alpha < P_3$. In this case $\frac{n_g}{r_g} | 3(q^6 + \cdots + q + 1)$ and $|L : P_3| = (q^6 + \cdots + q + 1)(q^6 + q^4 + q^3 + q^2 + 1)$. Thus $v > q^{12}$ and $\frac{n_g}{r_g} > q^5 \geq 243$.

Suppose first that $\frac{n_g}{r_g} < q^6 + \cdots + q + 1$. Then $u^2 - u + 1 = \frac{n_g}{r_g} \leq \frac{3}{5}(q^6 + \cdots + q + 1)$ and $u^2 + u + 1 = d_g < q^6 + q^4 + q^3 + q^2 + 1$ since $\frac{n_g}{r_g} > 243$. Thus $v < |L : P_3|$ which is a contradiction.

Then consider the case where $\frac{n_g}{r_g} \geq q^6 + \cdots + q + 1$. We must have $v \geq 3(q^6 + \cdots + q + 1)(q^6 + q^4 + q^3 + q^2 + 1)$. Suppose that $\frac{n_g}{r_g} = q^6 + \cdots + q + 1$. Then our lower bound on v implies that $d_g \geq 3(q^6 + q^4 + q^3 + q^2 + 1) > 2\frac{n_g}{r_g}$ which is impossible. The only other possibility is that $\frac{n_g}{r_g} = 3(q^6 + \cdots + q + 1) = u^2 - u + 1$. But then $u^2 + u + 1 = d_g < 7(q^6 + q^4 + q^3 + q^2 + 1)$ which again is impossible for $q \geq 7$. For $q = 3, 5$ we find that $3(q^6 + \cdots + q + 1) \neq u^2 - u + 1$ for integer u and so these cases can be excluded.

The second possibility that we need to consider is when $n = 14, p \not\equiv 1(3)$ is odd and $L_\alpha \leq P_6$, a parabolic subgroup stabilizing a 6-dimensional subspace in the vector space for G. We exclude this possibility as follows:

Refer to Section 6.2.2 when n is even and p is odd and observe that $v < 9q^{51}$ and $n_g < q^{49}$. Furthermore

$$L_\alpha \leq P_6 = \hat{}[q^{48}] : (\mathrm{SL}(6,q) \times \mathrm{SL}(8,q)).(q-1)$$

which has index greater than q^{48}. Thus $|P_6 : L_\alpha| < 9q^3$. Now $\mathrm{SL}(6,q)$ and $\mathrm{SL}(8,q)$ do not have any subgroups with index this small, hence $L_\alpha > \hat{}A.(\mathrm{SL}(6,q) \times \mathrm{SL}(8,q))$ where $A = [q^{48}] \cap L_\alpha$. Observe that $|[q^{48}] : A| \leq 3$. In fact, $A.(\mathrm{SL}(6,q) \times \mathrm{SL}(8,q))/A$ acts by conjugation on the non-identity elements of A with orbits of size divisible by $q^5 + \cdots + q + 1$, hence $A = [q^{48}]$. Then, for some α, $A : (\mathrm{SL}(6,q) \times \mathrm{SL}(8,q))$ (or its transpose) has the following form and contains the following conjugate of g^*:

$$h^* = \begin{pmatrix} -1 & & & \\ & I_{5\times 5} & & \\ & & -1 & \\ & & & I_{7\times 7} \end{pmatrix} \in \begin{pmatrix} \mathrm{SL}(6,q) & A \\ & \mathrm{SL}(8,q) \end{pmatrix}.$$

Let h be the projective image of h^*. Then $r_g > r_h(\hat{}(\mathrm{SL}(6,q) \times \mathrm{SL}(8,q))) > q^{10}.q^{14} = q^{24}$. Then h is certainly centralized by a subgroup of A of size no more than q^{36}. Hence $r_g > q^{36}$. This implies that $\frac{n_g}{r_g} < q^{13}$ and $v < q^{27}$ which is a contradiction.

7 $L = \mathrm{PSL}(2,q)$ or $L^\dagger = \mathrm{PSL}(3,q)$

In this section we prove firstly that if $L^\dagger = \mathrm{PSL}(3,q)$ then the hypothesis in Section 4.3 leads to a contradiction. In the case where $L = \mathrm{PSL}(2,q)$ we add two extra suppositions to the hypothesis. For $g \in G$ let $\mathrm{Fix}\, g$ be the set of fixed points of g; then our extra suppositions are as follows:

- Let $g, h \in G$ with g an involution, $h^2 = g$. Then $\mathrm{Fix}\, h = \mathrm{Fix}\, g$ or else $|\mathrm{Fix}\, h| = u + 1, u + 2$ or $u + \sqrt{u} + 1$.
- Let $g, h \in G$ with g an involution, $[g, h] = 1$. Then $\mathrm{Fix}\, h = \mathrm{Fix}\, g$ or else $|\mathrm{Fix}\, h \cap \mathrm{Fix}\, g| \leq u + \sqrt{u} + 1$.

We prove that, with the addition of these suppositions, if $L = \mathrm{PSL}(2,q)$, then the hypothesis in Section 4.3 leads to a contradiction.

To understand the implications of this, suppose for a moment that G is acting on a projective plane of order x. Recall that then g fixes a Baer subplane and so h, as described in our extra suppositions, either fixes this Baer subplane or else acts as an automorphism of this subplane. Then Lemma 3 implies that these suppositions must hold. Hence in proving a contradiction we prove the following proposition:

Proposition 25. *Suppose that G contains a minimal normal subgroup L isomorphic to $\mathrm{PSL}(2,q)$ with $q \geq 4$ or that G has a unique component L such that $L^\dagger$ is isomorphic to $\mathrm{PSL}(3,q)$ with $q \geq 2$. If G acts transitively on a projective plane $\mathcal{P}$ of order x then $\mathcal{P}$ is Desarguesian and $G \geq \mathrm{PSL}(3,x)$.*

7.1 Preliminary facts. We need some preliminary facts about $\mathrm{PSL}(2,q)$ and $\mathrm{PSL}(3,q)$. As before we assume that $(G/C_G(L))/Z(L) \leq \mathrm{P\Gamma L}(n,q)$ since $|\mathrm{Aut}(L) : \mathrm{P\Gamma L}(n,q)| \leq 2$ for $n = 2,3$. Observe that both $\mathrm{PSL}(2,q)$ and $\mathrm{PSL}(3,q)$ have a single conjugacy class of involutions of size, in odd characteristic, $\frac{1}{2}q(q \pm 1)$ and $q^2(q^2+q+1)$ respectively and, in even characteristic, q^2-1 and $(q^2-1)(q^2+q+1)$ respectively. Both also have the property that a Sylow 2-subgroup contains at least 2 such involutions. Since a point-stabilizer must contain such a Sylow 2-subgroup we conclude that $r_g \geq 2$. Note also that $\mathrm{PSL}(3,q)$ has a single conjugacy class of transvections and this class does not fuse with any other in $\mathrm{P\Gamma L}(3,q)$.

Liebeck and Saxl [26] assert that, for $\mathrm{PSL}(3,q)$, the maximal subgroups of odd degree lie, as before, in families $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_5$ for $q > 2$. Note that $\mathrm{PSL}(3,2) \cong \mathrm{PSL}(2,7)$ and so we will deal with this group in the $\mathrm{PSL}(2,q)$ case. We state a result of [29, 36] (outlined in [15]) which gives the structure of all the subgroups of $\mathrm{PSL}(2,q)$:

Theorem 26. *Let q be a power of the prime p. Let $d = (q-1,2)$. Then a subgroup of $\mathrm{PSL}(2,q)$ is isomorphic to one of the following groups.*

1. *The dihedral groups of order $2(q \pm 1)/d$ and their subgroups.*
2. *A parabolic group P_1 of order $q(q-1)/d$ and its subgroups. A Sylow p-subgroup P of P_1 is elementary abelian, $P \lhd P_1$ and the factor group P_1/P is a cyclic group of order $(q-1)/d$.*
3. *$\mathrm{PSL}(2,r)$ or $\mathrm{PGL}(2,r)$, where r is a power of p such that $r^m = q$.*
4. *A_4, S_4 or A_5.*

Note that when $p = 2$, the above list is complete without the final entry. Dickson also outlines the conjugacy classes of subgroups of $\mathrm{PSL}(2,q)$; in particular it is easy to see that there are unique $\mathrm{PSL}(2,q)$ conjugacy classes of the maximal dihedral subgroups of size $2(q \pm 1)/d$ as well as a unique $\mathrm{PSL}(2,q)$ conjugacy class of parabolic subgroups P_1.

The result of Liebeck and Saxl [26] asserts that all of the families of maximal subgroups can, for some q, contain a subgroup of odd index in $\mathrm{PSL}(2,q)$ thus, when $L = \mathrm{PSL}(2,q)$, we will simply go through the possibilities given in Theorem 26.

In the $\mathrm{PSL}(3,q)$ case we will also need to know the subgroups of $\mathrm{GL}(2,q)$ which can be easily obtained from the subgroups of $\mathrm{PSL}(2,q)$.

Theorem 27. *H, a subgroup of $\mathrm{GL}(2,q)$, $q = p^a$, is amongst the following up to conjugacy in $\mathrm{GL}(2,q)$. Note that the last two cases may be omitted when $p = 2$.*

1. *H is cyclic;*
2. *$H = AD$, where $A \leq \left\{ \left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix} \right) : \lambda \in \mathrm{GF}(q) \right\}$ and $D \leq N(A)$, is a subgroup of the group of diagonal matrices;*
3. *$H = \langle c, S \rangle$ where $c | q^2 - 1$, S^2 is a scalar 2-element in c;*
4. *$H = \langle D, S \rangle$ where D is a subgroup of the group of diagonal matrices, S is an anti-diagonal 2-element and $|H : D| = 2$;*
5. *$H = \langle \mathrm{SL}(2, p^b), V \rangle$ or contains $\langle \mathrm{SL}(2, p^b), V \rangle$ as a subgroup of index 2 and here $b | a$, V is a scalar matrix. In the second case, $p^b > 3$;*
6. *$H / \langle -I \rangle$ is isomorphic to $S_4 \times C$, $A_4 \times C$, or (with $p \neq 5$) $A_5 \times C$, where C is a scalar subgroup of $\mathrm{GL}(2, q)/\langle -I \rangle$;*
7. *$H / \langle -I \rangle$ contains $A_4 \times C$ as a subgroup of index 2 and A_4 as a subgroup with cyclic quotient group, C is a scalar subgroup of $\mathrm{GL}(2, q)/\langle -I \rangle$.*

Proof. In this proof and subsequently, we will refer to subgroups of $\mathrm{GL}(2, q)$ as being *of type y*, where y is a number between 1 and 7 corresponding to the list above.

When the characteristic is odd, the proof of this result is given in [6, Theorem 3.4]. When the characteristic is even we know that $\mathrm{GL}(2, q) \cong \mathrm{PSL}(2, q) \times (q - 1)$. Then, for $H < \mathrm{GL}(2, q)$ either $H \geq \mathrm{SL}(2, q)$ and we are in type 5 above, or we have $H \leq H_1 \times (q - 1)$ where H_1 is maximal in $\mathrm{PSL}(2, q)$.

If $H_1 = D_{2(q-1)}$ then H is clearly of type 1 or 4. Similarly if $H_1 = D_{2(q+1)}$ then H is of type 1 or 3; if $H_1 = P_1$ then H is of type 2 in $\mathrm{GL}(2, q)$.

Now consider $H \leq \mathrm{PSL}(2, q_0) \times (q - 1)$. Any maximal subgroup of $\mathrm{PSL}(2, q_0)$ must be an intersection with $D_{2(q \pm 1)}$ or P_1 (and so is already accounted for) or else equals $\mathrm{PSL}(2, q_1)$ where $q = q_1^b$.

Thus we must consider $H \leq \mathrm{PSL}(2, q_1) \times (q - 1)$ and $H \not\leq B \times (q - 1)$ for $B < \mathrm{PSL}(2, q_1)$. Provided $q_1 > 2$ this implies that H is a subgroup of $\mathrm{GL}(2, q)$ of type 5. If $q_1 = 2$ then $\mathrm{PSL}(2, q_1) \leq D_{2(q \pm 1)}$ and the case is already accounted for. □

Note that a subgroup of type 1 in $\mathrm{GL}(2, q)$ is never maximal in $\mathrm{GL}(2, q)$. Furthermore type 5 includes $\mathrm{GL}(2, q)$ itself. We now proceed with our analysis.

7.2 $L = \mathrm{PSL}(2, q)$. Assume that $L = \mathrm{PSL}(2, q)$, $q \geq 4$. Suppose first that $G/C_G(L)$ contains $\mathrm{PGL}(2, q)$. Then G has a normal subgroup N of index 2, $N/C_N(L)$ contains only field automorphisms and N acts transitively on our set of size $x^2 + x + 1$. Proving a contradiction for N will give a contradiction for G, hence it is enough to assume in general that $G/C_G(L)$ contains only field automorphisms and $|G/C_G(L)| \leq |\mathrm{PSL}(2, q)|.\log_p q$.

For $q = 4, 5$ or 9, L is isomorphic to an alternating group. This case has already been examined and so these values of q can be excluded. Observe that P_1, a parabolic subgroup of $\mathrm{PSL}(2, q)$, has odd index if and only if $p = 2$. Furthermore if $p = 2$ then $L_\alpha \leq P_1$ since L_α must contain a Sylow 2-subgroup of $\mathrm{PSL}(2, q)$. This implies that $n_g = q^2 - 1$, $r_g = q - 1$ and $u^2 - u + 1 = \frac{n_g}{r_g} = q + 1$. But then $u^2 - u = q$ which is impossible. Hence we assume L_α does not lie in a parabolic subgroup of $\mathrm{PSL}(2, q)$ and that p is odd.

Now the only maximal subgroups of $\mathrm{PSL}(2, q)$ which contain a Sylow p-subgroup of $\mathrm{PSL}(2, q)$ are the parabolic subgroups. Also, for $q = 3^a$ with $a \geq 3$, the only maximal subgroups containing a subgroup of index p in a Sylow p-subgroup of $\mathrm{PSL}(2, q)$ are the parabolic subgroups. Thus Lemma 8 implies that $p \equiv 1(3)$ and we assume this from here on. Note that, for an involution $g \in \mathrm{PSL}(2,q)$, $n_g = \frac{1}{2}q(q \pm 1)$.

We examine the non-parabolic subgroups of L as candidates to be L_α, using Theorem 26.

If $L_\alpha = A_4$ then $r_g = 3$ and, since $r_g \big| n_g$ and $p \equiv 1(3)$, we must have $n_g = \frac{1}{2}q(q-1)$ and $q \equiv 3(4)$. Similarly if $L_\alpha = A_5$ then $r_g = 15$ and $q \equiv 3(4)$. But then $\frac{q+1}{4}$ divides $|L : L_\alpha|$. Since $\frac{q+1}{4} \equiv 2(3)$ this contradicts Lemma 8.

If $L_\alpha = S_4$ then $r_g = 9$ and once more $q \equiv 3(4)$. In fact $\frac{n_g}{r_g} = \frac{q(q-1)}{18}$. Then in $\mathrm{PSL}(2,q)$ there is a unique conjugacy class of elements of order 4. Let h be such an element and observe that $r_h = 6$. Now the fixed set of h lies inside the fixed set of $g = h^2$ and $d_h = \frac{1}{3}d_g = \frac{1}{3}(u^2+u+1)$. Referring to our first extra supposition this implies that $|\operatorname{Fix} h| = u+1, u+2$ or $u+\sqrt{u}+1$. Since $|\operatorname{Fix} h|$ divides $|\operatorname{Fix} g|$ we have $\frac{1}{3}(u^2+u+1) = u+\sqrt{u}+1$ and $u = 4$. But then $\frac{q(q-1)}{18} = \frac{n_g}{r_g} = 13$ which is impossible.

Now suppose that $L_\alpha \leq D_{q\pm1}$ so $q \pm 1 \equiv 0(4)$. Then $\frac{n_g}{r_g} = \frac{\frac{1}{2}q(q\mp1)}{\frac{1}{2}|L_\alpha|+1}$. Now $|\frac{n_g}{r_g}|_p \neq 1$ and so $|\frac{n_g}{r_g}|_p = |v|_p = q$. Thus $|L_\alpha|+2$ divides $q \mp 1$.

Define $m := \frac{q\pm1}{|L_\alpha|}$ and assume first that $m > 1$. Observe that $v = q\frac{q\pm1}{|L_\alpha|}\frac{q\mp1}{2}a$ for some integer a and $d_g = \frac{|L_\alpha|+2}{2}\frac{q\pm1}{|L_\alpha|}a$. If $|L_\alpha| = 4$ then $\frac{n_g}{r_g} = \frac{q(q\mp1)}{6}$ and, in fact, since $q \equiv 1(3)$, $\frac{n_g}{r_g} = \frac{q(q-1)}{6}$. But then $d_g = \frac{3(q+1)}{4}$ and, since $\frac{q+1}{4} \equiv 2(3)$, this is a contradiction. Thus $|L_\alpha| > 4$.

