Gravity actions from matter actions

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Abstract

Starting from classical matter dynamics on a smooth manifold that are required to be predictive and quantizable, we derive a set of 'gravitational master equations' that determine the Lagrangian describing the dynamics of the geometry on which the matter dynamics are defined. We thus convert the physical problem of finding admissible gravitational dynamics for any tensorial geometry that can support physical matter equations into the clear mathematical task of solving a system of linear partial differential equations. This result builds on the insight that predictive and quantizable matter dynamics, on the one hand, restrict the class of admissible spacetime geometries to those that are bi-hyperbolic and energy-distinguishing, and, on the other hand, provide the necessary kinematical structure needed to formulate spacetime geometry dynamics as an initial value problem. The gravitational master equations then express the fact that the Lagrangian of the gravitational dynamics must arise as a representation of the algebra of hypersurface deformations—which can be calculated from the kinematical structure imprinted on the geometry by the matter field dynamics—on a suitable geometric phase space. We provide a general prescription of how to obtain the gravitational master equations for any candidate geometry and illustrate our procedure by way of four instructive examples. We solve the master equations for metric geometry supporting Maxwell theory, finding Einstein-Hilbert dynamics as the unique solution, and for a non-trivial composite geometry supporting modified Dirac dynamics. We also discuss generalized energy-momentum tensors of matter fields and their role as sources of the gravitational dynamics obtained from the gravitational master equations.

Keywords:
classical matter fields, dispersion relations, spacetime geometry, geometrodynamics

Zusammenfassung

Ausgehend von der Forderung, dass die Dynamik klassischer Materiefelder auf einer glatten Mannigfaltigkeit prädiktiv und quantisierbar sein muss, leiten wir einen Satz von „Mastergleichungen“ her, deren Lösungen die Dynamik (in Form einer Lagrangedichte) der den Materiegleichungen zugrundeliegenden Geometrie beschreiben. Es gelingt also das physikalische Problem der Suche nach geeigneten Gravitationsdynamiken für eine beliebige tensorielle Raumzeitgeometrie, die physikalische Materie tragen kann, in die bloß noch mathematische Frage nach der Lösung eines Systems von linearen partiellen Differentialgleichungen zu reformulieren. Dieses Ergebnis fußt auf der Einsicht, dass die Forderung nach der Prädiktivität und Quantisierbarkeit einer Materietheorie zunächst die möglichen Klassen der zugrundeliegenden Raumzeitgeometrien auf solche beschränkt, die bi-hyperbolisch sind und die Unterscheidung von positiven und negativen Energien zulassen. Gleichzeitig stellen solche Materietheorien bereits alle kinematischen Strukturen zur Verfügung, die nötig sind, um die Dynamik der Geometrie als Anfangswertproblem zu formulieren. Die Mastergleichungen stellen dann einen Ausdruck dafür dar, dass die Lagrangefunktion der Gravitationsdynamik, die die zeitliche Entwicklung von geometrischen Anfangsdaten beschreibt, eine Darstellung der Hyperflächendeformationsalgebra sein muss, welche sich ausgehend von der Dynamik der Materietheorie direkt berechnen lässt. Wir geben eine allgemeine Vorgehensweise an, mit der sich die Mastergleichungen für eine beliebige tensorielle Raumzeitgeometrie herleiten lassen und illustrieren dieses Verfahren anhand von vier physikalisch relevanten Beispielen. Im Fall von Maxwellscher Elektrodynamik auf einer metrischen Mannigfaltigkeit erhalten wir als eindeutige Lösung unserer Mastergleichungen die Einstein-Hilbert Wirkung. Außerdem lösen wir die Mastergleichungen für eine nichttriviale Erweiterung metrischer Geometrie, die eine modifizierte Form von Dirac Materie tragen kann. Die Arbeit wird abgerundet durch ein Studium von Energie-Impuls-Tensoren von Materie auf tensoriellen Raumzeiten.

Schlagwörter:
klassische Materiefelder, Dispersionsrelationen, Raumzeitgeometrie, Geometrodynamik

Contents

Chapter 1

Introduction

The main result of this thesis is the derivation of a set of 'gravitational master equations', which, once solved, determine the gravitational dynamics of any tensorial spacetime geometry G. The master equations require, as their only input, some classical matter dynamics that are both predictive and quantizable. Thus in precise and modern fashion, this result builds on the same insight put forward in Newton's first axiom and Einstein's identification of the physical spacetime structure [18], namely that the only way to learn something about the geometry of spacetime is by probing it with matter.

Over the past decades, several astrophyiscal surveys revealed—most recently the Planck mission [1]—that there is an enormous gap between our understanding of the visible matter in the universe—which we believe to understand very well—and the amount of matter, which we observe to interact only gravitationally—and whose dynamics are essentially unknown. According to the most common parametrization of the cosmological standard model, the ΛCDM model [67], only an (by previous hopes: remarkably low) amount of 5% of the matter in the universe is baryonic, while 27% is of entirely unknown (dark) origin and the remaining part of 68% consists of dark energy. Based on our current understanding of classical spacetime physics, there are two (at first sight: largely independent) screws one can adjust in order to attempt to remedy this theoretically rather unsatisfactory situation: either one complements the standard model by providing additional matter fields in the theory in order to account for the missing mass and energy density, or one modifies the dynamics of the underlying spacetime geometry to deviate from general relativity. Or, in fact, one attempts both at the same time. If one sets out to investigate both possibilities independently, however, one risks to run into inconsistencies. This is because the dynamics of matter on a spacetime are not only intimately related to the underlying gravitational dynamics, but the latter are in fact *determined* by the former—the constructive proof of which fact is the key result of this thesis.

Taking seriously the experimentally well-tested dynamics of Maxwell theory, as

Einstein did in order to identify the structure of physical spacetime, one inevitably concludes that spacetime must be modeled by a Lorentzian manifold. This conclusion remains unaltered if one substitutes the dynamics of any other standard model matter field—or all of them simultaneously, for that matter—because these are all conceived, by deliberate design, in the image of Maxwell theory. But having recognized the *kinematical* structure of physical spacetime to be encoded in a Lorentzian manifold, the *dynamics* of this geometry are suprisingly already determined under weak technical assumptions. More precisely, as first proven by Hojman, Kuchař and Teitelboim [30, 37], one can show that the Einstein-Hilbert action provides the unique covariant dynamics for the degrees of freedom encoded in a Lorentzian metric that can be cast into an initial value formulation with respect to the space and time directions singled out by the causal structure determined by the standard model matter dynamics. Requiring such an initial value formulation not only for the matter field dynamics but also for the gravitational dynamics is of course the essence of any physical theory that is able to make predictions based on currently available data—and thus is an essential theoretical requirement. But also from a very practical point of view, having an initial value formulation of the gravitational dynamics is a prerequisite for all kinds of numerical simulations [4], and also for canonical quantization attempts [68].

Since, so far, there is no evidence forcing—or even suggesting—that dark matter must be modelled again in the image of Maxwell theory (technically: that it must share the same causal structure) the pre-dark-matter era restriction of the underlying spacetime geometry to be a Lorentzian manifold ceases to be compelling. In order to not unduly adhere to conclusions that can no longer be drawn in the light of experimental facts, we must therefore take into technical account the very real possibility that the spacetime geometry to which dark matter fields couple be given by a type of tensor field that is not a Lorentzian metric. But this then prompts the immediate question as to what the analogues of the Einstein equations are for such a non-Lorentzian metric geometry. This question is answered in this thesis. Fortunately, the very matter dynamics whose formulation require a departure from Lorentzian geometry determine the desired gravitational dynamics of the geometry they couple to. This is the intimate link between the dynamics of a spacetime geometry and the dynamics of the matter fields it is designed to carry.

Instead of modifying matter dynamics, one could also try, in principle, to modify the gravitational dynamics first by introducing additional gravitational degrees of freedom, postulate their dynamics and then postulate matter dynamics on such spacetimes. There are various candidates for such theories, ranging from very simple models such as Brans-Dicke theory [11], over $f(R)$ theories [14], to more exotic models such as the relativistic versions of modified Newtonian dynamics [5] and general tensor-vector-scalar theories of gravity [47]. This search for consistent modifications of general relativity reaches back to Einstein and others who considered which dynamics could be given to a non-symmetric $(0, 2)$-tensor field [20, 46]. However, all these models need to prescribe how matter fields couple to the additional gravitational degrees of freedom in a way such that a solution to the gravitational field equations produces a background geometry on which the matter fields can indeed propagate in a causal way. It seems to be very difficult to obtain modified spacetime models this way around.

In this thesis, we will follow the first way, historically tried and tested by Einstein, and show how to determine the gravity action from a given matter action. Einstein demonstrated this—less systematically but physically spot-on—by conceiving general relativity starting from Maxwell electrodynamics, or, more precisely, from the causal structure of Maxwell electrodynamics. After all, matter field dynamics present the only access we have, experimentally and philosophically, to the geometry of spacetime, which makes them the natural point of departure for the construction of the dynamics of this geometry.

To keep things as conservative as possible, in this thesis, we technically restrict attention to matter field dynamics that can be formulated in terms of covariant partial differential equations of finite derivative order on a smooth manifold (M, G) where G, however, can be—a priori—any smooth tensor field, or, in fact, a collection of different tensor fields. On the one hand, this enables us to make use of the whole apparatus of continuous differential calculus. On the other hand, smooth manifolds are general enough to allow for a wide range of models. Now, the very least one must require of such matter dynamics to be physically admissible, however, is that the geometry, to which we would like the matter fields couple, is such that it renders the dynamics of the matter fields both *predictive* and *quantizable*. This will play an enormous role in this thesis, bot technically and conceptually. Since the matter equations we consider are given in terms of partial differential equations, fortunately, these physical conditions can be cast into clear-cut algebraic restrictions on the geometry by use of central results of the theory of partial differential equations [22, 31]. These restrictions constrain the spectrum of geometries G that can underly the matter field equations one starts with [58]: such geometries must be 'bi-hyperbolic' and 'energy-distinguishing'. Indeed, these two algebraic conditions restrict the admissible background geometries for Maxwell dynamics to Lorentzian metrics.

Remarkably, and most importantly for the programm executed in this thesis, bi-hyperbolicity and energy-distinguishability are not only necessary to render the matter dynamics under consideration predictive and quantizable, but they are also *sufficient* to give rise to an entire kinematical apparatus, including massless and

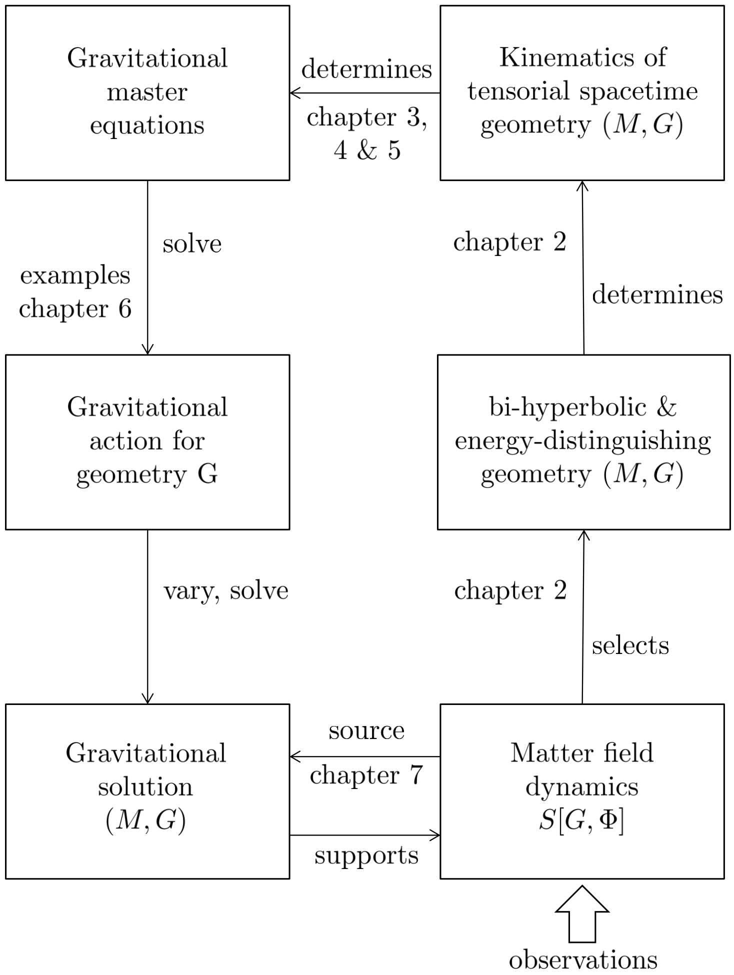


Figure 1.1: Logical outline of the present thesis.

massive point particle dynamics, a concise definition of observers and a completely fixed non-linear duality theory between tangent and cotangent spaces. One not only can, but must, deduce this kinematical structure directly from the matter field equations. This kinematical structure is, as we will see, key and physical input to the derivation of suitable gravitational dynamics for the underlying geometry. Because the kinematical structure on the manifold (M, G) depends on the matter field dynamics one employs (this was Einstein's big discovery), it is justified to only call the triple consisting of the manifold M, the geometry G and the matter field dynamics a tensorial spacetime if the geometry G is bi-hyperbolic and energy-distinguishing with respect to the matter field equations. This compatibility criterion is made precise in chapter 2.

The derivation of the gravitational dynamics for a tensorial geometry G then follows the same philosophy that led Hojman, Kuchař and Teitelboim—in the context of (as *we* would emphasize) Maxwell induced standard relativistic kinematics—to derive the Lagrangian of general relativity (up to an undetermined gravitational and cosmological constant, which emerge as integration constants) as the unique dynamics for a Lorentzian metric. Technically implementing the same philosophy for the case of any predictive and quantizable matter dynamics on a tensorial spacetime geometry (Maxwell theory on a metric background obviously presenting a very special case), which is the purpose of this thesis, requires to use an interplay of mathematical results of various fields (namely real algebraic geometry, convex analysis, and the theory of partial differential equations). Starting with the extraction of the kinematics from any given predictive and quantizable matter action (which needs to be provided as 'physical input') we first determine the algebra of hypersurface deformation operators in chapter 3.

The search for gravitational dynamics then takes the form of a clear-cut representation problem, since the canonical dynamics for the tensorial spacetime geometry G must be given in terms of constraint functionals on a suitable geometric phase space, whose Poisson constraint algebra takes the same form as the hypersurface deformation algebra. In chapter 4, we will be able to reformulate this representation problem into a linear functional differential equation, which we break down to a countable set of linear partial differential equations in chapter 5. At this point, in summary, we achieved to cast our problem—namely to construct gravitational dynamics whose solutions can carry initially prescribed matter dynamics—into a well-defined mathematical problem of solving a set of linear partial differential equations. Since these partial differential equations determine the entire gravity theory, they deserve the name *gravitational master equations*.

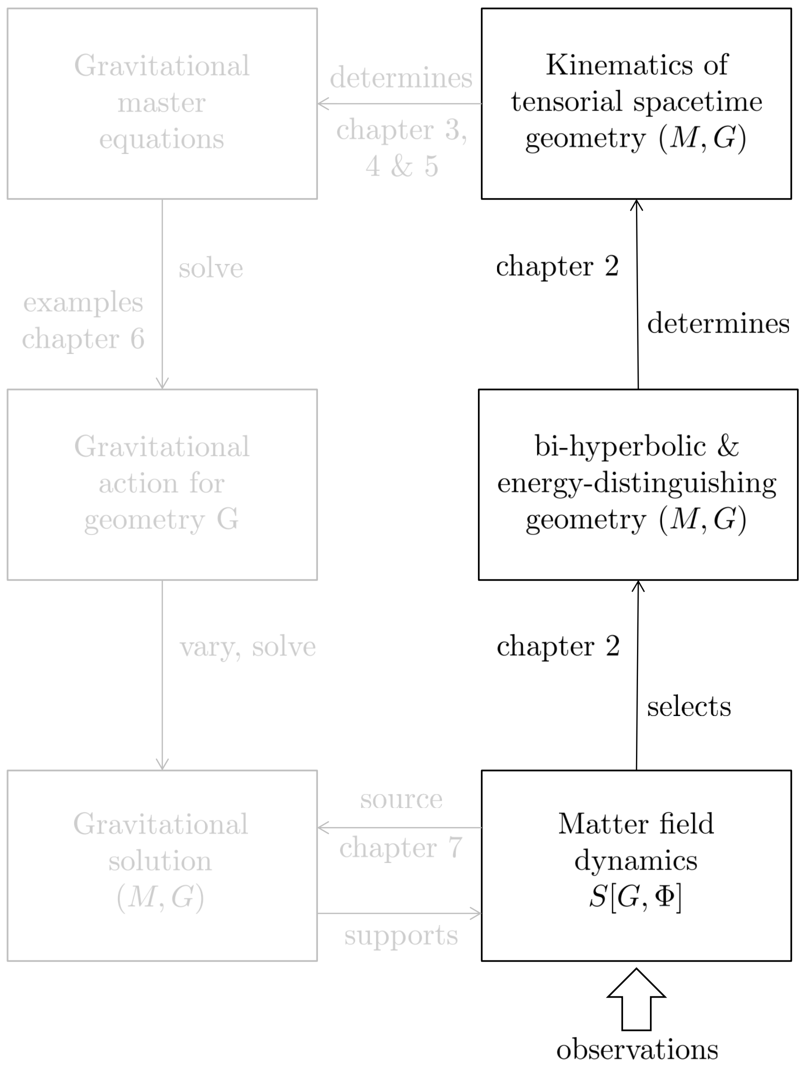
In chapter 6, we illustrate, by way of several interesting examples, how the gravitational master equations are obtained for any tensorial spacetime, starting from the matter dynamics one wishes to impose. As a particular example, we recover the result of Hojman, Kuchař and Teitelboim for metric geometry coupled to Maxwell electrodynamics as a solution to the corresponding master equations, which leads to the ADM form [2, 3] of the Einstein-Hilbert action, with Newton's constant and the cosmological constant arising as theoretically undetermined integration constants. We also derive and solve the master equations for a non-trivial tensor-vector geometry whose kinematics is determined by a modified Dirac equation as it is often used as a phenomenological matter model but without recognizing or heeding the need for correspondingly modified gravitational dynamics.

Finally, in chapter 7, we discuss matter fields as sources for the gravitational field of a tensorial spacetime geometry. This requires a proper understanding of energy-momentum tensors of matter fields on tensorial spacetimes, based on the seminal work of Gotay and Marsden [26]. This closes the circle from the gravitational dynamics, which are determined by the matter field dynamics one imposes, to the energy-momentum tenors of matter fields playing their role as sources to (at least parts of) the gravitational field equations. We illustrate these generalized energy-momentum tensors and the associated conservation laws with several examples.

A complete overview on the logical structure of this thesis is given in figure 1.1, which will accompany the reader at the beginning of each chapter with the topics in that particular chapter being highlighted.

The results peresented in this thesis were published in

- 'Gravitational dynamics for all tensorial spacetimes carrying predictive, interpretable and quantizable matter' by the author, K. Giesel, F.P. Schuller and M. N.R. Wohlfarth; Physical Review D 85 (2012), 104042 [24] and
- 'How quantizable matter gravitates: A practitioner's guide' by the author and F.P. Schuller; Physical Review D 89 (2014), 104061 [63].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Chapter 2

Kinematics of tensorial spacetimes: A review

In this chapter, we review the theory of tensorial spacetimes as it was developed in [58]. A tensorial spacetime $(M, G, S[G,\Phi])$ is a smooth manifold M equipped with an *a priori* arbitrary tensor field G that is bi-hyperbolic and energy-distinguishing with respect to an action $S[G,\Phi]$ for one or several classical matter fields Φ; the latter two properties present two essential physical consistency conditions—namely that the geometry G allows the matter theory to be predictive and quantizable—and take the form of simple algebraic restrictions on the tensor field G, which encodes the spacetime geometry. This will be explained in detail in this chapter. Furthermore, we review how tensorial spacetimes give rise to a complete kinematical framework, with the pivotal result being the duality maps between momenta and velocities of massless and massive point particles. This kinematical structure, which is imprinted on the spacetime geometry by the stipulated classical matter field action will be key to the derivation of the gravitational master equations that determine the dynamics of the tensorial spacetime geometry G.

2.1 From matter field equations to tensorial spacetimes

In the philosophy of this thesis, geometry is what can be detected by physical matter fields. More precisely, we consider the triple $(M, G, S[G,\Phi])$ consisting of a smooth manifold M equipped with a smooth tensor field G of arbitrary valence and symmetry, called the geometry on M, and, as crucial additional input, a matter action $S[G,\Phi]$ for a collection of classical matter fields $\Phi = (\Phi^A)$ that take their values in some tensor representation[1] space V on M—where we allow for reducible representations in order to be able to encode all matter fields of interest in the irreducible components of Φ. In developing the theory of tensorial spacetimes, one can restrict attention to matter actions $S[G, \Phi]$, which, upon variation with respect to the field components Φ^A and possibly required gauge fixing, give rise to linear partial differential equations of motion for the fields Φ. This does not mean that one shall not be able to consider non-linear matter equations on a tensorial spacetime (one can), but that they are most cleanly probed by linear matter dynamics since solutions of the latter can be scaled to arbitrarily small amplitudes and thus can serve as bona fide test matter. Thus, for such linear test matter, the equations of motion take the form

$$\sum_{n=0}^{N} Q[G]_{AB}^{a_1 \dots a_n}(x) \partial_{a_1} \dots \partial_{a_n} \Phi^A(x) = 0 \,, \tag{2.1}$$

where the small Latin indices run from 0 to $\dim M - 1$ and refer to a chosen set of spacetime coordinates $\{x^a\}$, and the coefficients Q are constructed from the geometry G and its partial derivatives only. If the coefficients $Q^{a_1 \cdots a_N}$ of the highest order derivatives of the field components Φ^A depend only on G but none of its derivatives, we say that the matter field is minimally coupled to the geometry. We will assume this to be the case from now on.

The specification of a matter action is the single point of physical input into the theory developed in this thesis—which achieves all its aims exploiting consistency conditions governing the interplay of matter and the underlying geometry. In other words, the whole theoretical machinery developed here must be fed by phenomenologically obtained matter field dynamics in order to claim relation to nature. But if indeed it is fed such matter field dynamics, it generates, purely due to the mentioned consistency conditions, the entire kinematics and dynamics of the underlying spacetime geometry.

Thoughout the present chapter, however, it is assumed that the geometry G is fixed everywhere on the manifold M, rather then being governed by its own equations of motion. It is then precisely the purpose of the remainder of this thesis to derive a set of master equations whose solutions provide Lagrangians for the dynamics of the geometry G, determined solely from the input triple $(M, G, S[G, \Phi])$.

Now, in order for the field components Φ^A to describe physical matter, one requires the matter dynamics (2.1) to be predictive and quantizable. Fortunately, these two physicality conditions can be reformulated as algebraic restrictions on the so-called principal tensor field P, which is a totally symmetric tensor field derived

[1]If there is additional structure available on the manifold M, one can also consider other types of matter fields taking values in representations different from $GL(\dim M, \mathbb{R})$ tensor representations. Indeed, we will do so in chapter 6.4, where we consider Dirac fields taking their values in the spinor representation of the local rotation group associated to a given metric.

from the leading-order coefficient of the (if necessary: gauge-fixed) matter field equations (2.1) by

$$P^{i_1\dots i_{\deg P}}(x)k_{i_1}(x)\dots k_{i_{\deg P}}(x) := \rho(G)(x)\det_{A,B}\left(Q(G)_{AB}^{a_1\dots a_N}(x)k_{a_1}(x)\dots k_{a_N}(x)\right) \tag{2.2}$$

for all points $x \in M$ and all covectors $k \in T_x^*M$; the scalar density $\rho(G)$ is solely constructed from the geometry G such that it cancels any density weight that results from the determinant over the tensor representation space V in which the field Φ takes its values. In constructing the principal tensor field P, one may cancel any repeated factors appearing from the construction on the right hand side of equation (2.2). The principal tensor field P then defines the homogeneous principal polynomial $P(x,k)$, which we use as a shorthand for the left hand side of (2.2), on the cotangent bundle T^*M of the manifold M, whose degree $\deg P$ is given by the rank of the tensor field P. Of course, any restriction on the principal tensor field P directly translates into restrictions on the geometry G that underlies the matter field equations (2.1). In fact, it has been shown in [58] that requiring predictivity and quantizability of the matter field dynamics (2.1) forces the principal polynomial $P(x,k)$, and thus the principal tensor field P, to be bi-hyperbolic and energy-distinguishing which notions we will define in the following section. Only those triples $(M, G, S[G,\Phi])$ that, by virtue of (2.2), give rise to a principal tensor field P that is bi-hyperbolic and energy-distinguishing are then called *tensorial spacetimes*.

As particular examples of matter field dynamics and the principal tensor field P they give rise to, we discuss Maxwell electrodynamics on a metric manifold, i.e. $(M, g, S_{\text{Maxwell}}[g,A])$, in section 6.2 and the reader may wish to jump at this point to section 6.2.1 in order to see how the principal tensor field associated with the tensorial spacetime $(M, g, S_{\text{Maxwell}}[g,A])$ arises as

$$P^{ab} = (g^{-1})^{ab}\,. \tag{2.3}$$

This principal tensor field is shared *a fortiori* by the dynamics of all other standard model matter fields on a metric manifold (M,g). A different but physically relevant example for a tensorial spacetime is provided by the triple $(M, G, S_{GLED}[G,A])$ consisting of a smooth four-dimensional manifold M, a fourth rank contravariant tensor field G^{abcd} featuring the symmetries $G^{abcd} = -G^{bacd} = G^{cdab}$ and the most general dynamics $S_{GLED}[G,A]$ of a covector gauge field A giving rise to linear field equations. The principal polynomial of such a matter theory, as is discussed in section 6.3, is given by

$$P_G^{abcd} = -\frac{24}{(G^{abcd}\epsilon_{abcd})^2}\,\epsilon_{mnpq}\,\epsilon_{rstu}\,G^{mnr(a}\,G^{b|ps|c}\,G^{d)qtu}\,. \tag{2.4}$$

A third example is provided by the triple $(M, g, W, \mathrm{Dirac})$, describing a Dirac type matter field propagating on a manifold M which is equipped with a metric g and a vector field W, and whose dynamics give rise to a second rank principal tensor field P of the form

$$P^{ab} = g^{ab} + W^a W^b\,, \tag{2.5}$$

as discussed in section 6.4.

Equipped with these concrete examples for principal tensor fields of specific tensorial spacetime candidates, in the following section, we review how the physicality conditions on the matter dynamics (2.1) are translated into algebraic conditions on the principal tensor field P.

2.2 Predictivity and quantizability

The first physicality requirement one imposes on the matter dynamics $S[G, \Phi]$ is that the matter field equations (2.1) be predictive, or, in other words, that they admit an initial value formulation. The weakest necessary condition for the predictivity of such field equations is that there exist non-trivial covector fields k on which the principal polynomial vanishes,

$$P(x, k(x)) = 0 \quad \text{for all } x \in M\,. \tag{2.6}$$

The physical role of these covector fields is revealed when considering the geometric optical limit of the partial differential equations (2.1), where they describe the wave fronts of solutions for the matter field Φ^A in a short wave approximation. Kinematically, one later identifies the covectors k, on which the principal polynomial vanishes, with the momenta of massless point particles [51].

The much more important and, in fact, stronger necessary condition for the predictivity of the matter field equations (2.1) is that the principal polynomial is hyperbolic [42, 22]. Hyperbolicity means that there exists at least one covector field h with $P(x, h(x)) > 0$ [2] such that for all covector fields r there exist only real functions λ on M satisfying

$$P(x, r(x) + \lambda(x)h(x)) = 0 \quad \text{for all } x \in M\,. \tag{2.7}$$

[2] Clearly, if we have $P(x, h(x)) < 0$, we can always arrange for $P(x, h(x)) > 0$ by absorbing an overall sign into the density $\rho(G)$ in the definition (2.2) of the principal tensor field P.

