# Information and Semimartingales

DISSERTATION

zur Erlangung des akademischen Grades
doctor rerum naturalium
(Dr. rer. nat.)
im Fach Mathematik

eingereicht an der
Mathematisch-Naturwissenschaftlichen Fakultät II
Humboldt-Universität zu Berlin

von
Herr Dipl.-Math. Stefan Ankirchner
geboren am 13.03.1976 in Mühldorf am Inn

Präsident der Humboldt-Universität zu Berlin:
Prof. Dr. Jürgen Mlynek

Dekan der Mathematisch-Naturwissenschaftlichen Fakultät II:
Prof. Dr. Uwe Küchler

Gutachter:

1. Prof. Dr. Peter Imkeller
2. Prof. Dr. Hans Föllmer
3. Prof. Dr. Arturo Kohatsu-Higa

| | |
|---|---|
| eingereicht am: | 22. März 2005 |
| Tag der mündlichen Prüfung: | 1. Juli 2005 |

**Abstract**

Stochastic Analysis provides methods to describe random numerical processes. The descriptions depend strongly on the underlying information structure, which is represented in terms of filtrations. The first part of this thesis deals with impacts of changes in the information structure on the appearance of a stochastic process. More precisely, it analyses the consequences of a filtration enlargement on the semimartingale decomposition of the process. From the martingale part a drift has to be subtracted in order to obtain a martingale in the enlarged filtration. Methods are given how one can compute and analyze this correcting drift.

The second and third part discuss the role of information in financial utility calculus: In the framework of the general semimartingale model of financial markets the link between information and utility is analyzed.

The second part is of a qualitative nature: It deals with implications of the assumption that the maximal expected utility of an investor is bounded. It is shown that finite utility implies some structure properties of the price process viewed from the intrinsic perspective: At first it follows that the price is a semimartingale. Moreover, one can show for continuous processes that the bounded variation part in the semimartingale decomposition is nicely controlled by the martingale part and does not explode. Thus the second part justifies these widespread assumptions.

The third part is of a quantitative nature: It analyzes the impact of information on utility. From an extrinsic point of view traders with different knowledge are compared. In particular, it is shown how additional information increases utility. If the preferences of the investor are described by the logarithmic utility function, then one can calculate the utility increment by means of the so-called information drift. Furthermore, the utility increment coincides with the mutual information between the additional knowledge and the original knowledge, 'mutual information' being defined in the sense of information theory. As a consequence the link between two different concepts of 'information' is established.

**Keywords:**
enlargement of filtrations, semimartingale, utility maximisation, mutual information

**Zusammenfassung**

Die stochastische Analysis gibt Methoden zur Erfassung und Beschreibung von zufälligen numerischen Prozessen an die Hand. Die Beschreibungen hängen dabei sehr stark von der Informationsstruktur ab, die den Prozessen in Gestalt von Filtrationen zugrunde gelegt wird. Der 1. Teil der vorliegenden Arbeit handelt davon, wie sich ein Wechsel der Informationsstruktur auf das Erscheinungsbild eines stochastischen Prozesses auswirkt. Konkret geht es darum, wie sich eine Filtrationsvergrößerung auf die Semimartingalzerlegung eines Prozesses auswirkt. Der Martingalteil muss um einen Drift korrigiert werden, um ein Martingal in der vergrößerten Filtration zu bleiben. Es werden Methoden beschrieben, mit denen dieser Korrekturdrift erzeugt und analysiert werden kann.

In dem 2. und 3. Teil der Arbeit wird die Rolle von Information im finanzmathematischen Nutzenkalkül untersucht: Im Rahmen des allgemeinen Semimartingalmodells für Finanzmärkte wird der Zusammenhang zwischen Information und Nutzen näher analysiert.

Im 2. Teil werden unter der Annahme, dass der maximale erwartete Nutzen eines Händlers beschränkt ist, qualitative Erkenntnisse über den Preisprozess hergeleitet. Es wird gezeigt, dass endlicher Nutzen einige strukturelle Implikationen für die intrinsische Sichtweise hat: Zunächst folgt, dass der Preisprozess ein Semimartingal ist. Des Weiteren lässt sich für stetige Prozesse zeigen, dass der Prozess mit beschränkter Variation in der Semimartingalzerlegung durch den Martingalteil kontrolliert wird und keine Explosionen zulässt. Diese Eigenschaften werden in vielen finanzmathematischen Modellen als gegeben angenommen. Somit liefert der 2. Teil eine Rechtfertigung für diese weitverbreiteten Annahmen.

Im 3. Teil wird quantitativ untersucht, wie sich Information auf den Nutzen auswirkt. Aus extrinsischer Sicht werden Händler mit unterschiedlichem Wissen verglichen. Vor allem wird analysiert, wie sich der Nutzen durch zusätzliche Information vergrößert. Falls die Präferenzen durch die logarithmische Nutzenfunktion beschrieben werden, lässt sich der Nutzenzuwachs mit dem sogenannten Informationsdrift berechnen. Darüber hinaus stimmt in diesem Fall der Nutzenzuwachs mit der gemeinsamen Information zwischen dem zusätzlichen Wissen und dem ursprünglichen Wissen überein, wobei 'gemeinsame Information' im Sinne der Informationstheorie verstanden wird. Somit ist die Verbindung zwischen zwei unterschiedlichen Konzepten von 'Information' hergestellt.

**Schlagwörter:**
Filtrationsvergrößerungen, Semimartingale, Nutzenmaximierung, gemeinsame Information

# Contents

# Chapter 1

# Introduction

The way an observer perceives a phenomenon or a procedure depends on his previous knowledge. The properties he attributes to an observed object are based on what he knows about the object before. Two people with different information at their disposal may give two completely different descriptions of the same observed process. This is often the case with phenomena in economics. Think of a slump of a stock price due to a bad business development. A board member of the related stock corporation knows about the problems in the firm and expects the price to fall. However, somebody who does not have any information in advance is surprised by the sudden announcement of the bad development and the consequential strong fall of the price. Hence his a priori expectation and description of the price process is totally different.

Stochastic Analysis allows us to describe processes from different point of views. It provides tools to analyze numerical processes which are random or appear random. The processes are described from the perspective of an observer following the development of the process, whereby the information flow of the observer is represented by a filtration $(\mathcal{F}_t)$. In this thesis we study the influence of information (filtrations) on the perception of stochastic processes. In particular we analyze what happens under a change of the filtration.

The analysis is restricted to the class of semimartingales: these are the processes which can be written as a sum of a martingale and a process of bounded variation. Now suppose that a filtration $(\mathcal{F}_t)$ is enlarged to $(\mathcal{G}_t)$. Under which conditions does every $(\mathcal{F}_t)$-semimartingale remain a semimartingale relative to $(\mathcal{G}_t)$? This implication, called Hypothèse (H'), has been studied intensively in the literature. If (H') is satisfied, one wants to know how the decompositions of the semimartingales change under an enlargement. The books [33], [32], [16] are only a few examples of the works giving answers to these questions.

The beginning of Part I is similar to the setting of a paper of Jacod [31]. Jacod considers the case of an initial enlargement by a random variable $G$ and shows the following: If regular conditional laws $Q_t$ of $G$ relative to the $\sigma$-algebras $\mathcal{F}_t$ exist and if for every $t \geq 0$ and almost all $\omega$ the measures $Q_t(\omega, \cdot)$ are all absolutely continuous with respect to one fixed distribution $\eta$, then (H') is satisfied. Jacod also provides formulas describing the semimartingale decomposition under the enlarged filtration. In Part I we deduce similar results in a more general setting. We suppose that the given filtration $(\mathcal{F}_t)$ is enlarged by another filtration $(\mathcal{H}_t)$, i.e. $\mathcal{G}_t = \bigcap_{s>t} \mathcal{F}_s \vee \mathcal{H}_s$. Moreover we replace Jacod's condition by a condition inspired from the notion of the decoupling measure. Under the decoupling measure the enlarging information is independent of the original information. The clue is that the enlargement of the filtration can be interpreted as a change from the decoupling measure to the original measure. In particular, the Girsanov transform can be used to obtain explicit Doob-Meyer decompositions relative to $(\mathcal{G}_t)$.

On the *original* probability space decoupling measures exist only under very restrictive assumptions. Therefore it is useful to switch to a *product space* where decoupling measures always exist: take the product measure. This idea is used in [20] in order to solve a 'paradox' on the Wiener space with a filtration enlarged by some random variable $G$. It is remarkable that a switching onto the product space has also been used by Yoeurp [47] in order to analyze progressive enlargements.

In the next step the natural questions arises whether the embedding of the $(\mathcal{F}_t)$-semimartingales into the space of the $(\mathcal{G}_t)$-semimartingales satisfies some continuity properties. Indeed, we will provide sufficient continuity criteria in terms of the entropy of the original measure relative to the decoupling measure. This entropy is known as 'mutual information' in information theory. Essentially, finite mutual information between the old and the new information implies continuity. This is a generalisation of a result by Yor [48]: Yor proves continuity for filtrations enlarged by a discrete random variable with finite absolute entropy.

In Chapter 5 we follow another way of computing Doob-Meyer decompositions under enlarged filtrations: We compute directly the information drift for a continuous $(\mathcal{F}_t)$-martingale $M$. The information drift is the density of the bounded variation process, which has to be subtracted in order that $M$ becomes a martingale relative to the bigger filtration $(\mathcal{G}_t)$. Information drifts can be obtained by diagonalizing density processes of kernels with respect to conditional probabilities. For initial enlargements on the Wiener space this has been shown for example by Imkeller in [26], [27], [28] and Imkeller, Pontier and Weisz in [29]. These quoted works are the paradigmas of the approach in Chapter 5. It is shown how information drifts can be obtained for non-initial enlargements. The results are then applied in order to calculate information drifts of enlargements of the Wiener filtration. Most of the drifts are well-known already and can be found for example in [49] and [6]. The derivations in Chapter 5 and 6 are different though.

In Part II we start studying the link between information and semimartingales in the framework of the so-called semimartingale model of equity markets. We consider a rational trader on a financial market who aims at maximizing the utility of his wealth at time $T > 0$. If the knowledge of a trader evolves in a way such that his expected maximal utility is finite, then from his point of view (filtration) the price process is a semimartingale. Moreover, it turns out that for continuous price processes the bounded variation part of the semimartingale decompostion is nicely controlled by the quadratic variation process. Finally finite utility implies that explosions of the drift are impossible. Thus, finite utility implies price structure properties which are commonly made in financial market models.

Part II is strongly motivated by a paper of Biagini and Oksendal ([7]). In the framework of the Black-Scholes model the authors suppose that one trader, the so-called insider, has more information than ordinary traders. The strategies of the insider need not be adapted to the Wiener filtration, but only to an enlargement. The wealth processes are interpreted by using anticipative calculus. For this the authors restrict the insider strategies to those integrands for which the forward integral is defined in the sense of Russo and Vallois [44]. They show that the boundedness of the logarithmic utility implies the price process to be a semimartigale with respect to the enlarged filtration.

In our approach we reduce the set of integrands to the set of buy-and-hold strategies. As a consequence the wealth processes are simple integrals and much easier to handle than the forward integrals used in [7]. Moreover, it directly leads us to the theorem of Bichteler, Dellacherie and Mokobodzki. This theorem is already used in mathematical finance in order to deduce the semimartingale property from no-arbitrage conditions. Ansel and Stricker [4] show that (NA) for simple integrands implies the price process to be a semimartingale. Delbaen and Schachermayer [10] restate this result by using the (NFLVR) condition. Inspired by their proofs, we show in Chapter 7 that finite expected utility implies the semimartingale property. Ansel and Stricker [4] not only show that the price process is a semimartingale, but they prove that it has a decomposition of the form $M + \alpha \cdot \langle M, M\rangle$, where $(\alpha^2 \cdot \langle M, M\rangle)_T < \infty$ almost surely. Delbaen and Schachermayer deduce these properties in [12]. Their work motivated us to derive similar results from the assumption of bounded expected utility.

In Part III the results on the relation between information and semimartingales are used in order to quantify the dependence between the maximal expected utility of an investor and his available information. In particular we compare the maximal expected utility of investors with different knowledge. Most of the times we suppose that one trader, an insider, has more information than other traders. The insider's knowledge is represented by a filtration $(\mathcal{G}_t)$ which is bigger than a filtration $(\mathcal{F}_t)$ representing the information flow of ordinary traders. This idea of modeling insiders on financial markets by using enlarged filtrations traces back to Duffie and Huang [17]. The model has been studied with increasing complexity by many authors, for example by Karatzas and Pikovsky [40], Grorud and Pontier [22] Amendinger, Imkeller and Schweizer in [2], Imkeller in [26], [27], [28] and Imkeller, Pontier and Weisz in [29]. Baudoin [5] modifies this model by introducing the concept of weak additional information consisting in the knowledge of the law of some random element. Kohatsu-Higa and Sulem [34] allow for price dynamics influenced by the insider. In most of these papers the model is based on the Wiener space. Moreover the insider is supposed to get extra information in the beginning of the trading interval, and hence his knowledge is represented by an initially enlarged filtration. Insiders with non-intial additional information have been studied only recently by Corcuera et al. in [9].

In the approach of Part III investors with different information are compared in the framework of the general semimartingale model of financial markets. A lot of the settings described in the quoted papers are special cases of the setting here. We allow for arbitrary enlargements of filtrations. Sometimes we make the restriction that the price process is continuous. As Part II shows, it is no restriction to further assume that the price process is a semimartingale with a bounded variation part which is nicely controlled by the quadratic variation: If these properties are not satisfied, then the considered maximal expected utility is infinite.

We represent the additional *logarithmic* utility by means of the information drift $\mu$ of the enlarged filtration $(\mathcal{G}_t)$ relative to the martingale part $M$ in the $(\mathcal{F}_t)$-decomposition: it is given by $\frac{1}{2}E\left(\mu^2 \cdot \langle M, M\rangle\right)_T$. All the properties of information drifts shown in Part I imply now similar results for the utility. Moreover, the mutual information used already in Chapter 2 reveals to be useful for calculating the additional logarithmic utility. Already the paper [2] indicates that there is a link between the additional logarithmic utility and information theory: The authors show that if the insider has access to a filtration which is initially enlarged by a discrete random variable $G$, then his additional logarithmic utility is given by the absolute entropy of $G$. By using the notion of mutual information we prove this relation for all initial enlargements. Moreover, we introduce the notion of information differences of filtrations in order to generalize this result to non-initial enlargements. Thus, the link between two completely different concepts of information is established.

The thesis is organized as follows.

In Chapter 2 we show that passing from $(\mathcal{F}_t)$ to an enlarged filtration $(\mathcal{G}_t)$ can be interpreted as a measure change on a product space. The main work consists of showing how objects can be translated from the original space into the product space and vice versa. Once this is done, an application of the Lenglart-Girsanov transform leads to explicit Doob-Meyer decompositions.

In Chapter 3 we provide sufficient criteria for the embedding of $(\mathcal{F}_t)$-semimartingales into the space of $(\mathcal{G}_t)$-semimartingales to be continuous relative to $\mathcal{S}^p$-norms.

In Chapter 4 we introduce metrics on the set of filtrations under which a given stochastic process $X$ is a special semimartingale. At first we define the distance between two filtrations as $L^p$-norm of the variation difference of the bounded variation parts in the related decompositions. In a second step we define metrics with the help of information drifts. In both cases we show that if $X$ is continuous, then the metrics are complete.

In Chapter 5 we prove a representation theorem for information drifts for general enlargements and apply it to some easy examples.

The representation theorem is further used in Chapter 6 in order to show how information drifts can be computed via Malliavin calculus on the Wiener space. We consider some concrete initial enlargements of the Wiener filtration and calculate explicitly their information drifts.

Part II starts with a short description of the general semimartingale model of financial markets. In Chapter 7 we want to justify the assumption of this model that the price process is a semimartingale. For this we consider traders with information $(\mathcal{F}_t)$ and suppose that the maximal expected utility by using simple strategies is finite. We distinguish between traders who are allowed to use all admissible simple strategies and traders who may only use investment strategies such that their wealth is bounded from below by some fixed constant. In both cases we will show that finite expected utility implies the semimartingale property of the underlying price process. In the end of Chapter 7 we even show that semimartingales can be characterized in terms of finite expected utility.

Once the semimartingale property is established, the question arises, whether the expected utility increases by taking the supremum over general strategies. In Chapter 8 we show that this is not the case if the utility function satisfies $\mathrm{dom}(U) = \mathbb{R}$ and if the price is continuous. We also discuss the case $\mathrm{dom}(U) \neq \mathbb{R}$, where similar results hold true.

In Chapter 9 we continue to investigate the consequences of finite expected utility. The starting point of our analysis is a price process which is a special semimartingale with decomposition $S = M + A$. We show that $A$ is absolutely continuous with respect to the quadratic variation $\langle M, M \rangle$. Consequently there is a predictable process $\alpha$ such that $S = M + \alpha \cdot \langle M, M \rangle$. Finally, bounded utility implies the integral $(\alpha^2 \cdot \langle M, M \rangle)_T$ to be finite almost everywhere.

Part III compares the maximal expected utility of traders with different information in the framework of the general semimartingale model. In order to give some economic motivation we begin with a reflection on utility based prices of additional information. We then proceed with a monotone convergence result in Chapter 11: If $(\mathcal{G}_t^n)$ is an increasing sequence of filtrations, then the utility suprema over all $(\mathcal{G}_t^n)$-predictable strategies converge to the supremum taken over all $(\bigvee_n \mathcal{G}_t^n)$-predictable strategies.

In Chapter 12 we prove duality results for the maximal expected utility under initially enlarged filtrations. We determine 'stochastic conjugates' as $f$-divergence of the decoupling measure on the product space.

If there is a portfolio which is optimal not only with respect to the fixed time horizon $T > 0$, but also with respect all times $0 < t < T$, then the underlying utility function is equal to the logarithm up to affine transformations. After describing this particular feature, we then concentrate on the logarithmic utility function. In Chapter 13 we give an explicit representation of the maximal expected logarithmic utility by means of the drift $\alpha$ appearing in the semimartingale decomposition of the price $S = M + \alpha \cdot \langle M, M \rangle$. If the wealth process must not be negative, then the maximal expected logarithmic utility is equal to $\log(x) + \alpha^2 \cdot \langle M, M \rangle$, where $x$ denotes the initial wealth. If (NFLVR) is satisfied, then this result is true without the restriction that wealth has to be positive.

In Chapter 14 we compare again an insider with information $(\mathcal{G}_t)$ with a normal trader with information represented by a smaller filtration $(\mathcal{F}_t)$. In contrast to Chapter 12 we do not assume the market to be free of arbitrage. The difference of the expected logarithmic utility is equal to $\frac{1}{2} E\left(\mu^2 \cdot \langle M, M \rangle\right)$, where $\mu$ is the information drift of $(\mathcal{G}_t)$ relative the martingale part in the $(\mathcal{F}_t)$-decomposition. Finally we discuss the link to information theory. We introduce the notion of *information difference* between two filtrations. If the market is complete with respect to the smaller filtration $(\mathcal{F}_t)$, then the additional utility is given by the information difference between the enlarged filtration $(\mathcal{G}_t)$ and the smaller filtration $(\mathcal{F}_t)$.

# Part I

# Enlargement of filtrations

# Chapter 2

# Enlarging filtrations equals changing measures

## 2.1 Embedding into a product space

Let $(\Omega, \mathcal{F}, P)$ be a probability space with right-continuous filtrations $(\mathcal{F}_t)_{t\geq 0}$ and $(\mathcal{H}_t)_{t\geq 0}$. Moreover, let $\mathcal{F}_\infty = \bigvee_{t\geq 0} \mathcal{F}_t$ and $\mathcal{H}_\infty = \bigvee_{t\geq 0} \mathcal{H}_t$.

Our objective is to study the enlarged filtration

$$\mathcal{G}_t = \bigcap_{s>t} (\mathcal{F}_s \vee \mathcal{H}_s), \qquad t \geq 0.$$

We relate this enlargement to a measure change on the product space

$$\bar{\Omega} = \Omega \times \Omega$$

equipped with the $\sigma$-field

$$\bar{\mathcal{F}} = \mathcal{F}_\infty \otimes \mathcal{H}_\infty.$$

We endow $\bar{\Omega}$ with the filtration

$$\bar{\mathcal{F}}_t = \bigcap_{s>t} (\mathcal{F}_s \otimes \mathcal{H}_s), \qquad t \geq 0.$$

$\Omega$ will be embedded into $\bar{\Omega}$ by the map

$$\psi : (\Omega, \mathcal{F}) \to (\bar{\Omega}, \bar{\mathcal{F}}),\ \omega \mapsto (\omega, \omega).$$

We denote by $\bar{P}$ the image of the measure $P$ under $\psi$, i.e.

$$\bar{P} = P_\psi.$$

Hence for all $\bar{\mathcal{F}}$-measurable functions $f : \bar{\Omega} \to \mathbb{R}$ we have

$$\int f(\omega, \omega') d\bar{P}(\omega, \omega') = \int f(\omega, \omega) dP(\omega). \tag{2.1}$$

We use notations and concepts of stochastic analysis as explained in the book by Protter [41]. Most of our results only hold for completed filtrations. Since we consider different measures relative to which completions are taken, we use the following notation. Let $(\mathcal{K}_t)$ be a filtration and $R$ a probability measure. We denote by $(\mathcal{K}_t^R)$ the filtration $(\mathcal{K}_t)$ completed by the $R$-negligible sets.

We start with a simple observation.

**Lemma 2.1.1.** *If $\bar{f} : \bar{\Omega} \to \mathbb{R}$ is $\bar{\mathcal{F}}_t^{\bar{P}}$-measurable, then the map $\bar{f} \circ \psi$ is $\mathcal{G}_t^P$-measurable.*

*Proof.* First observe that

$$\begin{aligned} \mathcal{G}_t &= \bigcap_{s>t} \sigma\left(A \cap B : A \in \mathcal{F}_s, B \in \mathcal{H}_s\right) \\ &= \bigcap_{s>t} \sigma\left(\psi^{-1}(A \times B) : A \in \mathcal{F}_s, B \in \mathcal{H}_s\right) \\ &= \psi^{-1}\left(\bigcap_{s>t} (\mathcal{F}_s \otimes \mathcal{H}_s)\right) = \psi^{-1}(\bar{\mathcal{F}}_t). \end{aligned}$$

Now let $\bar{f} = 1_A$ with $A \in \bar{\mathcal{F}}_t^{\bar{P}}$. There is a set $B \in \bar{\mathcal{F}}_t$ such that $\bar{P}(A \triangle B) = 0$. From the first part we deduce that the map $1_B \circ \psi$ is $\mathcal{G}_t$-measurable. Since we have $P$-almost surely

$$1_A \circ \psi = 1_B \circ \psi,$$

the map $1_A \circ \psi$ is $\mathcal{G}_t^P$-measurable. By standard arguments one can show the statement for arbitrary $\bar{\mathcal{F}}_t^{\bar{P}}$-measurable functions. □

**Lemma 2.1.2.** *If $\bar{X}$ is $(\bar{\mathcal{F}}_t^{\bar{P}})$-predictable, then $\bar{X} \circ \psi$ is $(\mathcal{G}_t^P)$-predictable.*

*Proof.* Let $0 < s \le t$, $A \in \bar{\mathcal{F}}_s^{\bar{P}}$ and

$$\bar{\theta} = 1_A 1_{]s,t]}$$

Then, by Lemma 2.1.1, $\bar{\theta} \circ \psi = (1_A \circ \psi) 1_{]s,t]}$ is $(\mathcal{G}_t^P)$-predictable. The proof may be completed by a monotone class argument. □

**Lemma 2.1.3.** *Let $\bar{Y}$ be $(\bar{\mathcal{F}}_t^{\bar{P}})$-adapted. Then the process*

$$Y = \bar{Y} \circ \psi$$

*is $(\mathcal{G}_t^P)$-adapted. Moreover, if $\bar{Y}$ is a $(\bar{\mathcal{F}}_t^{\bar{P}}, \bar{P})$-local martingale, then $Y$ is a $(\mathcal{G}_t^P, P)$-local martingale.*

*Proof.* The first statement follows immediately from Lemma 2.1.1. Now suppose that $\bar{Y}$ is a $(\bar{\mathcal{F}}_t^{\bar{P}}, \bar{P})$-martingale. Let $0 \leq s < t$ and $A \in \mathcal{G}_s$. Then there is a set $B \in \bar{\mathcal{F}}_s$ such that $\psi^{-1}(B) = A$ and hence

$$E^P[1_A(Y_t - Y_s)] = E^{\bar{P}}[1_B(\bar{Y}_t - \bar{Y}_s)] = 0.$$

Thus $Y$ is a $(\mathcal{G}_t^P)$-martingale.
Finally, let $\bar{Y}$ be a $(\bar{\mathcal{F}}_t^{\bar{P}})$-local martingale and $\bar{T}$ a localizing stopping time. The random time $T = \bar{T} \circ \psi$ is a $(\mathcal{G}_t^P)$-stopping time, since

$$\{T \leq t\} = \psi^{-1}\{\bar{T} \leq t\} \in \psi^{-1}(\bar{\mathcal{F}}_t^{\bar{P}}) \subset \mathcal{G}_t^P.$$

Now it is straightforward to show that $Y$ is a $(\mathcal{G}_t^P)$-local martingale. □

**Theorem 2.1.4.** *Let $\bar{Y}$ be a $(\bar{\mathcal{F}}_t^{\bar{P}}, \bar{P})$-semimartingale. Then the process $Y = \bar{Y} \circ \psi$ is a $(\mathcal{G}_t^P, P)$-semimartingale.*

*Proof.* Let $\bar{Y}$ be a $(\bar{\mathcal{F}}_t^{\bar{P}})$-semimartingale and $Y = \bar{Y} \circ \psi$. Obviously $Y$ has cadlag paths $P$-a.s. and Lemma 2.1.3 implies that $Y$ is $(\mathcal{G}_t^P)$-adapted. By the theorem of Bichteler-Dellacherie-Mokobodzki it is sufficient to show that if $(\theta^n)$ is a sequence of simple $(\mathcal{G}_t)$-adapted integrands converging uniformly to 0, then the simple integrals $(\theta^n \cdot Y)$ converge to 0 in probability relative to $P$. Recall that any $(\mathcal{G}_t)$-simple integrand is of the form $\sum_{1 \leq i \leq n} 1_{]t_i, t_{i+1}]} \theta_i$, where $\theta_i$ is $\mathcal{G}_{t_i}$-measurable. Since $\mathcal{G}_t = \psi^{-1}(\bar{\mathcal{F}}_t)$, one can find simple $(\bar{\mathcal{F}}_t)$-adapted processes $(\bar{\theta}^n)$ converging uniformly to 0 such that $\bar{\theta}^n \circ \psi = \theta^n$. The process $\bar{Y}$ being a semimartingale implies that the sequence $(\bar{\theta}^n \cdot \bar{Y})$ converges to 0 in probability relative to $\bar{P}$, and hence $(\theta^n \cdot Y)$ converges to 0 in probability relative to $P$. □

So far we have seen how objects can be translated from $\bar{\Omega}$ to $\Omega$. Now we look at the reverse transfer. For this we may use any product measure on $\bar{\Omega}$: let $R$ be a probability measure on $\mathcal{H}_\infty$, and

$$\bar{Q} = P\big|_{\mathcal{F}_\infty} \otimes R\big|_{\mathcal{H}_\infty}.$$

We will sometimes denote $\bar{Q}$ as *decoupling measure.*

**Lemma 2.1.5.** *Let $M$ be a right-continuous $(\mathcal{F}_t^P, P)$-local martingale. Then the process $\bar{M}(\omega, \omega') = M(\omega)$ is a $(\bar{\mathcal{F}}_t^{\bar{Q}}, \bar{Q})$-local martingale.*

*Proof.* It is immediate that $\bar{M}$ is $(\bar{\mathcal{F}}_t^{\bar{Q}})$-adapted. Assume at first that $M$ is a strict $(\mathcal{F}_t^P, P)$-local martingale. Then, for $0 \leq s < t$, and $A \in \mathcal{F}_s$, $B \in \mathcal{H}_s$ we have

$$E^{\bar{Q}}[1_A(\omega)1_B(\omega')(\bar{M}_t - \bar{M}_s)] = R(B)E^P[1_A(M_t - M_s)] = 0.$$

By the monotone class theorem, for all bounded $(\mathcal{F}_s \otimes \mathcal{H}_s)$-measurable functions $\theta$ we have

$$E^{\bar{Q}}[\theta(\bar{M}_t - \bar{M}_s)] = 0.$$

Since $\bar{M}$ is right-continuous, this remains true for all bounded $\bigcap_{u>s}(\mathcal{F}_u \otimes \mathcal{H}_u)$-measurable $\theta$, and hence $\bar{M}$ is a martingale with respect to $(\bar{\mathcal{F}}_t^{\bar{Q}})$.

Via $\bar{T}(\omega, \omega') = T(\omega)$ stopping times can be trivially extended to the product space. This finally shows that the local martingale property translates to $\bar{\Omega}$ with respect to $\bar{Q}$. □

In the sequel we will always assume that $\bar{P}$ is absolutely continuous with respect to $\bar{Q}$, i.e.

**Assumption 2.1.6.**

$$\bar{P} \ll \bar{Q} \text{ on } \bar{\mathcal{F}}.$$

Note that this assumption is always satisfied if $R \sim P$ and $(\mathcal{G}_t)$ is obtained by an initial enlargement by some discrete random variable $G$, i.e. $\mathcal{H}_t = \sigma(G)$ for all $t \geq 0$.

Now let $M$ be a $(\mathcal{F}_t^P, P)$-local martingale and $\bar{M}$ its extension to $\bar{\Omega}$ as in Lemma 2.1.5. Since $\bar{P} \ll \bar{Q}$, $\bar{M}$ is a $(\bar{\mathcal{F}}_t^{\bar{P}}, \bar{P})$-semimartingale and hence, by Theorem 2.1.4, $M$ is a $(\mathcal{G}_t^P, P)$-semimartingale. Thus, clearly hypothesis (H') is satisfied. But what is the Doob-Meyer decomposition of $M$ relative to $(\mathcal{G}_t^P, P)$?

Essentially the change of filtrations corresponds to changing the measure from $\bar{Q}$ to $\bar{P}$ on the product space $\bar{\Omega}$. Girsanov's theorem applies on $\bar{\Omega}$, since the measure $\bar{P}$ is absolutely continuous with respect to $\bar{Q}$. As a consequence we obtain a Girsanov-type result for the corresponding change of filtrations. For its explicit description we introduce the density process. Let $(\bar{Z}_t)$ denote a cadlag $(\bar{\mathcal{F}}_t^{\bar{Q}})$-adapted process with

$$\bar{Z}_t = \frac{d\bar{P}}{d\bar{Q}}\bigg|_{\bar{\mathcal{F}}_t^{\bar{Q}}}.$$

Note that we need to consider the completed filtration in order to assure the existence of a cadlag density process. Theorem 2.1.4 implies that the process $Z$ defined by

$$Z = \bar{Z} \circ \psi$$

is a $(\mathcal{G}_t^P, P)$-semimartingale. Before giving the Girsanov-type results, we show how the quadratic variation processes behave under the projection $\psi$.

**Lemma 2.1.7.** *Let $\bar{X}$ and $\bar{Y}$ be $(\bar{\mathcal{F}}_t^{\bar{P}}, \bar{P})$-semimartingales. If $X = \bar{X} \circ \psi$ and $Y = \bar{Y} \circ \psi$, then*

$$[\bar{X}, \bar{Y}] \circ \psi = [X, Y]$$

*up to indistinguishability relative to $P$.*

*Proof.* Put $X = \bar{X} \circ \psi$ and $Y = \bar{Y} \circ \psi$. Let $t > 0$ and $t_i^n = t\,\frac{i}{2^n}$ for all $i = 0, 1, \ldots, 2^n$. It is known that the sums

$$\bar{X}_0 \bar{Y}_0 + \sum_{0 \le i < 2^n} (\bar{X}_{t_{i+1}^n} - \bar{X}_{t_i^n})(\bar{Y}_{t_{i+1}^n} - \bar{Y}_{t_i^n})$$

converge to $[\bar{X}, \bar{Y}]_t$ in probability relative to $\bar{P}$ (see Theorem 20, Chapter VIII in [14]). Hence $[\bar{X}, \bar{Y}]_t \circ \psi$ is the limit (in probability) of the sums

$$X_0 Y_0 + \sum_{0 \le i < 2^n} (X_{t_{i+1}^n} - X_{t_i^n})(Y_{t_{i+1}^n} - Y_{t_i^n})$$

relative to $P$. Obviously the limit is also equal to $[X, Y]_t$, and hence we have

$$[\bar{X}, \bar{Y}]_t \circ \psi = [X, Y]_t.$$

Since both processes are cadlag, they coincide up to indistinguishability relative to $P$. □

Let $\bar{M}$ be a $(\mathcal{F}^{\bar{Q}}, \bar{Q})$-semimartingale and $M = \bar{M} \circ \psi$. Since $\bar{P}$ is absolutely continuous with respect to $\bar{Q}$, $\bar{M}$ is also a $(\bar{\mathcal{F}}_t^{\bar{P}}, \bar{P})$-semimartingale. Moreover, the bracket $[\bar{M}, \bar{Z}]$ relative to $\bar{Q}$ is $\bar{P}$-indistinguishable from the bracket relative to $\bar{P}$. Similarly, Lemma 2.1.7 implies that the bracket $[M, Z]$ of the $(\mathcal{G}_t^P, P)$-semimartingales $M$ and $Z$ coincides with $[\bar{M}, \bar{Z}] \circ \psi$.

We are now in a position to state the first Girsanov-type result. We begin with some definitions. Let

$$\bar{T} = \inf\{t > 0 : \bar{Z}_t = 0, \bar{Z}_{t-} > 0\}$$

and $\bar{U}_t = \Delta \bar{M}_{\bar{T}} 1_{\{t \ge \bar{T}\}}$. We further denote by $\tilde{U}$ the compensator of $\bar{U}$, i.e. the $(\bar{\mathcal{F}}_t^{\bar{Q}}, \bar{Q})$-predictable projection of $\bar{U}$. Moreover, we will use the abbreviation $\hat{U} = \tilde{U} \circ \psi$.

**Theorem 2.1.8.** *If $M$ is a $(\mathcal{F}_t^P, P)$-local martingale with $M_0 = 0$, then*

$$M - \frac{1}{Z} \cdot [M, Z] + \hat{U} \tag{2.2}$$

*is a $(\mathcal{G}_t^P, P)$-local martingale.*

*Proof.* Let $M$ be an $(\mathcal{F}_t^P, P)$-local martingale with $M_0 = 0$. We may assume that $M$ has cadlag paths. Lemma 2.1.5 implies that the process defined by

$$\bar{M}(\omega, \omega') = M(\omega)$$

is a $(\bar{\mathcal{F}}_t^{\bar{Q}})$-local martingale and the Lenglart-Girsanov Theorem yields that

$$\bar{M} - \frac{1}{\bar{Z}} \cdot [\bar{M}, \bar{Z}] + \tilde{U}_t$$

is a $(\bar{\mathcal{F}}_t^{\bar{P}}, \bar{P})$-local martingale (see Théorème 3 in [36] or Chapter III in [41]). Since the bracket process $[\bar{M}, \bar{Z}] \circ \psi$ is $P$-indistinguishable from $[M, Z]$ (see Lemma 2.1.7), we have

$$(\frac{1}{\bar{Z}} \cdot [\bar{M}, \bar{Z}]) \circ \psi = \frac{1}{Z} \cdot [M, Z]$$

up to indistinguishability. With Lemma 2.1.3 we conclude that

$$M - \frac{1}{Z} \cdot [M, Z] + \hat{U}_t$$

is a $(\mathcal{G}_t^P, P)$-local martingale. □

In case $M$ is continuous, the preceding decomposition in the larger filtration simplifies.

**Theorem 2.1.9.** *If $M$ is a continuous $(\mathcal{F}_t^P, P)$-local martingale with $M_0 = 0$, then*

$$M - \frac{1}{Z} \cdot [M, Z]$$

*is a $(\mathcal{G}_t^P, P)$-local martingale.*

*Proof.* Let $M$ be a continuous $(\mathcal{F}_t^P, P)$-local martingale with $M_0 = 0$ and put $\bar{M}(\omega, \omega') = M(\omega)$. The related process $\bar{U}$ vanishes, and hence $\hat{U}$ vanishes as well. The result follows now from Theorem 2.1.8. □

The preceding may also be formulated in terms of the stochastic logarithm of the density process $\bar{Z}$. To this end set $\bar{S} = \inf\{t > 0 : \bar{Z}_t = 0, \Delta\bar{Z}_t = 0\}$ and define

$$\bar{L} = \int_{0+}^{\cdot} \frac{1}{\bar{Z}_-} \, d\bar{Z} \quad \text{on} \quad [0, \bar{S}[. \tag{2.3}$$

So far, the process $\bar{L}$ is determined $\bar{P}$-, but *not* $\bar{Q}$-almost everywhere. (In order to define it everywhere we may put $\bar{L} = 0$ on $[\bar{S}, \infty[$.) Then $\bar{L}$ is an $(\bar{\mathcal{F}}_t^{\bar{P}}, \bar{P})$-semimartingale but not necessarily an $(\bar{\mathcal{F}}_t^{\bar{Q}}, \bar{Q})$-semimartingale. However, restricted to the time interval $[0, \bar{S}[$ it is an $(\bar{\mathcal{F}}_t^{\bar{Q}}, \bar{Q})$-local martingale. As usual we write $L = \bar{L} \circ \psi$. Alternatively, one can define $L$ through the stochastic integral

$$L = \int_{0+}^{\cdot} \frac{1}{Z_-} \, dZ.$$

Since the process $\bar{L}$ is a $(\bar{\mathcal{F}}_t^{\bar{Q}}, \bar{Q})$-local martingale on the interval $[0, \bar{S}[$, it can be decomposed into a unique local-martingale part $\bar{L}^c$ and a sum of compensated jumps $\bar{L}^d$. As before, we consider the processes $L^c = \bar{L}^c \circ \psi$ and $L^d = \bar{L}^d \circ \psi$.

Theorem 2.1.9 can now be reformulated as follows.

**Theorem 2.1.10.** *If $M$ is a continuous $(\mathcal{F}_t^P, P)$-local martingale with $M_0 = 0$, then*

$$M - [M, L]$$

*is a $(\mathcal{G}_t^P, P)$-local martingale.*

*Proof.* Let $M$ be a continuous $(\mathcal{F}_t^P, P)$-local martingale with $M_0 = 0$. Since $M$ is continuous, the bracket process $[M, Z]$ is continuous and Theorem 2.1.9 implies that

$$M - \frac{1}{Z} \cdot [M, Z] = M - \frac{1}{Z_-} \cdot [M, Z]$$

is a $(\mathcal{G}_t^P, P)$-local martingale. Moreover, the definition of $L$ implies that $\frac{1}{Z_-} \cdot [M, Z] = [M, L]$, $P$-a.s., so that $M - [M, L]$ is a $(\mathcal{G}_t^P, P)$-local martingale. □

Finally, we will need the following formula, in which the subtracted drift is represented in terms of the quadration variation of the given local martingale.

**Theorem 2.1.11.** *If $M$ is a continuous $(\mathcal{F}_t^P, P)$-local martingale with $M_0 = 0$, then there is a $(\mathcal{G}_t^P)$-predictable process $\alpha$ such that $P$-a.s.*

$$\int_0^\infty \alpha_t^2 \, d[M, M]_t \le [L, L]_\infty^c < \infty,$$

*and*

$$M - \alpha \cdot [M, M]$$

*is a* $(\mathcal{G}_t^P)$*-local martingale.*

*Proof.* Let $M$ be a continuous $(\mathcal{F}_t^P, P)$-local martingale with $M_0 = 0$. By the Kunita-Watanabe Inequality one has for $0 \le s < t$,

$$[M, L]_t - [M, L]_s \le [L, L]_t^{1/2} \left([M, M]_t - [M, M]_s\right)^{1/2}.$$

Since $[L, L]_t$ is finite for all $t \ge 0$, the measure $d[M, L]$ is absolutely continuous with respect to $d[M, M]$ and there exists a $(\mathcal{G}_t^P)$-predictable process $\alpha$ with

$$\alpha \cdot [M, M] = [M, L] = [M, L^c]$$

(see Lemme 1.36 in [30]). Moreover, the processes $M$ and $O = L^c - \alpha \cdot M$ are orthogonal w.r.t. $[\cdot, \cdot]$. Consequently,

$$\alpha^2 \cdot [M, M] = [\alpha \cdot M, \alpha \cdot M] \le [L^c, L^c] = [L, L]^c.$$

Recall that

$$[L, L] = \left(\frac{1}{\bar{Z}_-^2} \cdot [\bar{Z}, \bar{Z}]\right) \circ \psi$$

and that $\bar{Z}$ is a uniformly integrable nonnegative $(\bar{\mathcal{F}}_t^{\bar{Q}}, \bar{Q})$-martingale. Since $\bar{P}$-a.s. $\bar{Z}_\infty > 0$, one has also $\inf_{t \ge 0} \bar{Z}_t > 0$, $\bar{P}$-a.s. Moreover, $[\bar{Z}, \bar{Z}]_\infty < \infty$, $\bar{Q}$-a.s. Therefore, $[\bar{L}, \bar{L}]$ is $\bar{P}$-a.s. bounded and consequently $[L, L]_t^c$ converges as $t \to \infty$, $P$-a.s. to some real value which we denote by $[L, L]_\infty^c$. □

## 2.2 Girsanov transform for oblique brackets

In the previous section we have seen that an enlargement of the filtration $(\mathcal{F}_t)$ by $(\mathcal{H}_t)$ can be interpreted as a measure change from $\bar{Q}$ to $\bar{P}$. We then applied a theorem by Lenglart in order to derive Doob-Meyer decompositions of $(\mathcal{F}_t^P)$-martingales with respect to $(\mathcal{G}_t^P)$. The theorem we applied is formulated with the help of usual brackets $[\cdot, \cdot]$. There is also a version of Lenglart's theorem with oblique brackets $\langle \cdot, \cdot \rangle$. By using this version, we obtain $(\mathcal{G}_t^P)$-decompositions of $(\mathcal{F}_t^P)$-martingales with $(\mathcal{G}_t^P)$-*predictable* bounded variation part.

We start by recalling some basic facts. Let $X$ and $Y$ be two semimartingales such that $[X, Y]$ is locally integrable with respect to some probability measure $Q$. The oblique bracket $\langle X, Y \rangle^Q$ is defined to be the unique process satisfying

1) $[X,Y] - \langle X,Y\rangle^Q$ is a $Q$-local martingale, and

2) $\langle X,Y\rangle^Q$ is predictable.

(See Definition 39, Chapter VII in [14].)

We turn to a similar result like in Lemma 2.1.7 with oblique brackets.

**Lemma 2.2.1.** *Let $\bar{X}$ and $\bar{Y}$ be a $(\bar{\mathcal{F}}_t^{\bar{P}})$-semimartingales and put $X = \bar{X} \circ \psi$ and $Y = \bar{Y} \circ \psi$. If $[\bar{X},\bar{Y}]$ is locally $\bar{P}$-integrable, then $[X,Y]$ is locally $P$-integrable. In this case the oblique brackets exist and we have*

$$\langle \bar{X},\bar{Y}\rangle^{\bar{P}} \circ \psi = \langle X,Y\rangle^P$$

*up to indistinguishability relative to $P$.*

*Proof.* Suppose $(\bar{T}_n)$ is a localizing sequence of $(\bar{\mathcal{F}}_t^{\bar{P}})$-stopping times such that $[\bar{X},\bar{Y}]^{\bar{T}_n}$ is $\bar{P}$-integrable. Then the functions $T_n = \bar{T}_n \circ \psi$ are $(\mathcal{G}_t^P)$-stopping times satisfying $\lim_{n\to\infty} T_n = \infty$, $P$-almost surely and the brackets $[X,Y]^{T_n}$ are $P$-integrable.

The process $A = \langle \bar{X},\bar{Y}\rangle^{\bar{P}} \circ \psi$ is $(G_t^P)$-predictable (see Lemma 2.1.2) and up to indistinguishability we have

$$[X,Y] - A = ([\bar{X},\bar{Y}] - \langle \bar{X},\bar{Y}\rangle) \circ \psi,$$

proving that $[X,Y] - A$ is a $(\mathcal{G}_t^P)$-local martingale. Hence $A$ is equal to the bracket $\langle X,Y\rangle^P$ and the proof is complete. □

We still need the following notion, introduced by Lenglart in [36]. Recall that we are always assuming $\bar{P}$ to be absolutely continuous with respect to $\bar{Q}$.

**Definition 2.2.2.** *Let $\bar{X}$ and $\bar{Y}$ be $(\bar{\mathcal{F}}_t^{\bar{Q}})$-semimartingales. The bracket $\langle \bar{X},\bar{Y}\rangle^{\bar{Q}}$ is said to* exist $\bar{P}$-almost surely *if there exists an increasing sequence $(\bar{T}_n)$ of stopping times such that $[\bar{X},\bar{Y}]$ is $\bar{Q}$-integrable on $[0,\bar{T}_n]$ and such that $\bar{T}_n$ converges to $\infty$, $\bar{P}$-a.s.*
*In this case, for all $n \geq 0$ the predictable projection of $[\bar{X},\bar{Y}]^{\bar{T}_n}$ with respect to $\bar{Q}$ exists and by putting these projections together, we obtain a process $\bar{A}$ defined on $\bigcup_{n\geq 0}[0,\bar{T}_n]$. More precisely, if $\bar{A}^n$ is the predictable projection of $[\bar{X},\bar{Y}]^{\bar{T}_n}$ we set*

$$\bar{A} = \bar{A}^n \quad on \quad ]\bar{T}_{n-1},\bar{T}_n].$$

*If $\bar{T}_n$ converges $\bar{Q}$-a.s. to $\infty$, then $\bar{A}$ is equal to $\langle \bar{X},\bar{Y}\rangle^{\bar{Q}}$. If $\bar{T}_n$ converges only $\bar{P}$-a.s to $\infty$, then $\bar{A}$ is* only *determined up to $\bar{P}$-null sets. In any case, we will write $\langle \bar{X},\bar{Y}\rangle^{\bar{Q}}$ for the process $\bar{A}$.*

Now let $\bar{M}$ be a $(\bar{\mathcal{F}}_t^{\bar{Q}}, \bar{Q})$-local martingale and suppose the bracket $\langle \bar{M}, \bar{Z} \rangle$ of $\bar{M}$ with our density process $\bar{Z}$ exists $\bar{P}$-a.s. By a theorem of Lenglart (Théorème 2 in [36]),

$$\bar{M} - \frac{1}{\bar{Z}_-} \cdot \langle \bar{M}, \bar{Z} \rangle^{\bar{Q}}$$

is a $(\bar{\mathcal{F}}_t^{\bar{P}}, \bar{P})$-local martingale. This may be translated into the smaller world $\Omega$ if $\langle \bar{M}, \bar{Z} \rangle^{\bar{Q}} \circ \psi$ is $P$-a.s. equal to $\langle M, Z \rangle^P$. This is not always the case, but fortunately one can avoid this problem by writing the drift in terms of $\langle \bar{M}, \bar{M} \rangle^{\bar{Q}}$.