Now observe that $m(|L_\alpha|+2) > q \mp 1$; furthermore if $(m-1)(|L_\alpha|+2) = q \mp 1$ then $q \pm 1 - |L_\alpha| + 2m - 2 = q \mp 1$. Reducing modulo 4, this equation gives $2m \equiv 0(4)$ which is a contradiction since $m\big|v$. Thus $(m-2)(|L_\alpha|+2) \geq q \mp 1$. This implies that $m \geq |L_\alpha|+1$ and so $|L_\alpha|^2 + |L_\alpha| \leq q \pm 1$.

Since $\frac{n_g}{r_g} < d_g$ we have

$$\frac{q(q\mp1)}{|L_\alpha|+2} < \frac{|L_\alpha|+2}{2}\frac{q\pm1}{|L_\alpha|}a \text{ , thus } 2|L_\alpha|q(q\mp1) < (|L_\alpha|^2+4|L_\alpha|+4)(q\pm1)a.$$

We infer that $|L_\alpha| < (q+1)q^{-1}a$ by using the fact that $|L_\alpha| > 4$ and $|L_\alpha|^2+|L_\alpha| \leq q\pm1$. It then implies that $a > 3$.

Take h of maximal order in L_α. Since $|L_\alpha| > 4$ we know that h is not an involution and $n_h = q(q\mp1)$ and so $\frac{n_h}{r_h} = \frac{q(q\mp1)}{2}$. Thus $d_h = \frac{q\pm1}{|L_\alpha|}a$ which means that $d_h < d_g$. Now $[h,g] = 1$ and so, referring to our second extra supposition, $d_h^2 < 3d_g$ and so $\frac{(q\pm1)^2}{|L_\alpha|^2}a^2 < 3\frac{|L_\alpha|+2}{2}\frac{q\pm1}{|L_\alpha|}a$. This implies that $q \pm 1 < \frac{1}{2}|L_\alpha|^2 + |L_\alpha|$ which is a contradiction.

Hence $m = 1$ and $|L_\alpha| = q \pm 1$. We have two situations. If $q \equiv 3(4)$ then $n_g = \frac{1}{2}q(q-1)$ and $r_g = \frac{1}{2}(q+1)+1$. This means that $\frac{n_g}{r_g}$ is a not an integer, which is impossible. If $q \equiv 1(4)$ then $\frac{n_g}{r_g} = \frac{\frac{1}{2}q(q+1)}{\frac{1}{2}(q-1)+1} = q$. Since $|L : L_\alpha| = \frac{1}{2}q(q+1)$ we must have d_g a multiple of $\frac{q+1}{2}$. The only possibility is that $d_g = \frac{3(q+1)}{2}$ which means that $q = 13$ and $v = 273$.

In this case $|\operatorname{Fix} g| = 21$. But a Sylow 2-subgroup of $\mathrm{PSL}(2,q)$ which centralizes g fixes 9 points; this contradicts our second extra supposition.

Now suppose that $L_\alpha = \mathrm{PGL}(2,r)$ and $q = r^a$ where $a \equiv 2(4)$. Thus $q \equiv 1(4)$ and $\frac{n_g}{r_g} = \frac{\frac{1}{2}q(q+1)}{r^2}$. Now $\frac{q}{r^2} = |\frac{n_g}{r_g}|_p \neq |v|_p \geq \frac{q}{r}$ and so $|\frac{n_g}{r_g}|_p = 1$ and $r = \sqrt{q}$. Then $u^2 - u + 1 = \frac{n_g}{r_g} = \frac{1}{2}(q+1)$. Then $u = \frac{c+1}{2}$ where $c = \sqrt{2q-1}$. This implies that $u^2 + u + 1 = \frac{q+3+2c}{2}$. Now $|L : L_\alpha| = \frac{1}{2}(q+1)\sqrt{q}$ and so $\sqrt{q}$ divides $u^2 + u + 1$. Now observe that $\sqrt{q}(\frac{\sqrt{q}+5}{2}) > \frac{q+3+2c}{2}$. Furthermore $\sqrt{q}(\frac{\sqrt{q}-1}{2}) < \frac{n_g}{r_g}$. Thus $d_g = \sqrt{q}(\frac{\sqrt{q}+e}{2})$ where $e = 1$ or 3.

Now $2u = d_g - \frac{n_g}{r_g} = \frac{e\sqrt{q}-1}{2}$. We also know that $u = \frac{c+1}{2}$ and so we must have $e\sqrt{q} - 3 = 2\sqrt{2q-1}$. Since $e = 1$ or 3 we must have $e = 3$. Then $2\sqrt{2q-1} = 3\sqrt{q} - 3$, hence $2\sqrt{2q} > 3\sqrt{q} - 3$. Thus $q < 3^2(3-2\sqrt{2})^{-2} < 18^2$. This implies that $q = 7^2$ or 13^2. But neither of these satisfy the equality $2\sqrt{2q-1} = 3\sqrt{q} - 3$ and so can be excluded.

Now suppose that $L_\alpha = \mathrm{PSL}(2,r)$ and $q = r^a$ where a is odd. Then $\frac{n_g}{r_g} = \frac{\frac{1}{2}q(q\pm 1)}{\frac{1}{2}r(r\pm 1)}$ where $q \mp 1 \equiv 0(4)$. Now let h be an element of order $\frac{r\pm 1}{2}$. Then $\frac{n_h}{r_h} = \frac{q(q\mp 1)}{r(r\mp 1)}$. If $r \equiv 3(4)$ then

$$\frac{n_g}{r_g} = r^{a-1}(r^{a-1} + r^{a-2} + \cdots + r + 1) > r^{a-1}(r^{a-1} - r^{a-2} + \cdots - r + 1) = \frac{n_h}{r_h}.$$

Hence $d_g < d_h$ which is impossible.

Now if $r \equiv 1(4)$ then $u^2 - u + 1 = \frac{n_g}{r_g} = r^{a-1}(r^{a-1} - r^{a-2} + \cdots - r + 1)$ and so $r^{a-1} - r^{a-2} < u < r^{a-1}$. This means that

$$r^{2a-2} - r^{2a-3} + \cdots - r^a + 3r^{a-1} - 2r^{a-2} < d_g = \frac{n_g}{r_g} + 2u;$$
$$d_g = \frac{n_g}{r_g} + 2u < r^{2a-2} - r^{2a-3} + \cdots - r^a + 3r^{a-1}.$$

Now $r^{a-1} + r^{a-2} + \cdots + r + 1$ divides d_g. But observe that

$$\begin{aligned}
&(r^{a-1} + r^{a-2} + \cdots + r + 1)(r^{a-1} - 2r^{a-2} + 2r^{a-3} \cdots - 2r + 3)\\
&< r^{2a-2} - r^{2a-3} + \cdots - r^a + 3r^{a-1} - 2r^{a-2};\\
&(r^{a-1} + r^{a-2} + \cdots + r + 1)(r^{a-1} - 2r^{a-2} + 2r^{a-3} \cdots - 2r + 4)\\
&> r^{2a-2} - r^{2a-3} + \cdots - r^a + 3r^{a-1}.
\end{aligned}$$

This gives a contradiction and all possibilities are excluded.

7.3 $L^\dagger = \mathrm{PSL}(3,q)$. Once again we seek to show that the hypothesis in Section 4.3 leads to a contradiction; the usual action of $\mathrm{PSL}(3,q)$ on a Desarguesian projective plane $\mathrm{PG}(2,q)$ will not arise due to our restriction that all involutions fix $u^2 + u + 1$ points.

Recall that, for g an involution, $n_g = q^2(q^2+q+1)$ for q odd and $n_g = (q^2-1)(q^2+q+1)$ for q even. We assume here that $q > 2$ and we know that $L_\alpha \leq M$ where M is a member of $\mathcal{C}_1, \mathcal{C}_2$ or $\mathcal{C}_5$. We consider the latter two possibilities first. Observe that, in both cases, $p \equiv 1(3)$ since p^2 divides $|\operatorname{PSL}(3,q) : M|$.

Suppose that $M \in \mathcal{C}_2$. Then v is divisible by $q^3(q+1)(q^2+q+1)/6$. Now the highest power of q in $\frac{n_g}{r_g}$ is q^2. Since $v = \frac{n_g}{r_g}d_g$ and $(\frac{n_g}{r_g}, d_g) = 1$ we must have q^3 dividing d_g and q^2 dividing r_g. But then $u^2 - u + 1 = \frac{n_g}{r_g} \leq q^2 + q + 1$. This means that $v \leq (q^2+q+1)(q^2+3q+3)$ which is a contradiction.

Suppose that $M = N_{\operatorname{PSL}(3,q)}(\operatorname{PSL}(3,r)) \in \mathcal{C}_5$ where $q = r^a$ and $a \geq 3$ is an odd integer. Then $|v|_p = \frac{q^3}{r^3}$. Suppose first that $|v|_p = |\frac{n_g}{r_g}|_p \leq q^2$ and so $q \leq r^3$. Then we must have $a = 3$, $r_g|(q^2+q+1)$ and r^3 dividing $|L_\alpha|$. Since $r_g|(q^2+q+1)$ we cannot have $L_\alpha = \operatorname{PSL}(3,r)$ or $\operatorname{PSL}(3,r).3$. But since r^3 divides $|L_\alpha|$ we must have L_α inside a parabolic subgroup P of $\operatorname{PSL}(3,r).3$. But observe that then v is divisible by

$$|\operatorname{PSL}(3,q) : P| = \frac{q^3(q^3-1)(q^2-1)}{3r^3(r-1)(r^2-1)}$$

which is divisible by 9, a contradiction. The only other possibility is that $p \nmid \frac{n_g}{r_g}$ and $\frac{n_g}{r_g} \leq q^2+q+1$. But then $q^2 \leq r_g \leq r^2(r^2+r+1)$. This is impossible.

Hence we conclude that $M \in \mathcal{C}_1$. Thus $L_\alpha = \hat{}A.B$ where A is a subgroup of an elementary abelian unipotent subgroup, U, of order q^2 and B is a subgroup of odd index in $\operatorname{GL}(2,q)$. We will write $B \cap \operatorname{SL}(2,q) = (2, q-1).B_1$ where $B_1 \leq \operatorname{PSL}(2,q)$.

We will take α to be such that $L_\alpha \leq P_1$ where

$$P_1 = \hat{}\left\{ \begin{pmatrix} \frac{1}{\det Y} & ab \\ 0 & Y \end{pmatrix} : Y \in \operatorname{GL}_2(q), a, b \in \operatorname{GF}(q) \right\}.$$

Case $\boldsymbol{p \not\equiv 1(3)}$. In this case $|U : A| \leq 3$ and $|P : B_1 \cap P| \leq 3$ for some $P \in \operatorname{Syl}_p \operatorname{PSL}(2,q)$. If B_1 is a subgroup of P_1^*, a parabolic subgroup of $\operatorname{PSL}(2,q)$, then $q+1$ divides the index of B in $\operatorname{GL}(2,q)$ and $p = 2$. Then L_α is a subgroup of the Borel subgroup of $\operatorname{PSL}(3,q)$ and contains a normal Sylow 2-subgroup P. Thus $r_g = r_g(P) = 2q^2 - q - 1$ and so $r_g \nmid n_g$ which is a contradiction.

If $B_1 = \operatorname{PSL}(2,q)$ then $B \geq \operatorname{SL}(2,q)$. In fact, in odd characteristic, B must contain all matrices of determinant ± 1 since $|\operatorname{GL}(2,q) : B|$ is odd. Furthermore in its action by conjugation on the non-identity elements of U, $\operatorname{SL}(2,q)$ is transitive. Hence $A = U$. Thus, in both odd and even characteristic, L_α contains all involutions of the parabolic group: $q^2(q+2)$ of them in the odd case, $(q^2-1)(q+1)$ of them in the even case. In both cases $r_g \nmid n_g$ which is a contradiction.

For the remaining cases $p|v$ and so $p = 3$. If $B_1 \leq D_{q\pm1}$ then $q|v$ and we must have $q = 3$. In this case $n_g = 3^2 13$ and so $u^2 - u + 1 = \frac{n_g}{r_g} = 3$ or 13. If $\frac{n_g}{r_g} = 3$ then $v = 21$. This contradicts the fact that $|L : M| = 13$ and this divides v. So $\frac{n_g}{r_g} = 13, r_g = 9, d_g = 21$ and, since $B_1 \leq D_{q\pm1}$ we must have $L_\alpha = [3^2] : (8.2)$. But then L_α contains more than 9 involutions and this case is excluded.

If B_1 is a proper subgroup of $\operatorname{PSL}(2,q)$ isomorphic to A_4, S_4 or A_5 then $q = 3$ or 9. Now $\operatorname{PSL}(2,3) \cong A_4$ and so $q = 3$ is already excluded. If $q = 9$ then 5 divides $\operatorname{PSL}(2,q)$ and so $B_1 \cong A_5$, but $|\operatorname{PSL}(2,9) : A_5|$ is even which is impossible.

If $B_1 \cong \mathrm{PSL}(2,r)$ or $B_1 \cong \mathrm{PGL}(2,r)$ for $q = r^a, a > 1$ then $\frac{q}{r}|v$. Hence $q = 9$ and $r = 3$. but then 5 divides $|\,\mathrm{PSL}(2,9) : B_1|$ which is a contradiction.

Case $\boldsymbol{p \equiv 1(3)}$. In this case 3 divides $|\,\mathrm{PSL}(3,q) : M|$ and thus we assume that B contains both the Sylow 2 and Sylow 3-subgroups of $\mathrm{GL}(2,q)$. In fact $L = \mathrm{PSL}(3,q)$ since $Z(L)$ is semiregular (see Lemma 16.) Then B is a subgroup of $\mathrm{GL}(2,q)$ of type 4, 5, 6 or 7 in the list given earlier. Note that B contains the scalar subgroup of order 3 and so $|\,\mathrm{GL}(2,q) : B| = |\hat{\ }\mathrm{GL}(2,q) : \hat{\ }B|$.

Observe first that there are two P_1-conjugacy classes of involutions in P_1. Only one of these is centralized by a whole Sylow 2-subgroup, P, of P_1. Call this conjugacy class $\mathcal{A}$.

In the case where $L_\alpha = \hat{\ }A : B$, that is we have a split extension, we know that $\hat{\ }B$ contains a Sylow 2-subgroup of P_1 and so the involution in the centre of $\hat{\ }B$ must lie in $\mathcal{A}$. This implies that we can conjugate by elements of P_1 (i.e. choose α) such that this involution g is the projective image of

$$g^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We conclude that

$$B \leq \left\{ \begin{pmatrix} \frac{1}{\det Y} & \\ & Y \end{pmatrix} : Y \in \mathrm{GL}(2,q) \right\}.$$

We begin with two preliminary lemmas:

Lemma 28. *Let p be odd and $L_\alpha = \hat{\ }A : B \leq P_1$. Suppose that $|A| = q^2$ and that $(|B|, p) = 1$. Then $|B| > |\,\mathrm{GL}(2,q)|/(q^2+q+1)$.*

Proof. Let h be an element of order p. Then

$$v = \frac{n_h}{r_h} d_h = \frac{(q^2-1)(q^2+q+1)}{q^2-1} d_h = (q^2+q+1)d_h.$$

We have two possibilities:

1. Suppose that h is quasi-central. We must have $d_h = u^2+u+1$ where $v = u^4+u^2+1$. Then $u^2-u+1 = \frac{n_h}{r_h} = q^2+q+1$ and so $d_h = q^2+3q+3$. Thus $|B| = \frac{|\,\mathrm{GL}(2,q)|}{q^2+3q+3} a$ for some integer a. If $a = 1$ then $|B|$ is not an integer for $q > 1$. If $a \geq 2$ then $|B| > \frac{|\,\mathrm{GL}(2,q)|}{q^2+q+1}$ as required.

2. Suppose that h is not quasi-central. Then $d_h^2 < v$ and so, $v^2/(q^2+q+1)^2 < v$, which yields $v < (q^2+q+1)^2$. This implies that $|B| > |\,\mathrm{GL}(2,q)|/(q^2+q+1)$ as required. □

Lemma 29. *Let p be odd and $L_\alpha = \hat{\ }A : B \leq P_1$. Suppose that $(|B|, p) = 1$. Then $|A| \neq q$.*

Proof. Let h be an element of order p and suppose that $|A| = q$. Then

$$v = \frac{n_h}{r_h} d_h = \frac{(q^2-1)(q^2+q+1)}{q-1} d_h = (q+1)(q^2+q+1)d_h.$$

But, since v is odd and $q+1$ is even, this implies that d_h is not an integer. This is a contradiction. □

We now begin our analysis of the different possibilities for B. In the case where $B < \mathrm{GL}(2,q)$ is of type 4, 6 or 7 then Schur–Zassenhaus implies that $A.B$ is a split extension.