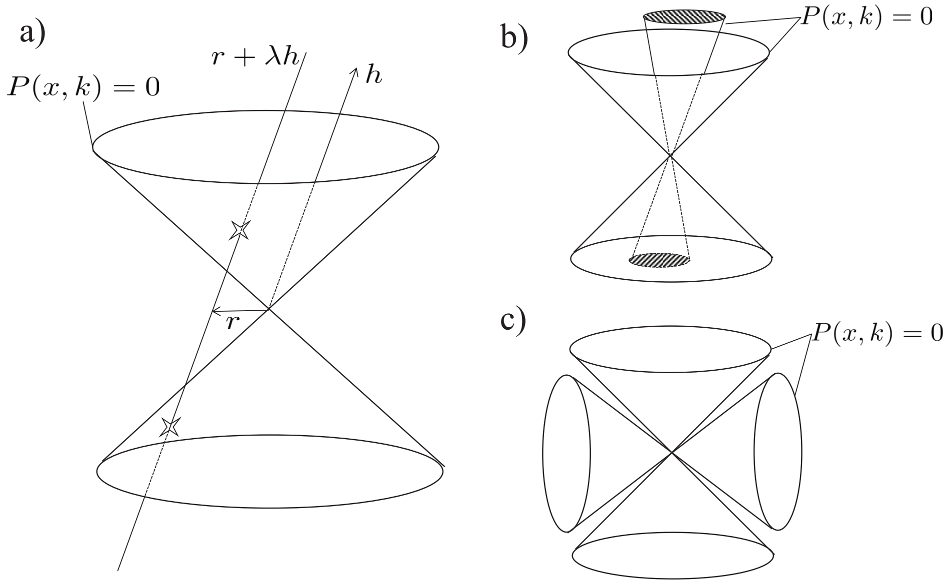


Figure 2.1: Illustration of a) a hyperbolic covector $h \in T_x^*M$ with respect to a hyperbolic polynomial $P(x,k)$ of second degree; b) a hyperbolic polynomial $P(x,k)$ of degree four where all covectors lying inside the shaded cones are hyperbolic; c) a fourth degree polynomial $P(x,k)$ which cannot be hyperbolic.

If such a covector field h indeed exists, it is called a hyperbolic covector field h. This abstract definition of hyperbolic covector fields can be interpreted geometrically by looking at the cotangent space T_x^*M at a particular point $x \in M$: a covector $h \in T_x^*M$ is called hyperbolic if the affine line $r + \lambda h$ going through an arbitrary point $r \in T_x^*M$ and being parametrized by λ intersects the vanishing set $\{k \in T_x^*M | P(x,k) = 0\}$ of the principal polynomial exactly $(\deg P)$-many times (see figure 2.1 for an illustration).

The various disconnected sets of hyperbolic covectors in each cotangent space of the manifold are called hyperbolicity cones of the principal tensor field P. They are open convex cones in the mathematical sense, i.e., taking two hyperbolic covectors from the same hyperbolicity cone C, any linear combination with positive coefficients will again lie in C. Hyperbolic covectors play a decisive role in the question of whether the matter field equations (2.1) can admit an initial value formulation, since they pre-select the possible initial data surfaces on which initial values for the matter fields can be specified: a hypersurface Σ in M can serve as an initial value surface only if the conormals n at every point of the hypersurface are hyperbolic covectors. Moreover, on the kinematical level, one can identify the set of hyperbolic covectors as the set of momenta of massive point particles, which will be discussed in the next section.

As a necessary conditon on the principal polynomial $P(x,k)$, hyperbolicity restricts the geometric tensor fields G that can provide a background for the matter field equations (2.1). For example, for Maxwell electrodynamics on a metric manifold, denoted by the triple $(M, g, S_{\text{Maxwell}}[g, A])$, whose principal tensor field is given by (2.3), hyperbolicity of the principal polynomial forces the inverse metric g^{-1} to be of Lorentzian signature. In other words, Lorentzian geometry is the only metric geometry that can render Maxwell's equations predictive. Going beyond metric geometry and thus beyond Lorentzian geometry may complicate matters, but still the same abstract principles apply; compare section 6.3 for general linear electrodynamics on area metric manifolds, and section 6.4, for the background geometry of modified Dirac matter.

The second requirement on the classical matter field dynamics $S[G, \Phi]$, namely quantizablity, can be translated into two additional algebraic conditions on the principal tensor field P and thus, again, into conditions on the geometry G. The first of these conditions is that the dual polynomial[3] $P^{\#}(x,v)$ (whose existence is guaranteed by the hyperbolicity of the principal polynomial $P(x,k)$, and which is defined by

$$P^{\#}(x, DP(x,k)) = 0 \qquad\qquad \forall k \in T_x^*M : P(x,k) = 0\,, \tag{2.8}$$

where DP denotes the derivative of the principal polynomial $P(x,k)$ with respect to its fibre argument) must be hyperbolic as well. This allows one to define a time-orientation on the manifold M by choosing an everywhere non-vanishing vector field H that is hyperbolic with respect to the dual polynomial $P^{\#}(x,v)$ for every $x \in M$. Since the dual polynomial $P^{\#}(x,v)$ is defined in terms of the principal polynomial $P(x,k)$, whose hyperbolicity does in general not imply the hyperbolicity of $P^{\#}(x,v)$, requiring the existence of a time-orientation further restricts the admissible tensor fields G. A geometry G, for which both the principal polynomial $P(x,k)$ (with respect to the matter dynamics $S[G,\Phi]$) and its dual polynomial $P^{\#}(x,v)$ are hyperbolic, is called bi-hyperbolic.

In order to formulate the third final algebraic condition on the polynomial P, it is convenient to consider, seperately in each tangent space T_xM, the connected set $C^{\#}_x$ of vectors, which are hyperbolic with respect to $P^{\#}(x,v)$ and to which the vector H_x of the chosen time-orientation vector field H belongs: quantizability of the matter dynamics requires that the geometry G be energy-distinguishing with respect to the time orientation H, which means that for every $x \in M$ and any

[3] Algebraic geometry provides the necessary tools to determine the dual polynomial. The calculation of the dual polynomial can either be done with the help of Groebner bases, which can be determined via Buchberger's algorithm, or, if one already has a guess on what the dual polynomial looks like, by direct verification of the definition (2.8). For more details we refer the reader to [58] and the references therein.

$k \in T_x^*M$, with $P(x,k) = 0$, we have either

$$k(X) > 0 \quad \text{or} \quad -k(X) > 0 \tag{2.9}$$

for all vectors $X \in C_x^\#$. This is indeed crucial for canonical quantization since only if this condition is satisfied for the geometry G, one can uniquely split the solutions to the matter field equations (2.1) into positive and negative energy parts[4]. There are several further technically important implications from bi-hyperbolicity and energy-distinguishability such as a reverse triangle equation and the insight that the principal tensor field must be of even rank.

In the following section we will explain why, remarkably, bi-hyperbolicity and energy-distinguishability do not only pose necessary conditions on the geometry G for the latter to support the matter field dynamics $S[G, \Phi]$, but are also sufficient to give rise to an entire kinematical apparatus, which allows, on the one hand, to associate the momenta of massless and massive point particles to the corresponding particle velocities, and allows, on the other hand, for the definition of observers. Only those triples $(M, G, S[G, \Phi])$ consisting of a manifold M, a tensorial geometry G and a matter theory $S[G, \Phi]$, for which the geometry G is bi-hyperbolic and energy-distinguishing with respect to a chosen time-orientation H, will therefore be justifiably called *tensorial spacetimes*.

2.3 Kinematics of tensorial spacetimes

In this section, we discuss the kinematical structure that is imprinted on a tensorial spacetime $(M, G, S[G, \Phi])$ by the matter dynamics $S[G, \Phi]$. Again, all kinematical constructions revolve around the principal tensor field P and the associated principal polynomial $P(x, k)$, which follow directly from the matter action $S[G, \Phi]$, as described theoretically in the previous section and illustrated in chapter 6 for several physically interesting examples.

Above, we found that the covectors k, on which the principal polynomial $P(x, k)$ vanishes, are identified with the momenta of massless point particles, and we thus have

$$P(x, k) = 0 \tag{2.10}$$

as the massless dispersion relation. But then hyperbolicity of the principal polynomial ensures that one can associate, to each of these massless covectors k, the corresponding ray vector[5] $\dot{x}$, tangent to the worldline $x(\tau)$ of the massless point

[4] Indeed, the quantization of general linear linear electrodynamics on area metric manifolds (M, G), which we discuss classically in section 6.3, was achieved in [59] by restricting attention to the bi-hyperbolic and energy-distinguishing subclasses of area metric tensor fields.

[5] Actually, one associates to each equivalence class $[k]$ of massless covectors that only differ by

particle. The underlying necessary duality relation can be deduced by variation of the totally constrained Hamiltonian action

$$S[x,k,\lambda] = \int d\tau [\dot{x}^a k_a - \lambda P(x,k)] \tag{2.11}$$

with respect to the momenta k, which yields

$$\dot{x}^a = \lambda (DP)^a(k)\,. \tag{2.12}$$

This can be solved for the momentum k if the principal polynomial $P(x,k)$ is hyperbolic[6]. Elimination of the momentum k from the action (2.11) leads to the equivalent action

$$S_{\text{massless}}[x,\mu] = \int d\tau\, \mu P^{\#}(x,\dot{x})\,, \tag{2.13}$$

whose stationary points yield the worldlines $x(\tau)$ of massless point particles, and which is given in terms of the dual polynomial $P^{\#}(x,v)$, as defined in (2.8). This illustrates the different roles the dual polynomial $P^{\#}(x,v)$ actually plays. On the hand, it is needed to test for quantizability of the matter dynamics $S[G,\Phi]$, and, on the other hand, it single-handedly provides the kinematical spacetime structure as seen by massless point particles.

Employing the chosen time-orientation vector field H on the manifold M, one may single out a particular hyperbolicity cone C_x of the principal polynomial $P(x,p)$ at each point $x \in M$ by requiring that all $p \in C_x$ satisfy $p(H_x) > 0$. Since we have chosen the principal polynomial to be positive on any of its hyperbolicity cones, one has, in particular, that any $p \in C_x$ satisfies

$$P(x,p) = m^{\deg P}\,, \tag{2.14}$$

for some positive real number m and one identifies such covectors p with the momenta of positive energy massive point particles with inertial mass m. As in the case of massless point particles, one would like to associate to each such massive particle momentum the corresponding particle velocity $\dot{x}$, and thus the corresponding point particle worldline $x(\tau)$. A suitable totally constrained Hamiltonian action to describe the dynamics of a massive point particle with worldline $x(\tau)$ is provided by

$$S[x,p,\lambda] = \int d\tau \left[\dot{x}^a p_a - \lambda m \ln P(x, p/m)\right]\,. \tag{2.15}$$

---

a scale factor, the corresponding equivalence class $[v]$ of vectors that only differ by a scale factor, for details see [58].

[6] Inverting such a system of polynomial equations is precisely what Buchberger's algorithm is taylored for. Unfortunately, already for moderate degrees of the principal polynomial, Buchberger's algorithm quickly exceeds the capabilities of modern computer algebra systems. Fortunately, it does not exceed human capability in case of several non-trivial examples.

This particular choice of the action—among the many classically equivalent choices—then proves most convenient when considering the equation of motion arising from the variation with respect to the momentum p,

$$\dot{x}^a(\tau) = \lambda \frac{(DP)^a(x,p/m)}{P(x,p/m)}\,. \tag{2.16}$$

The bi-hyperbolicty and energy-distinguishing properties of the principal tensor field P of a tensorial spacetime $(M, G, S[G,\Phi])$ then ensure that the Legendre map

$$L^a(x,p) := \frac{1}{\deg P}\frac{DP^a(x,p)}{P(x,p)} \tag{2.17}$$

exists[7] on the cones C_x of positive energy massive point particle momenta, and has a unique inverse L^{-1}. Thus, one can solve equation (2.16) for the momentum p, which yields

$$p_a = m\, L_a^{-1}(x, \dot{x}/(\lambda \deg P))\,. \tag{2.18}$$

Just as in the case of massless point particles, it is then possible to eliminate the momentum from the action (2.15), but, this time, also the Lagrange multiplier drops out and one is left with the action

$$S_{\text{massive}}[x] = m\int d\tau\,\Big[P(x, L^{-1}(x,\dot{x}))\Big]^{-1/\deg P}\,, \tag{2.19}$$

which describes the dynamics of a worldline of a massive point particle with inertial mass m. Note that since the inverse Legendre map can be shown to be a homogeneous function of degree -1 in its fibre argument, the action is reparametrization invariant. However, parametrizations with $P(x, L^{-1}(x,\dot{x})) = 1$ are distinguished, because then the relation between velocities and momenta, equation (2.16), simply becomes

$$m\,\dot{x}^a(\tau) = L^a(x,p)\,. \tag{2.20}$$

In this case, the curve parameter τ is called proper time. Using the proper time parametrization along the worldline $x(\tau)$, variation of the action (2.19) with respect

[7]Technically, bi-hyperbolicity and energy-distinguishability ensure that the so-called barrier function $f_x(p) = -(1/\deg P)\ln P(x,p)$, which one employs in the massive point particle action (2.15), is strictly convex on the cone of hyperbolic covectors. Together with the essential smoothness property of the barrier function, in convex analysis one then proves the existence of the Legendre map. For a more detailed account on the mathematical underpinnings, we refer the reader to [58] and the references therein.

to the worldline $x(\tau)$ yields the geodesic equation[8]

$$m\left[\ddot{x}^m g_{am}(x,\dot{x}) + \dot{x}^m\dot{x}^n\partial_n g_{am}(x,\dot{x}) - \frac{1}{2}\dot{x}^m\dot{x}^n\partial_a g_{mn}(x,\dot{x})\right] = 0\,, \tag{2.21}$$

which employs the Finsler metrics $g(x,v)$ defined by

$$g_{mn}(x,v)u^m w^n = \frac{1}{2}\frac{\partial^2\, P^*(x,v+su+tw)^{2/\deg P}}{\partial s\partial t}\bigg|_{s=t=0} \quad \text{for all } u,w\in T_xM\,, \tag{2.22}$$

in terms of the function $P^*(x,v) = [P(x,L^{-1}(x,v))]^{-1}$. In section 7.4, we recover the geodesic equation from the conservation of the energy-momentum tensor that can be associated to a massive point particle.

Finally, future-oriented observers have the property that the tangent vectors $\dot{o}$ to their worldlines o lie in the cones $C^\#_x$ to which the time-orientation vector field H belongs. This identification of observers by their worldline tangent vectors is physically well-motivated by observing the following three properties. First, since one has that $L^{-1}(x,C^\#_x)$ is contained in the cone C_x of positive energy hyperbolic covectors, observers are massive. Secondly, all such future-oriented observers at one point $x\in M$ agree on the sign of the energy $E=p(\dot{o})$ of any massless or massive point particle with momentum p, because of the energy-distinguishability property of tensorial spacetimes. Thirdly, observers are stable, meaning that it is kinematically impossible for their momentum $L^{-1}(\dot{o})$ to decay, i.e., to be a linear combination of some other positive energy massive momentum and a massless momentum. This is why the cone $L^{-1}(C^\#_x)$ is also called stability cone.

Particularly important for what follows is the construction of observer frames and coframes, which now is a straightfoward task using the kinematical framework above. Picking an observer worldline o one, first of all, normalizes the tangent vector $\dot{o}$ choosing proper time parametrization $P(L^{-1}(\dot{o}))=1$ along the worldline. This uniquely determines the dual $L^{-1}(\dot{o})$ of the observers tangent vector $\dot{o}$. Then, picking a basis on the set $\{v\in T_{o(\tau)}M\,|\,L^{-1}(\dot{o})(v)=0\}$, i.e., on the set of all vectors anihilated by $L^{-1}(\dot{o})$, provides the observer's spatial frame $\{e_\alpha\}$ (for $\alpha = 1\ldots\dim M-1$) and completes the observer tangent frame $\{\dot{o},e_\alpha\}$. The observer's coframe $\{L^{-1}(\dot{o}),\epsilon^\alpha\}$ is, in the end, completed by the duality relations $\epsilon^\alpha(\dot{o})=0$ and $\epsilon^\alpha(e_\beta)=\delta^\alpha_\beta$. A geometric illustration of the above kinematical structures, in particular the various cones, is given in figure 2.2.

As described above, any tensorial spacetime $(M,G,S[G,\Phi])$ gives rise to a complete kinematical framework, including massless and massive point particle dynamics and the identification of observers and their laboratory frames and coframes. Now,

[8] This justifies, *a posteriori*, the term inertial mass for the parameter m in the massive dispersion relation (2.14). If an external field exerts a force on the particle, the inertial mass mediates the deviation of the particle worldline from geodesic motion.

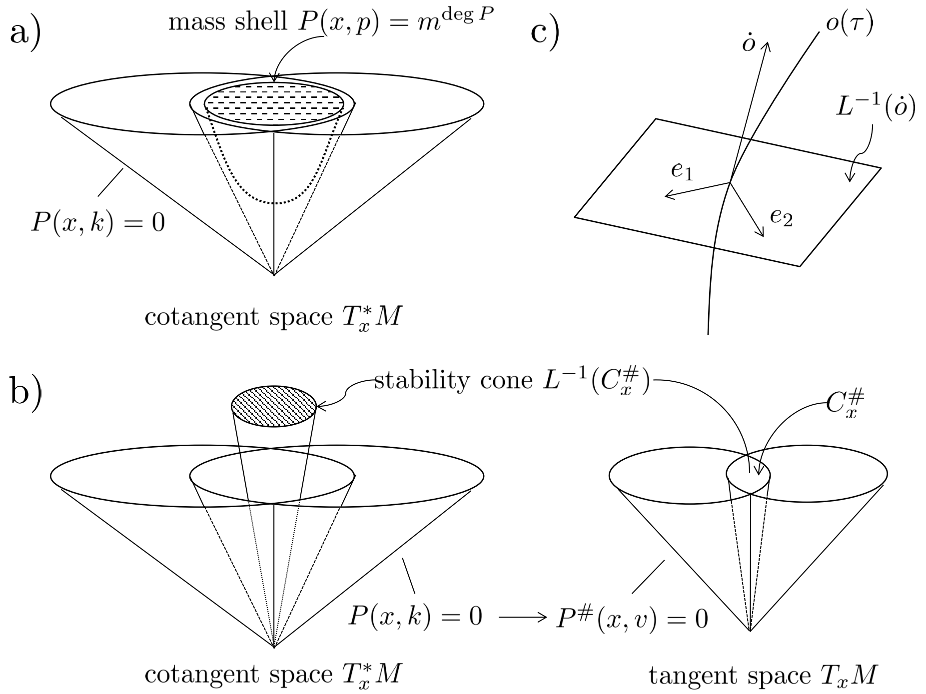


Figure 2.2: Illustration of a) the mass shell of some fourth degree hyperbolic principal polynomial $P(x,k)$; b) the relation of kinematical structures on cotangent space T^*_xM and tangent space T_xM, including the cone $C^\#_x$ of future-oriented observers and its image $L^{-1}(C^\#_x)$, the stability cone; c) the construction of observer frames with the help of the inverse Legendre map L^{-1} from an observer worldline $o(\tau)$ with $\dot{o} \in C^\#_{o(\tau)}$; the plotted plane contains all vectors anihilated by the covector $L^{-1}(\dot{o})$.

the crucial observation for the aim of this thesis is, that hypersurfaces Σ, which can carry initial data for the matter fields Φ and, at the same time, are accessible by future-oriented observers must be such that their conormals n lie in the stability cones $L^{-1}(C^\#_x)$ at every of its points. Only then, one can associate to these normal co-directions n, the corresponding normal directions $T = L(n)$, by making use of the Legendre map (2.17). Considering deformations of such accessible initial data hypersurfaces in their normal directions, which we will discuss in the next chapter, thus requires the kinematical structures imprinted on the geometry by the matter field dynamics. As we will see in chapter 4, such normal deformations hold the key for the derivation of the gravitational master equations that determine the dynamics of the tensorial spacetime geometry G.

2.4 Geometry seen by fields vs. geometry seen by point particles

Throughout this chapter, it was assumed that the geometry G is known on the entire manifold M, and that the matter fields Φ propagated on this fixed background.

What one ultimately wants from a complete theory of tensorial spacetimes, however, is that the geometry G is itself described by dynamical equations. It is the purpose of this work to derive gravitational master equations from which one can calculate, rather than postulate, such dynamical equations. The main idea of how to achieve this is the following: Just as the matter field dynamics, the dynamics for the underlying tensorial spacetime geometry G must be such that an observer can specify initial data for the latter on an accessible initial data hypersurface such that one can then predict its 'future' values, i.e., its values on another such hypersurface, with the help of the to-be-found dynamical equations; sweeping out the manifold M with the geometric data generated in this process, one must thus obtain a tensorial spacetime $(M, G, S[G, \Phi])$, i.e., a geometry G, which is bi-hyperbolic and energy-distinguishing.

However, instead of considering the fundamental tensorial geometry G to which fields couple, one may aim instead only at the induced point particle geometry encoded in the principal tensor field P. Indeed, as we have seen in this chapter, point particles really only see those aspects of the spacetime geometry that are encoded in the principal tensor field P defining the massless and massive dispersion relations. Consequently, using only point particle experiments, there is no way to actually capture all properties of a fundamental spacetime geometry G, which is seen directly only by the matter fields Φ via their dynamics (2.1) (It is a mere coincidence that in Lorentzian geometry, the tensor field P^{ab} is precisely given by the fundamental inverse metric geometry g^{ab} to which also matter fields couple). For a generic tensorial spacetime, the tensor field P can be expressed by the spacetime geometry G, but not necessarily vice versa. Hence, if one is interested in dynamics for the geometry to which fields Φ and point particles couple one has to find dynamics for the fundamental geometry G of the tensorial spacetime

$$(M, G, S[G, \Phi]). \tag{2.23}$$

If, on the other hand, one is interested only in those tensorial spacetimes

$$\left(M, P, S_{\text{massless}}[P, \mu, x], S_{\text{massive}}[P, x]\right), \tag{2.24}$$

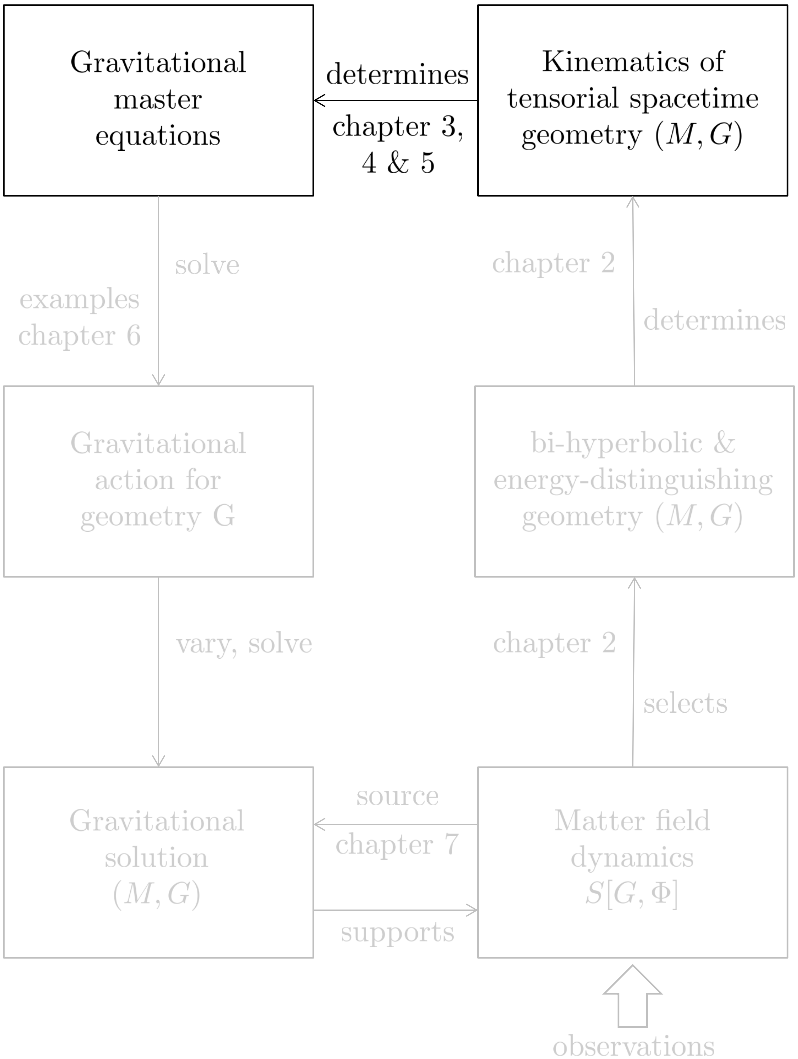
featuring a bi-hyperbolic and energy-distinguishing totally symmetric tensor field P, felt by, and only coupling to, massless and massive point particles, one can aim at finding dynamics only for the tensor field P. An advantage of considering such spacetime geometries seen by point particles is that they can only differ by the rank of the totally symmetric tensor field P, which due to the propetries of tensorial spacetimes must always be even[9]. Such a simple classification is not available for spacetime

[9]This is because bi-hyperbolicity and energy-distinguishability exclude the existence of planes

geometries G to which arbitrary types of matter fields can couple. Those have to be dealt with in a case-by-case analysis. We will derive the gravitational master equations for three fundamental spacetime geometries: metric geometry supporting Maxwell electrodynamics in section 6.2, area metric geometry carrying general linear electrodynamics in section 6.3, and for the background geometry of modified Dirac matter in section 6.4. In section 6.5, we will then also derive the master equations for any bi-hyperbolic and energy-distinguishing point particle geometry P. Thus we cover both points of view in this thesis.

The first ingredient for the derivation of gravitational master equations, both for the spacetime geometry seen by particular matter fields and the spacetime geometry seen by point particles, is provided by the theory of deformations of observer-accessible initial data hypersurfaces, which is the topic of the next chapter.

in the vanishing set of the principal polynomial $P(x, k)$, and homogeneous hyperbolic polynomials of odd degree can be shown to always contain at least one such null plane [58].



Gravitational master equations
determines
chapter 3, 4 & 5
Kinematics of tensorial spacetime geometry (M, G)
solve
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Chapter 3

Hypersurface deformations and induced change of hypersurface geometries

We now consider some arbitrary admissible tensorial spacetime $(M, G, S[G, \Phi])$, and, in it, an observer-accessible initial data hypersurface. We identify the hypersurface data one needs to collect in order to completely reconstruct the spacetime geometry G along this hypersurface. We then investigate the change of such hypersurface data if the hypersurface is deformed. It turns out that this change of hypersurface geometric data can be described by linear differential operators that bring about normal and tangential deformations of the observer-accessible initial data hypersurfaces. The key observation, in this chapter, is that these linear operators satisfy a so-called hypersurface deformation algebra which we will derive for any admissible tensorial spacetimes and which depends explicitly on the previously specified matter theory (2.1) via the thus induced Legendre map (2.17).

The results presented in this chapter have been published as
K. Giesel, F. P. Schuller, C. Witte and M. N. R. Wohlfarth,
Phys. Rev. D85 (2012), 104042,
whose section II.B., in particular, is presented here.

3.1 Initial data hypersurfaces

A hypersurface $X(\Sigma)$ in an ambient $(\dim M)$-dimensional manifold M is described by an embedding $X : \Sigma \hookrightarrow M$ of a $(\dim M - 1)$-dimensional manifold Σ into M. More precisely, using coordinates $\{x^a\}$ on the manifold M and a set of coordinates $\{y^\alpha\}$ on the manifold Σ, such an embedding is given by $\dim M$ functions $X^a : y^\alpha \mapsto X^a(y^\alpha)$. Here and in the remainder of this thesis, small greek indices always refer to hypersurface coordinates $\{y^\alpha\}$. Such an embedding immediately induces $\dim M - 1$ spacetime vectors

$$e_\alpha(y) = \frac{\partial X^a(y)}{\partial y^\alpha}\frac{\partial}{\partial x^a} \tag{3.1}$$

tangent to the hypersurface $X(\Sigma)$ where $\{\partial/\partial x^a\}$ is the coordinate-induced basis of the tangent space $T_{X(y)}M$, in other words, the e_α are the push-forwards of the coordinate-induced vector fields $\partial_\alpha = \partial/\partial y^\alpha$ on the manifold Σ to the manifold M under the embedding map X. In order to lighten the notation, we agree on considering spacetime quantities carrying solely the hypersurface label y, such as $e_\alpha(y)$, to actually mean $e_\alpha(X(y))$.