**Lemma 2.2.3.** *Let $M$ be a $(\mathcal{F}_t^P, P)$-local martingale and $\bar{M}(\omega, \omega') = M(\omega)$. Then we have*

1) $[\bar{M}, \bar{M}] \circ \psi = [M, M]$ *a.s. relative to* $\bar{P}$ and $\bar{Q}$, *and*

2) *if $[M, M]$ is locally $P$-integrable, then $[\bar{M}, \bar{M}]$ is locally integrable relative to* $\bar{P}$ and $\bar{Q}$. *Moreover,*

$$\langle \bar{M}, \bar{M} \rangle^{\bar{Q}} = \langle \bar{M}, \bar{M} \rangle^{\bar{P}}.$$

*Proof.* Since $\bar{M}$ does not depend on $\omega'$ we may choose $[\bar{M}, \bar{M}](\omega, \omega') = [M, M](\omega)$, and thus 1) is trivially satisfied.
If the bracket $\langle M, M \rangle^P$ exists, then

$$[\bar{M}, \bar{M}](\omega, \omega') - \langle M, M \rangle^P(\omega)$$

is a local martingale with respect to $\bar{P}$ and $\bar{Q}$, and hence we have shown 2). □

**Theorem 2.2.4.** *Suppose that the bracket $\langle \bar{Z}, \bar{Z} \rangle^{\bar{Q}}$ exists $\bar{P}$-a.s. in the sense of Lenglart. If $M$ is a $(\mathcal{F}_t^P, P)$-local martingale with $M_0 = 0$ for which $\langle M, M \rangle^P$ exists (i.e. [M,M] is locally $P$-integrable), then there is a $(\mathcal{G}_t^P)$-predictable process $\alpha$ such that*

$$M - \alpha \cdot \langle M, M \rangle^P$$

*is a $(\mathcal{G}_t^P, P)$-local martingale.*

*Proof.* Put $\bar{M}(\omega, \omega') = M(\omega)$. By the preceding lemma $[\bar{M}, \bar{M}]$ is locally integrable relative to $\bar{P}$ and $\bar{Q}$. Now let $(\bar{T}_n)$ be a sequence of stopping times such that $[\bar{Z}, \bar{Z}]^{\bar{T}_n}$ and $[\bar{M}, \bar{M}]^{\bar{T}_n}$ are $\bar{Q}$-integrable and $\bar{T}_n \nearrow \infty$, $\bar{P}$-a.s. By standard arguments (see e.g. page 264 in [14]) it follows that $[\bar{M}, \bar{Z}]^{\bar{T}_n}$ is

$\bar{Q}$-integrable and thus $\langle \bar{M}, \bar{Z} \rangle^{\bar{Q}}$ exists $\bar{P}$-a.s. in the sense of Lenglart. Now Kunita-Watanabe implies that $\bar{Q}$-a.e. $\langle \bar{M}, \bar{Z} \rangle^{\bar{Q}}_{\cdot \wedge \bar{T}_n}$ is absolutely continuous with respect to $\langle \bar{M}, \bar{M} \rangle^{\bar{Q}}_{\cdot \wedge \bar{T}_n}$. Consequently there are $(\bar{\mathcal{F}}_t^{\bar{Q}})$-predictable processes $\bar{\beta}^n$ such that on $[0, \bar{T}_n]$

$$\bar{\beta}^n \cdot \langle \bar{M}, \bar{M} \rangle^{\bar{Q}} = \langle \bar{M}, \bar{Z} \rangle^{\bar{Q}}$$

$\bar{Q}$-a.s, and in particular $\bar{P}$-a.s. By putting the $\bar{\beta}^n$ together we obtain a process $\bar{\beta}$ defined on $\bigcup_n [0, \bar{T}_n]$. Since $\bar{T}_n$ converges to infinity $\bar{P}$-a.s., $\bar{\beta}$ is defined $\bar{P}$-almost everywhere and hence may be chosen to be $(\bar{\mathcal{F}}_t^{\bar{P}})$-predictable. Observe that for all $t \geq 0$ we have

$$\int_0^t \bar{\beta}_s \, d\langle \bar{M}, \bar{M} \rangle_s^{\bar{Q}} = \langle \bar{M}, \bar{Z} \rangle_t^{\bar{Q}}$$

$\bar{P}$-almost surely. By Lenglart-Girsanov (see Théorème 2 in [36]) and Lemma 2.2.3

$$\begin{aligned} \bar{M} - \int_0^{\cdot} \frac{1}{\bar{Z}_{s-}} \, d\langle \bar{M}, \bar{Z} \rangle_s^{\bar{Q}} &= \bar{M} - \int_0^{\cdot} \frac{\bar{\beta}_s}{\bar{Z}_{s-}} \, d\langle \bar{M}, \bar{M} \rangle_s^{\bar{Q}} \\ &= \bar{M} - \int_0^{\cdot} \frac{\bar{\beta}_s}{\bar{Z}_{s-}} \, d\langle \bar{M}, \bar{M} \rangle_s^{\bar{P}} \end{aligned}$$

is a $(\bar{\mathcal{F}}_t^{\bar{P}}, \bar{P})$-local martingale. The process $\beta = \bar{\beta} \circ \psi$ is $(\mathcal{G}_t^P)$-predictable by Lemma 2.1.2. Now Lemma 2.1.3 implies that

$$M - \int_0^{\cdot} \frac{\beta_s}{Z_{s-}} \, d\langle M, M \rangle_s^P$$

is a $(\mathcal{G}_t^P, P)$-local martingale, and hence the result with $\alpha = \frac{\beta}{Z_-}$. □

## 2.3 Comparison with Jacod's condition

In Jacod's paper (see [31]) the filtration $(\mathcal{F}_t)$ is supposed to be enlarged by some random variable $G$ taking values in a Lusin space $(E, \mathcal{E})$. As a consequence, for $t \in [0, T]$ regular conditional distributions $Q_t$ of $G$ relative to $\mathcal{F}_t$ exist. The following condition is assumed to be satisfied:

**(A')** There exists a $\sigma$-finite measure $\eta$ such that $Q_t(\omega, \cdot)$ is absolutely continuous with respect to $\eta$ for all $t > 0$ and $\omega \in \Omega$.

We will show that condition (A') implies our Assumption 2.1.6. More precisely, with $\mathcal{H}_t = \sigma(G)$, we have the following.

**Lemma 2.3.1.** *Suppose $\eta$ is a probability measure and (A') is satisfied. If $R$ is a measure such that $RG^{-1} = \eta$, then $\bar{P}$ is absolutely continuous with respect to $\bar{Q} = P \otimes R$ on $\bar{\mathcal{F}}_s$ for all $s \geq 0$.*

*Proof.* Let $t \geq 0$, $A \in \mathcal{F}_t$ and $B \in \mathcal{E}$. We put

$$\tilde{P}(A \times G^{-1}[B]) = \int_A Q_t(\omega, B)\, dP(\omega),$$

and extend $\tilde{P}$ to a probability measure on $\mathcal{F}_t \otimes \sigma(G)$. Note that for $A \in \mathcal{F}_t$ and $B \in \mathcal{E}$

$$\tilde{P}(A \times G^{-1}[B]) = P(A \cap G^{-1}[B]) = \bar{P}(A \times G^{-1}[B]),$$

and hence $\tilde{P} = \bar{P}$ on $\mathcal{F}_t \otimes \sigma(G)$. Now let $s < t$ and $C \in \bar{\mathcal{F}}_s$ with $\bar{Q}(C) = 0$. We claim that $\tilde{P}(C) = \bar{P}(C) = 0$.
Choose a set $D \in \mathcal{F}_t \otimes \mathcal{E}$ such that $C$ is the inverse image of $D$ under the map $(\omega, \omega') \mapsto (\omega, G(\omega'))$. Then $\int 1_D(\omega, x)\, d\eta(x) = 0$ for $P$-a.a. $\omega$. With assumption (A') we conclude that $\int 1_D(\omega, x)\, Q_t(\omega, dx) = 0$ for $P$-a.a. $\omega$, and hence

$$\tilde{P}(C) = \int \int 1_D(\omega, x)\, Q_t(\omega, dx)\, dP(\omega) = 0.$$

Thus we have shown the result. □

Jacod does not use Girsanov's theorem in his paper [31]. However, he points out that his results could also be deduced by applying it to the conditional measures $P^x = P(\cdot|G = x)$, $x \in E$. Condition (A') implies that the conditional measures $P^x$ are absolutely continuous with respect to $P$. Hence, by Girsanov, for a given $(\mathcal{F}_t, P)$-local martingale there is a drift $A^x$ such that $M - A^x$ is a $(\mathcal{F}_t, P^x)$-local martingale. By combining the processes $A^x$ we obtain that

$$M - A^G$$

is a $(\mathcal{G}_t, P)$-local martingale. The main work consists in proving that the processes $A^x$ can be combined in a meaningful way. As far as we know, Jacod's sketch has never been worked out rigorously.

In our approach we embed every local martingale into the product space $\bar{\Omega}$. We apply Girsanov's theorem on the product space and then translate our results back into the original space. One of the advantages of our approach is that we do not have to assume regular conditional distributions to exist. And we do not need to show how processes can be combined. Instead we have to show how one can transfer objects from $\Omega$ to $\bar{\Omega}$ and vice versa. Moreover we are not restricted to initial enlargements, but only to enlargements of the form

$$\mathcal{G}_t = \bigcap_{s>t} (\mathcal{F}_s \vee \mathcal{H}_s), \quad t \geq 0.$$

Starting with Jacod's results one can obtain decompositions for filtrations of this kind by using predictable projections. For this suppose $A$ to be a bounded variation process such that $M - A$ is a local martingale with respect to the initially enlarged filtration $(\mathcal{F}_t \vee \mathcal{H}_\infty)$. If $B$ is the predictable projection of $A$ onto $(\mathcal{G}_t)$, then $M - B$ is a $(\mathcal{G}_t)$-local martingale.

## 2.4 Decoupling on the original space

Switching to a product space is not always necessary in order to decouple the new information, represented by the enlarging filtration $(\mathcal{H}_t)$, from the old information given by $(\mathcal{F}_t)$. If $\bar{P} \sim \bar{Q}$, then a decoupling measure exists on the original space:

**Theorem 2.4.1.** *If $\bar{P} \sim \bar{Q}$ on $\bar{\mathcal{F}}$, then there exists a unique probability measure $Q$ on $(\Omega, \mathcal{G}_\infty)$ such that*

1. $Q \sim P$,
2. $Q|_{\mathcal{F}_\infty} = P|_{\mathcal{F}_\infty}$,
3. $Q|_{\mathcal{H}_\infty} = R|_{\mathcal{H}_\infty}$,
4. *$\mathcal{F}_\infty$ and $\mathcal{H}_\infty$ are independent relative to $Q$.*

**Remark 2.4.2.** *Let $M$ be an $(\mathcal{F}_t^P, P)$-local martingale and $Q$ the measure with properties* (1)-(4). *Then $M$ is also a $(\mathcal{G}_t^P, Q)$-local martingale. Therefore $Q$ is sometimes called* martingale preserving probability measure *(see [1]). It is an equivalent local martingale measure (ELMM) of $M$ relative to the enlarged filtration $(\mathcal{G}_t^P)$.*

*Proof.* Let $\bar{U} = \frac{d\bar{Q}}{d\bar{P}}\big|_{\bar{\mathcal{F}}}$ and $U = \bar{U} \circ \psi$. Observe that $U$ is $\mathcal{G}_\infty$-measurable and $\int U \, dP = \int \bar{U} \, d\bar{P} = 1$. We claim that the probability measure

$$dQ = U \, dP$$

satisfies the required properties. Note that $\bar{U} > 0$, $\bar{P}$-almost surely, and therefore $U > 0$, $P$-almost surely, which implies $Q \sim P$. In order to show the other properties, let $A \in \mathcal{F}_\infty$ and $B \in \mathcal{H}_\infty$. Then

$$\begin{aligned} Q(A \cap B) &= \int U 1_A 1_B \, dP \\ &= \int \bar{U} 1_{A\times\Omega}(\omega,\omega') 1_{\Omega\times B}(\omega,\omega') \, d\bar{P}(\omega,\omega') \\ &= \int 1_{A\times B} \, d\bar{Q}(\omega,\omega') \\ &= P(A)R(B), \end{aligned}$$

By choosing $B = \Omega$, we obtain property (2), and by choosing $A = \Omega$, property (3). Moreover, for all $A \in \mathcal{F}_\infty$ and $B \in \mathcal{H}_\infty$,

$$Q(A \cap B) = Q(A)Q(B), \tag{2.4}$$

which shows that $\mathcal{F}_\infty$ and $\mathcal{H}_\infty$ are independent under $Q$. The ($\pi$-$\lambda$) Theorem implies that any measure satisfying equation (2.4) is unique, and thus the proof is complete. □

Note that for a lot of easy examples a decoupling measure $Q$ does *not* exist on the original space. Let for example $\mathcal{H}_t = \sigma(A)$, $t \geq 0$, where $A \in \mathcal{F}_\infty$ and $R(A) = P(A) \in (0,1)$. Then the independence property would require

$$Q(A \cap A^c) = Q(A)\, Q(A^c) = P(A)\, P(A^c) \neq 0,$$

which is of course impossible. Therefore, semimartingale decompostions via Girsanov's theorem can in general only be obtained by considering the corresponding product space.

# Chapter 3

# Continuous embeddings

Suppose $M$ is a continuous $(\mathcal{F}_t^P, P)$-local martingale with $M_0 = 0$. Under the assumptions of the previous chapter we know that there is a $(\mathcal{G}_t^P)$-predictable process $\alpha$ such that $M - \alpha \cdot [M, M]$ is $(\mathcal{G}_t^P, P)$-local martingale. Moreover, the information drift $\alpha$ satisfies

$$(\alpha^2 \cdot [M, M])_\infty \leq [L, L]_\infty^c. \tag{3.1}$$

In this chapter we provide bounds for

$$E\left[(\alpha^2 \cdot [M, M])_\infty^p\right]$$

for various moments $p \geq 1$ based on inequality (3.1). This will allow us to derive sufficient and necessary conditions for the embedding of $(\mathcal{F}_t^P, P)$-semimartingales into the set of $(\mathcal{G}_t^P, P)$-semimartingales to be continuous with respect to vector space topologies defined on the set of semimartingales.

Throughout this chapter we make similar assumptions as in the previous chapter. However, we define the measure $\bar{Q}$ as the product of the measure $P$ with itself, i.e.

$$\bar{Q} = P|_{\mathcal{F}_\infty} \otimes P|_{\mathcal{H}_\infty}.$$

Besides, we assume again that $\bar{P} \ll \bar{Q}$, and maintain the notation. In particular, we denote by $\bar{Z}_t = \frac{d\bar{P}}{d\bar{Q}}\Big|_{\bar{\mathcal{F}}_t^{\bar{Q}}}$ the density process, and by $\bar{L}$ the stochastic logarithm of $\bar{Z}$ (see Equation (2.3)). We use again the decomposition of $\bar{L}$ into a continuous part $\bar{L}^c$ and a part $\bar{L}^d$ consisting of compensated jumps. As before we denote by $Z$, $L$ and $L^c$ the corresponding $(\mathcal{G}_t)$-adapted processes obtained by a right side application of $\psi$.

## 3.1 Estimating the moment $p = 1$

Recall that the relative entropy of two probability measures $P$ and $Q$ on some $\sigma$-algebra $\mathcal{M}$ is defined by

$$H_{\mathcal{M}}(P\|Q) = \begin{cases} E^P\left(\log \frac{dP}{dQ}\Big|_{\mathcal{M}}\right), & \text{if } P \ll Q \text{ on } \mathcal{M} \\ \infty, & \text{if not } P \ll Q \text{ on } \mathcal{M}. \end{cases}$$

In our situation, the relative entropy $H_{\bar{\mathcal{F}}_\infty}(\bar{P}\|\bar{Q})$ provides an upper bound for the first moment of $[L, L]^c$:

**Lemma 3.1.1.**

$$\frac{1}{2} E^P[L, L]^c_\infty \leq H_{\bar{\mathcal{F}}_\infty}(\bar{P}\|\bar{Q}).$$

*If $(\bar{Z}_t)_{t \geq 0}$ is continuous and $\bar{Z}_0 = 1$, then one even has*

$$\frac{1}{2} E^P[L, L]_\infty = H_{\bar{\mathcal{F}}_\infty}(\bar{P}\|\bar{Q}).$$

**Remark 3.1.2.** *If the $\sigma$-field $\mathcal{F}_0$ is trivial, then the measures $\bar{P}$ and $\bar{Q}$ coincide on $\mathcal{F}_0 \otimes \mathcal{H}_0$. As will be shown in Lemma 3.4.1 we have $\bar{\mathcal{F}}_0^{\bar{Q}} = (\mathcal{F}_0 \otimes \mathcal{H}_0)^{\bar{Q}}$, and hence in this case $\bar{Z}_0 = 1$.*

*Proof.* To prove the first statement, we decompose $\bar{L}$ into its continuous and discontinuous part $\bar{L} = \bar{L}^c + \bar{L}^d$ and let $\bar{Z}^c_t = \mathcal{E}(\bar{L}^c)_t$ and $\bar{Z}^d_t = \bar{Z}_0\, \mathcal{E}(\bar{L}^d)_t$. Then $\bar{Z}_t = \bar{Z}^c_t\, \bar{Z}^d_t$ on $[0, \bar{S}[$. The following results are only valid when stopping all processes at a stopping time $\bar{T} = \inf\{t > 0 : \bar{Z}_t < \varepsilon\}$ for some $\varepsilon > 0$. To simplify notation, we omit the stopping times in the following computations. One has

$$\begin{aligned} \log \bar{Z}_t &= \log(\mathcal{E}(\bar{L}^c))_t + \log(\bar{Z}^d_t) \\ &= (\bar{L}^c_t - [\bar{L}, \bar{L}]^c_t) + \frac{1}{2}[\bar{L}, \bar{L}]^c_t + \log \bar{Z}^d_t, \end{aligned}$$

where the term in the first brackets is a $(\bar{\mathcal{F}}^{\bar{P}}_t, \bar{P})$-local martingale due to Girsanov's theorem. Consider the function $\xi(x) = x \log x$ $(x \in [0, \infty))$ and denote $\bar{A}_t = \bar{Z}_t \log \bar{Z}^d_t = \bar{Z}^c_t \xi(\bar{Z}^d_t)$. Then Itô's formula yields

$$\begin{aligned} \bar{A}_t = \xi(\bar{Z}_0) &+ \int_{0+}^t \xi(\bar{Z}^d_{s-})\, d\bar{Z}^c_s + \int_{0+}^t \bar{Z}^c_{s-} \xi'(\bar{Z}^d_{s-})\, d\bar{Z}^d_s \\ &+ \sum_{0 < s \leq t} \bar{Z}^c_{s-}\big(\xi(\bar{Z}^d_s) - \xi(\bar{Z}^d_{s-}) - \xi'(\bar{Z}^d_{s-})\, \Delta \bar{Z}^d_s\big), \end{aligned}$$

where all summands in the previous line are non-negative due to the convexity of $\xi$.

Let now $(\bar{T}^n)$ denote a $\bar{P}$-localizing sequence of bounded stopping times such that the integrals in the stopped processes $(\bar{A}_t^{\bar{T}^n})$ are $\bar{Q}$-submartingales and the stopped processes $(\bar{L}^c - [\bar{L}, \bar{L}]^c)^{\bar{T}^n}$ are $\bar{P}$-martingales. Then

$$E^{\bar{Q}}[\bar{A}_{\bar{T}^n}] \geq E^{\bar{Q}}[\xi(\bar{Z}_0)] \geq 0,$$

which leads to

$$\begin{aligned} E^{\bar{Q}}[\xi(\bar{Z}_{\bar{T}^n})] &\geq E^{\bar{Q}}[\bar{Z}_{\bar{T}^n} \log \bar{Z}^c_{\bar{T}^n}] = E^{\bar{P}}[\log \bar{Z}^c_{\bar{T}^n}] \\ &= \frac{1}{2} E^{\bar{P}}[\bar{L}, \bar{L}]^c_{\bar{T}^n}. \end{aligned} \tag{3.2}$$

Note that by Jensen's inequality for all stopping times $T$

$$E^{\bar{Q}}[\xi(\bar{Z}_\infty)|\bar{\mathcal{F}}_T] \geq \xi\left(E^{\bar{Q}}[\bar{Z}_\infty|\bar{\mathcal{F}}_T]\right) = \xi\left(\bar{Z}_T\right),$$

from which we deduce that $E^{\bar{Q}}[\xi(\bar{Z}_{\bar{T}^n})] \leq E^{\bar{Q}}[\xi(\bar{Z}_\infty)] = H_{\bar{\mathcal{F}}_\infty}(\bar{P}\|\bar{Q})$. With (3.2) we arrive at

$$\frac{1}{2} E^{\bar{P}}[\bar{L}, \bar{L}]^c_{\bar{T}^n} \leq H_{\bar{\mathcal{F}}_\infty}(\bar{P}\|\bar{Q})$$

and monotone convergence implies that $\frac{1}{2} E^{\bar{P}}[\bar{L}, \bar{L}]^c_\infty \leq H_{\bar{\mathcal{F}}_\infty}(\bar{P}\|\bar{Q})$.

It remains to show that in case $(\bar{Z}_t)$ is continuous with $\bar{Z}_0 = 1$ and $E^{\bar{P}}[\bar{L}, \bar{L}]_\infty < \infty$, we have $\frac{1}{2} E^{\bar{P}}[\bar{L}, \bar{L}]_\infty \geq H_{\bar{\mathcal{F}}_\infty}(\bar{P}\|\bar{Q})$. Indeed, then $(\bar{L}_t - [\bar{L}, \bar{L}]_t)$ is an $L^2$-bounded $(\bar{\mathcal{F}}_t^{\bar{P}}, \bar{P})$-martingale and one has

$$\begin{aligned} \frac{1}{2} E^{\bar{P}}[\bar{L}, \bar{L}]_t &= E^{\bar{P}}\left[\bar{L}_t - [\bar{L}, \bar{L}]_t + \frac{1}{2}[\bar{L}, \bar{L}]_t\right] \\ &= E^{\bar{P}} \log \bar{Z}_t = E^{\bar{Q}}[\bar{Z}_t \log \bar{Z}_t]. \end{aligned}$$

The left hand side converges to $\frac{1}{2} E^{\bar{P}}[\bar{L}, \bar{L}]_\infty$ as $t \to \infty$. On the other hand, $\xi(\bar{Z}_t)$ converges to $\xi(\bar{Z}_\infty)$ so that by Fatou's lemma

$$\liminf_{t \to \infty} E^{\bar{Q}}[\bar{Z}_t \log \bar{Z}_t] \geq E^{\bar{Q}} \xi(\bar{Z}_\infty) = H_{\bar{\mathcal{F}}_\infty}(\bar{P}\|\bar{Q}).$$

This completes the proof. $\square$

## 3.2 Estimating moments $p > 1$

Now we consider moments of order $p > 1$. In this case the $p$-th moment of $[L, L]_\infty$ can be compared to some generalized relative entropy. See [26] for elementary versions of the inequalities to be derived.

Our analysis requires some additional assumption. We suppose that $(\mathcal{G}_t)$ is an initial enlargement of $(\mathcal{F}_t)$, i.e.

$$\mathcal{G}_t = \bigcap_{s>t} (\mathcal{F}_s \vee \mathcal{A}), \qquad t \geq 0,$$

where $\mathcal{A}$ is some fixed sub-$\sigma$-algebra of $\mathcal{F}$. Moreover, we assume that $\mathcal{F}_0$ is trivial. Additionally, we need to make the following assumption.

**Condition 3.2.1** (C)**.** *Every $(\mathcal{F}_t^P, P)$-martingale has a continuous modification.*

We shall see that under this condition $\bar{L}$ is a continuous $(\bar{\mathcal{F}}_t^{\bar{Q}}, \bar{Q})$-local martingale.

We begin with the definition of the generalized relative entropy.

**Definition 3.2.2.** *For $p > 1$, and probability measures $P \ll Q$ on a $\sigma$-algebra $\mathcal{M}$, let*

$$H^p_{\mathcal{M}}(P\|Q) = E^P\Big(\log_+ \frac{dP}{dQ}\Big|_{\mathcal{M}}\Big)^p.$$

We provide now an upper bound of $E[L, L]^p_\infty$ with the help of the generalized entropy of $\bar{P}$ with respect to $\bar{Q}$ on the set $\bar{\mathcal{F}}_\infty$. To simplify notations, we omit the $\sigma$-algebra $\bar{\mathcal{F}}_\infty$, and write only $H^p(\bar{P}\|\bar{Q})$ and $H(\bar{P}\|\bar{Q})$. The aim of this section is to prove

**Theorem 3.2.3.** *For any $p \geq 1$ there exists a universal constant $C = C(p) < \infty$ such that under the above assumptions one has*

$$E[L, L]^p_\infty \leq C\left[H(\bar{P}\|\bar{Q}) + H^p(\bar{P}\|\bar{Q})\right].$$

For the proof we need some auxiliary results. We start by showing that there exists a continuous modification for $\bar{Z}$.

**Lemma 3.2.4.** *Let $\bar{M}$ be a uniformly integrable $(\bar{\mathcal{F}}_t^{\bar{Q}}, \bar{Q})$-local martingale. If condition (C) is satisfied, then for $P$-a.a. $\omega'$ the process $\bar{M}^{\omega'} = \bar{M}(\cdot, \omega')$ is a $(\mathcal{F}_t^P)$-local martingale.*

*Proof.* Choose a modification such that every path of $\bar{M}$ is cadlag. Now let $\hat{M}$ be an $\mathcal{A} \otimes \mathcal{O}(\mathcal{F})$-measurable process such that for all $\omega'$ and $s \geq 0$

$$\hat{M}_s^{\omega'} = E^P[\bar{M}_\infty^{\omega'} | \mathcal{F}_s].$$

For the existence of such a process we refer to [46], Proposition 3. Put $C_t = \{\hat{M}_t > \bar{M}_t\}$. Clearly $C_t \in \bar{\mathcal{F}}_t^{\bar{Q}}$ and $C_t(\cdot, \omega') \in \mathcal{F}_t^P$ for all $P$-a.a. $\omega'$ (recall that $(\mathcal{F}_t)$ is right-continuous). Moreover for $t \geq 0$

$$\begin{aligned} & \int\int 1_{C_t}(\omega, \omega')(\hat{M}_t^{\omega'} - \bar{M}_t^{\omega'})\ dP(\omega)\ dP(\omega') \\ = \ & E^{\bar{Q}}[1_{C_t}(\hat{M}_t - \bar{M}_t)] \\ = \ & E^{\bar{Q}}[1_{C_t}(\hat{M}_t - \bar{M}_\infty)] \\ = \ & \int\int 1_{C_t}(\omega, \omega')(\hat{M}_t^{\omega'} - \bar{M}_\infty^{\omega'})\ dP(\omega)\ dP(\omega') \\ = \ & \int 0\ dP(\omega') = 0, \end{aligned}$$

A similar result holds true on the set $\{\hat{M}_t < \bar{M}_t\}$, and as a consequence we have for $P$-a.a. $\omega'$

$$\hat{M}_t(\cdot, \omega') = \bar{M}_t(\cdot, \omega'),\ P\text{-a.s.}$$

Hence for $P$-a.a. $\omega'$ the process $(\bar{M}_q^{\omega'})_{q \in \mathbb{Q}^+}$ is a $(\mathcal{F}_t^P)$-martingale. Since $\bar{M}_t$ is cadlag and uniformly integrable we obtain that also

$$(\bar{M}_t^{\omega'})_{t \geq 0}$$

is a $(\mathcal{F}_t^P)$-martingale for $P$-a.a. $\omega'$. □

**Lemma 3.2.5.** *If (C) is satisfied, then every uniformly integrable $(\bar{\mathcal{F}}_t^{\bar{Q}}, \bar{Q})$-local martingale has a continuous modification.*

*Proof.* Let $\bar{M}$ be a $(\bar{\mathcal{F}}_t^{\bar{Q}}, \bar{Q})$-local martingale. We may suppose that $\bar{M}$ is cadlag everywhere, and hence, the set

$$N = \{(\omega, \omega') : t \mapsto \bar{M}_t(\omega, \omega') \text{ is not continuous}\}$$

is measurable. Fix $\omega'$ and suppose that $\bar{M}^{\omega'}$ is a $(\mathcal{F}_t^P)$-martingale. Then condition (C) implies that for $P$-a.a. $\omega$ the paths $t \mapsto \bar{M}_t^{\omega'}(\omega)$ are continuous, i.e. $P(N^{\omega'}) = 0$. Now Fubini's Theorem yields with Lemma 3.2.4

$$\begin{aligned} E^{\bar{Q}}(N) &= \int\int 1_{N^{\omega'}}(\omega)\ dP(\omega)\ dP(\omega') \\ &= \int 0\ dP(\omega') = 0, \end{aligned}$$

and hence the result. □

For the rest of the section we will suppose that $\bar{Z}$ is a continuous modification of our density process $\frac{d\bar{P}}{d\bar{Q}}\Big|_{\bar{\mathcal{F}}_t^{\bar{Q}}}$. Similarly, $\bar{L}$ will be assumed to be continuous.

*Proof of Theorem 3.2.3.* We assume that $H(\bar{P}\|\bar{Q})$ and $H^p(\bar{P}\|\bar{Q})$ are finite. Then $X_t = \bar{L}_t - [\bar{L},\bar{L}]_t$ is a continuous $L^2$-bounded $\bar{P}$-martingale by Lemma 3.1.1 and we write $\log \bar{Z}_t = X_t + \frac{1}{2}A_t$ where $A_t = [\bar{L},\bar{L}]_t = [X,X]_t$. Next, observe that

$$\begin{aligned} H^p(\bar{P}\|\bar{Q})^{1/p} &= E^{\bar{P}}\big[(X_\infty + \frac{1}{2}A_\infty)_+^p\big]^{1/p} \\ &\geq E^{\bar{P}}\Big[\Big(\frac{1}{2}A_\infty - (|X_\infty| \wedge \frac{1}{2}A_\infty)\Big)^p\Big]^{1/p} \\ &\geq \frac{1}{2}E^{\bar{P}}\big[A_\infty^p\big]^{1/p} - E^{\bar{P}}\big[|X_\infty|^p\big]^{1/p} \\ &\geq \frac{1}{2}E^{\bar{P}}[A_\infty^p]^{1/p} - C\,E^{\bar{P}}\big[A_\infty^{p/2}\big]^{1/p}, \end{aligned} \tag{3.3}$$

where the last inequality holds for some constant $C > 0$ due to the Burkholder-Davis-Gundy inequality. Now choose $\xi > 0$ such that for all $x \geq 0$

$$C^p x^{p/2} \leq \xi^p x + \frac{1}{4^p}x^p.$$

This leads to

$$C^p E^{\bar{P}} A_\infty^{p/2} \leq \xi^p E^{\bar{P}} A_\infty + \frac{1}{4^p} E^{\bar{P}} A_\infty^p$$

and hence to

$$C\,E^{\bar{P}}\big[A_\infty^{p/2}\big]^{1/p} \leq \xi E^{\bar{P}}\big[A_\infty\big]^{1/p} + \frac{1}{4}E^{\bar{P}}\big[A_\infty^p\big]^{1/p}.$$

With (3.3) we conclude that

$$H^p(\bar{P}\|\bar{Q})^{1/p} \geq \frac{1}{4}E^{\bar{P}}\big[A_\infty^p\big]^{1/p} - \xi E^{\bar{P}}\big[A_\infty\big]^{1/p} = \frac{1}{4}E^{\bar{P}}\big[A_\infty^p\big]^{1/p} - \xi H(\bar{P}\|\bar{Q})^{1/p}.$$

Consequently,

$$\begin{aligned} E^{\bar{P}}\big[A_\infty^p\big]^{1/p} &\leq 4\xi H(\bar{P}\|\bar{Q})^{1/p} + 4H^p(\bar{P}\|\bar{Q})^{1/p} \\ &\leq 8(\xi^p H(\bar{P}\|\bar{Q}) + H^p(\bar{P}\|\bar{Q}))^{1/p}, \end{aligned}$$

where the last step follows from the elementary inequality $a+b \leq 2(a^p+b^p)^{1/p}$, $a,b \geq 0$. □

**Remark 3.2.6.** *The above proof is based on the fact that there exists a constant $C_p$ such that for any continuous $L^2$-bounded $\bar{P}$-martingale $(X_t)$ with $X_0 = 0$ and quadratic variation process $(A_t)$ one has*

$$E^{\bar{P}} A_\infty^p \leq C_p \, E^{\bar{P}} \Big[ X_\infty + \frac{1}{2} A_\infty + \big( X_\infty + \frac{1}{2} A_\infty \big)_+^p \Big].$$

*Improving the estimate to*

$$E^{\bar{P}} A_\infty^p \leq C_p \, E^{\bar{P}} \big( X_\infty + \frac{1}{2} A_\infty \big)_+^p \tag{3.4}$$

*would lead to the better estimate $E^P [L, L]^p \leq C_p \, H^p(\bar{P} \| \bar{Q})$. However, an estimate stating (3.4) is not valid, as the following example shows.*

**Example 3.2.7.** *Let $W$ be a Wiener process and for fixed $\varepsilon > 0$, let $T$ denote the first hitting time of the slope $t \mapsto \varepsilon - t/2$. We consider $X_t = W_t^T$ and $A_t = [X, X]_t$. Then by the Lévy-Bachelier formula the law of $T = A_\infty$ has density*

$$1_{(0,\infty)}(t) \, \frac{\varepsilon}{t^{3/2}} \, \phi\Big( \frac{\varepsilon - t/2}{\sqrt{t}} \Big),$$

*where $\phi$ is the density of the standard normal law. Hence,*

$$E[A_\infty^p] = \varepsilon \int_0^\infty t^{p-3/2} \, \phi\Big( \frac{\varepsilon - t/2}{\sqrt{t}} \Big) \, dt.$$

*In particular, for $\varepsilon \downarrow 0$, one has $E[A_\infty^p] \approx \varepsilon$. On the other hand,*

$$E\Big[ \big( X_\infty + \frac{1}{2} A_\infty \big)_+^p \Big] = E[(W_T + T/2)^p] = \varepsilon^p$$

*such that one can always find a sufficiently small $\varepsilon > 0$ for which the inequality (3.4) is not valid.*

We next show a result which in a sense contains the inverse statement to Theorem 3.2.3.

**Lemma 3.2.8.** *For $p \geq 1$ there exists a universal constant $C = C(p) < \infty$ such that*

$$H^p(\bar{P} \| \bar{Q}) \leq C \left[ E^{\bar{P}} [\bar{L}, \bar{L}]_\infty^p + 1 \right].$$

*In particular finiteness of $E^{\bar{P}} \big( [\bar{L}, \bar{L}]_\infty^p \big)$ implies finiteness of the entropy $H^p(\bar{P} \| \bar{Q})$.*

*Proof.* We have, by Burkholder-Davis-Gundy, with a universal constant $C_1$

$$\begin{aligned} H^p(\bar{P}\|\bar{Q})^{1/p} &\leq E\left(|\bar{L}_\infty - \frac{1}{2}[\bar{L},\bar{L}]_\infty|^p\right)^{1/p} \\ &\leq E\left(|\bar{L}_\infty|^p\right)^{1/p} + E\left(\frac{1}{2}[\bar{L},\bar{L}]^p_\infty\right)^{1/p} \\ &\leq C_1 E\left([\bar{L},\bar{L}]^{p/2}_\infty\right)^{1/p} + E\left(\frac{1}{2}[\bar{L},\bar{L}]^p_\infty\right)^{1/p} \\ &\leq C_1\left(1+E[\bar{L},\bar{L}]^p_\infty\right)^{1/p} + E\left(\frac{1}{2}[\bar{L},\bar{L}]^p_\infty\right)^{1/p} \\ &\leq C_2\left(1+E\left([\bar{L},\bar{L}]^p_\infty\right)\right)^{1/p}, \end{aligned}$$

and thus the result. □

Suppose now that the enlargement $\mathcal{A}$ is induced by some discrete random variable $G$, i.e. $\mathcal{A} = \sigma(G)$. In that case one can estimate the moments of $[L, L]_\infty$ against some generalized *absolute* entropy of $G$.

**Definition 3.2.9.** *Let $(q_g)$ denote the probability weights of $G$. We denote by*

$$H^p(G) = \sum_g q_g(\log 1/q_g)^p.$$

*the generalized absolute entropy of order p.*

**Lemma 3.2.10.** *One has*

$$H^p(\bar{P}\|\bar{Q}) \leq H^p(G),$$

*and if $G$ is $\mathcal{F}_\infty$-measurable, then*

$$H^p(\bar{P}\|\bar{Q}) = H^p(G).$$

*Proof.* For the proof we need a monotonicity property of $f$-divergences. Due to Corollary 1.29 in [37] one has

$$\begin{aligned} H^p(\bar{P}\|\bar{Q}) &= H^p(P_{\mathrm{id}_{\mathcal{F}_\infty},\mathrm{id}_\mathcal{A}}\|P_{\mathrm{id}_{\mathcal{F}_\infty}} \otimes P_{\mathrm{id}_\mathcal{A}}) \\ &\leq H^p(P_{\mathrm{id}_{\mathcal{F}_\infty},G,\mathrm{id}_\mathcal{A}}\|P_{\mathrm{id}_{\mathcal{F}_\infty},G} \otimes P_{\mathrm{id}_\mathcal{A}}). \end{aligned}$$

Moreover, if $G$ is $\mathcal{F}_\infty$-measurable, then one even has equality in the previous line. We denote by $(q_g)$ the probability weights of $G$. One easily verifies that

$$\frac{dP_{\mathrm{id}_{\mathcal{F}_\infty},G,\mathrm{id}_\mathcal{A}}}{dP_{\mathrm{id}_{\mathcal{F}_\infty},G} \otimes P_{\mathrm{id}_\mathcal{A}}}(\omega, g, \omega') = 1_{\{g=G(\omega')\}}\,\frac{1}{q_g}.$$

Set $f(g,g') = 1_{\{g=g'\}}\,\frac{1}{q_g}$. Then

$$
\begin{aligned}
& H^p(P_{\mathrm{id}_{\mathcal{F}_\infty},G,\mathrm{id}_{\mathcal{A}}} \| P_{\mathrm{id}_{\mathcal{F}_\infty},G} \otimes P_{\mathrm{id}_{\mathcal{A}}}) \\
= & \int f(g,G(\omega'))\,(\log_+ f(g,G(\omega')))^p \, d(P_{\mathrm{id}_{\mathcal{F}_\infty},G} \otimes P_{\mathrm{id}_{\mathcal{A}}})(\omega,g,\omega') \\
= & \int_{\{(g,\omega'):g=G(\omega')\}} \frac{1}{q_g}\left(\log_+ \frac{1}{q_g}\right)^p d(P_G \otimes P_{\mathrm{id}_{\mathcal{A}}})(g,\omega'),
\end{aligned}
$$

since $f(g,G(\omega')) = 0$ if $g \neq G(\omega')$ and the integrand does not depend on $\omega$. Altogether, we arrive at

$$
H^p(\bar{P}\|\bar{Q}) \leq \sum_g q_g \left(\log \frac{1}{q_g}\right)^p = H^p(G)
$$

and equality holds if $G$ is $\mathcal{F}_\infty$-measurable. □

**Example 3.2.11.** *Let $M_t = W_t$ denote a Wiener process and consider the completed filtration $(\mathcal{F}_t) = (\mathcal{F}_t^W)$ generated by the Wiener process. We now consider an initial enlargement of the filtration $(\mathcal{F}_t)$ by some arbitrary $\sigma$-field $\mathcal{A}$, i.e. $\mathcal{G}_t = \bigcap_{s>t}(\mathcal{F}_s \vee \mathcal{A})$. Supposing that $\bar{P} \ll \bar{Q}$, the Doob-Meyer decomposition for $W$ with respect to $(\mathcal{G}_t)$ is of the form*

$$
W_t = \tilde{W}_t + \int_0^t \alpha_s \, ds,
$$

*where $\tilde{W}$ is a $(\mathcal{G}_t)$-Wiener process and $\alpha$ is a $(\mathcal{G}_t)$-adapted process. In fact, $\tilde{W}$ is continuous with quadratic variation process $[\tilde{W},\tilde{W}]_t = t$. Moreover, since $\mathcal{F}_0$ is trivial and all $(\mathcal{F}_t)$-martingales have continuous modifications, the results of this section lead to the estimate*

$$
E\Big(\int_0^t \alpha_s^2 \, ds\Big)^p \leq C_p\big[H(\bar{P}\|\bar{Q}) + H^p(\bar{P}\|\bar{Q})\big].
$$

*If in addition $\mathcal{A} = \sigma(G)$ is generated by some discrete random variable $G$, then*

$$
E\Big(\int_0^t \alpha_s^2 \, ds\Big)^p \leq C_p\big[H(G) + H^p(G)\big].
$$

## 3.3 Continuity of initial enlargements

In this section we analyze to which extent the embedding of $(\mathcal{F}_t^P)$-semimartingales into some space of $(\mathcal{G}_t^P)$-semimartingales is continuous. For simplicity we restrict to initial enlargements. It turns out that the embedding is continuous if and only if some generalized entropy of the measures $\bar{P}$ and $\bar{Q}$ is finite. We refer to [26] for similar results in the setting of an enlargement of the Wiener filtration by a Malliavin differentiable random variable.

Throughout this section we assume that $\mathcal{F}_0$ is trivial and we let

$$\mathcal{G}_t = \bigcap_{s>t} (\mathcal{F}_s \vee \mathcal{A}), \quad t \geq 0,$$

where $\mathcal{A}$ is some fixed sub-$\sigma$-algebra of $\mathcal{F}$. We remind the reader that $\bar{P} \ll \bar{Q}$ and $\bar{Z}_t = \frac{d\bar{P}}{d\bar{Q}}\big|_{\mathcal{F}_t^{\bar{Q}}}$, $t \geq 0$.

We now recall the definition of some basic norms on the set of semimartingales. For this let $X$ be a $(\mathcal{F}_t^P)$-semimartingale. Given a decomposition $X = M + A$ we define for all $1 \leq p < \infty$

$$j_p(M, A) = \left\| [M, M]_\infty^{\frac{1}{2}} + \int_{[0,\infty[} |dA_s| \right\|_{L^p}.$$

Now let $\mathcal{S}^p$ be the set of all $(\mathcal{F}_t)$-semimartingales $X$ such that

$$\inf_{X=M+A} j_p(M, A) < \infty,$$

where the infimum is taken over all semimartingale decompositions $X = M + A$. One can show that each semimartingale in $\mathcal{S}^p$ is special. This means there is a decomposition $X = M' + A'$ such that $A'$ is predictable and $A'_0 = 0$. Such a decomposition is unique and will be referred to as the *canonical decomposition*. Note that our terminology implies that the bounded variation process of the canonical decomposition starts in 0.

We define a norm on $\mathcal{S}^p$ by

$$\|X\|_{\mathcal{S}^p} = j_p(M', A').$$

One can show that there exists a constant $c > 0$, depending only on $p$, such that $c\|X\|_{\mathcal{S}^p} \leq \inf_{X=M+A} j_p(M, A) \leq \|X\|_{\mathcal{S}^p}$. In other words, $\| \cdot \|_{\mathcal{S}^p}$ is equivalent to the norm defined by

$$\|X\|_e = \inf_{X=M+A} j_p(M, A)$$

(see Remark 98 c), Chapter VII, in [14]). Moreover $\mathcal{S}^p$ is a Banach space with the following properties (see e.g. [14]):

- The space of all martingales in $\mathcal{S}^p$, denoted by $\mathcal{H}^p$, is a closed subspace.
- The set of all continuous semimartingales in $\mathcal{S}^p$, denoted by $\mathcal{S}^p_c$, and the set of all continuous martingales in $\mathcal{S}^p$, denoted by $\mathcal{H}^p_c$, are closed subspaces.

- The set of all predictable processes with integrable variation, vanishing in 0 and with norm $A \mapsto \| \int |dA_s| \|_{L^p}$ is a closed subspace of $\mathcal{S}^p$.

We are now in a position to prove the first main result.

**Theorem 3.3.1.** *Suppose $H_{\bar{\mathcal{F}}_\infty}(\bar{P}\|\bar{Q}) = C < \infty$. Then the embedding*

$$\mathcal{H}^2_c(\mathcal{F}_t) \to \mathcal{S}^1(\mathcal{G}_t), X \mapsto X,$$

*is a continuous linear mapping with norm $\leq 1 + \sqrt{2C}$.*

*Proof.* Let $M \in \mathcal{H}^2_c(\mathcal{F}_t)$. By Theorem 2.1.10, $(M - [M,L]) + [M,L]$ is a decomposition relative to $(\mathcal{G}_t)$. The Kunita-Watanabe inequality implies

$$\left\| \int_0^\infty |d[M,L]_t| \right\|_1 \leq \|[L,L]_\infty^{\frac{1}{2}}\|_2 \|[M,M]_\infty^{\frac{1}{2}}\|_2.$$

Hence by Lemma 3.1.1

$$\begin{aligned} \|M\|_{\mathcal{S}^1(\mathcal{G}_t)} &= \left\| [M,M]_\infty^{\frac{1}{2}} + \int_0^\infty |d[M,L]_t| \right\|_1 \\ &\leq \left(1 + \|[L,L]_\infty^{\frac{1}{2}}\|_2\right) \|[M,M]_\infty^{\frac{1}{2}}\|_2 \\ &\leq \left(1 + (E[L,L]_\infty)^{\frac{1}{2}}\right) \|M\|_{\mathcal{H}^2(\mathcal{F}_t)} \\ &\leq (1 + \sqrt{2C}) \|M\|_{\mathcal{H}^2(\mathcal{F}_t)}, \end{aligned}$$

and the proof is complete. □

As an immediate consequence we get the following

**Corollary 3.3.2.** *Suppose $H_{\bar{\mathcal{F}}_\infty}(\bar{P}\|\bar{Q}) < \infty$. Then the embedding*

$$\mathcal{S}^2_c(\mathcal{F}_t) \to \mathcal{S}^1(\mathcal{G}_t), X \mapsto X,$$

*is a continuous linear mapping.*

We aim at generalizing Theorem 3.3.1 and Corollary 3.3.2. Starting from the Banach space $\mathcal{S}^r(\mathcal{F}_t)$ with $r > 1$, what are sufficient criteria for the embedding into the space of $(\mathcal{G}_t)$-semimartingales to be continuous?

From now on we will make the additional assumption that condition (C) is satisfied. In other words, we will assume that $\mathcal{H}^p_c(\mathcal{F}_t) = \mathcal{H}^p(\mathcal{F}_t)$ for $p > 1$.

We begin by stating a result obtained by Yor.

**Lemma 3.3.3.** *(see Lemme 2 in [48]) Let $r \geq 1$ and $p, q > 0$ such that $\frac{1}{r} = \frac{1}{2p} + \frac{1}{q}$. Then the following conditions are equivalent:*

*1) There is a constant $C > 0$ such that every continuous $(\mathcal{G}_t)$-local martingale satisfies*

$$\left\| \int_0^\infty |d[M,L]_t| \right\|_r \leq C \| [M,M]_\infty^{\frac{1}{2}} \|_q.$$

*2)* $E[[L,L]_\infty^p] < \infty$.

We are now ready to state the main theorem.

**Theorem 3.3.4.** *Suppose Condition (C) is satisfied and let $p \geq 1$ and $q, r \geq 0$ such that $\frac{1}{r} = \frac{1}{2p} + \frac{1}{q}$. The generalized entropy $H^p(\bar{P} \| \bar{Q})$ is finite if and only if the embedding*

$$\mathcal{S}^q(\mathcal{F}_t) \to \mathcal{S}^r(\mathcal{G}_t), X \mapsto X,$$

*is a continuous linear mapping.*

*Proof.* Suppose $H^p(\bar{P} \| \bar{Q}) < \infty$. Theorem 3.2.3 implies that $[L,L]_\infty$ is $L^p$-integrable. Thus, by Lemma 3.3.3, there is a constant $C > 0$ such that for all continuous $(\mathcal{G}_t)$-local martingales we have

$$\left\| \int_0^\infty |d[M,L]_s| \right\|_{L^r} \leq C \left\| [M,M]_\infty^{\frac{1}{2}} \right\|_{L^q}.$$

Hence, for a martingale $M$ in $\mathcal{S}^q(\mathcal{F}_t)$ with decomposition $M = (M - [M,L]) + [M,L]$ relative to $(\mathcal{G}_t)$, we have

$$\begin{aligned} \|M\|_{\mathcal{S}^r(\mathcal{G}_t)} &= \left\| [M,M]_\infty^{\frac{1}{2}} + \int_0^\infty |d[M,L]_s| \right\|_{L^r} \\ &\leq \| [M,M]_\infty^{\frac{1}{2}} \|_{L^r} + \left\| \int_0^\infty |d[M,L]_s| \right\|_{L^r} \\ &\leq \| [M,M]_\infty^{\frac{1}{2}} \|_{L^r} + C \| [M,M]_\infty^{\frac{1}{2}} \|_{L^q} \\ &\leq (1+C) \| [M,M]_\infty^{\frac{1}{2}} \|_{L^q} \\ &\leq (1+C) \|M\|_{\mathcal{S}^q(\mathcal{F}_t)}. \end{aligned}$$

Therefore the map $\mathcal{S}^q(\mathcal{F}_t) \to \mathcal{S}^r(\mathcal{G}_t), X \mapsto X$, is continuous.

Now suppose the embedding to be continuous. Then Lemma 3.3.3 implies

$$E[[\bar{L}, \bar{L}]_\infty^p] < \infty.$$

So by Lemma 3.2.8 the proof is complete. ∎

**Example 3.3.5.** *Suppose $\mathcal{A}$ is generated by a countable partition $\mathcal{P} = \{A_1, A_2, \ldots\}$ of $\Omega$ into $\mathcal{F}_\infty$-measurable sets. Then the corresponding initial enlargement can be viewed as an enlargement by the discrete random variable $G(\omega) = \sum_n n\,1_{A_n}(\omega)$. Hence, for $p \geq 1$, we have by Lemma 3.2.10*

$$H^p_{\bar{\mathcal{F}}_\infty}(\bar{P}\|\bar{Q}) = \sum_{i\geq 1} P(A_i)\left(\log \frac{1}{P(A_i)}\right)^p.$$

*Now let $q, r \geq 0$ such that $\frac{1}{r} = \frac{1}{2p} + \frac{1}{q}$. Theorem 3.3.4 implies that the embedding $\mathcal{S}^q(\mathcal{F}_t) \to \mathcal{S}^r(\mathcal{G}_t), X \mapsto X$, is a continuous if and only if*

$$\sum_{i\geq 1} P(A_i)\left(\log \frac{1}{P(A_i)}\right)^p < \infty.$$

*This result was already shown by Marc Yor, using different arguments (see Théorème 2 in [48]).*

If the filtration $(\mathcal{F}_t)$ is generated by a fixed martingale $M$ with cadlag paths, then the relative entropy of $\bar{P}$ with respect to $\bar{Q}$ is equal to the so-called mutual information between $M$ and the enlarging $\sigma$-algebra $\mathcal{A}$. We recall this notion.

**Definition 3.3.6.** *Let $X$ and $Y$ be two random variables with values in the measure spaces $(M, \mathcal{M})$ and $(K, \mathcal{K})$ respectively. The* mutual information *between $X$ and $Y$ is defined by*

$$I(X, Y) = H_{\mathcal{M}\otimes\mathcal{K}}(P_{(X,Y)}\|P_X \otimes P_Y).$$

*Similarly, one can define the generalized mutual information to be*

$$I^p(X, Y) = H^p_{\mathcal{M}\otimes\mathcal{K}}(P_{(X,Y)}\|P_X \otimes P_Y), \quad p > 1.$$

*For a given $\sigma$-algebra $\mathcal{J} \subset \mathcal{F}$, let $\mathrm{id}_\mathcal{J}$ denote the map $(\Omega, \mathcal{F}) \to (\Omega, \mathcal{J}), \omega \mapsto \omega$. The mutual information between $X$ and $\mathcal{J}$ is defined by*

$$I(X, \mathcal{J}) = I(X, \mathrm{id}_\mathcal{J}).$$

We start with the following observation.

