Nonlinear differential-algebraic equations with properly formulated leading term

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1 Introduction

In [BaMä], a uniform theory for investigating linear differential-algebraic equations (DAEs) and their adjoint equations was proposed. By means of an additional coefficient matrix it is exactly fixed which derivatives of the solutions searched for are actually involved in the equation. Such a DAE is of the form

$$A(t)(D(t)x(t))' + B(t)x(t) = q(t), t \in \mathcal{I}, \tag{1.1}$$

where the coefficients $A(t)$ and $D(t)$ match well.
As a nonlinear version,

$$A(x(t), t)(D(t)x(t))' + b(x(t), t) = 0, \tag{1.2}$$

can be taken into account first (e.g. [HiMä]). However, with somewhat more consistency, we obtain equations of the form

$$A(x(t), t)(d(x(t), t))' + b(x(t), t) = 0, \tag{1.3}$$

which we want to investigate in this paper.
Note that it is just this sort of equations that result in the simulation of electric circuits, which is the origin of DAEs (e.g. [EsTi]). In this case, a transition to

$$A(x(t), t)d_x(x(t), t)x'(t) + b(x(t), t) + A(x(t), t)d_t(x(t), t) = 0$$

is problematical.
In the next section (§2) we will determine what we mean by a properly formulated leading term. In §3 DAEs with index μ, $\mu \in \{1, 2\}$ will be characterized by algebraic criteria. §4 is devoted to linearization and perturbation theorems. A special structure, which is important for electric circuits for instance, will be analyzed in §5. Finally, constraint sets will be investigated in §6.

2 Properly formulated leading terms and an equivalence theorem

We investigate the equation

$$A(x(t), t)(d(x(t), t))' + b(x(t), t) = 0, \tag{2.1}$$

with coefficient functions $A(x,t) \in L(\mathbb{R}^n, \mathbb{R}^m), d(x,t) \in \mathbb{R}^n$ and $b(x,t) \in \mathbb{R}^m, x \in \mathcal{D} \subseteq \mathbb{R}^m, t \in \mathcal{I} \subseteq \mathbb{R}$, that are continuous in their arguments, and which have the continuous partial derivatives A_x, d_x, b_x. Denote $D(x,t) = d_x(x,t)$.

Definition 2.1 *The leading term in* (2.1) *is said to be properly formulated if*

$$\ker A(x,t) \oplus \ im \ D(x,t) = \mathbb{R}^n, x \in \mathcal{D}, t \in \mathcal{I}, \tag{2.2}$$

and if there exists a projector function $R \in C^1(\mathcal{I}, L(\mathbb{R}^n))$ such that $R(t)^2 = R(t)$, $\ker A(x,t) = \ker R(t)$, $im \ D(x,t) = \ im \ R(t)$, and $d(x,t) = R(t)d(x,t)$ holds for $x \in \mathcal{D}, t \in \mathcal{I}$.

Consequently, the matrices $A(x,t)$ and $D(x,t)$ have constant rank for properly formulated leading terms. The subspaces $A(x,t)$ and $\operatorname{im} D(x,t)$ are independent of x and have bases from the class C^1. It holds that $A = AR, D = RD$.
If $A(x,t)$ and $D(x,t)$ fulfil the condition (2.2), but only ker $A(x,t)$ is independent of x and smooth, then we have, with a projector $P_A \in C^1(\mathcal{I}, L(\mathbb{R}^n))$, the relation $A(x,t) = A(x,t)P_A(t)$ and, hence

$$A(x(t),t)(d(x(t),t))' = A(x(t),t)(P_A(t)d(x(t),t))' - A(x(t),t)P_A'(t)d(x(t),t).$$

Then, with $\tilde{A}(x,t) := A(x,t)$, $\tilde{d}(x,t) := P_A(t)d(x,t)$, $\tilde{b}(x,t) := b(x,t) - -A(x,t)P_A'(t)d(x,t)$, the equation

$$\tilde{A}(x(t),t)(\tilde{d}(x(t),t))' + \tilde{b}(x(t),t) = 0 \tag{2.3}$$

has a proper leading term because of $\operatorname{im} \tilde{D}(x,t) = \operatorname{im} P_A(t)$. We can proceed analogously if only $\operatorname{im} D(x,t)$ is independent of x, or if the last condition of Definition 2.1 in not fulfilled. Hence, a proper formulation can be obtained if (2.2) holds and if one of the characteristic subspaces is independent of x and comes from C^1.

Definition 2.2 *A function $x \in C(\mathcal{I}_x, \mathbb{R}^m), \mathcal{I}_x \subseteq \mathcal{I}$, is said to be a solution of equation (2.1) if $x(t) \in \mathcal{D}, t \in \mathcal{I}_x$ and $d(x(.),.) \in C^1(\mathcal{I}_x, \mathbb{R}^n)$, and if equation* (2.1) *is fulfilled pointwisely.*

Unfortunately, regularity conditions do not define a linear function space here in general. In case $d(x,t) = D(t)x$ is linear itself (cf. (1.2)), a linear solution space is available by $C_D^1 := \{x \in C : Dx \in C^1\}$.

Fortunately, it is relatively simple to transform equation (2.1) into a (1.2) form. This allows the application of standard-notions and -methods (differentiability, linearization etc.) that are based on linear funtion spaces.
We form the natural extension for equation (2.1) with proper leading term

$$\begin{aligned} A(x(t),t)(R(t)y(t))' + b(x(t),t) &= 0, && (2.4) \\ y(t) - d(x(t),t) &= 0. && (2.5) \end{aligned}$$

With $\overline{x}=\binom{x}{y}$, $\overline{A}=\binom{A}{0}$, $\overline{d}(\overline{x},t)=R(t)y$, $\overline{D}(\overline{x},t)=\overline{D}(t)=(0,R(t))$, $\overline{b}(\overline{x},t)=\binom{b(x,t)}{y-d(x,t)}$ we can write (2.4), (2.5) as

$$\overline{A}(\overline{x}(t),t)(\overline{D}(t)\overline{x}(t))'+\overline{b}(\overline{x}(t),t)=0. \tag{2.6}$$

Due to $\ker\,\overline{A}(\overline{x},t)=\ker\,A(x,t)$, $\operatorname{im}\overline{D}(t)=\operatorname{im}\,R(t)=\operatorname{im}D(x,t)$ also (2.6) has a properly formulated leading part with $\overline{R}(t)=R(t)$ as the corresponding projector. Now the linear function space

$$C^1_{\overline{D}}=\left\{\overline{x}=\begin{pmatrix}x\\y\end{pmatrix}\in C:\overline{D}\overline{x}=Ry\in C^1\right\} \tag{2.7}$$

offers itself as solution space for (2.6). For (2.6) we seek functions $\overline{x}\in C^1_{\overline{D}}$ whose function values lie in the domain of definition $\mathcal{D}$ of the coefficients and fulfil equation (2.6) pointwisely.
If $x_*(.)$ is a solution of the original equation (2.1), then the pair $x_*(.),y_*(.)$ with $y_*(t):\equiv d(x_*(t),t)$ is obviously a solution of the class $C^1_{\overline{D}}$ for (2.6). If, reversely, a pair $x_*(.),y_*(.)$ from $C^1_{\overline{D}}$ forms a solution of (2.6), then $d(x_*(.),.)=Rd(x_*(.),.)=Ry_*\in C^1$ also holds because of $Ry_*\in C^1$, and $x_*(.)$ is a solution of equation (2.1).

Theorem 2.3 *Let the leading term of* (2.1) *be properly formulated.*

(i) The the leading term of the extension (2.6) *is properly formulated, too.*

(ii) The equations (2.1) *and* (2.6) *are equivalent via the relation* $y(.)=d(x(.),.)$.