In addition to the hypersurface tangent vectors, we may define, up to a so far undetermined scale, the conormals n to the hypersurface $X(\Sigma)$ by

$$n(y)(e_\alpha(y)) = 0 \quad \alpha = 1, \dots, \dim M\,. \tag{3.2}$$

Without any further structure, there is no way to complete the set of vectors $\{e_\alpha\}$ to form a tangent space basis, or, equivalently, the single covector n to form a cotangent space basis. In an admissible tensorial spacetime with geometry G, however, we have sufficient structure to effect presicely such a completion of bases. Since we want to restrict our attention to hypersurfaces which are accessible to observers, we must, first of all, require that the conormals n to the hypersurface lie in the stability cones $L^{-1}(C^\#_{X(y)})$ of the chosen time orientation $C^\#_x$. This fixes the orientation of the conormals n. But then we can, secondly, also normalize the covectors n by requiring

$$P(X(y), n(y)) = 1 \tag{3.3}$$

in order to eliminate the formerly arbitrary scale. Thirdly, but most importantly, we can convert the unique normal covector n into the unique normal vector T by virtue of the Legendre map (2.17):

$$T^a(y) = \frac{(P^{aa_2\dots a_{\deg P-1}}\, n_{a_2}\dots n_{\deg P-1})(y)}{P(X(y), n(y))}\,. \tag{3.4}$$

This vector T then completes the vectors e_α to form a basis of $T_{X(y)}M$. The respective coframes $\{n(y), \epsilon^\alpha(y)\}$ are then fully determined by the duality requirements

$$\epsilon^\alpha(y)(T(y)) = 0 \quad \text{and} \quad \epsilon^\alpha(y)(e_\beta(y)) = \delta^\alpha_\beta\,, \quad \alpha, \beta = 1, \dots \dim M - 1\,. \tag{3.5}$$

Thus, an observer-accessible initial data hypersurface $X(\Sigma)$ induces a complete observer frame, and observer coframe, at each of its points. The completeness relation $\delta^a_b = T^a n_b + e^a_\alpha \epsilon^\alpha_b$ is readily checked by contracting the left and right hand sides with all the individual frame fields. As we have seen in the previous chapter, the normalization condition (3.3) is equivalent to requiring the normal vector $T(y)$ to the hypersurface being the tangent to an observer worldline with proper time parametrization.

Given a particular admissible tensorial spacetime geometry G, we would now like to identify the hypersurface data one needs to collect in order to reconstruct the spacetime geometry along the hypersurface with the help of the induced observer frames and coframes.

3.2 Induced hypersurface geometries

With the help of the observer frames and coframes one can calculate the hypersurface geometry that is induced by the tensorial spacetime geometry G on the initial data hypersurface $X(\Sigma)$. In order to conversely reconstruct the spacetime geometry along the hypersurface $X(\Sigma)$, we need to identify the independent geometric inital data from these induced fields. Such parametrizations of the tensorial spacetime geometry G in terms of hypersurface initial data will be the starting point in the derivation of gravitational dynamics. Since the parametrization process highly depends on the particular tensorial spacetime geometry G and the matter dynamics it carries, we discuss three representative examples.

3.2.1 The hypersurface geometry induced by the spacetime geometry to which point particles couple

Let us first of all consider the case where the tensorial spacetime geometry G coincides with the totally symmetric tensor field P with components $P^{a_1 \dots a_{\deg P}}$ to which point particles couple according to (2.13) if they are masseless, and according to (2.19) if they are massive. Such a tensor field induces a set of $(\deg P + 1)$ spatial tensor fields

$$P^{\alpha_1 \dots \alpha_I}(y)[X] := P(X(y), \underbrace{\epsilon^{\alpha_1}(y), \dots, \epsilon^{\alpha_I}(y)}_{I}, \underbrace{n(y), \dots, n(y)}_{\deg P - I}) \qquad \text{for } I = 0, \dots, \deg P \tag{3.6}$$

on the hypersurface $X(\Sigma)$, which are all functionals of the embbeding map X. Here we made use of the observer coframes $\{n, \epsilon^\alpha\}$ defined in the previous section, in order to project the spacetime components of the tensor field P onto the hypersurface $X(\Sigma)$. Due to the frame conditions $P(X(y), n(y)) = 1$ and $T^a(y)\epsilon^\alpha_a(y) = 0$, however, not all of the above hypersurface tensor fields can be independent. In fact, for $I = 0$, where all spacetime indices on the tensor field P have been projected with the help of the conormal n, we have that $P = 1$, and, for $I = 1$, the functional P^α vanishes everywhere on $X(\Sigma)$. Thus only the remaining hypersurface tensor fields with $I = 2, \dots, \deg P$ encode the hypersurface geometry. This amounts to

$$\sum_{I=2}^{\deg P} \binom{I + \dim M - 2}{I} = \binom{\deg P + \dim M - 1}{\deg P} - \dim M \tag{3.7}$$

real degrees of freedom at every point of the hypersurface $X(\Sigma)$. Conversely, one can reconstruct the spacetime tensor field P along the hypersurface with the help of the hypersurface geometry and the frame fields $\{T^a, \epsilon^\alpha\}$. This reconstruction is afforded by the completeness relation

$$P^{a_1 \dots a_{\deg P}}(y) = T^{a_1} \dots T^{a_{\deg P}} + \sum_{I=2}^{\deg P} \binom{\deg P}{I} P^{\alpha_1 \dots \alpha_I} \, e^{(a_1}_{\alpha_1} \dots e^{a_I}_{\alpha_I} \, T^{a_{I+1}} \dots T^{a_{\deg P})} \,, \tag{3.8}$$

which may be checked by contracting both sides with all possible combinations of the frame fields.

3.2.2 The hypersurface geometry induced by a fundamental tensorial spacetime geometry

We now consider a fundamental tensorial spacetime geometry G to which fields Φ couple according to (2.1) and which does not coincide with the point particle geometry discussed in the previous subsection. For illustrational purposes, we consider an area metric geometry G^{abcd} in four dimensions and as matter dynamics on this geometry we take general linear electrodynamics, as discussed in section 6.3. Then the tensor field P, which defines the principal polynomial (6.44) of the matter field equations, is given by

$$P_G^{abcd} = -\frac{24}{(G^{ijkl}\epsilon_{ijkl})^2} \, \epsilon_{mnpq}\epsilon_{rstu} \, G^{mnr(a}G^{b|ps|c}G^{d)qtu}, \tag{3.9}$$

as is explained in section 6.3, to which the reader may refer at this point or alternatively take this result on trust for the time being. Using the complete coframe $\{n, \epsilon^1, \epsilon^2, \epsilon^3\}$ built with the help of the tensor field P_G, the area metric induces three different hypersurface tensor fields,

$$\begin{aligned}
G^{\alpha\beta}(y)[X] &= G(n(y), \epsilon^\alpha(y), n(y), \epsilon^\beta)\,, && (3.10)\\
G^{\alpha}{}_{\beta}(y)[X] &= \frac{1}{2}\omega_{G\beta\gamma\delta}\, G(n(y), \epsilon^\alpha(y), \epsilon^\gamma(y), \epsilon^\delta(y))\,, && (3.11)\\
G_{\alpha\beta}(y)[X] &= \frac{1}{4}\omega_{G\alpha\gamma\delta}\,\omega_{G\beta\mu\nu} G(\epsilon^\gamma(y), \epsilon^\delta(y), \epsilon^\mu(y), \epsilon^\nu(y)) && (3.12)
\end{aligned}$$

on the hypersurface $X(\Sigma)$. Here we assume that the functional $G^{\alpha\beta}(y)$ is invertible and may thus be formally employed as a metric on the hypersurface $X(\Sigma)$. This enables us to make use of the hypersurface volume form $\omega_{G\alpha\beta\gamma} = (|\det G^{\alpha\beta}|)^{-1/2}\epsilon_{\alpha\beta\gamma}$ in order to dualize antisymmetric pairs of indices for later convenience. However, because of the normalization condition (3.3) and the relation $T^a(y)\epsilon^\alpha_a(y) = 0$, which both explicitly contain the area metric tensor G through the tensor field P_G, not all components of the above functionals (3.10)-(3.12) can be independent. For one finds that $P_G(X(y), n(y)) = 1$ is equivalent to $G^\alpha{}_\alpha(y) = 3$, which fixes the trace of the second functional. Moreover, $T^a(y)\epsilon^\alpha_a(y) = 0$ reduces to $\omega_{G\,\alpha\beta\gamma}\, G^\beta{}_\rho\, G^{\rho\gamma} = 0$, which shows that the functional $G^\alpha{}_\beta[X(y)]$ must be symmetric with respect to the metric $G^{\alpha\beta}(y)$. Hence, taking these relations into account, one recognizes that the hypersurface geometry induced by the background of general linear electrodynamics on an observer-accessible initial data hypersurface consists of an invertible symmetric tensor field $G^{\alpha\beta}(y)$, a symmetric tensor field $G_{\alpha\beta}(y)$ and a tracefree tensor field $G^\alpha{}_\beta(y)$ which is symmetric with respect to the tensor field $G^{\alpha\beta}(y)$. This is also entirely plausible since the fields $(G^{\alpha\beta}, G^\alpha{}_\beta, G_{\alpha\beta})$ altogether feature 17 independent components, which together with the four frame conditions account for the 21 independent components of an area metric tensor G^{abcd} in four dimensions. Similar to the point particle case, the area metric tensor can be reconstructed from the fields $(G^{\alpha\beta}, G^\alpha{}_\beta, G_{\alpha\beta})$; the necessary completeness relation takes the form

$$\begin{aligned} G^{abcd}(X(y)) &= 4\, G^{\beta\delta}\, T^{[a}\, e^{b]}_\beta\, T^{[c} e^{d]}_\delta + G_{\rho\sigma}(\omega_G^{-1})^{\rho\alpha\beta}(\omega_G^{-1})^{\sigma\gamma\delta}\, e^a_\alpha\, e^b_\beta\, e^c_\gamma\, e^d_\delta \\ &\quad +2(G^\beta{}_\rho + \delta^\beta_\rho)(\omega_G^{-1})^{\rho\gamma\delta}\, T^{[a}\, e^{b]}_\beta e^c_\gamma e^d_\delta. \end{aligned} \tag{3.13}$$

As a second example for a fundamental spacetime geometry, we consider the tensorial spacetime $(M, g, S_{\mathrm{Maxwell}}[g, A])$, consisting of a Lorentzian manifold (M, g) and Maxwell theory as the matter dynamics, as discussed in section 6.2. Now the tensor field P, which defines the principal polynomial of the matter field equations, is given by

$$P^{ab} = g^{ab}\,. \tag{3.14}$$

The frame conditions $P(n) = 1$ and $T^a\epsilon^\alpha_a = 0$ imply that the only functionals comprising the spatial geometry on the hypersurface $X(\Sigma)$ are

$$g^{\alpha\beta}(y)[X] = g^{ab}\epsilon^\alpha_a\epsilon^\beta_b\,, \tag{3.15}$$

with $\dim M(\dim M - 1)/2$ independent components. The completeness relation $g^{ab} = T^aT^b + g^{\alpha\beta}e^a_\alpha e^b_\beta$ then reconstructs the inverse spacetime metric. We also note that, because of our normalization condition $P(X(y), n(y)) = 1$, hyperbolicity forces the spatial metric $g^{\alpha\beta}$ to be negative definite. This is of course the standard construction of the spatial geometry on a spacelike hypersurface in general relativity but here now understood as dictated by the causal structure of Maxwell theory on a metric background (which was exactly Einstein's point [18]).

In general, it is crucial to identify the independent degrees of freedom of the induced spatial geometry, as we have demonstrated above for area metric geometry carrying general linear electrodynamics and metric geometry carrying standard Maxwell electrodynamics. The reader be warned that depending on how the fundamental geometry G enters the corresponding principal tensor field P, it might not be possible to parametrize the frame conditions $P(X(y), n(y)) = 1$ and $T^a(y)\epsilon^\alpha_a(y) = 0$ with the help of the induced fields. This caveat might prevent the reader from finding gravitational dynamics for the geometry underlying his favourite matter model already at this stage—which would then outrule the matter model in the first place, since the coefficients of the matter equations of motion could not be determined dynamically from initial geometric data in this case. A particular example of such a prohibitive scenario is provided by the following variant of general linear electrodynamics, where the scalar density $G^{abcd}\epsilon_{abcd}$ in the principal polynomial (3.9) is not constructed from the non-cyclic part of the area metric tensor G, as described in section 6.3, but from the determinant of the area metric instead, considering the latter as a bilinear form on the space of two forms [64]. While the frame condition $T^a(z)\epsilon^\alpha_a(z) = 0$ then still leads to a symmetry property of the functional $G^\alpha{}_\beta(y)$, the normalization condition $P(X(y), n(y)) = 1$ does not provide a parametrization in terms of the properties of the induced fields $(G^{\alpha\beta}, G^\alpha{}_\beta, G_{\alpha\beta})$. This is because the mentioned determinant of the area metric tensor G^{abcd} is a highly complicated function of the induced fields which cannot be solved explicitly, so that the normalization condition cannot be implemented in this case by way of explicit conditions on the induced geometry.

3.3 Deformations of initial data hypersurfaces

We now wish to consider deformations of observer-accessible initial data hypersurfaces. Technically, this can be done by considering a one-paramter family of embeddings X_t, such that the original embedding map X is recovered for $t = 0$. We then require that the hypersurfaces $X_t(\Sigma)$ inherit their coordinates $\{y^\alpha \circ X_t^{-1}\}$ from

$$\begin{tikzcd} \Sigma \arrow[r, "X_t"] \arrow[d, "y"'] & X_t(\Sigma) \subset M \arrow[dl, "y \circ X_t^{-1}"] \\ \mathbb{R}^{\dim \Sigma} & \end{tikzcd}$$

In this way, we obtain a second system (t, y^α) of spacetime coordinates adapted to

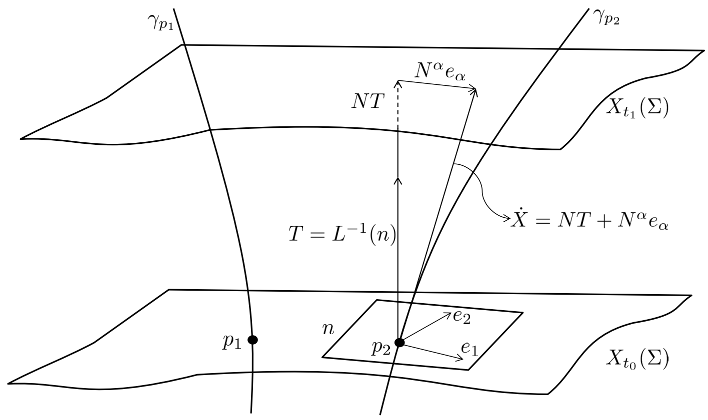

Figure 3.1: Deformation of an initial data hypersurface $X_{t_0}(\Sigma)$ into the nearby hypersurface $X_{t_1}(\Sigma)$. The curves γ_{p_i} are parametrized by the hypersurface label t and distinguished by their coordinate expression $\phi \circ \gamma_{p_i} = (t, y(p_i))$ in the chart ϕ adapted to the foliation X_t. The conormal n to the hypersurface $X_{t_0}(\Sigma)$ at the point p_2 is depicted by the plane of vectors it anihilates. The components of the connecting vector field $\dot{X}$ in the basis $\{T, e_\alpha\}$ are given by the lapse function N and the components N^α of the shift vector field.

the foliation X_t as the map $\phi(X_t(p)) = (t, y(p))$ with $p \in \Sigma$, in addition to the $\{x^a\}$ coordinates we used before. This construction then distinguishes a congruence consisting of the curves $\gamma_p = \phi^{-1}(t, y(p))$ for every point $p \in \Sigma$ that connects the points on each hypersurface $X_t(\Sigma)$ having the same hypersurface coordinates. Along the undeformed hypersurface $X(\Sigma)$, the connecting vector field $\dot{X}$ defined by $\dot{X}|_{X(p)} = (\partial\gamma_p/\partial t)|_{X(p)}$ for all $p \in \Sigma$, can be uniquely decomposed into a purely normal and a purely tangential part,

$$\dot{X}(y) = N(y)T(y) + N^\alpha(y)e_\alpha(y)\,, \tag{3.16}$$

where the hypersurface scalar field $N(y)$ and the hypersurface vector field components $N^\alpha(y)$ are given by

$$N(y) = n(y)(\dot{X}(y)) \quad \text{and} \quad N^\alpha(y) = \epsilon^\alpha(y)(\dot{X}(y)) \tag{3.17}$$

and thus completely parametrize any small deformation of the embedding map X into the deformed embedding map $X_0 + dt\,\dot{X}$ (see figure 3.1). In Lorentzian geometry, the hypersurface scalar field $N(y)$ is called the lapse function, whereas the the hypersurface vector field $\vec{N}(y) := N^\alpha(y)e_\alpha(y)$ is called the shift vector field and we will simply adopt this terminology.

Ultimately, we will be interested in how the hypersurface geometry, which is induced on the hypersurfaces $X_t(\Sigma)$ by the spacetime geometry G, changes along the foliation X_t. To this end, we consider first an arbitrary functional F of the initial embedding map X and investigate how it changes under hypersurface daformations. To linear order, this change can be encoded in the action of linear operators that act as functional derivatives. More precisely, we define the *normal* deformation of the functional F, parametrized by the lapse function N, as the quantity $\mathcal{H}(N)F$, where the linear functional differential operator $\mathcal{H}(N)$ is defined by

$$\mathcal{H}(N) := \int_\Sigma dy\, N(y)\, T^a(y)\, \frac{\delta}{\delta X^a(y)} \tag{3.18}$$

and $\delta/\delta X^a(y)$ denotes the functional derivative with respect to the embedding map X. Likewise the *tangential* deformation $\mathcal{D}(\vec{N})F$ of the functional F is defined by the action of the linear functional differential operator

$$\mathcal{D}(\vec{N}) := \int_\Sigma dy\, N^\alpha(y)\, e^a_\alpha(y)\, \frac{\delta}{\delta X^a(y)} \tag{3.19}$$

and parametized by the shift vector field $\vec{N}$.

Before we can proceed, we wish to fix some notation that will be heavily used in this and the following chapters; namely the definition of the Dirac delta distribution as the functional $\delta_z : f \mapsto \delta_x[f]$ mapping a smooth scalar test function f on the manifold Σ to the real numbers:

$$\delta_z[f] = \int_\Sigma dy f(y)\delta_z(y) := f(z)\,. \tag{3.20}$$

It follows from this definition that the delta distribution transforms as a *scalar density* of weight $+1$ in its second argument and as a *scalar* in its subscript argument. To lighten the notation, we will frequently follow standard custom and suppress the integral in the definition of the delta distribution writing $\delta_z(y)$ only. However, any equation containing such an expression is implicitly understood to only hold when integrated over appropriate test functions.

Now a check on the geometric meaning of the normal and tangential deformation operators is provided by letting them act on the embedding functions X, which, trivially, are functionals of themselves. Noting that $\delta X^a(z)/\delta X^b(y) = \delta^a_b\, \delta_z(y)$, and after performing the necessary integrations, one readily finds

$$\mathcal{H}(N)X^a(z) = N(z)T^a(z) \quad \text{and} \quad \mathcal{D}(\vec{N})X^a(z) = N^\alpha(z)e^a_\alpha(z)\,, \tag{3.21}$$

which are the normal and tangential components of the connecting vector field $\dot{X}$, which indeed describes the first order deformation of the initial hypersurface $X(\Sigma)$.

We are now in the position to determine how the hypersurface frame fields on the one hand, and the geometry induced by a tensorial spacetime geometry G on the other hand, change under hypersurface deformations.

3.3.1 Change of frame fields under hypersurface deformations

In order to calculate how functionals such as (3.6) and (3.10)-(3.12) change under hypersurface deformations generated by the operators (3.18) and (3.19), we first need to derive how the frame fields behave under changes of the embedding map X. For the tangent vector fields e_α, one sees directly from their definition (3.1) that

$$\frac{\delta e^a_\alpha(y)}{\delta X^b(z)} = -\delta^a_b\,\partial_\alpha \delta_y(z)\,, \tag{3.22}$$

where the partial derivative in front of the delta function is understood to act on the second argument of the latter, not the subscript. Secondly, equation (3.2) implies that

$$e^a_\alpha(y)\delta n_a(y) = -n_a(y)\delta e^a_\alpha(y), \tag{3.23}$$

while the normalization equation (3.3) yields

$$T^a(y)\delta n_a(y) = -\frac{1}{\deg P}\,\partial_b P^{j_1\dots j_{\deg P}}(y) n_{j_1}(y)\dots n_{j_{\deg P}}(y)\,\delta X^b(y). \tag{3.24}$$

Combining the last three equations (3.22)-(3.24), we obtain how the conormal n behaves under hypersurface deformations:

$$\frac{\delta n_a(y)}{\delta X^b(z)} = -\frac{1}{\deg P}(n_a n_{j_1}\dots n_{j_{\deg P}}\partial_b P^{j_1\dots j_{\deg P}})(y)\delta_y(z) + n_b(y)\epsilon^\alpha_a(y)\partial_\alpha\delta_y(z)\,. \tag{3.25}$$

One may proceed for the remaining frame fields T and ϵ^α in precisely the same fashion, using their defining equations (3.4) and (3.5). We spare the reader the intermediate steps and only present the result of the calculation:

$$\begin{aligned}\frac{\delta T^a(y)}{\delta X^b(z)} &= (\deg P - 1)(e^a_\alpha n_b P^{\alpha\beta})(y)\,\partial_\beta\delta_y(z) + \left(n_{j_2}\dots n_{j_{\deg P}}\partial_b P^{a j_2\dots j_{\deg P}}\right)(y)\,\delta_y(z)\\ &\quad -\left[\frac{\deg P - 1}{\deg P} T^a n_{j_1}\dots n_{j_{\deg P}}\partial_b P^{j_1\dots j_{\deg P}}\right](y)\,\delta_y(z), \qquad (3.26)\\ \frac{\delta \epsilon^\alpha_a(y)}{\delta X^b(z)} &= \epsilon^\alpha_b(y)\epsilon^\beta_a(y)\partial_\beta\delta_y(z) - (\deg P - 1)n_a(y)n_b(y)P^{\alpha\beta}(y)\partial_\beta\delta_y(z)\\ &\quad -(n_a\epsilon^\alpha_{j_1}n_{j_2}\dots n_{j_{\deg P}}\partial_b P^{j_1\dots j_{\deg P}})(y)\delta_y(z)\,. \qquad (3.27)\end{aligned}$$

In the last two equations, the appearance of the particular hypersurface functional $P^{\alpha\beta}$, defined by (3.6) for $I = 2$, can be traced back directly to the use of the Legendre map in the definition of the hypersurface frames.

Finally, we perform a consistency check on our construction, by testing whether the frame conditions $P(X(y), n(y)) = 1$ and $T^a(y)\epsilon^\alpha_a(y) = 0$ are preserved under arbitrary deformations parametrized by a lapse N and a shift vector field $\vec{N}$. We obtain

$$[\mathcal{H}(N)+\mathcal{D}(\vec{N})]P(X(z),n(z)) = \int_\Sigma dy\left[N(y)T^b(y)+N^\alpha(y)e^b_\alpha(y)\right]\times \\ \times\left[\deg P\, P^{a_1\ldots a_{\deg P}}\epsilon^\alpha_{(a_1}n_{a_2}\ldots n_{a_{\deg P})}\,n_b\right](z)\partial_\alpha\delta_z(y) \qquad (3.28)$$

and after elimination of the delta function, we indeed find

$$[\mathcal{H}(N)+\mathcal{D}(\vec{N})]P(X(z),n(z)) = -\deg P\, n_a(z)\,\partial_\alpha\dot{X}^a(z)P^\alpha[X(z)]\;\equiv 0\,, \qquad (3.29)$$

which implies that the normalization condition (3.3) is preserved under hypersurface deformations. The same can be shown for the second frame condition, where acting with the deformation operators on $T^a(z)\epsilon^\alpha_a(z)$ produces only terms proportional to the functional $P^\alpha[X(z)]$, which vanishes. This is indeed independent of whether one considers the spatial point particle geometry (3.6) or the more fundamental geometry G to which also fields can couple. Thus, if one has found a parametrization of the frame conditions in terms of the fields induced by the fundamental geometry G, this parametrization will be preserved. For example, considering the area metric geometry underlying general linear electrodynamics, the functional $G^\alpha{}_\beta$ remains tracefree and symmetric under hypersurface deformations. With these results, we may now derive how the normal and tangential deformation operators (3.18) and (3.19) act on the spatial geometry.

3.3.2 Induced deformation of hypersurface point particle geometries

We now start with the remaining hypersurface tensor fields for $I=2,\ldots,\deg P$ for the case of the pure point particle geometries given by (3.6) and calculate their change under hypersurface deformations. The case of a fundamental geometry is then treated in the immediately following subsection. The calculations are rather lengthy, but straightforward, and we only present the results here. Acting with the normal deformation operator $\mathcal{H}(N)$ one obtains

$$\mathcal{H}(N)P^{\alpha_1\ldots\alpha_I}(z) = N(z)\left[(\mathcal{L}_T P)^{a_1\ldots a_{\deg P}}\epsilon^{\alpha_1}_{a_1}\ldots\,\epsilon^{a_I}_{\alpha_I}\,n_{a_{I+1}}\ldots\,n_{a_{\deg P}}\right](z) \\ +\partial_\gamma N(z)\left[I(\deg P-1)P^{(\alpha_1\ldots\alpha_{I-1}}P^{\alpha_I)\gamma}-(\deg P-I)P^{\alpha_1\ldots\alpha_I\gamma}\right](z)\,, \qquad (3.30)$$

for $I=2,\ldots,\deg P$. One can see that the normal deformation operator generates two parts, one of which is local in the lapse function N and essentially given by the Lie derivative $\mathcal{L}_T$ of the spacetime tensor field P along the normal vector T projected to the hypersurface. The second part is non-local in the lapse function, and only contains the induced hypersurface geometry (3.6) in rather complicated fashion, but no derivatives. The explicit form of the first of the two terms in the non-local part can be traced back to the Legendre map (2.17) used in the definition of the tangent and cotangent frames[1]. The explicit form of the non-local terms will play a central role in our derivation of gravitational dynamics for admissible tensorial spacetimes.

In the case $\deg P = 2$, one can see from the above expression that the non-local term in $\mathcal{H}(N)P^{\alpha\beta}(z)$ vanishes. This is the case independent of whether the spacetime tensor field P is induced from a fundamental geometry or not[2]. The only other field for which there is no non-local part, when it is acted upon by the normal deformation operator, is a hypersurface scalar field induced by a spacetime scalar field[3]. This innocent observation will be put to good use when we derive the gravitational dynamics of the tensorial spacetime geometry supporting modified Dirac matter in chapter 6.4.

The first term in the above expression (3.30) can obviously not be expressed in terms of quantities that are intrinsic to the initial hypersurface $X_0(\Sigma)$, since the Lie derivative containing the partial spacetime derivative of the hypersurface normal vector T can only be calculated if the entire one-parameter family of embeddings X_t, and more importantly the spacetime geometry G away from X_0, are already known. Thus, in order to express the change of the fields (3.6) under normal hypersurface deformations purely from hypersurface tensor fields, i.e., without reference to data in the vicinity of $X_0(\Sigma)$, one has to specify the values of the totally symmetric normal velocities

$$K^{\alpha_1\dots\alpha_I}(z) = \left[(\mathcal{L}_T P)^{a_1\dots a_{\deg P}}\epsilon^{\alpha_1}_{a_1}\dots\epsilon^{a_I}_{\alpha_I}\, n_{a_{I+1}}\dots n_{a_{\deg P}}\right](z) \tag{3.31}$$

for $I = 2, \dots, \deg P$ by hand.