**Lemma 3.3.7.** *If $(\mathcal{F}_t)$ equals the filtration generated by $M$, then*

$$I(M, \mathcal{A}) = H_{\bar{\mathcal{F}}_\infty}(\bar{P}\|\bar{Q}),$$

*and for $p > 1$,*

$$I^p(M, \mathcal{A}) = H^p_{\bar{\mathcal{F}}_\infty}(\bar{P}\|\bar{Q}).$$

*Proof.* First observe that $\bar{\mathcal{F}}_\infty = \mathcal{F}_\infty \otimes \mathcal{A}$, because

$$\bar{\mathcal{F}}_\infty = \bigvee_t \bar{\mathcal{F}}_t \subset \bigvee_t (\mathcal{F}_t \otimes \mathcal{A}) \subset \mathcal{F}_\infty \otimes \mathcal{A} \subset \bar{\mathcal{F}}_\infty.$$

Now let $\mathbb{D}$ denote the Skorokhod space. We define a map $\phi$ by

$$\Omega \times \Omega \to \mathbb{D} \times \Omega, (\omega, \omega') \mapsto (M.(\omega), \omega').$$

Since $\mathcal{F}_\infty$ is generated by $M$, we have

$$\phi^{-1}(\mathcal{B}(\mathbb{D}) \otimes \mathcal{A}) = M^{-1}(\mathcal{B}(\mathbb{D})) \otimes \mathcal{A} = \mathcal{F}_\infty \otimes \mathcal{A},$$

and hence

$$H_{\bar{\mathcal{F}}_\infty}(\bar{P} \| \bar{Q}) = H_{\mathcal{B}(\mathbb{D}) \otimes \mathcal{A}}(\bar{P}_\phi \| \bar{Q}_\phi).$$

Now observe

$$\bar{P}_\phi = P_{\phi \circ \psi} = P_{(M, \mathrm{id}_\mathcal{A})}$$

and

$$\bar{Q}_\phi = P_M \otimes P_{\mathrm{id}_\mathcal{A}},$$

which yields the first claim. The second follows by similar arguments. □

As a consequence we obtain the following.

**Theorem 3.3.8.** *Suppose Condition (C) is satisfied and let $p \geq 1$ and $q, r \geq 0$ such that $\frac{1}{r} = \frac{1}{2p} + \frac{1}{q}$. If $(\mathcal{F}_t)$ equals the filtration generated by $M$, then the generalized mutual information $I^p(M, \mathcal{A})$ is finite if and only if the embedding*

$$\mathcal{S}^q(\mathcal{F}_t) \to \mathcal{S}^r(\mathcal{G}_t), X \mapsto X,$$

*is a continuous linear mapping.*

*Proof.* This follows by combining Theorem 3.3.4 with Lemma 3.3.7. □

**Example 3.3.9.** *Let $W$ be the standard Wiener process and $(\mathcal{F}_t)$ the filtration generated by $W$ and completed by the negligible sets relative to the Wiener measure. Moreover, let $V$ be a Gaussian element independent of $\mathcal{F}_\infty$, with zero mean and variance $w > 0$. Suppose the enlarging $\sigma$-algebra $\mathcal{A}$ is generated by the random variable*

$$W_1 + V.$$

*One can easily verify that three random variables $X, Y$ and $Z$ satisfy*

$$I^p(X, (Y, Z)) \leq I^p(X, Z) + I^p(X, Y|Z) \quad (p \geq 1).$$

*Consequently, we obtain for the mutual information between* $\mathrm{id}_{\mathcal{A}}$ *and* $W$

$$\begin{aligned} I^p(W, \mathrm{id}_{\mathcal{A}}) &= I^p(W_1 + V, (W_1, (W_t)_{0 \le t < 1})) \\ &\le I^p(W_1 + V, W_1) + I^p(W_1 + V, (W_t)_{0 \le t < 1} | W_1) \\ &= I^p(W_1, W_1 + V) \\ &< \infty. \end{aligned}$$

*Thus, for all* $p \geq 1$ *and* $q, r \geq 0$ *such that* $\frac{1}{r} = \frac{1}{2p} + \frac{1}{q}$, *the mapping* $\mathcal{S}^q(\mathcal{F}_t) \to \mathcal{S}^r(\mathcal{G}_t), X \mapsto X$, *is continuous.*

## 3.4 Appendix: Product filtrations satisfying the usual conditions

Let $(\Omega, \mathcal{F}, P)$ be a probability space with right-continuous filtrations $(\mathcal{F}_t)_{t \geq 0}$ and $(\mathcal{H}_t)_{t \geq 0}$. In Chapter 2 the filtration $(\mathcal{G}_t)$ has been defined as the smallest right-continuous filtration containing

$$\mathcal{G}_t^0 = \mathcal{F}_t \vee \mathcal{H}_t, \quad t \geq 0.$$

We will see that under suitable conditions the filtration $(\mathcal{G}_t^0)$ itself is already right-continuous.

We start by analyzing the filtration $(\bar{\mathcal{F}}_t)$ on our product space $\bar{\Omega}$.

**Lemma 3.4.1.** *The filtration*

$$\mathcal{F}_t \otimes \mathcal{H}_t, \quad t \geq 0,$$

*completed by the* $\bar{Q}$*-negligible sets, is right-continuous. In particular,* $\bar{\mathcal{F}}_t^{\bar{Q}} = (\mathcal{F}_t \otimes \mathcal{H}_t)^{\bar{Q}}$ *for* $t \geq 0$.

*Proof.* The filtrations $(\mathcal{F}_t \otimes \{\emptyset, \Omega\})$ and $(\{\emptyset, \Omega\} \otimes \mathcal{H}_t)$ are right-continuous and independent with respect to $\bar{Q}$. According to a result by Wu and Wang, the filtration

$$(\mathcal{F}_t \otimes \{\emptyset, \Omega\}) \vee (\{\emptyset, \Omega\} \otimes \mathcal{H}_t), \quad t \geq 0,$$

completed by the $\bar{Q}$-negligible sets, is right-continuous (see Theorem 1 in [24]). □

With the help of the preceding lemma we can easily derive a sufficient criterion for the filtration $(\mathcal{G}_t^0)$ to satisfy the usual conditions.

**Theorem 3.4.2.** *Suppose that* $(\mathcal{F}_t)_{t \geq 0}$ *and* $(\mathcal{H}_t)_{t \geq 0}$ *are completed by the* $P$*-negligible sets, and hence satisfy the usual conditions. If* $\bar{P} \ll \bar{Q}$, *then also* $(\mathcal{G}_t^0)$ *satisfies the usual conditions.*

*Proof.* Let $\mathcal{N}^{\bar{Q}}$ denote the set of $\bar{Q}$-negligible sets. By Lemma 3.4.1, the filtration

$$(\mathcal{F}_t \otimes \mathcal{H}_t) \vee \mathcal{N}^{\bar{Q}}, \quad t \geq 0,$$

is right-continuous. Consequently,

$$\begin{aligned} \psi^{-1}\left((\mathcal{F}_t \otimes \mathcal{H}_t) \vee \mathcal{N}^{\bar{Q}}\right) &= \mathcal{F}_t \vee \mathcal{H}_t \vee \psi^{-1}(\mathcal{N}^{\bar{Q}}) \\ &= \mathcal{G}_t^0 \vee \psi^{-1}(\mathcal{N}^{\bar{Q}}) \end{aligned}$$

is right-continuous, too. Since $\bar{P} \ll \bar{Q}$, we have $\psi^{-1}(\mathcal{N}^{\bar{Q}}) \subset \mathcal{N}^P$. This implies that $(\mathcal{G}_t^0 \vee \psi^{-1}(\mathcal{N}^{\bar{Q}}))$ is equal to $(\mathcal{G}_t^0)$, and hence the result. □

# Chapter 4

# Distances between filtrations

Let $(\mathcal{F}_t)$ be filtration with usual conditions on some fixed probability space. The vector space of all processes which are semimartingales with respect to $(\mathcal{F}_t)$ has been studied intensively in the literature. In this chapter we turn things around. Instead of fixing a filtration, we fix a stochastic process $X$ and consider the set of all filtrations under which $X$ is a semimartingale. It turns out that metrics can be defined very easily on this set. Moreover, we prove that our metrics are complete if the given process $X$ is continuous.

Throughout this chapter we make the convention that any filtration considered satisfies the usual conditions.

## 4.1 $\mathcal{H}^p$ and $\mathcal{S}^p$ revisited

In this section we consider again the spaces $\mathcal{H}^p$ and $\mathcal{S}^p$. We prove some properties allowing us to introduce metrics on the set of filtrations under which a given process is a semimartingale.

Let $(\mathcal{F}_t)$ be filtration. Recall that for all $p \in [1, \infty[$

$$\|M\|_{\mathcal{H}^p} = \|[M, M]_\infty^{\frac{1}{2}}\|_{L^p}$$

defines a norm on $\mathcal{H}^p$. By the Burkholder-Davis-Gundy Inequalty, for every process $M \in \mathcal{H}^p$ the supremum $M_\infty^* = \sup_t |M_t|$ belongs to $L^p$ and the norm defined by $M \mapsto \|M_\infty^*\|_{L^p}$ is equivalent to $\|\cdot\|_{\mathcal{H}^p}$. We will make use of the following well-known fact.

**Lemma 4.1.1.** *Any local martingale in $\mathcal{H}^p$ is a strict martingale.*

*Proof.* Let $(T_n)$ be a localizing sequence of stopping times such that for all $s < t$

$$E[M_{t\wedge T_n}|\mathcal{F}_s] = M_{s\wedge T_n}. \tag{4.1}$$

Observe that

$$|M_{t\wedge T_n}| \leq M^*_{t\wedge T_n} \leq M^*_\infty$$

and

$$E|M^*_\infty| \leq \|M^*_\infty\|_{L^p} < \infty,$$

which shows that $(M_{t\wedge T_n})$ is uniformly integrable. Obviously the same holds true for $(M_{s\wedge T_n})$. By taking limits in equation (4.1) we obtain

$$E[M_t|\mathcal{F}_s] = M_s.$$

□

Recall that $\mathcal{H}^p$ is a closed subspace of the Banach space $\mathcal{S}^p$ and that the norms $\|\cdot\|_{\mathcal{H}^p}$ and $\|\cdot\|_{\mathcal{S}^p}$ coincide on $\mathcal{H}^p$. Also the space of all predictable processes with integrable variation, vanishing in 0 and with norm $A \mapsto \|\int |dA_s|\|_{L^p}$ is a closed subspace of $\mathcal{S}^p$.

**Lemma 4.1.2.** *Let $X \in \mathcal{S}^p$. Then $[X,X]^{\frac{1}{2}}_\infty \in L^p$ and $X_t$ is integrable for all $t \geq 0$.*

*Proof.* Let $X = M + A$ be the canonical decomposition. Then

$$\begin{aligned} [X,X]^{\frac{1}{2}}_\infty &\leq [M,M]^{\frac{1}{2}}_\infty + [A,A]^{\frac{1}{2}}_\infty \\ &\leq [M,M]^{\frac{1}{2}}_\infty + \int |dA_s|, \end{aligned}$$

and hence $[X,X]^{\frac{1}{2}}_\infty \in L^p$.

Note that $[M,M]^{\frac{1}{2}}_\infty \in L^p$, and hence the Burkholder-Davis-Gundy Inequality implies that $M_t$ is integrable. Note that also $A_t$ is integrable, and therefore $X_t$, too. □

The norm $\|\cdot\|_{\mathcal{S}^p}$ depends of course on the underlying filtration. If we want to stress the filtration we are referring to, we write $\mathcal{S}^p = \mathcal{S}^p(\mathcal{F}_t)$.

We start with an easy observation.

**Lemma 4.1.3.** *Let $X$ be $(\mathcal{F}_t)$-adapted, $(\mathcal{G}_t)$ a filtration containing $(\mathcal{F}_t)$ and $X \in \mathcal{S}^p(\mathcal{G}_t)$ with canonical decomposition $X = M + A$. Then $X$ belongs also to $\mathcal{S}^p(\mathcal{H}_t)$, where*

$$\mathcal{H}_t = \bigcap_{s>t} (\mathcal{F}_s \vee \sigma(A_r : r \leq s)).$$

*Moreover, if $X$ is continuous, then $\|X\|_{\mathcal{S}^p(\mathcal{G}_t)} = \|X\|_{\mathcal{S}^p(\mathcal{H}_t)}$.*

*Proof.* Note that $M = X - A$ is a $(\mathcal{G}_t)$-local martingale adapted to $(\mathcal{H}_t)$. Since $\mathcal{H}_t \subset \mathcal{G}_t$ for all $t \geq 0$, it follows that $M$ is also a $(\mathcal{H}_t)$-local martingale. Hence $X = M + A$ is also a decomposition relative to $(\mathcal{H}_t)$ (probably not the canonical one), and we conclude that $X$ belongs to $\mathcal{S}^p(\mathcal{H}_t)$.

If $X$ is continuous, then $A$ is continuous and consequently predictable with respect to $(\mathcal{H}_t)$. Thus we have $\|X\|_{\mathcal{S}^p(\mathcal{G}_t)} = \|X\|_{\mathcal{S}^p(\mathcal{H}_t)}$ and the proof is complete. □

The following example shows that the norms $\|X\|_{\mathcal{S}^p(\mathcal{G}_t)}$ and $\|X\|_{\mathcal{S}^p(\mathcal{H}_t)}$ may not be equal if $X$ is not continuous.

**Example 4.1.4.** *Let $\xi$ be a random variable with distribution $P(\xi = 2) = P(\xi = -1) = \frac{1}{2}$. Consider now the stochastic process $X$ defined by*

$$X_t = \begin{cases} 0, & \text{if } 0 \leq t < 1, \\ \xi, & \text{if } t \geq 1. \end{cases}$$

*Put $\mathcal{G}_t = \sigma(\xi)$ for all $t \geq 0$. Observe that $X$ is predictable with respect to $(\mathcal{G}_t)$ and hence*

$$\|X\|_{\mathcal{S}^1(\mathcal{G}_t)} = \left\| \int |dX_t| \right\|_{L^1} = \frac{1}{2}2 + \frac{1}{2}1 = 1,5.$$

*Now let $(\mathcal{H}_t)$ be the filtration generated by $X$. With respect to $(\mathcal{H}_t)$, $X$ is no longer predictable. It is straightforward to show that the predictable projection of $X$ onto $(\mathcal{H}_t)$ is equal to the deterministic process $A_t = \frac{1}{2}1_{[1,\infty[}$, and hence*

$$\|X\|_{\mathcal{S}^1(\mathcal{H}_t)} = \left\| [X - A, X - A]_\infty^{\frac{1}{2}} + \int |dA_t| \right\|_{L^1} = \|1,5 + 0,5\|_{L^1} = 2.$$

*Note that $\|X\|_{\mathcal{S}^1(\mathcal{G}_t)} \neq \|X\|_{\mathcal{S}^1(\mathcal{H}_t)}$, which shows that the last statement in Lemma 4.1.3 does not hold for arbitrary processes with cadlag paths.*

Let $X$ be an $(\mathcal{F}_t)$-adapted process with cadlag paths and $(\mathcal{G}_t^n)$ a sequence of filtrations satisfying $\mathcal{G}_t^n \supset \mathcal{F}_t$ for all $t \geq 0$. Suppose that $X$ belongs to $\mathcal{S}^p(\mathcal{G}_t^n)$ for all $n \geq 1$ and denote by

$$X = M^n + A^n$$

the corresponding canonical decompositions, where $A^n$ is $(\mathcal{G}_t^n)$-predictable.

**Lemma 4.1.5.** *Suppose $(A^n)$ converges to some process $A$ relative to the norm $B \mapsto \| \int |dB_s| \|_{L^p}$. Then $X - A$ is a $(\mathcal{G}_t)$-martingale and $X$ belongs to $\mathcal{S}^p(\mathcal{G}_t)$, where*

$$\mathcal{G}_t = \bigcap_{s>t} (\mathcal{F}_s \vee \sigma(A_r : r \leq s)).$$

*Moreover, if $X$ is continuous, then $\lim_n \|X\|_{\mathcal{S}^p(\mathcal{G}_t^n)} = \|X\|_{\mathcal{S}^p(\mathcal{G}_t)}$.*

*Proof.* Suppose $(A^n)$ converges to $A$. Observe at first that $A$ is a finite variation process satisfying $\| \int |dA_s| \|_{L^p} < \infty$. Now fix $0 \leq s < t$. We start by showing that for all $k \geq 1$, $0 \leq r_1 \leq \ldots \leq r_k \leq s$, $C \in \mathcal{F}_s$ and open sets $U_1, \ldots, U_k \subset \mathbb{R}$,

$$E\left[1_C 1_{\{A_{r_1} \in U_1, \ldots, A_{r_k} \in U_k\}}(X_t - X_s)\right] = E\left[1_C 1_{\{A_{r_1} \in U_1, \ldots, A_{r_k} \in U_k\}}(A_t - A_s)\right]. \tag{4.2}$$

By a monotone class theorem it then follows that $X - A$ is a martingale with respect to the filtration $(\mathcal{F}_t \vee \sigma(A_r : r \leq t))$. Integrability and right-continuity imply that $X - A$ is also a $(\mathcal{G}_t)$-martingale.

Observe that $|A^n_{r_i} - A_{r_i}| \leq \int_0^{r_i} |d(A^n - A)_t|$ implies the sequence $(A^n_{r_i})$ to converge to $A_{r_i}$ in $L^p$. By taking subsequences, we may assume that the sequence is converging almost surely. Since $U_i$ is open, this implies

$$1_{\{A^n_{r_i} \in U_i\}} \longrightarrow 1_{\{A_{r_i} \in U_i\}} \text{ almost surely.}$$

Due to Lemma 4.1.1 the process $M^n = X - A^n$ is a strict $(\mathcal{G}^n_t)$-martingale, i.e. for all $n \geq 1$

$$E\left[1_C 1_{\{A^n_{r_1} \in U_1, \ldots, A^n_{r_k} \in U_k\}}(X_t - X_s)\right] = E\left[1_C 1_{\{A^n_{r_1} \in U_1, \ldots, A^n_{r_k} \in U_k\}}(A^n_t - A^n_s)\right].$$

By dominated convergence the left hand side converges to

$$E\left[1_C 1_{\{A_{r_1} \in U_1, \ldots, A_{r_k} \in U_k\}}(X_t - X_s)\right],$$

and due to uniform integrability the right hand side to

$$E\left[1_C 1_{\{A_{r_1} \in U_1, \ldots, A_{r_k} \in U_k\}}(A_t - A_s)\right],$$

which yields equation (4.2).

We still have to show that $\|X\|_{\mathcal{S}^p(\mathcal{G}_t)}$ is finite. For this recall that the quadratic variation process $[X, X]$ is the same under $(\mathcal{G}_t)$ and $(\mathcal{G}^n_t)$. By Lemma 4.1.2 we have $[X, X]^{\frac{1}{2}}_\infty \in L^p$. From

$$\begin{aligned} [X - A, X - A]^{\frac{1}{2}}_\infty &\leq [X, X]^{\frac{1}{2}}_\infty + [A, A]^{\frac{1}{2}}_\infty \\ &\leq [X, X]^{\frac{1}{2}}_\infty + \int |dA_s| \end{aligned}$$

we deduce that

$$\begin{aligned} \|X\|_e &\leq \left\| [X - A, X - A]^{\frac{1}{2}}_\infty + \int |dA_s| \right\|_{L^p} \\ &\leq \left\| [X, X]^{\frac{1}{2}}_\infty + 2 \int |dA_s| \right\|_{L^p} < \infty, \end{aligned}$$

with $\|\cdot\|_e$ defined as in Section 3.3. Since $\|\cdot\|_e$ is equivalent to $\|\cdot\|_{\mathcal{S}^p(\mathcal{G}_t)}$, we have $X \in \mathcal{S}^p(\mathcal{G}_t)$.

Finally suppose that $X$ is continuous. This implies the processes $A^n$ to be continuous and hence, the limit $A$, too. Therefore $(X - A) + A$ is the canonical decomposition of $X$ with respect to $(\mathcal{G}_t)$. Moreover, continuity implies $[X-A^n, X-A^n] = [X, X] = [X-A, X-A]$ up to indistinguishability. Thus

$$\lim_n \left\| [X - A^n, X - A^n]_\infty^{\frac{1}{2}} + \int |dA_s^n| \right\|_{L_p} = \left\| [X - A, X - A]_\infty^{\frac{1}{2}} + \int |dA_s| \right\|_{L_p},$$

and the proof is complete. □

**Remark 4.1.6.** *Example 4.1.4 shows that if $X$ is not continuous, then it may happen that* $\lim_n \|X\|_{\mathcal{S}^p(\mathcal{G}_t^n)} \neq \|X\|_{\mathcal{S}^p(\mathcal{G}_t)}$.

## 4.2 A metric on the set of filtrations under which a given process is a semimartingale

We now introduce a metric on the set of filtrations for which a given stochastic process is a semimartingale. Up to the end of this section let $X$ be a process with cadlag paths. We consider the set

$$\mathbb{F}^p(X) = \{(\mathcal{G}_t) \text{ with the usual conditions} : X \in \mathcal{S}^p(\mathcal{G}_t)\}$$

with $p \in [1, \infty[$. For simplicity we assume that $\mathbb{F}^p(X)$ is not empty.

We denote by $(\mathcal{F}_t^X)$ the smallest filtration with the usual conditions such that $X$ is adapted to it. Obviously every filtration $(\mathcal{G}_t) \in \mathbb{F}^p(X)$ contains $(\mathcal{F}_t^X)$.

**Definition 4.2.1.** *Let $p \geq 1$, $(\mathcal{G}_t)$ and $(\mathcal{H}_t)$ filtrations in $\mathbb{F}^p(X)$ with canonical decompositions $X = M + A$ and $X = N + B$ respectively. The* distance *$d_p$ between $(\mathcal{G}_t)$ and $(\mathcal{H}_t)$ relative to $X$ is defined by*

$$d_p((\mathcal{G}_t), (\mathcal{H}_t)) = \left\| \int |d(A_s - B_s)| \right\|_{L^p}.$$

**Theorem 4.2.2.** *$d_p$ is a semi-metric on the space $\mathbb{F}^p(X)$.*

*Proof.* This follows from the fact that the map $A \mapsto \| \int |dA_s| \|_{L^p}$ is a norm on the set of processes with integrable variation. □

**Theorem 4.2.3.** *If $X$ is continuous, then $(\mathbb{F}^p(X)/\sim, d_p)$ is a complete metric space.*

*Proof.* It is sufficient to show completeness. For this let $(\mathcal{G}_t^n)$ be a Cauchy sequence in $\mathbb{F}^p(X)$. This means that the sequence $(A^n)$ of the predictable bounded variation processes in the canonical decompositions with respect to $(\mathcal{G}_t^n)$ is Cauchy, and therefore converges to some process $A$ with integrable variation. Lemma 4.1.5 implies the sequence $(\mathcal{G}_t^n)$ to converge to the filtration $\mathcal{G}_t = \bigcap_{r>t}(\mathcal{F}_r^X \vee \sigma(A_s : s \le r))$, and hence the result. □

## 4.3 Monotone convergence of filtrations

In this section we describe the limit of a convergent sequence of *increasing* filtrations. For this let $p \in [1, \infty)$, $X$ an $(\mathcal{F}_t)$-adapted process with cadlag paths and $(\mathcal{G}_t^n)_{n\ge1}$ an increasing sequence of filtrations satisfying $\mathcal{G}_t^n \supset \mathcal{F}_t$ for all $t \ge 0$. We assume that for all $n \ge 1$ the process $X$ is in the space $\mathcal{S}^p(\mathcal{G}_t^n)$ and denote by

$$X_t = (X_t - A_t^n) + A_t^n$$

the canonical decomposition with respect to $(\mathcal{G}_t^n)$.

**Theorem 4.3.1.** *(Monotone convergence)*
*Let $X$ be locally bounded. If the sequence $(A^n)_{n\ge1}$ converges to a process $A$ relative to the norm $B \mapsto \| \int |dB_s| \|_{L^p}$, then $X - A$ is a local martingale with respect to the filtration*

$$\mathcal{G}_t = \bigcap_{s>t} \bigvee_{n\ge1} \mathcal{G}_s^n.$$

*In particular, $X$ belongs to the space $\mathcal{S}^p(\mathcal{G}_t)$ and the sequence $(\mathcal{G}_t^n)_n$ converges to $(\mathcal{G}_t)$, i.e.*

$$\lim_n \; d_p((\mathcal{G}_t^n), (\mathcal{G}_t)) = 0.$$

*Proof.* Suppose $T$ to be a stopping time $T$ such that $X^T$ is bounded. For simplicity we assume $X^T = X$.

Assume that $(A^n)$ converges to $A$. Observe that $A$ is of $L^p$-integrable variation. Moreover it is $(\mathcal{G}_t)$-predictable since all the processes $A^n$ are.

Choose a constant $C > 0$ such that

$$|X| \le C,$$

and

$$\left\| \int |dA_s^n| \right\|_{L^p} \le \frac{C}{2}$$

for all $n \geq 1$. We start by showing that $X - A$ is a martingale with respect to $\left(\bigvee_{n\geq 1} \mathcal{G}_t^n\right)_{t\geq 0}$. For this let $\varepsilon > 0$, $0 \leq s < t$ and $B \in \bigvee_{n\geq 1} \mathcal{G}_s^n$. It suffices to show

$$\Big| E\left[1_B(X_t - X_s)\right] - E\left[1_B(A_t - A_s)\right] \Big| \leq \varepsilon.$$

Observe that for all $r \geq 0$ the random variables $A_r^n$ converge in $L^1$ to $A_r$. Hence we may choose $n_0$ such that

$$\|A_t - A_t^n\|_{L^1} + \|A_s - A_s^n\|_{L^1} \leq \frac{\varepsilon}{3}$$

for all $n \geq n_0$. Moreover, $L^1$-convergence implies the sequence $(A_t^n - A_s^n)_{n\geq 1}$ to be uniformly integrable.

Now note that $\bigcup_{n\geq 1} \mathcal{G}_s^n$ is an algebra generating the sigma-algebra $\bigvee_{n\geq 1} \mathcal{G}_s^n$. Hence we can find a sequence $(B_i)$ of sets in $\bigcup_{n\geq n_0} \mathcal{G}_s^n$ such that $P(B \triangle B_i) \to 0$. Due to uniform integrability we may choose $n \geq n_0$ and $\tilde{B} \in \mathcal{G}_s^n$ satisfying

$$E(1_{\tilde{B}\triangle B}|X_t - X_s|) \leq \frac{\varepsilon}{3} \quad \text{and} \quad E(1_{\tilde{B}\triangle B}|A_t^n - A_s^n|) \leq \frac{\varepsilon}{3}$$

(see Cohn [8], page 139). Therefore,

$$\begin{aligned} \left| E\left[1_B(X_t - X_s)\right] - E\left[1_{\tilde{B}}(X_t - X_s)\right] \right| &= \left| E\left[(1_{\tilde{B}} - 1_B)(X_t - X_s)\right] \right| \\ &\leq \frac{\varepsilon}{3}. \end{aligned}$$

Moreover

$$\begin{aligned} & \Big| E\left[1_B(A_t - A_s)\right] - E\left[1_{\tilde{B}}(A_t^n - A_s^n)\right] \Big| \\ \leq\; & \Big| E\left[1_B(A_t - A_s - (A_t^n - A_s^n))\right] \Big| + \Big| E\left[(1_B - 1_{\tilde{B}})(A_t^n - A_s^n)\right] \Big| \\ \leq\; & E\left[|A_t - A_t^n| + |A_s - A_s^n|\right] + E[1_{B\triangle\tilde{B}}|A_t^n - A_s^n|] \\ \leq\; & \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}, \end{aligned}$$

and hence

$$\begin{aligned} & \Big| E\left[1_B(X_t - X_s)\right] - E\left[1_B(A_t - A_s)\right] \Big| \\ \leq\; & \Big| E\left[1_B(X_t - X_s)\right] - E\left[1_{\tilde{B}}(X_t - X_s)\right] \Big| \\ & + \left| E\left[1_{\tilde{B}}(X_t - X_s)\right] - E\left[1_{\tilde{B}}(A_t^n - A_s^n)\right] \right| \\ & + \Big| E\left[1_{\tilde{B}}(A_t^n - A_s^n)\right] - E\left[1_B(A_t - A_s)\right] \Big| \\ \leq\; & \frac{\varepsilon}{3} + 0 + \frac{2\varepsilon}{3} = \varepsilon. \end{aligned}$$

Thus we have shown that $X-A$ is a martingale with respect to $\left(\bigvee_{n\geq 1}\mathcal{G}_t^n\right)_{t\geq 0}$. Since the processes considered are right-continuous, $X-A$ is a martingale with respect to $(\mathcal{G}_t)$, which is the smallest filtration satisfying the usual conditions and containing $\left(\bigvee_{n\geq 1}\mathcal{G}_t^n\right)_{t\geq 0}$. □

## 4.4 Metrics via information drifts

Let $M$ be a continuous local martingale with respect to some filtration $(\mathcal{F}_t)$ and suppose $M_0 = 0$. We recall that in this chapter every filtration is supposed to fulfill the usual conditions.

We denote by $\mathbb{I}(M)$ the set of all filtrations $(\mathcal{G}_t)$ for which there exists a $(\mathcal{G}_t)$-predictable process $\alpha$ such that

$$M_t - \int_0^t \alpha_s \, d\langle M, M\rangle_s \quad \text{is a} \quad (\mathcal{G}_t) - \text{local martingale.}$$

The process $\alpha$ will be called *information drift* of $(\mathcal{G}_t)$ relative to $M$.

We define now metrics on $\mathbb{I}(M)$ which are based on the information drifts.

For $p \geq 1$ we denote by $\mathbb{I}^p(M)$ the set of filtrations $(\mathcal{G}_t) \in \mathbb{I}(M)$ with information drift $\alpha$ such that $\left\| \left(\int_0^\infty \alpha_t^2 \, d\langle M, M\rangle_t\right)^{\frac{1}{2}} \right\|_p$ is finite.

**Definition 4.4.1.** *Let $(\mathcal{G}_t)$ and $(\mathcal{H}_t)$ be two filtrations in $\mathbb{I}(M)$ with information drifts $\alpha$ and $\beta$ respectively. The* distance $\delta_p$ *between $(\mathcal{G}_t)$ and $(\mathcal{H}_t)$ is defined by*

$$\delta_p((\mathcal{G}_t), (\mathcal{H}_t)) = \left\| \left(\int_0^\infty (\alpha_t - \beta_t)^2 \, d\langle M, M\rangle_t\right)^{\frac{1}{2}} \right\|_p.$$

Note that $\delta_p$ is a semimetric on $\mathbb{I}^p(M)$. On $\mathbb{I}(M)$, however, $\delta_p$ is only a pseudo-semimetric, i.e. it may be infinite. As usual, one obtains a semimetric by putting

$$d_{\mathbb{I}}((\mathcal{G}_t), (\mathcal{H}_t)) = 1 - \frac{1}{1 + \delta_p((\mathcal{G}_t), (\mathcal{H}_t))}.$$

How is $\delta_p$ related to the distance $d_p$ defined in Section 4.2? Here is the answer.

**Theorem 4.4.2.** *Let $p, q, r > 0$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. If $M \in \mathcal{H}^q(\mathcal{F}_t)$, then*

$$\mathbb{I}^p(M) \subset \mathbb{F}^r(M),$$

*and there is a constant $C_q > 0$ such that*

$$d_r((\mathcal{G}_t), (\mathcal{H}_t)) \leq C_q \; \delta_p((\mathcal{G}_t), (\mathcal{H}_t)) \tag{4.3}$$

*for all $(\mathcal{G}_t)$ and $(\mathcal{H}_t)$ in $\mathbb{I}^p(M)$.*

*Proof.* Let $(\mathcal{G}_t)$ and $(\mathcal{H}_t)$ be two filtrations in $\mathbb{I}^p(M)$ with information drifts $\alpha$ and $\beta$ respectively. Then by Kunita-Watanabe and Hölder

$$
\begin{aligned}
d_r((\mathcal{G}_t),(\mathcal{H}_t)) &= \left\| \int |d((\alpha-\beta)\cdot\langle M,M\rangle)_t| \right\|_r \\
&= \left\| \int |\alpha-\beta|\, d\langle M,M\rangle_t| \right\|_r \\
&\leq \left\| \langle M,M\rangle_\infty^{\frac{1}{2}} \left( \int (\alpha_t-\beta_t)^2\, d\langle M,M\rangle_t \right)^{\frac{1}{2}} \right\|_r \\
&\leq \|\langle M,M\rangle_\infty^{\frac{1}{2}}\|_q \left\| \left( \int (\alpha_t-\beta_t)^2\, d\langle M,M\rangle_t \right)^{\frac{1}{2}} \right\|_p \\
&\leq \|\langle M,M\rangle_\infty^{\frac{1}{2}}\|_q\ \delta_p((\mathcal{G}_t),(\mathcal{H}_t)),
\end{aligned}
$$

from where we deduce inequality (4.3). By choosing $(\mathcal{H}_t) = (\mathcal{F}_t)$, this inequality also implies $\mathbb{I}^p(M) \subset \mathbb{F}^r(M)$, and thus the proof is complete. □

**Theorem 4.4.3.** *$(\mathbb{I}^p/\sim, \delta_p)$ is a complete metric space for all $p \geq 1$.*

In order to prove this we need

**Lemma 4.4.4.** *Let $p \geq 1$. If $\langle M,M\rangle$ is bounded, then*

$$\mathbb{I}^p(M) \subset \mathbb{F}^p(M),$$

*and there is a constant $C > 0$ such that*

$$d_p((\mathcal{G}_t),(\mathcal{H}_t)) \leq C\ \delta_p((\mathcal{G}_t),(\mathcal{H}_t)) \tag{4.4}$$

*for all $(\mathcal{G}_t)$ and $(\mathcal{H}_t)$ in $\mathbb{I}^p(M)$.*

*Proof.* Let $\langle M,M\rangle$ be bounded, for instance by $C > 0$. Then for any filtrations $(\mathcal{G}_t)$ and $(\mathcal{H}_t)$ in $\mathbb{I}^p(M)$ with information drifts $\alpha$ and $\beta$ respectively

$$
\begin{aligned}
d_p((\mathcal{G}_t),(\mathcal{H}_t)) &= \left\| \int |d((\alpha-\beta)\cdot\langle M,M\rangle)_t| \right\|_p \\
&= \left\| \int |\alpha-\beta|\, d\langle M,M\rangle_t| \right\|_p \\
&\leq \left\| \langle M,M\rangle_\infty^{\frac{1}{2}} \left( \int (\alpha_t-\beta_t)^2\, d\langle M,M\rangle_t \right)^{\frac{1}{2}} \right\|_p \\
&\leq C^{\frac{1}{2}}\ \delta_p((\mathcal{G}_t),(\mathcal{H}_t)),
\end{aligned}
$$

and thus the result. □

*Proof to Theorem 4.4.3.* Let $(\mathcal{G}_t^n)$ be a Cauchy sequence in $\mathbb{I}^p(M)$. This means that the corresponding sequence of information drifts $(\alpha^n)$ is Cauchy, and hence converges to some process $\alpha$ satisfying $\| \left(\int_0^\infty \alpha_t^2 \, d\langle M, M\rangle_t\right)^{\frac{1}{2}} \|_p < \infty$. Put

$$A_t^n = \int_0^t \alpha_s^n \, d\langle M, M\rangle_s$$

and

$$A_t = \int_0^t \alpha_s \, d\langle M, M\rangle_s.$$

We show at first that $M - A$ is a local martingale with respect to the filtration

$$\mathcal{G}_t = \bigcap_{s>t} (\mathcal{F}_s^M \vee \sigma(A_r : r \leq s)).$$

Note that $(\mathcal{G}_t)$ is the smallest filtration satisfying the usual conditions and containing the filtration generated by the joint process $X = (M, A)$, namely

$$\mathcal{G}_t^0 = \sigma(X_s : s \leq t).$$

Now let $T$ be a $(\mathcal{F}_t)$-stopping time such that $\langle M, M\rangle^T$ is bounded. By Lemma 4.4.4 it is not hard to find a $C > 0$ such that $d_p((\mathcal{K}_t), (\mathcal{H}_t)) \leq C \, \delta_p((\mathcal{K}_t), (\mathcal{H}_t))$ for all $(\mathcal{K}_t)$ and $(\mathcal{H}_t)$ in $\mathbb{I}^p(M^T)$. As a consequence $(A^n_{\cdot \wedge T})$ converges with respect to the norm $B \mapsto \| \int |dB_t| \|_p$ and the limit can only be $A_{\cdot \wedge T}$. It follows by Lemma 4.1.5 that $(M - A)^T$ is a martingale with respect to the filtration $(\sigma(X_s^T : s \leq t))_{t \geq 0}$. Basic results about stopping times imply that

$$\sigma(X_s^T : s \leq t) = \mathcal{G}^0_{T \wedge t}$$

(see for example Exercise 4.21, Chapter I in [42]). Therefore $(M - A)^T$ is a martingale with respect to $(\mathcal{G}^0_{T\wedge t})$, and hence with respect to $(\mathcal{G}_t^0)$ (see Remark to Corollary 3.6, Chapter II in [42]). Since $M$ and $A$ are continuous, $M - A = M - \alpha \cdot \langle M, M\rangle$ is a local martingale with respect to the filtration $(\mathcal{G}_t)$.

Note that $\alpha$ may not be predictable with respect to $(\mathcal{G}_t)$. However it coincides almost surely with a predictable version. To prove this, let ${}^p\alpha$ denote the predictable projection of $\alpha$ onto $(\mathcal{G}_t)$. By standard arguments one can show that $M - {}^p\alpha \cdot \langle M, M\rangle$ remains a $(\mathcal{G}_t)$-local martingale. As a consequence ${}^p\alpha \cdot \langle M, M\rangle$ is indistinguishable from $\alpha \cdot \langle M, M\rangle$, and hence $\alpha = {}^p\alpha$, $d\langle M, M\rangle \otimes P$-a.s. Since the limit of $(\alpha^n)$ is equal to ${}^p\alpha$, we have shown that $(\mathcal{G}_t^n)$ converges to $(\mathcal{G}_t)$ relative to $\delta_p$. □

In Chapter 2 we have seen sufficient criteria for filtrations of the form

$$\mathcal{G}_t = \bigcap_{s>t} (\mathcal{F}_s \vee \mathcal{H}_s)$$

to possess an information drift. More precisely, we have introduced two measures $\bar{P}$ and $\bar{Q}$ on the product space $\bar{\Omega} = \Omega \times \Omega$, and we have shown that absolute continuity of $\bar{P}$ with respect to $\bar{Q}$ implies the existence of an information drift $\alpha$ (see Theorem 2.1.11). Moreover, the relative entropy $\mathcal{H}(\bar{P}\|\bar{Q})$ provides an upper bound for

$$E \int_0^\infty \alpha^2 \, d\langle M, M\rangle = \delta_2((\mathcal{G}_t), (\mathcal{F}_t))^2$$

(see Lemma 3.1.1). Consequently we have the following.

**Lemma 4.4.5.** *If $\mathcal{H}(\bar{P}\|\bar{Q}) < \infty$, then $(\mathcal{G}_t) \in \mathbb{I}^2(M)$.*

Note that in the case of initially enlarged filtrations the entropy $\mathcal{H}(\bar{P}\|\bar{Q})$ is equal to the mutual information between $\mathcal{F}_\infty$ and the enlarging variable.

If $(\mathcal{G}_t)$ is obtained by an initial enlargement with a discrete random variable $G$, then $\bar{P}$ is always absolutely continuous with respect to $\bar{Q}$. Moreover, $\mathcal{H}(\bar{P}\|\bar{Q})$ is smaller than the absolute entropy of $G$ (see Lemma 3.2.10). Thus, we have the following.

**Theorem 4.4.6.** *Let $G$ be a discrete random variable and $\mathcal{G}_t = \bigcap_{s>t}(\mathcal{F}_s \vee \sigma(G))$. Then $(\mathcal{G}_t) \in \mathbb{I}(M)$. Moreover, if the absolute entropy of $G$ is finite, then $(\mathcal{G}_t) \in \mathbb{I}^2(M)$.*

*Proof.* Follows from Theorem 2.1.11 and Lemma 4.4.5. □

**Remark 4.4.7.** *Suppose $\mathcal{F}_0$ is trivial and Condition (C) is satisfied. In this case $\delta_2((\mathcal{F}_t), (\mathcal{G}_t))$ coincides with $\mathcal{H}(\bar{P}\|\bar{Q})$ (see Lemma 3.1.1). Thus, $(\mathcal{G}_t) \in \mathbb{I}^2(M)$ if and only if the absolute entropy of $G$ is finite.*

## 4.5 Monotone convergence of information drifts

Theorem 4.3.1 deals with increasing filtrations converging relative to the distance $d_p$. From this result we now deduce a monotone convergence property of information drifts.

Let $M$ be a continuous $(\mathcal{F}_t)$-local martingale starting in zero and let $(\mathcal{G}_t^n)_{n\geq 1}$ be an increasing sequence of filtrations, i.e. for all $t \geq 0$ we have

$$\mathcal{F}_t \subset \mathcal{G}_t^1 \subset \ldots \subset \mathcal{G}_t^n \subset \mathcal{G}_t^{n+1} \subset \ldots$$

We assume that for all $n \geq 1$ the process $M$ is a $(\mathcal{G}_t^n)$-semimartingale with Doob-Meyer decomposition

$$M = M^n + \int_0^{\cdot} \mu_s^n \, d\langle M, M\rangle_s,$$

where $\mu^n$ is $(\mathcal{G}_t^n)$-predictable and in $L^2(M)$, i.e. $E \int_0^\infty (\mu_t^n)^2 \, d\langle M, M\rangle_t < \infty$. We then have the following asymptotic property.

**Lemma 4.5.1.** *If the processes $(\mu^n)_{n\in\mathbb{N}}$ converge to some $\mu$ in $L^2(M)$, then*

$$M - \int_0^{\cdot} \mu_s \, d\langle M, M\rangle_s$$

*is a local martingale with respect to* $\mathcal{G}_t = \bigcap_{s>t} \bigvee_{n\geq 1} \mathcal{G}_s^n, \ t \geq 0.$

*Proof.* Put $A_t = \int_0^t \mu_s \, \langle M, M\rangle_s$ and $A_t^n = \int_0^t \mu_s^n \, \langle M, M\rangle_s$, $n \geq 1$. Choose a stopping time $T$ such that $\langle M, M\rangle^T$ is bounded, let's say by $C > 0$. To simplify notation we assume that $A^T = A$ and $(A^n)^T = A^n$, $n \geq 1$. The Kunita-Watanabe Inequality implies

$$\begin{aligned} \left\| \int |d(A_t^n - A_t)| \right\|_{L^2} &= \left\| \int |\alpha - \alpha^n| \, d\langle M, M\rangle \right\|_{L^2} \\ &\leq C^{\frac{1}{2}} \left\| \left( \int (\alpha - \alpha^n)^2 \, d\langle M, M\rangle \right)^{\frac{1}{2}} \right\|_{L^2}, \end{aligned}$$

and the result follows now from Theorem 4.3.1. □

**Remark 4.5.2.** *A direct proof of Lemma 4.5.1 can be found in [3].*

In the remainder of this section we give some sufficient criteria for a sequence of information drifts to converge in $L^2(M)$. We start with the following observation.

**Lemma 4.5.3.** *For $n, m \geq 1$ with $m \geq n \geq 1$ the processes $\mu^n$ and $\mu^m - \mu^n$ are orthogonal in the Hilbert space $L^2(M)$, i.e.*

$$\mathbb{E}\Big[\int_0^\infty \mu^n(\mu^m - \mu^n) \, d\langle M, M\rangle\Big] = 0.$$

*Proof.* Let $1 \leq n \leq m$. Recall that for all $i \geq 1$

$$\tilde{M}^i = M - \int_0^{\cdot} \mu_t^i \, d\langle M, M\rangle_t \tag{4.5}$$

is a local $(\mathcal{G}_t^i)$-martingale. Then

$$\int_0^\infty \mu^n(\mu^m - \mu^n)\, d\langle M, M\rangle = (\mu^n \cdot \tilde{M}^n)_\infty - (\mu^n \cdot \tilde{M}^m)_\infty.$$

Note that $\mu^n$ is $(\mathcal{G}_t^n)$- and $(\mathcal{G}_t^m)$-predictable. Moreover, $(\mu^n \cdot \tilde{M}^n)_t$ and $(\mu^n \cdot \tilde{M}^m)_t$ are $L^2-$martingales relative to $(\mathcal{G}_t^n)$ and $(\mathcal{G}_t^m)$ respectively. Therefore,

$$\mathbb{E}\Big[\int_0^\infty \mu^n(\mu^m - \mu^n)\, d\langle M, M\rangle\Big] = 0.$$

□

We are now in a position to prove the main result of the section.

**Theorem 4.5.4.** *If* $\sup_{n\geq 1} \|\mu^n\|^2_{L^2(M)} < \infty$, *then* $(\mu^n)$ *converges in* $L^2(M)$ *to a process* $\mu$. *Moreover,*

$$M - \int_0^\cdot \mu_t\, d\langle M, M\rangle_t$$

*is a local martingale with respect to* $\mathcal{G}_t = \bigcap_{s>t} \bigvee_{n\geq 1} \mathcal{G}_s^n,\ t \geq 0.$

*Proof.* Set $c = \sup_{n\geq 1} \|\mu^n\|^2_{L^2(M)}$. By the previous lemma for $m \geq n \geq 1$,

$$\|\mu^m\|^2_{L^2(M)} = \|\mu^n\|^2_{L^2(M)} + \|\mu^m - \mu^n\|^2_{L^2(M)}.$$

Thus $c = \lim_{n\to\infty} \|\mu^n\|^2_{L^2(M)}$ and

$$\|\mu^m - \mu^n\|^2_{L^2(M)} = \|\mu^m\|^2_{L^2(M)} - \|\mu^n\|^2_{L^2(M)} \leq c - \|\mu^n\|^2_{L^2(M)} \to 0$$

as $n \to \infty$. Therefore $\{\mu^n\}_{n\geq 1}$ is a Cauchy sequence in $L^2(M)$. By completeness of $L^2(M)$, there exists a unique $(\mathcal{G}_t)-$predictable process $\mu \in L^2(M)$ such that $\lim_{n\to\infty} \mu^n = \mu$ in $L^2(M)$. By Lemma 4.5.1 the process $M - \int \mu\, d\langle M, M\rangle$ is a $(\mathcal{G}_t)-$local martingale. □

In terms of filtration convergence the previous theorem can be restated in the following way.

**Corollary 4.5.5.** *If* $\sup_{n\geq 1} \delta_2((\mathcal{G}_t^n), (\mathcal{F}_t)) < \infty$, *then* $(\mathcal{G}_t^n)$ *converges to* $\mathcal{G}_t = \bigcap_{s>t} \bigvee_{n\geq 1} \mathcal{G}_s^n,\ t \geq 0$, *with respect to* $\delta_2$, *i.e.*

$$\lim_n \delta_2((\mathcal{G}_t^n), (\mathcal{G}_t)) = 0.$$

# Chapter 5

# Information drifts for general enlargements

Let $M$ be a continuous local martingale with respect to $(\mathcal{F}_t)$. Moreover, let $(\mathcal{G}_t) \in \mathbb{I}(M)$ be a larger filtration with information drift $\alpha$.

In this chapter we shall aim at describing the information drift $\alpha$ by conditional probabilities of the larger $\sigma$-algebras $\mathcal{G}_t$ with respect to the smaller ones $\mathcal{F}_t, t \geq 0$. Roughly, the relationship is as follows. Suppose for all $t \geq 0$ there is a regular conditional probability $P_t(\cdot, \cdot)$ of $\mathcal{F}$ given $\mathcal{F}_t$, which can be decomposed into a martingale component orthogonal to $M$, plus a component possessing a stochastic integral representation relative to $M$ with a kernel function $k_t(\cdot, \cdot)$. Then we shall see, that, provided $\alpha$ is square integrable with respect to $d\langle M, M\rangle \otimes P$, the kernel function at $t$ will be a signed measure in its set variable. Moreover, this measure is absolutely continuous with respect to the conditional probability if restricted to $\mathcal{G}_t$, and $\alpha$ coincides with their Radon-Nikodym density.

We shall even be able to show that this relationship also makes sense in the reverse direction: If absolute continuity of the stochastic integral kernel with respect to the conditional probabilities holds, then the Radon-Nikodym density turns out to provide an information drift $\alpha$ of $M$ in the larger filtration.

We shall finish the chapter with an illustration of this fundamental relationship by discussing some simple examples.

## 5.1 A representation theorem for information drifts

Our discussion requires some care with the underlying filtrations and state spaces. Of course, the need to work with conditional probabilities first of all confines us to spaces on which they exist. Let therefore $(\Omega, \mathcal{F}, P)$ be a standard Borel probability space (see [39]) with a filtration $(\mathcal{F}_t^0)_{t\geq 0}$ consisting of countably generated $\sigma-$algebras, and let $M$ be a $(\mathcal{F}_t^0)$-local martingale. We also deal with the smallest right-continuous and completed filtration containing $(\mathcal{F}_t^0)$, which we denote by $(\mathcal{F}_t)$. We suppose that $\mathcal{F}_0$ is trivial and that every $(\mathcal{F}_t)-$local martingale has a continuous modification. Since $\mathcal{F}_t^0$ is a subfield of a standard Borel space, there exist regular conditional probabilities $P_t$ relative to the $\sigma-$algebras $\mathcal{F}_t^0$. Then for any set $A \in \mathcal{F}$ the process

$$(t, \omega) \mapsto P_t(\omega, A)$$

is an $(\mathcal{F}_t^0)-$martingale with a continuous modification (see e.g. Theorem 4, Chapter VI in [14]). Note that the modification may not be adapted to $(\mathcal{F}_t^0)$, but only to $(\mathcal{F}_t)$. Furthermore it is no problem to assume that the processes $P_t(\cdot, A)$ are modified in a way such that $P_t(\omega, \cdot)$ remains a measure on $\mathcal{F}$ for $P_M-$almost all $(\omega, t)$, where $P_M$ is a measure on $\Omega \times \mathbb{R}_+$ defined by $P_M(\Gamma) = E\int_0^\infty 1_\Gamma(\omega, t) d\langle M, M\rangle_t$, $\Gamma \in \mathcal{F} \otimes \mathcal{B}_+$.

It is known that each of these martingales may be uniquely written (see e.g. [42], Chapter V)

$$P_t(\cdot, A) = P(A) + \int_0^t k_s(\cdot, A) dM_s + L_t^A, \tag{5.1}$$

where $k(\cdot, A)$ is $(\mathcal{F}_t)-$predictable and $L^A$ satisfies $\langle L^A, M\rangle = 0$.