The sets

$$\mathcal{M}_0(t):=\{x\in\mathcal{D}:b(x,t)\in\ \operatorname{im}\ A(x,t)\}$$

and

$$\begin{aligned}\overline{\mathcal{M}}_0(t)\ &:=\ \{\overline{x}=\begin{pmatrix}x\\y\end{pmatrix}\in\mathcal{D}\times\mathbb{R}^n:b(x,t)\in\ \operatorname{im}\ A(x,t),y=d(x,t)\}\\ &=\ \{\overline{x}\in\mathcal{D}\times\mathbb{R}^n:x\in\mathcal{M}_0(t),y=d(x,t)\}.\end{aligned}$$

are the geometrical location of the solutions of (2.1) and (2.6), respectively, It always holds that

$$x(t)\in\mathcal{M}_0(t),\ \ \overline{x}(t)\in\overline{\mathcal{M}}_0(t).$$

The problem in how far these sets are filled with solutions leads to notions of indices and corresponding solvability statements.

3 Subspaces, matrix chain and index

In this section we define characteristic subspaces and matrix chains for (2.1) and (2.6).
Further, let $B(y,x,t):=(A(x,t)y)_x+b_x(x,t)$ for $y\in\mathbb{R}^n,x\in\mathcal{D},t\in\mathcal{I}$ and

$\overline{B}(\overline{y},\overline{x},t) := (\overline{A}(\overline{x},t)\overline{y})'_{\overline{x}} + \overline{b}_{\overline{x}}(\overline{x},t)$ for $\overline{y} \in \mathbb{R}^n, \overline{x} \in \mathcal{D} \times \mathbb{R}^n, t \in \mathcal{I}$. More precisely, we have

$$\overline{B}(\overline{y},\overline{x},t) = \begin{bmatrix} A(x,t)\overline{y} \\ 0 \end{bmatrix}_{\overline{x}} + \begin{bmatrix} b_x(x,t) & 0 \\ -D(x,t) & I \end{bmatrix} = \begin{bmatrix} B(\overline{y},x,t) & 0 \\ -D(x,t) & I \end{bmatrix}$$

For the original equation (2.1) we form for $x \in \mathcal{D}, t \in \mathcal{I}, y \in \mathbb{R}^n$:

$$\begin{aligned} G_0(x,t) &= A(x,t)D(x,t), \\ N_0(x,t) &= \ker G_0(x,t), \\ S_0(y,x,t) &= \{z \in \mathbb{R}^m : B(y,x,t)z \in \ \operatorname{im}\ G_0(x,t)\}, \\ G_1(y,x,t) &= G_0(x,t) + B(y,x,t)Q_0(x,t) \end{aligned}$$

with a projector $Q_0(x,t) \in L(\mathbb{R}^m)$ onto $N_0(x,t)$,

$$\begin{aligned} P_0(x,t) &= I - Q_0(x,t), \\ N_1(y,x,t) &= \ker G_1(y,x,t), \\ S_1(y,x,t) &= \{z \in \mathbb{R}^m : B(y,x,t)P_0(x,t)z \in \ \operatorname{im}\ G_1(y,x,t)\}. \end{aligned}$$

For (2.6) this yields $\overline{G}_0(\overline{x},t) = \overline{A}(\overline{x},t)\overline{D}(\overline{x},t)$ etc. for $\overline{x} \in \mathcal{D} \times \mathbb{R}^n, t \in \mathcal{I}, \overline{y} \in \mathbb{R}^n$. $\overline{N}_0(t) = \mathbb{R}^m \times \ker R(t)$ depending on t only and being smooth is a special feature of (2.6).
Now, the relations among the subspaces of (2.1) and (2.6) are important, because the index will be defined via these subspaces later on.

Lemma 3.1 *Let equation* (2.1) *have a proper leading term, then*

$$\overline{N}_0(t) \cap \overline{S}_0(\overline{y},\overline{x},t) = (N_0(x,t) \cap S_0(\overline{y},x,t)) \times 0 \tag{3.1}$$

and

$$\begin{aligned} \overline{N}_1(\overline{y},\overline{x},t) \cap \overline{S}_1(\overline{y},\overline{x},t) = \Big\{ \tbinom{\xi}{\gamma} \in \mathbb{R}^m \times \mathbb{R}^n : \xi = Q_0(x,t)\xi, \gamma = R(t)\gamma, \\ D(x,t)^- \gamma + \xi \in N_1(\overline{y},x,t) \cap S_1(\overline{y},x,t) \Big\}. \end{aligned} \tag{3.2}$$

P r o o f. We determine

$$\begin{aligned} \overline{G}_0(\overline{x},t) &= \overline{A}(\overline{x},t)\overline{D}(\overline{x},t) = \begin{bmatrix} 0 & A(x,t) \\ 0 & 0 \end{bmatrix}, \\ \overline{N}_0(t) &= \left\{ \begin{pmatrix} \zeta \\ \gamma \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n : A(x,t)\gamma = 0 \right\} = \mathbb{R}^m \times \ker R(t), \end{aligned}$$

$$\begin{aligned} \overline{S}_0(\overline{y},\overline{x},t) &= \left\{ \begin{pmatrix} \zeta \\ \gamma \end{pmatrix} : B(\overline{y},x,t)\zeta \in \ \operatorname{im}\ A(x,t), -D(x,t)\zeta + \gamma = 0 \right\} \\ &= \left\{ \begin{pmatrix} \zeta \\ \gamma \end{pmatrix} : \zeta \in S_0(\overline{y},x,t), \gamma = d(x,t)\zeta \right\}, \end{aligned}$$

$$\begin{aligned}\overline{N}_0(t)\cap\overline{S}_0(\overline{y},\overline{x},t) &= \left\{\begin{pmatrix}\zeta\\ \gamma\end{pmatrix} : R(t)\gamma=0, \gamma=D(x,t)\zeta, \zeta\in S_0(\overline{y},x,t)\right\}\\ &= \left\{\begin{pmatrix}\zeta\\ \gamma\end{pmatrix} : \zeta\in N_0(x,t)\cap S_0(\overline{y},x,t), \gamma=0\right\}.\end{aligned}$$

Furthermore, by $\overline{Q}_0(t)=\begin{pmatrix} I & 0\\ 0 & I-R(t)\end{pmatrix}, \overline{P}_0(t)=\begin{pmatrix}0 & 0\\ 0 & R(t)\end{pmatrix}$ we also have

$$\overline{B}(\overline{y},\overline{x},t)\overline{P}_0(t)=\begin{bmatrix}0 & 0\\ 0 & R(t)\end{bmatrix}, \overline{G}_1(\overline{y},\overline{x},t)=\begin{bmatrix}B(\overline{y},x,t) & A(x,t)\\ -D(x,t) & I-R(t)\end{bmatrix}.$$

which implies

$$\begin{aligned}\overline{S}_1(\overline{y},\overline{x},t) &= \left\{\binom{\zeta}{\gamma} : \binom{0}{R(t)\gamma}\in \text{ im }\begin{bmatrix}B(\overline{y},x,t) & A(x,t)\\ -D(x,t) & I-R(t)\end{bmatrix}\right\}\\ &= \Big\{\binom{\zeta}{\gamma} : R(t)\gamma=-D(x,t)\alpha+(I-R(t))\beta\\ &\qquad \text{with } \ \alpha\in\mathbb{R}^m, \beta\in\mathbb{R}^n, B(\overline{y},x,t)\alpha+A(x,t)\beta=0\Big\}\\ &= \left\{\binom{\zeta}{\gamma} : R(t)\gamma=-D(x,t)\alpha \ \text{ with } \ \alpha\in S_0(\overline{y},x,t)\right\},\\ \overline{N}_1(\overline{y},\overline{x},t) &= \left\{\binom{\zeta}{\gamma} : B(\overline{y},x,t)\zeta+A(x,t)\gamma=0, D(x,t)\zeta=0, \gamma=R(t)\gamma\right\}\\ &= \Big\{\binom{\zeta}{\gamma} : \zeta\in N_0(x,t), \gamma=R(t)\gamma, 0=(A(x,t)D(x,t)+\\ &\qquad +B(\overline{y},x,t)Q_0(x,t))(D(x,t)^-\gamma+\zeta)\Big\}\\ &= \left\{\binom{\zeta}{\gamma} : \zeta=Q_0(x,t)\zeta, \gamma=R(t)\gamma, D(x,t)^-\gamma+\zeta\in N_1(\overline{y},x,t)\right\}.\end{aligned}$$