In contrast, the action of the tangential deformation operator (3.19) on the induced geometry (3.6) has a much simpler form. One readily calculates that

$$\mathcal{D}(\vec{N})P^{\alpha_1\dots\alpha_I}(z) = \mathcal{L}_{\vec{N}} P^{\alpha_1\dots\alpha_I}(z)\,, \tag{3.32}$$

for $I = 2, \dots, \deg P$, and in this case $\mathcal{L}_{\vec{N}}$ denotes the Lie derivative intrinsic to

[1]Kuchař was the first one to find such non-local contributions when he discussed the hypersurface dynamics of arbitrary tensor fields on metric manifolds [39, 40]. Due to the partial derivative of the lapse functions, he interpreted these terms as the reaction of the hypersurface fields under hypersurface tilts around some fixed points $X(\Sigma)$.

[2]In metric geometry, this allows one to interpret a hypersurface tilt as a Lorentz transformation of the tangent frame at one point of the hypersurface [39].

[3]For a spacetime scalar field, even the degree $\deg P$ of the spacetime geometry P is irrelevant. The hypersurface scalar field it induces will never exhibit a non-local part when acted upon by the normal deformation operator, simply because no frame vector is needed for its projection to the hypersurface.

the hypersurface along the hypersurface vector field $\vec{N}$. Thus, and as opposed to the normal deformation operator, the action of the tangential deformation operator can be described entirely by intrinsic hypersurface quantities. Moreover, from the explicit form of (3.32), we conclude that the tangential deformation operator generates infinitesimal hypersurface diffeomorphisms parametrized by the shift vector field when acting on hypersurface tensors. This is indeed a generic feature of tangential deformations; the operator $\mathcal{D}(\vec{N})$ merely reshuffles the geometric data on the hypersurface and does not care for how the hypersurface is embedded into spacetime.

Combining the two expressions (3.30) and (3.32) we denote by

$$\begin{aligned} \dot{P}^{\alpha_1 \dots \alpha_I}(z) \quad &:= \quad \left[H(N) + D(\vec{N})\right] P^{\alpha_1 \dots \alpha_I}(z) \\ &= \quad (N K^{\alpha_1 \dots \alpha_I} + \partial_\gamma N \, M^{\alpha_1 \dots \alpha_I \, \gamma} + \mathcal{L}_{\vec{N}} P^{\alpha_1 \dots \alpha_I})(z) \end{aligned} \tag{3.33}$$

for $I = 2, \dots, \deg P$ the first order change of the functionals (3.6) under arbitrary deformations of the hypersurface X; here the velocities $K^{\alpha_I \cdots \alpha_{\deg P}}$ are given by (3.31) and

$$M^{\alpha_1 \dots \alpha_I \, \gamma} := I(\deg P - 1) P^{(\alpha_1 \dots \alpha_{I-1}} P^{\alpha_I)\gamma} - (\deg P - I) P^{\alpha_1 \dots \alpha_I \gamma} \tag{3.34}$$

is used as a shorthand for the part that is non-local in the lapse function N. The induced fields (3.6) together with the velocities (3.31) may thus be regarded as initial data for the spatial geometry seen by point particles. Since in this chapter we still assume to have access to the entire spacetime geometry, we can of course *calculate* the values of the induced fields and their velocities on each hypersurface of the foliation X_t. The change of the geometry along the foliation is then simply the change of the values of these fields from one hypersurface to the next. It is the purpose of this thesis to find equations of motion that *generate* this change in the fields without knowing the spacetime geometry in the first place, and starting only from data (3.6) and (3.31) on one hypersurface $X(\Sigma)$.

3.3.3 Induced deformation of fundamental hypersurface geometries

The situation does not change significantly if one considers a fundamental tensorial spacetime geometry G (to which fields can couple) and the fields it induces on the hypersurface $X(\Sigma)$. Of course, the explicit form of the velocities K and the coefficients $M^{\cdots\gamma}$ depends on the geometry one considers and, more importantly, on the matter dynamics imposed on it. But they are calculated again by acting with the normal and tangential deformation operators on the induced fields. The action of the tangential deformation operator $\mathcal{D}(\vec{N})$ on the induced fields is always that of a spatial Lie derivative generating spatial diffeomorphisms. For illustrational purposes, we present the first order changes of the functionals (3.10)-(3.12) that describe the hypersurface geometry of general linear electrodynamics:

$$\begin{aligned}
\dot{G}^{\alpha\beta}(z) &= (NK^{\alpha\beta} - 2\partial_\gamma N(\omega_G^{-1})^{\delta\gamma(\alpha}G^{\beta)}{}_\delta + \mathcal{L}_{\vec{N}}G^{\alpha\beta})(z)\,, && (3.35)\\
\dot{G}^\alpha{}_\beta(z) &= (NK^\alpha{}_\beta - \partial_\gamma N\left[3\,\omega_{G\beta\sigma\tau}G^{\alpha\sigma}P_G^{\tau\gamma} + (\omega_G^{-1})^{\gamma\alpha\sigma}G_{\sigma\beta}\right] + \mathcal{L}_{\vec{N}}G^\alpha{}_\beta)(z)\,, && \\
& && (3.36)\\
\dot{G}_{\alpha\beta}(z) &= (NK_{\alpha\beta} - 6\,\partial_\gamma N\,\omega_{G\sigma\tau(\beta}G^\sigma{}_{\alpha)}P_G^{\tau\gamma} + \mathcal{L}_{\vec{N}}G_{\alpha\beta})(z)\,. && (3.37)
\end{aligned}$$

Here we have already introduced the velocities $(K^{\alpha\beta}, K^\alpha{}_\beta, K_{\alpha\beta})$ in order to represent the contributions local in the lapse function. We again stress the fact that these contributions actually depend on data on the entire foliation X_t and not only on quantities defined by the initial embedding X_0. The important difference to the point particle case is that the functional $P_G^{\alpha\beta}(z)$, which already appeared in the expressions (3.26) and (3.27) to describe the change of the frame vectors under changes of the embedding map X, now is a function of the induced fields $(G^{\alpha\beta}, G^\alpha{}_\beta, G_{\alpha\beta})$ that reads

$$P_G^{\alpha\beta} = \frac{1}{6}\left(G^{\alpha\gamma}G^{\delta\beta}G_{\gamma\delta} - G^{\alpha\beta}G^{\gamma\delta}G_{\gamma\delta} + 2G^{\alpha\beta}G^\gamma{}_\delta G^\delta{}_\gamma - 3G^{\gamma\delta}G^\alpha{}_\gamma G^\beta{}_\delta\right), \qquad (3.38)$$

which can be calculated directly from the tensor field (3.9) as $P_G^{abcd}n_a n_b \epsilon_c^\alpha \epsilon_d^\beta$.

As a second example for a fundamental tensorial spacetime geometry, we consider Lorentzian geometry, where matters become of course particularly simple. The change of the induced inverse spatial metric under hypersurface deformations is given by

$$\dot{g}^{\alpha\beta}(z) = (NK^{\alpha\beta} + \mathcal{L}_{\vec{N}}g^{\alpha\beta})(z)\,, \qquad (3.39)$$

where $K^{\alpha\beta} = (\mathcal{L}_T g)^{ab}\epsilon_a^\alpha \epsilon_b^\beta$. As we have already argued, there is no non-local contribution in this case. For the convenience of the reader, we would like to make contact to the standard literature on the $3+1$ formulation of general relativity at this point. There one usually considers the induced spatial metric $g_{\alpha\beta} = g_{ab}e_\alpha^a e_\beta^b$, instead of its inverse, as the relevant field on the hypersurface. Otherwise, the calculations proceed along the lines laid out above. The change of the metric becomes $\dot{g}_{\alpha\beta}(z) = (NK_{\alpha\beta} + \mathcal{L}_{N^\alpha\partial_\alpha}g_{\alpha\beta})(z)$ with $K_{\alpha\beta} = (\mathcal{L}_T g)_{ab}e_\alpha^a e_\beta^b$. It can be shown that the velocity $K_{\alpha\beta}$ is actually proportional by a factor of 2 to the extrinsic curvature $\mathcal{K}_{\alpha\beta} := \nabla_{(a}n_{b)}e_\alpha^a e_\beta^b$ of the hypersurface $X(\Sigma)$, where ∇ denotes the torsion-free spacetime metric compatible covariant derivative. It is easy to check that, now again using our terminology, the normal velocity of the inverse metric satisfies

$K^{\alpha\beta} = -2\, g^{\alpha\gamma} g^{\beta\delta} \mathcal{K}_{\alpha\beta}$.

For any other admissible tensorial spacetime geometry G supporting some matter dynamics of the form (2.1), we can summarize the above constructions as follows. Let G^A represent the remaining set of hypersurface fields induced by the tensorial spacetime geometry on the initial hypersurface $X_0(\Sigma)$ after the frame conditions $P(n) = 1$ and $T^a \epsilon_a^\alpha = 0$ have been solved. Here we introduced the capital multi-index A depicting the different sets of indices on the possibly different hypersurface tensor fields. Then, for any choice of lapse function N and shift vector field $\vec{N}$, the first order change of the hypersurface tensor fields G^A is given by

$$\dot{G}^A(z) = N(z) K^A(z) + \partial_\gamma N(z) M^{A\,\gamma} + \mathcal{L}_{\vec{N}} G^A(z)\,. \tag{3.40}$$

The last term is generated only by the tangential deformation operator. Only the coefficients $M^{A\,\gamma}$ have to be calculated by hand from the action of the normal deformation operator $H(N)$ on the fields G^A and depend on the type of tensor fields G^A and the Legendre map (2.17). The form of the other two terms, on the other hand, is always the same. It is the velocities K^A, which genuinely account for how the induced geometry G^A behaves away from the initial hypersurface $X_0(\Sigma)$.

We have seen in this section that the normal and tangential deformation operators (3.18) and (3.19) bring about a particular change of the induced hypersurface geometry under hypersurface deformations. In the next section we will derive the commutator algebra of the deformation operators, which encodes what happens if they are consecutively applied to an arbitrary hypersurface functional. This hypersurface deformation algebra will be key to the derivation of gravitational dynamics for the hypersurface geometry.

3.4 The hypersurface deformation algebra

We now want to derive the algebra satisfied by the linear operators $\mathcal{H}(N)$ and $\mathcal{D}(\vec{N})$. The procedure is well illustrated by the simplest of commutators $[\cdot, \cdot]$, namely the one between two of the tangential deformation operators:

$$\begin{aligned}[\mathcal{D}(\vec{N}), \mathcal{D}(\vec{M})]\, F[X] \quad = \quad & \int_\Sigma dy\, dz \Big[N^\alpha(y) e^a_\alpha(y) \frac{\delta}{\delta X^a(y)} \\ & \left(M^\beta(z) e^b_\beta(z) \frac{\delta}{\delta X^b(z)} F[X] \right) - (\vec{N} \leftrightarrow \vec{M}) \Big]\,. \end{aligned} \tag{3.41}$$

Using equation (3.22), the fact that functional derivatives commute and after integrating out the appearing delta functions, one finds that

$$
\begin{aligned}
[\mathcal{D}(\vec{N}), \mathcal{D}(\vec{M})]\, F[X] &= -\int_\Sigma dy \left[N^\alpha \partial_\alpha M^\beta - M^\alpha \partial_\alpha N^\beta\right] e^a_\beta \frac{\delta}{\delta X^a(y)} F[X] \\
&= -\mathcal{D}(\mathcal{L}_{\vec{N}} \vec{M})\, F[X]\,. \qquad (3.42)
\end{aligned}
$$

Here we see something we could have already expected from the action of the tangential deformation operators on the induced geometry in the previous section: the tangential deformation operators parametrized by different hypersurface vector fields $\vec{N}$ and $\vec{M}$ form a closed algebra mimicking the algebra of smooth vector fields generating hypersurface diffeomorphisms on the hypersurface Σ.

In order to obtain the expressions for the remaining two pairings of the deformation operators one has to work somewhat harder, because of the appearance of the normal vector T in the normal deformation operator (3.18), but the principle is the same. This way, one arrives at the hypersurface deformation algebra, which for any admissible tensorial spacetime reads

$$
\begin{aligned}
[\mathcal{H}(N), \mathcal{H}(M)] &= -\mathcal{D}((\deg P - 1) P^{\alpha\beta}(M\partial_\beta N - N\partial_\beta M)\partial_\alpha)\,, && (3.43)\\
[\mathcal{D}(\vec{N}), \mathcal{H}(M)] &= -\mathcal{H}(\mathcal{L}_{\vec{N}} M)\,, && (3.44)\\
[\mathcal{D}(\vec{N}), \mathcal{D}(\vec{M})] &= -\mathcal{D}(\mathcal{L}_{\vec{N}} \vec{M})\,. && (3.45)
\end{aligned}
$$

Again we emphasize that the explicit appearance of the particular projection $P^{\alpha\beta}$ in the first commutator-algebra equation can be traced back to the use of the Legendre map when defining the hypersurface frames. It is irrelevant for this result whether one considers the point particle geometry P or a fundamental spacetime geometry G. In the latter case, $P^{\alpha\beta}$ will be a function of the fields induced by the geometry G, such as for example the function (3.38) for general linear electrodynamics. It is precisely at this point where the matter theory (2.1) enters into the equations which will turn out to determine the dynamics of the geometry G. The other two algebra equations are indeed entirely independent of the hypersurface geometry and thus also independent of the specific matter field dynamics.

The hypersurface deformation algebra usually considered in the literature is of course the special case $\deg P = 2$, where the fundamental geometry is given by a metric so that $P^{\alpha\beta} = g^{\alpha\beta}$. Occasionally, one finds this algebra with varying signs on the right hand side of (3.43). This is due to different overall sign choices for the point particle geometry tensor P, and the sign shown in (3.43) results from our normalization condition $P(X(y), n(y)) = 1$. Had we chosen the normalisation $P(X(y), n(y)) = -1$, the relevant minus sign would disappear. This brings us to the following subtle point. As we already pointed out, choosing the $+1$ normalisation forces the hypersurface metric $g^{\alpha\beta}$ to be negative definite due to the required hyperbolicity of Maxwell's equations. Similarly a -1 normalization would force the hypersurface metric to have Riemannian signature. Hence, from the physical point of view, different signs in the first algebra relation (3.43) do not indicate different signatures of the spacetime metric but only a different sign convention in the Lorentzian case. This interpretation seems void in the metric case because one can, in principle, write down the hypersurface deformation algebra for all the different signatures of the spacetime metric[4]. However, this is only possible because a metric always provides a natural isomorphism between tangent and cotangent spaces, which can be used to construct the hypersurface frames independent of any matter field dynamics. In the light of our more general discussion, we must, nevertheless, reject any polynomial field P that is not bi-hyperbolic and energy distinguishing with respect to some specified matter field dynamics as unphysical. For any admissible tensorial spacetime (see section VIII in [58]), one can show that fixing the normalisation of the conormal n to either $+1$ or -1 necessarily fixes $P^{\alpha\beta}$ to be either negative or positive definite due to the bi-hyperbolicity and energy-distinguishing properties. In this sense, the matter theory $S[G, \Phi]$ also determines the signs in the hypersurface deformation algebra.

The first one to realize the importance of the hypersurface deformation algebra was Dirac [16] when investigating the canonical structure of general relativity. In the 1970's, Hojman, Kuchař and Teitelboim published a series of papers finally culminating in the conclusion that the hypersurface deformation algebra can be made the starting point of deriving Einstein's field equations [30]. The idea is to search for a representation of the algebra as the Poisson constraint algebra of suitable functionals on a geometrical phase space encoding the degrees of freedom of the hypersurface geometry. In the next chapter, we will follow this philosophy and derive a functional differential equation for these functionals on a geometric phase space that is equivalent to the requirement that the latter satisfy the Poisson algebra. In the second next chapter these functional differential equations will then be brought in the form of partial differential equations for these functionals. These 'gravitational master equations' determine then nothing more and nothing less than the Lagrangian of the gravity theory that must govern the dynamics of the underlying spactime geometry—which is the central result of this thesis.

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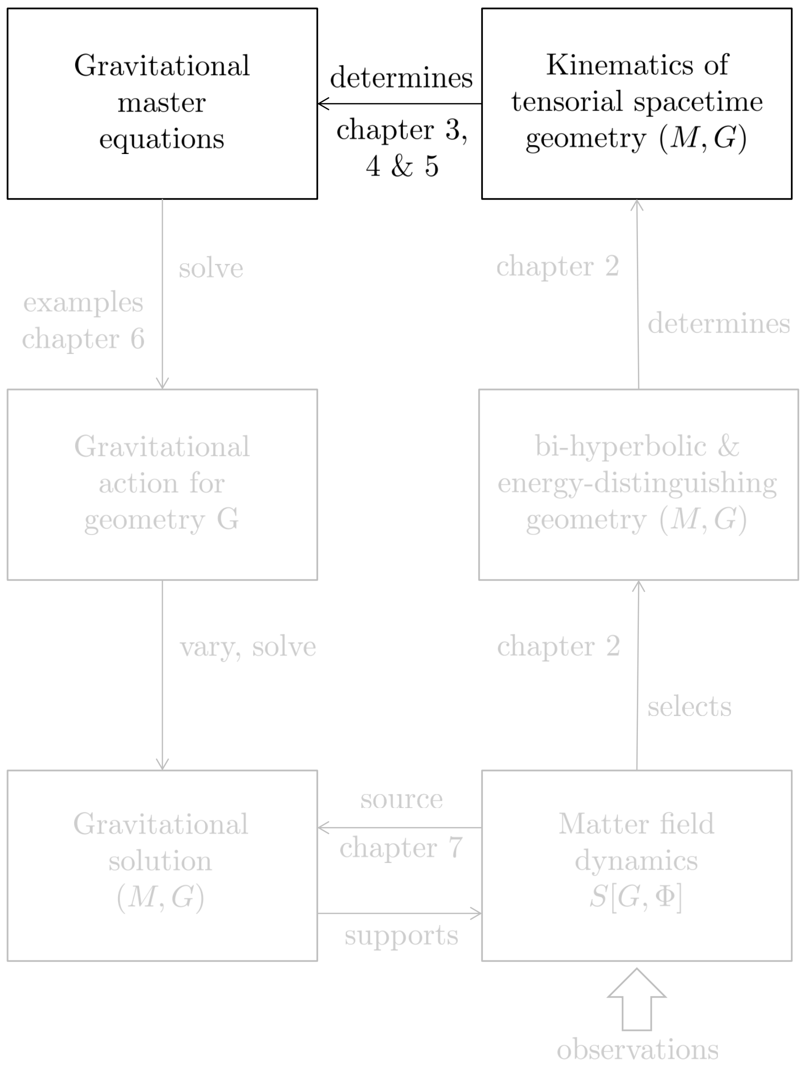
[4]For example, setting $P(X(y), n(y)) = 1$ and insisting on $g^{\alpha\beta}$ being positive definite would correspond to a Riemannian signature of the spacetime metric.


Gravitational master equations
determines
chapter 3, 4 & 5
Kinematics of tensorial spacetime geometry (M, G)
solve
examples chapter 6
chapter 2
determines
Gravitational action for geometry G
bi-hyperbolic & energy-distinguishing geometry (M, G)
vary, solve
chapter 2
selects
source
chapter 7
Gravitational solution (M, G)
Matter field dynamics $S[G, \Phi]$
supports
observations

Chapter 4

Gravitational master equations: functional differential form

In the previous chapter, we assumed the spacetime geometry G to be given and studied how the geometry induced by G on an observer-accessible initial data hypersurface changes if the hypersurface is deformed. In this chapter, we show how to derive dynamics for such geometric initial data on an initial data hypersurface $X(\Sigma)$, which *predict* the values of the hypersurface geometry on a hypersurface at a later time, such that stacking the evolved geometric data produces an admissible tensorial spacetime geometry G for which the fields induced on $X(\Sigma)$ behave exactly according to what the kinematical relation (3.40) dictates. This amounts to the requirement that the hypersurface deformation algebra must be represented as a Poisson algebra of suitable functionals on a to-be-specified geometric phase space. These considerations—which are the formal expression of the geometrodynamic point of view on gravity applied to any admissible tensorial spacetime geometry—culminate in a functional differential equation for the Lagrangian that generates the gravitational dynamics we are looking for by exploiting the information encoded in the deformation algebra.

The results presented in this chapter have been published as
K. Giesel, F. P. Schuller, C. Witte and M. N. R. Wohlfarth,
Phys. Rev. D85 (2012), 104042,
whose sections II.C. and III.A.-III.D., in particular, are discussed here.

4.1 Gravitational dynamics as representations of the kinematical hypersurface algebra

In the previous chapter we saw that the linear change $\dot{F}$ of an arbitrary functional F of the hypersurface embedding map $X : \Sigma \hookrightarrow M$ under hypersurface deformations (which we saw can be parametrized by a lapse function N and a shift vector field $\vec{N}$) is given by

$$\dot{F}[X] = \left(\int_\Sigma dz\, [N(z)\mathcal{H}(z) + N^\alpha(z)\mathcal{D}_\alpha(z)]F\right)[X]\,. \tag{4.1}$$

Here, $\mathcal{H}(z) := \mathcal{H}(\delta_z)$ and $\mathcal{D}_\alpha(z) := \mathcal{D}(\delta_z\partial_\alpha)$ are the localized versions of the hypersurface deformation operators (3.18) and (3.19). As a special case of such functionals, we calculated how fields G^A, which are induced by an admissible tensorial spacetime geometry G on a hyperbolic hypersurface, change under hypersurface deformations. However, in the previous chapter, we assumed the tensorial spacetime geometry G to be known throughout the entire spacetime manifold. Providing the first order changes of the induced hypersurface geometry additionally was, strictly speaking, completely redundant, since if the geometry is known everywhere throughout spacetime one can calculate the induced hypersurface geometry on any of the neighbouring hypersurfaces, simply by projecting the spacetime geometry G to it. Unfortunately, this divine point of view is none we can afford—for two reasons. Neither, do we usually have access to geometric data on an entire hypersurface, nor do we ever have access to geometric data on a hypersurface in the future. Thus, both spatially and temporally, our knowledge about the spacetime geometry is greatly restricted. The best we can hope to know are initial data $\hat{G}^A$ in some compact region of an initial hypersurface $X_0(\Sigma)$. In order to distinguish the induced fields G^A of a given spacetime geometry G from such initial data $\hat{G}^A$, we will always denote the latter with a hat. The lack of knowledge about the hypersurface geometry away from the initial hypersurface—which we aim at predicting by choice of appropriate dynamics—has to be compensated for, in an initial value formulation of such dynamics, by introducing velocities $\hat{K}^A$ in addition to the fields $\hat{G}^A$, so that the pair $(\hat{G}^A, \hat{K}^A)$ forms the velocity phase space of the gravitational dynamics we wish to derive. In order to evolve these initial data to later times, or, more precisely, to another hypersurface, we need to specify equations of motion so that the evolving data form a tensorial spacetime geometry G (see figure 4.1). In this chapter, we derive a *functional differential equation* for the Lagrangian L that precisely generates such equations of motion that ensure that the resulting spacetime geometry G reproduces the kinematics identified by hypersurface deformations.

The required initial data fields $\hat{G}^A$ of course depend on the spacetime geometry G we would like to describe. If, for example, one wants to represent a $\deg P = 4$ point particle geometry P, one has to choose the symmetric fields $(\hat{P}^{\alpha\beta}, \hat{P}^{\alpha\beta\gamma}, \hat{P}^{\alpha\beta\gamma\delta})$ as the initial configuration on $X(\Sigma)$, and provide a set of velocity fields $(\hat{K}^{\alpha\beta}, \hat{K}^{\alpha\beta\gamma}, \hat{K}^{\alpha\beta\gamma\delta})$. If, instead, one would like to determine a spacetime geometry for general linear electrodynamics—see section 6.3—the initial data consist of an invertible symmetric

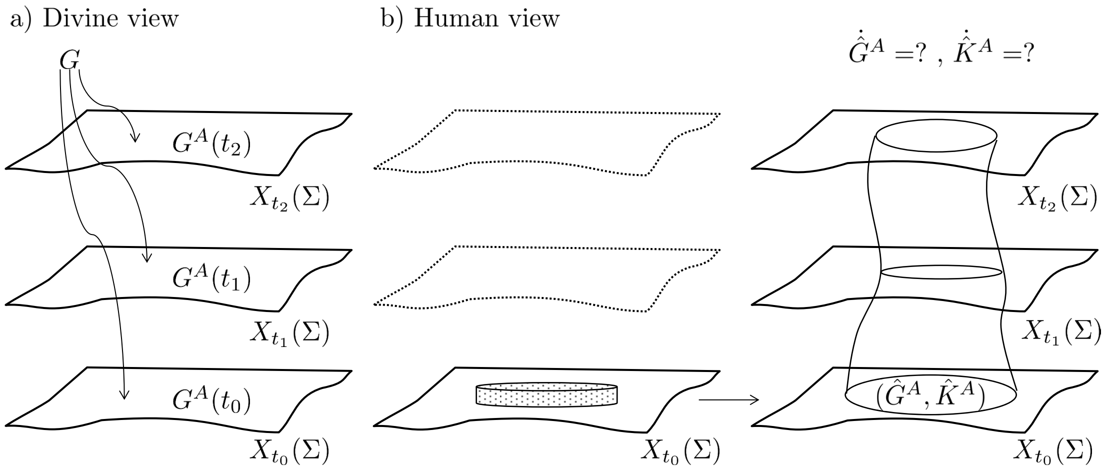


Figure 4.1: Illustration of a) The divine view on the geometry G defining an admissible tensorial spacetime: the geometry G^A induced on a family $X_t(\Sigma)$ of hypersurfaces can be calculated directly from the spacetime geometry G, which is already known everywhere; b) The human view on the geometry of spacetime. Realistically, one only has restricted access, both temporally and spatially, to the geometry in a small region of spacetime (dotted cell). This knowledge can be translated into a set $(\hat{G}^A, \hat{K}^A)$ of initial data on a single hypersurface $X_{t_0}(\Sigma)$, which has to be evolved with the help of to-be-determined equations of motion in order to reconstruct an admissible tensorial spacetime geometry G.

tensor field $\hat{G}^{\alpha\beta}$, a symmetric tensor field $\hat{G}_{\alpha\beta}$ and a tensor field $\hat{G}^\alpha{}_\beta$ that is tracefree and symmetric with respect to $\hat{G}^{\alpha\beta}$. Because of the kinematical relation (3.40), the velocities $\hat{K}^A$ must have the same index structure as the fields $\hat{G}^A$, as we already indicated by positioning of the capital multi-index A accordingly.

It turns out that rather than considering the velocity phase space $(\hat{G}^A, \hat{K}^A)$, it is advantageous to adjoin canonical momenta $\hat{\pi}_A$ to the fields $\hat{G}^A$ instead in order to form a canonical phase space $(\hat{G}^A, \hat{\pi}_A)$. The requirement that the $\hat{\pi}_A$ be canonically conjugate to the $\hat{G}^A$ is of course equivalent to adopting a Poisson bracket

$$\left\{\hat{C}, \hat{D}\right\} := \int_\Sigma dz \left[\frac{\delta \hat{C}}{\delta \hat{G}^A}\frac{\delta \hat{D}}{\delta \hat{\pi}_A} - \frac{\delta \hat{D}}{\delta \hat{G}^A}\frac{\delta \hat{C}}{\delta \hat{\pi}_A}\right] . \tag{4.2}$$

The advantage of the geometric phase space $(\hat{G}^A, \hat{\pi}_A)$ over the geometric *velocity* phase space $(\hat{G}^A, \hat{K}^A)$ lies in the fact that the canonical structure given by the Poisson bracket is completely independent of the dynamics we are looking for. The canonical structure one would have to employ on a velocity phase space, on the other hand, would depend on the Lagrangian we yet have to find[1]. We will only later see

[1] In classical mechanics, for example, a symplectic structure, which one can establish on the tangent bundle TM of some configuration manifold M, results from the pull-back of the canonical symplectic structure on the cotangent bundle T^*M with the help of the Legendre transform, which depends on the Lagrange function L on the tangent bundle, while the canonical symplectic

how the momenta $\hat{\pi}_A$ and the velocities $\hat{K}^A$ are related, namely by exploiting what we have learned from the hypersurface deformation picture.