Suppose first that B is a subgroup of type 4 in $\mathrm{GL}(2,q)$. Let α be such that $B \leq \langle D, S\rangle$ where D is the subgroup of diagonal matrices and S is an anti-diagonal 2-element. Note that we must have q dividing $|A|$.

Now observe that, since B contains a Sylow 2-subgroup of D, we can choose α such that

$$\begin{aligned}\begin{pmatrix}1 & e & f\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \in A \implies \begin{pmatrix}-1 & e & f\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}^2 \in A \\ \implies \begin{pmatrix}1 & -2e & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \in A \implies \begin{pmatrix}1 & e & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \in A.\end{aligned}$$

We conclude that $A = A_1 \times A_2$ where

$$A_1 \leq \left\{\begin{pmatrix}1 & e & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} : e \in \mathrm{GF}(q)\right\}, \quad A_2 \leq \left\{\begin{pmatrix}1 & 0 & f\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} : f \in \mathrm{GF}(q)\right\}.$$

Now consider an element, as given, of A_1. Then,

$$\begin{aligned}X = \begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & a\\ 0 & a^{-1} & 0\end{pmatrix} \in B &\implies \begin{pmatrix}-1 & e & 0\\ 0 & 0 & a\\ 0 & a^{-1} & 0\end{pmatrix}^2 \in A:B \\ &\implies \begin{pmatrix}1 & e & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & -e & -ae\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \in A:B \\ &\implies \begin{pmatrix}1 & 0 & ae\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \in A_2.\end{aligned}$$

Thus, for fixed X, we have an injection from A_1 into A_2. There is a similar injection from A_2 into A_1 and so $|A_1| = |A_2| = \sqrt{|A|}$. Now let

$$E = B \cap \left\{\begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & a\\ 0 & a^{-1} & 0\end{pmatrix} : a \in \mathrm{GF}(q)\right\}$$

and observe that

$$\begin{pmatrix}1 & e & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \in A_1, \begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & a\\ 0 & a^{-1} & 0\end{pmatrix} \in E, \implies \begin{pmatrix}1 & e & 0\\ 0 & 0 & a\\ 0 & a^{-1} & 0\end{pmatrix}^2 \in A:B$$

$$\implies \begin{pmatrix}-1 & e & ae\\ 0 & 0 & a\\ 0 & a^{-1} & 0\end{pmatrix} \in A:B$$

and this last element is an involution. We now count all the involutions in L_α as follows:

Pre-image of involution g in $\mathrm{SL}(3,q)$	Number of such involutions in L_α
$\begin{pmatrix}1 & c & d\\ & -1 & \\ & & -1\end{pmatrix}$	$\lvert A\rvert$
$\begin{pmatrix}-1 & 0 & d\\ & -1 & \\ & & 1\end{pmatrix}$	$\sqrt{\lvert A\rvert}$
$\begin{pmatrix}-1 & c & 0\\ & 1 & \\ & & -1\end{pmatrix}$	$\sqrt{\lvert A\rvert}$
$\begin{pmatrix}-1 & c & d\\ & & a\\ & a^{-1} & \end{pmatrix}$	$\lvert E\rvert\sqrt{\lvert A\rvert}$

Thus $r_g = \sqrt{|A|}(\sqrt{|A|}+|E|+2)$ and note that $r_g \leq q(2q+1)$ since $|E| \leq q-1$. Suppose that $(\frac{n_g}{r_g}, p) = 1$. Then $r_g \geq q^2$ and we must have $|A| = q^2$. Alternatively suppose that $(\frac{n_g}{r_g}, p) \neq 1$. Then

$$|\frac{n_g}{r_g}|_p = |v|_p \geq \frac{q^3}{|A|} \implies \frac{q^2}{\sqrt{|A|}} \geq |\frac{n_g}{r_g}|_p \geq \frac{q^3}{|A|} \implies |A| \geq q^2.$$

Thus, in either case, $|A| = q^2$. Then, by Lemma 28, $|B| > \frac{|\mathrm{GL}(2,q)|}{q^2+q+1}$. But $\frac{2(q-1)^2}{7} < \frac{|\mathrm{GL}(2,q)|}{q^2+q+1} = \frac{q(q-1)^2(q+1)}{q^2+q+1}$ for $q > 1$. Hence $|B| = 2(q-1)^2$ and $|E| = q-1$. Then $r_g = q(2q+1)$ which makes $\frac{n_g}{r_g}$ a non-integer unless $q = 1$. This is a contradiction.

Next assume that B is of type 6 or 7. To ensure that B has odd index in $\mathrm{GL}(2,q)$ we assume that $B \cong 2.(S_4 \times C)$ or $B \cong 2.(A_4 \times C).2$ where $C \leq Z(\mathrm{GL}(2,q))/\langle -I\rangle$.

Then we must have q dividing $|A|$ since $|v|_p \leq q^2$. We write $|A| = qp^a$ where $a \geq 1$ by Lemma 29. Since $\begin{pmatrix}1 & & \\ & -1 & \\ & & -1\end{pmatrix} \in B$ this means that $r_g > |A|$.

Suppose first that $q = p^a$ and $|A| = q^2$. By Lemma 28, $|\,\mathrm{GL}(2,q)|/(q^2+q+1) < |B| \leq 24(q-1)$, hence $24(q^2+q+1) > q^3 - q$ and $q < 30$. Then $q = 7, 13$ or 19. Note that in $\mathrm{GL}(2,7)$ subgroups of type 6 or 7 have even index and in $\mathrm{GL}(2,19)$ subgroups of type 6 and 7 have index divisible by 3. Hence we are left with $q = 13$. In this case $n_g = 3^2.13.61$ and v is divisible by $|L : M| = 3.7.13.61$. Now since $u^2 - u + 1 = \frac{n_g}{r_g}$ divides n_g we must have $u = 2, 4, 14$ or 23. But in all of these case $u^2 + u + 1$ is not divisible by both 7 and 61. Thus v is not divisible by both 7 and 61 which is a contradiction.

Thus assume now that $q > p^a$ and $|A| < q^2$. Then,

$$\begin{aligned}\frac{n_g}{r_g} < \frac{q^2(q^2+q+1)}{|A|} &\implies d_g < \frac{q^2(q^2+q+1)}{|A|} + 2\frac{q^2+q+1}{\sqrt{|A|}} + 2 \\ &\implies d_g < \frac{(q^2+2q+1)(q^2+q+1)}{|A|} \\ &\implies v < \frac{(q+1)^2q^2(q^2+q+1)^2}{|A|^2}.\end{aligned}$$

This implies that

$$\frac{(q^2+q+1)q^3(q-1)^2(q+1)}{|A||B|} \leq v < \frac{q^2(q^2+q+1)^2(q+1)^2}{|A|^2},$$

hence $|A| < (q+1)(q^2+q+1)q^{-1}(q-1)^{-2}|B|$, which yields $|A| < 2.|B|$ for $q \geq 7$.

Now elements from $\hat{}2.C$ do not centralize any element of $\hat{}A$. Thus let $m = \frac{(q-1)/2}{|C|}$ and observe that $\frac{q-1}{3m} = |\hat{}2.C|$ divides $|A| - 1 = qp^a - 1$. This in turn means that $\frac{q-1}{3m}$ divides $p^a - 1$. Since $q > p^a$ this means that $3m > p$. Then $|B| > |A|/2$, hence $48|C| > q.p^a/2$, which gives $48(q-1)/m > q.p^a$ and $p^{a+1} < 144$. Since $p \geq 7$, $a \geq 1$ we must have $p = 7$, $a = 1$. But when $p = 7$, $2.(A_4 \times C).2$ and $2.(S_4 \times C)$ have even index in $\mathrm{GL}(2,q)$ which is a contradiction.

Thus we are left with the possibility that B is of type 5 in $\mathrm{GL}(2,q)$. We want to show that $L_\alpha = \hat{}A.B$ is a split extension and we can choose α such that

$$B \leq \left\{ \begin{pmatrix} \frac{1}{\det Y} & \\ & Y \end{pmatrix} : Y \in B^* \right\} \cong B^* \leq \mathrm{GL}(2,q).$$

Observe first that each Sylow 2-subgroup of L_α contains a unique element of $\mathcal{A}$. Thus $\mathcal{A} \cap L_\alpha$ is a L_α conjugacy class. Furthermore there exist at least two non-conjugate maximal subgroups, M_1, M_2, of B which are of order not divisible by p and index in B not divisible by 2. Then, by Schur–Zassenhaus, $A : M_1$ and $A : M_2$ are subgroups of L_α. But M_1, M_2 must both have centres which are conjugate in L_α, in fact must lie in $\mathcal{A}$. This implies that there exist conjugates of M_1, M_2 which both lie in

$$\left\{ \begin{pmatrix} \frac{1}{\det Y} & \\ & Y \end{pmatrix} : Y \in B^* \right\} \cong B^* \leq \mathrm{GL}(2,q).$$

These conjugates must generate a complement to A as required.

Now note first that $\mathrm{SL}(2,r) \leq \mathrm{GL}(2,q)$ implies that $\mathrm{SL}(2,r) \leq \mathrm{SL}(2,q)$. Now write $q = r^f$ and observe that, for $f = p_1 \dots p_n$ where p_i is prime,

$$\mathrm{SL}(2,r) < \mathrm{SL}(2,r^{p_1}) < \cdots < \mathrm{SL}(2,r^{p_1 \cdots p_{n-1}}) < \mathrm{SL}(2,q).$$

Since B has odd index in $\mathrm{GL}(2,q)$ we assume that all of these primes are odd except, possibly, for p_1. What is more, the chain of subgroups given here is maximal except for the first inclusion when $p_1 = 2$. Now there is a unique conjugacy class in $\mathrm{SL}(2,q)$ of maximal subgroups isomorphic to $\mathrm{SL}(2,r)$ when $q = r^a$ for a an odd prime. Hence, stepping down the chain of inclusion, we assume that $\mathrm{SL}(2,r)$ has a unique conjugacy class in $\mathrm{SL}(2,q)$ except when $p_1 = 2$ in which case there are two conjugacy classes.

By examining [23, Action Table 3.5G]) we find that, when f is even, the two conjugacy classes are fused in $\mathrm{GL}(2,r^2)$ through conjugation by $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)$ where λ generates the group $\mathrm{GF}(r^2)^*$. Thus, in $\mathrm{GL}(2,q)$ there is a unique conjugacy class of $\mathrm{SL}(2,r)$ and we take α such that B^* contains the copy of $\mathrm{SL}(2,r)$ consisting of matrices of determinant 1 with entries in $\mathrm{GF}(r)$.

Observe that $B^* \ni \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and so

$$\begin{pmatrix} 1 & e & f \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in A \implies \begin{pmatrix} -1 & e & f \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 \in A \implies \begin{pmatrix} 1 & e & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in A$$

Once again we conclude that $A = A_1 \times A_2$ where

$$A_1 \leq \left\{ \begin{pmatrix} 1 & e & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : e \in \mathrm{GF}(q) \right\}, \quad A_2 \leq \left\{ \begin{pmatrix} 1 & 0 & f \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : f \in \mathrm{GF}(q) \right\}.$$

In the same way as earlier we also know that $|A_1| = |A_2| = \sqrt{|A|}$. We count involutions in L_α:

Pre-image of involution g in $\mathrm{SL}(3,q)$	Number of such involutions in L_α
$\begin{pmatrix} 1 & c & d \\ & -1 & \\ & & -1 \end{pmatrix}$	$\lvert A\rvert$
$\begin{pmatrix} -1 & c & d \\ & \pm 1 & \\ & & \mp 1 \end{pmatrix}$	$2\sqrt{\lvert A\rvert}$
$\begin{pmatrix} -1 & c & d \\ & \pm 1 & x \\ & & \mp 1 \end{pmatrix}$, $x \neq 0$	$2(r-1)\sqrt{\lvert A\rvert}$
$\begin{pmatrix} -1 & c & d \\ & v & w \\ & x & -v \end{pmatrix}$, $x \neq 0$	$r(r-1)\sqrt{\lvert A\rvert}$

Thus $r_g = \sqrt{|A|}(\sqrt{|A|} + r^2 + r)$. Now $\mathrm{SL}(2,r)$ has orbits of size r^2-1 in its action by conjugation on non-identity elements of A. Hence either $|A| = 1$ or $\sqrt{|A|} \geq r$. If $|A| = 1$ then, since q divides $|L_\alpha|$, we must have $r = q$ and so $\frac{n_g}{r_g} = q^2$. This contradicts Lemma 10. Hence $\sqrt{|A|} \geq r$ and so $|\frac{n_g}{r_g}|_p = \frac{q^2}{\sqrt{|A|}r}$.

Then either $|\frac{n_g}{r_g}|_p = 1$, $r = q$ and $\sqrt{|A|} = q$ or $|\frac{n_g}{r_g}|_p = |v|_p \geq \frac{q^3}{|A|r}p^a$ where $p^a = \frac{|G|/|L|}{|G_\alpha|/|L_\alpha|}$. In the latter case this means that

$$\frac{q^2}{\sqrt{|A|}r} \geq \frac{q^3}{|A|r}p^a$$

and so $|A| \geq q^2.p^{2a}$. This implies that $|A| = q^2$ and $a = 0$. In both cases we find that $|A| = q^2$ and so $r_g = qr(\frac{q}{r} + 1 + r)$. In order for this to divide n_g we find that we must have $r^4 + 2r^3 - r + 1$ divisible by $\frac{q}{r} + 1 + r$. For $q \geq r^6$ this is clearly a contradiction. Examining cases individually for $q \leq r^5$ we find only contradictions.

Thus Proposition 25 is proved.

8 $L^\dagger = \mathrm{U}(n,q)$

In this section we prove that, if $L^\dagger = \mathrm{U}(n,q)$, then the hypothesis in Section 4.3 leads to a contradiction. This implies the following proposition:

Proposition 30. *Suppose G contains a unique component L such that $L^\dagger$ is isomorphic to $\mathrm{U}(n,q)$. Then G does not act transitively on a projective plane.*

We may assume that $n \geq 3$ and $(n,q) \neq (3,2)$. We know ([23, Proposition 2.3.2]) that our unitary geometry (V,κ) has a hyperbolic basis. Unless stated otherwise, we will write all matrices representing elements of $\mathrm{SU}(n,q)$ according to this basis:

$$\begin{cases} \{e_1, f_1, \dots, e_m, f_m\}, & \text{if } n = 2m; \\ \{e_1, f_1, \dots, e_m, f_m, x\}, & \text{if } n = 2m+1. \end{cases}$$

where $\kappa(e_i,e_j) = \kappa(f_i,f_j) = 0$, $\kappa(e_i,f_j) = \delta_{ij}$, $\kappa(e_i,x) = \kappa(f_i,x) = 0$ for all i,j and $\kappa(x,x) = 1$.

We will also need to make use of an orthonormal basis for (V,κ). Let v_i, w_i with $i = 1,\dots,m$ be orthonormal vectors such that $\langle v_i, w_i\rangle = \langle e_i, f_i\rangle$ for all $i = 1,\dots,m$. Our orthonormal basis $\mathcal{B}$ will consist of these vectors v_i, w_i with $i = 1,\dots,m$, as well as the vector x in the case where n is odd.

Now the result of Liebeck and Saxl [26] implies that L_α lies inside a maximal subgroup M where

- for q odd, $M \in \mathcal{C}_1$, $M \in \mathcal{C}_2$, $M^\dagger = N_{\mathrm{U}(n,q)}(\mathrm{U}(n,q_0))$ where $q = q_0^a$ and a is odd, or $M^\dagger = M_{10}$ and $(n,q) = (3,5)$, or $n = 4$;
- for q even, $M \in \mathcal{C}_1$.

We show next that, in all cases, M must lie in $\mathcal{C}_1$:

Lemma 31. *L_α lies inside M, where M maximal in L lies inside $\mathcal{C}_1$.*

Proof. We may assume that p is odd. Define g to be the projective image of

$$g^* = \begin{pmatrix} -1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

For $n \neq 4$, g lies in the centre of a maximal subgroup $\hat{\ }(\mathrm{SU}(2,q) \times \mathrm{SU}(n-2,q)).(q+1)$. For $n = 4$, g lies in the centre of a maximal subgroup $\hat{\ }(\mathrm{SU}(2,q) \times \mathrm{SU}(2,q)).(q+1).2$. Furthermore, g has the same form under our orthonormal basis $\mathcal{B}$ and, under this basis, $\mathrm{P\Gamma U}(n,q) = \mathrm{U}(n,q).\langle\delta,\varphi\rangle$ where φ is a field automorphism and δ is conjugation by the projective image of

$$\begin{pmatrix} a & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

for some $a \in \mathrm{GF}(q^2)^*$, a primitive $(q+1)$-th root of unity. Then g is centralised by $\langle\sigma,\varphi\rangle$ hence $n_g | q^{2(n-2)}b$ where $(q,b) = 1$ and $b < q^{2(n-2)}$. Then, by Lemma 13, $|v|_p \leq q^{2(n-2)}$.