Now let $(\mathcal{G}_t^0)$ be another filtration on $(\Omega, \mathcal{F}, P)$ satisfying

$$\mathcal{F}_t^0 \subset \mathcal{G}_t^0$$

for all $0 \leq t \leq T$. We assume that each $\sigma-$field $\mathcal{G}_t^0$ is generated by a countable number of sets, and denote by $(\mathcal{G}_t)$ the smallest right-continuous and completed filtration containing $(\mathcal{G}_t^0)$. It is clear that each $\sigma-$field in the left-continuous filtration $(\mathcal{G}_{t-}^0)$ is also generated by a countable number of sets. We claim that the existence of an information drift of $(\mathcal{G}_t)$ relative to $(\mathcal{F}_t)$ for the process $M$ depends on whether the following condition is satisfied or not.

**Condition 5.1.1.** $k_t(\omega, \cdot)\big|_{\mathcal{G}^0_{t-}}$ *is a signed measure and satisfies*

$$k_t(\omega, \cdot)\Big|_{\mathcal{G}^0_{t-}} \ll P_t(\omega, \cdot)\Big|_{\mathcal{G}^0_{t-}}$$

*for* $P_M-$*a.a* $(\omega, t)$.

**Remark 5.1.2.** *Unfortunately, we have to distinguish between the filtrations* $(\mathcal{F}^0_t)$, $(\mathcal{G}^0_t)$ *and their extensions* $(\mathcal{F}_t)$, $(\mathcal{G}_t)$. *The reason is that the regular conditional probabilities considered exist only with respect to the smaller* $\sigma-$*fields. On the other hand, we use stochastic integration techniques which were developed only under the assumption that the underlying filtrations satisfy the usual conditions, and thus necessitates working also with the larger* $\sigma$*-fields.*

Let us next state some essential properties of the Radon-Nikodym density process existing according to our condition.

**Lemma 5.1.3.** *Suppose Condition 5.1.1 is satisfied. Then there exists an* $(\mathcal{F}_t \otimes \mathcal{G}_t)-$*predictable process* $\gamma$ *such that for* $P_M-$*a.a.* $(\omega, t)$

$$\gamma_t(\omega, \omega') = \frac{dk_t(\omega, \cdot)}{dP_t(\omega, \cdot)}\Big|_{\mathcal{G}^0_{t-}}(\omega').$$

**Remark 5.1.4.** *Note that* $\gamma_t(\omega, \cdot)$ *is* $\mathcal{G}_{t-}-$*measurable. This is due to the fact that the predictable* $\sigma-$*algebra does not change by taking the left-continuous version of the underlying filtration.*

*Proof.* Let $t^n_i = \frac{i}{2^n}$ for all $n \geq 0$ and $i \geq 0$. We denote by $\mathbb{T}$ the set of all $t^n_i$. It is possible to choose a family of finite partitions $(\mathcal{P}^{i,n})$ such that

- for all $t \in \mathbb{T}$ we have $\mathcal{G}^0_{t-} = \sigma(\mathcal{P}^{i,n} : i, n \geq 0 \text{ s.t. } t^n_i = t)$,
- $\mathcal{P}^{i,n} \subset \mathcal{P}^{i+1,n}$,
- if $i < j$, $n < m$ and $i\, 2^{-n} = j\, 2^{-m}$, then $\mathcal{P}^{i,n} \subset \mathcal{P}^{j,m}$.

We define for all $n \geq 0$

$$\gamma^n_t(\omega, \omega') = \sum_{i\geq 0} \sum_{A \in \mathcal{P}^{i,n}} 1_{]t^n_i, t^n_{i+1}]}(t) 1_A(\omega') \frac{k_t(\omega, A)}{P_t(\omega, A)}.$$

Note that $\frac{k_t(\omega,A)}{P_t(\omega,A)}$ is $(\mathcal{F}_t)-$predictable and $1_{]t_i^n,t_{i+1}^n]}(t)1_A(\omega')$ is $(\mathcal{G}_t)-$predictable. Hence the product of both functions, defined as a function on $\Omega^2 \times \mathbb{R}_+$, is predictable with respect to $(\mathcal{F}_t \otimes \mathcal{G}_t)$. It follows that each $\gamma^n$, and thus

$$\gamma = \liminf_{n\to\infty} \gamma^n$$

is $(\mathcal{F}_t \otimes \mathcal{G}_t)-$predictable.
Now fix $t \geq 0$. We claim that $k_t(\omega,\cdot) = \int_\cdot \gamma_t(\omega,\omega')P_t(\omega,d\omega')$, and hence that $\gamma_t(\omega,\cdot)$ is the density of $k_t(\omega,\cdot)$ with respect to $P_t(\omega,\cdot)$, $P_M-$a.s. For all $n \geq 0$ let $j = j(n)$ be the integer satisfying $t_j^n < t \leq t_{j+1}^n$ and denote by $\mathcal{Q}^n$ the corresponding partition $\mathcal{P}^{j,n}$. Observe that $(\mathcal{Q}^n)$ is an increasing sequence of partitions satisfying

$$\sigma(\mathcal{Q}^n : n \geq 0) = \mathcal{G}_{t-}^0$$

and hence

$$\begin{aligned}
\gamma_t(\omega,\omega') &= \liminf_n \gamma_t^n(\omega,\omega') \\
&= \liminf_n \sum_{A\in\mathcal{Q}^n} 1_A(\omega')\frac{k_t(\omega,A)}{P_t(\omega,A)} \\
&= \left.\frac{dk_t(\omega,\cdot)}{dP_t(\omega,\cdot)}\right|_{\mathcal{G}_{t-}^0}.
\end{aligned}$$

□

**Lemma 5.1.5.** *If $(t,\omega,\omega') \mapsto \theta_t(\omega,\omega')$ is $(\mathcal{F}_t \otimes \mathcal{G}_t)-$predictable and bounded, then*

$$\int\int\int \theta_t(\omega,\omega')\ P_t(\omega,d\omega')\ d\langle M,M\rangle_t\ dP(\omega) = \int\int \theta_t(\omega,\omega)\ d\langle M,M\rangle_t\ dP(\omega).$$

*Proof.* Let $0 \leq r < s$, $A \in \mathcal{F}_r$, $B \in \mathcal{G}_r$ and

$$\theta_t(\omega,\omega') = 1_{]r,s]}(t)1_A(\omega)1_B(\omega').$$

Then

$$\begin{aligned}
&\int\int\int \theta_t(\omega,\omega')\ P_t(\omega,d\omega')\ d\langle M,M\rangle_t\ dP(\omega) \\
=\ &\int\int_r^s 1_A(\omega)P_t(\omega,B)\ d\langle M,M\rangle_t\ dP(\omega) \\
=\ &\int\int_r^s 1_A(\omega)1_B(\omega)\ d\langle M,M\rangle_t\ dP(\omega) \\
=\ &\int\int \theta_t(\omega,\omega)\ d\langle M,M\rangle_t\ dP(\omega),
\end{aligned}$$

where the second equality holds due to results about optional projections (see Theorem 57, Chapter VI, in [14]). By a monotone class argument this can be extended to all bounded and $(\mathcal{F}_t \otimes \mathcal{G}_t)-$predictable processes. □

**Theorem 5.1.6.** *Suppose Condition 5.1.1 is satisfied and $\gamma$ is as in Lemma 5.1.3. Then*

$$\alpha_t(\omega) = \gamma_t(\omega, \omega)$$

*is the information drift of $(\mathcal{G}_t)$ relative to $M$.*

*Proof.* Suppose $\tau$ to be a stopping time such that $M^\tau$ is a martingale. For $0 \le s < t$ and $A \in \mathcal{G}_s^0$ we have to show

$$E\left[1_A(M_t^\tau - M_s^\tau)\right] = E\left[1_A \int_s^t \gamma_u(\omega, \omega)\, d\langle M, M\rangle_u^\tau\right].$$

For notational simplicity write $M^\tau = M$ and observe

$$\begin{aligned} E\left[1_A(M_t - M_s)\right] &= E\left[P_t(\cdot, A)(M_t - M_s)\right] \\ &= E\left[(M_t - M_s)\int_0^t k_u(\cdot, A)\, dM_u\right] + E[(M_t - M_s)L_t^A] \\ &= E\left[\int_s^t k_u(\cdot, A)\, d\langle M, M\rangle_u\right] \\ &= E\left[\int_s^t \int_A \gamma_u(\omega, \omega')\, dP_u(\omega, d\omega')\, d\langle M, M\rangle_u\right] \\ &= E\left[1_A(\omega)\int_s^t \gamma_u(\omega, \omega)\, d\langle M, M\rangle_u\right], \end{aligned}$$

where we used Lemma 5.1.5 in the last equation. □

We now look at the problem from the reverse direction: We show that the existence of a square integrable information drift implies Condition 5.1.1.

For this let $\alpha$ be the information drift of $(\mathcal{G}_t)$, i.e. $\tilde{M} = M - \int_0^\cdot \alpha_t\, d\langle M, M\rangle_t$ is a $(\mathcal{G}_t)$-local martingale. Moreover, suppose that $\alpha$ satisfies

$$E\int_0^\infty \alpha_t^2\, d\langle M, M\rangle_t < \infty.$$

To prove the main results (Theorems 5.1.9 and 5.2.1 below), we need the following lemma.

**Lemma 5.1.7.** *Let $0 \le s < t$ and $\mathcal{P} = \{A_1, \dots, A_n\}$ be a finite partition of $\Omega$ into $\mathcal{G}_s^0-$measurable sets. Then*

$$E\int_s^t \sum_{k=1}^n \left(\frac{k_u}{P_u}\right)^2 (\cdot, A_k)\, 1_{A_k}\, d\langle M, M\rangle_u \le 4E\left(\int_s^t \alpha_u^2\, d\langle M, M\rangle_u\right) < \infty.$$

*Proof.* Let $\mathcal{P} = \{A_1, \ldots, A_n\}$ be a finite $\mathcal{G}_s^0$−partition. An application of Ito's formula yields

$$\begin{aligned}
&\sum_{k=1}^{n} [1_{A_k} \log P_s(\cdot, A_k) - 1_{A_k} \log P_t(\cdot, A_k)] \\
= &\sum_{k=1}^{n} \Bigg[ -\int_s^t \frac{1}{P_u(\cdot, A_k)} 1_{A_k}\ dP_u(\cdot, A_k) \\
&+\frac{1}{2} \int_s^t \frac{1}{P_u(\cdot, A_k)^2} 1_{A_k}\ d\langle P(\cdot, A_k), P(\cdot, A_k)\rangle_u \Bigg] \\
= &\sum_{k=1}^{n} \Bigg[ -\int_s^t \frac{k_u}{P_u}(\cdot, A_k)\, 1_{A_k}\ d\tilde{M}_u - \int_s^t \frac{k_u}{P_u}(\cdot, A_k)\, 1_{A_k} \alpha_u\ d\langle M, M\rangle_u \\
&-\int_s^t \frac{1}{P_u(\cdot, A_k)} 1_{A_k}\ dL_u^{A_k} + \frac{1}{2} \int_s^t \left(\frac{k_u}{P_u}\right)^2 (\cdot, A_k)\, 1_{A_k}\ d\langle M, M\rangle_u \\
&+\frac{1}{2} \int_s^t \frac{1}{P_u(\cdot, A_k)^2}\, 1_{A_k}\ d\langle L^{A_k}, L^{A_k}\rangle_u \Bigg]
\end{aligned} \tag{5.2}$$

Note that $P_t(\cdot, A_k) \log P_t(\cdot, A_k)$ is a submartingale bounded from below for all $k$. Hence the expectation of the left hand side in the previous equation is at most 0.

A priori it is not clear whether

$$\sum_{k=1}^{n} \int_s^t \frac{k_u}{P_u}(\cdot, A_k)\, 1_{A_k}\ d\tilde{M}_u$$

is integrable or not. Consider therefore for all $\varepsilon > 0$ stopping times defined by

$$\tau_k^\varepsilon = \begin{cases} \infty & \omega \notin A_k \\ \inf\{t \geq s : P_t(\cdot, A_k) \leq \varepsilon\} & \text{else} \end{cases}$$

and

$$\tau^\varepsilon = \tau_1^\varepsilon \wedge \ldots \wedge \tau_n^\varepsilon.$$

Observe that $\tau^\varepsilon \to \infty$ as $\varepsilon \downarrow 0$ and that the stopped process

$$\sum_{k=1}^{n} \int_s^{t \wedge \tau^\varepsilon} \frac{k_u}{P_u}(\cdot, A_k)\, 1_{A_k}\ d\tilde{M}_u$$

has expectation zero, since

$$\begin{aligned} & E\left[\left(\int_s^{t\wedge\tau^\varepsilon}\sum_{k=1}^n \frac{k_u}{P_u}(\cdot,A_k)\,1_{A_k}\ d\tilde{M}_u\right)^2\right] \\ = \; & E\left[\int_s^{t\wedge\tau^\varepsilon}\sum_{k=1}^n \left(\frac{k_u}{P_u}\right)^2(\cdot,A_k)\,1_{A_k}\ d\langle M,M\rangle_u\right] \\ \leq \; & \frac{1}{\varepsilon^2}E\left[\int_s^{t\wedge\tau^\varepsilon}\sum_{k=1}^n (k_u)^2(\cdot,A_k)\,1_{A_k}\ d\langle M,M\rangle_u\right] \\ \leq \; & \frac{1}{\varepsilon^2}E\left[\sum_{k=1}^n\int_s^t d\langle P(\cdot,A_k),P(\cdot,A_k)\rangle_u\right] \\ < \; & \infty. \end{aligned}$$

Similarly, one can show that the expectation of

$$\int_s^{t\wedge\tau^\varepsilon}\frac{1}{P_u(\cdot,A_k)}1_{A_k}\ dL_u^{A_k}$$

vanishes. Consequently we may deduce from equation (5.2) and the Kunita-Watanabe inequality

$$\begin{aligned} & E\sum_{k=1}^n\frac{1}{2}\int_s^{t\wedge\tau^\varepsilon}\left(\frac{k_u}{P_u}\right)^2(\cdot,A_k)\,1_{A_k}\ d\langle M,M\rangle_u \\ \leq \; & E\sum_{k=1}^n\left[\int_s^{t\wedge\tau^\varepsilon}\frac{k_u}{P_u}(\cdot,A_k)\,1_{A_k}\alpha_u\ d\langle M,M\rangle_u\right] \\ \leq \; & E\left(\int_s^{t\wedge\tau^\varepsilon}\sum_{k=1}^n\left(\frac{k_u}{P_u}\right)^2(\cdot,A_k)\,1_{A_k}\ d\langle M,M\rangle_u\right)^{\frac{1}{2}}E\left(\int_s^{t\wedge\tau^\varepsilon}\alpha_u^2\ d\langle M,M\rangle_u\right)^{\frac{1}{2}}, \end{aligned}$$

which implies

$$E\int_s^{t\wedge\tau^\varepsilon}\sum_{k=1}^n\left(\frac{k_u}{P_u}\right)^2(\cdot,A_k)\,1_{A_k}\ d\langle M,M\rangle_u \leq 4E\left(\int_s^{t\wedge\tau^\varepsilon}\alpha_u^2\ d\langle M,M\rangle_u\right).$$

Now the proof may be completed by a monotone convergence argument. □

Let $\mathbb{T}$ and $(\mathcal{P}^{i,n})_{i,n\geq 0}$ be a family of partitions as in the proof of Lemma 5.1.3. We define for all $n \geq 0$

$$Z_t^n(\omega,\omega') = \sum_{i\geq 0}\sum_{A\in\mathcal{P}^{i,n}} 1_{]t_i^n,t_{i+1}^n]}(t)1_A(\omega')\frac{k_t(\omega,A)}{P_t(\omega,A)}.$$

Note that $Z^n$ is $(\mathcal{F}_t \otimes \mathcal{G}_t)-$predictable. We are now able to prove a converse statement to Theorem 5.1.6. Observe first

**Lemma 5.1.8.** *For $P_M-$almost all $(\omega, t) \in \Omega \times \mathbb{R}_+$ the discrete process $(Z_t^m(\omega, \cdot))_{m\geq 1}$ is an $L^2(P_t(\omega, \cdot))-$bounded martingale.*

*Proof.* Every statement in the sequel is meant to hold for $P_M-$a.a. $(\omega, t) \in \Omega \times \mathbb{R}_+$.

Let $m \geq 0$, $l \geq 0$ and $j$ be the natural number such that $]t_l^{m+1}, t_{l+1}^{m+1}] \subset ]t_j^m, t_{j+1}^m]$. We start by proving that on $]t_l^{m+1}, t_{l+1}^{m+1}]$ we have

$$E^{P_t(\omega,\cdot)}[Z_t^{m+1}(\omega, \cdot)|\mathcal{P}^{j,m}] = Z_t^m(\omega, \cdot).$$

For this, let $B \in \mathcal{P}^{j,m}$ and $A_1, \ldots, A_k \in \mathcal{P}^{l,m+1}$ such that $A_1 \cup \ldots \cup A_k = B$. Note that

$$\begin{aligned} E^{P_t(\omega,\cdot)}[1_B(\cdot)Z_t^{m+1}(\omega, \cdot)] &= E^{P_t(\omega,\cdot)}\left[\sum_{i=1}^k 1_{A_i}(\cdot)\frac{k_t}{P_t}(\omega, A_i)\right] \\ &= \sum_{i=1}^k k_t(\omega, A_i) \\ &= k_t(\omega, B) \\ &= E^{P_t(\omega,\cdot)}[1_B(\cdot)Z_t^m(\omega, \cdot)] \end{aligned}$$

on $]t_l^{m+1}, t_{l+1}^{m+1}]$. Consequently the process $(Z_t^m(\omega, \cdot))_{m\geq 1}$ is a martingale (with respect to a filtration depending on $t$). The martingale property implies that the sequence $\int (Z_t^n)^2(\omega, \omega')\ P_t(\omega, d\omega')$ is increasing, and hence, by monotone convergence,

$$\begin{aligned} &\sup_n E \int \int (Z_t^n)^2(\omega, \omega')\ P_u(\omega, d\omega')\ d\langle M, M\rangle_t \\ &= E \int \sup_n \int (Z_t^n)^2(\omega, \omega')\ P_u(\omega, d\omega')\ d\langle M, M\rangle_t. \end{aligned}$$

By Lemma 5.1.7 and Lemma 5.1.5 we have

$$\begin{aligned} &\sup_n E \int \int (Z_u^n)^2(\omega, \omega')\ P_u(\omega, d\omega')\ d\langle M, M\rangle_u \\ = \ &\sup_n E \int (Z_u^n)^2(\omega, \omega)\ d\langle M, M\rangle_u \\ = \ &\sup_n E \sum_{i\geq 0} \int_{t_i^n}^{t_{i+1}^n} \sum_{A\in\mathcal{P}^{i,n}} 1_A(\omega) \left(\frac{k_t(\omega, A)}{P_t(\omega, A)}\right)^2\ d\langle M, M\rangle_u \\ \leq \ &4E\left(\int \alpha_u^2\ d\langle M, M\rangle_u\right) < \infty. \end{aligned}$$

This shows that $(Z^n)_{n\geq 1}$ is an $L^2(P_t(\omega,\cdot))-$bounded martingale. $\square$

We now will show that $k$ can be chosen to be a signed measure. For this we identify $P_t(\omega,\cdot)$ with another measure on a countable generator of $\mathcal{G}^0_{t-}$. We then apply the result that two Banach space valued measures are equal if they coincide on a generator stable for finite intersections.

**Theorem 5.1.9.** *The kernel $k$ may be chosen such that*

$$\mathcal{G}^0_{t-} \ni A \mapsto k_t(\omega, A) \in \mathbb{R},$$

*is a signed measure which is absolutely continuous with respect to $P_t(\omega,\cdot)|_{\mathcal{G}^0_{t-}}$, for $P_M-$a.a. $(\omega,t)\in\Omega\times[0,\infty)$. This means that Condition 5.1.1 is satisfied.*

*Proof.* Lemma 5.1.8 implies that $(Z^m_t(\omega,\cdot))_{m\geq 1}$ is an $L^2(P_t(\omega,\cdot))-$bounded martingale and hence, for a.a. fixed $(\omega,t)$, $(Z^m_t(\omega,\cdot))_{m\geq 1}$ possesses a limit $Z$. It can be chosen to be $(\mathcal{F}_t\otimes\mathcal{G}_t)-$predictable. Take for example

$$Z_t = \liminf_n (Z^n_t \vee 0) + \limsup_n (Z^n_t \wedge 0).$$

Now define a signed measure by

$$\tilde{k}_t(\omega, A) = \int 1_A(\omega')Z_t(\omega,\omega')dP_t(\omega, d\omega').$$

Observe that $\tilde{k}_t(\omega,\cdot)$ is absolutely continuous with respect to $P_t(\omega,\cdot)$ and that we have for all $A\in\mathcal{P}^{j,m}$ with $j2^{-m}\leq t$

$$\tilde{k}_t(\omega, A) = k_t(\omega, A)$$

for $P_M-$a.a. $(\omega,t)\in\Omega\times\mathbb{R}_+$. One may also interpret $\mathcal{G}^0_{t-}\ni A\mapsto\tilde{k}_t(\omega,A)$, as an $L^2(M)-$valued measure. By applying the stochastic integral operator, we obtain an $L^2(\Omega)-$valued measure: $\mathcal{G}^0_{t-}\ni A\mapsto\int_0^t\tilde{k}_s(\omega,A)dM_s$. Moreover,

$$P_t(\omega, A) = P(A) + \int_0^t \tilde{k}_s(\omega, A)dM_s + L^A_t(\omega) \tag{5.3}$$

for all $A\in\bigcup_{j2^{-m}\leq t}\mathcal{P}^{j,m}$. Since the LHS and both expressions on the RHS are measures coinciding on a system which is stable for intersections, equation (5.3) holds for all $A\in\mathcal{G}^0_{t-}$. Hence, by choosing $k_t(\cdot,A)=\tilde{k}_t(\cdot,A)$ for all $A\in\mathcal{G}^0_{t-}$, the proof is complete. $\square$

**Remark 5.1.10.** *Since $k$ is determined up to $P_M-$null sets, we may assume that $k_t(\omega,\cdot)$ is absolutely continuous relative to $P_t(\omega,\cdot)$ everywhere.*

## 5.2 Calculating examples

We close this chapter with some examples showing how (well known) information drifts can be derived explicitly, based on the formalism of Theorem 5.1.6. To this end it is not always necessary to determine the signed measures $k_t(\omega, \cdot)$ on the whole $\sigma-$algebras $\mathcal{G}_t^0$, but only on some sub-$\sigma$-fields. This is the case, for example, if

$$\mathcal{G}_t^0 = \mathcal{F}_t^0 \vee \mathcal{H}_t^0, \; 0 \leq t \leq T,$$

where $(\mathcal{H}_t^0)$ is some countably generated filtration on $(\Omega, \mathcal{F})$.

Now suppose that $k_t(\omega, \cdot)$ is a signed measure on $(\mathcal{H}_{t-}^0)$ satisfying

$$k_t(\omega, \cdot)\Big|_{\mathcal{H}_{t-}^0} \ll P_t(\omega, \cdot)\Big|_{\mathcal{H}_{t-}^0}$$

for $P_M-$a.a $(\omega, t)$. Then we can show with the arguments of the proof of Lemma 5.1.3 that there is an $(\mathcal{F}_t \otimes \mathcal{H}_t)-$predictable process $\beta$ such that $P_M-$a.e.

$$\beta_t(\omega, \omega') = \frac{dk_t(\omega, \cdot)}{dP_t(\omega, \cdot)}\Big|_{\mathcal{H}_{t-}^0}.$$

The information drift of $(\mathcal{G}_t)$ relative to $(\mathcal{F}_t)$ is already determined by the trace of $(\beta_t)$. For the corresponding analogue of Theorem 5.1.6 we shall give a more explicit statement.

**Theorem 5.2.1.** *The process*

$$\alpha_t(\omega) = \beta_t(\omega, \omega)$$

*is the information drift of $(\mathcal{G}_t)$ relative to $M$.*

*Proof.* Suppose $T$ to be a stopping time such that $M^T$ is a martingale. For $0 \leq s < t$, $A \in \mathcal{H}_s^0$ and $B \in \mathcal{F}_s^0$ we have to show

$$E\left[1_A 1_B (M_t^T - M_s^T)\right] = E\left[1_A 1_B \int_s^t \beta_u(\omega, \omega) \; d\langle M, M\rangle_u^T\right].$$

For simplicity assume $M^T = M$ and observe

$$\begin{aligned}
E\left[1_A 1_B (M_t - M_s)\right] &= E\left[1_B P_t(\cdot, A)(M_t - M_s)\right] \\
&= E\left[1_B (M_t - M_s) \int_0^t k_u(\cdot, A)\ dM_u\right] \\
&\quad + E[1_B (M_t - M_s) L_t^A] \\
&= E\left[\int_s^t 1_B k_u(\cdot, A)\ d\langle M, M\rangle_u\right] \\
&= E\left[\int_s^t \int_A 1_B(\omega) \beta_u(\omega, \omega')\ dP_u(\omega, d\omega')\ d\langle M, M\rangle_u\right] \\
&= E\left[1_A(\omega) 1_B(\omega) \int_s^t \beta_u(\omega, \omega)\ d\langle M, M\rangle_u\right],
\end{aligned}$$

where we used Lemma 5.1.5 in the last equation. □

**Example 5.2.2.** *Let $(W_t)$ be the standard Wiener process and $(\mathcal{F}_t^0)$ the filtration generated by $(W_t)$. Moreover, let $(Y_t)$ be a Gaussian process independent of $\mathcal{F}_1$ such that for each pair $s, t$ with $0 \le s < t$ the difference $Y_t - Y_s$ is independent of $Y_t$. We denote by $w_t$ the variance of $Y_t$.*

*We enlarge our filtration by*

$$\mathcal{H}_t^0 = \sigma(W_1 + Y_s : 0 \le s \le t) = \sigma(W_1 + Y_t) \vee \sigma(Y_t - Y_s : 0 \le s \le t),$$

*and put $\mathcal{G}_t^0 = \mathcal{F}_t^0 \vee \mathcal{H}_t^0$, $0 \le t \le 1$. Now observe that for all $C \in \sigma(Y_t - Y_s : 0 \le s \le t)$ and Borel sets $B \in \mathcal{B}(\mathbb{R})$ we have*

$$\begin{aligned}
P_t(\cdot, \{W_1 + Y_t \in B\} \cap C) &= P(C) \int 1_B(x + W_1 - W_t + Y_t)\, dP \Big|_{x = W_t} \\
&= P(C) \int 1_B(y + x) \phi_{1-t+w_t}(y) dy \Big|_{x = W_t} \\
&= P(C) \int_B \phi_{1-t+w_t}(y - W_t) dy, \quad 0 \le t < 1,
\end{aligned}$$

*where*

$$\phi_v(y) = \frac{1}{(2\pi v)^{\frac{1}{2}}}\ e^{-\frac{y^2}{2v}}.$$

*The function $f(x,t) = P(C) \int_B \phi_{1-t+w_t}(y - x) dy$ is differentiable in $x$ and satisfies*

$$\frac{\partial}{\partial x} f(x,t) = P(C) \int_B \frac{y - x}{1 - t + w_t}\, \phi_{1-t+w_t}(y - x)\, dy$$

*for all* $0 \le t < 1$ *and* $x \in \mathbb{R}$. *By Ito's formula*

$$P_t(\cdot, \{W_1 + Y_t \in B\} \cap C) = f(0,0) + \int_0^t \frac{\partial}{\partial x} f(W_s, s)\, dW_s + A_t,\ 0 \le t < 1,$$

*where* $A$ *is a process of bounded variation. Note that* $A$ *is also a martingale, and thus* $A = 0$. *Hence*

$$\begin{aligned} & k_t(\cdot, \{W_1 + Y_t \in B\} \cap C) \\ = \;& P(C) \int_B \frac{y - W_t}{1 - t + w_t}\, \phi_{1-t+w_t}(y - W_t)\, dy \\ = \;& P(C) \int 1_B(y + x) \frac{y + x - x}{1 - t + w_t} \phi_{1-t+w_t}(y)\, dy \Big|_{x = W_t(\omega)} \\ = \;& \int_{\{W_1 + Y_t \in B\} \cap C} \frac{W_1(\omega') + Y_t(\omega') - W_t(\omega)}{1 - t + w_t}\, dP_t(\omega, d\omega') \end{aligned}$$

*As a consequence*

$$\beta_t(\omega, \omega') = \frac{k_t(\omega, d\omega')}{P_t(\omega, d\omega')}\Big|_{\mathcal{H}_t^0} = \frac{W_1(\omega') + Y_t(\omega') - W_t(\omega)}{1 - t + w_t},$$

*and by Theorem 5.2.1,*

$$W_t - \int_0^t \frac{W_1 + Y_s - W_s}{1 - s + w_s}\, ds,\ 0 \le t < 1,$$

*is a martingale relative to* $(\mathcal{G}_t)$.

*Similar examples can be found in [9], where the information drifts are derived in a completely different way, though.*

**Example 5.2.3.** *Let* $(W_t)$ *be the standard Wiener process and* $(\mathcal{F}_t)$ *the Wiener filtration. We use the abbreviation* $W_t^* = \sup_{0 \le s \le t} W_s$ *and consider the filtration enlarged by the random variable* $G = 1_{[0,c]}(W_1^*)$, $c > 0$. *Again we want to apply Theorem 5.2.1 in order to obtain the information drift of* $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(G)$. *To this end let* $Z_t = \sup_{t \le r \le 1}(W_r - W_t)$ *and denote by* $p_t$ *the density of* $Z_t$, $0 \le t < 1$. *Now,*

$$\begin{aligned} P_t(\cdot, G = 1) &= P(W_t^* \vee W_t + Z_t \le c | \mathcal{F}_t) \\ &= \int 1_{[0,c]}(y \vee x + Z_t) dP \Big|_{x = W_t, y = W_t^*} \\ &= 1_{[0,c]}(y) \int_0^{c-x} p_t(z) dz \Big|_{x = W_t, y = W_t^*}, \end{aligned}$$

*for all* $0 \le t < 1$. *Note that* $F(x,y,t) = 1_{[0,c]}(y)\int_0^{c-x} p_t(z)dz$ *is differentiable in* $x$ *for all* $0 \le t < 1$ *and* $x \in \mathbb{R}$, *and by Ito's formula*

$$P_t(\cdot, G=1) = F(0,0,0) + \int_0^t \frac{\partial}{\partial x} F(W_s, W_s^*, s)\, dW_s + A_t,\ 0 \le t < 1,$$

*where* $A$ *is a process of bounded variation. Hence*

$$k_t(\cdot, G=1) = \frac{\partial}{\partial x} F(W_t, W_t^*, t),\ 0 \le t < 1.$$

*Similarly, we have*

$$P_t(\cdot, G=0) = H(W_t, W_t^*, t),\ 0 \le t < 1,$$

*and*

$$k_t(\cdot, G=0) = \frac{\partial}{\partial x} H(W_t, W_t^*, t),\ 0 \le t < 1,$$

*where*

$$H(x,y,t) = 1_{(c,\infty)}(y) + 1_{[0,c]}(y)\int_{c-x}^{\infty} p_t(z)dz.$$

*As a consequence, for all* $0 \le t < 1$,

$$\begin{aligned}
\beta_t(\omega,\omega') &= \left.\frac{k_t(\omega, d\omega')}{P_t(\omega, d\omega')}\right|_{\sigma(G)} \\
&= 1_{\{1\}}(G(\omega'))\frac{\partial}{\partial x}\log F(W_t(\omega), W_t^*(\omega'), t) \\
&\quad + 1_{\{0\}}(G(\omega'))\frac{\partial}{\partial x}\log H(W_t(\omega), W_t^*(\omega'), t).
\end{aligned}$$

# Chapter 6

# Information drifts for the Wiener filtration

With the help of the representation theorem of the previous chapter we compute in this chapter explicit information drifts for enlargements of the Wiener filtration. We will concentrate on initial enlargements by the terminal value of some Markov process. The kernels appearing in the representation theorem will be derived with the Clark-Ocone formula. Therefore, we have to assume that the Markov process is in some sense Malliavin differentiable.

Let $(\Omega, \mathcal{F}, P)$ be the canonical Wiener space and $(W_t)_{t\geq 0}$ the Wiener process. More precisely, $\Omega = \mathcal{C}(\mathbb{R}_+, \mathbb{R})$, $\mathcal{F}$ is the $\sigma$-algebra of Borel sets with respect to uniform convergence on compact sets of $\mathbb{R}_+$, $P$ the Wiener measure and $W$ the coordinate process. We denote by $(\mathcal{F}_t^0)$ the natural filtration of $W$ and by $(\mathcal{F}_t)$ the natural filtration completed by the $P$-negligible sets. For the rest of the chapter we fix a time horizon $T > 0$ and denote by $\lambda_{[0,T]}$ the Lebesgue measure on $[0, T]$.

## 6.1 Malliavin traces of differentiable processes

We use notations and concepts of Malliavin calculus as described in the book by Nualart [38]. Only our definition of the 'trace' of a differentiable process is different.

Let $(X_t)$ be a process such that $X_t \in \mathbb{D}^{1,2}$ for all $t \in [0, T]$. For a partition $\Delta : 0 = t_0 \leq t_1 \leq \ldots \leq t_n = T$ we define

$$Z_t^\Delta = \sum_{i=1}^{n} 1_{]t_{i-1},t_i]}(t) D_t X_{t_i}.$$

**Definition 6.1.1.** *A process is called* Malliavin trace *of* $(X_t)$ *and denoted by* $(\mathrm{Tr}(X)_t)$ *if for any sequence of partitions* $(\Delta^n)_n$ *with meshsizes* $|\Delta^n|$ *tending to* 0 *the sequence* $(Z^{\Delta^n})_n$ *converges to* $\mathrm{Tr}(X)$ *in* $L^2(\Omega \times [0,T])$.

**Example 6.1.2.** (Chaos of first order)
*Let* $f \in L^2(R_+)$ *and* $X_t = \int_0^t f(s)dW_s$. *It is known that for fixed* $t > 0$

$$D_s X_t = f(s)$$

*almost everywhere on* $\Omega \times [0,t]$. *Therefore, for any partition* $\Delta$

$$Z_t^\Delta = f(t),$$

*and hence the Malliavin trace coincides with* $f$, *i.e.* $\mathrm{Tr}(X)_t = f(t)$.

**Example 6.1.3.** *Let* $X_t = |W_t|$. *We start by showing that* $D_s X_t = 1_{[0,t]}(s)\,\mathrm{sign}(W_t)$, *where*

$$\mathrm{sign}(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ -1, & \text{else.} \end{cases}$$

*It is easy to verify that the Malliavin derivative of* $W_t^2$ *satisfies* $D_s W_t^2 = 2\ 1_{[0,t]}(s) W_t$. *Now the idea is to apply the chain rule to* $|W_t| = \sqrt{W_t^2}$, *but since the square root is not differentiable in* 0 *we have to use approximations. For this let* $\psi^n$ *denote the function which is equal to the square root on* $[\frac{1}{n}, \infty)$, *and which is extended in a way such that it is continuously differentiable in* $\frac{1}{n}$ *and linear on* $(-\infty, \frac{1}{n}]$. *The derivative then satisfies*

$$\frac{d}{dx}\psi^n(x) = \begin{cases} \frac{1}{2\sqrt{x}}, & \text{if } x > \frac{1}{n}, \\ \frac{1\sqrt{n}}{2}, & \text{else,} \end{cases}$$

*and hence*

$$D_s\psi^n(W_t^2) = D_s W_t^2 \frac{d}{dx}\psi^n(W_t^2) = \begin{cases} 1_{[0,t]}(s)\mathrm{sign}(W_t), & \text{if } W_t^2 \geq \frac{1}{n}, \\ 1_{[0,t]}(s) W_t \sqrt{n}, & \text{else.} \end{cases}$$

*Now observe that for a constant* $C > 0$ *we have*

$$\begin{aligned} E\int_0^T \left(1_{\{W_t^2 \leq \frac{1}{n}\}} D_s\psi^n(W_t^2)\right)^2 ds &\leq t\ P(W_t^2 \leq \frac{1}{n})\frac{1}{n}\ n \\ &\leq t\ 2C\frac{1}{\sqrt{n}}, \end{aligned}$$

*and consequently* $D_s\psi^n(W_t^2)$ *converges to* $1_{[0,t]}(s)\,\mathrm{sign}(W_t)$ *in* $L^2(\Omega \times [0,T])$. *Since* $\psi^n(W_t^2)$ *converges to* $X_t = |W_t|$ *in* $L^2(\Omega)$, *we have* $D_s X_t = 1_{[0,t]}(s)\,\mathrm{sign}(W_t)$.

*We now show that the Malliavin trace of* $(X_t)$ *satisfies* $\mathrm{Tr}(X)_t = \mathrm{sign}(W_t)$. *For this note that the approximation*

$$\begin{aligned} Z_t^{\Delta} &= \sum_{i=1}^{n} 1_{]t_{i-1},t_i]}(t) D_t X_{t_i} \\ &= \sum_{i=1}^{n} 1_{]t_{i-1},t_i]}(t) 1_{[0,t_i]}(t) \mathrm{sign}(W_{t_i}) \\ &= \sum_{i=1}^{n} 1_{]t_{i-1},t_i]}(t) \mathrm{sign}(W_{t_i}) \end{aligned}$$

*is uniformly bounded by* 1. *Moreover let*

$$N = \{(\omega, t) \in \Omega \times [0, T) : \mathrm{sign}(\omega) \text{ is not right-continuous in } t\}.$$

*Then* $N$ *is contained in the measurable set* $\{(\omega, t) : \omega_t = 0\}$. *Since* $P(W_t = 0) = 0$ *for all* $t > 0$, *the set* $N$ *is negligible with respect to* $P \otimes \lambda_{[0,T]}$. *Dominated convergence implies that for any sequence of partitions* $(\Delta^n)_n$ *with meshsize tending to* 0 *the approximations* $(Z^{\Delta^n})_n$ *converge to* $\mathrm{sign}(W_t)$ *in* $L^2(\Omega \times [0, T])$, *and hence* $\mathrm{Tr}(X)_t = \mathrm{sign}(W_t)$.

## 6.2 Initial enlargements by Markov processes

Let $(X_t)$ be a Markov process with respect to $(\mathcal{F}_t)$, with values in $\mathbb{R}$ and with transition function $P_{s,t}$. We assume that $X_t$ belongs to $\mathbb{D}^{1,2}$ and that $X$ has a Malliavin trace $\mathrm{Tr}(X)$. In the sequel we compute explicitly the information drift of filtrations given by

$$\mathcal{G}_t = \bigcap_{s>t} \mathcal{F}_s \vee \sigma(X_T), \;\; 0 \le t \le T.$$

Most of the time we will work under the the following conditions:

**(A1)** $P_{s,t}(x, A)$ is continuously differentiable in $x$ with bounded derivative for all $A \in \mathcal{B}$, $0 \le s < t \le T$,

**(A2)** $t \mapsto \frac{d}{dx} P_{t,T}(X_t, A)$ is right-continuous on $[0, T)$, $A \in \mathcal{B}$,

**(A3)** the family $(D_s P_{r,T}(X_r, A))_{0 \le s \le t, 0 \le r \le t}$ is uniformly integrable, $0 \le t < T$,

**(A4)** the measure $P_{s,t}(x, \cdot)$ is absolutely continuous and has a density $p_{s,t}(x, \cdot)$ which is continuously differentiable in $x \in \mathbb{R}$, $0 \le s < t \le T$.

Property (A1) and the the chain rule (see f.e. Proposition 1.2.2 in [38]) imply that $P_{t,T}(X_t, A)$ belongs to $\mathbb{D}^{1,2}$ for all $t < T$ and $A \in \mathcal{B}$. Moreover we have

$$D_s P_{t,T}(X_t, A) = D_s X_t \frac{d}{dx} P_{t,T}(X_t, A).$$

With property (A4) this can be simplified to

$$D_s P_{t,T}(X_t, A) = D_s X_t \int_A \frac{d}{dx} p_{t,T}(X_t, y)\, dy.$$

**Theorem 6.2.1.** *Let Condition (A1) - (A4) be satisfied. If* $\mathcal{B}(\mathbb{R}) \to \mathbb{R}, A \mapsto \mathrm{Tr}(X)_t \frac{d}{dx} P_{t,T}(X_t, A)$ *is a signed measure which is absolutely continuous with respect to the measure* $P(X_T \in dy|\mathcal{F}_t)$, $P \otimes \lambda_{[0,T]}$*-almost surely, then* $(\mathcal{G}_t)$ *has an information drift* $\mu$ *relative to* $W$*, and* $\mu$ *is given by*

$$\mu_t = \mathrm{Tr}(X)_t \frac{d}{dx} \log p_{t,T}(X_t, X_T), \ \ 0 \le t \le T.$$

*Proof.* Let $t < T$ and choose $A \in \mathcal{B}(\mathbb{R})$. Then

$$P[X_T \in A|\mathcal{F}_t] = P_{t,T} 1_A(X_t)$$

The Clark-Ocone formula leads to the integral representation

$$P[X_T \in A|\mathcal{F}_t] = P(X_T \in A) + \int_0^t E\left[D_s P_{t,T} 1_A(X_t)|\mathcal{F}_s\right] dW_s,$$

where $E\left[D_s P_{t,T} 1_A(X_t)|\mathcal{F}_s\right]$ stands for the optional projection of the process $(D_s P_{t,T} 1_A(X_t))_{0 \le s \le t}$. We aim at simplifying this projection.

Observe at first that for all $r, q \in [0, T)$ we have

$$E\left[D_s P_{q,T} 1_A(X_q)|\mathcal{F}_s\right] = E\left[D_s P_{r,T} 1_A(X_r)|\mathcal{F}_s\right]$$

almost surely on $[0, r \wedge q] \times \Omega$. Hence, for any partition $\Delta^n$ of $[0, t]$ we have

$$E\left[D_s P_{t,T} 1_A(X_t)|\mathcal{F}_s\right] \ = \ \sum_{\Delta^n} 1_{]t_{i-1},t_i]}(s) E\left[D_s P_{t_i,T} 1_A(X_{t_i})|\mathcal{F}_s\right],$$

where the RHS is the optional projection of the sum

$$\sum_{\Delta^n} 1_{]t_{i-1},t_i]}(s) D_s P_{t_i,T} 1_A(X_{t_i}).$$

This sum satisfies

$$\begin{aligned}
&\sum_{\Delta^n} 1_{]t_{i-1},t_i]}(s) D_s P_{t_i,T} 1_A(X_{t_i}) \\
= &\sum_{\Delta^n} 1_{]t_{i-1},t_i]}(s) D_s X_{t_i} \frac{d}{dx} P_{t_i,T} 1_A(X_{t_i}) \\
= &\left( \sum_{\Delta^n} 1_{]t_{i-1},t_i]}(s) D_s X_{t_i} \right) \left( \sum_{\Delta^n} 1_{]t_{i-1},t_i]}(s) \frac{d}{dx} P_{t_i,T} 1_A(X_{t_i}) \right)
\end{aligned}$$

almost surely on the set $\Omega \times [0,t]$. If the meshsize of the partitions $\Delta^n$ converges to zero, then the factor on the left hand side in the last line of the preceding equation converges to the trace $\mathrm{Tr}(X)$ in $L^2(\Omega \times [0,T])$. Along an appropriate subsequence it converges almost surely and in order to simplify notation, we assume the sequence itself to converge almost surely. Then, due to (A2),

$$\lim_n \sum_{\Delta^n} 1_{]t_{i-1},t_i]}(s) D_s P_{t_i,T} 1_A(X_{t_i}) = \mathrm{Tr}(X)_s \frac{d}{dx} P_{s,T} 1_A(X_s)$$

a.e. on $\Omega \times [0,t]$. Moreover, property (A3) implies that the limit of the optional projections is equal to the optional projection of the limit, i.e.

$$E\left[D_s P_{t,T} 1_A(X_t) | \mathcal{F}_s\right] = \mathrm{Tr}(X)_s \frac{d}{dx} P_{s,T} 1_A(X_s)$$

and thus

$$P[X_T \in A | \mathcal{F}_t] = P(X_T \in A) + \int_0^t \mathrm{Tr}(X)_s \frac{d}{dx} P_{s,T} 1_A(X_s)\, dW_s.$$

Now observe that

$$\mathrm{Tr}(X)_s \frac{d}{dx} \log p_{s,T}(X_s, y), \quad y \in \mathbb{R},$$

is the density of the signed measure $\mathcal{B}(\mathbb{R}) \to \mathbb{R}, A \mapsto \mathrm{Tr}(X)_s \frac{d}{dx} P_{s,T} 1_A(X_s)$ with respect to the measure $\mathcal{B}(\mathbb{R}) \to \mathbb{R}, A \mapsto P_{s,T} 1_A(X_s)$. As a consequence

$$\left. \frac{k_s(\omega, d\omega')}{P_s(\omega, d\omega')} \right|_{\sigma(X_T)} = \mathrm{Tr}(X)_s(\omega) \frac{d}{dx} \log p_{s,T}(X_s(\omega), X_T(\omega'))$$

and by Theorem 5.2.1 we obtain as information on $[0,t]$

$$\mu_s = \mathrm{Tr}(X)_s \frac{d}{dx} \log p_{s,T}(X_s, X_T),$$

and the proof is complete. $\square$

**Remark 6.2.2.** *The result of Theorem 6.2.1 is not completely new. For example one can find similar results in [6], where conditions similar to (A1) - (A4) are used. Our proof, however, is based on Theorem 5.2.1, and therefore different.*

**Example 6.2.3.** (Chaos of first order) *Let* $f \in L^2(R_+)$, $X_t = \int_0^t f(s)dW_s$ *and* $(\mathcal{G}_t)$ *the filtration* $(\mathcal{F}_t)$ *enlarged by* $X_T$. *Note that* $(X_t)$ *is a Markov process with transition density*

$$p_{s,t}(x, y) = \frac{1}{\sqrt{2\pi v_{s,t}}} e^{-\frac{(y-x)^2}{2v_{s,t}}},$$

*where*

$$v_{s,t} = E\left[\left(\int_s^t f(u)dW_u\right)^2\right] = E\left[\int_s^t f^2(u)du\right].$$

*For simplicity we assume that* $v_{s,t} > 0$ *for each pair* $s, t$ *with* $0 \le s < t \le T$. *One can easily verify that conditions (A1) - (A4) are satisfied and that*

$$\frac{\partial}{\partial x} \log p_{s,t}(x, y) = \frac{\frac{\partial}{\partial x} p_{s,t}(x, y)}{p_{s,t}(x, y)} = \frac{y - x}{v_{s,t}}.$$

*Hence, by Theorem 6.2.1, the information drift* $\mu$ *of* $(\mathcal{G}_t)$ *relative to* $W$ *is given by*

$$\mu_t = \text{Tr}(X)_t \frac{X_T - X_t}{v_{t,T}} = f(t) \frac{\int_t^T f(s)dW_s}{v_{t,T}}, \ 0 \le t \le T.$$

**Example 6.2.4.** *Let* $f \in L^2(R_+)$ *and* $X_t = \int_0^t f(s)dW_s + Y$, *where* $Y$ *is* $\mathcal{N}(0, w)$*-distributed and independent of* $\mathcal{F}_T$. *The information drift of the filtration enlarged by* $X_T$ *is equal to*

$$\mu_t = f(t) \frac{\int_t^T f(s)dW_s + Y}{v_{t,T} + w}, \ 0 \le t \le T.$$

*This can be deduced from the preceding example by choosing a function* $g$ *such that* $\int_T^{2T} g^2(t)dt = w$. *The noise caused by* $Y$ *is equivalent to the noise caused by* $\int_T^{2T} g(t)dW_t$. *For simplicity we assume* $Y = \int_T^{2T} g(t)dW_t$. *Now Example 6.2.3 implies the result.*

**Example 6.2.5.** *Suppose* $X_t = |W_t|$ *and* $(\mathcal{G}_t)$ *is the filtration enlarged by the random variable* $|W_T|$.

*Observe at first that the transition function of the homogeneous Markov process* $(X_t)$ *has the density*

$$p_t(x,y) = \frac{1}{\sqrt{2\pi t}} \left[ e^{-\frac{(y-x)^2}{2t}} + e^{-\frac{(y+x)^2}{2t}} \right], \text{ for all } x,y > 0.$$

*Note that*

$$\frac{d}{dx} p_t(x,y) = \frac{1}{\sqrt{2\pi t}} \left[ \frac{y-x}{t} e^{-\frac{(y-x)^2}{2t}} - \frac{y+x}{t} e^{-\frac{(y+x)^2}{2t}} \right]$$

*and therefore*

$$\begin{aligned} \frac{d}{dx} \log p_t(x,y) &= \frac{-x}{t} + \frac{y}{t} \left[ \frac{e^{-\frac{(y-x)^2}{2t}} - e^{-\frac{(y+x)^2}{2t}}}{e^{-\frac{(y-x)^2}{2t}} + e^{-\frac{(y+x)^2}{2t}}} \right] \\ &= \frac{-x}{t} + \frac{y}{t} \tanh(\frac{xy}{t}). \end{aligned}$$

*Properties (A1) and (A2) are satisfied. The Malliavin derivative can be written as*

$$D_s P_{t,T} 1_A(X_t) = D_s X_t \frac{d}{dx} P_{t,T} 1_A(X_t) = D_s X_t \int_A \frac{d}{dx} p_{t,T}(X_t, y)\, dy,$$

*and (A3) is easily verified. As shown in Example 6.1.3, the Malliavin trace satisfies* $\mathrm{Tr}(|W|)_t = \mathrm{sign}(W_t)$*, and hence the information drift is given by*

$$\begin{aligned} \mu_t &= \mathrm{sign}(W_t) \left( \frac{-|W_t|}{T-t} + \frac{|W_T|}{T-t} \tanh\left( \frac{|W_t||W_T|}{T-t} \right) \right) \\ &= W_t \left( -\frac{1}{T-t} + \frac{|W_T|}{|W_t|(T-t)} \tanh\left( \frac{|W_t||W_T|}{T-t} \right) \right). \end{aligned}$$

# Part II

# Finite utility and semimartingales

# The general semimartingale model for financial markets

How do people take decisions when investing in some financial market? What is the best way to invest money? Should I buy shares of stock corporations or put my money in a bank account? Economists have developed a lot of models in order to answer these questions. The common way to do this is to determine the optimal portfolio in the framework of these models. A very popular and very abstract model for which this has been done is the so-called general semimartingale model for financial markets. We refer to the work of Kramkov and Schachermayer for the most general results (see [35]). In the following we present the main ideas of this model.