In the following calculations in this chapter, and in fact throughout this thesis, repeated capital multi-indices are to be understood as a sum over the various patterns of small greek hypersurface indices the capital multi-index stands for. If, for instance, we would like to describe a $\deg P = 4$ point particle geometry, we have that

$$\hat{G}^A \hat{\pi}_A = \hat{P}^{\alpha\beta}\hat{\pi}_{\alpha\beta} + \hat{P}^{\alpha\beta\gamma}\hat{\pi}_{\alpha\beta\gamma} + \hat{P}^{\alpha\beta\gamma\delta}\hat{\pi}_{\alpha\beta\gamma\delta}\,.$$

It is important to note that in order for the Poisson brackets to be invariant under diffeomorphisms on the hypersurface Σ, the canonical momenta must be tensor densities of weight one. Moreover, we have to remark that there is actually some freedom in the definition of the canonical momenta: we can always change the latter by adding a functional derivative $\delta\Lambda/\delta\hat{G}^A$ of some scalar density $\Lambda[\hat{G}^A]$ of weight one with respect to the canonical variables $\hat{G}^A$, since this does not change the Poisson brackets. This observation will be quite useful later on.

We now stipulate that the dynamics of the phase space variables $(\hat{G}^A, \hat{\pi}_A)$ be generated by a suitable Hamiltonian functional H_t according to Hamilton's equations

$$\dot{\hat{G}}^A(y) = \{\hat{G}^A(y), H_t\} \quad \text{and} \quad \dot{\hat{\pi}}_A(y) = \{\hat{\pi}_A(y), H_t\}\,, \tag{4.3}$$

where the dot denotes the derivative with respect to an evolution parameter t. It can be shown abstractly [52] that such a Hamiltonian functional, which generates spacetime diffeomorphism invariant dynamics for the hypersurface fields $(\hat{G}^A, \hat{\pi}_A)$, must be necessarily of the form

$$H_t = \int_\Sigma dz[N_t(z)\hat{\mathcal{H}}(z) + N_t^\alpha(z)\hat{\mathcal{D}}_\alpha(z)]\,, \tag{4.4}$$

for two—at this point—undetermined functionals $\hat{\mathcal{H}}(z)$ and $\hat{\mathcal{D}}(z)$ of the phase space $(\hat{G}^A, \hat{\pi}_A)$. The lapse function N_t and shift vector field $\vec{N}_t$ enter the theory as Lagrange multipliers, reflecting the freedom of choice of the foliation of the spacetime manifold. Moreover, they enforce the two constraint equations

$$\hat{\mathcal{H}}(z) \approx 0 \quad \text{and} \quad \hat{\mathcal{D}}_\alpha(z) \approx 0 \tag{4.5}$$

that are to hold only on solutions of the equations of motion. This restriction we here denoted by employing the weak equal sign $\approx$. Both the equations of motion and the constraint equations may of course be derived from the canonical action

structure on the cotangent bundle is independent of the dynamics.

functional

$$S[\hat{G}, \hat{\pi}, N, N^\alpha] = \int_{\mathbb{R}} dt \left[\int_\Sigma dz\, (\dot{\hat{G}}^A \pi_A)(z) - H_t \right] , \tag{4.6}$$

by variation with respect to the fields $\hat{G}^A, \hat{\pi}_A, N$ and N^α.

After the equations of motion have been solved, yielding a solution $\hat{G}^A_t$, one can reconstruct the spacetime geometry with the help of the completeness relations one has obtained from the hypersurface deformation point of view presented in the previous chapter. Using again the example of a $\deg P = 4$ point particle geometry, the spacetime geometry P^{abcd} is reconstructed as

$$P^{abcd} = T_t^a\, T_t^b\, T_t^c\, T_t^d + 6\hat{P}_t^{\alpha\beta} e_{t\alpha}^{(a}\, e_{t\beta}^{\ b}\, T_t^c\, T_t^{d)} + 4\hat{P}_t^{\alpha\beta\gamma} e_{t\alpha}^{(a}\, e_{t\beta}^{\ b}\, e_{t\gamma}^{\ c}\, T_t^{d)} + \hat{P}_t^{\alpha\beta\gamma\delta} e_{t\alpha}^{\ a}\, e_{t\beta}^{\ b}\, e_{t\gamma}^{\ c}\, e_{t\delta}^{\ d} , \tag{4.7}$$

but with the normal vector $T_t = N_t^{-1}(\partial_t - N_t^\alpha e_\alpha)$ given entirely in terms of the coordinate-induced hypersurface frame vectors ∂_t and e_α determined by the foliation X_t. Thus the lapse function and the shift vector field directly determine some of the components of the spacetime geometry P in the coordinates (t, y). To keep the notation short, we will drop the label t on the lapse function and the shift vector field in the following, but both are understood to depend on it.

It should be clear now that if the change of the tensor fields $\hat{G}^A$ generated by the Hamiltonian (4.4) is to coincide with what the hypersurface deformation would yield for the reconstructed spacetime geometry independent of any choice of (N, N^α), we must have that on the initial hypersurface $X(\Sigma)$

$$\mathcal{H}(z) G^A(y)[X] \;=\; \left\{ \hat{G}^A(y), \hat{\mathcal{H}}(z) \right\} \quad \text{and} \tag{4.8}$$

$$\mathcal{D}_\alpha(z) G^A(y)[X] \;=\; \left\{ \hat{G}^A(y), \hat{\mathcal{D}}_\alpha(z) \right\} . \tag{4.9}$$

This is the only requirement we can impose, given the definition of the deformation operators $\mathcal{H}$ and $\mathcal{D}$ on the left hand side. In particular, there is no way to extend this compatibility condition also to the canonical momenta $\hat{\pi}_A$, because the latter have no counterpart as functionals of the embedding map, so that there would be no corresponding expression on the left hand side. Nevertheless, the right hand sides of the above equations are of course equivalent to the functional derivatives $\delta\hat{\mathcal{H}}(z)/\delta\hat{\pi}_A(y)$ and $\delta\hat{\mathcal{D}}_\alpha(z)/\delta\hat{\pi}_A(y)$ of the superhamiltonian and the supermomentum with respect to the canonical momenta. Hence, the application of the normal and tangential deformation operators on the functionals G^A, which we have exemplarily calculated in equations (3.30) and (3.32) for the point particle geometry P^A, teaches us—namely in the form of a functional differential equations—how the superhamiltonian and the supermomentum functionally depend on the canonical momenta $\hat{\pi}_A$. Unfortunately, the functional dependence of the superhamiltonian and supermomentum on the configuration variables $\hat{G}^A$ cannot be immediately deduced. However, the missing piece of information is encoded in the hypersurface deformation algebra derived in the previous chapter, which can be translated into a functional differential equation that precisely contains the necessary functional derivatives of the supermomentum and superhamiltonian with respect to the configuration variables $\hat{G}^A$. In order to come to this conclusion, one first realizes that the compatibility conditions (4.8) and (4.9) imply, together with the hypersurface deformation algebra, that the smeared superhamiltonian and supermomentum,

$$\hat{\mathcal{H}}(N) = \int_\Sigma dz\, \hat{\mathcal{H}}(z)N(z) \quad \text{and} \quad \hat{\mathcal{D}}_\alpha(\vec{N}) = \int_\Sigma dz\, \hat{\mathcal{D}}_\alpha(z)N^\alpha(z) \tag{4.10}$$

have to satisfy the Poisson algebra

$$\{\hat{\mathcal{H}}(N), \hat{\mathcal{H}}(M)\} = \hat{\mathcal{D}}((\deg P - 1)\hat{P}^{\alpha\beta}(M\partial_\beta N - N\partial_\beta M)\partial_\alpha)\,, \tag{4.11}$$

$$\{\hat{\mathcal{D}}(\vec{N}), \hat{\mathcal{H}}(M)\} = \hat{\mathcal{H}}(\mathcal{L}_{\vec{N}}M)\,, \tag{4.12}$$

$$\{\hat{\mathcal{D}}(\vec{N}), \hat{\mathcal{D}}(\vec{M})\} = \hat{\mathcal{D}}(\mathcal{L}_{\vec{N}}\vec{M})\,. \tag{4.13}$$

This can be seen by translating the commutators in the hypersurface deformation algebra acting on the functionals $G^A[X]$ into the equivalent nested Poisson brackets using the compatibility rules (4.8) and (4.9). After rearranging the latter with the help of the Jacobi identity, the above Poisson algebra relations can be read off directly from the resulting expressions[2]. Note that the Poisson algebra differs by an overall sign from the hypersurface deformation algebra, which is due to our convention for the Poisson brackets which makes the supermomentum and superhamiltonian to act 'from the right' as $\{\cdot, \mathcal{H}(N)\}$ and $\{\cdot, \mathcal{D}(\vec{N})\}$ on any phase space functional. Had we chosen the canonical structure with the configuration variables and the canonical momenta interchanged, both algebras would look exactly the same.

The task now is clear: In order to find canonical gravitational dynamics on a geometric phase space $(\hat{G}^A, \hat{\pi}_A)$ representing the spacetime geometry G that underlies the matter field dynamics (2.1), we have to find all phase space functionals $\hat{\mathcal{H}}$ and $\hat{\mathcal{D}}_\alpha$ that satisfy the Poisson algebra equations (4.11)-(4.13) and the compatibility requirements (4.8) and (4.9).

In the present and the following chapter, we will derive a set of differential equations that, once solved, precisely yields such phase space functionals for any hypersurface geometry $\hat{G}^A$. In chapter 6, we then illustrate this procedure for concrete examples. The first step in the construction of such gravitational master equations is the explicit determination of the supermomentum and, partially, the

[2] In the context of metric geometry, this was shown, for example, by Kouletsis in [34] also in the history formalism using non-equal time Poisson brackets.

superhamiltonian.

4.2 The canonical supermomentum

The supermomentum $\hat{\mathcal{D}}_\alpha(y)$ can be determined completely, and directly, due to two facts. First, the action of the tangential deformation operator (3.32) on hypersurface functionals can be described entirely by intrinsic hypersurface quantities so that the compatibility requirement (4.9) already yields

$$\{\hat{G}^A(z), \hat{\mathcal{D}}(\vec{N})\} = \frac{\delta \hat{\mathcal{D}}(\vec{N})}{\delta \hat{\pi}_A(z)} = \mathcal{L}_{\vec{N}} \hat{G}^(z)\,. \tag{4.14}$$

Secondly, we can extend this relation to also determine the functional dependence of the supermomentum on the configuration variables $\hat{G}^A$. Using the Jacobi identity for $\{\hat{\pi}_B(y), \{\hat{G}^A(z), \hat{\mathcal{D}}(\vec{N})\}\}$ followed by a functional integration with respect to $\hat{\pi}_B$, we obtain

$$\{\hat{\pi}_B(y), \hat{\mathcal{D}}(\vec{N})\} = -\frac{\delta \hat{\mathcal{D}}(\vec{N})}{\delta \hat{G}^A(y)} = \mathcal{L}_{\vec{N}} \hat{\pi}_A(y)\,. \tag{4.15}$$

Here, we had to use the closure (4.13) of the Poisson algebra of two supermomenta to eliminate an integration constant of the form $F[\hat{G}^A]_B$. Also note that since the momenta $\hat{\pi}_A$ are tensor densities of weight $+1$, the Lie derivative of the latter contains an additional term $+(\partial_\beta N^\beta)\hat{\pi}_A$ compared to the Lie derivative of a tensor of the same valence. We thus see that the supermomentum really merely reshuffles the entire phase space data $(\hat{G}^A, \hat{\pi}_A)$ along the flow lines of the shift vector field $\vec{N}$ and this is precisely what a spatial diffeomorphism is supposed to do.

Equations (4.14) and (4.15) present a coupled system of functional differential equations which is integrable since all second functional derivatives indeed commute. The general solution of this system of equations is then given by

$$\hat{\mathcal{D}}(\vec{N}) = \int_\Sigma dz\, \hat{\pi}_A(z) \mathcal{L}_{\vec{N}} \hat{P}^A(z)\,. \tag{4.16}$$

where, again, one must use the closure of the supermomentum subalgebra (4.13) to force an emerging integration constant to vanish. For later use, we remark here that it is always possible to free the shift vector field—which is assumed to be compactly supported—of derivatives by an integration by parts of the above expression of the supermomentum. However, the resulting expression will depend on the explicit geometry $\hat{G}^A$, and since we will only need to use the supermomentum in this form when it appears in the first algebra equation (4.11), we leave this issue open for now.

Having determined the supermomentum $\hat{\mathcal{D}}(\vec{N})$, we can turn our attention to the superhamiltonian $\hat{\mathcal{H}}(N)$, which must be determined from the remaining algebra equations (4.11) and (4.12) and the compatibility requirement (4.8).

4.3 The non-local part of the superhamiltonian

In the previous section, we derived that the supermomentum generates spatial diffeomorphisms of the phase space data $(\hat{G}^A, \hat{\pi}_A)$. The way it acts on any phase space functional is thus completely determined by the tensorial nature of the latter. In particular, we may deduce from the second algebra equation (4.12) how the supermomentum acts on the superhamiltonian: setting the function $M = \delta_z$ and performing an integration by parts on the right hand side, we find that

$$\{\hat{\mathcal{H}}(z), \hat{\mathcal{D}}(\vec{N})\} = \partial_\alpha(\hat{\mathcal{H}} N^\alpha)(z)\,. \tag{4.17}$$

This implies that the superhamiltonian is a scalar density of weight one. Of course, this is just consistent with the form (4.4), where we implictly assumed that the Hamiltonian H_t is invariant under hypersurface diffeomorphisms. Nevertheless, later, we will further exploit this fact by translating it into a set of differential *covariance* equations the superhamiltonian must satisfy.

As for the supermomentum, we can immediately learn something about the superhamiltonian from the compatibility requirement (4.8). Combining the latter with what we obtained from the hypersurface deformations in equation (3.40), we conclude that the functional derivative $\delta\hat{\mathcal{H}}/\delta\hat{\pi}_A$ is of the form

$$\frac{\delta\hat{\mathcal{H}}(N)}{\delta\hat{\pi}_A(z)} = N(z)\hat{K}^A[\hat{G}^A](\hat{\pi}_B)(z) + \partial_\gamma N(z) M^{A\gamma}(z)\,, \tag{4.18}$$

with the coefficients $M^{A\gamma}$ depending on the type of tensor fields G^A and the Legendre map (2.17), but here being considered as functions of the configuration variables $\hat{G}^A$. We also had to replace the the velocities K^A, which in the hypersurface deformation point of view were functionals $\hat{K}^A$ of the foliation X_t, by so far unknown functionals on the geometric phase space $(\hat{G}^A, \hat{\pi}_A)$. However, because the lapse function appears only in its undifferentiated form in front of the velocities K^A, we know that the velocity functionals $\hat{K}^A$ must be functions only of the momenta $\hat{\pi}_A$, but not of any of their derivatives. In fact, equation (4.18) implies that the superhamiltonian can be split into a 'local' and a 'non-local' part,

$$\hat{\mathcal{H}}(N) = \int_\Sigma dz\, N(z) \left[\hat{\mathcal{H}}_{\text{local}}[\hat{G}](\hat{\pi})(z) + \hat{\mathcal{H}}_{\text{non-local}}[\hat{G}, \hat{\pi}](z)\right](N)\,, \tag{4.19}$$

which two are well-distinguished by the condition that $\hat{\mathcal{H}}_{\text{local}}[\hat{G}](\hat{\pi})(z)$ be a function of the momenta $\hat{\pi}_A$ but possibly a functional of the configuration variables $\hat{G}^A$, whereas the other part, $\hat{\mathcal{H}}_{\text{non-local}}[\hat{G},\hat{\pi}](z)$, depend on the derivatives of both. The non-local part $\hat{\mathcal{H}}_{\text{non-local}}$ generates the second term in equation (4.18), and since the coefficients $M^{A\gamma}$ are known explicitly as functions of the configuration variables $\hat{G}^A$, a functional integration with respect to the momenta $\hat{\pi}_A$ yields

$$\hat{\mathcal{H}}_{\text{non-local}}(z) = -\partial_\gamma(M^{A\gamma}\hat{\pi}_A)(z). \tag{4.20}$$

Note that thus the non-local superhamiltonian is the divergence of a vector density of weight one and hence is a scalar density of the same weight. This is to be expected if the splitting of the superhamiltonian into a sum of two parts is supposed to be well-defined. The non-local superhamiltonian is thus always completely determined by the behaviour of the induced spatial geometry under hypersurface tilts, as described in the previous chapter. This was already anticipated by Kuchař in his discussion of the canonical dynamics of arbitrary tensor fields on metric manifolds [38]. It is once again noteworthy that the Legendre map (2.17) hugely influences the non-local superhamiltonian through the coefficients $M^{A\gamma}$. We will come back to this observation later.

It remains for the local part $\hat{\mathcal{H}}_{\text{local}}$ to generate the first part of equation (4.18),

$$\hat{K}^A[\hat{G}^A](\hat{\pi}_A)(z) = \frac{\partial\hat{\mathcal{H}}_{\text{local}}[\hat{G}^A](\hat{\pi}_A)(z)}{\partial\hat{\pi}_A(z)}\,, \tag{4.21}$$

which then yields a relation between the velocity functionals $\hat{K}^A$ and the canonical momenta $\hat{\pi}_A$. Since the velocities are the partial derivatives of a scalar density of weight one with respect to tensor densitites of the same weight, they are hypersurface tensors, as expected. Unfortunately, this is all one can deduce about the local superhamiltonian from the kinematical relation (4.8). This time, we cannot extend this equation to the functional derivatives of the superhamiltonian with respect to the configuration variables $\hat{G}^A$ (as we have done for the supermomentum), simply because we know nothing further about the velocity functionals. However, the local superhamiltonian $\hat{\mathcal{H}}_{\text{local}}$ is the last missing piece in the determination of the gravitational dynamics for the hypersurface geometry and indeed the remainder of this chapter is devoted to find the equations that determine $\hat{\mathcal{H}}_{\text{local}}$. The only relation we have not made use of, yet, is the Poisson algebra equation (4.11). Its local nature makes it tempting to use a series expansion for the local superhamiltonian in terms of the momenta in the remaining algebra equation, so that one would only have to determine the expansion coefficients as functionals of the configuration variables $\hat{G}^A$. This, however, appears not very promising (at least without making further assumptions which we do not wish to introduce), since the superhamiltonian enters the algebra equation quadratically. Fortunately, as we will show in the next section, a reformulation of the dynamics in terms of an equivalent Lagrangian functional solves this problem by converting the said quadratic condition into a linear one.

4.4 Lagrangian formulation of gravitational dynamics

When we discussed the first order change $\dot{G}^A$ of the induced spatial geometries under hypersurface deformations via equation (3.40), we already saw that $\dot{G}^A$ can be specified completely in terms of the induced fields G^A, the lapse function N, the shift vector field N^α and in terms of additional hypersurface data in form of the velocities K^A. In the phase space formulation of the dynamics, it is indeed advantageous to promote these velocities to independent dynamical variables instead of considering them as functionals of the phase space variables $(\hat{G}^A, \hat{\pi}_A)$, and to correspondingly demote the momenta. In fact, we have already seen how the velocities are related to the canonical momenta in the previous section. Equation (4.21) implies that the velocities $\hat{K}^A$ are actually the Legendre dual variables of the canonical momenta with respect to the local superhamiltonian $\hat{\mathcal{H}}_{\text{local}}$. This shows that one may reformulate the gravitational dynamics entirely in terms of the variables $\hat{G}^A$ and $\hat{K}^A$ by means of a Legendre transformation of the local superhamiltonian and rewriting the constraints (4.5) accordingly. Apart from being fully equivalent to the canonical formulation, the resulting Lagrangian dynamics have the benefit of simplifying the remaining algebra equation (4.11) into a *linear* functional differential equation for the Lagrangian, which is simply the Legendre dual function of the local superhamiltonian[3].

The definition of the variables $\hat{K}^A$ dual to the canonical momenta $\hat{\pi}_A$ is given in equation (4.21) and must be inverted to yield $\hat{\pi}_A[\hat{G}^A](\hat{K}^A)$. We may then define the Lagrangian

$$L[\hat{G}](\hat{K})(y) := \hat{\pi}_A[\hat{G}](\hat{K})(y)\,\hat{K}^A(y) - \hat{\mathcal{H}}_{\text{local}}[\hat{G}]\,(\hat{\pi}[\hat{G}](\hat{K}))(y) \tag{4.22}$$

as the corresponding Legendre transform of the local superhamiltonian. It is clear that the Lagrangian L is a scalar density of weight one, like the local superhamiltonian. The functional dependence of the Lagrangian on the configuration variables $\hat{G}^A$ is inherited from the local superhamiltonian, so that we have

$$\left.\frac{\delta\hat{\mathcal{H}}_{\text{local}}(z)}{\delta\hat{G}^A(y)}\right|_{\hat{\pi}[\hat{G}](\hat{K})} = -\frac{\delta L(z)}{\delta\hat{G}^A(y)}\,. \tag{4.23}$$

[3]The idea to use the Legendre transformation in order to simplify the remaining algebra equation for the superhamiltonian goes back to a seminal paper by Kuchař [37].

Conversely, given the Lagrangian L one recovers the momenta $\hat{\pi}_A$ as the Legendre dual variables of the velocities:

$$\frac{\partial L(z)}{\partial \hat{K}^A(z)} = \hat{\pi}_A(z)[\hat{G}](\hat{K})\,. \tag{4.24}$$

Again, this must be inverted to find the velocities as functions of the momenta in order to obtain the local superhamiltonian by

$$\hat{\mathcal{H}}_{\text{local}}[\hat{G}](\hat{\pi})(x) = \hat{\pi}_A(x)\hat{K}^A[\hat{G}](\hat{\pi})(x) - L[\hat{G}](\hat{K}^A[\hat{G}](\hat{\pi}))(x)\,. \tag{4.25}$$

The original equations of motion (4.3) have to be transformed accordingly to obtain their Lagrangian counterparts. The first order equation for the configuration variables becomes particularly simple, namely

$$\dot{\hat{G}}^A(z) = N(z)\hat{K}^A(z) + \partial_\gamma N(z)\, M^{A\,\gamma}(z) + \mathcal{L}_{\vec{N}}\hat{G}^A(z), \tag{4.26}$$

which simplicity shows that the Lagrangian picture is actually closer to the hypersurface deformation description (3.40) than the canonical formulation. In order to also rewrite the second equation of motion, which originally described the dynamics of the momenta, in terms of the Lagrangian, we introduce the quantities

$$Q_A{}^{B\,\mu}(z) = -\frac{\partial M^{B\,\mu}(z)}{\partial \hat{G}^B(z)}\,. \tag{4.27}$$

We may then express the functional derivatives of the non-local part of the superhamiltonian with respect to the phase space variables $(\hat{G}^A, \hat{\pi}_A)$ in terms of the coefficients $M^{A\,\gamma}$ and $Q_A{}^{B\,\beta}$ as

$$\begin{aligned}
\frac{\delta\hat{\mathcal{H}}_{\text{non-local}}(y)}{\delta\hat{G}^A(z)} &= Q_A{}^{B\beta}(y)\partial_\beta\hat{\pi}_B(y)\delta_y(z) - Q_A{}^{B\beta}(y)\hat{\pi}_B(y)\partial_\beta\delta_y(z) \\
&\quad + \partial_\beta Q_A{}^{B\beta}(y)\,\pi_B(y)\,\delta_y(z)\,, &(4.28)\\
\frac{\delta\hat{\mathcal{H}}_{\text{non-local}}(y)}{\delta\hat{\pi}_A(z)} &= M^{A\varsigma}(y)\partial_\varsigma\delta_y(z) - \partial_\varsigma M^{A\varsigma}(y)\delta_y(z)\,. &(4.29)
\end{aligned}$$

Using this, the split of the superhamiltonian $\mathcal{H}(N)$ into the local and non-local parts and the Legendre transformation formulae, we obtain the second equation of motion $\dot{\hat{\pi}}(z) = \{\hat{\pi}_A(z), H\}$ in the Lagrangian formulation:

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\partial L(z)}{\partial \hat{K}^A(z)}\right) &= \int_\Sigma dx\,\left[N(x)\frac{\delta L(x)}{\delta \hat{G}^A(z)}\right] + \mathcal{L}_{\vec{N}}\left(\frac{\partial L(z)}{\partial \hat{K}^B(z)}\right) \\
&\quad + \partial_\beta N(z)\, Q_A{}^{B\,\beta}(z)\,\frac{\delta L(z)}{\delta K^B(z)}\,. &(4.30)
\end{aligned}$$

The two equations of motion (4.26) and (4.30) have to be supplemented by the two constraint equations (4.5), but with the momenta $\hat{\pi}_A[\hat{G}^A](\hat{K}^A)$ expressed in terms of the velocities $\hat{K}^A$.

It remains to show what we claimed earlier, namely that the Poisson algebra equation (4.11), which is quadratic in the unknown local superhamiltonian $\hat{\mathcal{H}}(x)_{\text{local}}$, can be turned into a linear equation in the newly obtained Lagrangian L. To this end, we will use equation (4.11) in its local version, setting $N = \delta_x$ and $M = \delta_y$. It is thus turned into a distributional equation in two variables, and its left hand side,

$$\int_\Sigma dz \left[\frac{\delta\hat{\mathcal{H}}(x)_{\text{local}}}{\delta\hat{G}^A(z)} + \frac{\delta\hat{\mathcal{H}}(x)_{\text{non-local}}}{\delta\hat{G}^A(z)}\right]\left[\frac{\delta\hat{\mathcal{H}}(y)_{\text{local}}}{\delta\hat{\pi}_A(z)} + \frac{\delta\hat{\mathcal{H}}(y)_{\text{non-local}}}{\delta\hat{\pi}_A(z)}\right] - (x \leftrightarrow y)\,, \quad (4.31)$$

can be rewritten using the Legendre transformation and the definitions of the coefficients $Q_A{}^{B\beta}$ and $M^{A\gamma}$. We will use the shorthand $(x \leftrightarrow y)$ to indicate that the same terms are repeated with the arguments x and y interchanged. In a first step, the left hand side of the algebra equation takes the rather complicated form

$$\begin{aligned}
&-\frac{\delta L(x)}{\delta\hat{G}^A(y)}\hat{K}^A(y) + \partial_{y\varsigma}\left[\frac{\delta L(x)}{\delta\hat{G}^A(y)}M^{A\varsigma}(y)\right] - M^{A\zeta}(x)Q_A{}^{B\beta}(y)\partial_\beta\hat{\pi}_B(y)\partial_\zeta\delta_x(y)\\
&-\hat{K}^A(y)Q_A{}^{B\beta}(x)\hat{\pi}_B(x)\partial_\beta\delta_x(y) - Q_A{}^{B\beta}(y)\hat{\pi}_B(y)M^{A\xi}(x)\partial^2_{\beta\xi}\delta_x(y)\\
&+Q_A{}^{B\beta}(x)\hat{\pi}_B(x)\partial_\xi M^{A\xi}(y)\partial_\beta\delta_x(y) - \partial_\beta Q_A{}^{B\beta}(y)\pi_B(y)M^{A\zeta}(x)\partial_\zeta\delta_x(y)\\
&-(x \leftrightarrow y)\,,
\end{aligned} \quad (4.32)$$

where again the momenta are to be regarded as functions of the velocities $\hat{K}^A$. The expression above can now be simplified using its distributional nature. We first observe that because of the antisymmetry in x and y, we can set all arguments of the functions in front of the first partial derivatives of the delta function in the third, fourth and sixth term to x, without changing the resulting distribution. This is not possible in the term containing the second derivatives of the delta distribution. However, one straightforwardly checks the identity

$$\begin{aligned}
&Q_A{}^{B\,(\beta|}(y)\hat{\pi}_A(y)M^{A|\,\xi)}(x)\partial^2_{\beta\xi}\delta_x(y) - (x \leftrightarrow y) =\\
&\quad = -\,Q_A{}^{B\,(\beta|}(x)M^{A|\,\xi)}(x)\hat{\pi}_A(x)\partial^2_{\beta\xi}\delta_x(y)\\
&\qquad + 2\,Q_A{}^{B\,(\beta|}(x)\partial_\beta M^{A|\xi)}(x)\hat{\pi}_B(x)\partial_\xi\delta_x(y) - (x \leftrightarrow y)\,,
\end{aligned} \quad (4.33)$$

which achieves just the same.