Finally, we obtain

$$\zeta=Q_0(x,t)\zeta, \gamma=R(t)\gamma=-D(x,t)\alpha, \alpha\in S_0(\overline{y},x,t), D(x,t)^-\gamma+\zeta\in N_1(\overline{y},x,t)$$

for $\binom{\zeta}{\gamma}\in\overline{N}_1(\overline{y},\overline{x},t)\cap\overline{S}_1(\overline{y},\overline{x},t)$. Thus,

$$\begin{aligned}B(\overline{y},x,t)P_0(x,t)(D(x,t)^-\gamma+\zeta) &= B(\overline{y},x,t)P_0(x,t)D(x,t)^-\gamma\\ &= -B(\overline{y},x,t)P_0(x,t)\alpha\in \text{ im } G_1(\overline{y},x,t)\end{aligned}$$

because of $B(\overline{y},x,t)\alpha=G_0(x,t)w$, i. e.,

$$B(\overline{y},x,t)P_0(x,t)\alpha=(G_0(x,t)+B(\overline{y},x,t)Q_0(x,t))(P_0(x,t)w-Q_0(x,t)\alpha). \ \square$$

Conclusion 3.2 *For* (2.1) *and* (2.6) *it holds that*

$$\dim(\overline{N}_0(t)\cap\overline{S}_0(\overline{y},\overline{x},t))=\dim(N_0(x,t)\cap S_0(\overline{y},x,t))$$

and

$$\dim(\overline{N}_1(\overline{y},\overline{x},t)\cap\overline{S}_1(\overline{y},\overline{x},t))=\dim(N_1(\overline{y},x,t)\cap S_1(\overline{y},x,t)).$$

Definition 3.3 *An equation* (2.1) *with properly formulated leading term is called a DAE of index 1 if*

$$N_0(x,t) \cap S_0(y,x,t) = 0 \ \textit{for} \ x \in \mathcal{D}, t \in \mathcal{I}, y \in \mathbb{R}^n,$$

or a DAE of index 2 if

$$\begin{aligned} \dim(N_0(x,t) \cap S_0(y,x,t)) &= \textit{const}, \\ N_1(y,x,t) \cap S_1(y,x,t) &= 0 \ \textit{for} \ x \in \mathcal{D}, t \in \mathcal{I}, y \in \mathbb{R}^n. \end{aligned}$$

Theorem 3.4 *The original equation* (2.1) *with proper leading term and its natural extension* (2.6) *have the index* $\mu \in \{1,2\}$ *simultaneously.*

Now we continue the above matrix chain with

$$G_2(y,x,t) := G_1(y,x,t) + B(y,x,t)P_0(x,t)Q_1(y,x,t)$$

and

$$\overline{G}_2(\overline{y},\overline{x},t) := \overline{G}_1(\overline{y},\overline{x},t) + \overline{B}(\overline{y},\overline{x},t)\overline{P}_0(t)\overline{Q}_1(\overline{y},\overline{x},t),$$

where $Q_1(y,x,t) \in L(\mathbb{R}^m)$, $\overline{Q}_1(\overline{y},\overline{x},t) \in L(\mathbb{R}^{m+n})$ are projectors onto $N_1(y,x,t)$ and $\overline{N}_1(\overline{y},\overline{x},t)$, respectively. We will investigate only problems (2.1) with index $\mu, \mu \in \{1,2\}$ here, hence,

$$N_1(y,x,t) \oplus S_1(y,x,t) = \mathbb{R}^m, \tag{3.3}$$

holds on principle, in fact for $\mu = 1$ with $N_1(y,x,t) = 0$, $S_1(y,x,t) = \mathbb{R}^m$ trivially, and for $\mu = 2$ due to [GrMä], Theorem A.13. Hence, we may assume that $Q_1(y,x,t)$ projects onto $N_1(y,x,t)$ along $S_1(y,x,t)$.
Analogously, let $\overline{Q}_1(\overline{y},\overline{x},t)$ be the projector onto $\overline{N}_1(\overline{y},\overline{x},t)$ along $\overline{S}_1(\overline{y},\overline{x},t)$. Simple computation now yield

Lemma 3.5 *For* $\mu \in \{1,2\}$ *it holds that*

$$\overline{Q}_1 = \begin{pmatrix} 0 & Q_0Q_1D^- \\ 0 & DQ_1D^- \end{pmatrix}, \overline{D}\,\overline{P}_1\,\overline{D}^- = DP_1D^-, \overline{D}\,\overline{Q}_1\,\overline{D}^- = DQ_1D^-,$$

and $\overline{D}\,\overline{Q}_1\overline{G}_2^{-1} = (DQ_1G_2^{-1} \ \ DQ_1D^-)$.

4 Linearization and perturbation theorems

First, we consider the equation

$$A(x(t),t)(D(t)x(t))' + b(x(t),t) = 0, \tag{4.1}$$

i.e., the case that $D(x,t) = D(t)x, x \in \mathcal{D}, t \in \mathcal{I}$. Later on, we apply the obtained results via (2.6) to the general form (2.1).
We fix $x_* \in C^1_D(\mathcal{I}_*, \mathbb{R}^m)$ with $x_*(t) \in \mathcal{D}, t \in \mathcal{I}_*$. Let $\mathcal{I}_* \subseteq \mathcal{I}$ be compact. For all

x from a sufficiently small neighbourhood $\mathcal{U}(x_*)$ of x_* in $C^1_D(\mathcal{I}_*, \mathbb{R}^m)$ we can define the map

$$\begin{aligned} \mathcal{F} \quad &: \quad \mathcal{U}(x_*) \subseteq C^1_D(\mathcal{I}_*, \mathbb{R}^m) \to C(\mathcal{I}_*, \mathbb{R}^m), \\ \mathcal{F}(x) \quad &:= \quad A(x(.),.)(Dx)'(.) + b(x(.),.), x \in \mathcal{U}(x_*), \end{aligned} \tag{4.2}$$

and we can write equation (4.1) as

$$\mathcal{F}(x) = 0,$$

where $\mathcal{F}$ is Frechét-differentiable. The derivative

$$\mathcal{F}_x(x) \in L_b(C^1_D(\mathcal{I}_*, \mathbb{R}^m), C(\mathcal{I}_*, \mathbb{R}^m))$$

is given by

$$\begin{aligned} \mathcal{F}_x(x)\triangle x \quad &= \quad A(x(.),.)(D\triangle x)'(.) + B((Dx)'(.), x(.),.)\triangle x(.) \\ &\quad\quad \text{for } \triangle x \in C^1_D(\mathcal{I}_*, \mathbb{R}^m), \end{aligned} \tag{4.3}$$

where we assume natural norms on $C^1_D(\mathcal{I}_*, \mathbb{R}^m)$ and $C(\mathcal{I}_*, \mathbb{R}^m)$.
For applying the implicit function theorem (e.g.[KaAk]), the property of the image $\operatorname{im} \mathcal{F}_x(x_*)$ is of crucial importance.
With $A_*(t) = A(x_*(t), t), B_*(t) = B((D(t)x_*(t))', x_*(t), t), t \in \mathcal{I}_*$, the linear DAE

$$A_*(t)(D(t)\triangle x(t))' + B_*(t)\triangle x(t) = q(t), t \in \mathcal{I}_* \tag{4.4}$$

is nothing else but the equation

$$\mathcal{F}_x(x_*)\triangle x = q.$$

Let $S_{*0}(t), N_{*0}(t), G_{*1}(t)$ etc. denote the chain of subspaces and matrices generated for (4.4).