Suppose that $L_\alpha \leq M$ where $M \in \mathcal{C}_2$, or $M^\dagger = N_{\mathrm{U}(n,q)}(\mathrm{U}(n,q_0))$ where $q = q_0^a$ and a is odd, or $M^\dagger = M_{10}$ and $(n,q) = (3,5)$, or $n = 4$. Observe that $|\,\mathrm{U}(n,q)|_p = q^{\frac{1}{2}n(n-1)}$ while, for $n \neq 4$, $|M|_p \leq q^{\frac{1}{4}n(n-1)}$. Thus we must have $\frac{1}{2}n(n-1) - 2(n-2) = \frac{1}{2}(n^2 - 5n + 8) \leq \frac{1}{4}n(n-1)$. This implies that $n \leq 6$. We assume this from here on.

Note that we may also assume that $p \equiv 1(3)$ since, in all given cases, $|\,\mathrm{U}(n,q) : M^\dagger|$ odd implies that p^2 divides $|\,\mathrm{U}(n,q) : M^\dagger|$. We may immediately rule out the possibility that $M^\dagger = M_{10}$.

Consider first the case where $n \neq 4$. If $M \in \mathcal{C}_2$ then $|\,\mathrm{U}(n,q) : M^\dagger|_p > q^{2(n-2)}$ for $n = 3, 5$ and 6 which is a contradiction. If $M = N_{\mathrm{U}(n,q)}(\mathrm{U}(n,q_0))$ then $q = q_0^a$ where a is an odd prime. Then $|M|_p \leq q^{\frac{1}{2a}n(n-1)}$ hence we have $\frac{1}{2}(n^2 - 5n + 8) \leq \frac{1}{2a}n(n-1)$ which implies that $n = 3$ and $q = q_0^3$. Now, when $n = 3$, $n_g = q^2(q^2 - q + 1)$ and L_α contains a Sylow p-subgroup of M. If $L_\alpha \geq \mathrm{U}(3,q_0)$ then $r_g = q_0^2(q_0^2 - q_0 + 1)$ but then $r_g \not| n_g$ which is a contradiction. The only other possibility is that $L_\alpha \cap \mathrm{U}(3,q_0) \leq P_1^*$, where P_1^* is a parabolic subgroup of $\mathrm{U}(3,q_0)$. But this has even index in $\mathrm{U}(3,q_0)$ which is a contradiction.

Now suppose that $n = 4, p \equiv 1(3)$. Note that here $L = \mathrm{U}(4,q)$ and that $n_g = \frac{1}{2}q^4(q^2 - q + 1)(q^2 + 1)$. We need to consider the cases where M is a maximal subgroup of odd index not lying in $\mathcal{C}_1$. Furthermore we need $|\,\mathrm{U}(4,q) : M|_p \leq q^4$. We go through the possibilities in turn.

- Suppose that $M \in \mathcal{C}_2$. There exist two subgroups $M \in \mathcal{C}_2$ such that $|\,\mathrm{U}(4,q):M|_p \leq q^4$ but only one has odd index. We need to rule out this possibility, when $M \cong \hat{}(\mathrm{SU}(2,q)\times \mathrm{SU}(2,q)).(q+1).2$ and $|\,\mathrm{U}(4,q):M|_p = q^4$. Then L_α must contain a Sylow p-subgroup of M. But the parabolic subgroup of $\mathrm{SU}(2,q)$ has even index hence we may conclude that, for some α,
$$L_\alpha > \hat{} \begin{pmatrix} \mathrm{SU}(2,q) & \\ & \mathrm{SU}(2,q) \end{pmatrix}.$$
Then L_α contains h, the projective image of
$$\begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}.$$
Now h is a $\mathrm{U}(4,q)$-conjugate of g, thus $r_g \geq \frac{1}{2}(q^2-q)^2$. Hence $\frac{n_g}{r_g} < q^2(q+1)(q+2)$. If $q^4 \big| \frac{n_g}{r_g}$ then we must have $\frac{n_g}{r_g} = q^4$ which is a contradiction of Lemma 10. The only other possibility is that $\frac{n_g}{r_g} \leq \frac{1}{2}(q^2-q+1)(q^2+1) < \frac{1}{2}q^4$. But then $d_g < q^4$ and so $v < \frac{1}{2}q^4(q^2-q+1)(q^2+1)$ which contradicts $L_\alpha \leq M$.
- Suppose that $M \in \mathcal{C}_6$ or $M \in S$. The only odd index subgroup is $M = 2^4.A_6$ where $q \equiv 3(8)$. But then $|\,\mathrm{U}(4,q):M|_p > q^4$ which is a contradiction.
- Suppose that $M \in \mathcal{C}_5$. If $M = N_{\mathrm{U}(4,q)}(\mathrm{U}(4,q_0))$ then $q = q_0^a$ where a is an odd prime. Then $|M|_P \leq q^{\frac{6}{a}}$ hence we must have $\frac{1}{2}(n^2-5n+8) = 2 \leq \frac{6}{a}$ which implies that $q = q_0^3$. However this implies that 9 divides $|\,\mathrm{U}(n,q):M|$ which is a contradiction.
The only other odd index subgroup in $\mathcal{C}_5$ is $M = \mathrm{PGSp}(4,q)$ when $q \equiv 1(4)$. Now, given our original basis $\{e_1,f_1,e_2,f_2\}$ and our original hermitian form κ, define the form $\kappa_\sharp = \zeta^{-1}\kappa$ over the $\mathrm{GF}(q)$-vector space $V_\sharp$ spanned by $\{\zeta e_1, f_1, \zeta e_2, f_2\}$. Here ζ is an element of $\mathrm{GF}(q^2)$ such that $\zeta^q = -\zeta$. Then $\kappa_\sharp$ is a symplectic form over $V_\sharp$.
Clearly if g^* is an isometry for $(\kappa_\sharp, V_\sharp)$ then g^* is an isometry for (κ, V) and we have an embedding $\mathrm{Sp}(4,q) < \mathrm{SU}(4,q)$. This embedding corresponds to a maximal subgroup $\mathrm{PSp}(4,q) < \mathrm{U}(4,q)$ when $q \not\equiv 1(4)$ and $\mathrm{PGSp}(4,q) < \mathrm{U}(4,q)$ when $q \equiv 1(4)$. In the latter case, there are two conjugacy classes of $\mathrm{PGSp}(4,q)$ in $\mathrm{U}(4,q)$; it is this case which concerns us.
Under the orthonormal basis $\{v_1,w_1,v_2,w_2\}$, the two conjugacy classes of $\mathrm{PGSp}(4,q)$ in $\mathrm{U}(4,q)$ are fused by x, the projective image of
$$\begin{pmatrix} \lambda & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$
where $\lambda \in \mathrm{GF}(q^2)$ is a $(q+1)$-primitive element. Thus r_g is the same no matter which of the two conjugacy classes we lie in. Assume from here on that $L_\alpha \leq M = \mathrm{PGSp}(4,q)$ preserving $(\kappa_\sharp, V_\sharp)$.

Then $|\operatorname{U}(4,q):M|_p = q^2$, thus $|M:L_\alpha|_p \leq q^2$. The only maximal subgroup, M_1, of $\operatorname{PSp}(4,q)$ such that $|\operatorname{PSp}(4,q):M_1|$ is odd and $|\operatorname{PSp}(4,q):M_1|_p \leq q^2$ is $(\operatorname{Sp}(2,q)\circ\operatorname{Sp}(2,q)).2$. Thus either

– $L_\alpha = M$ with v divisible by $\frac{1}{2}q^2(q+1)(q^2-q+1)$; or
– $L_\alpha \cap \operatorname{PSp}(4,q) \leq B = (\operatorname{Sp}(2,q)\circ\operatorname{Sp}(2,q)).2$. Note that $|(\operatorname{U}(4,q):B|_p = q^4$. Since the parabolic subgroups of $\operatorname{Sp}(2,q)$ are of even index we must have $L_\alpha \cap \operatorname{PSp}(4,q) = B$ and so $L_\alpha = B.2$ with v divisible by $\frac{1}{4}q^4(q+1)(q^2-q+1)(q^2+1)$.

Under our original basis this implies that, for some α,

$$L_\alpha > \hat{}\begin{pmatrix} \operatorname{SU}(2,q) & \\ & \operatorname{SU}(2,q) \end{pmatrix}.$$

Now $\operatorname{PSp}(4,q)$ is normalized in $\operatorname{U}(4,q)$ by h, the projective image of

$$\begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}.$$

Thus h lies in L_α and, as before, we know that h is a $\operatorname{U}(n,q)$-conjugate of g. We may conclude that $r_g \geq \frac{1}{2}(q^2-q)^2$ and so $\frac{n_g}{r_g} < q^2(q+1)(q+2)$. As in the case where $M \in \mathcal{C}_2$ this contradicts $L_\alpha = B.2$. We conclude that $M = \operatorname{PGSp}(4,q)$. Now observe that $C_{\operatorname{PSp}(4,q)}(h) \cong \hat{}\operatorname{GL}(2,q).2$; thus $r_g \geq \frac{1}{2}q^3(q+1)(q^2+1)$ and $\frac{n_g}{r_g} < q^2$. This implies that $v < q^2(q+1)(q+2)$ which is a contradiction for $q > 4$. $\square$

Thus L_α lies inside a maximal subgroup $M \in \mathcal{C}_1$. There are two types of $M \in \mathcal{C}_1$ [23, Table 3.5B]:

- The parabolic subgroups, $P_m, 1 \leq m \leq \left\lfloor \frac{n}{2} \right\rfloor$. Observe that $(q+1)^m$ divides $|L:P_m|$. This implies that $p=2$. If $q \equiv 1(3)$ then $(q+1) \equiv 2(3)$ and $q+1$ divides v. If $m > 1$ and $q \equiv 2(3)$ then $9|v$. Neither of these situations are allowed. Hence $m=1$ and we must have $q = 2^a$, a odd.
- The subgroups B_m of type $\operatorname{GU}(m,q) \perp \operatorname{GU}(n-m,q)$ with $1 \leq m < n/2$. In this case $q^{m(n-m)}$ divides $|L:B_m|$ and we must have $p \equiv 1(3)$. Observe that $q^{m(n-m)} > q^{2(n-2)}$ for $\frac{n}{2} > m > 2$. But we know, by the argument in the previous lemma, that $|v|_p \leq q^{2(n-2)}$ hence $m \leq 2$

We now examine these two situations in turn and seek a contradiction.

8.1 Case $p=2, q=2^a$, a odd, $L_\alpha \leq P_1$. Let n_e be the even element of $\{n, n-1\}$ while n_o is the odd element. Then $i := |\operatorname{U}(n,q):P_1| = \frac{(q^{n_e}-1)(q^{n_o}+1)}{q^2-1}$. We know that $3|(q+1)\big|i$. In addition, $q^{n_e-2}+\cdots+q^2+1|i$ and so for all $r|\frac{n_e}{2}, q^{2r-2}+\cdots+q^2+1|i$ which means that for all $r|\frac{n_e}{2}, r \equiv 1(3)$. A similar argument allows us to conclude from the fact that $(q^{n_o-1}-\cdots+q^2-q+1)|i$ that for all $r|n_o, r \equiv 1(3)$. We may conclude from this that n is even and $n \equiv 2(12)$. Thus $n \geq 14$.

Now $L_\alpha = [q^{2n-3}] : B \leq P_1$ where $B \leq \hat{}((q^2-1) \times \mathrm{SU}(n-2,q))$. We consider the two possibilities given by Lemma 14:

- $B \leq \hat{}((q^2-1) \times B_1)$ for some $B_1 < \mathrm{SU}(n-2,q)$. We know that B_1 must lie in a parabolic subgroup of $\mathrm{SU}(n-2,q)$ by Liebeck, Saxl [26]. However any parabolic subgroup of $\mathrm{SU}(n-2,q)$ has index divisible by $q+1$ which would result in $9|v$ which is a contradiction.
- $B = \hat{}(A_1 \times \mathrm{SU}(n-2,q))$ for some $A_1 \leq (q^2-1)$. For some α

$$L_\alpha \geq \hat{} \begin{pmatrix} \mathrm{SU}(n-2,q) & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

Now consider transvections in $\mathrm{SU}(n,q)$. All transvections are conjugate to

$$g^* : V \to V, v \mapsto v + s\kappa(v, e_1)e_1$$

for some $s \in \mathrm{GF}(q^2), s + s^q = 0$ [33, p. 119]. For $W = \langle e_1 \rangle$, define $X_{W,W^\perp}$ to be the subgroup of $\mathrm{SU}(n,q)$ consisting of all transvections of this form. Now suppose that $h \in \mathrm{SU}(n,q)$ preserves W. Then, for $v \in V$,

$$\begin{aligned} v(h^{-1}g^*h) &= (vh^{-1} + s\kappa(vh^{-1}, e_1)e_1)h \\ &= v + s\kappa(vh^{-1}, e_1hh^{-1})e_1h \\ &= v + s\kappa(v, e_1h)e_1h \\ &= v + stt^q\kappa(v, e_1)e_1 \end{aligned}$$

where $t \in \mathrm{GF}(q)^*$ is defined via $e_1h = te_1$. Then $(stt^q)^q + stt^q = tt^q(s + s^q) = 0$. Thus $X_{W,W^\perp}$ is normal in the parabolic subgroup of $\mathrm{SU}(n,q)$ stabilizing W. Since $|X_{W,W^\perp}| = q$ [33, p. 114], we may conclude that, for g the projective image of g^*, $\frac{|P_1|}{q-1}$ divides $C_L(g)$. Then, since the only maximal subgroup of $\mathrm{U}(n,q)$ whose order is divisible by $\frac{|P_1|}{q-1}$ is P_1, we find that $n_g \leq \frac{|\mathrm{U}(n,q)|(q-1)(n,q+1)2\log_2 q}{|P_1|}$.
Furthermore, $g \in L_\alpha$ and, by the same argument, $r_g \geq \frac{|\mathrm{SU}(n-2,q)|}{|P_1^*|}$ where P_1^* is a parabolic subgroup of $\mathrm{SU}(n-2,q)$. Thus,

$$\frac{n_g}{r_g} \leq \frac{|\mathrm{U}(n,q)|(q-1)(n,q+1)2\log_2 q}{|P_1|} \frac{|P_1^*|}{|\mathrm{SU}(n-2,q)|} < q^8.$$

Then $v < q^{17}$ which is a contradiction.

8.2 Case $p \equiv 1(3)$, $L_\alpha \leq B_m$, $m \leq 2$.

Observe that

$$|L : B_m| = q^{m(n-m)} \frac{(q^n - (-1)^n) \dots (q^{n-m+1} - (-1)^{n-m+1})}{(q+1) \dots (q^m - (-1)^m)}.$$

Consider two situations:

Suppose n is odd. Then L contains the projective image, g, of

$$g^* = \begin{pmatrix} -1 & & & \\ & \ddots & & \\ & & -1 & \\ & & & 1 \end{pmatrix}.$$

Then g is centralized in $\mathrm{U}(n,q)$ by $\hat{}\mathrm{GU}(n-1,q)$. Furthermore, as in Lemma 31, g has the same form, under the basis $\mathcal{B}$, as above and so is centralised by $\langle \sigma, \varphi \rangle$. Hence $n_g|(q^{n-1})(q^{n-1} - \cdots - q + 1)$. Thus $|v|_p \leq q^{n-1}$. Suppose that $m \geq 2$, in which case $|L : B_m|$ is divisible by $q^{2(n-2)}$. Thus we need $2(n-2) \leq n-1$ which gives $n \leq 3$. For $n = 3$ we know that $m = 1$. Thus, in general, $L_\alpha \leq B_1 = \hat{}\mathrm{GU}(n-1,q)$. Furthermore L_α contains a Sylow p-subgroup of $\hat{}\mathrm{GU}(n-1,q)$.

Thus either $L_\alpha \geq \hat{}\mathrm{SU}(n-1,q)$ or L_α lies in a parabolic subgroup of $\hat{}\mathrm{GU}(n-1,q)$. But $(q+1)$ divides $|\hat{}\mathrm{GU}(n-1,q) : P|$ for P a parabolic subgroup of $\hat{}\mathrm{GU}(n-1,q)$ which is impossible. Thus $L_\alpha \geq \hat{}\mathrm{SU}(n-1,q)$ and L_α contains all the involutions of $\hat{}\mathrm{GU}(n-1,q)$.