The right hand side of the algebra equation (4.11) only contains the already determined supermomentum $\hat{\mathcal{D}}(\vec{N})$. Freeing the shift vector field from any derivative by an integration by parts in the general expression (4.16), and inserting it into the localized version of the algebra equation (4.11), the right hand side of the algebra equation can be rewritten in the form

$$(-U^{A\,\mu\nu}\,\partial_\mu\hat{\pi}_A + V^{A\,\nu}\hat{\pi}_A)(x)\,\partial_\nu\delta_x(y) - (x \leftrightarrow y)\,, \tag{4.34}$$

where we have introduced the coefficients $U^{A\,\mu\nu}$ and $V^{A\,\nu}$. We now make the following observation. The terms containing the first partial derivatives of the momentum variables on the left hand side (4.32) and right hand side (4.34) may be combined into an expression of the form $T^{A\,\mu\nu}(x)\partial_\mu\hat{\pi}_A(x)\partial_\nu\delta_x(y)$, with tensor coefficients

$$T^{A\,\mu\nu} = -\,Q_B{}^{A\,\mu}M^{B\,\nu} + U^{A\,\mu\nu}. \tag{4.35}$$

The terms containing the symmetric part $T^{A\,(\mu\nu)}$ of these tensor coefficients can be further simplified by means of the distributional identity

$$\begin{aligned} &T^{A\,(\mu\nu)}(x)\partial_\mu\hat{\pi}_A(x)\partial_\nu\delta_x(y) - (x \leftrightarrow y)\\ &\quad= \Big[T^{A\,(\mu\nu)}(x)(\hat{P})\partial^2_{\mu\nu}\delta_x(y) - \partial_\mu T^{A\,(\mu\nu)}(x)(\hat{P})\partial_\nu\delta_x(y)\Big]\,\hat{\pi}_A(x) - (x \leftrightarrow y)\,, \end{aligned} \tag{4.36}$$

while the terms containing the antisymmetric part $T^{A\,[\mu\nu]}$ stay as they are. Combining the two expressions (4.32) and (4.34) with the identities (4.33) and (4.36) and substituting the momenta by $\hat{\pi}_A(x) = \partial L(x)/\partial\hat{K}^A$ we finally arrive at the *linear* functional differential master equation for the Lagrangian $L[\hat{G}](\hat{K})$:

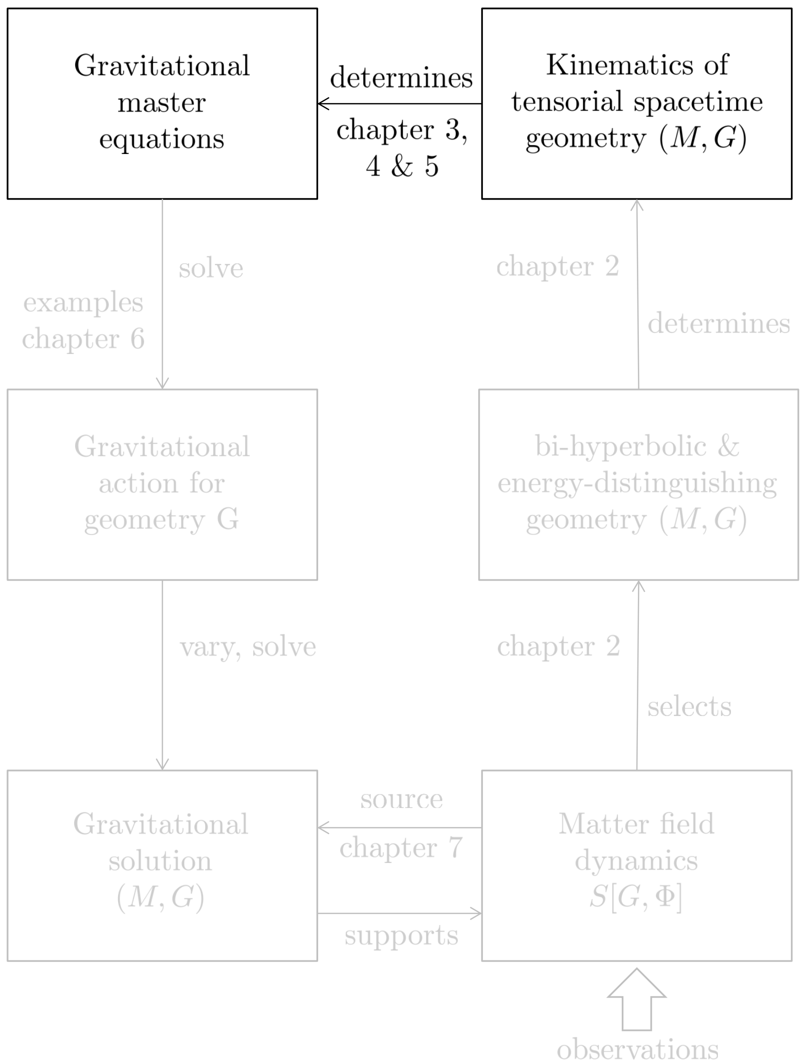
Master equation (functional differential form)

$$\begin{aligned} 0 \;=\; & -\frac{\delta L(x)}{\delta\hat{G}^A(y)}\hat{K}^A(y) + \partial_{y\varsigma}\left[\frac{\delta L(x)}{\delta\hat{G}^A(y)}M^{A\varsigma}(y)\right] - \frac{\partial L(x)}{\partial\hat{K}^A(x)}\hat{K}^B(x)Q_B{}^{A\beta}(x)\partial_\beta\delta_x(y)\\ & + \frac{\partial L(x)}{\partial\hat{K}^A(x)}\Big[U^{A\,(\mu\nu)}(x)\partial^2_{\mu\nu}\delta_x(y) + S^{A\,\mu}(x)\partial_\mu\delta_x(y)\Big]\\ & + T^{A\,[\mu\nu]}(x)\partial_\mu\left(\frac{\partial L(x)}{\partial\hat{K}^A(x)}\right)\partial_\nu\delta_x(y) - (x \leftrightarrow y)\,, \end{aligned} \tag{4.37}$$

where we have introduced the coefficients

$$S^{A\gamma} = -\partial_\beta(Q_B{}^{A[\beta|}M^{B\,|\gamma]}) - \partial_\beta U^{A\,(\beta\gamma)} - V^{A\,\gamma}\,. \tag{4.38}$$

In order to find the Lagrangian for any hypersurface geometry $\hat{G}^A$, we now have to extract all information contained in the linear *functional differential* equation (4.37) and the fact that the Lagrangian is a scalar density of weight one. In the next chapter, we will show that this leads to an equivalent set of linear *partial differential* master equations and thus the problem of finding gravitational dynamics can be expressed as the clear-cut mathematical problem of solving this system of equations.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Chapter 5

Gravitational master equations: partial differential form

In this chapter, we will convert the linear *functional* differential master equation, which we derived in the previous chapter, into an equivalent set of linear *partial* differential master equations and thus cast the physical problem of finding viable gravitational dynamics into an easy formulated mathematical task. This conversion proceeds in two steps. First we make use of the linear nature of the functional differential master equation by performing a series expansion of the Lagrangian in the velocities, and, secondly, we extract all independent information in the resulting set of functional differential equations. We will then partly solve the differential master equations as far as it is possible for an arbitrary hypersurface geometry $\hat{G}^A$. Beyond this, a case-by-case analysis is required, which we execute for several examples in chapter 6.

The results presented in this chapter have been published as
K. Giesel, F. P. Schuller, C. Witte and M. N. R. Wohlfarth,
Phys. Rev. D85 (2012), 104042,
whose section II.E, in particular, is further elaborated here.

5.1 Linear differential equations determine gravitational dynamics

The extraction of all information contained in the functional differential equation (4.37) proceeds in two steps. In the first one, we exploit the linear structure of the equation by performing a series expansion of the Lagrangian in the velocities $\hat{K}^A$,

$$L(x)[\hat{G}](\hat{K}) = \sum_{i=0}^{\infty} C(x)[\hat{G}]_{A_1 \dots A_i} \hat{K}^{A_1}(x) \dots \hat{K}^{A_i}(x) \,. \tag{5.1}$$

The coefficients $C(x)[\hat{G}]_{A_1\ldots A_i}$ must be tensor densities of weight one because the Lagrangian is a scalar density of the same weight and the velocities are tensors. The zeroth order coefficient C will turn out to require slightly different treatment than the higher order coefficients $C_{A_1\ldots A_i}$ and in order to talk about this distinction, it will be useful refer to C as the gravitational potential. The coefficients $C_{A_1\ldots A_i}$ (for $i \geq 0$) carry the complete functional dependence of the Lagrangian on the configuration variables $\hat{G}^A$. The series expansion is justified since we saw from equation (4.22) in section 4.4 that the Lagrangian must be a local function of the velocities. Inserting the ansatz (5.1) into the functional differential equation (4.37) turns the latter into a functional differential equation for the coefficients $C(x)[\hat{G}]_{A_1\ldots A_i}$, which can then be split into a countable set of equations by comparing the terms in front of the different powers of the velocities. The N-th order equation can be extracted by applying the N-th order functional derivative operator

$$\frac{\delta^N}{\delta \hat{K}^{B_1}(x_1)\ldots\delta\hat{K}^{B_N}(x_N)} \tag{5.2}$$

to equation (4.37), subsequently setting all remaining $\hat{K}^A$ to zero. The final result of this procedure may be summarized by the zeroth order equation

$$\begin{aligned} 0 \;=\; & \partial_{y^\varsigma}\left[M^{A\varsigma}(y)\frac{\delta C(x)}{\delta\hat{G}^A(y)}\right] + C(x)_A\left[U^{A\,(\mu\nu)}(x)\partial^2_{\mu\nu}\delta_x(y) + S^{A\,\mu}(x)\partial_\mu\delta_x(y)\right] \\ & +(\partial_\mu C_A)(x)\,T^{A\,[\mu\nu]}(x)\partial_\nu\delta_x(y) - (x\leftrightarrow y)\,, \end{aligned} \tag{5.3}$$

together with the N-th order contributions ($N \geq 1$),

$$\begin{aligned} 0 \;=\; & \Big\{(N+1)!\,C(x)_{AB_1\ldots B_N}\Big(U^{A\,(\mu\nu)}(x)\partial^2_{\mu\nu}\delta_x(y) + S^{A\,\mu}(x)\partial_\mu\delta_x(y)\Big) \\ & + N!\,\partial_{y^\varsigma}\left[M^{A\varsigma}(y)\frac{\delta C(x)_{B_1\ldots B_N}}{\delta\hat{G}^A(y)}\right] \\ & - N N!\,Q_{(B_1}{}^{M\,\beta}(x)C(x)_{B_2\ldots B_N)M}\partial_\beta\delta_x(y)\Big\}\,\delta_x(x_1)\ldots\delta_x(x_N) \\ & -(N-1)!\sum_{j=1}^{N}\frac{\delta C(x)_{B_1\ldots\widetilde{B_j}\ldots B_N}}{\delta\hat{G}^{B_j}(y)}\delta_y(x_j)\,\delta_x(x_1)\ldots\widetilde{\delta_x(x_j)}\ldots\delta_x(x_N) \\ & + (N+1)!\,(\partial_\mu C_{AB_1\ldots B_N})(x)\,T^{A\,[\mu\nu]}(x)\delta_x(x_1)\ldots\delta_x(x_N)\partial_\nu\delta_x(y) \\ & - (N+1)!\,T^{A\,[\mu\nu]}(x)\,C_{AB_1\ldots B_N}(x)\sum_{i=1}^{N}\delta_x(x_1)\ldots\widetilde{\delta_x(x_i)}\ldots\delta_x(x_N)\,\times \\ & \qquad\qquad \times\partial_\mu\delta_x(x_i)\,\partial_\nu\delta_x(y) - (x\leftrightarrow y)\,, \end{aligned} \tag{5.4}$$

where $\sim$ instructs us to omit the corresponding term. We can then proceed to extract all information from the above distributional differential equations by multiplying them with test functions $f(x,y,x_1,\ldots,x_N)$ of $N+2$ variables and integrating out all the delta distributions. The only additional, but very conservative, assumption we have to make is that the coefficients $C(x)[\hat{G}]_{A_1\dots A_N}$ are uniquely determined, at every point of the hypersurface, by the configuration variables and all their partial derivatives. In other words, the functional derivative of the coefficients with respect to the configuration variables, which appear in equation (5.3) and (5.4), can be expanded as

$$\frac{\delta C(x)[\hat{G}]_{B_1\dots B_i}}{\delta \hat{G}^A(y)} = \sum_{j=0}^{\infty}(-1)^j \frac{\partial C(x)_{B_1\dots B_i}(\hat{G}, \partial\hat{G}, \dots)}{\partial \partial^j_{\alpha_1\dots\alpha_j}\hat{G}^A(x)} \partial^j_{\alpha_1\dots\alpha_j}\delta_x(y)\,. \tag{5.5}$$

We can then start with equation (5.3), multiply it with an arbitrary test function[1] $f(x, y)$ and eliminate all delta distributions. This results in the integral equation

$$\begin{aligned}
0 = \int dx \Big\{ & C_A U^{A\,(\mu\nu)}(x)(\partial^2_{2\,\mu\nu} f)(x,x) \\
& -(C_A\, S^{A\mu} + \partial_\nu C_A T^{A\,[\nu\mu]})(x)(\partial_{2\,\mu} f)(x,x) \\
& -\sum_{j=0}^{\infty}\sum_{s=0}^{j}\binom{j}{s}\frac{\partial C(x)}{\partial\partial^j_{\alpha_1\dots\alpha_j}\hat{G}^A(x)}(\partial^{s+1}_{2\,\zeta(\alpha_1\dots\alpha_s}f)(x,x)(\partial^{j-s}_{\alpha_{s+1}\dots\alpha_j)}M^{A\zeta})(x)\Big\} - \\
& - (\partial_2 \to \partial_1)\,,
\end{aligned} \tag{5.6}$$

where the subscript '1' or '2' indicates that the partial derivatives act only on the first or second entry of the test function $f(x, y)$. One may be tempted to conclude, from the vanishing of the integrand, that the coefficients in front of the different higher order derivatives ∂_1^m and ∂_2^n of the arbitrary test function already have to vanish inividually. However, these derivatives are evaluated at the point (x, x) rather than the point (x, y), and since we have, for example

$$\partial_\mu f(x,x) = (\partial_{1\,\mu} f)(x,x) + (\partial_{2\,\mu} f)(x,x)\,, \tag{5.7}$$

they cannot be independent. In order to obtain all non-redundant information encoded in equation (5.6), we first eliminate one of the two types of derivatives in favour of a combination of the other type of derivative and some total derivatives, which may subsequently be eliminated by an integration by parts. More precisely, consider the simple example provided by the identity

$$\begin{aligned}
& \int dx \Big\{ A(x) f(x,x) + B^\mu(x)(\partial_{1\,\mu} f)(x,x) + C^\mu(x)(\partial_{2\,\mu} f)(x,x)\Big\} \\
= & \int dx \Big\{ [A(x) - \partial_\mu C^\mu(x)] f(x,x) + [B^\mu(x) - C^\mu(x)](\partial_{1\,\mu} f)(x,x)\Big\}\,,
\end{aligned} \tag{5.8}$$

where we have eliminated the partial derivatives on the second entry of the test

[1] Test functions, by definition, are always compactly supported. That means that one can drop all boundary terms when performing an integration by parts.

function in favour of those on the first entry and some total derivatives. One can see that only $A - \partial_\mu C^\mu = 0$ and $B^\mu - C^\mu = 0$ can be deduced from the left hand side of the equation rather than that A, B^μ and C^μ vanish individually. This can be generalised to higher order derivatives: one easily proves by induction that

$$(\partial^n_{2\,\alpha_1\ldots\alpha_n} f)(x,x) = \sum_{t=0}^{n} \binom{n}{t} (-1)^t (\partial^{n-t}_{(\alpha_1\ldots\alpha_t}\partial^t_{1\,\alpha_{t+1}\ldots\alpha_n)} f)(x,x)\,, \tag{5.9}$$

so that we can eliminate all partial derivatives acting on the second entry of the test function in equation (5.6). After reordering multiple sums, this results in an expression of the form

$$0 = \int dx \left\{ f(x,x)A(x) + \sum_{w=1}^{\infty} (\partial^w_{1\,\beta_1\ldots\beta_w} f)(x,x) B^{\beta_1\ldots\beta_w}(x) \right\}. \tag{5.10}$$

The vanishing of the coefficient A results in the equation

$$\begin{aligned} 0 = \partial^2_{\mu\nu}(C_A U^{A\,(\mu\nu)}) + \partial_\mu(C_A S^{A\,\mu}) + \partial_\nu C_A \partial_\mu T^{A\,[\nu\mu]} + \\ + \sum_{j=0}^{\infty}\sum_{s=0}^{j} (-1)^s \binom{j}{s} \partial^{s+1}_{\zeta\alpha_1\ldots\alpha_s} \left[\frac{\partial C}{\partial\partial^j_{\alpha_1\ldots\alpha_j}\hat{G}^A} \partial^{j-s}_{\alpha_{s+1}\ldots\alpha_j} M^{A\,\zeta} \right], \end{aligned} \tag{5.11}$$

whereas $B^{\beta_1} = 0$ implies

$$\begin{aligned} 0 = 2\partial_\mu(C_A U^{A\,(\beta\mu)}) + 2C_A S^{A\,\beta} + 2\,\partial_\nu C_A T^{A\,[\nu\beta]} + \sum_{j=0}^{\infty} \partial^j_{\gamma_1\ldots\gamma_j} M^{A\,\beta} \frac{\partial C}{\partial\partial^j_{\gamma_1\ldots\gamma_j}\hat{G}^A} \\ + \sum_{j=0}^{\infty}\sum_{s=0}^{j} (-1)^s \binom{j}{s} (s+1)\,\partial^s_{\alpha_1\ldots\alpha_s} \left[\partial^{j-s}_{\gamma_1\ldots\gamma_{j-s}} M^{A\,(\beta|} \frac{\partial C}{\partial\partial^j_{|\alpha_1\ldots\alpha_s)\gamma_1\ldots\gamma_{j-s}}\hat{G}^A} \right]. \end{aligned} \tag{5.12}$$

The vanishing of all other coefficients $B^{\beta_1\ldots\beta_w}$ with $w \geq 2$ can be summarized in only two equations. There is one equation for all even $w \geq 2$, which reads

$$0 = \sum_{j=w}^{\infty}\sum_{s=w+1}^{j+1} (-1)^s \binom{j+1}{s} \binom{s}{w} \partial^{s-w}_{\alpha_w\ldots\alpha_{s-1}} \left(\frac{\partial C}{\partial\partial^j_{(\beta_1\ldots\beta_{w-1}|\alpha_w\ldots\alpha_j}\hat{G}^A} \partial^{j+1-s}_{\alpha_s\ldots\alpha_j} M^{A\,|\beta_0)} \right). \tag{5.13}$$

The last non-trivial equation is valid for all odd $w \geq 3$:

$$0 = 2\sum_{j=w-1}^{\infty}\binom{j+1}{w}\partial^{j+1-w}_{\alpha_w\ldots\alpha_j}M^{A\,(\beta_0|}\frac{\partial C}{\partial\partial^{j}_{|\beta_1\ldots\beta_{w-1})\alpha_w\ldots\alpha_j}\hat{G}^A}$$
$$-\sum_{j=w}^{\infty}\sum_{s=w+1}^{j+1}(-1)^s\binom{j+1}{s}\binom{s}{w}\times$$
$$\times\partial^{s-w}_{\alpha_w\ldots\alpha_{s-1}}\left[\partial^{j+1-s}_{\alpha_s\ldots\alpha_j}M^{A\,(\beta_0|}\frac{\partial C}{\partial\partial^{j}_{|\beta_1\ldots\beta_{w-1})\alpha_w\ldots\alpha_j}\hat{G}^A}\right]. \tag{5.14}$$

With slight modification, the same procedure, which above led from equation (5.3) to the equivalent set of equations (5.11)-(5.14), can be applied to equation (5.4). After multiplication of the latter with a test function $f(x, y, x_1, \ldots, x_N)$ and integrating out all delta distributions, one obtains, for any $N \geq 1$,

$$\begin{aligned}
0 \;=\; & \int dx \Big\{(N+1)!C_{AB_1\ldots B_N}\left(U^{A\,(\gamma\delta)}\partial^2_{2\,\gamma\delta}f - S^{A\,\gamma}\partial_{2\,\gamma}f\right) \\
& -(N+1)!\,\partial_\nu C_{AB_1\ldots B_N}T^{A\,[\nu\mu]}\partial_{2\mu}f + NN!Q_{B_1}{}^{M\beta}C_{B_2\ldots B_N)M}\partial_{2\,\beta}f \\
& -(N+1)!\,C_{AB_1\ldots B_N}T^{A\,[\mu\nu]}\left(\sum_{i=3}^{N+1}[\partial_{2\mu}\partial_{i\nu}f] - 2\partial_\mu\partial_{(1,3\ldots N+1)\nu}f\right) \\
& -N!\sum_{s=0}^{\infty}\sum_{j=s}^{\infty}\binom{j}{s}\frac{\partial C_{B_1\ldots B_N}}{\partial\partial^j_{\alpha_1\ldots\alpha_j}\hat{G}^A}\left(\partial^{s+1}_{2\,\zeta(\alpha_1\ldots\alpha_{s-1}}f\,\partial^{j-s}_{\alpha_{s+1}\ldots\alpha_j)}M^{A\,\zeta}\right) \\
& -(N-1)!\sum_{s=1}^{\infty}\sum_{j=s}^{\infty}\sum_{i=1}^{N-1}\frac{\partial C_{B_1\ldots\widetilde{B}_i\ldots B_N}}{\partial\partial^j_{\alpha_1\ldots\alpha_j}\hat{G}^{B_i}}\partial^s_{2\,(\alpha_1\ldots\alpha_s}\partial^{j-s}_{(i+2)\,\alpha_{s+1}\ldots\alpha_j)}f \\
& +(N-1)!\sum_{t=1}^{\infty}\sum_{k=0}^{\infty}\sum_{j=k+t}^{\infty}(-1)^j\frac{j!}{t!k!(j-t-k)!}\partial^{j-t-k}_{\alpha_1\ldots\alpha_{j-t-k}}\frac{\partial C_{B_1\ldots B_{N-1}}}{\partial\partial^j_{\alpha_1\ldots\alpha_j}\hat{G}^{B_N}}\times \\
& \qquad\times\partial^t_{2\,\alpha_{j-t-k}\ldots\alpha_{j-k+1}}\partial^k_{(3,\ldots,N+1)\alpha_{j-k+2}\ldots\alpha_j}f\Big\} - \\
& \qquad -\Big\{\partial_2 \leftrightarrow \partial_1\Big\},
\end{aligned} \tag{5.15}$$

where $\partial_{(3,\ldots,N+1)}f$ denotes a derivative acting only on entries three to $N+1$ of the test function. These arise since, again, the test function and its various derivatives are evaluated at the point $(x, \ldots, x)$ (with $N+2$ entries) and thus they cannot all be independent. Without loss of generality, we have chosen to eliminate the partial derivative acting on the $(N+2)$-nd entry, making use of the identity

$$\partial^j_{(2,N+2)\,\alpha_1\ldots\alpha_j}f = \sum_{s=0}^{j}\binom{j}{s}(-1)^{j-s}\partial^s_{(\alpha_1\ldots\alpha_s}\partial^{j-s}_{(1,3,\ldots,N+1)\,\alpha_{s+1}\ldots\alpha_j}f\,, \tag{5.16}$$

which led to the above result. After reordering multiple sums, equation (5.15) with

$N \geq 1$ can be brought to the form

$$0 = \int dx \sum_{s=1}^{\infty} \sum_{j=0}^{\infty} \sum_{\mathrm{Part}_m(j)} {}^{(s;j)}B^{\beta_1 \ldots \beta_{s+j}}_{B_1 \ldots B_N} (\partial_2^s \partial_3^{m_3} \ldots \partial_{N+1}^{m_{N+1}})_{(\beta_1 \ldots \beta_{s+j})} f \; - \; (\partial_2 \to \partial_1)\,, \tag{5.17}$$

where the third sum is meant as a summation over partitions $j = m_3 + \ldots m_{N+1}$ of the label j into $N-1$ summands. If, for instance, $j = 0$ for any $N \geq 1$ then there is only the trivial partition $j = 0 + \cdots + 0$ of j into $N-1$ zeros, whereas for $j = 2$ and $N = 3$, we may have $j = 2 + 0$, $j = 1 + 1$ and $j = 0 + 2$. Now the coefficients ${}^{(s;j)}B^{\beta_s \ldots \beta_{s+j}}_{B_1 \ldots B_N}$ labeled by the three numbers s, j and N must indeed vanish since all the corresponding derivatives of the test function are independent. This way one finds that, at the level of $j = 0$, there are three types of nontrivial equations. First, for any $N \geq 1$, we find that the vanishing of the coefficient ${}^{(1;0)}B$ implies that

$$\begin{aligned} 0 \;\; = \;\; & (N+1)!\, C_{AB_1 \ldots B_N} (S^{A\,\beta} + 2\,\partial_\mu T^{A\,[\mu\beta]}) + (N+1)!\,\partial_\mu C_{AB_1 \ldots B_N} T^{A\,[\mu\beta]} \\ & - N N!\, Q_{(B_1}{}^{M\,\beta} C_{B_2 \ldots B_N)M} + \; N! \sum_{n=0}^{\infty} \frac{\partial C_{B_1 \ldots B_N}}{\partial \partial^n_{\alpha_1 \ldots \alpha_n} \hat{G}^A} \partial^n_{\alpha_1 \ldots \alpha_n} M^{A\,\beta} \\ & + (N-1)! \sum_{i=1}^{N-1} \frac{\partial C_{B_1 \ldots \widetilde{B}_i \ldots B_N}}{\partial \partial_\beta \hat{G}^{B_i}} - \; (N-1)! \sum_{n=1}^{\infty} (-1)^n\, n\, \partial^{n-1}_{\alpha_2 \ldots \alpha_n} \frac{\partial C_{B_1 \ldots B_{N-1}}}{\partial \partial^n_{\beta \alpha_2 \ldots \alpha_n} \hat{G}^{B_N}}\,. \end{aligned} \tag{5.18}$$