For simplicity we consider only a financial market with two assets: A stock and a money market account. An investor is supposed to have the possibility to buy or sell shares of the stock and to put a part of his money on the money market account. The money market account is supposed to be non-risky. For simplicity we assume the price of it to be constant. In other words, we use the money market account as numeraire. We denote the price process of the stock by $S$ and we assume it to be a semimartingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. Any investment strategy can be represented by the number of shares $\theta_t$ the trader holds at time $t$. We assume any strategy $\theta$ to be predictable relative to $(\mathcal{F}_t)$, and we interpret the stochastic integral process $(\theta \cdot S)$ as the gain process. For a given initial wealth $x \in \mathbb{R}$, $x + (\theta \cdot S)_t$ is then the total wealth at time $t$.

In order to generate an optimal strategy, the model has to be completed by some assumptions on the preferences of the investor. This is done in the most common way: The investor's preferences are supposed to be described by some von Neumann-Morgenstern utility function $U$. More precisely, given two pay-off functions $X$ and $Y$, the trader will choose $X$ if

$$E[U(X)] > E[U(Y)].$$

Recall that the utility function $U$ is only unique up to affine transformations: Given the linear map $\psi_{a,b} : \mathbb{R} \to \mathbb{R}, x \mapsto ax + b$, with $a > 0$ and $b \in \mathbb{R}$, the utility function $\psi_{a,b} \circ U$ gives rise to the same preference ordering. The trader is supposed to maximize the utility of his wealth at some fixed time $T > 0$. Hence, his maximal expected utility may be written as

$$u(x) = \sup\{EU(x + (\theta \cdot S)_T) : \theta \text{ is an allowed strategy}\},$$

where we still have to specify which strategies to allow. The investments $\theta^*$ of the trader will be optimal if they satisfy $EU(x + (\theta^* \cdot S)_T) = u(x)$. Note that the value $u(x)$ does not have any economic meaning. By choosing an affine transformation of $U$, the value of $u(x)$ may change. Nevertheless there are some reasons for studying $u(x)$.

The first reason is that it helps to find the optimal strategy $\theta^*$, provided it exists. Note that $\theta^*$ remains optimal under affine transformations of $U$. If there is no optimal strategy, then $u(x)$ helps to find strategies being close to the supremum. In this sense $u(x)$ is a point of orientation indicating how good a chosen strategy is.

Suppose the trader is given some additional information such that he has not only access to $(\mathcal{F}_t)$, but that his information flow is given by some bigger filtration $(\mathcal{G}_t)$. The maximal expected utility can help to calculate some fair price of the additional information. For instance, let $u^{\mathcal{F}}(x)$ and $u^{\mathcal{G}}(x)$ be the maximal expected utility for a trader whose strategies have to be adapted to $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$ respectively. $q \in \mathbb{R}$ will be called *utility based price* if $u^{\mathcal{F}}(x) = u^{\mathcal{G}}(x - q)$. We will come back to this topic in the beginning of Chapter 10.

The model presented so far makes the crucial mathematical assumption that the price process of the stock is a semimartingale. This assumption allows us to use the powerful tool of stochastic calculus. However, are there any economic or mathematical reasons justifying this assumption? Indeed, there are some reasons which we give in this part of the thesis. To this end, we will at first drop the assumption that $S$ is a semimartingale. Moreover, we allow only simple strategies, since we do not have general stochastic integration.

The main result of Chapter 7 will be the following: If the maximal expected utility $u(x)$ is finite, then the process $S$ is a semimartingale. Note that the property of $u(x)$ to be finite does not change under affine transformations of the utility function $U$. Hence this is a property depending only on the preference ordering of our investor.

Once the semimartingale property is established, we can work with general integrands again. In Chapter 8 we therefore compare simple with general investment strategies. We will see that for continuous price processes the general utility maximum can be attained by simple strategies.

In Chapter 9 we continue to investigate consequences of finite utility and we derive further structure properties of the price process. For example, finite utility implies the bounded variation part in the semimartingale decomposition to be nicely controlled by the martingale part.

# Chapter 7

# On the link between finite utility, the no-arbitrage and the semimartingale property

Delbaen and Schachermayer [10] establish a link between the (NFLVR) condition and the semimartingale property of an asset price process on a financial market. In this chapter we shall compare these two properties with a third one: the boundedness of expected utility with respect to wealth processes based on simple admissible integrands, and non-bounded utility functions. For this we will distinguish two cases. At first we allow the wealth process to be unbounded from below. After that we analyze the case where the wealth has to stay above a certain level during the trading period. Our main result roughly shows that boundedness of utility implies the semimartingale property of the price process, regardless of whether his wealth has to be bounded from below or not.

## 7.1 Basic definitions and properties

Here we collect the most important definitions, notations and conventions needed throughout this part. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(\mathcal{F}_t)_{0 \leq t \leq T}$ an arbitrary filtration satisfying the usual conditions, where $T$ is the finite time horizon. Suppose that $S : [0, T] \times \Omega \to \mathbb{R}$ is a stochastic process. $S$ will take the role of the asset price process on our financial market. Throughout this chapter, we let $S$ have cadlag paths and be adapted to $(\mathcal{F}_t)$. For the moment we do not need any more assumptions. Only in the end of the Section 7.2 we shall sometimes assume local boundedness of $S$.

The wealth of the agent on our market with information horizon $(\mathcal{F}_t)$

will be determined in the subsequent sections by simple investment strategies (integrands) of the following form. A simple integrand is a linear combination of processes of the form $f1_{]T_1,T_2]}$ where $f$ is a bounded and $\mathcal{F}_{T_1}$-measurable random variable and $T_1$ and $T_2$ are finite stopping times with respect to the filtration $(\mathcal{F}_t)$. The collection of simple integrands will be denoted by $\mathcal{S}$ and the stochastic integral process of simple or more general predictable integrands with respect to a cadlag process $X$ by $\theta \cdot X$. We now recall some terminology introduced in [10].

**Definition 7.1.1.** *Let $\theta$ be a strategy.*

*a) Let $a \geq 0$. $\theta$ is called* $a$-admissible *if for all $t \in [0,T]$ we have $(\theta \cdot S)_t \geq -a$ almost surely. It will be called* $a$-superadmissible *if $(\theta \cdot S)_t > -a$ almost surely.*

*b) $\theta$ is said to be* admissible *if it is a-admissible for some $a \geq 0$.*

We also recall some notions of arbitrage. For this we put

$$\mathcal{K}^s = \{(\theta \cdot S)_T | \theta \in \mathcal{S} \text{ admissible}\}$$

and write $C^s$ for the set of functions dominated by elements of $\mathcal{K}^s$, i.e. $C^s = \mathcal{K}^s - L^0_+$. Now let $C = C^s \cap L^\infty$.

**Definition 7.1.2.** *The process $S$ is said to satisfy the* no free lunch with vanishing risk *(NFLVR) property for simple integrands if*

$$\bar{C} \cap L^\infty_+ = \{0\},$$

*where $\bar{C}$ denotes the closure of $C$ in $L^\infty$. If the intersection contains more than the trivial element* 0*, we say that $S$ satisfies (FLVR) for simple integrands.*

For the general (NFLVR) condition, we refer to $\mathcal{K}$ defined as $\mathcal{K}^s$ just with general $(\mathcal{F}_t)$-predictable $\theta$ and with well defined stochastic integral.

If not stated otherwise in the sequel, we mean by a *utility function* a function $U : \mathbb{R} \to [-\infty, \infty)$ which is strictly concave and strictly increasing on $\text{dom}(U) = \{y : U(y) > -\infty\}$. We always assume that $\text{dom}(U)$ is non-empty, and we interpret the integral $EU(x + (\theta \cdot S)_T)$ as the expected utility from terminal wealth of a trader possessing initial wealth $x$ and choosing the strategy $\theta$. Note that the integral might not exist. For ease of notation, we use the convention $EU(x + (\theta \cdot S)_T) = -\infty$ if both the positive and the negative part of $U(x + (\theta \cdot S)_T)$ have infinite expectation. Moreover, some abbreviations will be helpful.

**Definition 7.1.3.**

a) The maximal expected utility by using simple admissible strategies is given by

$$s(x) = \sup\{EU(x + (\theta \cdot S)_T) : \theta \in \mathcal{S} \text{ is admissible}\}.$$

b) If losses must not be smaller than $a \geq 0$, we write

$$s_a(x) = \sup\{EU(x + (\theta \cdot S)_T) : \theta \in \mathcal{S} \text{ is } a - \text{admissible}\}.$$

c) Sometimes we will consider the maximum taken over all strategies, such that the utility never becomes $-\infty$. More precisely, if $\mathrm{dom}(U) \neq \mathbb{R}$, $c = \sup\{y \in \mathbb{R} : U(y) = -\infty\}$ and $x > c$, we set

$$s_+(x) = \sup\{EU(x + (\theta \cdot S)_T) : \theta \in \mathcal{S} \text{ is } (x - c) - \text{superadmissible}\},$$

and if $\mathrm{dom}(U) = \mathbb{R}$, then $s_+(x) = s(x)$.

The following auxiliary results will be frequently used.

**Lemma 7.1.4.** *Let* $x > \sup\{y : U(y) = -\infty\}$ *and* $a > 0$. *Then*

$$s_a(x) = \sup_{\varepsilon > 0} s_{a-\varepsilon}(x).$$

*Proof.* Let $x > \sup\{y : U(y) = -\infty\}$ and choose an $a$-admissible simple strategy $\theta$ such that $EU(x + (\theta \cdot S)_T) > -\infty$. Put

$$\theta^n = (1 - \frac{1}{n})\theta$$

for all $n \geq 1$. Clearly $\theta^n$ is $(a - \frac{a}{n})$-admissible. Now observe that by monotone convergence

$$\lim_n E\left([U(x + (\theta^n \cdot S)_T) \wedge U(x)] - U(x)\right)^- \\ = E\left([U(x + (\theta \cdot S)_T) \wedge U(x)] - U(x)\right)^-,$$

and

$$\lim_n E\left([U(x + (\theta^n \cdot S)_T) \vee U(x)] - U(x)\right)^+ \\ = E\left([U(x + (\theta \cdot S)_T) \vee U(x)] - U(x)\right)^+.$$

Thus we obtain

$$\lim_n EU(x + (\theta^n \cdot S)_T) = EU(x + (\theta \cdot S)_T),$$

and this implies

$$s_a(x) \le \sup_{\varepsilon > 0} s_{a-\varepsilon}(x).$$

Since the right hand side does obviously not exceed the left hand side, the proof is complete. □

As an immediate consequence we obtain the following.

**Corollary 7.1.5.** *If* $\mathrm{dom}(U) \neq \mathbb{R}$, $c = \sup\{y \in \mathbb{R} : U(y) = -\infty\}$ *and* $x > c$, *then*

$$s_+(x) = s_{x-c}(x).$$

## 7.2 Unbounded wealth

In this section we explore the relationship between finiteness of $s(x)$ and the semimartingale property of the price dynamics.

We start with a useful reformulation of the (FLVR) property.

**Lemma 7.2.1.** *S satisfies the (FLVR) property for simple integrands if and only if there is a sequence* $(\theta^n)_{n\ge 0}$ *of admissible simple integrands such that the following two conditions are satisfied*

*i)* $f_n = (\theta^n \cdot S)_T, n \in \mathbb{N}$, *converges a.s. to a nonnegative function* $f$ *satisfying* $P(f > 0) > 0$ *and*

*ii)* $\|f_n^-\|_\infty \rightarrow 0.$

*Proof.* Let $(f_n)$ be a sequence satisfying i) and ii), and let $\alpha > 0$ such that $P(f > \alpha) > \alpha$. Egoroff's theorem implies that there is a measurable set $B$ that satisfies $P(B^c) \le 1 - \frac{\alpha}{2}$, and is such that $(1_B f_n)$ converges uniformly to $1_B f$. Then $(1_B f_n^+ - f_n^-)$ belongs to $C^s$, and converges uniformly to $1_B f$. Since $P(1_B f > \alpha) > \frac{\alpha}{2}$, $S$ satisfies (FLVR).

For the reverse direction, suppose that the (FLVR) property holds. Then there is a sequence $(\theta^n)_{n\in\mathbb{N}}$ of simple integrands such that the integrals $g_n = (\theta^n \cdot S)_T, n \in \mathbb{N}$, satisfy

i') $\|g_n^-\|_\infty \rightarrow 0$ and

ii') $g_n^+ \nrightarrow 0$ in probability.

One can find an $\alpha > 0$ such that for any $n \geq 0$ there exists a $k \geq n$ with $P(g_k > \alpha) > \alpha$. By taking a subsequence, still denoted by $(g_n)_{n\in\mathbb{N}}$, we assume that $P(g_n > \alpha) > \alpha$ holds for all $n \geq 0$. From Lemma A.1.1 in [10] we know that there are $f_n \in conv(g_k : k \geq n)$ converging almost surely to some $f$ with $P(f > 0) > 0$. Observe that every $f_n$ is still an integral of some simple process with respect to $S$. i) and ii) follow and the claim is proven. □

The following proposition provides the link between the boundedness of the agent's utility for simple strategies and the (NFLVR) condition.

**Proposition 7.2.2.** *Let* $U : \mathbb{R} \to [-\infty, \infty)$ *be a utility function with* $\lim_{y\to\infty} U(y) = \infty$. *Then for all* $x > \sup\{y \in \mathbb{R} : U(y) = -\infty\}$ *(recall* $\sup \emptyset = -\infty$*) the following implication holds.*

$$\textit{If} \quad s(x) < \infty, \quad \textit{then} \quad (NFLVR)\, \textit{for simple integrands.}$$

*Proof.* Let $x > \sup\{y \in \mathbb{R} : U(y) = -\infty\}$. Then there is a $\delta > 0$ for which $x - \delta > \sup\{y \in \mathbb{R} : U(y) = -\infty\}$. We put $D = U(x - \delta) \wedge 0 > -\infty$. Suppose that the (NFLVR) property for simple integrands is violated. By the preceding lemma we can find a sequence $(\theta^n)_{n\in\mathbb{N}}$ of admissible simple integrands such that the final payoffs $f_n = (\theta^n \cdot S)_T, n \in \mathbb{N}$, satisfy

**i)** $f_n = (\theta^n \cdot S)_T \to f$ a.s. , where $f$ is nonnegative with $P(f > 0) > 0$ and

**ii)** $\|f_n^-\|_\infty \to 0$.

For $n \in \mathbb{N}$ we set $\varepsilon_n = \|f_n^-\|_\infty$. For all but finitely many $n \in \mathbb{N}$ we have $\varepsilon_n < \delta$. To simplify notation we assume that this holds for all $n \in \mathbb{N}$. We now define new simple integrands

$$\pi^n = \frac{\delta}{\varepsilon_n}\theta^n$$

for all $n \in \mathbb{N}$. It is clear that all the integrals $(\pi^n \cdot S)_T$ exceed the bound $-\delta$. Furthermore the random variables $U(x + (\pi^n \cdot S)_T)$ are bounded from below by the constant $D$. More formally,

$$\begin{aligned} U(x + (\pi^n \cdot S)_T) &= U(x + \frac{\delta}{\varepsilon_n}(\theta^n \cdot S)_T) \\ &\geq U(x + \frac{\delta}{\varepsilon_n}(-\varepsilon_n)) \\ &= U(x - \delta) \\ &= D > -\infty. \end{aligned}$$

Since $f_n$ converges to the nontrivial nonnegative function $f$, one can find an integer $n_0$ and real numbers $\alpha > 0$ and $\beta > 0$ such that

$$P((\theta^n \cdot S)_T > \alpha) > \beta$$

for all $n \geq n_0$. This implies

$$\begin{aligned}
\liminf_{n\to\infty} E[U(x + (\pi^n \cdot S)_T)] &= \liminf_{n\to\infty} E[U(x + \frac{\delta}{\varepsilon_n}(\theta^n \cdot S)_T)] \\
&\geq \liminf_{n\to\infty} E[D 1_{\{(\theta^n \cdot S)_T \leq \alpha\}} \\
&\quad + U(x + \frac{\delta}{\varepsilon_n}\alpha) 1_{\{(\theta^n \cdot S)_T > \alpha\}}] \\
&\geq \liminf_{n\to\infty} [D(1-\beta) + U(x + \frac{\delta}{\varepsilon_n}\alpha)\beta] \\
&= \infty.
\end{aligned}$$

Thus $s(x) = \infty$, and hence the result. □

**Remark 7.2.3.** *Proposition* 7.2.2 *holds in particular for all increasing functions* $U$ *with* $\lim_{y\to\infty} U(y) = \infty$.

Combining Proposition 7.2.2 with the results of the fundamental paper by Delbaen and Schachermayer [10] we obtain the intuitively plausible relationship between boundedness of the expected utility and the semimartingale property for the asset price process.

**Corollary 7.2.4.** *Let* $S$ *be a cadlag and locally bounded adapted process,* $U : \mathbb{R} \to [-\infty, \infty)$ *a utility function with* $\lim_{y\to\infty} U(y) = \infty$ *and* $x > \sup\{y \in \mathbb{R} : U(y) = -\infty\}$. *If* $s(x) < \infty$, *then* $S$ *is a semimartingale with respect to* $(\mathcal{F}_t)$.

*Proof.* By Proposition 7.2.2, the process $S$ satisfies the (NFLVR) property for simple integrands. Theorem 7.2 in Delbaen and Schachermayer [10] states that in this case $S$ is already a semimartingale. Note that Delbaen and Schachermayer [10] use a slight different definition of simple integrands. They admit unbounded processes. But one can show that (NFLVR) for bounded simple integrands is equivalent to (NFLVR) for all (possibly unbounded) simple processes. □

## 7.3 Wealth bounded from below

We can sharpen the result of the preceding Corollary 7.2.4. In fact, even if the wealth process has to be positive at any time in the trading interval $[0, T]$, we can show that boundedness of expected utility over all simple strategies is sufficient for the semimartingale property of $S$ to follow. To this end, we need the next result, which is of interest for its own.

**Theorem 7.3.1.** *Let $S$ be a cadlag and locally bounded adapted process indexed by $[0, T]$, $U$ a utility function satisfying $\lim_{y\to\infty} U(y) = \infty$, $x > \sup\{y \in \mathbb{R} : U(y) = -\infty\}$ and $a > 0$. If*

$$\sup\{E[U(x + (\theta \cdot S)_T)] : \theta \in \mathcal{S},\ a-\textit{admissible and}\ \ |\theta| \leq 1\} < \infty,$$

*then $S$ is a semimartingale.*

Our proof is similar to the one of Theorem 7.2 in Delbaen and Schachermayer [10]. Since $S$ is locally bounded we can find a sequence of stopping times $(T_n)_{n\in\mathbb{N}}$ such that the stopped processes $S^{T_n}$ are bounded. It is sufficient to prove that each $S^{T_n}$ is a semimartingale. To put it simply we assume that $S$ is already bounded by some constant $C$. Our proof shall proceed in two lemmas for which we suppose that the assumptions of Theorem 7.3.1 hold.

**Lemma 7.3.2.** *Let $\Theta$ to be a set of simple integrands $\theta$ satisfying $|\theta| \leq 1$. If $\{\sup_{0\leq t\leq T}(\theta \cdot S)_t^- : \theta \in \Theta\}$ is bounded in $L^0$, then the set $\{\sup_{0\leq t\leq T}(\theta \cdot S)_t^+ : \theta \in \Theta\}$ is also bounded in $L^0$.*

*Proof.* Suppose that $\{\sup_{0\leq t\leq T}(\theta \cdot S)_t^+ : \theta \in \Theta\}$ is not bounded in $L^0$. Then one can find a sequence $(c_n)_{n\in\mathbb{N}}$ of real numbers and $(\theta_n)_{n\in\mathbb{N}}$ in $\Theta$ satisfying $c_n \to \infty$ and $P(\sup_{0\leq t\leq T}(\theta^n \cdot S)_t > c_n + 2C) \geq \varepsilon > 0$ for all $n \in \mathbb{N}$. Since $\{\sup_{0\leq t\leq T}(\theta \cdot S)_t^- : \theta \in \Theta\}$ is bounded in $L^0$, there is a constant $K$ for which

$$\sup_{\theta\in\Theta} P(\sup_{0\leq t\leq T}(\theta \cdot S)_t^- \geq K) < \frac{\varepsilon}{2}.$$

Consider the stopping times

$$T_n = \inf\{t > 0 : (\theta^n \cdot S)_t^- \geq K \text{ or } (\theta^n \cdot S)_t \geq c_n + 2C\} \wedge T, \quad n \in \mathbb{N}.$$

We then have for $n \in \mathbb{N}$

i) $(\theta^n 1_{[0,T_n]} \cdot S)_T \geq -K - 2C$

ii) $P((\theta^n 1_{[0,T_n]} \cdot S)_T \geq c_n) \geq \frac{\varepsilon}{2}$.

We choose $\delta \in (0,a)$ such that $U(x-\delta) > -\infty$ still holds and a sequence of real numbers $(\gamma_n)_{n\in\mathbb{N}}$ with $\gamma_n \in (0,1)$ converging to 0 slowly enough to guarantee $\gamma_n c_n \to \infty$. Now define new simple integrands by

$$\pi^n = \frac{\delta\gamma_n}{K+2C} 1_{[0,T^n]}\theta^n, \quad n \in \mathbb{N}.$$

For all but finitely many $n \in \mathbb{N}$ we remark $|\frac{\delta\gamma_n}{K+2C}| \leq 1$, because $\gamma_n \to 0$. Without loss of generality we suppose $|\pi^n| \leq 1, n \in \mathbb{N}$. The integrands $\pi^n$ satisfy for all $t \in [0,T]$

$$\begin{aligned}(\pi^n \cdot S)_t &= \frac{\delta\gamma_n}{K+2C}(1_{[0,T^n]}\theta^n \cdot S)_t \\ &\geq \frac{\delta\gamma_n}{K+2C}(-K-2C) \geq -\delta\gamma_n \\ &\geq -\delta.\end{aligned} \tag{7.1}$$

In particular, this means that $\pi^n$ is $a$-admissible and $U(x+(\pi^n \cdot S)_T)$ is bounded from below by the constant $D = U(x-\delta) > -\infty$. Now observe

$$P((\pi^n \cdot S)_T \geq \frac{\delta\gamma_n c_n}{K+2C}) = P((1_{[0,T^n]}\theta^n \cdot S)_T \geq c_n) > \frac{\varepsilon}{2}.$$

Put $a_n = \frac{\delta\gamma_n c_n}{K+2C}, n \in \mathbb{N}$, and note that $a_n \to \infty$ as $n \to \infty$. Hence

$$\begin{aligned}E[U(x+(\pi^n \cdot S)_T)] &= E[U(x+(\pi^n \cdot S)_T)1_{\{(\pi^n \cdot S)_T < a_n\}}] \\ &\quad + E[U(x+(\pi^n \cdot S)_T)1_{\{(\pi^n \cdot S)_T \geq a_n\}}] \\ &\geq DP[(\pi^n \cdot S)_T) < a_n] + U(x+a_n)P[(\pi^n \cdot S)_T \geq a_n] \\ &\geq D(1-\frac{\varepsilon}{2}) + U(x+a_n)\frac{\varepsilon}{2} \longrightarrow \infty.\end{aligned}$$

But this is in contradiction with the hypothesis $\sup\{EU(x+(\theta \cdot S)_T) : \theta \in \mathcal{S},\ a-\text{adm. and } |\theta| \leq 1\} < \infty$. □

As in [10] one can show that the preceding lemma implies

**Lemma 7.3.3.** *(Lemma 7.4. in [10]) The set*

$$\{\sum_{k=0}^{n}(S_{T_{k+1}} - S_{T_k})^2 | n \in \mathbb{N},\ 0 \leq T_0 \leq \ldots \leq T_{n+1} \leq T\}$$

*is bounded in* $L^0$.

*Proof of Theorem 7.3.1.* We can now complete the proof of Theorem 7.3.1 as in [10]. The two preceding lemmas imply that $S$ is a semimartingale (proof of Theorem 7.2 in [10]). □

As an immediate consequence we obtain

**Corollary 7.3.4.** *Let $S$ be a cadlag and locally bounded adapted process, $U : \mathbb{R} \to [-\infty, \infty)$ a utility function with $\lim_{y\to\infty} U(y) = \infty$, $x > \sup\{y \in \mathbb{R} : U(y) = -\infty\}$ and $a > 0$. If $s_a(x) < \infty$, then $S$ is a semimartingale with respect to $(\mathcal{F}_t)$. In particular, if $s_+(x) < \infty$, then $S$ is a semimartingale with respect to $(\mathcal{F}_t)$.*

## 7.4 Finite utility characterizes semimartingales

In Theorem 7.3.1 we saw that bounded utility via simple strategies $\theta$, such that $|\theta| \leq 1$, implies the process $S$ to be a semimartingale. In this section we show the converse: If $S$ is a semimartingale, then there is a utility function $U$ with $\lim_{y\to\infty} U(y) = \infty$ such that by using simple strategies $\theta$ with $|\theta| \leq 1$ the maximal expected utility is finite.

Assume $S$ to be a semimartingale and fix an initial wealth $x$. For simplicity we assume $x > 0$. In the following we abbreviate

$$\beta = \{(\theta \cdot S)_T : \theta \in \mathcal{S}, \text{ such that } |\theta| \leq 1\}.$$

Since $S$ is a semimartingale, the set $\beta$ is bounded in $L^0$, i.e.

$$\lim_{c\to\infty} \sup_{Y\in\beta} P(|Y| \geq c) = 0,$$

(see e.g. Section 9, Chapter III in [41]). Now choose a strictly decreasing and convex function $f : [0, \infty) \to \mathbb{R}$, such that $\lim_{y\to\infty} f(y) = 0$ and

$$f(c) \geq \sup_{Y\in\beta} P(|Y| \geq c)$$

for all $c \geq 0$. We define

$$U(y) = \begin{cases} \sqrt{\frac{1}{f(y)} - \frac{1}{f(0)}}, & \text{if } y \geq 0, \\ -\infty, & \text{if } y < 0. \end{cases}$$

**Lemma 7.4.1.** *$U$ is strictly increasing, concave, and satisfies $\lim_{y\to\infty} U(y) = \infty$. Moreover, on the interval $[0, \infty)$, $U$ is the inverse of the function $f^{-1} \circ \xi$, where*

$$\xi : [0, \infty) \to (0, f(0)],\ y \mapsto \frac{1}{y^2 + \frac{1}{f(0)}}.$$

*Proof.* Obviously $U$ is strictly increasing. In order to prove concavity note that the map $\psi : (0, f(0)] \to \mathbb{R}$, $y \mapsto -\sqrt{\frac{1}{y} - \frac{1}{f(0)}}$ is convex and nondecreasing. By standard arguments, see e.g. Theorem 5.1 in Rockafellar [43], also the composition $\psi \circ f$ is convex, and hence $U = -\psi \circ f$ is concave.
The last assertion follows from the fact that the inverse $(f^{-1} \circ \xi)^{-1}$ is equal to $\xi^{-1} \circ f$. □

**Theorem 7.4.2.** *$U$ is a utility function such that for all $a > 0$*

$$\sup\{E[U(x + (\theta \cdot S)_T)] : \theta \in \mathcal{S},\ a - \textit{admissible and}\ \ |\theta| \leq 1\} < \infty.$$

*Proof.* Let $a > 0$ and $\theta$ be an $a$-admissible simple strategy such that $|\theta| \leq 1$. Since $U(y) = -\infty$ for $y < 0$, we may assume that a.s. $X = x + (\theta \cdot S)_T \geq 0$. This implies

$$\begin{aligned} EU(X) &= \int_0^\infty P(U(X) \geq c)\, dc \\ &= \int_0^\infty P(X \geq U^{-1}(c))\, dc \\ &\leq \int_0^\infty f(U^{-1}(c))\, dc \\ &= \int_0^\infty \xi(c)\, dc \\ &= \int_0^\infty \frac{1}{c^2 + \frac{1}{f(0)}}\, dc \\ &< \infty, \end{aligned}$$

and hence the result. □

Combining this with the results of the previous section we obtain a characterization of semimartingales in terms of finite maximal expected utility.

**Theorem 7.4.3.** *Suppose $S$ to be a cadlag and locally bounded adapted process indexed by $[0, T]$, $x > 0$ and $a > 0$. Then $S$ is a semimartingale if and only if there exists a utility function $U$, such that $\lim_{y\to\infty} U(y) = \infty$, $\sup\{y \in \mathbb{R} : U(y) = -\infty\} = 0$, and*

$$\sup\{E[U(x + (\theta \cdot S)_T)] : \theta \in \mathcal{S},\ a - \textit{admissible and}\ \ |\theta| \leq 1\} < \infty.$$

*Proof.* This follows from Theorem 7.4.2 and Theorem 7.3.1. □

# Chapter 8

# Simple versus general strategies

In the preceding chapter we have seen that if the expected utility maximized over the set of simple strategies is finite, the price process $S$ is a semimartingale. As a consequence, $S$ is a stochastic integrator, and its stochastic integral is defined not only for simple integrands, but for a much wider class of $(\mathcal{F}_t)$-predictable processes. A natural question arising in this context is the following: can a trader increase his optimal utility by using general $S$-integrands? While this may be the case for discontinuous $S$, as is shown by an example at the end of this chapter, its main result will prove that for continuous asset price processes $S$ the answer is no.

Besides the assumptions of the previous chapter we suppose throughout this chapter that $S$ is an $(\mathcal{F}_t)$-semimartingale. We denote by $\mathcal{A}$ the set of all $(\mathcal{F}_t)$-predictable processes $\theta$ which satisfy $\theta_0 = 0$ and which are integrable with respect to $S$ in the sense of Protter (see Section 2, Chapter IV in [41]). From now on we mean by *strategy* an element of $\mathcal{A}$.

As in the previous chapter we use for all $\theta \in \mathcal{A}$ the convention $E[U(x + (\theta \cdot S)_T)] = -\infty$ if both the negative and the positive part are not integrable.

Before stating the main result of this chapter, some preliminary steps are in order.

## 8.1 Approximation by simple strategies

We next define quantities to be compared to the maximal expected utility taken over simple strategies.

**Definition 8.1.1.** *Let $c = \sup\{y : U(y) = -\infty\} \in [-\infty, \infty)$, and fix an initial wealth $x > c$. Then let*

*a)* $u(x) = \sup\{EU(x + (\theta \cdot S)_T) : \theta \in \mathcal{A}$ *is admissible*$\}$, *b)* $u_a(x) = \sup\{EU(x + (\theta \cdot S)_T) : \theta \in \mathcal{A} \text{ is } a - \text{admissible}\}$, $a \geq 0$.

*c) If* $\mathrm{dom}(U) \neq \mathbb{R}$, *we set*

$$u_+(x) = \sup\{EU(x + (\theta \cdot S)_T) : \theta \in \mathcal{A} \text{ is } (x - c) - \text{superadmissible}\},$$

*and if* $\mathrm{dom}(U) = \mathbb{R}$, *then* $u_+(x) = u(x)$.

Sometimes we will also write $u^{\mathcal{F}}(x)$, and similarly $u_a^{\mathcal{F}}(x)$ and $u_+^{\mathcal{F}}(x)$, in order to stress the filtration we are referring to.

Like for the maximal utility via simple integrands we can deduce the following simplifications.

**Lemma 8.1.2.** *Let* $x > \sup\{y : U(y) = -\infty\}$ *and* $a > 0$. *Then*

$$u_a(x) = \sup_{\varepsilon > 0} u_{a-\varepsilon}(x).$$

*Proof.* Can be shown like Lemma 7.1.4. □

This implies immediately the following.

**Corollary 8.1.3.** *If* $\mathrm{dom}(U) \neq \mathbb{R}$, $c = \sup\{y \in \mathbb{R} : U(y) = -\infty\}$ *and* $x > c$, *then*

$$u_+(x) = u_{x-c}(x).$$

It is known that stochastic integrals can be approximated by simple integrals (since they can even be defined as their limit). If the underlying process is continuous, then the approximating simple integrands can be chosen to be admissible. This will be shown now.

**Lemma 8.1.4.** *Suppose* $S$ *is a continuous semimartingale. Let* $a > 0$ *and* $\theta$ *be an* $a$*-admissible strategy. For all* $\varepsilon > 0$ *there is a sequence* $(\theta^n)$ *of* $(a+\varepsilon)$*-admissible simple strategies such that a.e.* $(\theta^n \cdot S)$ *converges to* $(\theta \cdot S)$ *uniformly on* $[0, T]$.

*Proof.* Let $\theta \in \mathcal{A}$ be $a$-admissible and $\varepsilon > 0$. There is a sequence $(\pi^n)$ of simple strategies such that a.e. the integrals $(\pi^n \cdot S)$ converge to $(\theta \cdot S)$ uniformly on $[0, T]$ (see e.g. Theorem 2.12, Chapter IV in [42]). Now put

$$T^n = \inf\{t \geq 0 : (\pi^n \cdot S)_t \leq -a - \varepsilon\} \wedge T$$

for all $n \in \mathbb{N}$. Then the strategies

$$\theta^n = 1_{[0,T_n]}\pi^n$$

are simple and $(a+\varepsilon)$-admissible. Observe that $\lim_n T_n = T$ a.s. and hence $(\theta^n \cdot S)$ converges to $(\theta \cdot S)$ uniformly on $[0, T]$. □

## 8.2 $\mathrm{dom}(U) = \mathbb{R}$

Throughout this section we assume that $U$ is a utility function with $\mathrm{dom}(U) = \mathbb{R}$.

We now state and prove the first main result in this chapter.

**Theorem 8.2.1.** *Let $x \in \mathbb{R}$ and $a > 0$. If $S$ is continuous, then*

$$u_a(x) = s_a(x).$$

*Proof.* By Lemma 8.1.2 it suffices to show that $s_a(x) \geq u_{a-\varepsilon}(x)$ for all $\varepsilon > 0$. For this fix $\varepsilon > 0$ and choose an $(a - \varepsilon)$-admissible strategy $\theta$. By Lemma 8.1.4 there is a sequence $(\theta^n)$ of $a$-admissible simple strategies such that $(\theta^n \cdot S)$ converges to $(\theta \cdot S)$ uniformly on $[0, T]$. From Fatou's lemma we get

$$\begin{aligned} EU(x + (\theta \cdot S)_T) &\leq \liminf_n EU(x + (\theta^n \cdot S)_T) \\ &\leq s_a(x), \end{aligned}$$

and hence the result. □

**Corollary 8.2.2.** *If $S$ is continuous, then for all $x \in \mathbb{R}$*

$$u(x) = s(x).$$

*Proof.* This follows from Theorem 8.2.1, since

$$\begin{aligned} u(x) &= \sup_{a>0} u_a(x) \\ &= \sup_{a>0} s_a(x) \\ &= s(x). \end{aligned}$$

□

## 8.3 $\mathrm{dom}(U) \neq \mathbb{R}$

Throughout this section let $U$ be a utility function with $\mathrm{dom}(U) \neq \mathbb{R}$. For simplicity we assume that $\sup\{y : U(y) = -\infty\} = 0$.

**Theorem 8.3.1.** *Let $x > 0$. If $S$ is continuous, then*

$$u_a(x) = s_a(x) \text{ for all } a \in (0, x].$$

*Proof.* Let $\theta$ be $a$-admissible. Then it must satisfy

$$(\theta \cdot S)_T \geq -x, \text{ a.s.}$$

Consider now the strategies

$$\theta^n = (1 - \frac{1}{n})\theta, \quad n \in \mathbb{N}.$$

Notice that $\theta^n$ is $(x - \frac{x}{n})$-admissible. In particular

$$(\theta \cdot S)_T \geq -x + \frac{x}{n}, \text{ a.s.}$$

and hence $U(x + (\theta^n \cdot S)_T)$ is bounded from below for all $n \in \mathbb{N}$. As in the proof of Lemma 7.1.4 one can deduce that

$$\lim_n EU(x + (\theta^n \cdot S)_T) = EU(x + (\theta \cdot S)_T).$$

Therefore it is sufficient to show for all $n \in \mathbb{N}$

$$EU(x + (\theta^n \cdot S)_T) \leq s_a(x).$$

To this end fix $n \in \mathbb{N}$. By Lemma 8.1.4 we can choose a sequence $(\pi^k)$ of $(x - \frac{x}{2n})$-admissible simple strategies such that $(\pi^k \cdot S)_{k \geq 1}$ converges to $(\theta^n \cdot S)$ uniformly on $[0, T]$. Note that for all $k \geq 1$

$$x + (\pi^k \cdot S)_T \geq \frac{x}{2n} > 0,$$

and hence by Fatou's Lemma

$$\begin{aligned} EU(x + (\theta^n \cdot S)_T) &\leq \liminf_k EU(x + (\pi^k \cdot S)_T) \\ &\leq s_a(x). \end{aligned}$$

Thus the proof is complete. □

From the preceding theorem one can deduce the following.

**Corollary 8.3.2.** *Let $x > 0$. If $S$ is continuous, then*

$$u_+(x) = s_+(x).$$

*Proof.* The preceding theorem implies together with Corollaries 7.1.5 and 8.1.3

$$u_+(x) = u_x(x) = s_x(x) = s_+(x),$$

and thus the result. □

For utility functions with $\mathrm{dom}(U) \neq \mathbb{R}$ it may happen that $u(x)$ is *not* equal to $s(x)$. This will be shown in Example 13.3.3. However, if $S$ does not allow arbitrage, then under all utility functions the maximal utility $u(x)$ can be attained by simple strategies. To this end we need the following auxiliary result.

**Proposition 8.3.3.** *Let $S$ be a continuous semimartingale satisfying (NFLVR). If $(\theta \cdot S)_T \geq -a$ a.s, then the process $\theta$ is a-admissible.*

*Proof.* For every $\varepsilon > 0$ define a stopping time by

$$\tau_\varepsilon = \inf\{t > 0 : (\theta \cdot S)_t = -a - \varepsilon\} \wedge T.$$

Suppose $\theta$ is not $a$-admissible. Then for some $\varepsilon > 0$ we must have $P(\tau_\varepsilon < T) > 0$. The strategy $\pi = 1_{]\tau_\varepsilon, T]}\theta$ satisfies

$$\begin{aligned} (\pi \cdot S)_T &= 1_{\{\tau_\varepsilon < T\}}[(\theta \cdot S)_T - (\theta \cdot S)_{\tau_\varepsilon}] \geq 0, \\ P((\pi \cdot S)_T > 0) &= P(\tau_\varepsilon < T) > 0. \end{aligned}$$

Hence $\pi$ is an arbitrage opportunity. But this is a contradiction to (NFLVR). □

**Theorem 8.3.4.** *Suppose $S$ satisfies the (NFLVR) condition and $x > 0$. If $S$ is continuous, then*

$$u(x) = s(x).$$

*Proof.* Let $\theta$ be admissible and $EU(x + (\theta \cdot S)_T) > -\infty$. Then it must satisfy

$$(\theta \cdot S)_T \geq -x, \text{ a.s.}$$

Proposition 8.3.3 implies that $\theta$ is $x$-admissible, and thus we have $u(x) = u_x(x)$ and $s(x) = s_x(x)$. Now Theorem 8.3.1 yields the result. □

We close this section with an example inspired by Example 7.5 in [12] and showing that in the previous results the requirement that $S$ is continuous cannot be dropped.

**Example 8.3.5.** *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of Gaussian unit variables and $(\phi_n)_{n\in\mathbb{N}}$ a sequence of random variables satisfying $P(\phi_n = 1) = 2^{-n}$ and $P(\phi_n = 0) = 1 - 2^{-n}$. Furthermore suppose that $Z$ is a random variable with distribution $P(Z = a) = P(Z = b) = \frac{1}{2}$, where $0 < a < 1$ and $b >$*

1. *We assume that all these random variables are independent. Choose an enumeration* $(q_n)_{n\in\mathbb{N}}$ *of the rationals in* $[0,1[$. *The process defined by*

$$S = 1_{[0,1[}(t) + Z\,1_{\{1\}}(t) + \sum_{\{n:q_n\leq t\}} \phi_n X_n, \quad 0 \leq t \leq 1,$$

*is cadlag. We start by showing that* $S$ *is a semimartingale satisfying the (NFLVR) property. For this purpose denote by* $\tilde{P}$ *the restriction of* $P$ *to* $\sigma(Z)$. *It is obvious, that there is a probability measure* $\tilde{Q} \sim \tilde{P}$ *on* $\sigma(Z)$ *such that the expectation of* $Z$ *with respect to* $\tilde{Q}$ *is equal to* $1$. *Note that the extension* $dQ = \frac{d\tilde{Q}}{d\tilde{P}}dP$ *is a probability measure such that*

**i)** $Q = \tilde{Q}$ *on* $\sigma(Z)$,

**ii)** $Q = P$ *on* $\sigma(\phi_n X_n, n \in \mathbb{N})$ *and*

**iii)** $Q \sim P$.

*Hence the process* $S$ *is a* $Q$*-martingale with respect to its natural filtration. By the fundamental theorem of asset pricing (see Corollary 1.2 in [12]) this implies that* $S$ *is a semimartingale satisfying the (NFLVR) property.*

*As in Example 7.5 in [10] one can show that* $\theta = 0$ *is the only simple integrand which is admissible for* $S$. *Hence, for* $x > \sup\{y : U(y) = -\infty\}$ *and* $a > 0$ *we have*

$$s_a(x) = U(x).$$

*However, the non-simple strategy* $\theta = 1_{\{1\}}$ *has as final payoff*

$$x + (\theta \cdot S)_1 = x + (S_1 - S_{-1}) = x + (Z - 1) \quad a.s.$$

*Now choose* $x$, $a$ *and* $b$ *such that*

$$EU(x + (\theta \cdot S)_1) = \frac{1}{2}U(x + a - 1) + \frac{1}{2}U(x + b - 1) > U(x).$$

*For example if* $U = \log$, $x = 1$, $a$ *and* $b$ *are such that* $ab = e^2$, *then*

$$EU(x + (\theta \cdot S)_1) = \frac{1}{2}\log(a) + \frac{1}{2}\log(b) = \frac{1}{2}\log(ab) = 1 > 0 = U(x).$$

*Thus we have*

$$u_a(x) \neq s_a(x), \quad u_+(x) \neq s_+(x)$$

*and*

$$u(x) \neq s(x).$$

# Chapter 9

# Finite utility and drift densities

In Chapter 7 we have seen that finite utility via simple strategies implies the price process to be a semimartingale. In this chapter we continue to investigate the implications of bounded utility. We will see that one can deduce even more structure properties of the price dynamics. With the help of the results in Chapter 8 we will derive that continuous price processes can be decomposed into $S = M + \alpha \cdot \langle M, M \rangle$ such that $\int_0^T \alpha^2 \, d\langle M, M \rangle < \infty$, almost surely.

## 9.1 Existence of drift densities

According to the previous chapters bounded utility for an agent implies the semimartingale property. Ansel and Stricker show in [4] that under the (NA) condition for simple integrands, the process of bounded variation in the Doob-Meyer decomposition of $S$ must be controlled by the martingale *(uncertainty)* part $M$ of $S$, i.e. there is an $(\mathcal{F}_t)-$predictable process $\alpha$ such that

$$S = M + \alpha \cdot \langle M, M \rangle. \tag{9.1}$$

Delbaen and Schachermayer [12] extend this result and they show, although they do not mention it explicitly, that the implication still holds if one can exclude only 0-admissible arbitrage strategies. From this result we will deduce that boundedness of maximal utility implies a decomposition of the form (9.1) to exist.

Throughout this section we suppose $S$ to be a locally bounded semimartingale. As a consequence $S$ is a special semimartingale with unique Doob-Meyer decomposition

$$S = S_0 + M + A,$$

where $A$ is a predictable process of locally bounded variation with $A_0 = 0$. Moreover, the bracket process $[M, M]$ is locally bounded, and hence the oblique bracket $\langle M, M\rangle$ exists. Note that, since $M_0 = 0$, we have $[M, M]_0 = \langle M, M\rangle_0 = 0$.

We start with a useful result, which can be deduced from Theorem 2.3 in [12].

**Lemma 9.1.1.** *The process $A$ is of the form*

$$dA = \alpha\, d\langle M, M\rangle,$$

*for some predictable process $\alpha$ if and only if for each strategy $\theta$, such that $|\theta| \in \{0, 1\}$, the relation $\theta \cdot \langle M, M\rangle = 0$ implies $\theta \cdot A = 0$.*

This result allows us to state the following lemma.

**Lemma 9.1.2.** *If $A$ is not of the form $dA = \alpha\, d\langle M, M\rangle$, then there exists a 0-admissible arbitrage strategy.*

*Proof.* If $A$ is not of the form $dA = \alpha\, d\langle M, M\rangle$, then by Lemma 9.1.1 we can find a strategy $\theta$ with values in $\{-1, 0, 1\}$ such that $\theta \cdot \langle M, M\rangle = 0$ and $(\theta \cdot A) \neq 0$. Obviously this implies the stochastic integral $(\theta \cdot M)$ to vanish. An application of Theorem 2.1, ii) in [12] yields that there are two disjoint, predictable sets $C$ and $D$ in $\Omega \times \mathbb{R}_+$ such that

$$(\theta \cdot A)_t = \int_0^t (1_C - 1_D)|d(\theta \cdot A)|.$$

Now put

$$\zeta = 1_C\theta - 1_D\theta,$$

and observe that $(\zeta \cdot S) = (\zeta \cdot A)$. Moreover, for all $t \geq 0$,

$$\begin{aligned}
(\zeta \cdot A)_t &= (1_C\theta \cdot A)_t - (1_D\theta \cdot A)_t \\
&= \int_0^t 1_C(1_C - 1_D)\,|d(\theta \cdot A)| + \int_0^t 1_D(1_C - 1_D)\,|d(\theta \cdot A)| \\
&= \int_0^t 1_C\,|d(\theta \cdot A)| + \int_0^t 1_D\,|d(\theta \cdot A)| \\
&= \int_0^t |d(\theta \cdot A)| \\
&\geq 0,
\end{aligned}$$

and hence $\zeta$ is 0-admissible. Since $(\zeta \cdot A)_T = \int_0^T |d(\theta \cdot A)| > 0$ with positive probability, the proof is complete. □

**Remark 9.1.3.** *The proof of the preceding lemma is inspired by Theorem 3.5 in [12].*

Lemma 9.1.2 allows to establish the link between boundedness of maximal expected utility and the existence of a decomposition of the form (9.1).

**Theorem 9.1.4.** *Let $U$ be a utility function with $\lim_{y\to\infty} U(y) = \infty$ and $x > \sup\{y : U(y) = -\infty\}$. If $u_0(x) < \infty$, then $S$ has a decomposition of the form (9.1). In particular this implication holds true if $u_+(x)$ is finite.*

*Proof.* Assume that there is no decomposition of the form (9.1). Then, by Lemma 9.1.2, one can find a 0-admissible arbitrage strategy $\theta$. The blown up strategies

$$\theta^n = n\,\theta$$

remain 0-admissible and satisfy

$$\lim_n EU(x + (\theta^n \cdot S)_T) = \infty.$$

This implies $u_0(x) = \infty$, and hence the result. □

## 9.2 Instantaneous infinite utility

In this section, we establish a relationship between the intensity of the intrinsic drift $\alpha \cdot \langle M, M\rangle$ of $S$ and the boundedness of expected utility. We shall prove that if this drift has an instantaneously infinite increase at some stopping time $T'$, then at this same time there is an equally infinite increase of expected utility with respect to unbounded utility functions. Due to close connections between (NFLVR) and finite utility, explained in Chapter 7, our treatment will in some parts heavily rely on similar arguments in Delbaen and Schachermayer [12].

This is the case in the following lemma in which a link between infinite intrinsic drift and the existence of admissible strategies inducing large wealths is established.

**Lemma 9.2.1.** *Suppose $P(\int_0^T \alpha^2 d\langle M, M\rangle = \infty) = \eta > 0$. Then for all $a, \xi > 0$ we can find an $a$-admissible strategy $\theta$ such that $P((\theta \cdot S)_T \geq 1) \geq \eta - \xi$.*

*Proof.* Let $a, \xi > 0$. We set $R = \frac{2(1+a)^2}{\xi\, a^2}$. By monotone convergence

$$\lim_{n\to\infty} P\left(\int_0^T 1_{\{|\alpha|\leq n\}} \alpha_s^2 \; d\langle M, M\rangle_s \geq R\right) \geq \eta.$$

Hence we may choose $n$ such that

$$R \leq \int_0^T 1_{\{|\alpha| \leq n\}} \alpha_s^2 \, d\langle M, M\rangle_s < \infty$$

on a measurable set $B$ with $P(B) \geq \eta - \frac{\xi}{2}$. Define

$$T_1 = \inf\left\{ t \geq 0 \middle| \int_0^t 1_{\{|\alpha| \leq n\}} \alpha_s^2 \, d\langle M, M\rangle_s \geq R \right\} \wedge T.$$

Then the strategy $\theta = \frac{1+a}{R} \alpha 1_{[0,T_1]} 1_{\{|\alpha| \leq n\}}$ satisfies

$$\int_0^T \theta^2 \, d\langle M, M\rangle = \frac{(1+a)^2}{R^2} \int_0^{T_1} 1_{\{|\alpha| \leq n\}} \alpha_s^2 \, d\langle M, M\rangle_s \tag{9.2}$$

$$\leq \frac{(1+a)^2}{R}, \tag{9.3}$$

$$\int_0^T |\theta_s \alpha_s| \, d\langle M, M\rangle_s \leq \frac{1+a}{R} \int_0^{T_1} 1_{\{|\alpha| \leq n\}} \alpha_s^2 \, d\langle M, M\rangle_s \tag{9.4}$$

$$\leq 1 + a, \tag{9.5}$$

$$\int_0^T \theta_s \alpha_s \, d\langle M, M\rangle_s = 1 + a \quad \text{on the set } B. \tag{9.6}$$

The first two properties show that $\theta$ is $S$-integrable. By Doob's $L^2$-inequality (see Proposition 1.7, chapter II in Revuz, Yor [42]) we obtain

$$\begin{aligned} P((\theta \cdot M)_T^* \geq a) &\leq \frac{1}{a^2} \sup_{0 \leq t \leq T} E|(\theta \cdot M)_t|^2 \\ &\leq \frac{(1+a)^2}{R a^2} = \frac{\xi}{2}. \end{aligned}$$

We will now stop $\theta$ at

$$T_2 = \inf\{t \geq 0 | (\theta \cdot M)_t \leq -a\} \wedge T_1.$$

Then, according to the third property,

$$\begin{aligned} P((\theta 1_{[0,T_2]} \cdot S)_T \geq 1) &\geq P\left( \int_0^{T_2} \theta_s \alpha_s \, d\langle M, M\rangle \geq 1 + a, (\theta \cdot M)_T^* < a \right) \\ &\geq P(B) - P((\theta \cdot M)_T^* \geq a) \\ &\geq (\eta - \frac{\xi}{2}) - \frac{\xi}{2} = \eta - \xi. \end{aligned}$$

Thus the claim is proved. □

As an immediate consequence of the preceding lemma, an infinite drift with positive probability entails free lunches.