Now, for $n > 3$, consider a different involution g as in Lemma 31. Then $n_g = q^{2(n-2)} \frac{(q^n+1)(q^{n-1}-1)}{(q+1)(q^2-1)}$ and $r_g \geq r_g(\hat{}\mathrm{GU}(n-1,q)) = q^{2(n-3)} \frac{(q^{n-1}-1)(q^{n-2}+1)}{(q+1)(q^2-1)}$. This implies that $\frac{n_g}{r_g} \leq q^4$ and so $\frac{n_g}{r_g} \leq q^4 - q^2 + 1$ and $v < q^8 + q^4 + 1$. But $|L : B_1| = q^{n-1}(q^{n-1} - \cdots - q + 1)$ which is greater than $q^8 + q^4 + 1$ for $n \geq 7$. For $n = 5$, $2|\,\mathrm{U}(5,q) : B_1| > q^8 + q^4 + 1$ and so have $L = \mathrm{U}(5,q)$, $L_\alpha = B_1$ and $v = q^4(q^4 - q^3 + q^2 - q + 1)$. But, since $q^4 > \sqrt{v}$, this implies that $d_g = q^4$ which contradicts Lemma 10.

For $n = 3$ there is a unique conjugacy class of involutions of size $q^2(q^2 - q + 1)$. Since $\hat{}\mathrm{SU}(2,q) \leq L_\alpha \leq \hat{}\mathrm{GU}(2,q)$, L_α must contain precisely the involutions lying in $\hat{}\mathrm{GU}(2,q)$ of which there are $q^2 - q + 1$. Then $\frac{n_g}{r_g} = q^2$ which contradicts Lemma 10.

Suppose n is even and let g be as in the proof of Lemma 31. Now $|\,\mathrm{U}(n,q) : B_1|$ is even and thus $L_\alpha < B_2 \cong \hat{}(\mathrm{SU}(n-2,q) \times \mathrm{SU}(2,q)).(q+1)$ and, since $|v|_p \leq q^{2(n-2)}$, L_α contains a Sylow p-subgroup of $\hat{}(\mathrm{SU}(n-2,q) \times \mathrm{SU}(2,q))$. Note that, since B_2 is non-maximal in $L = \mathrm{U}(4,q)$, we may assume that $n \geq 6$.

Now the index of the parabolic subgroups of $\mathrm{SU}(n-2,q)$ in $\mathrm{SU}(n-2,q)$ is even. Hence we must have $L_\alpha > \hat{}\mathrm{SU}(n-2,q)$. For some α, we may assume that

$$L_\alpha \geq \hat{} \begin{pmatrix} \mathrm{SU}(n-2,q) & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

Now g is centralized in L by some conjugate of B_2. This implies that

$$n_g = q^{2(n-2)} \frac{(q^n-1)(q^{n-1}+1)}{(q+1)(q^2-1)} \quad \text{and} \quad r_g \geq q^{2(n-4)} \frac{(q^{n-2}-1)(q^{n-3}+1)}{(q+1)(q^2-1)}.$$

Thus $\frac{n_g}{r_g} \leq q^6(q^2+1)$ and $v \leq q^{16} + q^{15}$ and, for $n \geq 8$, this contradicts $L_\alpha \leq B_2$.

We are left with the possibility that $n = 6$. But $2|\,\mathrm{U}(6,q) : B_2| > q^{16} + q^{15}$, thus $L_\alpha = B_2$ and $v = q^8(q^4 + q^2 + 1)(q^4 - q^3 + q^2 - q + 1)$. But then $q^8 \geq \sqrt{v}$ and so $d_g = q^8$ which contradicts Lemma 10.

Thus Proposition 30 is proven.

9 $L = \mathrm{PSp}(n, q)$

In this section we prove that, if $L = \mathrm{PSp}(n,q)$, then the hypothesis in Section 4.3 leads to a contradiction. This implies the following proposition:

Proposition 32. *Suppose G contains a minimal normal subgroup isomorphic to $\mathrm{PSp}(n,q)$ with $n \geq 4$. Then G does not act transitively on a projective plane.*

We know [23, Proposition 2.4.1] that our symplectic geometry (V, κ) has a symplectic basis. Unless stated otherwise, we will write all matrix representations of $\mathrm{Sp}(n,q)$ according to this basis, $\{e_1, f_1, \ldots, e_m, f_m\}$, where $n = 2m$. Here $\kappa(e_i, e_j) = \kappa(f_i, f_j) = 0$ and $\kappa(e_i, f_j) = \delta_{ij}$.

By Liebeck and Saxl [26], we know that L_α lies inside a maximal subgroup M where

- for q odd, $M \in \mathcal{C}_1, \mathcal{C}_2$ or $M = N_{\mathrm{PSp}(n,q)}(\mathrm{PSp}(n, q_0))$ or $n = 4$;
- for q even, $M \in \mathcal{C}_1$.

Note that when $n = 4$ we can assume that $q > 3$ since $\mathrm{PSp}(4,3) \cong \mathrm{U}(4,2)$ which has already been covered.

Lemma 33. *L_α lies inside a maximal subgroup from family $\mathcal{C}_1$.*

Proof. Assume that q is odd and that $L_\alpha \leq M$ where M is a maximal subgroup of $\mathrm{PSp}(n,q)$ that does not lie in $\mathcal{C}_1$. Observe that in $\mathrm{PSp}(n,q)$ there exists a subgroup $B \cong \mathrm{Sp}(2,q) \circ \mathrm{Sp}(n-2,q)$.

For $n \neq 4$, by [23, Lemma 3.2.1 and Table 3.5.c], B is normal in a $\mathrm{P\Gamma Sp}(n,q)$-maximal subgroup B_Γ such that $|\,\mathrm{P\Gamma Sp}(n,q) : B_\Gamma| = |L : B|$. Thus, for $n \neq 4$, the involution $g \in Z(B)$ has $n_g = |L : B| = q^{n-2}(q^{n-2} + \cdots + q^2 + 1)$.

When $n = 4$ the same argument applies to $B \cong (\mathrm{Sp}(2,q) \circ \mathrm{Sp}(2,q)).2$ and the involution $g \in Z(B)$ has $n_g = \frac{1}{2}q^2(q^2+1)$.

Therefore the highest power of p in v is at most q^{n-2}. The lowest index of p among maximal subgroups $M \in \mathcal{C}_2$ or $M = N_{\mathrm{PSp}(n,q)}(\mathrm{PSp}(n,q_0))$ is $q^{\frac{1}{8}n^2}$. This implies that $n - 2 \geq \frac{1}{8}n^2$ which is a contradiction for $n > 4$.

Now suppose that M is maximal in $\mathrm{PSp}(4,q)$, $M \notin \mathcal{C}_1$, $|\,\mathrm{PSp}(4,q) : M|$ is odd and $|\,\mathrm{PSp}(4,q) : M|_p \leq q^2$. We must have $M = (\mathrm{Sp}(2,q) \circ \mathrm{Sp}(2,q)).2$. Then $L_\alpha \leq M$ and $L_\alpha \geq P$ for some P a Sylow p-subgroup of M. Since the parabolic subgroups of $\mathrm{Sp}(2,q)$ have even index in $\mathrm{Sp}(2,q)$ we must have $L_\alpha = (\mathrm{Sp}(2,q) \circ \mathrm{Sp}(2,q)).2$.

Now we can choose α such that

$$L_\alpha = \hat{}\left\langle \begin{pmatrix} \mathrm{Sp}(2,q) & \\ & \mathrm{Sp}(2,q) \end{pmatrix}, h^* := \begin{pmatrix} & I_{2\times 2} \\ I_{2\times 2} & \end{pmatrix} \right\rangle.$$

Observe that h is conjugate to g in $\mathrm{PSp}(4,q)$.

Now h has at least $\frac{1}{2}q^2(q^2-1)$ L_α-conjugates in L_α, thus $\frac{n_g}{r_g} \leq \frac{\frac{1}{2}q^2(q^2+1)}{\frac{1}{2}q(q^2-1)} \leq 2q$. Then $v \leq 8q^2$. But $v > |L : L_\alpha| = \frac{1}{2}q^2(q^2+1)$ which is a contradiction for $q > 3$.

Hence in all cases $M \in \mathcal{C}_1$. $\square$

In $\mathcal{C}_1$ we have subgroups of two types:

- Parabolic subgroups, $P_m \cong [q^a].(\frac{q-1}{(q-1,2)}).(\mathrm{PGL}(m,q) \times \mathrm{PSp}(n-2m,q))$ where $1 \leq m \leq \frac{n}{2}, a = \frac{m}{2} - \frac{3m^2}{2} + mn$. If $L_\alpha \leq P_m$ then $(q+1) \big| |\,\mathrm{PSp}(n,q) : P_m|$ divides v. Hence we must have $p = 2$.
- Subgroups, B_m, of type $\mathrm{Sp}_m \perp \mathrm{Sp}_{n-m}$ isomorphic to $\mathrm{Sp}(m,q) \circ \mathrm{Sp}(n-m,q)$ where $2 \leq m < \frac{n}{2}$ and m is even. In this case q^2 divides $|\,\mathrm{PSp}(n,q) : B_m|$ which in turn divides v. Hence we must have $p \equiv 1(3)$.

9.1 Case $\boldsymbol{p = 2, L_\alpha \leq P_m}$. The index of P_m in $\mathrm{Sp}(n,q)$ is divisible by q^2+1 for all $m > 1$ which is impossible and so $m = 1$. Then $P_1 \cong [q^{n-1}] : ((q-1) \times \mathrm{Sp}(n-2,q))$ and $|\,\mathrm{Sp}(n,q) : P_1| = (q+1)(q^{n-2} + \cdots + q^2 + 1)$. We conclude that $q \equiv 2(3)$ and that every prime dividing $\frac{n}{2}$ is congruent to $1(3)$. Hence $n \geq 14$ and $n \equiv 2(4)$. This implies that $n - 2 \equiv 0(4)$ and every parabolic subgroup of $\mathrm{Sp}(n-2,q)$ has index divisible by q^2+1. Thus $L_\alpha = [q^{n-1}] : (A \times \mathrm{Sp}(n-2,q))$ for some $A \leq q-1$.

Now consider $\mathrm{Sp}(n,q)$ acting on a vector space V preserving a symplectic form κ. For $u \in V, a \in \mathrm{GF}(q)$ we have transvections in $\mathrm{Sp}(n,q)$ defined by,

$$g_{a,u} : V \to V, v \mapsto v + a\kappa(v,u)u.$$

Set $W = \langle u \rangle$ and let $X_{W,W^\perp} = \{g_{a,u} : a \in \mathrm{GF}(q)\}$. Then $X_{W,W^\perp} < \mathrm{Sp}(n,q)$ is of size q. The parabolic subgroup of $\mathrm{Sp}(n,q)$ which preserves W normalizes $X_{W,W^\perp}$.

Now let $g = g_{1,u}$. Then, since the only maximal subgroup whose order is divisible by $\frac{|P_1|}{q-1}$ is P_1, we have

$$n_g \leq \frac{|\,\mathrm{Sp}(n,q)|}{|P_1|}(q-1)\log_2 q.$$

Similarly $r_g \geq \frac{|\,\mathrm{Sp}(n-2,q)|}{|P_1^*|}$ where P_1^* is a parabolic subgroup of $\mathrm{Sp}(n-2,q)$. Then

$$\frac{n_g}{r_g} \leq \frac{|\,\mathrm{Sp}(n,q)||P_1^*|(q-1)\log_2 q}{|\,\mathrm{Sp}(n-2,q)||P_1|} \leq q^4.$$

Thus $v \leq q^9$ which contradicts $n \geq 14$ and this case is excluded.

9.2 Case $\boldsymbol{p \equiv 1(3), L_\alpha < B_m}$. We know that the maximum power of p in v is at most q^{n-2}. Now $|\,\mathrm{PSp}(n,q) : B_m|_p = q^{n^2/4 - m^2/4 - (n-m)^2/4}$. Thus we need,

$$n - 2 \geq \frac{1}{4}(n^2 - m^2 - (n-m)^2) = \frac{1}{2}m(n-m).$$

This implies that $m = 2$ and so $L_\alpha \leq \mathrm{Sp}(2,q) \circ \mathrm{Sp}(n-2,q)$. If $n = 4$ then B_2 is not maximal and so we assume that $n > 4$. Furthermore we know that L_α must contain a

Sylow p-subgroup of $\mathrm{Sp}(2,q)\circ\mathrm{Sp}(n-2,q)$. But the indices of a parabolic subgroup of $\mathrm{Sp}(2,q)$ in $\mathrm{Sp}(2,q)$ and of a parabolic subgroup of $\mathrm{Sp}(n-2,q)$ in $\mathrm{Sp}(n-2,q)$ are both divisible by $q+1$, hence are even. Thus we conclude that $L_\alpha=\mathrm{Sp}(2,q)\circ\mathrm{Sp}(n-2,q)$.

Now $r_g\geq\frac{1}{2}q^{n-4}(q^{n-4}+\cdots+q^2+1)$ and so $\frac{n_g}{r_g}\leq 2q^2(q^2+1)$ and $v\leq 8q^4(q^2+1)^2$. But $v>|L:L_\alpha|=q^{n-2}(q^{n-2}+\cdots+q^2+1)$ which is a contradiction for $n>6$.

Thus we must assume that $n=6$ and $|L:L_\alpha|=q^4(q^4+q^2+1)$ and $\frac{n_g}{r_g}\leq 2q^2(q^2+1)$. If $|\frac{n_g}{r_g}|_p=|v|_p\geq q^4$ then $\frac{n_g}{r_g}=q^4$ which contradicts Lemma 10. Thus $|\frac{n_g}{r_g}|_p=1$ and so $\frac{n_g}{r_g}\big|q^4+q^2+1$. If $\frac{n_g}{r_g}=q^4+q^2+1$ then d_g is not divisible by q^4 which contradicts the fact that $|L:L_\alpha|$ divides v. If $\frac{n_g}{r_g}<\frac{1}{2}(q^4+q^2+1)$ then $v<|L:L_\alpha|$ which is also a contradiction.

10 $L=\Omega(n,q)$, nq odd

Throughout the next two sections, Greek letters such as ε,η and ζ will stand for either $+,-$ or $\circ$. We will write polynomials such as $x-\varepsilon$ to mean $x-\varepsilon 1$. We write $\Omega^\circ(n,q)$ to mean $\Omega(n,q)$ when n is odd.

In this section we assume that $n\geq 7$ and q is odd and we prove that, if $L=\Omega(n,q)$, then the hypothesis in Section 4.3 leads to a contradiction. This implies the following proposition:

Proposition 34. *Suppose that n is odd, $n\geq 7$ and G has a minimal normal subgroup isomorphic to $\Omega(n,q)$. Then G does not act transitively on a projective plane.*

Observe that L contains $\Omega^\varepsilon(n-1,q).2$ for $\varepsilon=-$ and $\varepsilon=+$. One of these groups contains a central involution and hence L contains an involution g such that $r_g(L)=\frac{1}{2}q^{\frac{n-1}{2}}(q^{\frac{n-1}{2}}+\varepsilon)$. Examining [23, Table 3.5.D] for fusion of conjugacy classes, we see that $n_g=r_g(L)$ and thus $|v|_p\leq q^{\frac{n-1}{2}}$.

We begin by proving that L_α must lie in a maximal subgroup $M\in\mathcal{C}_1$:

Lemma 35. *L_α does not lie inside a subgroup $M\in\mathcal{C}_i, i>1$.*

Proof. We examine the list of odd index maximal subgroups in G as given by Liebeck and Saxl [26]. The following possibilities are available for a maximal subgroup M of odd index. We exclude them in turn.

- $L=\Omega(7,q)$ and $M=\Omega(7,2)$. We know that $|v|_p\leq q^3$ and so $|L_\alpha|$ must be divisible by q^6. This is impossible for $L_\alpha\leq M$.
- $M\in\mathcal{C}_2$ or $M=N_{\Omega(n,q)}(\Omega(n,q_0))$ where $q=q_0^c$ for c an odd prime. In both cases $|M|_p\leq\sqrt{|\Omega^\epsilon(n,q)|_p}$. Now $|\Omega^\varepsilon(n,q)|_p=q^{\frac{1}{4}(n-1)^2}$ and so we must have,

$$\frac{1}{8}(n-1)^2+\frac{1}{2}(n-1)\geq\frac{1}{4}(n-1)^2.$$

This is impossible for $n\geq 7$. $\square$

Thus L_α lies inside a parabolic subgroup or a subgroup B_m of type $O(m,q) \perp O^\eta(n-m,q)$ for some odd $m < n$. In fact parabolic subgroups have even index in $P\Omega(n,q)$ hence we may assume that $L_\alpha \le B_m$ for some m.

Since $|v|_p \le q^{\frac{n-1}{2}}$ we know that $L_\alpha \le B_1 = \Omega^\eta(n-1,q).2$ and that L_α contains a Sylow p-subgroup of $\Omega^\eta(n-1,q)$. Now the parabolic subgroups of $\Omega^\eta(n-1,q)$ have even index. Hence we must have $L_\alpha = \Omega^\eta(n-1,q)$ and v is divisible by $|\Omega(n,q) : \Omega^\eta(n-1,q).2| = \frac{1}{2}q^{\frac{n-1}{2}}(q^{\frac{n-1}{2}}+\eta)$.