Lemma 4.1 *Let the DAE* (4.1) *have a properly formulatd leading term. Let* $DP_{*1}D^-, DQ_{*1}D^- \in C^1(\mathcal{I}_*, L(\mathbb{R}^n))$.

(i) The index-μ-property, $\mu \in \{1, 2\}$, transforms itself from (4.1) *onto the linearization* (4.4).

(ii) For $\mu = 1$, $\mathcal{F}_x(x_*)$ is surjective.

(iii) For $\mu = 2$ it holds that $\operatorname{im} \mathcal{F}_x(x_*) = C^1_{DQ_{*1}G^{-1}_{*2}}(\mathcal{I}_*, \mathbb{R}^m)$.

P r o o f. The assumption (i) immediately results from the construction of the subspaces and the matrices in §3. The assumptions (ii) and (iii) are concluded from the existence theorems for linear DAEs in [BaMä]. $\square$

Conclusion 4.2 *In the index-2 case,* $\operatorname{im} \mathcal{F}_x(x_*)$ *is a non-closed proper subset in* $C(\mathcal{I}_*, \mathbb{R}^m)$.

If the DAE (4.1) has a dynamic degree of freedom, it has to be completed by initial or boundary conditions. Since A and D are singular, we cannot expect a degree of freedom m as in the case of regular DAEs, but a lower one. An IVP with the initial condition $x(t_0) = x_0 \in \mathbb{R}^m$ is not solvable in general. Hence, x_0 has to be consistent to an extend.
Basing on our experience with linear DAEs ([BaMä]), we impose an initial condition for (4.1) with $t_0 \in \mathcal{I}_*$ in the form

$$D(t_0)\pi(x(t_0) - x^0) = 0 \text{ with } x^0 \in \mathbb{R}^m, \tag{4.5}$$

where $\pi \in L(\mathbb{R}^m)$ will still have to be fixed. By means of the mapping

$$\begin{aligned} \mathcal{F}_{IVP} &: \mathcal{U}(x_*) \subseteq C_D^1(\mathcal{I}_*, \mathbb{R}^m) \to C(\mathcal{I}_*, \mathbb{R}^m) \times L_{IC}, \\ L_{IC} &:= \text{im } D(t_0)\pi, \\ \mathcal{F}_{IVP}(x) &:= (\mathcal{F}x, D(t_0)\pi x(t_0)), \quad x \in \mathcal{U}(x_*), \end{aligned} \tag{4.6}$$

we can describe the IVP (4.1), (4.5) in a compact way by

$$\mathcal{F}_{IVP}(x) = (0, D(t_0)\pi x^0).$$

Now, the equation

$$\mathcal{F}_{IVP}(x) = (q, D(t_0)\pi x^0)$$

corresponds to the perturbed IVP

$$A(x(t), t)(D(t)x(t))' + b(x(t), t) = q(t), \quad t \in \mathcal{I}_*, \tag{4.7}$$

$$D(t_0)\pi(x(t_0) - x^0) = 0. \tag{4.8}$$

Theorem 4.3 *Let* (4.1) *be a DAE with proper leading term and index* $\mu \in \{1, 2\}$, *and let* $x_* \in C_D^1(\mathcal{I}_*, \mathbb{R}^m)$ *be a solution of* (4.1) *and*

$$DP_{*1}D^-, DQ_{*1}D^- \in C^1(\mathcal{I}_*, L(\mathbb{R}^n)), \pi := P_{*1}(t_0).$$

Further, for $x \in \mathcal{U}(x_*)$, *let*

$$D(t)Q_{*1}(t)G_{*2}(t)^{-1}b(x(t), t) \text{ be continuously differentiable w.r.t. } t. \tag{4.9}$$

(i) If $\mu = 1$, *then the IVP* (4.7), (4.8) *is uniquely solvable on* $\mathcal{I}_*$ *for arbitrary* $x^0 \in \mathbb{R}^m$ *with* $|D(t_0)(x^0 - x_*(t_0))| \leq \sigma$, *and* $q \in C(\mathcal{I}_*, \mathbb{R}^m)$ *with* $\| q \|_\infty \leq \tau$, $\sigma, \tau > 0$ *sufficiently small. For the solutions* $x \in C_D^1(\mathcal{I}_*, \mathbb{R}^m)$ *it holds that*

$$\| x - x_* \|_{C_D^1} \leq \text{ const } (|D(t_0)(x^0 - x_*(t_0))| + \| q \|_\infty).$$

(ii) If $\mu = 2$, *then the IVP* (4.7), (4.8) *is uniquely solvable on* $\mathcal{I}_*$ *for arbitrary* $x^0 \in \mathbb{R}^m$ *with*
$|D(t_0)\pi(x^0 - x_*(t_0))| \leq \sigma$ *and* $q \in C^{-1}_{DQ_{*1}G_{*2}}(\mathcal{I}_*, \mathbb{R}^m), \| q \|_\infty +$
$\| (DQ_{*1}G_{*2}^{-1}q)' \|_\infty \leq \tau$, $\sigma, \tau > 0$ *sufficiently small.*
For the solutions $x \in C_D^1(\mathcal{I}_*, \mathbb{R}^m)$ *it holds that*

$$\begin{aligned} \| x - x_* \|_{C_D^1} \leq & \text{ const } (|D(t_0)\pi(x^0 - x_*(t_0))| + \| q \|_\infty \\ & + \| (DQ_{*1}G_{*2}^{-1}q)' \|_\infty). \end{aligned}$$

(iii) The solution x of the IVP (4.1), (4.8) *is continuously differentiable w.r.t. x^0. The sensitivity matrix $x_{x^0}(t) =: X(t) \in L(\mathbb{R}^m)$ satisfies the IVP*

$$\begin{aligned} A(x(t),t)(D(t)X(t))' + B((D(t)x(t))', x(t), t)X(t) &= 0, \quad t \in \mathcal{I}_*, \\ D(t_0)\pi(X(t_0) - I) &= 0. \end{aligned}$$

Remark: If we take into account that $P_{*1}(t) = I$, $Q_{*1}(t) = 0$ for $\mu = 1$, then the smoothness assumptions of Theorem 4.3. are always trivially given in this case. For $\mu = 2$ the regularity condition (4.9) implies restrictions of the admissible structure of (4.1). The following condition is sufficient for (4.9):

$$D^- \in C^1(\mathcal{I}_*, L(\mathbb{R}^n, \mathbb{R}^m)),$$

$\beta(x,t) := D(t)Q_{*1}(t)G_{*2}(t)^{-1}b(x,t)$ is continuously differentiable and

$$b'_x(P_0(t)x + sQ_0(t)x, t)Q_0(t)x \in \text{ im } G_{*1}(t), s \in [0,1], x \in \mathcal{D}, t \in \mathcal{I}_*. \tag{4.10}$$

Namely, then $\beta(x,t) \equiv \beta(P_0(t)x,t)$ is true and for $x(.) \in \mathcal{U}(x_*)$ it holds that $\frac{d}{dt}\beta(x(t),t) = \frac{d}{dt}\beta(P_0(t)x,t) = \frac{d}{dt}\beta(D(t)^- D(t)x(t),t) = \beta_x(D(t)^- D(t)x(t),t)$ $\{D(t)^{-'} D(t)x(t) + D(t)^-(D(t)x(t))'\}$ $+\beta_t(D(t)^- D(t)x(t),t)$. The condition (4.10) is equivalent to

$$W_{*0}(t)b'_x(P_0(t)x + sQ_0(t)x, t)Q_0(t)x \in \text{ im } W_{*0}(t)B_{*0}(t)Q_0(t). \tag{4.11}$$

If the derivative free part of (4.1) is linear in x or if (4.1) is a DAE in Hessenberg form, then (4.11) is given.