**Corollary 9.2.2.** *If* $\int_0^T \alpha^2 d\langle M, M\rangle = \infty$ *on a set with positive probability, then* $S$ *satisfies (FLVR).*

We are mainly interested in another consequence of the lemma: Infinite drift with positive probability also implies that the expected utility becomes infinite.

**Theorem 9.2.3.** *Suppose* $U$ *is a utility function satisfying* $\lim_{y\to\infty} U(y) = \infty$. *If*

$$\int_0^T \alpha^2 d\langle M, M\rangle = \infty$$

*on a set with positive probability, then for all* $a > 0$ *and* $x > \sup\{y : U(y) = -\infty\}$ *we have*

$$u_a(x) = \infty.$$

*Proof.* Choose $a > 0$ and $x > \sup\{y : U(y) = -\infty\}$. By possibly reducing $a$ we may assume that $D = U(x - a) > -\infty$. By Lemma 9.2.1 there is an $\alpha > 0$ and a sequence $(\theta^n)_{n\in\mathbb{N}}$ of $a$-admissible strategies satisfying

$$P((\theta^n \cdot S)_T \geq n) > \alpha.$$

Since $U(x) \to \infty$, we obtain

$$\begin{aligned} \liminf_{n\to\infty} EU(x + (\theta^n \cdot S)_T) &\geq \liminf_{n\to\infty} U(x + n)\,\alpha \;+\; D\,(1-\alpha) \\ &= \infty, \end{aligned}$$

which proves the theorem. □

**Remark 9.2.4.** *Theorem 9.2.3 neither follows from the preceding corollary nor from the 'Immediate Arbitrage Theorem' of Delbaen and Schachermayer in [12]. This is because there are situations where (NA) is violated, but* $u_a(x)$ *is finite for some* $a$.

In the preceding findings about infinite utility the agent may need an arbitrarily long time to obtain unbounded utility. For completeness, we shall now generalize this to arbitrarily short time intervals after a stopping time. The following notion is related to the notion of *immediate arbitrage* (Definition 3.2 in [12]).

**Definition 9.2.5.** *Let $U$ be a utility function with $\lim_{y\to\infty} U(y) = \infty$ and $x > \sup\{y : U(y) = -\infty\}$. The semimartingale $S$ admits* instantaneous infinite utility *at the stopping time $\tilde{T}$, where we suppose $P(\tilde{T} < T) > 0$, if for all $\varepsilon > 0$*

$$\sup_{\mathcal{A}\ni\theta\, adm.} EU(x + (\theta 1_{]\tilde{T},\tilde{T}+\varepsilon]} \cdot S)_T) = \infty.$$

**Theorem 9.2.6.** *(Instantaneous infinite utility)*
*Suppose $\tilde{T}$ is a stopping time with $P(\tilde{T} < T) > 0$. If*

$$\int_{\tilde{T}}^{(\tilde{T}+\varepsilon)\wedge T} \alpha^2 \, d\langle M, M\rangle = \infty \qquad \text{for all } \varepsilon > 0,$$

*then $S$ admits immediate infinite utility at time $\tilde{T}$.*

*Proof.* Let $\varepsilon > 0$. We define $S'_t = S_{t\wedge\tilde{T}+\varepsilon} - S_{t\wedge\tilde{T}}, 0 \le t \le T$. The semimartingale $S'$ satisfies the conditions of Theorem 9.2.3. Hence it admits infinite utility. Let $(\theta^n)_{n\in\mathbb{N}}$ be a sequence with $\lim_{n\to\infty} EU(x+(\theta^n \cdot S')_T) = \infty$. On the interval $]\tilde{T} \wedge T, (\tilde{T} + \varepsilon) \wedge T]$ the process $S'$ coincides with $S$. Hence $\theta^n \cdot S' = \theta^n 1_{]\tilde{T}\wedge T,(\tilde{T}+\varepsilon)\wedge T]} \cdot S$. The claim is now obvious. □

# Conclusion

In this chapter we have seen other properties to follow from finite utility. With the results of all three chapters in Part II we now know that the maximal expected utility can only be finite if $S$ is semimartingale and has a decomposition with a bounded variation part nicely controlled by the martingale part.

The next theorem summarizes the main results obtained so far.

**Theorem 9.2.7.** *(Structure properties implied by finite utility)*
*Let $S$ be an arbitrary continuous stochastic process indexed by $[0, T]$, $U$ a utility function with $\lim_{y\to\infty} U(y) = \infty$ and $x > \sup\{y : U(y) = -\infty\}$. If $s_+(x)$ is finite, then $S$ is a semimartingale with decomposition of the form $S = S_0 + M + \alpha \cdot \langle M, M\rangle$, where $M$ is a local martingale starting in zero and $\int_0^T \alpha^2 \, d\langle M, M\rangle < \infty$, a.s. Moreover expected utility maximized over general admissible strategies is also finite and given by $s_+(x)$, i.e. we have $u_+(x) = s_+(x)$.*

*Proof.* This follows by combining Corollary 7.3.4, Theorem 9.1.4, Theorem 9.2.3 and, depending on the domain of $U$, Theorem 8.2.1 or Corollary 8.3.2. □

Recall that $\theta_0 = 0$ for all $\theta \in \mathcal{A}$. Therefore $(\theta \cdot S)_t = (\theta \cdot (S - S_0))_t$, $t \geq 0$, and we assume without loss of generality that $S$ starts in zero.

In the remaining chapters we will always suppose the price process to be of the form $S = M + \alpha \cdot \langle M, M \rangle$, where $M$ is a continuous local martingale starting in zero and $\int_0^T \alpha^2 \, d\langle M, M \rangle < \infty$, a.s. This, of course, makes our analysis easier.

# Part III

# Comparing investors

# Chapter 10

# Comparison of investors with different utility

In the framework of the general semimartingale model for financial markets we now compare investors who possess different information. Again the information or more precisely, the information flow is represented by filtrations. Thus, an investor with information $(\mathcal{F}_t)$ will only use investment strategies being predictable with respect to $(\mathcal{F}_t)$. As in Part II we denote by $u^{\mathcal{F}}(x)$ the maximal expected utility of an investor whose information is given by $(\mathcal{F}_t)$.

We first give some economic motivation for comparing the maximal expected utility of different traders. We then explain why this is possible in the framework of the general semimartingale model.

## 10.1 Why a comparison is useful

Consider the following situation on a market of assets whose prices are influenced by the climate: A trader investing on this market asks a climate expert for some prognosis of the weather. How much can the expert demand for his consultations? What is the value of the expert's knowledge - from the trader's point of view?

Suppose that in the beginning the trader has initial wealth $x$ and information $(\mathcal{F}_t)$. The expert proposes to advise him for the price $p$. Let $(\mathcal{G}_t)$ denote the filtration the trader would have if the expert gave him the extra information. Our trader will pay $p$ if $u^{\mathcal{G}}(x-p) > u^{\mathcal{F}}(x)$, and he will reject the offer if $u^{\mathcal{G}}(x-p) < u^{\mathcal{F}}(x)$. The price $q$ satisfying

$$u^{\mathcal{G}}(x-q) = u^{\mathcal{F}}(x),$$

is called *utility based price* of the extra information. It is the monetary value of the expert's knowledge from the trader's point of view.

Another typical situation with asymmetric information among traders is given if insider trading takes place. An insider is a trader who has access to secret information concerning, for example, the development of a firm or a stock corporation. Think of a member of the board. If he had the permission of buying or selling shares of his firm, how big would his advantage be compared to a normal trader? Again, the utility based price measures the additional utility in terms of money.

In order to determine the utility based price, one has to calculate the maximal expected utility under different filtrations. Essentially, this is what this part is about. We will provide methods to calculate the maximal expected utility for enlarged filtrations. To this end we will give formulas describing the additional utility $\Delta u(x) = u^{\mathcal{G}}(x) - u^{\mathcal{F}}(x)$, when the information of a trader increases from $(\mathcal{F}_t)$ to $(\mathcal{G}_t)$. Under affine transformations of the utility function $U$ the value of the additional utility changes. Therefore, it has no absolute economic meaning, only relative to the choice of the utility function. The calculation of $\Delta u(x)$ can be understood as an auxiliary step in the determination of the monetary value of the extra information. Examples for the calculation of the monetary value of extra information can be found in Amendinger et al. [1].

## 10.2 Why a comparison is possible

Let $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$ be two filtrations satisfying the usual conditions and suppose $\mathcal{F}_t \subset \mathcal{G}_t$, $t \geq 0$. Let $S$ be a continuous stochastic process being a semimartingale for both $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$, and let $\theta$ be $(\mathcal{F}_t)$-predictable such that the stochastic integral $(\theta \cdot S)$ exists with respect to $(\mathcal{F}_t)$. In contrast to what one would expect, it can happen that under the bigger filtration $(\mathcal{G}_t)$ the stochastic integral of $(\theta \cdot S)$ does *not* exist. An example can be found in Jeulin's book ([32], page 46-47).

In the context of financial markets with insiders this would mean: There are strategies which can be used by the normal trader, but not by the insider. This is of course a paradoxical situation which has to be excluded.

We give now a sufficient condition under which every $(\mathcal{F}_t)$-integrable process is also integrable with respect to $(\mathcal{G}_t)$. For this let $M$ be a continuous $(\mathcal{F}_t)$-local martingale. We write $(\theta \cdot_{\mathcal{F}} M)$ and $(\theta \cdot_{\mathcal{G}} M)$ for the stochastic integral processes computed under the filtrations $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$ respectively.

**Theorem 10.2.1.** *Suppose an information drift $\mu$ of $(\mathcal{G}_t)$ relative to $M$ exists and that $\int_0^\infty \mu_t^2 \, d\langle M, M\rangle_t < \infty$, almost surely. Let $\theta$ be an $(\mathcal{F}_t)$-predictable process for which the stochastic integral $(\theta \cdot_{\mathcal{F}} M)$ exists. Then also $(\theta \cdot_{\mathcal{G}} M)$*

*exists, and we have*

$$(\theta \cdot_{\mathcal{F}} M) = (\theta \cdot_{\mathcal{G}} M).$$

*Proof.* If $\theta$ is bounded, then the result follows immediately from Theorem 33, Chapter IV in [41].

Now suppose that $\theta$ is an arbitrary $(\mathcal{F}_t)$-predictable process such that $(\theta \cdot_{\mathcal{F}} M)$ exists. By stopping appropriately we may assume $\theta \in L^2(M)$, i.e. $E \int_0^\infty \theta_t^2 \, d\langle M, M\rangle_t < \infty$. For $n \geq 1$ let

$$\theta^n = (\theta \wedge n) \vee -n.$$

By the Burkholder-Davis-Gundy Inequality there exists a constant $C > 0$ such that

$$\begin{aligned} E\left[\sup_{t\geq 0} |((\theta - \theta^n) \cdot_{\mathcal{F}} M)_t|^2\right] &\leq C\, E\Big\langle ((\theta - \theta^n) \cdot_{\mathcal{F}} M), ((\theta - \theta^n) \cdot_{\mathcal{F}} M)\Big\rangle_\infty \\ &= C\, E \int_0^\infty (\theta - \theta^n)^2 \, d\langle M, M\rangle, \end{aligned}$$

and monotone convergence implies

$$\lim_n (\theta^n \cdot_{\mathcal{F}} M) = (\theta \cdot_{\mathcal{F}} M).$$

Similarly one obtains $\lim_n (\theta^n \cdot_{\mathcal{G}} \tilde{M}) = (\theta \cdot_{\mathcal{G}} \tilde{M})$, where $\tilde{M} = M - \mu \cdot \langle M, M\rangle$ is a $(\mathcal{G}_t)$-local martingale.

By the Kunita-Watanabe Inequality we have

$$\int_0^\infty \theta\mu \, d\langle M, M\rangle \leq \left(\int_0^\infty \theta^2 \, d\langle M, M\rangle\right)^{\frac{1}{2}} \left(\int_0^\infty \mu^2 \, d\langle M, M\rangle\right)^{\frac{1}{2}}.$$

Therefore the integral of $\theta$ with respect to the bounded variation process $\mu \cdot \langle M, M\rangle$ exits. One can even show that $\lim_n \int_0^\infty \theta^n \mu \, d\langle M, M\rangle = \int_0^\infty \theta\mu \, d\langle M, M\rangle$ which implies that $\lim_n (\theta^n \cdot_{\mathcal{G}} M) = (\theta \cdot_{\mathcal{G}} M)$. Since by the first part of the proof $(\theta^n \cdot_{\mathcal{G}} M) = (\theta^n \cdot_{\mathcal{F}} M)$ for all $n \geq 1$, we obtain the result. □

Suppose again that the continuous stock price process $S$ is a semimartingale for both filtrations $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$. Let the bounded variation part in both decompositions be of the form $\alpha \cdot \langle M, M\rangle$, where $\int_0^T \alpha^2 \, d\langle M, M\rangle < \infty$ almost surely. By the preceding theorem a trader with information $(\mathcal{G}_t)$ can use every strategy of a trader with information $(\mathcal{F}_t)$. Moreover, the wealth process resulting from an $(\mathcal{F}_t)$-strategy is the same under both filtrations.

Recall that by Theorem 9.2.7 finite expected utility implies the drift to be locally square integrable. Hence among traders with finite utility the interpretation of wealth processes under different filtrations causes no problem.

**Remark 10.2.2.** *We applied Theorem 10.2.1 already in Lemma 4.5.3 without mentioning it explicitly.*

*In the following we omit again the filtration in the definition of the stochastic integrals.*

# Chapter 11

# Monotone utility convergence

In this chapter we study sequences of increasing strategy sets and observe the related maximal expected utility. We show, indeed, that the maximal utility satisfies a monotone continuity property if the price process is continuous. In the last section we sketch some possible applications of this property.

Let $(\mathcal{G}_t^n)$ be a sequence of increasing filtrations satisfying the usual conditions. Moreover let $S$ be a continuous process such that $S$ is a semimartingale relative to $(\mathcal{G}_t^n)$, for all $n \geq 0$. The smallest filtration satisfying the usual conditions and containing every filtration $(\mathcal{G}_t^n)$ is given by

$$\mathcal{G}_t = \bigcap_{s>t} \sigma\left(\mathcal{G}_s^n : n \geq 1\right).$$

Throughout this chapter we suppose that $S$ is a continuous $(\mathcal{G}_t)$-semimartingale with decomposition

$$S_t = M_t + \int_0^t \alpha_s \, d\langle M, M\rangle_s,$$

where $M$ is a $(\mathcal{G}_t)$-local martingale starting in zero and $\alpha$ a $(\mathcal{G}_t)$-predictable process satisfying $\int_0^T \alpha_t^2 \, d\langle M, M\rangle_t < \infty$, a.s. For the existence of such a decomposition we refer to Theorem 9.2.7.

Since the sequence $(\mathcal{G}_t^n)$ is increasing, the sequence of the related maximal expected utility is also increasing. And, as will be shown below, it satisfies a monotone convergence property. In order to make our analysis easier, we distinguish between the domains of our utility functions. We will at first consider the case $\mathrm{dom}(U) = \mathbb{R}$, and then $\mathrm{dom}(U) \neq \mathbb{R}$. Recall that $u^{\mathcal{F}}(x)$ denotes the maximal expected utility of a trader who possesses initial wealth $x$ and who is allowed to use only $(\mathcal{F}_t)$-predictable admissible strategies. Similarly, we write $u_a^{\mathcal{F}}(x)$ and $u_+^{\mathcal{F}}(x)$, in order to stress the filtration we are referring to.

## 11.1 Convergence in the case $\mathrm{dom}(U) = \mathbb{R}$

Throughout this section we assume $\mathrm{dom}(U) = \mathbb{R}$.

We start with the observation that the utility maximum can be attained by using strategies in $L^2(M)$. We denote by $L^2_{\mathcal{F}}(M)$ the set of all $(\mathcal{F}_t)$-predictable processes $\theta \in L^2(M)$.

**Lemma 11.1.1.** *Let $x \in \mathbb{R}$ and $a \in (0, \infty)$. Then*

$$u^{\mathcal{F}}_a(x) = \sup\{EU(x+(\theta \cdot S)_T) : \theta \in L^2_{\mathcal{F}}(M) \cap \mathcal{A},\ (a-\varepsilon)-\textit{adm. for some } \varepsilon > 0\}.$$

*Proof.* Obviously the RHS is not bigger than the LHS. For the reverse inequality choose $\varepsilon > 0$ and an $(a-\varepsilon)$-admissible strategy $\theta$ satisfying $EU(x+(\theta \cdot S)_T) > -\infty$. By Lemma 8.1.2 it is sufficient to show that $EU(x+(\theta \cdot S)_T)$ is not greater than the RHS. Since $\theta$ is $S$-integrable, the stopping times

$$T_n = T \wedge \inf\{t \geq 0 : \int_0^t \theta_r^2 \, d\langle M, M\rangle_r \leq n\}$$

converge almost surely to $T$ for $n \to \infty$. Note that the strategies

$$\theta^n = 1_{[0,T_n]}\theta$$

are $(a - \varepsilon)$-admissible and belong to $L^2_{\mathcal{F}}(M)$. Fatou's Lemma implies

$$\liminf_n EU(x + (\theta^n \cdot S)_T) \geq EU(x + (\theta \cdot S)_T),$$

and thus the result. □

We start by showing that the sequence $\left(u^{\mathcal{G}^n}_a(x)\right)$ satisfies a monotone convergence property.

**Theorem 11.1.2.** *Let $x \in \mathbb{R}$ and $a \in (0, \infty)$. Then*

$$\lim_n u^{\mathcal{G}^n}_a(x) = u^{\mathcal{G}}_a(x).$$

*Proof.* Let $\theta \in L^2_{\mathcal{G}}(M)$ be $(a - \varepsilon)$-admissible. The stopping times

$$\tau_k = T \wedge \inf\{t \geq 0 : \int_0^t \alpha_s^2 \, d\langle M, M\rangle_s \geq k\}$$

converge to $T$, a.s, and hence

$$\liminf_k EU(x + (\theta \cdot S)_{\tau_k}) \geq EU(x + (\theta \cdot S)_T).$$

By Lemma 11.1.1 it suffices to show that for all $k \geq 1$, $EU(x + (\theta \cdot S)_{\tau_k})$ is not greater than $\sup_n u_a^{\mathcal{G}^n}(x)$. To simplify notation we assume that $\tau_k = T$ for some $k$.

Now let $\theta^n$ be the projection of $\theta$ onto $L^2_{\mathcal{G}^n}$. Note that by Doob's inequality there is a constant $C > 0$, such that

$$\begin{aligned} E((\theta^n - \theta) \cdot S)^*_T &\leq E((\theta^n - \theta) \cdot M)^*_T + E((\theta^n - \theta)\alpha \cdot \langle M, M\rangle)^*_T \\ &\leq \|((\theta^n - \theta) \cdot M)^*_T\|_2 + E((\theta^n - \theta)\alpha \cdot \langle M, M\rangle)^*_T \\ &\leq C\,\|((\theta^n - \theta) \cdot M)_T\|_2 + E(|\theta^n - \theta||\alpha| \cdot \langle M, M\rangle)_T. \end{aligned}$$

The first summand in the preceding line goes to 0, because $(\theta^n)$ converges to $\theta$ in $L^2_{\mathcal{G}}(M)$. The second vanishes due to Kunita-Watanabe and due to our assumption that $\int_0^T \alpha_s^2 \, d\langle M, M\rangle_s$ is bounded. Consequently, by choosing a subsequence if necessary, almost everywhere the sequence $(\theta^n \cdot S)$ converges uniformly to $(\theta \cdot S)$ on $[0, T]$. Now put

$$T_n = T \wedge \inf\{t \geq 0 : (\theta^n \cdot S)_t \leq -a + \frac{\varepsilon}{2}\}$$

and

$$\pi^n = 1_{[0,T_n]}\theta^n.$$

The strategies $\pi^n$ are $(a - \frac{\varepsilon}{2})$-admissible and satisfy almost surely

$$\lim_n (\pi^n \cdot S)_T = (\theta \cdot S)_T.$$

With Fatou's Lemma we obtain

$$\liminf_n EU(x + (\pi^n \cdot S)_T) \geq EU(x + (\theta \cdot S)_T),$$

and hence the result. □

We obtain immediately the following.

**Corollary 11.1.3.** *For all $x \in \mathbb{R}$ we have*

$$\lim_n u^{\mathcal{G}^n}(x) = u^{\mathcal{G}}(x).$$

## 11.2 Convergence in the case $\mathrm{dom}(U) \neq \mathbb{R}$

Throughout this section we assume $\mathrm{dom}(U) \neq \mathbb{R}$. To simplify notation we suppose that $\sup\{y : U(y) = -\infty\} = 0$. The analogue to Lemma 11.1.1 is the following.

**Lemma 11.2.1.** *For $x > 0$ and $a \in (0, x]$ we have*

$u_a^{\mathcal{F}}(x) = \sup\{EU(x+(\theta\cdot S)_T) : \theta \in L^2_{\mathcal{F}}(M)$ *and* $(a-\varepsilon)-$*adm. for some* $\varepsilon > 0\}$.

From this we can deduce the analogue to Theorem 11.1.2:

**Theorem 11.2.2.** *For $x > 0$ and $a \in (0, x]$ we have*

$$\lim_n u_a^{\mathcal{G}^n}(x) = u_a^{\mathcal{G}}(x).$$

By applying Lemma 8.1.3, we obtain immediately the following.

**Corollary 11.2.3.** *Let $x > 0$. Then*

$$\lim_n u_+^{\mathcal{G}^n}(x) = u_+^{\mathcal{G}}(x).$$

**Example 11.2.4.** *For the logarthmic utility function we have an alternative proof of the previous result. Let $U = \log$ and $x > 0$. We will see that in this case the maximal expected utility is uniquely determined by the drift in the semimartingale decomposition $S = M + \alpha \cdot \langle M, M\rangle$. More precisely, we have $u_+^{\mathcal{G}}(x) = \log x + \frac{1}{2} E \int_0^T \alpha^2 d\langle M, M\rangle$ (see Theorem 13.2.4).*

*Now let $S = M^n + \alpha^n \cdot \langle M, M\rangle$ be the semimartingale decomposition relative to $(\mathcal{G}_t^n)$. It follows from Theorem 4.5.4 that $(\alpha^n)$ converges to $\alpha$ in $L^2(M)$, and hence $\lim_n u_+^{\mathcal{G}^n}(x) = u_+^{\mathcal{G}}(x)$.*

If we make the additional assumption of (NFLVR), then even the result of Corollary 11.1.3 is valid for utility functions with $\mathrm{dom}(U) \neq \mathbb{R}$.

**Corollary 11.2.5.** *If $S$ satisfies (NFLVR) relative to $(\mathcal{G}_t)$, then for all $x > 0$ we have*

$$\lim_n u^{\mathcal{G}^n}(x) = u^{\mathcal{G}}(x).$$

*Proof.* Let $\theta$ be a $(\mathcal{G}_t)$-predictable strategy such that $EU(x + (\theta \cdot S)_T) > -\infty$. Then $(\theta \cdot S)_T \geq -x$, a.s. Lemma 8.3.3 implies that $\theta$ is $x$-admissible, and thus we have $u(x) = u_x(x)$. Similarly, we obtain $u^{\mathcal{G}^n}(x) = u_x^{\mathcal{G}^n}(x)$, $n \geq 1$. The claim now follows from Theorem 11.2.2. □

We close this section with an example showing that the previous results are not valid without the requirement that $S$ is continuous.

**Example 11.2.6.** *Let $T > 1$ and $\phi$ a standard normal random variable. Suppose the price process is given by*

$$S_t = \begin{cases} 1, & \text{if } 0 \leq t < 1, \\ 1 + \phi + \frac{1}{2}, & \text{if } 1 \leq t \leq T, \end{cases}$$

*and let* $(\mathcal{F}_t)$ *be the completed filtration generated by* $S$. *Moreover let* $(\varepsilon_n)$ *be a sequence of independent normal random variables with mean zero and* $Var(\varepsilon_n) = \frac{1}{n}$. *We define*

$$\mathcal{G}_t^n = \mathcal{F}_t \vee \sigma(1_{\{|\phi|\geq 1\}} + \varepsilon_n), \quad 0 \leq t \leq T,$$

*and claim that*

$$u_a^{\mathcal{G}^n}(x) = U(x)$$

*for all utility functions* $U$, $a > 0$ *and* $x > \sup\{y : U(y) = -\infty\}$. *For this let* $\theta$ *be* $(\mathcal{G}_t^n)$*-predictable and* $S$*-integrable. If* $\theta_1 \neq 0$ *a.s., then the integral* $(\theta \cdot S)_1$ *is unbounded from below and hence* $\theta$ *is not admissible. Since the process* $S$ *is constant on the remaining part of the trading interval, we have* $u_a^{\mathcal{G}^n}(x) = U(x)$.

*A trader having access to*

$$\mathcal{G}_t = \bigvee_{n\geq 1} \mathcal{G}_t^n$$

*knows whether the absolute value of* $\phi$ *is bigger or smaller than* 1. *Therefore he has access to non-trivial admissible trading strategies. As a consequence* $u_a^{\mathcal{G}}(x) > U(x)$, *and hence*

$$\lim_n u_a^{\mathcal{G}^n}(x) \neq u_a^{\mathcal{G}}(x).$$

*It is staightforward to show that we also have* $\lim_n u^{\mathcal{G}^n}(x) \neq u^{\mathcal{G}}(x)$. *Note that the price process* $S$ *satisfies the (NFLVR) property with respect to* $(\mathcal{G}_t)$. *Thus, also in Corollary 11.2.5 the assumption that* $S$ *is continuous cannot be dropped.*

## 11.3 Convex analysis of utility limits

The maximal expected utility $u_+^{\mathcal{G}}(x)$, interpreted as a function depending on the initial wealth $x$, is concave. Sometimes, it is easier to determine the conjugate of the concave function $u_+^{\mathcal{G}}(x)$, than the function itself. For example, if no arbitrage is possible, then the conjugate can be represented in terms of the ELMMs.

In this section we consider the conjugate functions of the approximations $u_+^{\mathcal{G}^n}(x)$, and we prove that the conjugate of the limit is equal to the limit of the conjugates. For simplicity we consider only the case $\mathrm{dom}(U) \neq \mathbb{R}$ and assume that $\sup\{y : U(y) = -\infty\} = 0$. Moreover, we set $u_+^{\mathcal{G}}(x) = -\infty$ for $x \leq 0$.

It is straightforward to show that $u_+^{\mathcal{G}}$ is concave. The conjugate function is given by

$$v^{\mathcal{G}}(y) = \sup_{x>0}[u_{\mathcal{G}}(x) - xy], \quad y \in \mathbb{R}.$$

Note that $v^{\mathcal{G}}(y)$ is convex and

$$u_{\mathcal{G}}(x) = \inf_{y>0}[v_{\mathcal{G}}(y) + xy], \quad x > 0. \tag{11.1}$$

**Lemma 11.3.1.** *The conjugate functions $v^{\mathcal{G}^n}$ of $u_+^{\mathcal{G}^n}$ converge pointwise to $v^{\mathcal{G}}$, i.e.*

$$\lim_n v^{\mathcal{G}^n}(y) = v_{\mathcal{G}}(y), \quad y \in \mathbb{R}.$$

*Proof.* Note that $v^{\mathcal{G}^n}(y)$ is increasing in $n$. Then Theorem 11.1.2 implies

$$\begin{aligned} v^{\mathcal{G}}(y) &= \sup_{x>0}[u_+^{\mathcal{G}}(x) - xy] \\ &= \sup_{x>0}[\sup_n u_+^{\mathcal{G}^n}(x) - xy] \\ &= \sup_n \sup_{x>0}[u_+^{\mathcal{G}^n}(x) - xy] \\ &= \sup_n v^{\mathcal{G}^n}(y), \end{aligned}$$

and thus the result. □

# 11.4 Robust information

The monotone convergence property of the maximal expected utility has some useful applications. For example it can be used in order to approximate continuous filtration enlargements by initial enlargements. Moreover, it may also be applied to situations where (additional) information is disturbed by some noise: monotone convergence shows that if the noise is small enough, then it has only a small impact on the maximal expected utility. In other words, the maximal expected utility is robust with respect to small changes in the information structure.

## Initial versus continuous enlargements

Let $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$ be two filtrations satisfying the usual conditions and suppose $\mathcal{F}_t \subset \mathcal{G}_t$, $t \geq 0$. Let $S$ be a continuous price process being a semimartingale for both $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$. Moreover we suppose that the bounded variation parts in the related decompositions have square integrable densities. The maximal expected utility $u_+^{\mathcal{G}}(x)$ can be approximated by using piecewise initial enlargements of $(\mathcal{F}_t)$.

Let $\Delta : 0 = s_0 \le \cdots \le s_n = T$, $n \in \mathbb{N}$, be a partition of the interval $[0,T]$, and let for $r \in [s_i, s_{i+1})$, $i = 0, \ldots, n-1$,

$$\mathcal{G}_r^{\Delta} = \bigcap_{u>r} \mathcal{G}_{s_i} \vee \mathcal{F}_u.$$

Monotone convergence implies immediately the following result.

**Theorem 11.4.1.** *Let* $\mathrm{dom}(U) \neq \mathbb{R}$ *and* $x > \sup\{y : U(y) = -\infty\}$. *Let* $(\Delta_n)$ *be a sequence of partitions such that* $\Delta_{n+1}$ *is a refinement of* $\Delta_n$ *and* $\lim_n |\Delta_n| = 0$. *Then*

$$\lim_n u_+^{\mathcal{G}^{\Delta_n}}(x) = u_+^{\mathcal{G}}(x).$$

We will use this result implicitly in Chapter 14.3: We will see that the additional logarithmic utility under initially enlarged filtrations can be expressed as mutual information. Monotone convergence allows us then to find a similar representation for non-initial enlargements.

## Diminishing noise

Let $(\mathcal{F}_t)$ be a filtration satisfying the usual conditions and $S$ a continuous price process being a $(\mathcal{F}_t)$-semimartingale. Suppose a trader, possibly an insider, knows the value of some random variable $G$. His information flow is then given by $\mathcal{G}_t = \bigcap_{s>t} (\mathcal{F}_s \vee \sigma(G))$. Let again $S$ be a $(\mathcal{G}_t)$-semimartingale such that the bounded variation part has a square integrable density.

What happens if the trader is not completely sure whether the information $G$ is right? Indeed, he could have misunderstood the information. Or somebody gave him deliberately a wrong information. Or a mistake occured by transmitting the information.

Fortunately we need not change our analysis if the noise is small enough: the monotone convergence property implies that the maximal expected utility is approximately the same. We now give some details.

Let $(Y_t)_{t\ge 0}$ be a Brownian motion or a standard Poisson process which is independent of $(\mathcal{F}_t)$ and $G$. The disturbed information can be represented by

$$\mathcal{G}_t^{\varepsilon} = \bigcap_{s>t} \left(\mathcal{F}_s \vee \sigma(G + Y_u : u \ge \varepsilon)\right),$$

where $\varepsilon > 0$. Note that $(\mathcal{G}_t^{\frac{1}{n}})_n$ is an increasing sequence of filtrations. It is straightforward to show that monotone convergence implies the following.

**Theorem 11.4.2.** *Let* $\operatorname{dom}(U) \neq \mathbb{R}$ *and* $x > \sup\{y : U(y) = -\infty\}$. *Then*

$$\lim_n u_+^{\mathcal{G}^{\frac{1}{n}}}(x) = u_+^{\mathcal{G}}(x).$$

# Chapter 12

# $f$-divergences and utility under initially enlarged filtrations

## 12.1 Starting from complete markets

Let $(\mathcal{F}_t)$ be a filtration satisfying the usual conditions and $S$ a continuous price process starting in zero and being a $(\mathcal{F}_t)$-semimartingale with decomposition $S = M + \alpha \cdot \langle M, M \rangle$. In contrast to the previous chapters we assume here the market to be complete. This means that there exists a unique equivalent local martingale measure $R$ and that the maximal expected utility can be explicitly calculated by means of the density $\frac{dR}{dP}$.

We restrict the class of utility functions in order to simplify our analysis: let $U$ be strictly increasing, strictly concave and continuously differentiable on $(0, \infty)$. Furthermore we assume that $U$ satisfies the Inada conditions

$$\lim_{x \to 0^+} U'(x) = \infty \qquad \text{and} \qquad \lim_{x \to \infty} U'(x) = 0, \tag{12.1}$$

and that

$$u_+^{\mathcal{F}}(x) < \infty \qquad \text{for some } x > 0. \tag{12.2}$$

On $(0, \infty)$ the derivative of $U$ has an inverse function, which we will denote by $I$. Observe that $I$ is a function with domain $(0, \infty)$ and with range $(0, \infty)$.

The maximal expected utility can be determined by studying the dual problem. For this let

$$V(y) = \sup_{x > 0} [U(x) - xy]$$

be the convex conjugate function of $-U(-x)$. It can be shown that $V$ is a continuously differentiable, decreasing and strictly convex function satisfying $\lim_{y \downarrow 0} V'(y) = -\infty$ and $\lim_{y \to \infty} V'(y) = 0$.

Note that also the function $u_+^{\mathcal{F}}(x)$ is concave on $(0,\infty)$. We can therefore again define the conjugate

$$v^{\mathcal{F}}(y) = \sup_{x>0} \left[ u_+^{\mathcal{F}}(x) - xy \right], \quad y \in \mathbb{R}.$$

One has the following results, a proof of which can be found in Kramkov, Schachermayer [35].

**Theorem 12.1.1.** *(Theorem 2.0 in [35]) Assume that the Conditions (12.1) and (12.2) are satisfied.*

*(i)* $u_+^{\mathcal{F}}(x) < \infty$*, for all* $x > 0$*, and* $v^{\mathcal{F}}(y) < \infty$ *for* $y$ *sufficiently large. Letting* $y_0 = \inf\{y | v(y) < \infty\}$*, the function* $v^{\mathcal{F}}(y)$ *is continuously differentiable and strictly convex on* $(y_0, \infty)$*. Defining* $x_0 = \lim_{y \downarrow y_0}(-(v^{\mathcal{F}})'(y))$ *the function* $u_+^{\mathcal{F}}$ *is continuously differentiable on* $(0,\infty)$ *and strictly concave on* $(0, x_0)$*. The conjugate function* $v^{\mathcal{F}}$ *of* $u_+^{\mathcal{F}}$ *satisfies*

$$v^{\mathcal{F}}(y) = EV\left(y\frac{dR}{dP}\right).$$

*(ii) For all* $0 < x < x_0$ *we have*

$$u_+^{\mathcal{F}}(x) = EU(I(y\frac{dR}{dP})),$$

*where* $y = (u_+^{\mathcal{F}})'(x)$*.* $I(y\frac{dR}{dP})$ *is replicable, i.e. there is an admissible strategy* $\theta$ *such that* $x + (\theta \cdot S)_T = I(y\frac{dR}{dP})$*, and the process* $x + (\theta \cdot S)$ *is a uniformly integrable martingale under* $Q$*.*

We now consider the price process $S$ under the enlarged filtration

$$\mathcal{G}_t = \bigcap_{s>t} \mathcal{F}_s \vee \sigma(G),\ t \geq 0,$$

where $G$ is an arbitrary random variable with values in a Polish space $\Gamma$. Suppose that $\sigma(G)$ and $\mathcal{F}_T$ can be decoupled by a measure $Q$ such that both $\sigma$-fields are independent. Then, conditioned on the value of $G$, the process $S$ is also complete relative to $(\mathcal{G}_t)$. Therefore Theorem 12.1.1 applies again, and one may deduce a representation of the maximal expected utility $u_+^{\mathcal{G}}(x)$ with the help of the density $\frac{dQ}{dP}$ (see Amendinger, Becherer and Schweizer [1]).

What can we do if a decoupling of $\sigma(G)$ and $\mathcal{F}_T$ is impossible? In other words: how can we determine $u_+^{\mathcal{G}}(x)$ and its conjugate $v^{\mathcal{G}}(y)$ if no ELMM $Q$ relative to $(\mathcal{G}_t)$ exists?

Again the answer can be given by switching to the product space $\bar{\Omega} = \Omega \times \Omega$ and by decoupling both $\sigma$-fields on $\bar{\Omega}$ with a product measure. As we will see, this allows a representation of the dual function with the help of measures on the product space. However, we will not compute the conjugate of the function $u_+^{\mathcal{G}}(x)$ itself. Instead we will compute a 'stochastic conjugate function' of the maximal expected utility conditioned on the enlarging variable $G$. We give an exact definition of stochastic conjugates, although it is straightforward.

**Definition 12.1.2.** *Let $\xi : \Omega \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a stochastic process such that almost every path is convex. Then a process $\xi^*$ is called* stochastic conjugate *of $\xi$ if $\xi^*(\omega)$ is the conjugate of $\xi(\omega)$, for almost all $\omega$.*

As in Chapter 2 we consider the embedding $\psi : \Omega \to \bar{\Omega}, \omega \mapsto (\omega, \omega)$. Moreover let $\bar{P}$ be the image of $P$ under $\psi$, and

$$\bar{Q} = R \otimes P.$$

Note that $\bar{S}(\omega, \omega') = S(\omega)$ is a $\bar{Q}$-local martingale with respect to the filtration $\bar{\mathcal{F}}_t = \bigcap_{s>t} \mathcal{F}_s \otimes \sigma(G)$. We will prove that $\bar{P} \ll \bar{Q}$ on $\mathcal{A} = \mathcal{F}_T \otimes \sigma(G)$ implies that the stochastic conjugate can be written as an $f$-divergence, where we integrate only the first variable and add a random multiplier $W$:

$$\int f\left(W(\omega') \, \frac{d\bar{P}}{d\bar{Q}}(\omega, \omega')\right) \, dR(\omega).$$

We give the definition of $f$-divergences and some basic results in the next section. We then proceed by adding some noise to the enlarging random variable $G$. We can thus approximate $(\mathcal{G}_t)$ with filtrations under which $S$ is conditionally complete and determine the stochastic conjugate function through convergence.

**Remark 12.1.3.** *The measure $\bar{Q}$ is not the only martingale measure of the embedded process $\bar{S}$ with respect to the filtration $(\bar{\mathcal{F}}_t)$: if $P' \sim P$ on $\sigma(G)$, then $R \otimes P'$ is a martingale measure equivalent to $\bar{Q}$. However, among all these measures $R \otimes P'$, the measure $\bar{Q}$ minimises the entropy relative to $\bar{P}$.*

*In order to sketch the proof, let $\mathcal{P}$ denote the set of probability measures $P'$ equivalent to $P$ and defined on $\sigma(G)$. Then $R \otimes P' \sim \bar{Q}$ and, since*

*$\bar{P} \ll R \otimes P$, we also have $\bar{P} \ll R \otimes P'$ for all $P' \in \mathcal{P}$. Note that*

$$\begin{aligned}
\inf_{P' \in \mathcal{P}} \mathcal{H}(\bar{P} \| R \otimes P') &= \inf_{P' \in \mathcal{P}} \int \log \left( \frac{d\bar{P}}{dR \otimes P} \right) + \log \left( \frac{dR \otimes P}{dR \otimes P'} \right) \, d\bar{P}, \\
&= \mathcal{H}(\bar{P} \| R \otimes P) + \inf_{P' \in \mathcal{P}} \int \log \left( \frac{dP}{dP'} \right) \, dP.
\end{aligned}$$

*It is straightforward to show that the right hand side attains its infimum if $P' = P$, and hence $\bar{Q}$ is the* entropy minimising martingale measure *for $\bar{S}$ with respect to the enlarged filtration $(\bar{\mathcal{F}}_t)$.*

## 12.2 $f$-divergences

Let $P$ and $Q$ be probability measures on the measurable space $(\Omega, \mathcal{F})$. Throughout this section let $f : (0, \infty) \to \mathbb{R}$ be a convex function and $f(0) = \lim_{x \downarrow 0} f(x)$.

**Definition 12.2.1.** *Let $\mathcal{A}$ be a sub-$\sigma$-algebra of $\mathcal{F}$. The $f$-*divergence *of $P$ relative to $Q$ on $\mathcal{A}$ is defined as*

$$f_{\mathcal{A}}(P \| Q) = \begin{cases} \int f \left( \left. \frac{dP}{dQ} \right|_{\mathcal{A}} \right) dQ, & \text{if } P \ll Q \text{ on } \mathcal{A} \text{ and the integral exists,} \\ \infty, & \text{else.} \end{cases}$$

**Example 12.2.2.** *Let $f(x) = x \log x$. Then $f(P \| Q)$ coincides with the entropy of $P$ relative to $Q$, i.e.*

$$f(P \| Q) = \mathcal{H}(P \| Q).$$

We will make use of the following approximation result.

**Lemma 12.2.3.** *(see Theorem 1.30 in [37])*
*Let $(\mathcal{A}_n)$ be a sequence of increasing sub-$\sigma$-fields and $\mathcal{A} = \bigvee_n \mathcal{A}_n$. Then $(f_{\mathcal{A}_n}(P \| Q))$ is an increasing sequence and*

$$\lim_n f_{\mathcal{A}_n}(P \| Q) = f_{\mathcal{A}}(P \| Q).$$

By interchanging the measures $P$ and $Q$ in $f(P \| Q)$ one obtains again a divergence, namely with respect to the so-called reverse function of $f$.

**Definition 12.2.4.** *The* reverse function *of the convex function $f$ is defined by*

$$\hat{f}(x) = x f \left( \frac{1}{x} \right), \quad x \in (0, \infty).$$

*Again we set $\hat{f}(0) = \lim_{x \downarrow \infty} \hat{f}(x)$.*

**Lemma 12.2.5.** *If $f$ is strictly convex and differentiable on $(0, \infty)$, then also the reverse function $\hat{f}$ is strictly convex and differentiable on $(0, \infty)$. Moreover, if $P \sim Q$, then*

$$f_{\mathcal{A}}(P\|Q) = \hat{f}_{\mathcal{A}}(Q\|P).$$

*Proof.* For a proof of these properties see Lemma 1 in [23], or Theorem 1.13 in [37]. □

Note the conjugate function $v^{\mathcal{F}}(y)$ in Thereom 12.1.1 is given as divergence with respect to the convex function $x \mapsto V(yx)$. Therefore, we will use the notation

$$f_y(x) = f(yx), \quad x \geq 0.$$

Obviously, the function $f_y(x)$ is also convex in $y$, i.e. for all $\lambda \in (0, 1)$, and $y, z > 0$ we have

$$f_{\lambda y + (1-\lambda) z}(x) = \lambda f_y + (1 - \lambda) f_z.$$

We will often have to deal with random multipliers: if $Y \geq 0$ is random, then let

$$f_Y(P\|Q) = \int f\left(Y\, \frac{dP}{dQ}\right) dQ, \quad x \geq 0.$$

## 12.3 Solving the problem for discrete $G$

Let us come back to our aim at determining the function $u_+^{\mathcal{G}}(x)$. We will assume that

$$u_+^{\mathcal{G}}(x) < \infty \qquad \text{for some } x > 0.$$

One can easily show that this implies $u_+^{\mathcal{G}}(x) < \infty$ for all $x \in \mathbb{R}$. In terms of convex analysis this means that $-u_+^{\mathcal{G}}(x)$ is proper (see [43]).

Monotone utility convergence implies that we can approximate $u_+^{\mathcal{G}}(x)$ by using filtrations enlarged by discrete random variables. Therefore let us assume throughout this section that $G$ is discrete. In other words, we assume that the state space $\Gamma$ is countable. Moreover, we denote by $\mu$ the distribution of $G$, and, for simplicity, we assume $\mu(g) > 0$ for all $g \in \Gamma$.

We start by solving the dual problem for each $g \in \Gamma$. To this end let $u_+(x, g)$ be the maximal expected utility relative to $(\mathcal{G}_t)$ and the measure $P^g = P(\cdot|G = g)$; more precisely

$$\begin{aligned} u_+(x, g) \;=\; & \sup\{E[(U(x + (\theta \cdot S)_T)|G = g] : \theta \ (\mathcal{G}_t)\text{-predictable}, \\ & x\text{-adm. and } S\text{-integrable relative to } P^g\}. \end{aligned}$$

**Lemma 12.3.1.** *Let $g \in \Gamma$. $u_+(x,g)$ is equal to the maximal expected utility where the supremum is taken only over all $(\mathcal{F}_t)$-predictable processes which are $x$-admissible and integrable with respect to $P$. Moreover*

$$u_+^{\mathcal{G}}(x) = \sum_{g \in \Gamma} \mu(g)\ u_+(x,g).$$

*Proof.* We define at first

$$\begin{aligned}\tilde{u}_+(x,g) &= \sup\{E[(U(x + (\theta \cdot S)_T)|G = g] : \theta\ (\mathcal{F}_t)\text{-predictable,} \\ &\quad x\text{-adm. and } S\text{-integrable relative to } P\},\end{aligned}$$

Obviously $\tilde{u}_+(x,g) \le u_+(x,g)$. Thus both statements follow, if we show (i) $u_+^{\mathcal{G}}(x) \le \sum_g \mu(g)\tilde{u}_+(x,g)$ and (ii) $\sum_g \mu(g) u_+(x,g) \le u_+^{\mathcal{G}}(x)$.

Denote by $\mathcal{P}(\mathcal{F})$ and $\mathcal{P}(\mathcal{G})$ the predictable $\sigma$-fields with respect to $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$ respectively. We show at first

$$\mathcal{P}(\mathcal{G}) = \left\{ \bigcup_g (A^g \cap (\{G = g\} \times \mathbb{R}_+)) : A^g \in \mathcal{P}(\mathcal{F}) \right\}. \tag{12.3}$$

Note that the RHS is a $\sigma$-algebra which is contained in $\mathcal{P}(\mathcal{G})$. Moreover, each set of the form $(A \cap \{G = g\}) \times ]s,t]$, $A \in \mathcal{F}_s$, belongs to the RHS. Therefore, $\mathcal{P}(\mathcal{G})$ is a subset of the RHS, and hence equation (12.3) holds.

A monotone class argument implies that every $(\mathcal{G}_t)$-predictable process may be written as a sum of the form $\sum_g 1_{\{G=g\}}\zeta^g$, where all $\zeta^g$ are $(\mathcal{F}_t)$-predictable.

Now let $\theta = \sum_g 1_{\{G=g\}}\zeta^g$, with $\zeta^g$ predictable relative to $(\mathcal{F}_t)$. Moreover, assume that $\theta$ is an $x$-admissible and bounded strategy. It is straightforward to show, via stopping for example, that $\zeta^g$ may be chosen to be bounded, $x$-admissible and hence $S$-integrable relative to $P$.

As a consequence, $EU(x + (\theta \cdot S)_T) = \sum_g \mu(g) E[U(x + (\zeta^g \cdot S)_T)|G = g] \le \sum_g \mu(g)\tilde{u}_+(x,g)$. Thus (i) holds.

Now let $\theta^g$ be $(\mathcal{G}_t)$-predictable, $x$-admissible relative to $P^g$, and such that $X_T^g = x + (\theta^g \cdot S)_T$ satisfies

$$E[U(X_T^g)|G = g] \ge u_+(x,g) - \varepsilon.$$

Observe that $\theta^G(\omega) = \theta^{G(\omega)}(\omega)$ is $(\mathcal{G}_t)$-predictable, $x$-admissible and integrable with respect to $P$. Moreover

$$\begin{aligned} EU(X_T^G) &= \sum_g \mu(g)\, E[U(X_T^g)|G = g] \\ &\ge -\varepsilon + \sum_g \mu(g)\, u_+(x,g). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this implies (ii), and the proof is complete. □

In order to determine the conjugate of $u_+(x,g)$ we disturb $G$ in the following way: Let $(N_t)_{t\geq 0}$ be an independent counting process induced by a standard Poisson process and let $(H_k)$ be a sequence of independent random variables with distribution $\mu$. Moreover let $H_0 = G$ and $G^n = H_{N_{\frac{1}{n}}}$. Note that $G^n$ also has the distribution $\mu$.

We approximate $(\mathcal{G}_t)$ with the filtrations

$$\mathcal{G}_t^n = \bigcap_{s>t} (\mathcal{F}_s \vee \sigma(G^n)).$$

As before we denote by $u_+^n(x,g)$ the maximal expected utility relative to $(\mathcal{G}_t^n)$ and the measure $P(\cdot|G^n = g)$. A reasoning as in Lemma 12.3.1 shows that

$$\begin{aligned} u_+^n(x,g) &= \sup\{E[(U(x+(\theta\cdot S)_T)|G^n = g] : \theta \ (\mathcal{F}_t)\text{-predictable}, \\ &\qquad x\text{-adm. and } S\text{-integrable relative to } P\} \end{aligned}$$

Moreover we have:

**Lemma 12.3.2.** *Let $x > 0$ and $g \in \Gamma$. Then*

$$\lim_n u_+^n(x,g) = u_+(x,g).$$

*Proof.* Let $\theta$ be $(\mathcal{F}_t)$-predictable and $x$-admissible relative to $P$. Put $X_T = x + (\theta \cdot S)_T$ and assume that $U(X_T)$ is integrable relative to $P$. Then $\lim_n E[U(X_T)|G^n = g] = E[U(X_T)|G = g]$, and therefore $\liminf_n u_+^n(x,g) \geq u_+(x,g)$, for $\mu$-almost all $g$.