Now consider the involution h centralized in L by $(\Omega^\zeta(2,q)\times\Omega(n-2,q)).[4]$. Then $n_h = \frac{q^{n-2}(q^{n-1}-1)}{2(q-\zeta)}$. Now $\Omega^\eta(n-1,q)$ contains a conjugate of h centralized by, at most, $(\Omega^\zeta(2,q)\times\Omega^{\zeta\eta}(n-3,q)).[4]$. then $r_h \ge \frac{q^{n-3}(q^{\frac{n-3}{2}}+\eta\zeta)(q^{\frac{n-1}{2}}-\eta)}{2(q-\zeta)}$. This implies that $\frac{n_h}{r_h} \le q(q+1)$ and so $v \le 2q^2(q+1)^2$. But then $v < |L : L_\alpha|$ which is a contradiction.

Hence we have proved Proposition 34.

11 $L = P\Omega^\varepsilon(n,q)$, n even

In this section we assume that $n \ge 8$ and we prove that, if $L = P\Omega^\varepsilon(n,q)$, then the hypothesis in Section 4.3 leads to a contradiction. This implies the following proposition:

Proposition 36. *Suppose that n is even, $n \ge 8$ and G has a minimal normal subgroup isomorphic to $P\Omega^\varepsilon(n,q)$. Then G does not act transitively on a projective plane.*

First we examine what happens when $p = 2$:

Lemma 37. *Suppose $n \ge 8$ is even and G has a minimal normal subgroup isomorphic to $P\Omega^\varepsilon(n,2^a)$. Then G does not act transitively on a set of size x^2+x+1.*

Proof. Write $q = 2^a$. We know that $L_\alpha \le P_m$ for some integer m. If $m > 1$ then q^b+1 divides $|P\Omega^\varepsilon(n,q) : P_m|$ where b is some even integer. Since $q^b+1 \equiv 2(3)$ this is impossible. Thus L_α lies inside some parabolic subgroup P_1. Now

$$|P\Omega^\varepsilon(n,q) : P_1| = \frac{(q^{\frac{n}{2}}-\epsilon)(q^{\frac{n-2}{2}}+\epsilon)}{q-1}.$$

If $q \equiv 2(3)$ then $q^{\frac{n-2}{2}}+1 \equiv q^{\frac{n}{2}}+1 \equiv 2(3)$. Since one of these divides $|P\Omega^\varepsilon(n,q) : P_m|$, this is impossible. Hence $q \equiv 1(3)$. Now let n_e be the even one of $\frac{n}{2}$ and $\frac{n-2}{2}$, n_o the odd one. Then one of the following holds:

- $|\Omega^\varepsilon(n,q) : P_1| = \frac{q^{n_e}-1}{q-1}(q^{n_0}+1)$ and 9 divides $|\Omega^\varepsilon(n,q) : P_1|$; or
- $|\Omega^\varepsilon(n,q) : P_1| = \frac{q^{n_o}-1}{q-1}(q^{n_e}+1)$ and $q^{n_e}+1 \equiv 2(3)$.

Both of these cases are impossible. □

Throughout the rest of the section p is an odd prime. Now L contains maximal subgroups in $\mathcal{C}_1$ of type $O^\zeta(2,q) \perp O^\eta(n-2,q)$ for $\zeta\eta = \varepsilon$. One of these groups contains a central involution and hence L contains an involution g with $|L : C_L(g)| =$

$\frac{1}{2}q^{n-2}(q^{\frac{n-2}{2}}+\eta)(q^{\frac{n}{2}}-\epsilon)/(q-\zeta)$. Examining for fusion of conjugacy classes in [23, Tables 3.5.E and 3.5.F] we see that, except when $(n,\varepsilon)=(8,+)$, $n_g=|L:C_L(g)|$. When $(n,\varepsilon)=(8,+)$, we know that $n_g\leq 3|L:C_L(g)|$ and so, in all cases, $|v|_p\leq q^{n-2}$.

We begin by proving that L_α must lie in a maximal subgroup $M\in\mathcal{C}_1$:

Lemma 38. *L_α does not lie inside a subgroup $M\in\mathcal{C}_i, i>1$.*

Proof. We examine the list of odd index maximal subgroups in G as given by Liebeck and Saxl [26]. The following possibilities are available for a maximal subgroup of odd index $M\notin\mathcal{C}_1$. We exclude them in turn.

- $L=P\Omega^+(8,q)$ and either $M=\Omega^+(8,2)$ or $M=2^3.2^6.\,\mathrm{PSL}(3,2)$. We know that $|v|_p\leq q^6$ and so $|L_\alpha|_p\geq q^6$. This is impossible for $L_\alpha\leq M$ in both cases.
- $M\in\mathcal{C}_2$ or $M=N_{P\Omega^\varepsilon(n,q)}(P\Omega^\varepsilon(n,q_0))$ where $q=q_0^c$ for c an odd prime. In both cases $|M|_p\leq\sqrt{|P\Omega^\epsilon(n,q)|_p}$. Now $|P\Omega^\varepsilon(n,q)|_p=q^{n(n-2)/4}$ and so we must have

$$\frac{1}{8}n(n-2)+n-2\geq\frac{1}{4}n(n-2).$$

  This is impossible for $n>8$. When $n=8$, no subgroup M of odd index has $|M|_p\geq 6$ so the result stands. □

Thus L_α lies inside a parabolic subgroup P_m or a subgroup B_m of type $O(m,q)^{\zeta_1}\perp O^{\eta_1}(n-m,q)$ for some $m<\frac{n}{2}$. In fact parabolic subgroups have even index in $P\Omega^\varepsilon(n,q)$ for p odd. Hence we assume that $L_\alpha\leq B_m$ for some integer m. We know that $|v|_p\leq q^{n-2}$ and so $|P\Omega^\varepsilon(n,q):B_m|_p\leq q^{n-2}$. This implies that $m=1$ or $m=2$. Note also that $p\equiv 1(3)$.

Suppose first that $L_\alpha\leq B_2$ where B_2 is of type $O^{\zeta_1}(2,q)\perp O^{\eta_1}(n-2,q)$ for $\zeta_1\eta_1=\varepsilon$. Then $|P\Omega^\varepsilon(n,q):B_2|=\frac{1}{2}q^{n-2}(q^{\frac{n-2}{2}}+\eta_1)(q^{\frac{n}{2}}-\epsilon)/(q-\zeta_1)$ and so L_α must contain a Sylow p-subgroup of B_2. Since the parabolic subgroups of $P\Omega^{\eta_1}(n-2,q)$ have even index we must have $L_\alpha>\Omega^{\eta_1}(n-2,q)$.

In the case where $L_\alpha\leq B_1$ then $L_\alpha\leq\Omega(n-1,q).c_1$ where $c_1\in\{1,2\}$. Now $|P\Omega^\varepsilon(n,q):B_1|_p=q^{\frac{n-2}{2}}$ hence $|B_1:L_\alpha|_p\leq q^{\frac{n-2}{2}}$. Examining the proof of Lemma 35 this means that $L_\alpha\cap\Omega(n-1,q)$ lies inside a maximal subgroup of $\Omega(n-1,q)$ in family $\mathcal{C}_1$.

Since the parabolic subgroups of $\Omega(n-1,q)$ have even index in $\Omega(n-1,q)$ this means that $L_\alpha\cap\Omega(n-1,q)\leq B^*_{m_1}$; here $B^*_{m_1}$ is a maximal subgroup of $\Omega(n-1,q)$ of type $O_{m_1}(q)\perp O^\gamma(n-1-m_1,q)$ for some odd $m_1<n-1$. In fact $|B_1:L_\alpha|_p\leq q^{\frac{n-2}{2}}$ implies that $m_1=1$ and that L_α contains a Sylow p-subgroup of $B_1^*=\Omega^{\eta_1}(n-2,q).c_2$ where $c_2\in\{1,2\}$. Once again, since the parabolic subgroups of $\Omega^{\eta_1}(n-2,q)$ have even index we must have $L_\alpha>\Omega^{\eta_1}(n-2,q)$.

Thus in both cases, when $m=1$ and when $m=2$, we see that $L_\alpha>\Omega^{\eta_1}(n-2,q)$ is a subgroup of $P\Omega^\varepsilon(n,q)$ which preserves a decomposition of the associated vector space V into subspaces, $V=W_2\perp W_{n-2}$, where $\dim W_i=i$ and the W_i are non-degenerate subspaces of V.

Then $H = \Omega^{\eta_1}(n-2,q)$ contains h a conjugate of g, and $C_H(h)$ is isomorphic to either $(\Omega^{\gamma_1}(2,q) \times \Omega^{\gamma_2}(n-4,q)).2$ or $2.(P\Omega^{\gamma_1}(2,q) \times P\Omega^{\gamma_2}(n-4,q)).[4]$ (see [23, Proposition 4.1.6]). In either case $r_g \geq \frac{1}{2}q^{n-4}(q^{\frac{n-4}{2}} + \gamma_2)(q^{\frac{n-2}{2}} - \eta_1)/(q-\gamma_1)$.

If $n > 8$ this means that $\frac{n_g}{r_g} \leq \frac{q^2(q+1)^3}{(q-1)^2}$ and so $v \leq 2q^4(q+1)^4$. Since $|L : L_\alpha| < v$ we must have $n = 10, q = 7$ and $L_\alpha = B_1$. But then $|L : B_1|$ is divisible by $\frac{1}{2}7^4(7^5 \pm 1)$. This is impossible since then $|L : B_1|$ is divisible by a prime $s \equiv 2(3)$.

If $n = 8$ then $\frac{n_g}{r_g} < 4q^2(q+1)^2$. Then $v < 28q^4(q+1)^4$ which is less than $|L : B_2|$. Thus $L_\alpha = B_1$. But then $|L : L_\alpha|$ is even which is a contradiction.

Proposition 36 is now proven.

12 *L* is an exceptional group of Lie type in odd characteristic

In this section we prove that, if L is an exceptional group of Lie type in odd characteristic, then the hypothesis in Section 4.3 leads to a contradiction. This implies the following proposition:

Proposition 39. *Suppose that G has a minimal normal subgroup L where L is an exceptional group of Lie type in odd characteristic or that G has a unique component L such that $L^\dagger$ is isomorphic to a simple group $E_6(q)$ or ${}^2E_6(q)$ where q is odd. Then G does not act transitively on a projective plane.*

We introduce some extra notation for this section and the following one. We will write E_6^- for ${}^2E_6$, E_6^+ for E_6. Similarly SL^- will stand for SU, SL^+ for SL. We will use ε to denote either ± 1 or $\pm$ depending on the context. Generally our notation refers to the adjoint version of the exceptional group, any variation on this will be specified. For a group G, we will write $\frac{1}{2}G$ to mean a subgroup in G of index 2. We define $P(G) := \min\{|G : H| : H < G\}$. Finally, for a group H we write $O^{p'}H$ to mean the unique smallest normal subgroup N of H such that $|H/N|_p = 1$.

We have eight possibilities for L which we will examine in turn. As usual we will examine odd-index maximal subgroups of L, treating these as candidates to contain a stabilizer L_α, and seek to show a contradiction.

We immediately exclude the case where $L = {}^2G_2(q)$, $q > 3$, by examining the list of maximal subgroups of ${}^2G_2(q)$ given in [24, Theorem C] (see also [35]). We see that any maximal subgroup of odd index must have index divisible by 9 and hence cannot contain a point-stabilizer. Hence this case is excluded. Note that the list given by Kleidman [24] contains a maximal subgroup of odd index (with structure $(2^2 \times D_{\frac{1}{2}(q+1)}) : 3$) which has been omitted by Liebeck and Saxl [26] and by Kantor [22].

For the remaining cases we will refer to the results of Liebeck and Saxl [26] giving the maximal subgroups $M^\dagger$ of odd index in $L^\dagger$. These maximal subgroups $M^\dagger$ take one of two forms: Either $M^\dagger = N_{L^\dagger}(L^\dagger(q_0))$, where $q = q_0^a$ for a an odd prime and the subgroup $L^\dagger(q_0)$ of $L^\dagger(q)$ corresponds to the centralizer of a field automorphism of $L^\dagger(q)$ (see [22, Theorem C]), or $M^\dagger$ is enumerated in [26, Table 1].

Note that, by [23, Table 5.1.B], $\mathrm{Out}\, L$, the outer automorphism group of L, has order strictly less than q provided $L \neq {}^3D_4(3), {}^2E_6(5)$. We also use the following lemma:

Lemma 40. *Let φ be a field automorphism of $L(q)$ of prime order a. Let $L(q_0) = O^{p'}C_{L(q)}(\varphi)$ where $q = q_0^a$. Then $N_{L(q)}(L(q_0)) \lesssim \mathrm{Inndiag}(L(q_0))$ and, furthermore, $\mathrm{Inndiag}(L(q_0)) = L(q_0).d$ where*

$$d = \begin{cases} (3, q_0 - \varepsilon) & L = E_6^\varepsilon \\ (2, q_0 - 1) & L = E_7 \\ 1 & \textit{otherwise} \end{cases}$$

Proof. Our notation is consistent with that in [19]. Write $L(q) = O^{p'}C_{\overline{L}}(\sigma)$ where $\overline{L}$ is a simple adjoint $\overline{\mathbb{F}_p}$-algebraic group, $\overline{\mathbb{F}_p}$ is the algebraic closure of $\mathrm{GF}(q)$ and σ is a Steinberg automorphism [19, Definition 2.2.1].

By [19, Proposition 2.5.17], there exists a Steinberg automorphism τ of $\overline{L}$ such that $\tau^a = \sigma$ and τ induces φ on L. Then $L(q_o) = O^{p'}C_{\overline{L}}(\tau)$ and, by [19, Proposition 2.5.9], $N_{\overline{L}}(L(q_0)) = C_{\overline{L}}(\tau)$. Thus $N_{L(q)}(L(q_0)) = C_{L(q)}(\tau) \leq C_{L(q)}(\varphi) \lesssim \mathrm{Inndiag}(L(q_0))$ by [19, Proposition 4.9.1]. The structure of the group $\mathrm{Inndiag}(L(q_0))$ is given in [19, Theorem 2.5.12]. □

12.1 Case $L = E_8(q)$. Referring to [19, Table 4.5.1], we see that $E_8(q)$ contains an involution g such that $C_L(g) \geq 2.(\mathrm{PSL}(2,q) \times E_7(q))$. There is one such $E_8(q)$ conjugacy class of involutions in L and so n_g divides

$$2q^{56}(q^{10}+1)(q^{12}+1)(q^6+1)(q^{30}-1)(q^2-1)^{-1}.$$

Using Lemma 13 this implies that $|v|_p \leq q^{56}$ and hence that $|L_\alpha|_p \geq q^{64}$. The list in [26, Table 1] contains no maximal subgroups M such that $|M|_p \geq q^{64}$. Similarly Lemma 40 implies that $|N_L(E_8(q_0))|_p = |E_8(q_0)|_p = q_0^{120}$. Since $q = q_0^a$ where a is an odd prime, $q_0^{120} \leq q^{40}$ and so this possibility is excluded.

12.2 Case $L = E_7(q)$. Referring to [19, Table 4.5.1], we see that $E_7(q)$ contains an involution g such that $C_L(g)$ contains $\mathrm{SL}^\varepsilon(8,q)/(4, q-\varepsilon)$ for ε either $+$ or $-$. There is one such $\mathrm{Inndiag}(E_7(q))$ conjugacy class of involutions in L and so n_g divides

$$(4, q-1)q^{35}(q^7+\varepsilon)(q^5+\varepsilon)(q^3+\varepsilon)(q^8+q^4+1)(q^{12}+q^6+1).$$

This implies that $|v|_p \leq q^{35}$ and hence that $|L_\alpha|_p \geq q^{28}$. The list in [26, Table 1] contains one maximal subgroup such that $|M|_p \geq q^{28}$, namely $M = N_L(2.(\mathrm{PSL}(2,q) \times P\Omega^+(12,q))$. Then $|L:M|_p = q^{32}$ and so $p \equiv 1(3)$. But this implies that 9 divides $|L:M|$ and so it is not possible that $L_\alpha \leq M$.

Similarly Lemma 40 implies that $|N_L(E_7(q_0))|_p \leq |E_7(q_0).2|_p = q_0^{63}$. Since $q = q_0^a$ where a is an odd prime, $q_0^{63} \leq q^{21}$ and so this possibility is excluded.

12.3 Case $L^\dagger = E_6^\varepsilon(q)$. Referring to [19, Table 4.5.1], we see that L contains an involution g such that $C_L(g)$ contains $\mathrm{Spin}^\varepsilon_{10}(q)$. Here $\mathrm{Spin}^\varepsilon_{10}(q) \cong (4, q-\varepsilon).P\Omega^\varepsilon(10, q)$. There is only one such $\mathrm{Inndiag}(E_6^\varepsilon(q))$ conjugacy class of involutions in L and so,

$$n_g = q^{16}(q^6 + \varepsilon q^3 + 1)(q^2 + \varepsilon q + 1)(q^8 + q^4 + 1).$$

This implies that $|v|_p \leq q^{16}$ and hence that $|L_\alpha|_p \geq q^{20}$. Then Lemma 40 implies that $|N_{L^\dagger}(L^\dagger(q_0))|_p \leq |L^\dagger(q_0).(3, q-\varepsilon)|_p$ which divides $3q_0^{36}$. Since $q = q_0^a$ where a is an odd prime, $q_0^{36} \leq q^{12}$ and so this possibility is excluded.