P r o o f of Theorem 4.3:
$\mathcal{F}_{IVP\,x}(x_*)$ is a bijection from $C^1_D(\mathcal{I}_*, \mathbb{R}^m)$ onto $C(\mathcal{I}_*, \mathbb{R}^m) \times L_{\mathcal{IC}}$ for $\mu = 1$.
For $\mu = 2$, $\mathcal{F}_{IVP\,x}(x_*)$ is injective but not surjective. According to Lemma 4.1, $\operatorname{im} \mathcal{F}_{IVP\,x}(x_*) \times L_{\mathcal{IC}} = C^1_{DQ_{*1}G_{*2}^{-1}}(\mathcal{I}_*, \mathbb{R}^m) \times L_{\mathcal{IC}} =: X$.
Equipped with a natural norm, X is a Banach space. We summarize the two cases that $\mu = 1, \mu = 2$ by using $Q_{*1} = 0$, $P_{*1} = I$ for $\mu = 1$.
For the mapping

$$\mathcal{H}(x,d,q) := \mathcal{F}_{IVP}(x) - (q,d), \quad x \in \mathcal{U}(x_*), (q,d) \in X$$

it holds, due to the condition (4.9), that

$$\mathcal{H}(x,d,q) \in X.$$

With $d_* := D(t_0)\pi x_*(t_0)$ we obtain

$$\mathcal{H}(x_*, d_*, 0) = 0, \quad \mathcal{H}_x(x_*, d_*, 0) = \mathcal{F}_{IVP\,x}(x_*).$$

$\mathcal{H}_x(x_*, d_*, 0)$ is a homeomorphism for $\mu = 1$ as well as for $\mu = 2$. Due to the implicit function theorem there exists a uniquely determined continuously differentiable mapping

$$f : \overline{B}(d_*, \sigma) \times \overline{B}(0, \tau) \subseteq X \to C^1_D(\mathcal{I}_*, \mathbb{R}^m)$$

with $f(d_*,0)=x_*$, $\mathcal{H}(f(d,q),d,q)=0$, $\| f(d,q)-f(d_*,0) \|_{C^1_D} \leq K(|d-d_*|+\| q \|_X)$ for $|d-d_*|\leq\sigma, \| q \|_X := \| q \|_\infty + \| (DQ_{*1}G_{*2}^{-1}q)' \|_\infty \leq \tau$.
Then the two assumption (i) and (ii) follow because of $d=D(t_0)\pi x^0$.
In particular, $f(d,0)$ is continuously differentiable w.r.t. d, i.e., the solution $x(.)$ of the IVP (4.1) with $D(t_0)\pi x(t_0)=d$ has a continuous derivative w.r.t. d. With $d=D(t_0)\pi x^0$ this implies $x_{x^0}(t)=x_d(t)D(t_0)\pi$. As usually, the variational equation (4.10) now results by differentiation w.r.t. x^0. □

If we introduce a perturbation index for (4.1.) analogously to the standard case of DAEs, then the inequalities in (i) and (ii) mean that a DAE (4.1) with index μ has the perturbation index $\mu, \mu\in\{1,2\}$, too.

For the extended system (2.4), (2.5) and for (2.6), respectively, it holds with $\overline{x}=\binom{x}{y}$, $\triangle\overline{x}=\binom{\triangle x}{\triangle y}$ that

$$\overline{\mathcal{F}}_{\overline{x}}(\overline{x})\triangle\overline{x}=\begin{pmatrix} A(x(.),.)(R\triangle y)'(.)+B((Ry)'(.),x(.),.)\triangle x(.) \\ \triangle y(.)-D(x(.),.)\triangle x(.) \end{pmatrix}.$$

In particular, the equation $\overline{\mathcal{F}}_{\overline{x}}(\overline{x}_*)\triangle\overline{x}=(q,r)$ with $\overline{x}_*=\binom{x_*}{y_*}$ is nothing else but the linear DAE

$$\begin{aligned} A(x_*(t),t)(R(t)\triangle y(t))'+B((R(t)y_*(t))',x_*(t),t)\triangle x(t) &= q(t), \\ \triangle y(t)-D(x_*(t),t)\triangle x(t) &= r(t). \end{aligned} \tag{4.12}$$

Now, let $x_*(.)$ be a solution of the nonlinear equation (2.1) and $y_*=d(x_*(.),.)$. Then $\overline{x}_*(.)$ solves the extended form (2.4), (2.5) with $y_*=Ry_*=d(x_*(.),.)$.
With the coefficients

$$\begin{aligned} A_*(t) &= A(x_*(t),t), D_*(t)=D(x_*(t),t), \\ B_*(t) &= B((d(x_*(t),t))',x_*(t),t) \end{aligned} \tag{4.13}$$

(4.12) can be reduced to

$$\begin{aligned} A_*(t)(R(t)\triangle y(t))'+B_*(t)\triangle x(t) &= q(t), \\ \triangle y(t)-D_*(t)\triangle x(t) &= r(t). \end{aligned} \tag{4.14}$$

For $r(t)=0$ this yields

$$A_*(t)(D_*(t)\triangle x(t))'+B_*(t)\triangle x(t)=q(t), \tag{4.15}$$

which can be regarded as a linearization of the initial equation (2.1).
By Theorem 3.4 and Lemma 4.1 the index $\mu\in\{1,2\}$ is transformed from (2.1) to (4.14) and (4.15).
Now, let N_{*0}, S_{*0}, G_{*1} etc. be the subspaces and matrices of the chain formed for A_*, D_* and B_* from (4.13).
We investigate the perturbed IVP

$$A(x(t),t)(d(x(t),t))'+b(x(t),t)=q(t), \tag{4.16}$$

$$D_*(t_0)\pi D_*(t_0)^-(d(x(t_0),t_0)-y^0)=0, \tag{4.17}$$
$$y^0\in\mathbb{R}^n, q\in C(\mathcal{I}_*,\mathbb{R}^m).$$

For $\pi=I$ the initial condition (4.17) simplifies to $R(t_0)(d(x(t_0),t_0)-y^0)=0$, i.e.,

$$d(x(t_0),t_0)=R(t_0)y^0. \tag{4.18}$$

Theorem 4.4 *Let the DAE* (2.1) *with proper leading part be of index $\mu=1$. Let $x_*(.)$ be the solution of (2.1) on the interval $\mathcal{I}_*$. Further, let $\pi=I$. Then the IVP* (4.16), (4.18) *is uniquely solvable with a solution $x(.)$ defined on $\mathcal{I}_*$ for $q\in C(\mathcal{I}_*,\mathbb{R}^m)$, $\| q \|_\infty\le\tau$, $y^0\in\mathbb{R}^n$, $|R(t_0)(y^0-d(x_*(t_0),t_0))|\le\sigma$, $\tau,\sigma>0$ sufficiently small. $x(.)$ depends continuously differentiably on y^0. It holds that*

$$\begin{aligned}\| x-x_* \|_\infty &+ \| d(x(.),.)-d(x_*(.),.) \|_{C^1}\\ &\le const(|R(t_0)(d(x(t_0),t_0)-d(x_*(t_0),t_0))|+\| q \|_\infty).\end{aligned}$$

P r o o f of the theorem: We form the extended system for the DAE (4.16) and apply Theorem 4.3 for $\mu=1$. □

In the nature of things DAEs of index 2 require a higher regularity of some components. Here it is essential in how far we can assume (4.9) for the extended equation (2.6). We have (cf. §2)

$$\begin{aligned}\overline{n}(\overline{x},t) &:= \overline{D}(t)\overline{Q}_{*1}(t)\overline{G}_{*2}(t)^{-1}\overline{b}(\overline{x},t)\\ &= n(x,t)+D_*(t)Q_{*1}(t)D_*(t)^-(y-d(x,t))\end{aligned} \tag{4.19}$$

with

$$n(x,t):=D_*(t)Q_{*1}(t)G_{*2}(t)^{-1}b(x,t). \tag{4.20}$$

In the case of a linear original equation (2.1) with $A(x,t)\equiv A(t)$, $b(x,t)\equiv B(t)x+q(t)$, $d(x,t)\equiv D(t)x$ the expressions (4.19), (4.20) simplify to

$$\begin{aligned}\overline{n}(\overline{x},t) &= D(t)Q_1(t)G_2(t)^{-1}q(t)+D(t)Q_1(t)D(t)^-y\\ n(x,t) &= D(t)Q_1(t)x+D(t)Q_1(t)G_2(t)^{-1}q(t).\end{aligned}$$

It is natural to demand that $DQ_1G_2^{-1}q\in C^1$ for index-2 DAEs, and likewise that $DQ_1D^-\in C^1$. Thus, condition (4.9), which requires that $\overline{n}(\overline{x}(.),.)$ has to belong to the class C^1 for continuous $x(.),y(.)$ and continuously differentiable $(Ry)(.)$, is fulfilled for (2.1) in the linear case.