Furthermore, the coefficient ${}^{(2;0)}B$ yields for any $N \geq 1$ that

$$\begin{aligned} 0 \;\; = \;\; & (N+1)!\, C_{AB_1 \ldots B_N} U^{A\,(\beta_1 \beta_2)} - \; N! \sum_{n=1}^{\infty} n \, \frac{\partial C_{B_1 \ldots B_N}}{\partial \partial^n_{(\beta_1 | \alpha_2 \ldots \alpha_n} \hat{G}^A} \, \partial^{n-1}_{\alpha_2 \ldots \alpha_n} M^{A\,|\beta_2)} \\ & - (N-1)! \sum_{i=1}^{N-1} \frac{\partial C_{B_1 \ldots \widetilde{B}_i \ldots B_N}}{\partial \partial^2_{\beta_1 \beta_2} \hat{G}^{B_i}} \\ & + (N-1)! \sum_{n=2}^{\infty} (-1)^n \binom{n}{2} \partial^{n-2}_{\alpha_3 \ldots \alpha_n} \frac{\partial C_{B_1 \ldots B_{N-1}}}{\partial \partial^n_{\beta_1 \beta_2 \alpha_3 \ldots \alpha_n} \hat{G}^{B_N}}\,, \end{aligned} \tag{5.19}$$

and all other coefficients of the form ${}^{(s \geq 3;0)}B$ let us conclude for any $N \geq 1$ that

$$\begin{aligned} 0 \;\; = \;\; & N! \sum_{n=s-1}^{\infty} \binom{n}{s-1} \frac{\partial C_{B_1 \ldots B_N}}{\partial \partial^n_{(\beta_1 \ldots \beta_{s-1} | \alpha_s \ldots \alpha_n} \hat{G}^A} \partial^{n-s+1}_{\alpha_s \ldots \alpha_n} M^{A\,|\beta_s)} \\ & + (N-1)! \sum_{i=1}^{N-1} \frac{\partial C_{B_1 \ldots \widetilde{B}_i \ldots B_N}}{\partial \partial^s_{\beta_1 \ldots \beta_s} \hat{G}^{B_i}} \\ & - (N-1)! \sum_{n=s}^{\infty} (-1)^n \binom{n}{s} \partial^{n-s}_{\alpha_{s+1} \ldots \alpha_n} \frac{\partial C_{B_1 \ldots B_{N-1}}}{\partial \partial^n_{\beta_1 \ldots \beta_s \alpha_{s+1} \ldots \alpha_n} \hat{G}^{B_N}}\,. \end{aligned} \tag{5.20}$$

Fortunately, there are only three more types of equations for any $j \geq 1$ that are valid for all $N \geq 2$. The first two types involve partitions of the label j with $m_a = j$

for some $a = 3, \ldots, N+1$. The vanishing of the antisymmetric part of the coefficient ${}^{(s\geq 1; j=\cdots+m_a+\ldots)}B$ with $m_a = 1$ implies

$$0 = C_{AB_1 \ldots B_N} T^{A\,[\mu\nu]}\,. \tag{5.21}$$

Furthermore, the vanishing of the symmetric part of the coefficient ${}^{(s\geq 1; j=\cdots+m_a+\ldots)}B$ with $m_a \geq 1$ implies

$$\begin{aligned} 0 \;\; = \;\; & -(N-1)! \sum_{q=s+j}^{\infty} (-1)^q \frac{q!}{s!j!(q-j-s)!} \partial^{q-j-s}_{\alpha_{s+j+1} \ldots \alpha_q} \frac{\partial C_{B_1 \ldots B_{N-1}}}{\partial\, \partial^q_{\beta_1 \ldots \beta_{s+j} \alpha_{s+j+1} \ldots \alpha_q} \hat{G}^{B_N}} \\ & + (N-1)! \binom{s+j}{s} \frac{\partial C_{B_1 \ldots \widetilde{B}_i \ldots B_N}}{\partial \partial^{s+j}_{\beta_1 \ldots \beta_{s+j}} \hat{G}^{B_i}} \quad \text{for } i = 1 \ldots N-1. \end{aligned} \tag{5.22}$$

Finally, the last type covers all remaining partitions of the label $j \geq 2$ into at least two non-vanishing summands, and the corresponding equation reads

$$0 \;\; = \;\; -\frac{(N-1)!}{m_3! \ldots m_{N+1}!} \sum_{q=s+j}^{\infty} (-1)^q \frac{q!}{s!(q-j-s)!} \partial^{q-j-s}_{\alpha_{s+j+1} \ldots \alpha_q} \frac{\partial C_{B_1 \ldots B_{N-1}}}{\partial\, \partial^q_{\beta_1 \ldots \beta_{s+j} \alpha_{s+j+1} \ldots \alpha_q} \hat{G}^{B_N}}\,. \tag{5.23}$$

Equations (5.11)-(5.13) and (5.18)-(5.23) are all we can extract about the gravitational dynamics from the first algebra relation (4.11). In the next sections, we will show that we can further reduce these equations by observing that the coefficients $C_{B_1 \ldots B_N}$ for $N \geq 1$ can only depend on the derivatives of the fields $\hat{G}^A$ up to the second order, while the potential term C can depend at most on the third partial derivatives of the fields $\hat{G}^A$. Because of these slightly differerent dependencies, it pays off to split the above equations into those that do not contain the potential C, and those that do.

5.2 Master equations *not* containing the potential C

In this section, we simplify the master equations that contain only the coefficients $C_{B_1 \ldots B_N}$ for $N \geq 1$. This simplification is achieved by the observation that these coefficients can at most depend on the second derivatives of the fields $\hat{G}^A$, so that $C_{B_1 \ldots B_N} = C_{B_1 \ldots B_N}(\hat{G}, \partial\hat{G}, \partial^2 G)$. We also derive a set of differential equations that follows from the second algebra equation (4.12) and express the fact that the coefficients $C_{B_1 \ldots B_N}$ are tensor densities of weight one. We will show in which cases these equations can be solved explicitly.

5.2.1 Collapse to second derivative order

In order to simplify the master equations that contain the coefficients $C_{B_1\ldots B_N}$ for all $N \geq 1$, we first observe that equation (5.22) and (5.23) together imply that

$$\frac{\partial C_{B_1\ldots\widetilde{B}_i\ldots B_N}}{\partial\partial^{s+j}_{\gamma_1\ldots\gamma_{s+j}}\hat{G}^{B_i}} = 0 \quad \text{for all } N \geq 2 \text{ and } s+j \geq 3\,. \tag{5.24}$$

This shows that none of the coefficients $C_{B_1\ldots B_N}$ with $N \geq 1$ can depend on partial derivatives of the fields $\hat{G}^A$ higher than the second, and thus we have that $C_{B_1\ldots B_N} = C_{B_1\ldots B_N}(\hat{G}^A, \partial\hat{G}^A, \partial^2\hat{G}^A)$. This greatly simplifies the remaining master equations, which only contain these coefficients. More precisely, equations (5.19) and (5.18) imply, for all $N \geq 2$, that

$$\begin{aligned} 0 \;=\;& (N+1)!\, C_{AB_1\ldots B_N} U^{A\,(\alpha\beta)} - N!\,\frac{\partial C_{B_1\ldots B_N}}{\partial\partial_{(\beta|}\hat{G}^A} M^{A\,|\alpha)} - 2N!\,\frac{\partial C_{B_1\ldots B_N}}{\partial\partial^2_{(\beta|\gamma}\hat{G}^A}\partial_\gamma M^{A\,|\alpha)} \\ & - (N-2)(N-1)!\frac{\partial C_{B_1\ldots B_{N-1}}}{\partial\partial^2_{\alpha\beta}\hat{G}^{B_N}}\,, \end{aligned} \tag{5.25}$$

$$\begin{aligned} 0 \;=\;& (N+1)! C_{AB_1\ldots B_N}(S^{A\,\alpha} + 2\,\partial_\mu T^{A\,[\mu\alpha]}) \\ & +(N+1)!\,\partial_\mu C_{AB_1\ldots B_N} T^{A\,[\mu\alpha]} - N N!\, Q_{(B_1}{}^{M\,\alpha}\, C_{B_2\ldots B_N)M} \\ & +N!\,\frac{C_{B_1\ldots B_N}}{\partial\hat{G}^A} M^{A\,\alpha} + N!\,\frac{\partial C_{B_1\ldots B_N}}{\partial\partial_\gamma\hat{G}^A}\partial_\gamma M^{A\,\alpha} + N!\,\frac{\partial C_{B_1\ldots B_N}}{\partial\partial^2_{\gamma\delta}\hat{G}^A}\partial^2_{\gamma\delta} M^{A\,\alpha} \\ & +(N-1)!\sum_{a=1}^{N}\frac{\partial C_{B_1\ldots\widetilde{B_a}\ldots B_N}}{\partial\partial_\alpha\hat{G}^{B_a}} - 2(N-1)!\partial_\gamma\frac{\partial C_{B_1\ldots B_{N-1}}}{\partial\partial^2_{\alpha\gamma}\hat{G}^{B_N}}\,. \end{aligned} \tag{5.26}$$

The other non-trivial equations are given by (6.94), which implies

$$0 = \frac{\partial C_{B_1\ldots B_N}}{\partial\partial^2_{(\alpha\beta|}\hat{G}^A} M^{A\,|\gamma)} \qquad \text{for all } N \geq 2\,, \tag{5.27}$$

while equation (5.21) still reads

$$0 = C_{AB_1\ldots B_N} T^{A\,[\mu\nu]} \quad \text{for all } N \geq 2\,. \tag{5.28}$$

Finally, equation (5.22) yields, for any $i = 1, \ldots, N-1$, the symmetry condition

$$\frac{\partial C_{B_1\ldots\widetilde{B}_i\ldots B_N}}{\partial\partial^2_{\mu\nu}\hat{G}^{B_a}} = \frac{\partial C_{B_1\ldots\ldots B_{N-1}}}{\partial\partial^2_{\mu\nu}\hat{G}^{B_N}} \qquad \text{for all } N \geq 2\,. \tag{5.29}$$

Equations (5.25)-(5.29) are all that we can deduce for the coefficients $C_{B_1\ldots B_N}$ with $N \geq 1$ from the first algebra equation (4.11). In addition to these differential equations, the coefficients $C_{B_1\ldots B_N}$ (for $N \geq 1$) have to satisfy another set of three differential equations that express the fact that they are tensor densities of weight one. These equations can either be derived by extracting the information contained in the second algebra equation (4.12) (the way we did it for the first algebra equation) or by direct calculation [62]. Unfortunately, the explicit form of these invariance equations highly depends on the type of tensor fields $\hat{G}^A$. In order to illustrate the direct method, we choose $\hat{G}^A = \hat{G}^\alpha{}_\beta$ for an arbitrary $(1,1)$-tensor field $\hat{G}^\alpha{}_\beta$. The generalisation to other types of tensor fields then follows suit. Under an arbitrary change of coordinates $\bar{x}^\alpha = \bar{x}^\alpha(x)$ on $X_0(\Sigma)$, the field $\hat{G}^\alpha{}_\beta$ transforms as

$$\bar{G}^\gamma{}_\delta = \hat{G}^\alpha{}_\beta \, A^\gamma_\alpha \, (A^{-1})^\beta_\delta, \tag{5.30}$$

where $A^\beta_\alpha = \partial \bar{x}^\beta / \partial x^\alpha$ is the Jacobian of the transformation, and A^{-1} denotes its inverse. The transformation behaviour can then be extended to the first partial derivatives of the fields:

$$\begin{aligned} \bar{G}^\gamma{}_{\delta\,,\mu} \;\; = \;\; & \hat{G}^\alpha{}_{\beta\,,\nu} \, A^\gamma_\alpha \, (A^{-1})^\beta_\delta (A^{-1})^\nu_\mu + \hat{G}^\alpha{}_\beta A^\gamma_{\alpha\,,\nu} (A^{-1})^\beta_\delta (A^{-1})^\nu_\mu \\ & - \hat{G}^\alpha{}_\beta A^\gamma_\alpha A^\rho_{\sigma\,,\nu} A^\nu_\mu (A^{-1})^\beta_\rho (A^{-1})^\sigma_\delta \,, \end{aligned} \tag{5.31}$$

where for brevity we denote partial derivatives with a comma. Similarly, one calculates the transformation behaviour of the second partial derivatives of $\hat{G}^\alpha{}_\beta$. Since the coefficients $C_{B_1\ldots B_N}$ are tensor densities of weight one, they must transform as

$$\bar{C}_{C_1\ldots C_N}(\bar{G}, \bar{\partial}\bar{G}, \bar{\partial}^2\bar{G}) = \det(A) \, A^{B_1}_{C_1} \ldots A^{B_N}_{C_N} \, C_{B_1\ldots B_N}(\hat{G}, \partial\hat{G}, \partial^2\hat{G}), \tag{5.32}$$

where $A^B_C = (A^{-1})^{\beta_1}_{\gamma_1} A^{\gamma_2}_{\beta_2}$ denotes the transformation of the capital multi-indices. Taking the derivative of equation (5.32) with respect to the variables $A^\rho_{\sigma\,,\mu\nu} = \partial^3 \bar{x}^\rho / (\partial x^\sigma \partial x^\mu \partial x^\nu)$, and observing that the right hand side is in fact independent of these, one directly obtains the first invariance equation

$$0 = \hat{G}^{(\alpha}{}_\mu \frac{\partial C_{B_1\ldots B_N}}{\partial \partial^2_{\beta\gamma)} \hat{G}^\rho{}_\mu} - \hat{G}^\mu{}_\rho \frac{\partial C_{B_1\ldots B_N}}{\partial \partial^2_{(\alpha\beta} \hat{G}^\mu{}_{\gamma)}} \quad \text{for all } N \geq 1. \tag{5.33}$$

One has to work somewhat harder to derive the second invariance identity, which follows from differentiation of equation (5.32) with respect to the variables $A^\rho_{\mu\,,\nu} = \partial^2 \bar{x}^\rho / (\partial x^\mu \partial x^\nu)$. In order to cast the resulting equation into a form that holds in any coordinate chart, one makes use of the identity

$$A^\alpha_{\beta\,,\gamma} = \Gamma^\mu_{\beta\gamma} A^\alpha_\mu - \bar{\Gamma}^\alpha_{\sigma\tau} A^\sigma_\beta A^\tau_\gamma \,, \tag{5.34}$$

which holds for any choice of connection coefficients $\Gamma^{\alpha}_{\beta\gamma}$, and thus one temporarily converts partial derivatives into covariant derivatives with respect to the connection Γ. It does not matter which connection one employs here. After all, these quantities are merely used for book-keeping reasons and can be forgotten after the calculation. The derivative of (5.32) with respect to $A^{\rho}_{\mu,\nu} = \partial^2\bar{x}^{\rho}/(\partial x^{\mu}\partial x^{\nu})$ then implies, together with equation (5.33) and after several pages of algebra, the second invariance identity

$$\begin{aligned} 0 \;=\;& \hat{G}^{(\alpha}{}_{\mu}\frac{\partial C_{B_1\dots B_N}}{\partial\partial_{\beta)}\hat{G}^{\rho}{}_{\mu}} - \hat{G}^{\mu}{}_{\rho}\frac{\partial C_{B_1\dots B_N}}{\partial\partial_{(\alpha}\hat{G}^{\mu}{}_{\beta)}} + 2\,\partial_{\nu}\hat{G}^{(\alpha}{}_{\mu}\frac{\partial C_{B_1\dots B_N}}{\partial\partial^2_{\beta)\nu}\hat{G}^{\rho}{}_{\mu}} - 2\,\partial_{\nu}\hat{G}^{\mu}{}_{\rho}\frac{\partial C_{B_1\dots B_N}}{\partial\partial^2_{\nu(\alpha}\hat{G}^{\mu}{}_{\beta)}} \\ -\;& \partial_{\nu}\hat{G}^{\mu}{}_{\rho}\frac{\partial C_{B_1\dots B_N}}{\partial\partial^2_{\alpha\beta}\hat{G}^{\mu}{}_{\nu}} \quad \text{for all } N \geq 1\,. \qquad (5.35) \end{aligned}$$

The final invariance equation results from the derivative of equation (5.32) with respect to A^{α}_{β} and is even harder to obtain. Since we will never have to make use of this final equation, we spare the reader the details. We will display such a last invariance equation, for illustrational purposes, in the special case where $\hat{G}^A$ denotes the hypersurface geometry induced by an area metric spacetime geometry coupled to general linear electrodynamics in section 6.3.

In the next section, we show that the master equations and the invariance equations can actually be further reduced to a new set of equations, which only contain derivatives of the coefficients $C_{B_1\dots B_N}$ (for $N \geq 1$) with respect to at most the first derivatives of the fields $\hat{G}^A$.

5.2.2 Reduction to first derivative order

In what follows, we will see which general conclusions we are able to draw from the master equations we derived in the previous section. We already showed that the relevant coefficients $C_{B_1\dots B_N}$ (for $N \geq 1$), which together with the potential C determine the gravitational dynamics of the fields $\hat{G}^A$, can only depend on at most second partial derivatives of the latter. We will now see that the master equations imply that the coefficients $C_{B_1\dots B_N}$ (for $N \geq 1$) can only depend polynomially, and then at most to the power of $\dim M - 1$, on the second derivatives of the fields $\hat{G}^A$.

In order to arrive at this result, we first write out the divergence in the last term in equation (5.26):

$$\begin{aligned} \partial_{\gamma}\frac{\partial C_{B_1\dots B_{N-1}}}{\partial\partial^2_{\alpha\gamma}\hat{G}^{B_N}} = \frac{\partial^2 C_{B_1\dots B_{N-1}}}{\partial\hat{G}^A\,\partial\,\partial^2_{\alpha\gamma}\hat{G}^{B_N}}\,\partial_{\gamma}\hat{G}^A \;&+\; \frac{\partial^2 C_{B_1\dots B_{N-1}}}{\partial\,\partial_{\delta}\hat{G}^A\,\partial\,\partial^2_{\alpha\gamma}\hat{G}^{B_N}}\,\partial^2_{\gamma\delta}\hat{G}^A \\ &+\; \frac{\partial^2 C_{B_1\dots B_{N-1}}}{\partial\,\partial^2_{\delta\mu}\hat{G}^A\,\partial\,\partial^2_{\alpha\gamma}\hat{G}^{B_N}}\,\partial^3_{\gamma\delta\mu}\hat{G}^A\,, \qquad (5.36) \end{aligned}$$

which holds for $N \geq 2$. Since none of the coefficients $C_{B_1\dots B_N}$ can depend on the third partial derivatives of the fields, we may conclude that the last term must already vanish,

$$\frac{\partial^2 C_{B_1\dots B_{N-1}}}{\partial\,\partial^2_{(\delta\mu}\hat{G}^A\,\partial\,\partial^2_{\gamma)\alpha}\hat{G}^{B_N}} = 0 \quad \text{for all } N \geq 2. \tag{5.37}$$

This is because the only other term, which also contains third partial derivatives of the fields $\hat{G}^A$, arises from $T^{A\,[\mu\alpha]}\partial_\mu C_{AB_1\dots B_N}$, but, writing out the total derivative, the resulting expression vanishes identically because of equation (5.28).

We now conduct the following argument, which is based on an observation by Lovelock [44]. For definiteness, we first restrict our attention to the case of a four-dimensional spacetime manifold, i.e., $\dim\Sigma = 3$. Nevertheless, it is straightforward to generalize the reasoning to any dimension. First of all, we introduce the quantities

$$\Lambda_{B_1\dots B_N}{}_{Q}^{\alpha\beta}{}_{R}^{\gamma\delta}{}_{S}^{\kappa\lambda}{}_{T}^{\rho\sigma} := \frac{\partial^4 C_{B_1\dots B_N}}{\partial\partial^2_{\alpha\beta}\hat{G}^Q\,\partial\partial^2_{\gamma\delta}\hat{G}^R\,\partial\partial^2_{\kappa\lambda}\hat{G}^S\,\partial\partial^2_{\rho\sigma}\hat{G}^T}\,. \tag{5.38}$$

It is easy to see that these must be tensor densities of the same weight as the coefficients $C_{B_1\dots B_N}$, by taking the derivative of their transformation law (5.32) under changes of the coordinates on $X_0(\Sigma)$ with respect to the second derivatives of the fields $\hat{G}^A$. The individual pairs of small greek indices are of course symmetric. Moreover, from the symmetry condition (5.37) we conclude that the quantities $\Lambda_{B_1\dots B_N}{}_{Q}^{\alpha\beta}{}_{R}^{\gamma\delta}{}_{S}^{\kappa\lambda}{}_{T}^{\rho\sigma}$ vanish whenever we symmetrize over three adjacent greek indices. This also implies that the latter are symmetric under exchange of any of the pairs $\alpha\beta$, $\gamma\delta$, $\kappa\lambda$ and $\rho\sigma$. We can now investigate the numerical value of all the components of the coefficients $\Lambda_{B_1\dots B_N}{}_{Q}^{\alpha\beta}{}_{R}^{\gamma\delta}{}_{S}^{\kappa\lambda}{}_{T}^{\rho\sigma}$. In three dimensions, we know that at least three of the eight small greek indices must take the same value. Using the symmetries of the small greek indices, we can, however, always arrange for equal indices to stand in a row. But the symmetry condition (5.37) implies that the latter vanish, and thus we may conclude that

$$\Lambda_{B_1\dots B_N}{}_{Q}^{\alpha\beta}{}_{R}^{\gamma\delta}{}_{S}^{\kappa\lambda}{}_{T}^{\rho\sigma} = 0\,. \tag{5.39}$$

Or, in other words, the coefficients $C_{B_1\dots B_N}$ can depend on the second derivatives of the fields $\hat{G}$ only up to the third power. We may thus expand

$$\begin{aligned} C_{B_1\dots B_N} \;=\;& {}^{(3)}\Lambda_{B_1\dots B_N}{}_{Q}^{\alpha\beta}{}_{R}^{\gamma\delta}{}_{S}^{\kappa\lambda}\hat{G}^Q{}_{,\alpha\beta}\,\hat{G}^R{}_{,\gamma\delta}\,\hat{G}^S{}_{,\kappa\lambda} + {}^{(2)}\Lambda_{B_1\dots B_N}{}_{Q}^{\alpha\beta}{}_{R}^{\gamma\delta}\hat{G}^Q{}_{,\alpha\beta}\,\hat{G}^R{}_{,\gamma\delta} \\ &+ {}^{(1)}\Lambda_{B_1\dots B_N}{}_{Q}^{\alpha\beta}\hat{G}^Q{}_{,\alpha\beta} + {}^{(0)}\Lambda_{B_1\dots B_N} \end{aligned} \tag{5.40}$$

where all of the expansion coefficients ${}^{(i)}\Lambda$ only depend on at most the first derivatives of the fields $\hat{G}^A$, and only the first coefficient is necessarily a tensor density of the same weight as $C_{B_1\dots B_N}$. We may then insert this expansion back into the remaining master equations in order to split them into a new set of differential equations for all the coefficients ${}^{(i)}\Lambda$ of the expansion by comparing powers of the second derivatives of the fields $\hat{G}^A$.

In order to generalize this argument to higher dimensions, we must add more derivatives in equation (5.38). One quickly realises that the polynomial dependence is always of degree $\dim \Sigma$ or, equivalently, $(\dim M - 1)$. In special situations, this result can be further refined in conjunction with the help of the invariance equations for the coefficients $C_{B_1 \ldots B_N}$. We will see this explicitly in section 6.2, where we solve the master equations in the case of a metric spacetime geometry that supports Maxwell electrodynamics.

In the next section, we solve the invariance equations in the special case where one of the fields $\hat{G}^A$ can be formally employed as a hypersurface metric.

5.2.3 Solutions to the invariance equations

We now discuss what can be extracted from the invariance equations in general, and in the special case that one of the tensor fields $\hat{G}^A$ can be formally employed as a hypersurface metric. The invariance equations reflect the tensor-density nature of the coefficients $C_{B_1 \ldots B_N}$ (for $N \geq 1$), which are functions of the form $C_{B_1 \ldots B_N}(\hat{G}^A, \partial \hat{G}^A, \partial^2 \hat{G}^A)$. The partial derivatives of the tensor fields $\hat{G}^A$ are of course not tensor fields, and hence the invariance equations encode how those non-tensorial fields have to be combined in order to produce the weight-one tensor densities $C_{B_1 \ldots B_N}$.

A thorough inspection of the structure of the invariance equations reveals that each of the latter can generally be intepreted as the coordinate expression of a vector field Y acting on the functions $C_{B_1 \ldots B_N}$ [35, 48]. Here the coordinate chart is spanned by the fields $(\hat{G}^A, \partial_\gamma \hat{G}^A, \partial^2_{\gamma\delta} \hat{G}^A)$. Indeed, this observation can be formalized mathematically using the theory of jet bundles, but we do not need to go into more detail here[2]. The important point here is that one can simplify the invariance equations by choosing a coordinate system different from the coordinates $(\hat{G}^A, \partial_\gamma \hat{G}^A, \partial^2_{\gamma\delta} \hat{G}^A)$, which can be thought of as a normal coordinate system. This is possible if one of the fields $\hat{G}^A$ can be employed as a hypersurface metric. Thus, let us assume that the hypersurface geometry is only given by an inverse metric, so that $\hat{G}^A = \hat{P}^{\alpha\beta}$. For simplicity, we discuss this particular case first, and then generalize it to all cases where, apart from a hypersurface metric, we have an arbitrary number of additional hypersurface tensor fields $\hat{G}^A$. The invariance equations for this case can be derived

[2] The vector fields that appear in the invariance equations are the result of lifting the generators of infinitesimal hypersurface diffeomorphisms, which are vector fields on $X(\Sigma)$, to the second jet bundle of the tensor bundle spanned by the tensor fields $\hat{G}^A$. All functions that lie in the kernel of these vector fields are then invariant under spatial diffeomorphism.

in analogy to the invariance equations (5.33) and (5.35). Explicitly those are

$$0 = \hat{P}^{\alpha(\sigma} \frac{\partial C_{B_1 \dots B_N}}{\partial \partial^2_{\mu\nu)} \hat{P}^{\alpha\rho}} \,, \tag{5.41}$$

as well as

$$0 = 2\,\hat{P}^{\alpha(\mu} \frac{\partial C_{B_1 \dots B_N}}{\partial \partial_{\nu)} \hat{P}^{\alpha\rho}} - \partial_\rho \hat{P}^{\alpha\beta} \frac{\partial C_{B_1 \dots B_N}}{\partial \partial^2_{\mu\nu} \hat{P}^{\alpha\beta}} + 4\,\partial_\sigma \hat{P}^{\alpha(\mu} \frac{\partial C_{B_1 \dots B_N}}{\partial \partial^2_{\nu)\sigma} \hat{P}^{\alpha\rho}} \,. \tag{5.42}$$