12.3.1 Subcase $\varepsilon = +$. In this case the list in [26, Table 1] contains two maximal subgroups $M^\dagger$ such that $|M^\dagger|_p \geq q^{20}$: $M^\dagger = N_{L^\dagger}((4, q-1).P\Omega^+(10, q))$ or $M^\dagger$ is parabolic of type D_5. If $p \equiv 1(3)$ in either case then 9 divides $|L : M|$ which is a contradiction. Hence $p \not\equiv 1(3)$, the universal and adjoint versions coincide and L is simple.

In the non-parabolic case, $|L : M|_p > p^2$ which is impossible for $p \not\equiv 1(3)$. Hence M is a parabolic subgroup of $E_6^+(q)$ of type D_5 and we have $|L : M| = (q^6 + q^3 + 1)(q^2 + q + 1)(q^8 + q^4 + 1)$.

Now $M \cong [q^{16}] : (\mathrm{Spin}^+_{10}(q)H)$ where H is a Cartan subgroup of $E_6(q)$ and H normalizes $\mathrm{Spin}^+_{10}(q)$. Here $\mathrm{Spin}^+_{10}(q) \cong (4, q-1).P\Omega^+(10, q)$ and $P\Omega^+(10, q)$ has parabolic subgroups of even index. This implies that $L_\alpha \geq [q^{16}] : (\mathrm{Spin}^+_{10}(q).2)$ for $p \neq 3$.

Furthermore, for $p = 3$, every non-parabolic subgroup of $P\Omega^+(10, q)$ has index divisible by 9 [23]. This means that $L_\alpha \geq [\frac{q^{16}}{3}].(\mathrm{Spin}^+_{10}(q).2)$. Now E, the commutator subgroup of the Levi complement in M, is isomorphic to $\mathrm{Spin}^+_{10}(q)$ and $|E : L_\alpha \cap E|$ is at most $\frac{3}{2}(q-1)$. But $P(\mathrm{Spin}^+_{10}(q)) > \frac{3}{2}(q-1)$ [23, Table 5.2.A]. Thus $L_\alpha > E$.

Now if $q = 3^a$ then $|E|$ is divisible by $3^{8a} - 1$; in particular, $|E|$ is divisible by the primitive prime divisors of $3^{8a} - 1$. This implies that if $\varphi : E \to \mathrm{GL}(m, 3)$ is a non-trivial representation of E over $\mathrm{GF}(3)$ then $m \geq 8a$. Now consider the action of E on the unipotent radical of the full parabolic group, $[q^{16}]$, considered as a module over $\mathrm{GF}(3)$. We know that E does not act trivially on any submodule of the unipotent radical (otherwise $Z(E)$ would have too large a centralizer; see [19, Table 4.5.1]). Thus the action must be either irreducible or split into two modules both of size q^8. In either case we must have $L_\alpha \geq [q^{16}] : (\mathrm{Spin}^+_{10}(q).2)$.

We return to the general case where $p \not\equiv 1(3)$ and assume that M contains $C_L(g) = \mathrm{Spin}^+_{10}(q)H$. Furthermore we know that L acts on the cosets of M as a rank 3 permutation group with subdegrees $1, q(q^3+1)(q^8-1)/(q-1)$ and $q^8(q^4+1)(q^5-1)/(q-1)$ ([22]). Then we have two possibilities:

- Suppose $C_M(h) \geq \mathrm{Spin}^+_{10}(q)$ for all h in L_α where h is L-conjugate to g. Now if $M = [q^{16}] : C_L(g)$ then M contains q^{16} M-conjugates of $C_L(g)$ each containing a unique copy of $\mathrm{Spin}^+_{10}(q)$. Any other L-conjugate of $C_L(g)$ lies inside a non-trivial conjugate of M. But these intersect M with non-trivial indices as above. These intersections cannot contain $\mathrm{Spin}^+_{10}(q)$. Hence M contains only M-conjugates of g and, in fact, all these must lie in L_α. Thus $r_g = q^{16}$ and $\frac{n_g}{r_g} = (q^8 + q^4 + 1)(q^6 +$

$q^3+1)(q^2+q+1)$. Set

$$u = q^8 + \frac{1}{2}q^7 + \frac{3}{8}q^6 + \frac{5}{16}q^5\frac{99}{128}q^4 + \frac{127}{256}q^3 + \frac{423}{1024}q^2 + \frac{749}{2048}q + \frac{39587}{32768}.$$

Then $u^2-u+1 > \frac{n_g}{r_g}$ for $q \geq 47$. If we set $u_1 = u - \frac{1}{32768}$ then $u_1^2 - u_1 + 1 < \frac{n_g}{r_g}$ for $q > 1$. Thus we need to check $q < 47$ but no such q satisfies $u^2 - u + 1 = \frac{n_g}{r_g}$ for integer u.

- Suppose there exists h in L_α which is L-conjugate to g and $C_M(h)$ does not contain a copy of $\mathrm{Spin}_{10}^+(q)$. Then $C_L(h)$ lies inside a non-trivial conjugate of M. Hence $|M : C_M(h)|$ is a multiple of $q(q^3+1)(q^8-1)/(q-1)$ or $q^8(q^4+1)(q^5-1)/(q-1)$. Furthermore we know that q^{16} divides $|M : C_M(h)|$ since $|M|_p = q^{16}|C_L(g)|_p$. Hence $|M : C_M(h)| \geq q^{16}(q^4+1)(q^5-1)/(q-1)$.
  Now, if $L_\alpha \geq [q^{16}] : (\mathrm{Spin}_{10}^+(q).2)$ then $r_g = r_g(M)$ since $L_\alpha \trianglelefteq M$ and $|M : L_\alpha|$ is odd. Thus $r_g \geq q^{16}(q^4+1)(q^5-1)/(q-1)$ and $\frac{n_g}{r_g} < q^8+q^4+1$. Then $d_g \leq q^8+q^4+1 < (q^6+q^3+1)(q^2+q+1)$. Thus $v < |L : M|$ which is a contradiction.

12.3.2 Subcase $\varepsilon = -$. In this case the list in [26, Table 1] contains one maximal subgroup $M^\dagger$ in $L^\dagger$ such that $|M^\dagger|_p \geq q^{20}$, namely $M^\dagger = N_{L^\dagger}((4, q+1).P\Omega^-(10,q))$. In fact $|M|_p = q^{20}$ and so $p \equiv 1(3)$ and the universal and adjoint versions of E_6^- coincide and L is simple. Then $M = N_L(\mathrm{Spin}_{10}^-(q)) \cong \mathrm{Spin}_{10}^-(q).(q+1)$ ([19, Table 4.5.2]). Furthermore L_α must contain a Sylow p-subgroup of M. But the parabolic subgroups of $P\Omega_{10}^-(q)$ have even index, hence $\mathrm{Spin}_{10}^-(q).2 \leq L_\alpha \leq \mathrm{Spin}_{10}^-(q).(q+1)$.

Now, using [19, Table 4.5.2], we see that $E_6^-(q)$ contains two conjugacy classes of involutions: those conjugate to g, centralized by $\mathrm{Spin}_{10}^-(q)$, and those conjugate to g_1 say, centralized by $\mathrm{SL}(2,q) \circ \mathrm{SU}(6,q)$. Then $n_g = q^{16}(q^2-q+1)(q^6-q^3+1)(q^8+q^4+1)$ and $N_{g_1} = q^{20}(q^4+1)(q^2+1)(q^6-q^3+1)(q^8+q^4+1)$.

We examine the involutions lying in $\mathrm{Spin}_{10}^-(q)$ using [19, Table 4.5.2]. Apart from the central involution, $\mathrm{Spin}_{10}^-(q)$ contains two conjugacy classes of involutions. Let h be an involution in $\mathrm{Spin}_{10}^-(q)$ centralized by $\mathrm{Spin}_4^+(q) \circ \mathrm{Spin}_6^-(q)$. Then L_α contains at least $\frac{1}{4}q^{12}(q^4+q^3+q^2+q+1)(q^2-q+1)(q^4+1)(q^2+1)$ conjugates of h. If h is L-conjugate to g, then $\frac{n_g}{r_g} < 4q^8$ which is a contradiction. Thus assume that h is L-conjugate to g_1.

In this case $\frac{n_g}{r_g} \leq 4q^{16} + 4q^{12} + 4q^8$. Then

$$d_g < \frac{n_g}{r_g} + 2\sqrt{\frac{n_g}{r_g}} + 2 < 4q^{16} + 4q^{12} + 6q^8 + 2q^4 + 2.$$

This implies that $v < 19|L : M|$ for $q \geq 7$.

Suppose that q^{16} does not divide $\frac{n_g}{r_g}$. Then $\frac{n_g}{r_g}$ divides the product $(q^2-q+1)(q^6-q^3+1)(q^8+q^4+1)$ and so $d_g < 3q^{16}$ and $v = |L : M|$. This contradicts Lemma 11. Thus $v = 7|L : M|$ or $v = 13|L : M|$ and $q^{16} | \frac{n_g}{r_g}$.

If $\frac{n_g}{r_g} \geq 7q^{16}$ then $v > 49q^{32} > 13|L : M|$ which is a contradiction. Thus, by Lemma 10, $\frac{n_g}{r_g} = 3q^{16}$. This implies that $3q^{16} < d_g < 3q^{16} + 2\sqrt{3}q^8 + 2$ and so

$9q^{32} < v < 9q^{32} + 12q^{24} + 6q^{16}$. But then $7|L : M| < v < 13|L : M|$ which is a contradiction.

12.4 Case $L = {}^3D_4(q)$. We know that ${}^3D_4(q)$ has a single conjugacy class of involutions [19] which is centralized by a maximal subgroup isomorphic to $(\mathrm{SL}(2, q^3) \circ \mathrm{SL}(2, q)).2$ [25]. Hence, for g an involution in L, $n_g = q^8(q^8 + q^4 + 1)$ and so $|v|_p \leq q^8$ and $|L_\alpha|_p \geq q^4$.

If $L_\alpha < M = N_L({}^3D_4(q_0))$ then this condition implies that $q = q_0^3$. No such subfield subgroup exists.

There are two other odd index maximal subgroups M such that $|M|_p \geq q^4$; see [26]. The first possibility is that $M = G_2(q)$ and $|L : M|_p = q^6$. But then odd index subgroups of $G_2(q)$ have p-index strictly greater than q^2; see [26]. Thus $L_\alpha = G_2(q)$. Now $r_g(G_2(q)) = q^4(q^4 + q^2 + 1)$ and so $\frac{n_g}{r_g} = q^4(q^4 - q^2 + 1)$. But this implies that $|v|_p \leq q^4$ which is impossible.

The second possibility is that $L_\alpha \leq M = 2.(\mathrm{PSL}(2, q) \times \mathrm{PSL}(2, q^3)).2$. Then $|L : M| = q^8(q^8 + q^4 + 1)$ and so $p \equiv 1(3)$ and L_α contains a Sylow p-subgroup of M. But the parabolic subgroups of $\mathrm{PSL}(2, q)$ have even index, hence we conclude that $L_\alpha = M$.

Now $r_g(2.(\mathrm{PSL}(2, q) \times \mathrm{PSL}(2, q^3))) \geq 1 + \frac{1}{2}q^3(q^3 - 1)\frac{1}{2}q(q - 1)$. This implies that $\frac{n_g}{r_g} < 7q^8$. Suppose that $|\frac{n_g}{r_g}|_p = 1$ and hence $\frac{n_g}{r_g} \leq q^8 + q^4 + 1$. Then $d_g < 3q^8$ and so $d_g = q^8$. This contradicts Lemma 11.

Thus $|\frac{n_g}{r_g}|_p > 1$ and so we must have either $\frac{n_g}{r_g} = q^8$ (contradicting Lemma 10) or $\frac{n_g}{r_g} = 3q^8$. If $\frac{n_g}{r_g} = 3q^8$ then $d_g < \frac{13}{3}(q^8 + q^4 + 1)$ which is the smallest possibility for d_g that is larger than $\frac{n_g}{r_g}$. Thus we have a contradiction.

12.5 Case $L = G_2(q)$. Referring to [19, Table 4.5.1], we see that $G_2(q)$ contains an involution g such that $C_L(g)$ contains $\mathrm{SL}(2, q) \circ \mathrm{SL}(2, q)$. There is one such conjugacy class of involutions in L and, examining [24], we see that $C_L(g) \cong (\mathrm{SL}(2, q) \circ \mathrm{SL}(2, q)).2$. Hence $n_g = q^4(q^4 + q^2 + 1)$. Using Lemma 13, we may conclude that $|v|_p \leq q^4$ and hence that $|L_\alpha|_p > q^2$.

Examining the odd-index maximal subgroups [23], we find that all have p-index divisible by p^2 and so $p \equiv 1(3)$. We have a number of possibilities for M an odd-index maximal subgroup, $|M|_p \geq q^2$, M containing L_α:

- Suppose $M = N_L(G_2(q_0))$. Then using Lemma 40 we find that $q = q_0^3$. But this means that 9 divides $|L : M|$ which is impossible.
- Suppose $M = (\mathrm{SL}(2, q) \circ \mathrm{SL}(2, q)).2$. Then $L_\alpha \geq 2.P.2$ where P is a Sylow p-subgroup of $\mathrm{PSL}(2, q) \times \mathrm{PSL}(2, q)$. Since the parabolic subgroup of $\mathrm{PSL}(2, q)$ have even index we must have $L_\alpha = M$ and $v = q^4(q^4 + q^2 + 1)a$ for some integer a. Then Lemma 11 implies that $a \neq 1$ and so $a \geq 7$.
  Now $\mathrm{PSL}(2, q) \times \mathrm{PSL}(2, q)$ has at least $\frac{1}{4}q^2(q \pm 1)^2$ involutions and thus so does $\mathrm{SL}(2, q) \circ \mathrm{SL}(2, q)$. Then

$$\frac{n_g}{r_g} < 4q^2\frac{q^4 + q^2 + 1}{q^2 - 2q + 1} < 7q^4$$

for $q \geq 7$. Thus either $\frac{n_g}{r_g} = q^4$ (contradicting Lemma 10) or $\frac{n_g}{r_g} = 3q^4$ or $\frac{n_g}{r_g}$ divides $q^4 + q^2 + 1$.
If $u^2 - u + 1 = \frac{n_g}{r_g} = 3q^4$ then $u^2 + u + 1 = d_g < 3q^4 + 2\sqrt{3q^4} + 2 < 4q^4 + 4q^2 + 4$. This implies that $v < 12q^4(q^4 + q^2 + 1)$ and so $a = 7$. But then $d_g = \frac{7}{3}(q^4 + q^2 + 1)$ which is less than $\frac{n_g}{r_g}$ for $q \geq 7$. This is impossible.
If $u^2 - u + 1 = \frac{n_g}{r_g} = q^4 + q^2 + 1$ then $u = q^2 + 1$ and $d_g = q^4 + 3q^2 + 3$. But then $(v, p) = 1$ which is impossible. If $\frac{n_g}{r_g} < q^4 + q^2 + 1$ then $u \leq q^2$ which implies that $\frac{n_g}{r_g} \leq q^4 - q^2 + 1$ and $d_g \leq q^4 + q^2 + 1$. Then $\frac{n_g}{r_g} d_g < |L : M|$ which is a contradiction.