Theorem 4.5 *Let the DAE* (2.1) *with proper leading term be of index $\mu=2$. Let $x_*\in C(\mathcal{I}_*,\mathbb{R}^m)$ be the solution of* (2.1)*, and $D_*P_{*1}D_*^-$, $D_*Q_{*1}D_*^-$ be continuously differentiable. Moreover, let $\pi:=P_{*1}(t_0)$.*
Let $\overline{n}(\overline{x}(.),.)\in C^1(\mathcal{I}_*,\mathbb{R}^n)$ for all $x\in C(\mathcal{I}_*,\mathbb{R}^m)$ from a neighbourhood of x_* and for $y\in C^1_R(\mathcal{I}_*,\mathbb{R}^n)$ from a neighbourhood of $y_*=d(x_*(.),.)$.

(i) For

$$q \in C^1_{D_* Q_{*1} G_{*2}^{-1}}(\mathcal{I}_*, \mathbb{R}^m), \|q\|_\infty + \|(D_* Q_{*1} G_{*2}^{-1} q)'\|_\infty \le \tau,$$
$$y^0 \in \mathbb{R}^n, |D_* P_{*1} D_*^- (y^0 - d(x_*(t_0), t_0))| \le \sigma,$$

$\tau, \sigma > 0$ sufficiently small, the IVP (4.16), (4.17) is uniquely solvable and for the solution $x \in C(\mathcal{I}_*, \mathbb{R}^m)$ it holds that

$$\|x - x_*\|_\infty + \|d(x(.),.) - d(x_*(.),.)\|_{C^1}$$
$$\le \; const \; \{\|q\|_\infty + \|(D_* Q_{*1} G_{*2}^{-1} q)'\|_\infty$$
$$+ \; |D_*(t_0) \pi D_*(t_0)^- (d(x(t_0), t_0) - d(x_*(t_0), t_0))|\}.$$

(ii) The solution $x(.)$ of the IVP depends contiuously differentiably on y^0.

P r o o f. We use the extended form (2.6) for (2.1) and write the IVP (4.16), (4.17) in the following way (cf. §§1,2)

$$\overline{A}(\overline{x}, (t), t)(\overline{D}(t)\overline{x}(t))' + b(\overline{x}(t), t) = \overline{q}(t), \tag{4.21}$$

$$\overline{D}(t_0)\overline{P}_{*1}(t_0)(\overline{x}(t_0) - \overline{x}^0) = 0, \tag{4.22}$$

with $\overline{D}(t_0)\overline{P}_{*1}(t_0) = (0 \; D_* P_{*1} D_*^-)$ and $\overline{q}(t) = \binom{q(t)}{0}$. For (2.6), the condition (4.9) is given by the assumption.
$\overline{q} \in C^1_{\overline{DQ_{*1}} \, \overline{G}_{*2}^{-1}}(\mathcal{I}_*, \mathbb{R}^{m+n})$ holds if and only if $q \in C^1_{D_* Q_{*1} G_{*2}^{-1}}(\mathcal{I}_*, \mathbb{R}^m)$. Consequently, Theorem 4.3 yields the assertion. □
We formulate a further perturbation theorem, whose assumptions are possibly easier to be checked.

Theorem 4.6 *Let the DAE (2.1) with proper leading term be of index $\mu = 2$. Let $x_* \in C(\mathcal{I}_*, \mathbb{R}^m)$ be the solution of (2.1), and let $D_* P_{*1} D_*^-$, $D_* Q_{*1} D_*^-$ as well as $D_* x_*$ be continuously differentiable. Let $\pi := P_{*1}(t_0)$.*
For all $x \in C^1_{D_*}(\mathcal{I}_*, \mathbb{R}^m)$ from a neighbourhood of x_* let $n(x(.),.)$ and $d(x(.),.)$ be continuously differentiable. Then the assumptions from Theorem 4.5 remain true, where the solutions of the IVP (4.16), (4.17) are even from $C^1_{D_*}(\mathcal{I}_*, \mathbb{R}^m)$.

P r o o f. We check immediately whether the linear index-2 DAE (4.12) has solutions $\triangle x \in C^1_{D_*}(\mathcal{I}_*, \mathbb{R}^m)$, $\triangle y \in C^1_R(\mathcal{I}_*, \mathbb{R}^n)$ for right-hand sides $q \in C^1_{D_* Q_{*1} G_{*2}^{-1}}(\mathcal{I}_*, \mathbb{R}^m)$, $r \in C^1_R(\mathcal{I}_*, \mathbb{R}^n)$. With $\overline{X} := C^1_{D_*}(\mathcal{I}_*, \mathbb{R}^m) \times C^1_R(\mathcal{I}_*, \mathbb{R}^m)$ and $\overline{Y} := C^1_{D_* Q_{*1} G_{*2}^{-1}}(\mathcal{I}_*, \mathbb{R}^m) \times C^1_R(\mathcal{I}_*, \mathbb{R}^n)$ we obtain $\overline{\mathcal{F}}_{\overline{x}}(\overline{x}_*) \in L_b(\overline{X}, \overline{Y}), im \overline{\mathcal{F}}_{\overline{x}}(\overline{x}_*) = \overline{Y}$.
Then the related mappping $\overline{\mathcal{F}}_{IVP\,x}(\overline{x}_*)$ acts bijectively between the spaces $\overline{X}$ and $\overline{Y} \times \overline{L}_{IC}$. By assumption it holds that $\overline{x}_* = \binom{x_*}{y_*} \in \overline{X}$.
The regularity conditions for $n(x,t)$ and $d(x,t)$ ensure that $\overline{\mathcal{F}}(\overline{x}) \in \overline{Y}$ is always true for $\overline{x} \in \overline{X}$ from a neighbourhood of $\overline{x}_*$. We can further argue analogously to the proof of Theorem 4.3, where the mapping $\overline{\mathcal{H}}(\overline{x}, d, \overline{q}) := \overline{\mathcal{F}}_{IVP}(\overline{x}) - (\overline{q}, d)$ operates in the spaces $\overline{X} \times L_{IC} \times \overline{Y}$ and $\overline{Y} \times L_{IC}$ now. □
The additional regularity conditions in case of DAEs of index $\mu = 2$, which shall

guarantee certain properties of the mappings (the images $\overline{\mathcal{F}}(\overline{x})$ have to lie in the "right" space), imply restrictions on the admissible structure.
An interesting special class of DAEs, which is important for applications, consists of DAEs for which $N_0(x,t)$ does not depend on x, i.e.,

$$\ker D(x,t) = N_0(t), \quad x \in \mathcal{D}, t \in \mathcal{I}, \tag{4.23}$$

and $P_0(t)$ is continuously differentiable w.r.t. t. Quite often $N_0(x,t)$ is even independent of x and t.
Then it holds that

$$d(x,t) = d(P_0(t)x,t), \quad x \in \mathcal{D}, t \in \mathcal{I}, \tag{4.24}$$

and further $d_x(x,t) = d_x(P_0(t)x,t)$, in particular,

$$D_*(t) = d_x(P_0(t)x_*(t),t).$$

Lemma 4.7 *Let* (2.1) *be a DAE with proper leading term. Let* (4.3) *be valid and* $d \in C^1(\mathcal{D}\times\mathcal{I},\mathbb{R}^n)$, $P_0 \in C^1(\mathcal{I},L(\mathbb{R}^m))$. *Let* $x_* \in C(\mathcal{I}_*,\mathbb{R}^m)$ *be a solution of the DAE* (2.1).