On the other hand

$$E[U(X_T)|G^n = g] = P(N_{\frac{1}{n}} \geq 1)EU(X_T) + P(N_{\frac{1}{n}} = 0)E[U(X_T)|G = g],$$

which implies

$$u_+^n(x,g) \leq P(N_{\frac{1}{n}} \geq 1)u_+^{\mathcal{F}}(x) + P(N_{\frac{1}{n}} = 0)u_+(x,g).$$

Consequently

$$\limsup_n u_+^n(x,g) \leq u_+(x,g),$$

and thus the result. □

Now let $v(y,g)$ be the conjugate function of $u_+(x,g)$. We will represent $v(x,g)$ as an $f$-divergence between the measures $\bar{P}$ and $\bar{Q} = R \otimes P$. For the moment we assume that $\bar{Q}$ and $\bar{P}$ are restricted to $\mathcal{A} = \mathcal{F}_T \otimes \sigma(G)$ and that $(V_y)\hat{}_{\mathcal{A}} \left(\bar{P}\|\bar{Q}\right) < \infty$ for some $y > 0$.

Moreover, let $\bar{G}$ denote the random variable on $(\bar{\Omega}, \mathcal{A})$ defined by $\bar{G}(\omega, \omega') = G(\omega')$. Now let $\bar{P}^g = \bar{P}(\cdot|\bar{G} = g)$ and $\bar{Q}^g = \bar{Q}(\cdot|\bar{G} = g)$, where both measures are supposed to be defined on $\mathcal{A}$. For all $g \in \Gamma$ we define

$$[0, \infty) \ni y \mapsto c(y, g) = (V_y)\hat{}_{\mathcal{A}} \left(\bar{P}^g \| \bar{Q}^g\right).$$

The function $c(y, g)$ has the following properties.

**Lemma 12.3.3.** *Let $g \in \Gamma$. Then $c(y, g) < \infty$ for $y > 0$ sufficiently large. Letting $y(g) = \inf\{y : c(y, g) < \infty\}$, then $c(y, g)$ is decreasing, differentiable and strictly convex on $(y(g), \infty)$.*

*Proof.* Since we assumed $(V_y)\hat{}_{\mathcal{A}} \left(\bar{P} \| \bar{Q}\right) < \infty$ for some $y > 0$, and $\mu(g) > 0$, we also have $c(y, g) < \infty$ for $y > 0$ sufficiently large.

Our assumptions on $U$ imply that $V$, and thus $y \mapsto (V_y)\hat{}(z)$ (with $z > 0$), is strictly convex, decreasing and differentiable on $(0, \infty)$. Therefore, also $c(y, g)$ is strictly convex on $(y(g), \infty)$.

It is known from convex analysis that for any convex function the left and right derivatives exist on the interior of the domain. Thus $c(y, g)$ has a right derivative $\frac{d}{dy+} c(y, g)$ and a left derivative $\frac{d}{dy-} c(y, g)$ on $(y(g), \infty)$. Moreover

$$\begin{aligned} \frac{d}{dy+} c(y, g) &= \int \frac{d}{dy+} (V_y)\hat{} \left( \frac{d\bar{P}^g}{d\bar{Q}^g} \right) d\bar{Q}^g \\ &= \int \frac{d}{dy-} (V_y)\hat{} \left( \frac{d\bar{P}^g}{d\bar{Q}^g} \right) d\bar{Q}^g \\ &= \frac{d}{dy-} c(y, g), \end{aligned}$$

showing that $c(y, g)$ is differentiable on $(y(g), \infty)$. □

We are now ready to state and prove the first duality result. To simplify the analysis we assume that $(V_y)\hat{}_{\mathcal{A}} \left(\bar{P} \| \bar{Q}\right)$ is finite for all $y > 0$. As a consequence of the preceding lemma, the derivative of $c(y, g)$ is invertible on $(0, \infty)$, and we denote the inverse by $w(z, g) = (c')^{-1}(z, g)$. Since $\lim_{y \downarrow 0} V'(y) = -\infty$ and $\lim_{y \to \infty} V'(y) = 0$, the domain of $w(z, g)$ is given by $(-\infty, 0)$.

**Theorem 12.3.4.** *The function $c(y, g) = (V_y)\hat{}_{\mathcal{A}} \left(\bar{P}^g \| \bar{Q}^g\right)$ is the conjugate of $u_+(x, g)$. Moreover, for all $x > 0$,*

$$u_+(x, g) = xw(-x, g) + c(w(-x, g), g) = xw(-x, g) + (V_{w(-x,g)})\hat{}_{\mathcal{A}} \left(\bar{P}^g \| \bar{Q}^g\right).$$

*Proof.* Let $\mathcal{A}_n = \mathcal{F}_T \otimes \sigma(G^n)$, $\bar{G}^n(\omega, \omega') = G(\omega')$. Note that $\bar{Q} \sim \bar{P}$ on $\mathcal{A}_n$. With Theorem 2.4.1 we deduce that there is a decoupling measure $Q^n$ on $(\Omega, \mathcal{G}^n_T)$ satisfying

$$\frac{dQ^n}{dP} = \left( \frac{d\bar{Q}}{d\bar{P}}\Big|_{\mathcal{A}_n} \circ \psi \right).$$

Denote by $P^n$ the restriction of $P$ to the $\sigma$-field $\mathcal{G}^n_T$. Then the measure $Q^n$ is an ELMM of $S$ relative to the enlarged filtration $(\mathcal{G}^n_t)$ and the measure $P^n$.

Fix $g \in \Gamma$, and recall $\mu(g) > 0$. Note that for all measures $P' \sim P$ on $\sigma(G^n)$, we have $R \otimes P'(\cdot|\bar{G}^n = g) = \bar{Q}^n(\cdot|\bar{G}^n = g)$. Therefore the measure $Q^n(\cdot|G^n = g)$ is the unique ELMM with respect to $P^n(\cdot|G^n = g)$ (see also Theorem 3.2 in [1]). Moreover, relative to $Q^n(\cdot|G^n = g)$ the initial $\sigma$-field $\mathcal{G}_0$ is trivial. Therefore, $S$ satisfies the (PRP) with respect to $Q^n(\cdot|G^n = g)$ and the filtration $(\mathcal{G}^n_t)$. Hence we may apply Theorem 12.1.1 to the conjugate function $v^n(y, g)$ of $u^n_+(x, g)$. This yields, for $y > 0$,

$$\begin{aligned}
v^n(y,g) &= E^{P^n(\cdot|G^n=g)} V\left( y \frac{dQ^n(\cdot|G^n = g)}{dP^n(\cdot|G^n = g)} \right) \\
&= E^{\bar{P}^n(\cdot|\bar{G}^n=g)} V\left( y \frac{d\bar{Q}^n(\cdot|\bar{G}^n = g)}{d\bar{P}^n(\cdot|\bar{G}^n = g)} \right) \\
&= (V_y)\hat{}\big(\bar{P}^n(\cdot|\bar{G}^n = g) \| \bar{Q}^n(\cdot|\bar{G}^n = g)\big) \\
&= \frac{1}{\mu(g)} \int 1_{\{\bar{G}^n = g\}}\, (V_y)\hat{}\left( \frac{d\bar{P}}{d\bar{Q}}\Big|_{\mathcal{A}_n} \right) d\bar{Q}.
\end{aligned}$$

Although $(\mathcal{A}_n)$ is not a filtration, the density process $\left(\frac{d\bar{P}}{d\bar{Q}}\big|_{\mathcal{A}_n}\right)_n$ is a convergent martingale (relative to a bigger filtration). Let $(\Delta_n)$ be a sequence of new symbols and define random variables by

$$f_n = \begin{cases} \Delta_n, & \text{if } N_{\frac{1}{n}} = N_{\frac{1}{n+1}} \\ G^n, & \text{else.} \end{cases}$$

Moreover, let $\mathcal{B}_n = \mathcal{F}_T \otimes \sigma(G^n) \vee \sigma(f_k : 1 \le k < n)$. Note that $\sigma(G^n)$ is independent of $\sigma(f_k : 1 \le k < n)$, and therefore $\frac{d\bar{P}}{d\bar{Q}}\big|_{\mathcal{B}_n} = \frac{d\bar{P}}{d\bar{Q}}\big|_{\mathcal{A}_n}$. Similarly, with $\mathcal{B} = \bigvee_{k\ge1} \mathcal{B}_k$, we have $\frac{d\bar{P}}{d\bar{Q}}\big|_{\mathcal{A}} = \frac{d\bar{P}}{d\bar{Q}}\big|_{\mathcal{B}}$. On the one hand Jensen's inequality yields that the integrands $1_{\{\bar{G}^n=g\}}\, (V_y)\hat{}\left(\frac{d\bar{P}}{d\bar{Q}}\big|_{\mathcal{A}_n}\right)$ are uniformly integrable with respect to $\bar{Q}$, and on the other hand the martingale property relative to $(\mathcal{B}_n)$

implies that they converge almost surely to $1_{\{\bar G=g\}}\ (V_y)^{\hat{}}\Big(\frac{d\bar P}{d\bar Q}\big|_{\mathcal A}\Big)$. Therefore,

$$\begin{aligned}\lim_n v^n(y,g) &= \frac{1}{\mu(g)}\int 1_{\{\bar G(\omega')=g\}}\ (V_y)^{\hat{}}\left(\frac{d\bar P}{d\bar Q}\bigg|_{\mathcal A}\right)\, d\bar Q\\ &= (V_y)^{\hat{}}_{\mathcal A}\,(\bar P^g\|\bar Q^g)\\ &= c(y,g).\end{aligned}$$

It remains to show that the limit of the conjugates, $c(y,g)$, is indeed equal to the conjugate of $u_+(x,g)$. By Theorem 12.1.1, for all $g$, the functions $v^n(y,g)$ are differentiable and strictly convex on $(0,\infty)$. Let $w^n(y,g)$ denote the inverse function of the derivative of $v^n(y,g)$. By Lemma 12.3.3, also $c(y,g)$ is differentiable on $(0,\infty)$. Therefore Theorem 25.7 in [43] implies that on $(0,\infty)$ the derivatives of $v^n(y,g)$ converge pointwise to the derivative of $c(y,g)$, and hence

$$\lim_n w^n(z,g) = w(z,g).$$

Note that $u_+^n(x,g) = \inf_y[v^n(y,g)+xy]$, and that $v^n(y,g)+xy$ achieves its minimum if $-x = \frac{d}{dy}v^n(y,g)$ (see Theorem 23.5 in [43]). Therefore

$$u_+^n(x,g) = v^n(w^n(-x,g),g) + xw^n(-x,g).$$

Moreover, $v^n(y,g)$ converges uniformly on each closed bounded set of $(0,\infty)$ (see Theorem 10.8 in [43]). Therefore, by letting $n\to\infty$, we obtain with Lemma 12.3.2

$$u_+(x,g) = c(w(-x,g),g) + xw(-x,g).$$

This shows that $u_+(x,g)$ is the dual function of $c(y,g)$, and finally, that $c(y,g)$ is dual to $u_+(x,g)$. Thus the proof is complete. □

In terms of stochastic conjugates we may reformulate the previous result as follows.

**Theorem 12.3.5.** *The process $c(y,G)$ is the stochastic conjugate of the concave process $u_+(x,G)$, called the* conditional expected utility *relative to $G$. Moreover*

$$c(y,G) = \int\ (V_y)^{\hat{}}\left(\frac{d\bar P}{d\bar Q}\bigg|_{\mathcal A}\right)\ dR,$$

*and*

$$u_+^{\mathcal G}(x) = xEw(-x,G) + \int (V_{w(-x,G)})^{\hat{}}\left(\frac{d\bar P}{d\bar Q}\bigg|_{\mathcal A}\right)\ d\bar Q,\quad x>0.$$

*Proof.* Note that

$$\begin{aligned}
&\sum_g 1_{\{G(\omega')=g\}}(V_y)\hat{}(\bar{P}^g\|\bar{Q}^g) \\
=& \sum_g 1_{\{G(\omega')=g\}}\frac{1}{P(g)}\int (V_y)\hat{}\left(\frac{d\bar{P}}{d\bar{Q}}\right) 1_{\{G(\omega')=g\}}\ d\bar{Q} \\
=& \sum_g 1_{\{G(\omega')=g\}}\int (V_y)\hat{}\left(\frac{d\bar{P}}{d\bar{Q}}\right) 1_{\{G(\omega')=g\}}\ dR(\omega) \\
=& \int (V_y)\hat{}\left(\frac{d\bar{P}}{d\bar{Q}}\right)\ dR(\omega).
\end{aligned}$$

By the preceding theorem, $u_+^{\mathcal{G}}(x,g) = xw(-x,g) + c(w(-x,g),g)$, for all $x > 0$, $g \in \Gamma$. Moreover, by Lemma 12.3.1, $u_+^{\mathcal{G}}(x) = Eu_+(x,G)$, which implies the result. □

If $G$ is $\mathcal{F}_T$-measurable and $R = P$, then the conjugate $c(y,G)$ can be interpreted as a generalised *absolute* entropy with respect to $V$:

**Lemma 12.3.6.** *If $G$ is $\mathcal{F}_T$-measurable and $R = P$, then*

$$c(y,G) = V(y\,\mu(G)) = \sum_g 1_{\{G=g\}}V(y\,\mu(g)).$$

*Proof.* Under the assumptions of the lemma, it is straightforward to show that

$$\frac{d\bar{P}}{dP\otimes P} = \sum_g 1_{\{G(\omega)=G(\omega')\}}\frac{1}{\mu(g)}.$$

Therefore

$$\begin{aligned}
c(y,G) &= \sum_g 1_{\{G=g\}}\mu(g)(V_y)\hat{}\left(\frac{1}{\mu(g)}\right) \\
&= \sum_g 1_{\{G=g\}}V(y\,\mu(g)).
\end{aligned}$$

□

## 12.4 Solving the problem for non-discrete $G$

In the previous section we considered only discrete random variables $G$. This made it easier to compute the maximal expected utility under initial enlargements. An approximation through discretisation will allow us to derive similar results for *general* initial enlargements.

Throughout this section let $G$ be an arbitrary random variable with values in a Polish space $\Gamma$. We assume that $u_+^{\mathcal{G}}(x)$ and $(V_y)\hat{}_{\mathcal{F}_T\otimes\sigma(G)}\left(\bar{P}\|\bar{Q}\right)$ are finite for all $x > 0$ and $y > 0$ respectively.

Let $(\mathcal{P}^n)$ be a sequence of countable partitions of $\sigma(G)$ such that $\sigma(G) = \bigvee_n \sigma(\mathcal{P}^n)$, and $\mathcal{P}^n \subset \mathcal{P}^{n+1}$. We denote by $u_+^n(x)$ the maximal expected utility under the filtration enlarged by $\sigma(\mathcal{P}^n)$. Then monotone utility convergence implies that $\lim_n u_+^n(x) = u_+^{\mathcal{G}}(x)$. Moreover, for all $A \in \mathcal{P}^n$, we denote by $u_+(x, A)$ the maximal expected utility under the filtration enlarged by $\sigma(\mathcal{P}^n)$ and the measure $P$ restricted to $A$. By Lemma 12.3.1 we have

$$u_+^n(x) = \sum_{A\in\mathcal{P}^n} P(A)\, u_+(x, A).$$

The process $Z_n(x) = \sum_{A\in\mathcal{P}^n} 1_A\, u_+(x, A)$ is the conditional expected utility with respect to $\mathcal{P}^n$.

**Lemma 12.4.1.** *Let $x > 0$. Then $(Z_n(x))$ is a submartingale with respect to the filtration $\mathcal{H}_n = \sigma(\mathcal{P}^n)$, $n \geq 1$. Moreover, $(Z_n(x))$ is uniformly integrable and convergent in $L^1$.*

*Proof.* Let $n \geq 1$, and $A \in \mathcal{P}^n$. Since $\mathcal{P}^n \subset \mathcal{P}^{n+1}$, there are sets $B_1, \ldots, B_k$ in $\mathcal{P}^{n+1}$ such that $A = B_1 \cup \ldots \cup B_k$. Obviously

$$\begin{aligned} E[1_A Z_n(x)] &= P(A)u_+(x, A) \\ &\leq P(B_1)u_+(x, B_1) + \ldots + P(B_k)u_+(x, B_k) \\ &= E[1_A Z_{n+1}(x)]. \end{aligned}$$

Therefore $E[Z_{n+1}(x)|\mathcal{H}_n] \geq Z_n(x)$, for all $n \geq 1$, which means that $(Z_n(x))$ is a submartingale.

Note that $u_+(x, A) \geq U(x)$, and hence $Z_n(x) \geq U(x)$, a.s. Therefore $(Z_n(x))$ is uniformly integrable from below. Moreover $\sum_{C\in\mathcal{C}} P(A)\, u_+(x, A) \leq u_+^{\mathcal{G}}(x) < \infty$, for every subset $\mathcal{C} \subset \mathcal{P}^n$. In particular

$$E\left[Z_n(x); Z_n(x) \geq M\right] \leq u_+^{\mathcal{G}}(x),$$

showing that the submartingale $(Z_n(x))$ is uniformly integrable. As a consequence, it converges in $L^1$ (see f.e. Chapter 4 in [18]). □

**Definition 12.4.2.** *The $L^1$-limit $Z(x) = \lim_n Z_n(x)$ will be called* conditional expected utility *relative to $G$. We choose this name, because $Z(x)$ is $\sigma(G)$-measurable, and for all $B \in \sigma(G)$ with $P(B) > 0$ we have*

$$u_+(x, B) = \int_B Z(x)\; dP.$$

We apply now the results of the previous section to our approximations $Z_n(x)$: Theorem 12.3.5 implies that the stochastic conjugate of $Z_n(x)$ is given by

$$Y_n(y) = \int (V_y)\hat{}\left(\frac{d\bar{P}}{d\bar{Q}}|_{\mathcal{F}_T\otimes\sigma(\mathcal{P}^n)}\right) dR(\omega), \quad y > 0,$$

$P$-almost surely. We claim that $Y_n(y)$ converges to

$$Y(y) = \int (V_y)\hat{}\left(\frac{d\bar{P}}{d\bar{Q}}|_{\mathcal{F}_T\otimes\sigma(G)}\right) dR(\omega).$$

More precisely:

**Lemma 12.4.3.** *The processes $Y(y)$ and $Y_n(y)$ are strictly convex, decreasing and differentiable on $(0,\infty)$ almost surely. Moreover, for almost all $\omega$ the functions $y \mapsto Y_n(y)$ converge pointwise to $y \mapsto Y(y)$ on $(0,\infty)$, and the derivatives $\frac{d}{dy}Y_n(y)$ converge to the derivative $\frac{d}{dy}Y(y)$.*

*Proof.* It is straightforward to show the first statement. For the second, let $y > 0$. Note that $(V_y)\hat{}_{\mathcal{F}_T\otimes\sigma(\mathcal{P}^n)}\left(\bar{P}\|\bar{Q}\right)$ converges to $(V_y)\hat{}_{\mathcal{F}_T\otimes\sigma(G)}\left(\bar{P}\|\bar{Q}\right)$ (see Lemma 12.2.3). This implies that the sequence $(V_y)\hat{}\left(\frac{d\bar{P}}{d\bar{Q}}|_{\mathcal{F}_T\otimes\sigma(\mathcal{P}^n)}\right)$ is uniformly integrable, and therefore it converges to $(V_y)\hat{}\left(\frac{d\bar{P}}{d\bar{Q}}|_{\mathcal{F}_T\otimes\sigma(G)}\right)$ in $L^1(\bar{Q})$. Moreover, $(Y_n(y))$ converges to $Y(y)$ in $L^1(P)$, and thus, there is a subsequence, for which $(Y_n(y))$ converges almost surely to $Y(y)$. One can also show that $(Y_n(y))$ is a uniformly integrable submartingale. Hence $\lim Y_n(y) = Y(y)$ in general, and not only for a subsequence.

We now apply the following well-known fact from convex analysis: If $(f_n)$ is sequence of convex functions converging on a dense subset to a finite function $f$, then $(f_n)$ converges to $f$ everywhere and $f$ is convex. Moreover $(f_n)$ converges uniformly on every bounded set (see Theorem 10.8 in [43]).

In order to apply this result to our processes $(Y_n)$, let $\mathcal{D}$ be a countable dense subset of $(0,\infty)$. For any $q \in \mathcal{D}$ we have $\lim_n Y_n(q) = Y(q)$ almost surely, and therefore, for almost all $\omega$, the functions $y \mapsto Y_n(y)$ converge pointwise to $y \mapsto Y(y)$ on $(0,\infty)$.

Finally, another result from convex analysis implies that the derivatives converge almost surely:

$$\lim_n \frac{d}{dy}Y_n(y) = \frac{d}{dy}Y(y)$$

(see Theorem 25.7 in [43]). □

Since the processes $Y$, $Y_n$ are strictly convex and decreasing, the derivatives are strictly increasing and invertible. Hence we may define

$$W_n(z) = \left(\frac{d}{dy}Y_n\right)^{-1}(z),$$

and

$$W(z) = \left(\frac{d}{dy}Y\right)^{-1}(z), \quad z < 0.$$

Note that Lemma 12.4.3 implies that $\lim_n W_n(z) = W(z)$, a.s. We will see that $W$ plays the role of a Lagrangian multiplier.

**Theorem 12.4.4.** *The process $Y$ is the stochastic conjugate of the conditional expected utility $Z$. Therefore, for $x > 0$,*

$$\begin{aligned} u_+^{\mathcal{G}}(x) &= E[xW(-x) + Y(W(-x))] \\ &= xEW(-x) + \int (V_{W(-x)})^{\hat{}} \left(\frac{d\bar{P}}{d\bar{Q}}|_{\mathcal{F}_T\otimes\sigma(G)}\right) d\bar{Q} \\ &= xEW(-x) + (V_{W(-x)})^{\hat{}}_{\mathcal{F}_T\otimes\sigma(G)}(\bar{P}\|\bar{Q}), \end{aligned}$$

*Proof.* We have to show that $Z(x,\omega) = xW(-x,\omega) + Y(W(-x,\omega),\omega)$ for almost all $\omega$.

According to Theorem 12.3.4

$$Z_n(x,\omega) = xW_n(-x,\omega) + Y_n(W_n(-x,\omega),\omega),$$

almost surely. Moreover, by Lemma 12.4.3, for almost all $\omega$ the functions $y \mapsto Y_n(y)$ converge pointwise to $y \mapsto Y(y)$ on $(0,\infty)$. Since these functions converge uniformly on every closed bounded subset of $(0,\infty)$, we have

$$\lim_n Y_n(W_n(x,\omega),\omega) = Y(W(x,\omega),\omega),$$

and hence $Z(x) = xW(x) + Y(W(x))$, almost surely.

Finally,

$$\begin{aligned} u_+^{\mathcal{G}}(x) &= E[Z(x)] \\ &= xEW(-x) + \int (V_{W(-x)})^{\hat{}} \left(\frac{d\bar{P}}{d\bar{Q}}|_{\mathcal{F}_T\otimes\sigma(G)}\right) d\bar{Q} \\ &= xEW(-x) + (V_{W(-x)})^{\hat{}}_{\mathcal{F}_T\otimes\sigma(G)}(\bar{P}\|\bar{Q}), \end{aligned}$$

and the proof is complete. □

## 12.5 Examples

In this section we will apply the results obtained so far to some concrete examples. So let again $(\mathcal{G}_t)$ be a filtration initially enlarged by an arbitrary random variable $G$. We start with the power utility function.

**Proposition 12.5.1.** *Let $0 < p < 1$ and $U(x) = \frac{1}{p}x^p$ for all $x \geq 0$. Then*

$$u_+^{\mathcal{G}}(x) = U(x) \int \left( \int \left( \frac{d\bar{P}}{d\bar{Q}} \right)^{\frac{1}{1-p}} dR \right)^{1-p} dP.$$

*Proof.* Observe that $f(z) = (U')^{-1}(z) = (z)^{\frac{1}{p-1}}$ and

$$V(y) = U(f(y)) - yf(y) = \frac{1-p}{p}\, y^{-\frac{p}{1-p}},$$

and consequently

$$\begin{aligned} (V_y)\hat{}(x) &= \frac{1-p}{p}\, x \left(\frac{y}{x}\right)^{-\frac{p}{1-p}} \\ &= \left( \frac{1-p}{p}\, y^{-\frac{p}{1-p}} \right) x^{\frac{1}{1-p}}. \end{aligned}$$

Hence the stochastic conjugate $Y$ of the conditional expected utility $Z$ satisfies

$$Y(y) = \frac{1-p}{p}\, y^{-\frac{p}{1-p}} \int \left( \frac{d\bar{P}}{d\bar{Q}} \right)^{\frac{1}{1-p}} dR.$$

Hence $\frac{d}{dy}Y(y) = y^{-\frac{1}{1-p}} \int \left( \frac{d\bar{P}}{d\bar{Q}} \right)^{\frac{1}{1-p}} dR$, and thus

$$W(z) = \left( \left( \int \left( \frac{d\bar{P}}{d\bar{Q}} \right)^{\frac{1}{1-p}} dR \right)^{-1} (-z) \right)^{-(1-p)}, \quad z < 0.$$

Therefore, by Theorem 12.4.4,

$$\begin{aligned} u_+^{\mathcal{G}}(x) &= E\left[xW(-x) + Y(W(-x)\right] \\ &= x^p \int \left( \int \left( \frac{d\bar{P}}{d\bar{Q}} \right)^{\frac{1}{1-p}} dR \right)^{1-p} dP \\ &\quad + \frac{1-p}{p}\, x^p \int \left( \int \left( \frac{d\bar{P}}{d\bar{Q}} \right)^{\frac{1}{1-p}} dR \right)^{1-p} dP \\ &= U(x) \int \left( \int \left( \frac{d\bar{P}}{d\bar{Q}} \right)^{\frac{1}{1-p}} dR \right)^{1-p} dP. \end{aligned}$$

□

Let us turn now to the logarithmic utility function.

**Proposition 12.5.2.** *Let $U(x) = \log x$ for all $x > 0$. Then $W(-x) = \frac{1}{x}$ and the maximal expected utility is equal to the relative entropy of $\bar{P}$ with respect to $\bar{Q}$, i.e.*

$$u_+^{\mathcal{G}}(x) = \log(x) + \mathcal{H}(\bar{P}\|\bar{Q}).$$

*Proof.* Observe that the dual of the logarithm is given by $V(y) = -\log(y) - 1$, hence

$$(V_y)\hat{}(x) = x(-\log(\frac{y}{x}) - 1).$$

The conjugate is given by

$$\begin{aligned} Y(y) &= -\log(y) \int \frac{d\bar{P}}{d\bar{Q}}\, dR + \int \frac{d\bar{P}}{d\bar{Q}} \left(-1 + \log \frac{d\bar{P}}{d\bar{Q}}\right) dR \\ &= -\log(y) - 1 + \int \log \frac{d\bar{P}}{d\bar{Q}}\, dR \end{aligned}$$

and therefore $W(z) = -\frac{1}{z}$. Moreover, Theorem 12.4.4 implies

$$\begin{aligned} u_+^{\mathcal{G}}(x) &= xW(-x) - 1 - \log(W(-x)) + \int \frac{d\bar{P}}{d\bar{Q}} \log\left(\frac{d\bar{P}}{d\bar{Q}}\right) d\bar{Q} \\ &= \log(x) + \mathcal{H}(\bar{P}\|\bar{Q}). \end{aligned}$$

□

**Remark 12.5.3.** *For the logarithm we need not consider the* conditional *expected utility, in order to generate the conjugate function. The usual conjugate of $u_+^{\mathcal{G}}(x)$ can be derived easily: Note that*

$$Z(x) = \inf_{y>0}[xy + Y(y)].$$

*By taking expectations we obtain*

$$EZ(x) = \mathcal{H}(\bar{P}\|\bar{Q}) - 1 + \inf_{y>0}[xy - \log(y)],$$

*and hence the conjugate of $u_+^{\mathcal{G}}(x)$ is given by*

$$v^{\mathcal{G}}(y) = -\log(y) + \mathcal{H}(\bar{P}\|\bar{Q}) - 1.$$

## 12.6 Additional logarithmic utility

The properties of the logarithm lead to a simple formula for the additional logarithmic utility of an investor with information $(\mathcal{G}_t)$ compared to an investor having only access to $(\mathcal{F}_t)$.

**Theorem 12.6.1.** *If $U = \log$, then the utility difference $\Delta u = u_+^{\mathcal{G}}(x) - u_+^{\mathcal{F}}(x)$ does not depend on $x$, and it is equal to the mutual information between $\mathcal{F}_T$ and $G$, i.e.*

$$\Delta u = \mathcal{H}_{\mathcal{F}_T \otimes \sigma(G)}(\bar{P} \,\|P \otimes P) = I(\mathcal{F}_T, G).$$

*In particular, if $G$ is discrete and $\mathcal{F}_T$-measurable, the additional utility is equal to the absolute entropy of $G$ relative to $P$,*

$$\Delta u = -\sum_g P(G = g) \log P(G = g).$$

*Proof.* Let $f(\omega, \omega') = \frac{d\bar{P}}{d\bar{Q}}\big|_{\mathcal{F}_T \otimes \sigma(G)}$ and $g(\omega) = \frac{dR}{dP}\big|_{\mathcal{F}_T}$. We show at first that

$$f(\omega, \omega')g(\omega) = \frac{d\bar{P}}{d(P \otimes P)}\big|_{\mathcal{F}_T \otimes \sigma(G)}. \tag{12.4}$$

For this let $A \in \mathcal{F}_T$ and $B \in \sigma(G)$. Note that

$$\begin{aligned}
\int 1_{A \times B}(\omega, \omega') f(\omega, \omega') g(\omega)\, d(P \otimes P) &= \int 1_{A \times B}(\omega, \omega') f(\omega, \omega')\, d(R \otimes P) \\
&= \int 1_{A \times B}(\omega, \omega') f(\omega, \omega')\, d\bar{Q} \\
&= \bar{P}(A \times B),
\end{aligned}$$

which implies (12.4).

Recall that $u_+^{\mathcal{G}}(x) = \log(x) + \mathcal{H}(\bar{P}\|\bar{Q})$ and $u_+^{\mathcal{F}}(x) = \log(x) + \mathcal{H}(P\|R)$. Thus

$$\begin{aligned}
u_+^{\mathcal{G}}(x) - u_+^{\mathcal{F}}(x) &= \mathcal{H}(\bar{P}\|\bar{Q}) - \mathcal{H}(P\|R) \\
&= \int \left(\log f(\omega, \omega') - \log g^{-1}(\omega)\right)\, d\bar{P} \\
&= \int \log\left(f(\omega, \omega')g(\omega)\right)\, d\bar{P} \\
&= \int \log\left(\frac{d\bar{P}}{d(P \otimes P)}\Big|_{\mathcal{F}_T \otimes \sigma(G)}\right)\, d\bar{P} \\
&= \mathcal{H}_{\mathcal{F}_T \otimes \sigma(G)}(\bar{P} \,\|P \otimes P).
\end{aligned}$$

Finally, if $G$ is discrete and $\mathcal{F}_T$-measurable, then Lemma 3.2.10 implies that $\Delta u$ is equal to the absolute entropy of $G$. $\square$

**Example 12.6.2.** *Let $(\Omega, \mathcal{F}, P)$ be the 1-dimensional canonical Wiener space equipped with the Wiener process $(W_t)_{0\le t\le 1}$. More precisely, $\Omega = \mathcal{C}([0,1], \mathbb{R})$ is the set of continuous functions on $[0,1]$ starting in $0$, $\mathcal{F}$ the $\sigma$-algebra of Borel sets with respect to uniform convergence, $P$ the Wiener measure and $W$ the coordinate process. Let $(\mathcal{F}_t)_{0\le t\le 1}$ be the completed natural filtration generated by $W$. It is known that $W$ satisfies (PRP) relative to $(\mathcal{F}_t)$.*

*Suppose the price process $S$ is of the form*

$$S_t = \exp(W_t + bt), \qquad 0 \le t \le 1,$$

*with $b \in \mathbb{R}$. We want to calculate the additional utility of an investor knowing whether the price exceeds a certain level or not. Thus let*

$$G = 1_{(c,\infty)}(S_1^*),$$

*where $c > 0$ and $S_1^* = \max_{0\le t\le 1} S_t$. By Theorem 12.6.1 the additional utility is equal to the entropy*

$$H(G) = p \log p + (1-p)\log(1-p)$$

*where*

$$p = P(S_1^* > c).$$

*This may be calculated via the Girsanov Theorem. Namely we have*

$$\begin{aligned} P(S_1^* > c) &= P(\forall t \in [0,1] : \max_{t\in[0,1]} W_t + bt > \log c) \\ &= \int_0^1 \exp\left(b \log c - \frac{b^2}{2} s\right) \frac{|\log c|}{\sqrt{2\pi s^3}} \exp\left(-\frac{|\log c|^2}{2s}\right) ds. \end{aligned}$$

**Remark 12.6.3.** *Let $\mathcal{H}_t = \bigcap_{s>t} \mathcal{F}_s \vee \sigma(H)$ be another initially enlarged filtration such that $\sigma(H)$ is a sub-$\sigma$-field of $\sigma(G)$. Then the utility difference $u^{\mathcal{G}} - u^{\mathcal{H}}$ is equal to the mutual information of $\mathcal{F}_T$ and $G$* conditioned *on $H$. See Section 14.3 for details.*

*Let $S = M + \alpha \cdot \langle M, M\rangle$ be the Doob-Meyer decomposition of $S$ relative to $(\mathcal{F}_t)$. According to Lemma 3.1.1, the entropy $\mathcal{H}_{\mathcal{F}_T\otimes\sigma(G)}(\bar{P}\,\|P\otimes P)$ is equal to $\frac{1}{2}E\int_0^T \mu^2 d\langle M, M\rangle$, where $\mu$ is the information drift of $(\mathcal{G}_t)$ relative to $M$. Therefore,*

$$\Delta u = \frac{1}{2} E \int_0^T \mu^2 d\langle M, M\rangle.$$

*We will show that this relation remains true under non-intial enlargements.*

## Preview: maximal expected utility for non-initial enlargements

So far we considered *initial* enlargements of a given filtration and we determined the conjugate function of the maximal expected utility conditioned on the enlarging random variable. What can we do, if the filtration is not only enlarged in the beginning, but at any moment during the trading period? Can we still determine a conjugate of the maximal expected utility?

As pointed out in the previous chapter, one may approximate general enlargements by piecewise initial enlargements of the filtration: the trading interval is divided into small subintervals, and in the beginning of each subinterval the filtration is enlarged initially. Naturally, the idea arises to apply the results of this chapter to each subinterval, and thus derive again a representation of the maximal expected utility via $f$-divergences. Unfortunately there is the following problem: Let $t$ be a point in the interval $(0, T)$. The maximal utility up to time $T$ is in general *not* the sum of the maximal utility up to time $t$ and the maximal utility between $t$ and $T$. Utility functions satisfying this property will be said to be *time-homogeneous*. We will see in the next chapter that the logarithmic utility function is essentially the only utility function to be time-homogeneous.

# Chapter 13

# Logarithmic utility of an investor

Under logarithmic preferences the optimal investments depend linearly on the initial wealth (see for example Theorem 12.1.1). As pointed out in the previous chapter, the logarithmic utility is also homogeneous with respect to time: the optimal strategy relative to a fixed time horizon $T$ is also optimal, if any time $t$ before $T$ may be chosen as terminal, and is therefore in a way an *always optimal strategy*.

The first aim of this chapter is to show that the logarithm is essentially the *only* utility function to be time-homogeneous. After this, the homogenity properties of the logarithm will allow us to determine the maximal expected *logarithmic* utility in a very general way.

## 13.1 Always optimal strategies

Let $(\mathcal{F}_t)$ be a filtration satisfying the usual conditions and $S$ a continuous price process starting in zero and being a semimartingale for $(\mathcal{F}_t)$. As in the previous chapter we assume here the market to be complete, so that we may invoke the general results by Kramkov and Schachermayer [35] quoted in Theorem 12.1.1. Let $R$ denote the unique ELMM, and suppose that $S$ is decomposed into

$$S = M + \alpha \cdot \langle M, M \rangle,$$

where $M$ is a $(\mathcal{F}_t)$-local martingale starting in zero and $\alpha$ an $(\mathcal{F}_t)$-predictable process. The Radon-Nikodym density of the martingale measure given $P$ is known to be described by the exponential of $(\alpha \cdot M)$:

$$\frac{dR}{dP}\Big|_{\mathcal{F}_t} = \mathcal{E}(-\alpha \cdot M)_t, \quad t \in [0, T], \tag{13.1}$$

(see [13]). In the following we shall abbreviate

$$Z = \mathcal{E}(-\alpha \cdot M).$$

We restrict the class of utility functions like in Chapter 12. More precisely, let $U$ be strictly increasing, strictly concave and continuously differentiable on $(0,\infty)$, and assume that $U$ satisfies properties 12.1 and 12.2. Denote again by $I$ the inverse function of the derivative of $U$ on $(0,\infty)$, and observe that $I$ is a function with domain $(0,\infty)$ and range $(0,\infty)$.

In general, the maximal expected utility $u(x)$ depends on the time interval in which the traders are allowed to act. We denote by $u_t(x)$ the maximal expected utility of a trader of initial wealth $x$ who is not allowed to hold any shares of the stock after time $t \leq T$ , i.e.

$$u_t(x) = \sup_{\theta\in\mathcal{A}} EU(x + (\theta 1_{[0,t]} \cdot S)_T) = \sup_{\theta\in\mathcal{A}} EU(x + (\theta \cdot S)_t).$$

**Definition 13.1.1.** *A strategy* $\theta^* \in \mathcal{A}$ *is called* always optimal*, if for all* $t \in [0,T]$ *and* $x > 0$

$$EU(x + (\theta^* \cdot S)_t) = u_t(x).$$

We will now analyze to which extent always optimal strategies exist. Consider at first the case where the drift $\alpha$ is equal to 0. In this case the price process $S$ is a $P$-local martingale and intuitively one would expect that a risk averse trader will not trade at all. Theorem 12.1.1 confirms that the maximal expected utility is the utility of the initial capital $U(x)$. Hence in this case the trivial strategy $\theta = 0$ is always optimal, whatever the utility function $U$ looks like.

If the drift $\alpha$ is not trivial, however, the situation is different. It turns out that in general always optimal strategies exist only for logarithmic utility functions. Before proving this we define

$$\bar{Z}_T = \sup_{0\leq t\leq T} Z_t$$

and

$$\underline{Z}_T = \inf_{0\leq t\leq T} Z_t.$$

We will only consider the case where

$$\text{ess inf } \underline{Z}_T = 0 \qquad \text{and} \qquad \text{ess sup } \bar{Z}_T = \infty. \tag{13.2}$$

**Theorem 13.1.2.** *Assume that* $I = (U')^{-1}$ *is twice continuously differentiable on* $(0,\infty)$ *and that the conditions (12.1), (12.2) and (13.2) are satisfied.*

*Then an always optimal strategy exists if and only if $U$ is the logarithm up to affine transformations, i.e.*

$$U(x) = a\log(x) + b$$

*for some constants $a > 0$ and $b \in \mathbb{R}$.*

*Proof.* Suppose at first that $U(x) = \log(x)$. By Theorem 12.1.1 we have for any $t \in [0,T]$

$$\begin{aligned} u_t(x) &= EU(I(yZ_t)) = EU(\frac{1}{yZ_t}) \\ &= E\log(xZ_t^{-1}) + c \;=\; E\log[x\mathcal{E}(\alpha\cdot S)_t] + c \\ &= E\log[x + (x\alpha\mathcal{E}(\alpha\cdot S)\cdot S)_t] + c. \end{aligned}$$

This shows that $\theta^* = x\alpha\mathcal{E}(\alpha\cdot S)$ is always optimal.

We now prove the converse statement. Let $\theta^*$ be an always optimal strategy. By Theorem 12.1.1 the process

$$x + (\theta^*\cdot S) = I(yZ)$$

is a $R$-martingale. Hence

$$ZI(yZ)$$

is a $P$-martingale. Since the function $\phi : (0,\infty) \to \mathbb{R}, \phi(x) = xI(yx)$ is twice continuously differentiable, we may apply Itô's formula and obtain for $t \in [0,T]$

$$Z_tI(yZ_t) = \phi(Z_t) = \phi(1) + \int_0^t \phi'(Z_s)\; dZ_s + \frac{1}{2}\int_0^t \phi''(Z_s)\; d\langle Z,Z\rangle_s.$$

From this equation we can deduce that the continuous process of bounded variation

$$\int_0^\cdot \phi''(Z_s)\; d\langle Z,Z\rangle_s = \int_0^\cdot \phi''(Z_s)\alpha_s^2 Z_s^2\; d\langle M,M\rangle_s$$

is a local $P$-martingale and hence vanishes. We now show that $\phi''(z) = 0$ for all $z > 0$. Suppose that this is not true. Then there exist $0 < p < q$ such that $\phi''$ does not vanish on the interval $(p,q)$. Observe that on the set

$$A = \{(t,\omega) : Z_t(\omega) \in (p,q)\}$$

we have $\alpha = 0$, $P_M$-a.s. This means that the process $\int_0^\cdot \alpha^2\; d\langle M,M\rangle$ is constant on $A$. Hence also the processes $\int_0^\cdot \alpha\; dM$ and $Z = \mathcal{E}(\alpha\cdot M)$ are constant on $A$ (see [21]), i.e.

$$1_A(t,\omega)Z_t(\omega) \quad \text{is constant} \qquad a.s.$$

In other words, the trajectories $t \mapsto Z_t(w)$ are a.s. constant on $(p, q)$.

Suppose first that $q < 1$ or $p > 1$. Since $Z_0 = 1$, it follows that the entire trajectories of $Z$ are above $q$ or below $p$, respectively. This contradicts (13.2).

Suppose next that $p < 1 < q$. Since $Z$ is constant on $(p, q)$, we must have $Z = 1$, which also contradicts property (13.2). Thus we have shown $\phi'' = 0$.

On the other hand

$$\phi'(x) = I(yx) + yxI'(yx)$$

and

$$\phi''(x) = 2yI'(yx) + xy^2 I''(yx).$$

Hence $I'$ solves the differential equation

$$2I'(z) = -zI''(z), \qquad z > 0.$$

By assumption (12.1) the function $I' : (0, \infty) \to (-\infty, 0)$ satisfies

$$\lim_{z \to 0^+} I'(z) = -\infty.$$

Hence $I'(z) = -\frac{a}{z^2}$, and

$$I(z) = \frac{a}{z} + c_1$$

for some constants $a > 0$ and $c_1 \in \mathbb{R}$. It follows

$$U'(x) = \frac{a}{x - c_1}$$

and

$$U(x) = a \log(x - c_1) + c_2$$

for some $c_2 \in \mathbb{R}$. Note that $c_1 = 0$, because $\lim_{x \to 0^+} U(x) = -\infty$. This completes the proof. □

## 13.2 Maximal utility if wealth stays positive

From now on we uniquely consider the logarithmic utility function. So let

$$U(x) = \begin{cases} \log x & \text{if } x > 0, \\ -\infty & \text{if } x \leq 0 \end{cases}$$

throughout the remaining chapters. Moreover, we do not any longer assume that the market is complete, or even free of arbitrage. As usual, we only assume the asset price process to be continuous, to start in zero, and that an agent with information horizon $(\mathcal{F}_t)$ has bounded logarithmic utility. According to the conclusion in Theorem 9.2.7, $S$ is therefore a semimartingale with Doob-Meyer decomposition

$$S = M + \alpha \cdot \langle M, M \rangle, \qquad (13.3)$$

where $M$ is a continuous local martingale starting in zero and $\int_0^T \alpha^2 d\langle M, M\rangle < \infty$, a.s.

The aim of this section consists in computing explicitly the expected logarithmic utility of the agent. In fact, it only depends on the drift density $\alpha$, i.e.

$$u_+(x) = \log x + \frac{1}{2} E \int_0^T \alpha_s^2 \, d\langle M, M\rangle_s, \quad x > 0. \qquad (13.4)$$

Equation (13.4) is valid irrespective of whether (NFLVR) holds, provided (13.3) is guaranteed.

We start by proving some auxiliary results which will turn out to present the optimal portfolio as the unique solution of a linear stochastic equation.

**Proposition 13.2.1.** *If $\pi$ is a predictable and $S$-integrable process, then the product $\mathcal{E}(\pi \cdot S)\mathcal{E}(-\alpha \cdot M)$ is a local martingale.*

*Proof.* We use Yor's addition formula

$$\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X + Y + \langle X, Y\rangle),$$

for two continuous semimartingales (see e.g. [15], p. 374). It implies

$$\mathcal{E}(\pi \cdot S)\mathcal{E}(-\alpha \cdot M) = \mathcal{E}((\pi - \alpha) \cdot M),$$

and hence, the result. □

**Remark 13.2.2.** *Proposition 13.2.1 states that $\mathcal{E}(-\alpha \cdot M)$ is a strict martingale density for $\mathcal{E}(\pi \cdot S)$ in the sense of Schweizer [45].*

**Lemma 13.2.3.** *Let $x > 0$. The process $\theta^* = x\alpha\mathcal{E}(\alpha \cdot S)$ is $x$-superadmissible and solves the integral equation*

$$\theta_t^* = \alpha_t(x + \int_0^t \theta_r^* \, dS_r), \quad 0 \le t \le T. \qquad (13.5)$$

*Proof.* We observe that the process $\theta^* = x\alpha\mathcal{E}(\alpha \cdot S)$ is predictable and satisfies for all $t \in [0,T]$

$$\begin{aligned} x + (\theta^* \cdot S)_t &= x + x\int_0^t \alpha_r \mathcal{E}(\alpha \cdot S)_r dS_r \\ &= x(1 + \int_0^t \alpha_r \mathcal{E}(\alpha \cdot S)_r dS_r) \\ &= x\mathcal{E}(\alpha \cdot S)_t \; > \; 0. \end{aligned}$$

This yields that $\theta^*$ is $x$-superadmissible. At the same time, multiplying both extreme terms by $\alpha_t$ shows that $\theta^*$ solves (13.5). □

We now state the main result of this section. It generalizes Theorem 3.5. of [2], where it was proved in the special case of a semimartingale given by an SDE.

**Theorem 13.2.4.** *For any $x > 0$ the following equation holds*

$$u_+(x) = \log x + \frac{1}{2} E \int_0^T \alpha_s^2 \; d\langle M, M\rangle_s. \tag{13.6}$$

*If $E\int_0^T \alpha_s^2 d\langle M, M\rangle_s < \infty$, then the process $\theta^* = x\alpha\mathcal{E}(\alpha \cdot S)$ is the unique optimal portfolio.*

*Proof.* We first assume that $E\int_0^T \alpha_s^2 d\langle M, M\rangle_s < \infty$.

Let $\theta \in \mathcal{A}$ be $x$-superadmissible. Then $x + (\theta \cdot S)_t > 0$ a.s. for all $t \in [0,T]$ and hence we can define a new process by

$$\pi_t = \frac{\theta_t}{x + (\theta \cdot S)_t}, \quad 0 \le t \le T.$$

Since $\pi$ is predictable, the integral $\pi \cdot S$ is defined.

The SDE

$$\begin{aligned} Y_0 &= x, \\ dY_t &= \pi_t Y_t dS_t = Y_t d(\pi \cdot S)_t \end{aligned}$$

is uniquely solved by the process $Y = x\mathcal{E}(\pi \cdot S)$. On the other hand the process $x + (\theta \cdot S)_t$ is also easily seen to be a solution. By uniqueness this implies

$$x + (\theta \cdot S) = x\mathcal{E}(\pi \cdot S). \tag{13.7}$$

In the next step we will show that the expected logarithmic utility of $x + (\theta \cdot S)_T$ is not greater than $\log x + \frac{1}{2}E\int_0^T \alpha_s^2 d\langle M, M\rangle_s$. Applying the inequality $\log z \leq z - 1$, valid for positive $z$, to the product of two positive numbers $a, b$ we get the inequality

$$\log a \leq ab - \log b - 1.$$

If we take $a = x\mathcal{E}(\pi \cdot S)$ and $b = \frac{1}{x}\mathcal{E}(-\alpha \cdot M)$ we obtain

$$\log x\mathcal{E}(\pi \cdot S) \leq \mathcal{E}(\pi \cdot S)\mathcal{E}(-\alpha \cdot M) - \log \frac{1}{x}\mathcal{E}(-\alpha \cdot M) - 1.$$

By Proposition 13.2.1 the product $\mathcal{E}(\pi \cdot S)\mathcal{E}(-\alpha \cdot M)$ is a local martingale. Since it is nonnegative, it is also a supermartingale and therefore by (13.7),

$$\begin{aligned}
E[\log(x + (\theta \cdot S)_T)] &= E[\log x\mathcal{E}(\pi \cdot S)_T] \\
&\leq E[\mathcal{E}(\pi \cdot S)_T\mathcal{E}(-\alpha \cdot M)_T - \log \frac{1}{x}\mathcal{E}(-\alpha \cdot M)_T - 1] \\
&\leq -E[\log \frac{1}{x}\mathcal{E}(-\alpha \cdot M)_T] \\
&= \log x - E\left[-\int_0^T \alpha_t dM_t - \frac{1}{2}\int_0^T \alpha^2 d\langle M, M\rangle\right] \\
&= \log x + \frac{1}{2}E\int_0^T \alpha^2 d\langle M, M\rangle.
\end{aligned}$$

This implies

$$u_+(x) \leq \log x + \frac{1}{2}E\int_0^T \alpha^2 d\langle M, M\rangle.$$

Before we prove that in fact equality holds, we note

$$E\log(x\mathcal{E}(\alpha \cdot S)_T) = \log x + \frac{1}{2}E\int_0^T \alpha^2 d\langle M, M\rangle.$$

Therefore it is enough to show that there is a process $\theta$ such that $E\log(x + (\theta \cdot S)_T) = E\log(x\mathcal{E}(\alpha \cdot S)_T)$.
According to Lemma 13.2.3 the process $\theta^* = x\alpha\mathcal{E}(\alpha \cdot S)$ belongs to $\mathcal{A}$, is $x$-superadmissible and satisfies

$$\alpha = \frac{\theta^*}{x + (\theta^* \cdot S)},$$

from which we deduce

$$x + (\theta^* \cdot S)_t = x\mathcal{E}(\alpha \cdot S)_t.$$

This proves the theorem in the case where $E\int_0^T \alpha_s^2 d\langle M, M\rangle_s < \infty$.