- Suppose $M = \mathrm{SL}^\varepsilon(3, q).2$ and so $p \equiv 1(3)$. Consider first the situation where $L_\alpha = M$. When $\varepsilon = +$, $M = \langle \mathrm{SL}(3, q), \varphi \rangle$ where φ is a graph automorphism [11, (2.6)]. When $\varepsilon = -$, $M \leq \mathrm{P\Gamma U}(3, q)$ [24, Proposition 2.2]. In both cases M is equal to a universal version of $A_2^\varepsilon(q)$ extended by a graph automorphism [19, Definition 2.5.13].
Examining [19, Table 4.5.2] we see that M has 2 conjugacy classes of involutions. These have size $q^2(q^2 + \varepsilon q + 1)$ and $q^2(q^2 + \varepsilon q + 1)(q - \varepsilon)$. When $\varepsilon = +$ this gives $r_g = q^3(q^2 + q + 1)$ and $\frac{n_g}{r_g} = q(q^2 - q + 1)$. This is impossible since either $|\frac{n_g}{r_g}|_p = 1$ or $|\frac{n_g}{r_g}|_p \geq q^3$. When $\varepsilon = -$ we have $r_g = q^2(q^2 - q + 1)(q + 2)$ and $\frac{n_g}{r_g} = \frac{q^2(q^2+q+1)}{q+2}$. This is not an integer for $q > 1$ hence can be excluded.
Thus we must have $L_\alpha < M$ and we know that $|M : L_\alpha|_p \leq q$. Examining the subgroups of $\mathrm{SL}^\varepsilon(3, q)$ we find that $L_\alpha \cap \mathrm{SL}^\varepsilon(3, q) \leq P_1$, a parabolic subgroup of $\mathrm{SL}^\varepsilon(3, q)$.
When $\varepsilon = -$, $|\mathrm{SL}^\varepsilon(3, q) : P_1|$ is even hence this possibility can be excluded.
When $\varepsilon = +$, $M = \langle \mathrm{SL}(3, q), m \rangle$ where m is a graph automorphism of $\mathrm{SL}(3, q)$. Since L_α has odd index in $G_2(q)$, L_α must contain a graph automorphism. Examining [23, Table 3.5.A] we find that $L_\alpha \cap \mathrm{SL}(3, q)$ lies inside a subgroup M_1 of $\mathrm{SL}(3, q)$ of type $\mathrm{GL}(2, q) \oplus \mathrm{GL}(1, q)$ or of type $P_{1,2}$. In the former case we find that $|v|_p \geq q^5$. Since $|n_g|_p = q^4$ we must have $|\frac{n_g}{r_g}|_p = 1$ which implies that $\frac{n_g}{r_g} \leq q^4 + q^2 + 1$ and $|d_g|_p \geq q^5$ which contradicts Lemma 12. In the latter case, we find that $|\mathrm{SL}(3, q) : M_1|$ is even and this case can be excluded.

We have covered all possible odd-index maximal subgroups in $G_2(q)$.

12.6 Case $L = F_4(q)$. Referring to [19, Table 4.5.1], we see that $F_4(q)$ contains an involution g such that $C_L(g)$ contains $\mathrm{Spin}(9, q)$. There is one such conjugacy class of involutions in L and so $n_g = q^8(q^8 + q^4 + 1)$.

This implies that $|v|_p \leq q^8$ and hence that $|L_\alpha|_p \geq q^{16}$. Then Lemma 40 implies that $|N_L(F_4(q_0))|_p \leq |F_4(q_0)|_p = q_0^{24}$. Since $q = q_0^a$ where a is an odd prime, $q_0^{24} \leq q^8$ and so L_α does not lie in $|N_L(F_4(q_0))$.

The list in [26, Table 1] contains one maximal subgroup M such that $|M|_p \geq q^{16}$. Then $M \cong 2.\Omega(9, q)$, L_α must contain a Sylow p-subgroup of M since $|L : M|_p = q^{16}$. Furthermore, $p \equiv 1(3)$. Now the parabolic subgroups of $\Omega(9, q)$ have even index, hence we may conclude that $L_\alpha = M$ and $v = q^8(q^8 + q^4 + 1)a$ for some integer a. Lemma 11 implies that $a \neq 1$ and hence $a \geq 7$.

Now suppose $r_g \geq \frac{1}{2}q^4(q^4-1)$. Then $\frac{n_g}{r_g} \leq 2q^4(q^4+3) < \frac{7}{3}q^8$. Then $d_g < \frac{14}{3}q^8$ and $v < 7q^{16}$ which is a contradiction. Also r_g is clearly greater than 1. Thus there is an involution $g \in 2.\Omega(9,q)$ such that

$$1 < |2.\Omega(9,q) : C_{2.\Omega(9,q)}(g)| < q^4(q^4-1)/2.$$

Now let B be the central subgroup of L_α of order 2, so that $L_\alpha/B \cong P\Omega(9,q)$. Let $h = Bg$ an involution in $P\Omega(9,q)$. Then we must have

$$|\Omega(9,q) : C_{\Omega(9,q)}(h)| < q^4(q^4-1)/2.$$

Examining [19, Table 4.5.1] we see that all involution centralizers in $\Omega(9,q)$ have index at least $\frac{1}{2}q^4(q^4-1)$. Hence we have a contradiction.

Proposition 39 is now proven.

13 L is an exceptional group of Lie type in characteristic 2

In this section we prove that, if L is an exceptional group of Lie type in characteristic 2, then the hypothesis in Section 4.3 leads to a contradiction. This implies the following proposition:

Proposition 41. *Suppose G has a minimal normal subgroup L where L is an exceptional group of Lie type in characteristic 2 or that G has a unique component L such that $L^\dagger$ is isomorphic to $E_6(q)$ or ${}^2E_6(q)$ where $q = 2^a$. Then G does not act transitively on a projective plane.*

We have nine possibilities for L and, by Tits' Lemma [31, 1.6], we know that L_α must lie in a parabolic subgroup M of L. We demonstrate that this is impossible, generally by showing a contradiction with Lemma 8.

13.1 Case $L = {}^3D_4(q), G_2(q), q > 2$. In each case, for any parabolic subgroup M, $|L : M|$ is divisible by $(q^4+q^2+1)(q+1)$. If $q \equiv 1(3)$ then $|L : M|$ is divisible by $q+1 \equiv 2(3)$, while if $q \equiv 2(3)$ then 9 divides $|L : M|$. Thus M cannot contain L_α (Lemma 8) and we are done.

13.2 Case $L = {}^2B_2(q), q > 2, {}^2F_4(q)', F_4(q), E_7(q), E_8(q)$. Examining the indices of the parabolic subgroups M in L in these cases, we find that they are nearly always divisible by q^m+1 for some even integer m. Since $q^m+1 \equiv 2(3)$ these cases are excluded. We deal with the exceptions which are as follows:

(1) $L = E_7(q)$ and M is of type E_6. Then $|L : M|$ is divisible by $(q^5+1)(q^9+1)$. If $q \equiv 1(3)$ then $q^5+1 \equiv 2(3)$ and if $q \equiv 2(3)$ then 9 divides $|L : M|$. Both of these are impossible hence M cannot contain L_α.

(2) $L = E_7(q)$ and M is of type D_6. Then $|L : M|$ is divisible by $(q^8+q^4+1)(q^{12}+q^6+1)$ which is in turn divisible by 9. Hence M cannot contain L_α.

(3) $L = E_7(q)$ and M is of type $D_5 \times A_1$. Then $|L : M|$ is divisible by $(q^5+1)(q^8+q^4+1)$. If $q \equiv 1(3)$ then $q^5+1 \equiv 2(3)$ and if $q \equiv 2(3)$ then 9 divides $|L : M|$. Both of these are impossible hence M cannot contain L_α.

Note that Kantor's argument to exclude the last two cases ($L = E_7(q)$ and M of type D_6 or $D_5 \times A_1$) when the action is primitive is incorrect [22].

13.3 Case $L^\dagger = E_6^\varepsilon(q)$. We proceed as in Section 13.2; we need only examine the parabolic subgroups M in L which are not divisible by q^m+1 for some even integer m. There are two such possibilities:

1. $L^\dagger = E_6^+(q)$ and M is of type D_5. Then $|L : M| = (q^6+q^3+1)(q^8+q^4+1)(q^2+q+1)$. For $q \equiv 1(3)$, $|L : M|$ is divisible by 9 hence M cannot contain L_α. Thus we assume that $q \equiv 2(3)$ and so L is simple.
   Now we know that $M' := [q^{16}].\Omega_{10}^+(q) \leq L_\alpha \leq M \cong [q^{16}] : (\Omega_{10}^+(q)H)$ where H is the Cartan subgroup of L. This is because all parabolic subgroups of $\Omega_{10}^+(q)$ have index divisible by $q^4+1 \equiv 2(3)$.
   By [4, (15.1),(15.5)], L contains an involution g such that $C_L(g) = [q^{21}] : \mathrm{SL}(6,q)$ and so $n_g = (q^6+q^3+1)(q^8+q^4+1)(q^8-1)$. Now if $r_g \geq (q^6+q^3+1)(q^8-1)$ then $\frac{n_g}{r_g} \leq (q^4+1)^2-(q^4+1)+1$ and so $d_g \leq (q^4+1)^2+(q^4+1)+1$. But then $\frac{n_g}{r_g}d_g < |L : M|$ which is a contradiction. Thus, for all $h \in L_\alpha$ conjugate in G to g, $|K : C_K(h)| < (q^6+q^3+1)(q^8-1)$.
   Now $\Omega_{10}^+(q) \not\leq C_L(g)$. Furthermore the only maximal subgroups of $\Omega_{10}^+(q)$ with index less than $(q^6+q^3+1)(q^8-1)$ are the parabolic subgroups and $\mathrm{Sp}_8(q)$. All but one type of parabolic subgroups have index divisible by q^3+1. Since q^3+1 does not divide n_g, there must be $h \in L_\alpha$ conjugate in G to g such that $C_K(h)$ lies in either $[q^{16}].([q^8] : \frac{1}{2}((q-1) \times \mathrm{SO}_8^+(q)))$ or $[q^{16}].\,\mathrm{Sp}_8(q)$.
   Consider the first possibility. Now $\mathrm{SO}_8^+(q) \not\leq C_L(g)$ and so
   $$r_g \geq P(\mathrm{SO}_8^+(q))\frac{|\Omega_{10}^+(q)|}{|[q^8] : \frac{1}{2}((q-1) \times \mathrm{SO}_8^+(q))|}.$$
   Using the value for $P(\mathrm{SO}_8^+(q))$ given in [23, Table 5.2.A] we conclude that $r_g > (q^6+q^3+1)(q^8-1)$ which is impossible.
   Similarly $\mathrm{Sp}_8^+(q) \not\leq C_L(g)$ and so $r_g \geq P(\mathrm{Sp}_8^+(q))|\Omega_{10}^+(q)|/|\,\mathrm{Sp}_8^+(q))|$. Once again we find that $r_g > (q^6+q^3+1)(q^8-1)$ which is impossible.
2. $L^\dagger = E_6^-(q)$ and M is of type ${}^2D_4(q)$. Then $|L : M|$ is divisible by $(q^5+1)(q^9+1)$; we exclude this possibility in the same way as in Section 13.2, when $L = E_7(q)$ and M is of type E_6.

Theorem A is now proven.

References

[1] M. Aschbacher, On the maximal subgroups of the finite classical groups. *Invent. Math.* **76** (1984), 469–514. MR746539 (86a:20054) Zbl 0537.20023

[2] M. Aschbacher, Overgroups of Sylow subgroups in sporadic groups. *Mem. Amer. Math. Soc.* **60** (1986), iv+235. MR831891 (87e:20037) Zbl 0585.20005

[3] M. Aschbacher, *Finite group theory*. Cambridge Univ. Press 2000. MR1777008 (2001c:20001) Zbl 0997.20001

[4] M. Aschbacher, G. M. Seitz, Involutions in Chevalley groups over fields of even order. *Nagoya Math. J.* **63** (1976), 1–91. MR0422401 (54 #10391) Zbl 0359.20014

[5] R. E. Block, On the orbits of collineation groups. *Math. Z.* **96** (1967), 33–49. MR0209171 (35 #74) Zbl 0163.42304

[6] D. M. Bloom, The subgroups of $\mathrm{PSL}(3,q)$ for odd q. *Trans. Amer. Math. Soc.* **127** (1967), 150–178. MR0214671 (35 #5520) Zbl 0153.03702

[7] A. R. Camina, Projective planes with a transitive automorphism group. *Innov. Incidence Geom.* **1** (2005), 191–196. MR2213959 (2007f:51006) Zbl 1099.51004

[8] A. R. Camina, P. M. Neumann, C. E. Praeger, Alternating groups acting on finite linear spaces. *Proc. London Math. Soc.* (3) **87** (2003), 29–53. MR1978569 (2004b:05030) Zbl 1031.05133

[9] A. R. Camina, C. E. Praeger, Line-transitive automorphism groups of linear spaces. *Bull. London Math. Soc.* **25** (1993), 309–315. MR1222720 (94f:20005) Zbl 0792.05018

[10] R. W. Carter, *Simple groups of Lie type*. Wiley-Interscience 1989. MR1013112 (90g:20001) Zbl 0723.20006

[11] B. Chang, The conjugate classes of Chevalley groups of type (G_2). *J. Algebra* **9** (1968), 190–211. MR0227258 (37 #2843) Zbl 0285.20043

[12] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson, *Atlas of finite groups*. Oxford Univ. Press 1985. MR827219 (88g:20025) Zbl 0568.20001

[13] B. N. Cooperstein, Maximal subgroups of $G_2(2^n)$. *J. Algebra* **70** (1981), 23–36. MR618376 (82h:20055) Zbl 0459.20007

[14] P. Dembowski, *Finite geometries*. Springer 1997. MR1434062 (97i:51005) Zbl 0865.51004

[15] L. E. Dickson, *Linear groups: With an exposition of the Galois field theory*. Dover Publications Inc., New York 1958. MR0104735 (21 #3488) Zbl 0082.24901

[16] N. Gill, Nilpotent Singer groups. *Electron. J. Combin.* **13** (2006), Research Paper 94, 6 pp. (electronic). MR2274309 Zbl 05132807

[17] A. Gonçalves, C. Y. Ho, Alternating groups as collineation groups. *J. Algebra* **225** (2000), 581–601. MR1741553 (2001a:51004) Zbl 0951.51002

[18] D. Gorenstein, R. Lyons, The local structure of finite groups of characteristic 2 type. *Mem. Amer. Math. Soc.* **42** (1983), vii+731. MR690900 (84g:20025) Zbl 0519.20014

[19] D. Gorenstein, R. Lyons, R. Solomon, *The classification of the finite simple groups*, volume 40 of *Mathematical Surveys and Monographs*. Amer. Math. Soc. 1994. MR1303592 (95m:20014) Zbl 0816.20016

[20] C. Y. Ho, Finite projective planes with abelian transitive collineation groups. *J. Algebra* **208** (1998), 533–550. MR1655465 (2000c:51004) Zbl 0918.51012

[21] D. R. Hughes, F. C. Piper, *Projective planes*. Springer 1973. MR0333959 (48 #12278) Zbl 0267.50018

[22] W. M. Kantor, Primitive permutation groups of odd degree, and an application to finite projective planes. *J. Algebra* **106** (1987), 15–45. MR878466 (88b:20007) Zbl 0606.20003

[23] P. Kleidman, M. Liebeck, *The subgroup structure of the finite classical groups*. Cambridge Univ. Press 1990. MR1057341 (91g:20001) Zbl 0697.20004

[24] P. B. Kleidman, The maximal subgroups of the Chevalley groups $G_2(q)$ with q odd, the Ree groups ${}^2G_2(q)$, and their automorphism groups. *J. Algebra* **117** (1988), 30–71. MR955589 (89j:20055) Zbl 0651.20020

[25] P. B. Kleidman, The maximal subgroups of the Steinberg triality groups ${}^3D_4(q)$ and of their automorphism groups. *J. Algebra* **115** (1988), 182–199. MR937609 (89f:20024) Zbl 0642.20013

[26] M. W. Liebeck, J. Saxl, The primitive permutation groups of odd degree. *J. London Math. Soc.* (2) **31** (1985), 250–264. MR809946 (87c:20007) Zbl 0573.20004

[27] W. Ljunggren, Einige Bemerkungen über die Darstellung ganzer Zahlen durch binäre kubische Formen mit positiver Diskriminante. *Acta Math.* **75** (1943), 1–21. MR0017303 (8,135k) Zbl 0060.09104

[28] G. Malle, The maximal subgroups of ${}^2F_4(q^2)$. *J. Algebra* **139** (1991), 52–69. MR1106340 (92d:20068) Zbl 0725.20014

[29] H. E. Moore, *The subgroups of the generalized finite modular group*. Volume 9, Chicago, the Decennial Publications 1903. Zbl 34.0172.02

[30] I. Schur, Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen. *J. fr Math.* **139** (1911), 155–250. Zbl 42.0154.02

[31] G. M. Seitz, Flag-transitive subgroups of Chevalley groups. *Ann. of Math.* (2) **97** (1973), 27–56. MR0340446 (49 #5201) Zbl 0338.20052

[32] M. Suzuki, On a class of doubly transitive groups. *Ann. of Math.* (2) **75** (1962), 105–145. MR0136646 (25 #112) Zbl 0106.24702

[33] D. E. Taylor, *The geometry of the classical groups*. Heldermann 1992. MR1189139 (94d:20028) Zbl 0767.20001

[34] A. Wagner, On perspectivities of finite projective planes. *Math. Z* **71** (1959), 113–123. MR0110974 (22 #1842) Zbl 0085.14301

[35] H. N. Ward, On Ree's series of simple groups. *Trans. Amer. Math. Soc.* **121** (1966), 62–89. MR0197587 (33 #5752) Zbl 0139.24902

[36] A. Wiman, Bestimmung aller Untergruppen einer doppelt unendlichen Reihe von einfachen Gruppen. *Stockh. Akad. Bihang* **25** (1899), 47. JFM 30.0197.01

Received 28 September, 2005; revised 26 February, 2007

N. Gill, The Institute of Mathematical Sciences, CIT Campus, Taramani, Chennai 600 113, India
Email: nickgill@cantab.net