(i) Then P_0x_*, D_*, D_*^- *and* D_*x_* *are continuously differentiable on* $\mathcal{I}_*$.

(ii) $d(x(.),.)$ *is continuously differentiable on* $\mathcal{I}_*$ *for all* $x \in C^1_{D_*}(\mathcal{I}_*,\mathbb{R}^m)$.

P r o o f. Provided that P_0x_* is continuously differentiable, then $D_*(t) = d_x(P_0(t)x_*(t),t)$ is so, too. As a reflexive generalized inverse with C^1-projectors $P_0(t) = D_*(t)^-D_*(t)$, $R(t) = D_*(t)D_*(t)^-$ also $D_*(t)^-$ is continuously differentiable. Furthermore, $D_*x_* = D_*P_0x_*$ belongs to the class C^1. Then $x \in C^1_{D_*}(\mathcal{I}_*,\mathbb{R}^m)$ implies

$$d(x(.),.) = d(D_*(.)^-D_*(.)x(.),.) \in C^1(\mathcal{I}_*,\mathbb{R}^n).$$

It remains to show that P_0x_* is actually C^1. Therefore, we investigate the function

$$\mathcal{K}(x,y,t) := y - d(x,t), \;\; x \in \mathcal{D}, y \in \mathbb{R}^n, t \in \mathcal{I}.$$

We have $\mathcal{K}(x_*(t),y_*(t),t) = 0$, and $\mathcal{K}_x(x_*(t),y_*(t),t) = -D_*(t)$ acts bijectively between $\operatorname{im} P_0(t)$ and $\operatorname{im} R(t)$.
For each $t \in \mathcal{I}_*$ the equation $\mathcal{K}(x,y,t) = 0$ provides a solution function $\zeta(.,t)$ with $\mathcal{K}(\zeta(y,t),y,t) = 0$, $\zeta(y,t) = P_0(t)\zeta(y,t)$, $\zeta(y,t) = \zeta(R(t)y,t)$, $P_0(t)x_*(t) = \zeta(y_*(t),t)$. Then the regularity of P_0x_* results from that of ζ, since $y_*(t)$ is continuously differentiable. □
For a better understanding let us remark that, in case of $d(x,t) \equiv D(t)x + \delta(t)$, the equation $y - D(t)x - \delta(t) = 0$ leads to $y = R(t)y$ and $P_0(t)x = D(t)^-y - D(t)^-\delta$, i.e., $\zeta(y,t) = D(t)^-y - D(t)^-\delta(t)$.
For nonlinear DAEs (2.1) satisfying the assumptions of Lemma 4.7, i.e., DAEs with a smooth function d and an only time-dependent smooth subspace N_0, it is convenient to work with the function space $C^1_{P_0}(\mathcal{I}_*,\mathbb{R}^m) = C^1_{D_*}(\mathcal{I}_*,\mathbb{R}^m)$. (2.1) can be left in the original form or we may take

$$(d(x(t),t))' = d_x(P_0(t)x(t),t)(P_0(t)x(t))' + d_t(P_0(t)x(t),t).$$

5 Special systems of circuit simulation

The modified nodal analysis (MNA) used in industrial simulation packages generates, for large classes of circuits, systems of the form (cf. [EsTi])

$$\left.\begin{array}{rll} A_c(q(A_c^\top e(t),t)' & +b_c(e(t),j_L(t),j_V(t)) & =0, \\ (\phi(j_L(t),t))' & +b_L(e(t),j_L(t),j_V(t)) & =0, \\ & b_V(e(t),j_L(t),j_V(t)) & =0, \end{array}\right\}, \tag{5.1}$$

where $e(t)$, $j_L(t)$, $j_V(t)$ denote the nodal potentials and the currents of inductances and the voltage sources, respectively. The functions $q(z,t)$ and $\phi(w,t)$ are continuously differentiable and the Jacobians $q_z(z,t)$, $\phi_w(w,t)$ are positive definit. With

$$x := \begin{pmatrix} e \\ j_L \\ j_V \end{pmatrix}, \quad \tilde{A} := \begin{pmatrix} A_c & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}, \quad \tilde{d}(x,t) := \begin{pmatrix} q(A_c^\top e,t \\ \phi(j_L,t) \end{pmatrix},$$

$$M(x,t) := \begin{pmatrix} q_z(A_c^\top e,t) & 0 \\ 0 & \phi_w(j_L,t) \end{pmatrix}, \quad \tilde{b}(x,t) := \begin{pmatrix} b_c(e,j_L,j_V,t) \\ b_L(e,j_L,j_V,t) \\ b_V(e,j_L,j_V,t \end{pmatrix}$$

it holds that

$$\tilde{D} := d_x = M\tilde{A}^\top, G_0 := \tilde{A}\tilde{D} = \tilde{A}M\tilde{A}^\top,$$

$$\operatorname{im} G_0 = \operatorname{im}\tilde{A}, \ker G_0 = \ker\tilde{D} = \ker\tilde{A}^\top.$$

The matrix $G_0(x,t)$ has a nullspace N_0 that is independent of x and t. Obviously, (5.1) is nothing else but

$$\tilde{A}(\tilde{d}(x(t),t))' + \tilde{b}(x(t),t) = 0. \tag{5.2}$$

If $\operatorname{im}\tilde{D}(x,t)$ is independent of x, then (5.2) has a proper leading term. Because of the constant nullspace N_0, Lemma 4.7 is relevant.
If $\operatorname{im}\tilde{D}(x,t)$ changes with x, we can put $\tilde{A} = \tilde{A}P_{\tilde{A}}$ in (5.2) and shift the constant projector $P_{\tilde{A}}$ with $\ker P_{\tilde{A}} = \ker\tilde{A}$ below the derivative. The DAE

$$\tilde{A}(P_{\tilde{A}}\tilde{d}(x(t),t))' + \tilde{b}(x(t),t) = 0 \tag{5.3}$$

has a proper leading term with constant subspace N_0 and $\tilde{R} = P_{\tilde{A}}$. Here, too, Lemma 4.7 may be applied. Moreover, $P_{\tilde{A}}\tilde{D}(x,t)$ is also constant now.
On the other hand, we can differentiate

$$\tilde{A}(\tilde{d}(x(t),t))' = \tilde{A}\{M(x(t),t)(\tilde{A}^\top x(t))' + d_t(x(t),t)\}$$

and investigate the equation

$$\tilde{A}M(x(t),t)(\tilde{A}^\top x(t))' + \tilde{b}(x(t),t) + \tilde{A}d_t(x(t),t) = 0. \tag{5.4}$$

If $\ker(\tilde{A}M(x,t))$ is independent of x, (5.4) represents a DAE with proper leading term. If $\ker(\tilde{A}M(x,t))$ depends on x, we proceed to

$$\tilde{A}M(x(t),t)P_{\tilde{A}}^\top(\tilde{A}^\top x(t))' + \tilde{b}(x(t),t) + \tilde{A}d_t(x(t),t) = 0. \tag{5.5}$$

Combined version are also possible, the choice depends, among other things, on the subspaces $\operatorname{im} D(x,t)$, $D(x,t)S_1(x,t)$, $D(x,t)N_1(x,t)$ being constant in case of

$$A(x(t),t)(d(x(t),t))' + b(x(t),t) = 0.$$

Both versions, (5.3) as well as (5.5), have a constant $\operatorname{im} D(x,t)$. For them, it holds uniformly that

$$S_1(x,t) = \{z : \tilde{b}_x(x,t)P_0 z \in \text{ im } (\tilde{A}\tilde{D}(x,t) + \tilde{b}_x(x,t)Q_0)\}.$$

Then we have $D(x,t)S_1(x,t) = P_{\tilde{A}}\tilde{D}(x,t)S_1(x,t)$ for (5.3), whereas $D(x,t)S_1(x,t) = \tilde{A}^\top S_1(x,t)$ is true for (5.5).