We now claim that equation (13.6) is still true if $E\int_0^T \alpha_s^2 d\langle M,M\rangle_s = \infty$.Since $\int_0^T \alpha_s^2 d\langle M,M\rangle_s < \infty$ almost surely, we can find an increasing sequence of stopping times $(T_n)_{n\in\mathbb{N}}$ such that $T_n \to T$ and

$$E\int_0^{T_n} \alpha_s^2 \, d\langle M,M\rangle_s < \infty.$$

With the first part of the proof we deduce

$$u_+(x) \geq \log x + \frac{1}{2}E\int_0^{T_n} \alpha_s^2 \, d\langle M,M\rangle_s$$

for all $n \in \mathbb{N}$. By Beppo-Levi the right hand side goes to infinity as $n \to \infty$. Hence $u_+(x) = \infty$, which completes the proof. □

## 13.3 Maximal utility if wealth may become negative

Here we allow the wealth process to take negative values and we deduce a representation for $u(x)$. If $S$ allows arbitrage, then $u(x)$ is infinite. Therefore, we assume in this section that $S = M + \alpha \cdot \langle M,M\rangle$ satisfies (NFLVR).

If $\theta \in \mathcal{A}$ is not $x$-superadmissible, then by Proposition 8.3.3

$$(\theta \cdot S)_T \leq -x$$

on a set with positive probability. This implies $E\log(x + (\theta \cdot S)_T) = -\infty$, and therefore $u(x) = u_+(x)$. Hence we have shown:

**Theorem 13.3.1.** *Let $S$ be a continuous semimartingale satisfying (NFLVR). The maximal expected logarithmic utility is given by*

$$u(x) = \log x + \frac{1}{2}E\int_0^T \alpha_s^2 \, d\langle M,M\rangle_s.$$

**Remark 13.3.2.** *Kramkov and Schachermayer [35] show that under the assumption of (NFLVR) a more general result can be obtained. They give explicit formulas for the maximal expected utility not only for the logarithm but for a large class of utility functions.*

*We mention that $E\int_0^T \alpha_s^2 d\langle M,M\rangle_s < \infty$ does not imply the (NFLVR) property. In the following examples the integral of the drift is finite, but arbitrage is possible and hence $u(x)$ is infinite (see Proposition 7.2.2). Hence the assumption of (NFLVR) in Theorem 13.3.1 cannot be dropped.*

**Example 13.3.3.** *Let $S$ be a $BES^3$ process starting in $x > 0$. It is known that $S$ solves the equation*

$$S_t = x + B_t + \int_0^t S_u^{-1}\, du, \quad 0 \leq t,$$

*where $(B_t)$ is a Brownian motion (see Proposition 3.3, Chapter VI in [42]). It is straightforward to show that*

$$E \int_0^T S_u^{-2}\, du < \infty,$$

*and hence, by Theorem 13.2.4, $u_+(x)$ is finite, too. On the other hand Delbaen and Schachermayer prove in [11] that $S$ allows arbitrage.*

*Moreover, this example shows that the assumption (NFLVR) cannot be dropped in Theorem 8.3.4: It is known that there are no simple arbitrage strategies (see [11]). Hence every simple strategy $\theta$ satisfying $U(x+(\theta\cdot S)_T) > 0$, a.s, must be $x$-superadmissible (else one can construct a simple arbitrage strategy). Consequently*

$$\sup_{\mathcal{S}\ni\theta\, adm.} E[U(x + (\theta \cdot S)_T)] \leq u_+(x) < \infty.$$

*Since $S$ allows arbitrage for general strategies, we have $u(x) = \infty$. Thus Theorem 8.3.4 does not hold without the assumption (NFLVR).*

Situations where the trader has finite utility $u_+(x)$, but (NFLVR) is not satisfied, can easily arise on markets with insiders. An insider acts using information from an enlarged filtration. As in the following example, this produces sources for possible arbitrage which, in contrast to the previous example, are very explicit.

**Example 13.3.4.** *Let $W$ be a Brownian motion on some probability space $(\Omega, F, P)$. We denote by $(\mathcal{F}_t)_{t\geq 0}$ the completed filtration generated by $W$. We will study the price process*

$$S_t = \mathcal{E}(W)_t, \quad t \geq 0,$$

not *under $(\mathcal{F}_t)_{t\geq 0}$, but with respect to a larger filtration. Choose for example $T = 1$, let $a, b \in \mathbb{R}$ such that $a < b$, let $G = 1_{[a,b]}(W_1)$, and take the right continuous and completed version of $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(G)$, $t \in [0, 1]$. By Theorem 12.6.1, an agent in this filtration possesses finite logarithmic utility, if wealth has to be positive. $u_+(x)$ is given by the entropy of $G$, or, alternatively, by $\frac{1}{2}E\int_0^1 \alpha_s^2 ds$ with the corresponding information drift $\alpha$.*

*We will see now that there are arbitrage strategies. Define a stopping time by*

$$T = \inf\{t \geq 0 : W_t \leq a - 1\} \wedge 1.$$

*The strategy* $\theta = 1_{\{W_1 \in [a,b]\}} 1_{]T,1]}$ *is admissible, because*

$$(\theta \cdot S)_t \geq -e^{a-1}, \; 0 \leq t \leq 1.$$

*Furthermore* $\theta$ *satisfies*

**i)** $(\theta \cdot S)_1 = 1_{\{W_1 \in [a,b]\}}(S_1 - S_T) \geq 0$ *and*

**ii)** $P((\theta \cdot S)_1 > 0) = P(T < 1, W_1 \in [a,b]) > 0,$

*which shows that* $\theta$ *is an arbitrage strategy. In particular* $S$ *doesn't have the (NFLVR) property.*

# Chapter 14

# Additional logarithmic utility

We now return to the setting of a financial market with agents possessing asymmetric information. We assume that each investor takes his portfolio decisions on the basis of his individual information horizon, given by different filtrations $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$. We just suppose that $\mathcal{F}_t \subset \mathcal{G}_t$, $t \geq 0$, but do not specify at all what the sources for the additional information in $(\mathcal{G}_t)$ are. The asset price process $S$ is again a continuous semimartingale with $S_0 = 0$, and as in the preceding chapter we shall consider logarithmic utility $U = \log$. Our main result shows that in this general setting the utility difference $u_+^{\mathcal{G}}(x) - u_+^{\mathcal{F}}(x)$ is given by

$$\frac{1}{2} E \int_0^T \gamma_s^2 d\langle M, M\rangle_s,$$

where $M$ is the local martingale part in the $(\mathcal{F}_t)$-decomposition and $\gamma$ the information drift obtained by passing from $(\mathcal{F}_t)$ to $(\mathcal{G}_t)$.

## 14.1 Orthogonalizing utility

Let us first specify those agents who possess finite utility on the basis of their knowledge. For this let $(\mathcal{H}_t)$ be a filtration satisfying the usual conditions, and recall that $L^2_{\mathcal{H}}(M)$ is the Hilbert space of all $(\mathcal{H}_t)$-predictable processes $\alpha$ such that $E \int_0^T \alpha^2 \, d\langle M, M\rangle < \infty$.

By Theorem 9.2.7, finite utility $s_+^{\mathcal{H}}(x)$ implies $S$ to be a $(\mathcal{H}_t)$-semimartingale with decomposition $S = M + \alpha \cdot \langle M, M\rangle$ such that $\int_0^T \alpha^2 \, d\langle M, M\rangle < \infty$, a.s. Moreover, Theorem 13.2.4 implies $\alpha \in L^2_{\mathcal{H}}(M)$. Therefore we introduce the following notion.

**Definition 14.1.1.** *A filtration $(\mathcal{H}_t)$ satisfying the usual conditions will be* *called* finite utility filtration *for $S$, if $S$ is a $(\mathcal{H}_t)$-semimartingale with decomposition $S = M + \alpha \cdot \langle M, M\rangle$, and if $\alpha$ belongs to $L^2_{\mathcal{H}}(M)$.*

**Remark 14.1.2.** *In a strict sense, $\alpha \in L^2_{\mathcal{H}}(M)$ is not a process, but a class of processes. In the following $\alpha$ will sometimes refer to the whole class and sometimes only to a single representative.*

*A finite utility filtration for $S$ may not be a finite utility filtration for a different process. Nevertheless, we will often omit the process in the definition since we are always referring to a fixed $S$.*

*Note that $u^{\mathcal{H}}_+(x)$ is finite for all $x > 0$, if $(\mathcal{H}_t)$ is a finite utility filtration (see Theorem 13.2.4).*

We can use elementary Hilbert space methods in order to compare traders with different information. For this let $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$ be two finite utility filtrations and suppose $\mathcal{F}_t \subset \mathcal{G}_t$, $t \geq 0$. We denote by

$$S = M + \alpha \cdot \langle M, M\rangle$$

the semimartingale decomposition with respect to $(\mathcal{F}_t)$ and by

$$S = N + \beta \cdot \langle N, N\rangle$$

the decomposition with respect to $(\mathcal{G}_t)$. Note that

$$M = N - (\alpha - \beta) \cdot \langle M, M\rangle$$

is the $(\mathcal{G}_t)$-semimartingale decomposition of $M$. Moreover, the process $\mu = (\beta - \alpha)$ is the information drift of $(\mathcal{G}_t)$ with respect to $M$.

The utility difference $u^{\mathcal{G}}_+(x) - u^{\mathcal{G}}_+(x)$ depends only on $\mu$. This will follow from the next result.

**Theorem 14.1.3.** *The information drift $\mu$ of $(\mathcal{G}_t)$ relative to $M$ is orthogonal to $L^2_{\mathcal{F}}(M)$. Moreover, $\alpha$ is the orthogonal projection of $\beta$ onto $L^2_{\mathcal{F}}(M)$.*

*Proof.* Let $\theta \in L^2_{\mathcal{F}}(M)$. Since $\theta$ is adapted to both $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$, the integrals $(\theta \cdot M)$ and $(\theta \cdot N)$ are square integrable martingales with expectation zero. Therefore, with Theorem 10.2.1,

$$\begin{aligned} E\int_0^T \theta\mu \; d\langle M, M\rangle &= E\left[\int_0^T \theta\beta \; d\langle M, M\rangle - \int_0^T \theta\alpha \; d\langle M, M\rangle\right] \\ &= E\left[\int_0^T \theta \; dM - \int_0^T \theta \; dN\right] \\ &= 0. \end{aligned}$$

Thus we have shown that $\mu$ is orthogonal to $L^2_{\mathcal{F}}(M)$. It follows immediately that $\alpha$ is the orthogonal projection of $\beta$ onto $L^2_{\mathcal{F}}(M)$. $\square$

An application of the Pythagoras Theorem yields:

**Theorem 14.1.4.** *The utility difference* $\Delta u = u_+^{\mathcal{G}}(x) - u_+^{\mathcal{F}}(x)$ *does not depend on the initial wealth* $x > 0$, *and it satisfies*

$$\begin{aligned} \Delta u &= \frac{1}{2} E \int_0^T (\beta - \alpha)^2 \, d\langle M, M \rangle \\ &= \frac{1}{2} E \int_0^T \mu^2 \, d\langle M, M \rangle . \end{aligned}$$

We see that the additional logarithmic utility depends only on the information drift $\mu$. Therefore, we may write the utility increment in terms of the metric $\delta_2$ introduced in Chapter 4:

**Lemma 14.1.5.** *Let* $\delta_2$ *be defined with respect to the* $(\mathcal{F}_t)$*-local martingale* $M$. *Then*

$$\sqrt{\Delta u} = \delta_2((\mathcal{F}_t), (\mathcal{G}_t)).$$

Moreover, we may characterize finite utility filtrations with the help of the metric $\delta_2$.

**Lemma 14.1.6.** *The filtration* $(\mathcal{G}_t)$ *is a finite utility filtration if and only if* $(\mathcal{F}_t)$ *is a finite utility filtration and* $\delta_2((\mathcal{F}_t), (\mathcal{G}_t)) < \infty$.

*Proof.* The equivalence follows immediately from the definition of finite utility filtrations and Theorem 14.1.4. □

## 14.2 Measuring utility

The results of the previous section allow us to study *how* utility is increasing. The additional logarithmic utility is given by $\frac{1}{2} E \int_0^T \mu^2 \, d\langle M, M \rangle$, where $\mu$ is the information drift. Differentiating with respect to time $t$ shows that the additional utility is increasing with speed $\frac{1}{2} E \left( \mu_t^2 \, \frac{d\langle M, M \rangle_t}{d\lambda} \right)$ at time $t$. We now introduce a measure describing the impact of the new information during the trading interval.

We use again initially enlarged approximations of the filtration $(\mathcal{G}_t)$ modeling the knowledge of the better-informed trader. For $s \in [0, T]$ we set

$$\mathcal{G}_t^s = \begin{cases} \mathcal{F}_t, & t < s \\ \bigcap_{u > t} \mathcal{F}_u \vee \mathcal{G}_s, & t \geq s. \end{cases}$$

In the following, we assume that $(\mathcal{G}_t^s)$ is a finite utility filtration for arbitrary $s \in [0, T]$. Let for $0 \leq s \leq t \leq T$

$$\pi_0([0, s) \times (t, T]) = F(s, t) = \frac{1}{2} E \int_t^T \left( \mu_r^s \right)^2 d\langle M, M \rangle_r ,$$

where $\mu^s$ is a $(\mathcal{G}_t^s)-$information drift. $\pi_0$ is defined only on the set $J = \{[0,s) \times (t,T] : s \leq t\}$. As the next theorem shows, $\pi_0$ can be extended to a measure on the Borel sets of $D = \{(s,t) \in \mathbb{R}^2 : 0 \leq s < t \leq T\}$.

**Theorem 14.2.1.** *There exists a unique measure $\pi$ on the Borel sets $\mathcal{B}(D)$ of $D$ satisfying $\pi|_J = \pi_0$.*

*Proof.* In order to show that $\pi_0$ can be extended to a unique measure on $\mathcal{B}(D)$ it suffices to verify the following statements (see for instance Elstrodt [19] Satz 4.5, Paragraph 4). For any $(s,t) \in \overline{D}$ and any sequence $(s_n,t_n)_{n\in\mathbb{N}}$ in $\overline{D}$ with $s_n \leq s$, $t_n \geq t$ and $\lim_{n\to\infty}(s_n,t_n) = (s,t)$ we have $\lim_{n\to\infty} F(s_n,t_n) = F(s,t)$. Moreover, $F(s_n,t_n) \leq F(s,t) < \infty$.

Let $s_n$, $t_n$, $s$ and $t$ as above. Without loss of generality we assume that $(s_n)$ is monotonically increasing. For $u \in [t,T]$ we consider the filtrations $(\mathcal{G}_r^{s_n})_{r\in[u,T]}$, $n \in \mathbb{N}$, over the time interval $[u,T]$. Since $(\mu_r^{s_n})_{r\in[u,T]}$ are $(\mathcal{G}_r^{s_n})-$information drifts, it follows with Theorem 14.1.3 that

$$E\int_u^T \big(\mu^s - \mu^{s_n}\big)\mu^{s_n}\, d\langle M,M\rangle = 0.$$

In particular,

$$E\int_u^T \big(\mu^{s_n}\big)^2\, d\langle M,M\rangle \leq E\int_t^T \big(\mu^{s}\big)^2\, d\langle M,M\rangle < \infty.$$

By Theorem 4.5.4 the processes $(\mu_r^{s_n})_{r\in[u,T]}$ converge to the information drift $(\mu_r^s)_{r\in[u,T]}$ in $L^2(M;[u,T])$. Therefore, for any $u \in (t,T]$,

$$\liminf_{n\to\infty} E\int_{t_n}^T \big(\mu^{s_n}\big)^2\, d\langle M,M\rangle \geq E\int_u^T \big(\mu^{s}\big)^2\, d\langle M,M\rangle.$$

Due to the continuity of $M$ the right hand side of the previous equation tends to
$E\int_t^T \big(\mu^s\big)^2\, d\langle M,M\rangle$ as $u \downarrow t$. Consequently, we obtain $\lim_{n\to\infty} F(s_n,t_n) = F(s,t)$. □

The measure $\pi$ describes the utility increase. For example, $\pi([0,s)\times(u,t])$, $s \leq u \leq t$, is the impact of the additional information $\mathcal{G}_s$ during the interval $(u,t]$. Moreover, $\pi(D)$ is equal to the total utility increment $\Delta u = u_+^{\mathcal{G}}(x) - u_+^{\mathcal{F}}(x)$. To show this we now approximate the general filtration $(\mathcal{G}_t)$ like in Section 11.4 by successive initial enlargements. Let $\Delta : 0 = s_0 \leq \cdots \leq s_n = T$, $n \in \mathbb{N}$, be a partition of the interval $[0,T]$. We let for $r \in [s_i, s_{i+1})$, $i = 0,\ldots,n-1$,

$$\mathcal{G}_r^\Delta = \bigcap_{u>r} \mathcal{G}_{s_i} \vee \mathcal{F}_u.$$

Similarly, let $\mu^{s_i}$ be a $(\mathcal{G}_r^{s_i})-$information drift for $i = 0, \dots, n-1$ and set $\mu_r^\Delta = \mu_r^{s_i}$ for $r \in [s_i, s_{i+1})$. Obviously $\mu^\Delta$ is the $(\mathcal{G}_t^\Delta)$-information drift, and therefore $u_+^{\mathcal{G}^\Delta} - u_+^{\mathcal{F}} = \frac{1}{2} E \int_0^T (\mu^\Delta)^2 \, d\langle M, M\rangle$. Moreover, it is straightforward to show that

$$u_+^{\mathcal{G}^\Delta} - u_+^{\mathcal{F}} = \pi(D^\Delta),$$

where $D^\Delta = \{(s,t) \in D : \exists i \in \{0, \dots, n-1\} \text{ with } s < s_i \text{ and } t > s_i\}$. We can now express the total utility gain with the help of $\pi$.

**Theorem 14.2.2.**

$$\Delta u = \pi(D).$$

*Proof.* Let $\Delta_n$, $n \in \mathbb{N}$, be a sequence of partitions such that for all $n \in \mathbb{N}$, $\Delta_{n+1}$ is a refinement of $\Delta_n$, and $\lim_n |\Delta_n| = 0$. Then $\mathcal{G}_t^{\Delta_n} \subset \mathcal{G}_t^{\Delta_{n+1}}$, $t \in [0,T]$, and Theorem 11.4.1 implies $\lim_n u_+^{\mathcal{G}^{\Delta_n}} = u_+^{\mathcal{G}}$. Therefore, by using monotone convergence,

$$\pi(D) = \lim_{n\to\infty} \pi(D^{\Delta_n}) = \lim_n u_+^{\mathcal{G}^{\Delta_n}} - u_+^{\mathcal{F}} = \Delta u.$$

□

The preceding theorem shows that the convergence result in Theorem 11.4.1 is true not only for sequences $(\Delta_n)$ for which $\Delta_{n+1}$ is a refinement of $\Delta_n$, $n \geq 1$.

**Proposition 14.2.3.** *Let $(\Delta_n)$ be an arbitrary sequence of partitions of the interval $[0,T]$ satisfying $\lim_n |\Delta_n| = 0$. Then, for $x > 0$,*

$$\lim u_+^{\mathcal{G}^{\Delta_n}}(x) = u_+^{\mathcal{G}}(x).$$

*Proof.* If the mesh of the partitions $\Delta_n$ tends to zero, then by dominated convergence, $\lim_{n\to\infty} \pi(D^{\Delta_n}) = \pi(D)$, and therefore the result. □

## 14.3 Shannon information and additional utility

We continue now to analyze the link between the additional logarithmic utility and the entropy of the new information.

Throughout this chapter let $S$ be a continuous price process, $(\mathcal{F}_t)$ a finite utility filtration for $S$, and $M$ the continuous $(\mathcal{F}_t)$-local martingale part. Moreover, we assume that $M$ satisfies the (PRP) relative to $(\mathcal{F}_t)$.

The expected logarithmic utility increment is given by an integral version of relative entropies of conditional probabilities, which we will interpret as *Shannon information difference* between filtrations.

Our techniques require the existence of conditional probabilities. Therefore, we make the same assumptions as in Chapter 5. In particular, we assume that our probability space is standard and we distinguish again between the countably generated filtration $(\mathcal{F}_t^0)$ and the completed right-continuous filtration $(\mathcal{F}_t)$. Since we assume (PRP), the regular conditional probabilities relative to the $\sigma$-fields $\mathcal{F}_t^0$ satisfy

$$P_t(\cdot, A) = P(A) + \int_0^t k_s(\cdot, A)\, dM_s,$$

where $k(\cdot, A)$ is $(\mathcal{F}_t)$-predictable. Let again $(\mathcal{G}_t^0)$ be a filtration satisfying $\mathcal{F}_t^0 \subset \mathcal{G}_t^0$ and being generated by countably many sets. To simplify notation we assume the filtration $(\mathcal{G}_t^0)$ to be left-continuous. Let $(\mathcal{G}_t)$ be the smallest completed and right-continuous filtration containing $(\mathcal{G}_t^0)$. In the following, we assume that $(\mathcal{G}_t)$ is a finite utility filtration and denote by $\mu$ its information drift. Recall that by Theorem 5.1.9 we may assume that $k_t(\omega, \cdot)$ is a signed measure. For fixed $r > 0$, let $\mu^r$ be the information drift of the initially enlarged filtration $(\mathcal{G}_t^r)$, defined as in the end of Chapter 11. For stating the main result we need the following lemma.

**Lemma 14.3.1.** *Let $0 \le s < t$ and $(\mathcal{P}^m)_{m \ge 0}$ an increasing sequence of finite partitions such that $\sigma(\mathcal{P}^m : m \ge 0) = \mathcal{G}_s^0$. Then*

$$\lim_m E \int_s^t \sum_{A \in \mathcal{P}^m} \left( \frac{k_u}{P_u} \right)^2 (\cdot, A)\, 1_A \; d\langle M, M\rangle_u \quad = \quad E \int_s^t (\mu_u^s)^2 \; d\langle M, M\rangle_u$$

*and*

$$\lim_m E \int_s^t \sum_{A \in \mathcal{P}^m} \frac{k_u}{P_u}(\cdot, A)\, 1_A\, \mu_u^s \; d\langle M, M\rangle_u \quad = \quad E \int_s^t (\mu_u^s)^2 \; d\langle M, M\rangle_u.$$

*Proof.* By Lemma 5.1.8 the process

$$Y_u^m(\omega, \omega') = \sum_{A \in \mathcal{P}^m} \frac{k_u}{P_u}(\omega, A) 1_A(\omega'), \; m \ge 1,$$

is a $L^2(P_u(\omega, \cdot))$-bounded martingale for $P_M-$a.a. $(\omega, u) \in \Omega \times [s, t]$. Hence $(Y_u^m(\omega, \cdot))$ converges to the density

$$\gamma_u = \left. \frac{k_u(\omega, d\omega')}{P_u(\omega, d\omega')} \right|_{\mathcal{G}_s^0}$$

for $P_M$-a.a. $(\omega, u) \in \Omega \times [s,t]$, and uniform integrability implies

$$\lim_m \int (Y_u^m)^2(\cdot,\omega')\, P_u(\cdot, d\omega') = \int \gamma_u^2(\cdot,\omega')\, P_u(\cdot, d\omega').$$

By Theorem 5.1.6 we have

$$\gamma_u(\omega,\omega) = \mu_u^s(\omega)$$

$P_M-$a.s. on $\Omega \times [s,t]$. The martingale property implies that the sequence $\left(\int (Y_u^m)^2(\cdot,\omega')\, dP_u(\cdot, d\omega')\right)_m$ is increasing and hence, by monotone convergence,

$$\begin{aligned} & \lim_m E \int_s^t \int (Y_u^m)^2(\cdot,\omega')\, P_u(\cdot, d\omega')\, d\langle M, M\rangle_u\, dP \\ = \;& E \int_s^t \int \gamma_u^2(\cdot,\omega')\, P_u(\cdot, d\omega')\, d\langle M, M\rangle_u\, dP \\ = \;& E \int_s^t (\mu_u^s)^2\, d\langle M, M\rangle_u\, dP. \end{aligned}$$

In a similar way one can prove the second statement. □

We next discuss the concept of the additional information of a $\sigma-$field relative to a filtration.

**Definition 14.3.2.** *The* additional information *of $\mathcal{A}$ relative to the filtration $(\mathcal{F}_r)$ on $[s,t]$ $(0 \le s < t \le T)$ is defined by*

$$H_{\mathcal{A}}(s,t) = \int \mathcal{H}_{\mathcal{A}}(P_t(\omega,\cdot) \| P_s(\omega,\cdot))\, dP(\omega).$$

The following lemma establishes the basic link between the entropy of a filtration enlargement and additional logarithmic utility of a trader possessing this information advantage.

**Lemma 14.3.3.** *For $0 \le s < t$ we have*

$$H_{\mathcal{G}_s^0}(s,t) = \frac{1}{2} E \int_s^t (\mu_u^s)^2 \;\, d\langle M, M\rangle_u.$$

*Proof.* Let $(\mathcal{P}^m)_{m\ge 0}$ be an increasing sequence of finite partitions such that $\sigma(\mathcal{P}^m : m \ge 0) = \mathcal{G}_s^0$. Recall that by equation (5.2) in Chapter 5.2

$$\begin{aligned} & \sum_{A\in\mathcal{P}^m} [1_A \log P_s(\cdot, A) - 1_A \log P_t(\cdot, A)] \\ = \;& \sum_{A\in\mathcal{P}^m} \Bigg[ -\int_s^t \frac{k_u}{P_u}(\cdot, A)\, 1_A\; d\tilde{M}_u - \int_s^t \frac{k_u}{P_u}(\cdot, A)\, 1_A \mu_u\; d\langle M, M\rangle_u \\ & + \frac{1}{2} \int_s^t \left(\frac{k_u}{P_u}\right)^2 (\cdot, A)\, 1_A\; d\langle M, M\rangle_u \Bigg] \end{aligned}$$

Since $\tilde{M}$ is a local martingale, we obtain by stopping and taking limits if necessary

$$
\begin{aligned}
& E \sum_{A \in \mathcal{P}^m} P_s(\cdot, A) \log \frac{P_t(\cdot, A)}{P_s(\cdot, A)} \\
= \ & E \sum_{A \in \mathcal{P}^m} \left[ \int_s^t \frac{k_u}{P_u}(\cdot, A)\, 1_A \mu_u \; d\langle M, M \rangle_u - \frac{1}{2} \int_s^t \left( \frac{k_u}{P_u} \right)^2 (\cdot, A)\, 1_A \; d\langle M, M \rangle_u \right].
\end{aligned}
$$

Note that in the previous line $\mu$ may be replaced by $\mu^s$, because $(\mu - \mu^s)$ is orthogonal to $L^2(M)(\mathcal{G}^s)$ (see Theorem 14.1.3). Applying Lemma 14.3.1 yields

$$\lim_m H_{\mathcal{P}^m}(s,t) = \frac{1}{2} E \int_s^t (\mu_u^s)^2 \; d\langle M, M \rangle_u .$$

Fatou's Lemma implies

$$\liminf_m H_{\mathcal{P}^m}(s,t) \geq H_{\mathcal{G}_s^0}(s,t).$$

On the other hand we have $H_{\mathcal{P}^m}(s,t) \leq H_{\mathcal{G}_s^0}(s,t)$, since $\mathcal{P}^m \subset \mathcal{G}_s^0$, and thus

$$\lim_m H_{\mathcal{P}^m}(s,t) = H_{\mathcal{G}_s^0}(s,t),$$

which completes the proof. □

Let us now return to the approximation of a filtration by initial enlargements.

**Definition 14.3.4.** *Let $\Delta : 0 = s_0 \leq \cdots \leq s_n = T$, $n \in \mathbb{N}$, be a partition of the interval $[0,T]$ and let $\mu^\Delta$ be the information drift of $(\mathcal{G}_r^\Delta)$. The additional information of $(\mathcal{G}_r^\Delta)$ relative to $(\mathcal{F}_r)$ is defined as*

$$H_\Delta = \sum_{i=0}^{n-1} H_{\mathcal{G}_{s_i}^0}(s_i, s_{i+1}).$$

**Theorem 14.3.5.** *We have*

$$\lim_{|\Delta| \to 0} H_\Delta = \frac{1}{2} E \int_0^T \mu_u^2 \; d\langle M, M \rangle_u .$$

*Proof.* This follows from Proposition 14.2.3 and Lemma 14.3.3. □

**Definition 14.3.6.** *Let $X$, $Y$ and $Z$ be three random variables in some measurable spaces. The* conditional mutual information *of $X$ and $Y$ given $Z$ is defined by*

$$I(X,Y|Z) = \mathbb{E}\Big[\mathcal{H}(P_{X,Y|Z} \| P_{X|Z} \otimes P_{Y|Z})\Big],$$

*provided the regular conditional probabilities exist.*

*Remember that if $\mathcal{A}$ is a sub-$\sigma$-algebra of $\mathcal{F}$, then we write $\mathrm{id}_{\mathcal{A}}$ for the measurable map $(\Omega, \mathcal{F}) \to (\Omega, \mathcal{A})$, $\omega \mapsto \omega$. For two sub-$\sigma$-algebras $\mathcal{A}$ and $\mathcal{D}$ we abbreviate*

$$I(\mathcal{A}, \mathcal{D}) = I(\mathrm{id}_{\mathcal{A}}, \mathrm{id}_{\mathcal{D}}).$$

*Since our probability space is standard, for any sub-$\sigma$-fields $\mathcal{A}, \mathcal{D}, \mathcal{E}$ of $\mathcal{F}$, there exists a regular conditional probability $P_{\mathrm{id}_{\mathcal{A}},\mathrm{id}_{\mathcal{D}}|\mathrm{id}_{\mathcal{E}}}$, and we define*

$$I(\mathcal{A}, \mathcal{D}|\mathcal{E}) = I(\mathrm{id}_{\mathcal{A}}, \mathrm{id}_{\mathcal{D}}|\mathrm{id}_{\mathcal{E}}).$$

**Theorem 14.3.7.**

$$\lim_{|\Delta|\to 0} \sum_i I(\mathcal{G}^0_{s_i}, \mathcal{F}^0_{s_{i+1}}|\mathcal{F}^0_{s_i}) = \frac{1}{2} E \int_0^T \mu_u^2 \; d\langle M, M\rangle_u.$$

*Proof.* Note that for three random variables $X, Y$ and $Z$ we have

$$\frac{dP_{(X,Y)|Z}}{d(P_{X|Z} \otimes P_{Y|Z})} = \frac{dP_{X|(Y,Z)}}{dP_{X|Z}}.$$

This property implies that one has for $0 \le s < t \le T$,

$$\begin{aligned} I(\mathcal{G}^0_s, \mathcal{F}^0_t|\mathcal{F}^0_s) &= \int\int \log \frac{dP_{\mathrm{id}_{\mathcal{G}^0_s}|\mathrm{id}_{\mathcal{F}^0_t}}}{dP_{\mathrm{id}_{\mathcal{G}^0_s}|\mathrm{id}_{\mathcal{F}^0_s}}} \; dP(\omega') \; dP(\omega) \\ &= \int\int \log \frac{P_t(\cdot, d\omega')}{P_s(\cdot, d\omega')}\bigg|_{\mathcal{G}^0_s} \; dP(\omega') \; dP(\omega) \\ &= H_{\mathcal{G}^0_s}(s,t). \end{aligned}$$

Thus the assertion is an immediate consequence of Theorem 14.3.5. □

This result motivates the following notion.

**Definition 14.3.8.** *The* information difference *of $(\mathcal{G}^0_r)$ relative to $(\mathcal{F}^0_r)$ up to time $T$ is defined as*

$$A(\mathcal{G}^0, \mathcal{F}^0) = \lim_{|\Delta|\to 0} \sum_i I(\mathcal{G}^0_{s_i}, \mathcal{F}^0_{s_{i+1}}|\mathcal{F}^0_{s_i}).$$

**Remark 14.3.9.** *Note that we did not use $M$ in our definition of the information difference of $(\mathcal{G}_r^0)$ relative to $(\mathcal{F}_r^0)$. However, by Theorem 14.3.7, the information difference may be represented in terms of any local martingale satisfying the (PRP).*

*The information difference of two filtrations is related to the metric $\delta_2$ introduced in Chapter 4. Namely,*

$$\sqrt{A(\mathcal{G}^0, \mathcal{F}^0)} = \delta_2((\mathcal{G}_t), (\mathcal{F}_t)).$$

Theorem 14.3.7 can be reformulated in the following way.

**Theorem 14.3.10.** *The additional utility of an agent with information $(\mathcal{G}_t)$ is equal to the information difference of $(\mathcal{G}_r^0)$ relative to $(\mathcal{F}_r^0)$, i.e.*

$$\Delta u = A(\mathcal{G}^0, \mathcal{F}^0).$$

If $(\mathcal{G}_t)$ is initially enlarged by some random variable $G$, then the information difference of $(\mathcal{G}_r^0)$ relative to $(\mathcal{F}_r^0)$ coincides with the Shannon information between $G$ and $(\mathcal{F}_T^0)$.

**Lemma 14.3.11.** *Let $\mathcal{G}_t^0 = \mathcal{F}_t^0 \vee \sigma(G)$, where $G$ is a random variable with values in some Polish space. Then*

$$A(\mathcal{G}^0, \mathcal{F}^0) = I(G, \mathcal{F}_T^0 | \mathcal{F}_0^0).$$

*Proof.* Let $0 \leq s \leq t$. By standard arguments we have

$$I(\mathcal{G}_s^0, \mathcal{F}_t^0 | \mathcal{F}_s^0) = I(G, \mathcal{F}_t^0 | \mathcal{F}_s^0)$$

and

$$\begin{aligned} I(G, \mathcal{F}_t^0 | \mathcal{F}_0^0) &= I(G, (\mathcal{F}_t^0, \mathcal{F}_s^0) | \mathcal{F}_0^0) \\ &= I(G, \mathcal{F}_t^0 | \mathcal{F}_s^0) + I(G, \mathcal{F}_s^0 | \mathcal{F}_0^0) \end{aligned}$$

(see e.g. [25] Theorem 1.6.3.) By iteration we obtain for all partitions $\Delta$

$$\sum_i I(\mathcal{G}_{s_i}^0, \mathcal{F}_{s_{i+1}}^0 | \mathcal{F}_{s_i}^0) = I(G, \mathcal{F}_T^0 | \mathcal{F}_0^0),$$

and hence the result. □

The results of Thereom 12.6.1 may again be deduced from Lemma 14.3.11 and Theorem 14.3.10. However, here we do not assume $\mathcal{F}_0$ to be trivial, and therefore we obtain a representation in terms of the conditional information:

the additional logarithmic utility of an agent with information $(\mathcal{G}_t)$ is equal to the Shannon information between $G$ and $(\mathcal{F}_T^0)$ *conditioned* on $\mathcal{F}_0$, i.e.

$$\Delta u = I(\mathcal{F}_T^0, G|\mathcal{F}_0^0).$$

Moreover, if $\mathcal{G}_t^0 = \mathcal{F}_t^0 \vee \sigma(G)$ and $G$ is $\mathcal{F}_T^0-$measurable, then the mutual information $I(\mathcal{F}_T^0, G|\mathcal{F}_0^0)$ is equal to the conditional absolute entropy of $G$ (see also [2]).

Let us finish the section with an example for a non-inital enlargement.

**Example 14.3.12.** *We consider the classical stock market model with one asset. Let $(\mathcal{F}_t^0)_{t\in[0,1]}$ be a Brownian filtration generated by the Brownian motion $(B_t)_{t\in[0,1]}$ and denote by $(\mathcal{F}_t)$ its completion. The stock price is modeled by the process*

$$S_t = S_0 \exp\{B_t + bt\},$$

*where $S_0 > 0$ is the deterministic stock price at time 0 and $b \in \mathbb{R}$.*

*The knowledge of the insider at time $t$ is modeled by $\mathcal{G}_t = \bigcap_{r>t} \mathcal{F}_r \vee \sigma((G_s)_{s\in[0,r]})$, where $G_t := B_1 + \tilde{B}_{g(1-t)}$, $(\tilde{B}_t)$ is a Brownian motion independent of $(B_t)$ and $g : [0,1] \to [0,\infty)$ is a decreasing function. We are therefore in a setting similar to Example 5.2.2. We now calculate the utility increment from the perspective of the notion of information difference of filtrations. For $0 \le s \le t \le 1$ we have*

$$\begin{aligned} I(\mathcal{G}_s^0, \mathcal{F}_t^0|\mathcal{F}_s^0) &= I((G_u)_{u\in[0,s]}, \mathcal{F}_t^0|\mathcal{F}_s^0) \\ &= I(G_s, \mathcal{F}_t^0|\mathcal{F}_s^0) = I(G_s, B_t|\mathcal{F}_s^0) + I(G_s, \mathcal{F}_t^0|\mathcal{F}_s^0, B_t) \\ &= I(B_1 + \tilde{B}_{g(1-s)}, B_t - B_s|\mathcal{F}_s^0) \\ &= I(B_1 - B_s + \tilde{B}_{g(1-s)}, B_t - B_s). \end{aligned}$$

*Recall that the differential entropy $h(Y)$ of a random variable $Y$ is defined as the relative entropy of its distribution with respect to Lebesgue measure. Using the formula for the differential entropy for Gaussian measures we obtain*

$$\begin{aligned} I(\mathcal{G}_s^0, \mathcal{F}_t^0|\mathcal{F}_s^0) &= h(B_1 - B_s + \tilde{B}_{g(1-s)}) - h(B_1 - B_t + \tilde{B}_{g(1-s)}) \\ &= \frac{1}{2}\log(2\pi e(1 - s + g(1-s))) - \frac{1}{2}\log(2\pi e(1 - t + g(1-s))) \\ &= \frac{1}{2}\log\frac{1 - s + g(1-s)}{1 - t + g(1-s)} \end{aligned}$$

*Alternatively one can express $I(\mathcal{G}_s^0, \mathcal{F}_t^0|\mathcal{F}_s^0)$ as*

$$I(\mathcal{G}_s^0, \mathcal{F}_t^0|\mathcal{F}_s^0) = \frac{1}{2}\int_s^t \frac{1}{1 - u + g(1-s)}\, du.$$

*For a partition* $\Delta : 0 = t_0 \le \cdots \le t_m = 1$ *(*$m \in \mathbb{N}$*) one has*

$$I_\Delta = \sum_{i=0}^{m-1} I(\mathcal{G}^0_{t_i}, \mathcal{F}^0_{t_{i+1}} | \mathcal{F}^0_{t_i}) = \frac{1}{2}\int_0^1 \frac{1}{1 - u + g(1 - \max\{t_i : t_i \le u\})}\, du$$

*Next, choose a sequence of refining partitions* $(\Delta_n)$ *such that their mesh tends to* 0*. Then the term in the latter integral is monotonically increasing in* $n$ *and convergent. Hence, one obtains*

$$A(\mathcal{G}^0, \mathcal{F}^0) = \lim_{n\to\infty} I_{\Delta_n} = \frac{1}{2}\int_0^1 \frac{1}{1 - u + g(1-u)}\, du.$$

*On the other hand,*

$$A(\mathcal{G}^0, \mathcal{F}^0) = \Delta u = u_+^{\mathcal{G}}(x) - u_+^{\mathcal{F}}(x).$$

*Consequently the insider has finite utility if and only if* $\int_0^1 \frac{1}{1-u+g(1-u)}\, du < \infty$*. Now suppose* $g(y) = Cy^p$ *for some* $C > 0$ *and* $p > 0$*. It is straightforward to show that the integral, and hence the additional utility, is finite if and only if* $p \in (0, 1)$*. This equivalence follows also from results in [9], where the authors compute explicitly the information drift.*

# Conclusion

In this chapter we have seen how the additional logarithmic utility can be calculated or estimated by means of the ‘mutual information’. We have shown roughly the following: If the market is complete, then the additional utility is equal to the mutual information between the old and the new information. If the market is incomplete, then the mutual information is only an upper bound, since it may include information being irrelevant for traders.

As a consequence, we now face the problem that we can choose between two methods of determining the additional utility: either we determine the information drift or we calculate the mutual information. In practice it can be very hard to determine the information drift and to calculate its integral. In these cases it may be more appropriate to determine the mutual information, although it may only provide an upper bound.

The link to information theory can be further exploited in order to gain more insight into how utility increases due to additional information. This has partly been done in [3], where solutions to the entropy maximization problem are used in order to obtain bounds for the mutual information and thus the additional logarithmic utility.

# Acknowledgement

It was a pleasure to work under the supervision of P. Imkeller. I gratefully acknowledge his assistance, his very inspiring and fruitful remarks and his willingness for always taking a lot of time for discussions. I am very grateful to H. Föllmer and to A. Kohatsu-Higa for accepting the task of being co-examiners and for showing interest in my work. Special thanks go to S. Dereich. A lot of the results presented in this thesis are the result of long, intense and very joyful discussions with him. I incredibly appreciated working with friends and collegues who were always willing to give help and encouragement: Matthias an der Heiden, Peter Arts, André Depperschmidt, Georgi Dimitroff, Felix Esche, Anne Gundel, Mesrop Janunts, Anton Klimovsky, Matthias Müller, Stefan Sturm, Sina Tutsch, Christian Vormoor, Stefan Weber, Wiebke Wittmüß, Henryk Zähle. Financial support by the Deutsche Forschungsgemeinschaft via Graduiertenkolleg "Stochastische Prozesse und Probabilistische Analysis" is gratefully acknowledged.

# Bibliography

[1] J. Amendinger, D. Becherer, and M. Schweizer. A monetary value for initial information in portfolio optimization. *Finance Stoch.*, 7(1):29–46, 2003.

[2] J. Amendinger, P. Imkeller, and M. Schweizer. Additional logarithmic utility of an insider. *Stochastic Process. Appl.*, 75(2):263–286, 1998.

[3] S. Ankirchner, S. Dereich, and P. Imkeller. The Shannon information of filtrations and the additional logarithmic utility of insiders. *to appear in Annals of Probability*, 2005.

[4] J.-P. Ansel and Ch. Stricker. Lois de martingale, densités et décomposition de Föllmer-Schweizer. *Ann. Inst. H. Poincaré Probab. Statist.*, 28 (3):375–392, 1992.

[5] F. Baudoin. Conditioning of brownian functionals and applications to the modelling of anticipations on a financial market. PhD thesis, Université Pierre et Marie Curie, 2001.

[6] F. Baudoin and M. Thieullen. Pinning class of the Wiener measure by a functional: related martingales and invariance properties. *Probab. Theory Related Fields*, 127(1):1–36, 2003. ISSN 0178-8051.

[7] F. Biagini and B. Oksendal. A general stochastic calculus approach to insider trading. Preprint, 2003.

[8] D. L. Cohn. *Measure theory.* Birkhäuser Boston, Mass., 1980. ISBN 3-7643-3003-1.

[9] J. Corcuera, P Imkeller, A. Kohatsu-Higa, and D. Nualart. Additional utility of insiders with imperfect dynamical information. *Preprint*, September 2003.

[10] F. Delbaen and W. Schachermayer. A general version of the fundamental theorem of asset pricing. *Mathematische Annalen*, 300:463–520, 1994.

[11] F. Delbaen and W. Schachermayer. Arbitrage possibilities in Bessel processes and their relations to local martingales. *Probab. Theory Related Fields*, 102(3):357–366, 1995. ISSN 0178-8051.

[12] F. Delbaen and W. Schachermayer. The existence of absolutely continuous local martingale measures. *Ann. Appl. Probab.*, 5(4):926–945, 1995. ISSN 1050-5164.

[13] F. Delbaen and W. Schachermayer. The variance-optimal martingale measure for continuous processes. *Bernoulli*, 2(1):81–105, 1996.

[14] C. Dellacherie and P.A. Meyer. *Probabilités et potentiel. Chapitres V à VIII*, volume 1385 of *Actualités Scientifiques et Industrielles [Current Scientific and Industrial Topics]*. Hermann, Paris, revised edition, 1980. ISBN 2-7056-1385-4. Théorie des martingales. [Martingale theory].

[15] C. Dellacherie, P.A. Meyer, and B. Maisonneuve. *Probabilities and Potential. Chapitres XVII - XXIV*. Hermann, Paris, 1992.

[16] C. Dellacherie and M. Weil, editors. *Séminaire de Probabilités. XII*, volume 649 of *Lecture Notes in Mathematics*. Springer, Berlin, 1978. ISBN 3-540-08761-3. Held at the University of Strasbourg, Strasbourg, 1976/1977.

[17] D. Duffie and C. Huang. Multiperiod security markets with differential information: martingales and resolution times. *J. Math. Econom.*, 15 (3):283–303, 1986. ISSN 0304-4068.

[18] R. Durrett. *Probability: theory and examples.* Duxbury Press, Belmont, CA, second edition, 1996. ISBN 0-534-24318-5.

[19] J. Elstrodt. *Maß- und Integrationstheorie. (Measure and integration theory).* Springer-Lehrbuch. Berlin: Springer, 1996.

[20] H. Föllmer and P. Imkeller. Anticipation cancelled by a Girsanov transformation: a paradox on Wiener space. *Ann. Inst. H. Poincaré Probab. Statist.*, 29(4):569–586, 1993. ISSN 0246-0203.

[21] R. K. Getoor and M. J. Sharpe. Conformal martingales. *Invent. Math.*, 16:271–308, 1972.

[22] A. Grorud and M. Pontier. Insider trading in a continuous time market model. *International Journal of Theoretical and Applied Finance*, 1: 331–347, 1998.

[23] A. Gundel. Robust utility maximization for complete and incomplete market models. *Finance Stoch.*, 9, 2005.

[24] S. W. He and J. G. Wang. The property of predictable representation of the sum of independent semimartingales. *Z. Wahrsch. Verw. Gebiete*, 61(1):141–152, 1982. ISSN 0044-3719.

[25] S. Ihara. *Information theory for continuous systems*. Singapore: World Scientific, 1993.

[26] P. Imkeller. Enlargement of the Wiener filtration by an absolutely continuous random variable via Malliavin's calculus. *Probab. Theory Related Fields*, 106(1):105–135, 1996. ISSN 0178-8051.

[27] P. Imkeller. Random times at which insiders can have free lunches. *Stochastics and Stochastics Reports*, 74:465–487, 2002.

[28] P. Imkeller. Malliavin's calculus in insider models: additional utility and free lunches. *Math. Finance*, 13(1):153–169, 2003. Conference on Applications of Malliavin Calculus in Finance (Rocquencourt, 2001).

[29] P. Imkeller, M. Pontier, and F. Weisz. Free lunch and arbitrage possibilities in a financial market model with an insider. *Stochastic Process. Appl.*, 92(1):103–130, 2001. ISSN 0304-4149.

[30] J. Jacod. *Calcul stochastique et problèmes de martingales*, volume 714 of *Lecture Notes in Mathematics*. Springer, Berlin, 1979. ISBN 3-540-09253-6.

[31] J. Jacod. Grossissement initial, hypothese (H'), et théorème de Girsanov. In Th. Jeulin and M. Yor, editors, *Grossissements de filtrations: exemples et applications*, pages 15–35. Springer-Verlag, 1985.

[32] Th. Jeulin. *Semi-martingales et grossissement d'une filtration*, volume 833 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980. ISBN 3-540-10265-5.

[33] Th. Jeulin and M. Yor, editors. *Grossissements de filtrations: exemples et applications*, volume 1118 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1985. ISBN 3-540-15210-5. Papers from the seminar on stochastic calculus held at the Université de Paris VI, Paris, 1982/1983.

[34] A. Kohatsu-Higa and A. Sulem. Utility maximization in an insider influenced market. Preprint, September 2004.

[35] D. Kramkov and W. Schachermayer. The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Ann. Appl. Probab.*, 9(3):904–950, 1999. ISSN 1050-5164.

[36] E. Lenglart. Transformation des martingales locales par changement absolument continu de probabilités. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 39(1):65–70, 1977.

[37] F. Liese and I. Vajda. *Convex statistical distances*, volume 95 of *Teubner-Texte zur Mathematik [Teubner Texts in Mathematics]*. BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1987. ISBN 3-322-00428-7. With German, French and Russian summaries.

[38] D. Nualart. *The Malliavin calculus and related topics*. Probability and its Applications (New York). Springer-Verlag, New York, 1995. ISBN 0-387-94432-X.

[39] K.R. Parthasarathy. *Introduction to probability and measure.* Delhi etc.: MacMillan Co. of India Ltd. XII, 1977.

[40] I. Pikovsky and I. Karatzas. Anticipative portfolio optimization. *Adv. in Appl. Probab.*, 28(4):1095–1122, 1996. ISSN 0001-8678.

[41] Ph. Protter. *Stochastic Integration and Differential Equations*. Springer-Verlag, New York, second edition, 2004.

[42] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

[43] R. T. Rockafellar. *Convex analysis*. Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970.

[44] F. Russo and P. Vallois. Forward, backward and symmetric stochastic integration. *Probab. Theory Related Fields*, 97(3):403–421, 1993. ISSN 0178-8051.

[45] M. Schweizer. On the minimal martingale measure and the Föllmer-Schweizer decomposition. *Stochastic Anal. Appl.*, 13(5):573–599, 1995. ISSN 0736-2994.

[46] C. Stricker and M. Yor. Calcul stochastique dépendant d'un paramètre. *Z. Wahrsch. Verw. Gebiete*, 45(2):109–133, 1978. ISSN 0044-3719.

[47] C. Yoeurp. Grossissement de filtration et théorème de girsanov généralisé. In Th. Jeulin and M. Yor, editors, *Grossissements de filtrations: exemples et applications*. Springer-Verlag, 1985.

[48] M. Yor. Entropie d'une partition, et grossissement initial d'une filtration. In Th. Jeulin and M. Yor, editors, *Grossissements de filtrations: exemples et applications*, pages 45–58. Springer-Verlag, 1985.

[49] M. Yor. *Some aspects of Brownian motion. Part II.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1997. ISBN 3-7643-5717-7. Some recent martingale problems.

# Selbständigkeitserklärung

Hiermit erkläre ich, die vorliegende Arbeit selbständig ohne fremde Hilfe verfaßt und nur die angegebene Literatur und Hilfsmittel verwendet zu haben.
Stefan Ankirchner

22. März 2005