6 Constraints

Obviously, all solutions $x_* \in C^1_D(\mathcal{I}_*, \mathbb{R}^m)$ of the DAE

$$A(x(t),t)(D(t)x(t))' + b(x(t),t) = 0 \tag{6.1}$$

have to satisfy the condition $x(t) \in \mathcal{M}_0(t), t \in \mathcal{I}_*$, with the constraint set

$$\mathcal{M}_0(t) := \{x \in \mathcal{D} : b(x,t) \in \text{ im } A(x,t)\}. \tag{6.2}$$

For DAEs of index $\mu = 1$, $\mathcal{M}_0(t)$ is completely filled with solutions and $\mathcal{M}_0(t_0)$ is the set of consistent initial values at time $t_0 \in \mathcal{I}$ (cf. [HiMä]).

Theorem 6.1 *Let* (6.1) *be a DAE of index* $\mu = 1$. *Let* $t_0 \in \mathcal{I}$ *and* $x_0 \in \mathcal{M}_0(t_0)$. *Then there exists a unique (maximal) solution* $x_* \in C^1_D(\mathcal{I}_*, \mathbb{R}^m)$ *with* $\mathcal{I}_* \ni t_0$, $x_*(t_0) = x_0$.

P r o o f. Because of $x_0 \in \mathcal{M}_0(t_0)$ there exists a $y_0 = R(t_0), y_0 \in \mathbb{R}^n$, with

$$A(x_0,t_0)D(t_0)D(t_0)^- y_0 + b(x_0,t_0) = 0.$$

With $x = D(t)^- D(t)x + Q_0(t)x = D(t)^- u + Q_0(t)w$, $w := Q_0(t)x + D(t)^- y$, $u := D(t)x$ we find

$$\begin{aligned} A(x,t)D(t)D(t)^- y \quad &+ \quad b(x,t) \\ &= \quad A(D(t)^- u + Q_0(t)w, t)D(t)w + b(D(t)^- u + Q_0(t)w, t) \\ &=: \quad \mathcal{F}(u,w,t). \end{aligned}$$

It holds that $\mathcal{F}(u_0,w_0,t_0) = 0$ for $u_0 := D(t_0)x_0$, $w_0 := Q_0(t_0)x_0 + D(t_0)^- y_0$. Moreover,

$$\mathcal{F}_w(u_0,w_0,t_0) = G_1(D(t_0)w_0, D(t_0)^- u_0 + Q_0(t_0)w_0, t_0) = G_1(y_0,x_0,t_0)$$

is regular in case of index $\mu = 1$. Consequently, the implicit function theorem implies the local equivalence of the relations $\mathcal{F}(u,w,t) = 0$ and $w = \omega(u,t)$ with a

continuous function $\omega(u,t)$, which has a continuous partial derivative $\omega_u(u,t)$ and for which $w_0 = \omega(u_0,t_0)$ is true.
The regular IVP

$$u'(t) - R'(t)u(t) = D(t)\omega(u(t),t), u(t_0) = u_0 \tag{6.3}$$

has a solution $u \in C^1(\mathcal{I}_*, \mathbb{R}^m)$ for which $u(t) = R(t)u(t), t \in \mathcal{I}_*$, holds. Then the function $x(t) := D(t)^- u(t) + Q_0(t)\omega(u(t),t), t \in \mathcal{I}_*$, is the desired solution. □
Behind Theorem 6.1 is the idea that the dynamics of a DAE is dominated by an inherent regular differential equation (here, in case of $\mu = 1$, the differential equation (6.3)), and representations of the solution result from the constraints.
In the case of DAEs (6.1) of index $\mu = 2$, hidden constraints have to be taken into account. Under the assumptions of the following lemma these hidden constraints can be described relatively easy.

Lemma 6.2 *For the DAE* (6.1) *of index* $\mu = 2$ *let* $im\, G_1(y,x,t)$ *not depend on* y *and* $Q_0(t)x$. *For a continuously differentiable projector* $W_1(x,t) \in L(\mathbb{R}^m)$, $\ker W_1(x,t) = im\ G_1(y,x,t)$, *let the relation*

$$W_1(x,t) = W_1(P_0(t)x,t), x \in \mathcal{D}, t \in \mathcal{I}, \tag{6.4}$$

be valid. Then

$$(W_1 b)(x,t) = (W_1 b)(P_0(t)x,t), x \in \mathcal{D}, t \in \mathcal{I}, \tag{6.5}$$

is also true.

P r o o f. By construction, $W_1 B Q_0 = 0$ and $W_1 (Ay)_x Q_0 z = 0$ for all $z \in \mathbb{R}^m$, hence,

$$W_1 b_x Q_0 = W_1(B - (Ay)_x)Q_0 = 0.$$

Finally, we have

$$\begin{aligned}(W_1 b)(x,t) \quad &- \quad (W_1 b)(P_0(t)x,t) \\ &= \int_0^1 (W_1 b)_x(sx + (1-s)P_0(t)x)Q_0(t)x ds \\ &= \int_0^1 \{W_{1x}(...)Q_0(t)xb(...) + W_1 b_x(...)Q_0(t)x\} ds = 0. \qquad \Box\end{aligned}$$

The structure introduced in Lemma 6.2 allows us to differentiate the equation $(W_1 b)(x,t) = 0$ along a solution and thus to find the hidden constraints. More precisely, let $D(t)$ and $D(t)^-$ be from C^1. Then the function $(W_1 b)(x_*(t),t)$ is continuously differentiable for each solution $x_* \in C_D^1(\mathcal{I}_*, \mathbb{R}^m)$. $x_*(t) \in \mathcal{M}_0(t)$ implies $(W_1 b)(x_*(t),t) = 0, t \in \mathcal{I}_*$ and $(W_1 b)(D(t)^- D(t)x_*(t),t) = 0, t \in \mathcal{I}_*$, respectively, and differentiation yields

$$(W_1 b)_x(D(t)^- D(t)x_*(t),t)(D(t)^- D(t)x_*(t))' + (W_1 b)'_t(D(t)^- D(t)x_*(t),t) = 0.$$

From (6.1) we obtain immediately

$$D(t)^-(D(t)x_*(t))^- = -D(t)^- A(x_*(t),t)^- b(x_*(t),t).$$

This makes clear that

$$x_*(t) \in \mathcal{H}_1(t), t \in \mathcal{I}_*,$$

with

$$\begin{aligned}\mathcal{H}_1(t) := \quad \{x \in \mathcal{D} \quad & : (W_1 b)_x(x,t)(D(t)^{-'} D(t)x - D(t)^- A(x,t)^- b(x,t)) \\ & +(W_1 b)(P_0(t)x,t) = 0\}\end{aligned}$$

must be true. Thus, $\mathcal{H}_1(t)$ describes a hidden constraints. $A(x,t)^-$ is a reflexive generalized inverse with $A(x,t)^- A(x,t) = R(t)$.
It can be supposed that there are no further hidden constraints for $\mu = 2$ and that

$$\mathcal{M}_1(t) = \mathcal{M}_0(t) \cap \mathcal{H}_1(t), t \in \mathcal{I},$$

represents the sharp geometrical location of the solution of the DAE (6.1) that is of index $\mu = 2$.
Furthermore, it can be supposed that, under regularity conditions admitting the existence of the tangent space $T_x\mathcal{M}_1(t)$, the relations

$$\begin{aligned} P_0(t)T_x\mathcal{M}_1(t) &= P_0(t)S_1(y,x,t), \\ D(t)T_x\mathcal{M}_1(t) &= D(t)S_1(y,x,t) \end{aligned}$$

hold for $x \in \mathcal{M}_1(t)$, $y = -A(x,t)^- b(x,t)$.

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