Since the field $\hat{P}^{\alpha\beta}$ can be employed as a hypersurface metric, we can now perform a coordinate change from $(\hat{P}^{\alpha\beta}, \partial_\gamma \hat{P}^{\alpha\beta}, \partial^2_{\gamma\delta} \hat{P}^{\alpha\beta})$ to a new set of coordinates $(\hat{P}^{\alpha\beta}, \Gamma^\alpha_{\beta\gamma}, R_{\alpha\beta\gamma\delta}, S_{\alpha\beta\gamma\delta})$, trading the first partial derivatives of the field $\hat{P}^{\alpha\beta}$ for the Levi-Civita connection coefficients Γ of $\hat{P}^{\alpha\beta}$, and its second partial derivatives for the corresponding Riemann-Christoffel tensor R and another variable S. Explicitly this coordinate transformation is given by

$$\begin{aligned}
\Gamma^\alpha_{\beta\gamma} &= {}^{\hat{P}}\Gamma^{\alpha\rho}{}_{\beta\gamma\lambda\kappa} \hat{P}^{\lambda\kappa}{}_{,\rho} && (5.43)\\
R_{\alpha\beta\gamma\delta} &= {R_1}^{\mu\nu}{}_{\kappa\tau\alpha\beta\gamma\delta}\, \hat{P}^{\kappa\tau}{}_{,\mu\nu} + {R_2}^{\sigma\tau}{}_{\mu\nu\kappa\epsilon\alpha\beta\gamma\delta}\, \hat{P}^{\mu\nu}{}_{,\sigma} \hat{P}^{\kappa\epsilon}{}_{,\tau} && (5.44)\\
S_{\alpha\beta\gamma\delta} &= {S_1}^{\mu\nu}{}_{\kappa\tau\alpha\beta\gamma\delta}\, \hat{P}^{\kappa\tau}{}_{,\mu\nu} + {S_2}^{\sigma\tau}{}_{\mu\nu\kappa\epsilon\alpha\beta\gamma\delta}\, \hat{P}^{\mu\nu}{}_{,\sigma} \hat{P}^{\kappa\epsilon}{}_{,\tau}\,, && (5.45)
\end{aligned}$$

where for brevity we used a comma to denote partial derivatives. The coefficients in the above expressions are

$$\begin{aligned}
{}^{\hat{P}}\Gamma^{\alpha\rho}{}_{\beta\gamma\kappa\tau} &= \frac{1}{2} \hat{P}_{\beta(\kappa} \hat{P}_{\tau)\gamma} \hat{P}^{\alpha\rho} - \delta^\alpha_{(\kappa} \hat{P}_{\tau)(\beta} \delta^\rho_{\gamma)}\,, && (5.46)\\
{R_1}^{\mu\nu}{}_{\kappa\tau\alpha\beta\gamma\delta} &= 2\delta^{(\mu}_{[\beta} \hat{P}_{\alpha](\kappa} \hat{P}_{\tau)[\gamma} \delta^{\nu)}_{\delta]}\,, && (5.47)\\
{R_2}^{\sigma\tau}{}_{\mu\nu\kappa\epsilon\alpha\beta\gamma\delta} &= \delta^\tau_{(\nu} \hat{P}_{\mu)[\alpha} \hat{P}_{\beta](\kappa} \hat{P}_{\epsilon)[\delta} \delta^\sigma_{\gamma]} + \delta^\tau_{(\nu} \hat{P}_{\mu)[\delta} \hat{P}_{\gamma](\kappa} \hat{P}_{\epsilon)[\alpha} \delta^\sigma_{\beta]} + \delta^\sigma_{[\alpha} \hat{P}_{\beta](\kappa} \hat{P}_{\epsilon)(\mu} \hat{P}_{\nu)[\gamma} \delta^\tau_{\delta]}\\
&\quad + 2\,\delta^\sigma_{[\alpha} \hat{P}_{\beta](\mu} \hat{P}_{\nu)(\kappa} \hat{P}_{\epsilon)[\gamma} \delta^\tau_{\delta]} + \frac{1}{2}\,\hat{P}_{(\mu|[\alpha} \hat{P}_{\beta](\kappa} \hat{P}_{\epsilon)|\nu)} \delta^\sigma_{[\delta} \delta^\tau_{\gamma]}\\
&\quad + \frac{1}{2}\,\hat{P}_{(\mu|[\delta} \hat{P}_{\gamma](\kappa} \hat{P}_{\epsilon)|\nu)} \delta^\sigma_{[\alpha} \delta^\tau_{\beta]} + \frac{1}{2}\,\hat{P}^{\sigma\tau} \hat{P}_{(\mu|[\alpha} \hat{P}_{\beta](\kappa} \hat{P}_{\epsilon)[\gamma} \hat{P}_{\delta]|\nu)}\,, && (5.48)\\
{S_1}^{\mu\nu}{}_{\kappa\tau\alpha\beta\gamma\delta} &= -\hat{P}_{\alpha(\kappa} \hat{P}_{\tau)(\beta} \delta^{(\mu}_\gamma \delta^{\nu)}_{\delta)} + \frac{1}{2} \hat{P}_{(\kappa|(\beta} \delta^{(\mu}_\gamma \hat{P}_{\delta)|\tau)} \delta^{\nu)}_\alpha \quad \text{and} && (5.49)\\
{S_2}^{\sigma\tau}{}_{\mu\nu\kappa\epsilon\alpha\beta\gamma\delta} &= 2\hat{P}_{\alpha(\mu} \hat{P}_{\kappa)(\beta} \delta^\sigma_\gamma \delta^\tau_{\delta)} \hat{P}_{\nu\epsilon} - \hat{P}_{(\beta|(\mu} \hat{P}_{\kappa)|\gamma} \delta^\sigma_{\delta)} \hat{P}_{\nu\epsilon} \delta^\tau_\alpha\,. && (5.50)
\end{aligned}$$

The variable $S_{\alpha\beta\gamma\delta}$ is needed since the Riemann tensor does not contain all the second partial derivatives of the field $\hat{P}^{\alpha\beta}$. Without this variable, the change of coordinates is not invertible. We note that the variables $S_{\alpha\beta\gamma\delta}$ are not components of a tensor and feature the symmetry $S_{\alpha\beta\gamma\delta} = S_{\alpha(\beta\gamma\delta)}$. In order to transform the original invariance equations to the new coordinates, we also need the inverse coordinate transformation:

$$
\begin{aligned}
\hat{P}^{\alpha\beta}{}_{,\gamma} &= -2\hat{P}^{\mu(\alpha}\Gamma^{\beta)}_{\mu\gamma} & (5.51)\\
\hat{P}^{\mu\nu}{}_{,\gamma\delta} &= \frac{1}{3}\hat{P}^{\mu\alpha}\hat{P}^{\nu\beta}\left(R_{\alpha\gamma\beta\delta}+R_{\beta\gamma\alpha\delta}\right)-\hat{P}^{\mu\alpha}\hat{P}^{\nu\beta}\left(S_{\alpha\beta\gamma\delta}+S_{\beta\alpha\gamma\delta}\right)\\
&\quad+\frac{1}{3}\hat{P}_{\rho\sigma}\hat{P}^{\mu\alpha}\hat{P}^{\nu\beta}\left(\Gamma^{\rho}_{\beta(\gamma}\Gamma^{\sigma}_{\delta)\alpha}+2\Gamma^{\rho}_{\gamma\delta}\Gamma^{\sigma}_{\alpha\beta}\right)+2\hat{P}^{\rho(\mu}\Gamma^{\nu)}_{\sigma(\gamma}\Gamma^{\sigma}_{\delta)\rho}+\hat{P}^{\rho\sigma}\Gamma^{\mu}_{\rho(\gamma}\Gamma^{\nu}_{\delta)\sigma}\,.\\
&& (5.52)
\end{aligned}
$$

With the help of the transformation formulae, we can then cast the first invariance equation (5.41) into the form

$$\frac{\partial C_{B_1\dots B_N}}{\partial S_{\alpha\beta\gamma\delta}}=0\,, \tag{5.53}$$

and the second invariance equation (5.42) can be rewritten in the new coordinates as

$$\frac{\partial C_{B_1\dots B_N}}{\partial \Gamma^{\alpha}_{\beta\gamma}}=0\,. \tag{5.54}$$

In other words, the coefficients $C_{B_1\dots B_N}$ cannot explicitly depend on the new non-tensorial variables $\Gamma^{\alpha}_{\beta\gamma}$ and $S_{\alpha\beta\gamma\delta}$, but we have that $C_{B_1\dots B_N}=C_{B_1\dots B_N}(\hat{P}^{\alpha\beta},R_{\alpha\beta\gamma\delta})$. This is of course what one would expect according to the well-known theorem that the Riemann tensor is the only tensor that can be formed from a metric, and its first and second derivatives.

This procedure of changing the coordinates can be generalized to all cases where, in addition to a metric, one has an arbitrary set of other hypersurface tensor fields $\hat{G}^A$. The first and second *partial* derivatives of the additional fields $\hat{G}^A$ can then be replaced by the first and the symmetrized second *covariant* derivatives of $\hat{G}^A$ using the torsion-free and metric compatible Levi-Civita connection of the metric at hand. The antisymmetric part of the second covariant derivatives of the fields $\hat{G}^A$ does not have to be considered, because it can always be expressed by the Riemann tensor and the undifferentiated fields $\hat{G}^A$. After rewriting the respective invariance equations, one again ends up with equations (5.53) and (5.54). In particular, this can be done for all hypersurface point particle geometries of arbitrary degree by formally employing the particular field $\hat{P}^{\alpha\beta}$ as a metric, and treating all other tensor fields $\hat{P}^{\alpha_1\dots\alpha_I}$, for $I=3,\dots,\deg P$, as additional fields. It can also be done for area metric geometry by employing the tensor field $\hat{G}^{\alpha\beta}$ as a metric, with respect to which one defines the Levi-Civita connection and the Riemann tensor. It might, however, not always be possible to perform such a change of coordinates for the induced hypersurface geometry $\hat{G}^A$ of other generic spacetime geometries G. Although we are always guaranteed by the bi-hyperbolicity and the energy-distinguishing property that the tensor field $\hat{P}^{\alpha\beta}$, which is distinguished by the matter field equations one employs, can be formally used as a metric tensor on the hypersurface $X_0(\Sigma)$, it might not be possible to find an invertible coordinate transformation from the actual fields $\hat{G}^A$ to a new set of coordinates, which contains $\hat{P}^{\alpha\beta}$. Nevertheless, if such a coordinate transformation exists, one can proceed to rewrite the master equations, and solve them in the new coordinates. We present two particular examples of this procedure in sections 6.2 and 6.4.

Combining the results of this section with the result of the previous section, where we have learned that the cofficients $C_{B_1\ldots B_N}$ (for $N \geq 1$) can at most depend cubically on the second derivatives of the fields $\hat{G}^A$, one can straightforwardly rewrite the expansion (5.40) in covariant form. This has the advantage that the coefficients in front of the resulting different powers of the Riemann tensor and the symmetrized second covariant derivatives of the remaining fields $\hat{G}^A$ must then be weight-one tensor densities, which thus can only depend on the employed hypersurface metric, the remaining fields $\hat{G}^A$ and their first covariant derivatives.

5.3 Master equations containing the potential C

In this section we will discuss the remaining master equations that contain the potential C. We collapse the respective equations, observing that the potential C can at most depend on the third partial derivatives of the fields $\hat{G}^A$. We also discuss the solution of the invariance equations for the potential C, which, again, reflect the fact that the latter is a scalar density of weight one.

5.3.1 Collapse to third derivative order

Although it is more difficult to restrict the dependence of the potential C on the derivatives of the fields $\hat{G}^A$, we can still deduce that the potential can at most depend on the third derivatives of $\hat{G}^A$, so that $C = C(\hat{G}^A, \partial\hat{G}^A, \partial^2 G^A, \partial^3\hat{G}^A)$, and, moreover, it can only be linear in $\partial^3\hat{G}^A$. In order to arrive at this result, we impose the very weak technical assumption that the potential C only depends on the partial derivatives of the fields $\hat{G}^A$ to some *finite* derivative order. For then we can conclude recursively from equation (5.20), for $N = 1$, that the potential C can at most depend on the third partial derivatives of the fields $\hat{G}^A$. This is because we already learned that the coefficient C_{B_1} can at most depend on the second partial derivatives $\partial^2\hat{G}^A$. The only remaining piece of information from equation (5.20) is given by the case $s = 3$ and $N = 1$, which yields

$$0 = \frac{\partial C_{B_1}}{\partial\, \partial^2_{(\beta_1\beta_2|}\hat{G}^A}\, M^{A\,|\beta_3)} + \frac{\partial C}{\partial\, \partial^3_{\beta_1\beta_2\beta_3}\hat{G}^{B_1}}\,. \tag{5.55}$$

Taking the derivative of this equation with respect to $\partial^3\hat{G}^B$ directly implies that the potential can at most be linear in $\partial^3\hat{G}^B$. We now reduce the remaining master equations using this result. Equations (5.19) and (5.18) imply, for $N=1$,

$$\begin{aligned} 0 &= 2\,C_{AB_1}U^{A\,(\alpha\beta)} - \frac{\partial C_{B_1}}{\partial\partial_{(\beta|}\hat{G}^A}M^{A\,|\alpha)} - 2\,\frac{\partial C_{B_1}}{\partial\partial^2_{(\beta|\gamma}\hat{G}^A}\partial_\gamma M^{A\,|\alpha)} \\ &\quad + \frac{\partial C}{\partial\partial^2_{\alpha\beta}\hat{G}^{B_N}} - 3\,\partial_\gamma\frac{\partial C}{\partial\,\partial^3_{\alpha\beta\gamma}\hat{G}^{B_1}} \quad \text{and} \end{aligned} \tag{5.56}$$

$$\begin{aligned} 0 &= 2\,C_{AB_1}(S^{A\,\alpha} + 2\,\partial_\mu T^{A\,[\mu\alpha]}) + 2\,\partial_\mu C_{AB_1}T^{A\,[\mu\alpha]} - Q_{B_1}{}^{M\,\alpha}\,C_M \\ &\quad + \frac{C_{B_1}}{\partial\hat{G}^A}M^{A\,\alpha} + \frac{\partial C_{B_1}}{\partial\partial_\gamma\hat{G}^A}\partial_\gamma M^{A\,\alpha} + \frac{\partial C_{B_1}}{\partial\partial^2_{\gamma\delta}\hat{G}^A}\partial^2_{\gamma\delta}M^{A\,\alpha} \\ &\quad + \frac{\partial C}{\partial\partial_\alpha\hat{G}^{B_1}} - 2\,\partial_\gamma\frac{\partial C}{\partial\partial^2_{\alpha\gamma}\hat{G}^{B_1}} + 3\,\partial^2_{\beta\gamma}\frac{\partial C}{\partial\,\partial^3_{\alpha\beta\gamma}\hat{G}^{B_1}}\,. \end{aligned} \tag{5.57}$$

Combining equations (5.14) for $w=3$ with equations (5.11) and (5.12), one learns that equation (5.11) is the divergence of (5.12), and can thus be dropped. Equation (5.12) yields

$$\begin{aligned} 0 &= 2\partial_\mu(C_A U^{A\,(\beta\mu)}) + 2C_A S^{A\,\beta} + 2\partial_\nu C_A T^{A\,[\nu\beta]} \\ &+ 2\frac{\partial C}{\partial\hat{G}^A}\,M^{A\,\beta} + 2\frac{\partial C}{\partial\partial_\mu\hat{G}^A}\,\partial_\mu M^{A\,\beta} + 2\frac{\partial C}{\partial\partial^2_{\mu\nu}\hat{G}^A}\,\partial^2_{\mu\nu}M^{A\,\beta} + 2\frac{\partial C}{\partial\partial^3_{\mu\nu\rho}\hat{G}^A}\,\partial^2_{\mu\nu\rho}M^{A\,\beta} \\ &- \partial_\mu\left(2\frac{\partial C}{\partial\partial_{(\mu|}\hat{G}^A}M^{A\,|\beta)} + 4\frac{\partial C}{\partial\partial^2_{(\mu|\nu}\hat{G}^A}\partial_\nu M^{A\,|\beta)} + 6\frac{\partial C}{\partial\partial^3_{(\mu|\nu\rho}\hat{G}^A}\partial^2_{\nu\rho}M^{A\,|\beta)}\right) \\ &+ \partial^2_{\mu\nu}\left(3\frac{\partial C}{\partial\,\partial^2_{(\mu\nu|}\hat{G}^A}M^{A\,|\beta)} + 9\frac{\partial C}{\partial\,\partial^2_{(\mu\nu|\rho}\hat{G}^A}\partial_\rho M^{A\,|\beta)}\right) \\ &- 4\partial^3_{\mu\nu\rho}\left(\frac{\partial C}{\partial\,\partial^3_{(\mu\nu\rho|}\hat{G}^A}M^{A\,|\beta)}\right)\,. \end{aligned} \tag{5.58}$$

The only other non-trivial master equations are then given by equation (5.13), which, for $w=2$, takes the form

$$0 = \partial_\alpha\left(\frac{\partial C}{\partial\,\partial^2_{(\beta_1|\alpha}\hat{G}^A}\,M^{A\,|\beta_2)} + 4\,\frac{\partial C}{\partial\,\partial^3_{(\beta_1|\alpha\gamma}\hat{G}^A}\,\partial_\gamma M^{A\,|\beta_2)} - 2\partial_\delta\left\{\frac{\partial C}{\partial\,\partial^3_{\alpha\delta(\beta_1}\hat{G}^A}\,M^{A\,|\beta_2)}\right\}\right)\,, \tag{5.59}$$

and equation (5.14), which, for $w=3$, yields

$$0 = 2\,\frac{\partial C}{\partial\,\partial^2_{(\beta_1\beta_2|}\hat{G}^A}\,M^{A\,|\beta_3)} + 6\,\frac{\partial C}{\partial\,\partial^3_{(\beta_1\beta_2|\gamma}\hat{G}^A}\,\partial_\gamma M^{A\,|\beta_3)} - 4\,\partial_\gamma\left(\frac{\partial C}{\partial\,\partial^3_{(\beta_1\beta_2|\gamma}\hat{G}^A}\,M^{A\,|\beta_3)}\right)\,. \tag{5.60}$$

Equations (5.55)-(5.60) present the gravitational master equations that contain the potential C. They have to be complemented by the respective invariance equations for the potential C. These invariance equations are calculated in exactly the same fashion as for the coefficients $C_{B_1\dots B_N}$, the only difference being that the potential C can also depend on the third partial derivatives of the fields $\hat{G}^A$. The explicit form of the invariance equations, again, depends on the type of tensor fields $\hat{G}^A$ one considers. From the transformation law

$$\bar{C}(\bar{G},\partial\bar{G},\partial^2\bar{G},\partial^3\bar{G}) = \det(A)\, C(\hat{G},\partial\hat{G},\partial^2\hat{G},\partial^3\hat{G})\,, \tag{5.61}$$

which describes the change of the weight-one scalar density C under a change of coordinates $\bar{x}^\alpha = \bar{x}^\alpha(x)$ on the hypersurface $X_0(\Sigma)$, with $A^\alpha_\beta = \partial\bar{x}^\alpha/\partial x^\beta$, one obtains, for a hypersurface geometry $\hat{G}^A = \hat{G}^\alpha{}_\beta$ for example, the first invariance equation

$$0 = \hat{G}^{(\alpha}{}_\mu \frac{\partial C}{\partial\partial^3_{\beta\gamma\delta)}\hat{G}^\rho{}_\mu} - \hat{G}^\mu{}_\rho \frac{\partial C}{\partial\partial^3_{(\alpha\beta\gamma}\hat{G}^\mu{}_{\delta)}} \tag{5.62}$$

by differentiation with respect to $A^\rho_{\alpha,\beta\gamma\delta} = \partial^4\bar{x}^\rho/\partial x^\alpha\partial x^\beta\partial x^\gamma\partial x^\delta$. The generalization to other hypersurface geometries is straightforward.

We will not display the other invariance equations here, but directly describe how they can be solved in the next section.

5.3.2 Solutions to the invariance equations

The invariance equations for the potential C can be dealt with in the same fashion as with those for the coefficients $C_{B_1\dots B_N}$. In particular, we can, again, solve them explicitly if one of the hypersurface tensor fields $\hat{G}^A$ can be formally employed as a hypersurface metric as it was shown in [36, 71]. We will not repeat the calculations here, but only describe the results. More precisely, assume that the hypersurface geometry is solely given by an inverse metric $\hat{P}^{\alpha\beta}$, so that the potential C formally depends on the coordinates $(\hat{P}^{\alpha\beta},\partial_\mu\hat{P}^{\alpha\beta},\partial^2_{\mu\nu}\hat{P}^{\alpha\beta},\partial^3_{\mu\nu\rho}\hat{P}^{\alpha\beta})$. Then one can choose new coordinates

$$\hat{P}^{\alpha\beta},\, \Gamma^\rho_{\alpha\beta},\, \Gamma^\rho_{(\alpha\beta\,,\gamma)},\, R_{\alpha\beta\gamma\delta},\, \Gamma^\rho_{(\alpha\beta\,,\gamma\delta)},\, R_{\alpha\beta\gamma\delta\,;\rho}\,, \tag{5.63}$$

where $\Gamma^\alpha_{\beta\gamma}$ denotes the Levi-Civita connection of $\hat{P}^{\alpha\beta}$, $R_{\alpha\beta\gamma\delta}$ the Riemann tensor of $\hat{P}^{\alpha\beta}$, a comma denotes partial derivatives, and a semi-colon denotes the covariant derivative with respect to the connection $\Gamma^\alpha_{\beta\gamma}$. This change of coordinates is indeed invertible, and it can then be shown that, in analogy to the coefficients $C_{B_1\dots B_N}$, the potential C can only depend on the tensor components $\hat{P}^{\alpha\beta}$, $R_{\alpha\beta\gamma\delta}$ and $R_{\alpha\beta\gamma\delta\,;\rho}$:

$$C = C(\hat{P}^{\alpha\beta}, R_{\alpha\beta\gamma\delta}, R_{\alpha\beta\gamma\delta\,;\rho}) \,. \tag{5.64}$$

This result can, again, be extended to all cases where, apart from a metric $\hat{P}^{\alpha\beta}$, the hypersurface geometry $(\hat{P}^{\alpha\beta}, \hat{G}^B)$ contains an arbitrary set of additional tensor fields $\hat{G}^B$. One only has to extend the coordinate transformation (5.63) to the first covariant derivatives and the completely symmetrized second and third covariant derivatives of the additional fields. It is indeed sufficient to only consider symmetrized second covariant derivatives of any additional hypersurface field $\hat{G}^B$ because the antisymmetric part $\hat{G}^B{}_{;[\mu\nu]}$ can always be expressed in terms of the Riemann tensor and the undifferentiated field $\hat{G}^B$. The same holds for the symmetrized third covariant derivatives $\hat{G}^B{}_{;(\mu\nu\rho)}$ because any other component of $\hat{G}^B{}_{;\mu\nu\rho}$ can be expressed in terms of $\hat{G}^B$, $\hat{G}^B{}_{;\mu}$, $R_{\alpha\beta\gamma\delta}$ and $R_{\alpha\beta\gamma\delta\,;\rho}$. It follows that

$$C = C(\hat{P}^{\alpha\beta}, R_{\alpha\beta\gamma\delta}, R_{\alpha\beta\gamma\delta\,;\rho}, \hat{G}^B, \hat{G}^B{}_{;\mu}, \hat{G}^B{}_{;(\mu\nu)}, \hat{G}^B{}_{;(\mu\nu\rho)}) \,. \tag{5.65}$$

Moreover, we already learned from equation (5.55) that the potential can at most be linear in the third partial derivatives of the fields $(\hat{P}^{\alpha\beta}, \hat{G}^B)$, and this implies that the potential C can be decomposed as

$$C = {}^{(2)}C + {}^{(3)}C^{\alpha\beta\gamma\delta\rho} R_{\alpha\beta\gamma\delta\,;\rho} + C_B^{\alpha\beta\gamma} \hat{G}^B{}_{;(\alpha\beta\gamma)} \tag{5.66}$$

where the coefficients ${}^{(2)}C$, ${}^{(3)}C^{\alpha\beta\gamma\delta\rho}$ and ${}^{(3)}C_B^{\alpha\beta\gamma}$, which must be tensor densities of weight one, can at most depend on the second derivatives of the fields $(\hat{P}^{\alpha\beta}, \hat{G}^B)$. Thus, following the same reasoning as in section 5.2.3, the latter can only depend on $\hat{P}^{\alpha\beta}$, $R_{\alpha\beta\gamma\delta}$, $\hat{G}^B$, $\hat{G}^B{}_{;\mu}$ and $\hat{G}^B{}_{;(\mu\nu)}$.

If none of the hypersurface tensor fields $\hat{G}^A$ can be employed as a hypersurface metric, one would have to find another way to extract information from the invariance equations. Unfortunately, to the best knowledge of the author, there is, so far, no general scheme to do so.

5.4 The role of the non-local part of the superhamiltonian in the master equations

Although it is straightforward to calculate the non-local part of the superhamiltonian for any spacetime geometry, a failure of the latter to vanish seriously complicates the solution of the master equations, which determine the Lagrangian L. In principle, there are two complementary strategies to deal with this issue. It will be instructive to briefly comment on both strategies before we see both of them at work in the concrete example of a spacetime geometry that supports modified Dirac matter in section 6.4.

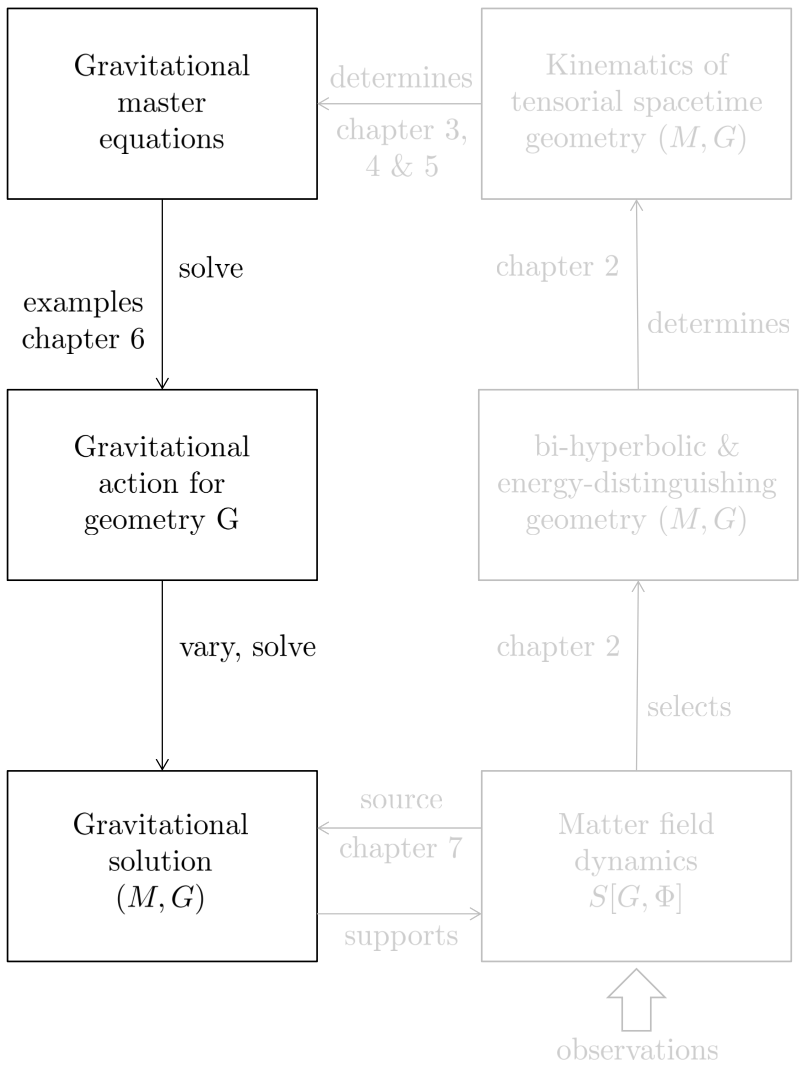
The non-local superhamiltonian enters the master equations explicitly in form of the coupling terms that contain the coefficients M and Q and implicitly in form of contributions to the coefficients S and T. Now, the first strategy to simplify the master equations in the presence of a non-vanishing non-local part of the superhamiltonian is based on the observation that its direct contributions to the master equations through the coefficients M and Q is, at least structurally, very similar to the invariance equations. Take for example equation (5.27),

$$0 = \frac{\partial C_{B_1 \dots B_N}}{\partial \partial^2_{(\alpha\beta|} \hat{G}^A} M^{A|\gamma)} \qquad \text{for all } N \geq 2\,, \tag{5.67}$$

and compare it with the example (5.33) of the first invariance equation. Similarly, one recognizes striking similarities between the of second invariance equation (5.35) and the non-local coupling terms involving the coefficient M in equation (5.25). Now, as we have discussed in sections 5.2.3 and 5.3.2, one may solve the invariance equations by making a clever choice of coordinates parametrizing the fields $\hat{G}^A$ and their first and second partial derivatives, in order to cast the invariance equations into a particular normal form. The same procedure can also be productive in order to simplify the non-local contributions to the master equations. Indeed, we will see in 6.4 how such a clever choice of coordinates on the configuration space of the spatial geometry $\hat{G}^A$ can significantly simplify the master equations.

There are two types of circumstances in which it is easy to find such well-behaved coordinates. First, if possible, one should choose a parametrization of the spatial geometry that contains as many hypersurface scalar fields as possible, which are induced from spacetime scalar fields. For such hypersurface fields, we already know that they do not contribute to the non-local part of the superhamiltonian. Secondly, if the degree of the principal polynomial of the matter field equations (2.1) is two, one should try to use, as one of the hypersurface fields, the hypersurface metric $\hat{P}^{\alpha\beta}$ because, again, we know that there will be no contribution to the non-local part of the superhamiltonian for this field and hence no contribution from the coefficients M and Q in the master equations. However, if the spatial geometry is not at least in part parametrizable by such fields, one has to find a different way to simplify the master equations.

The second strategy to simplify the master equations is applicable if among the geometric variables $\hat{G}^A$ there is one that can formally serve as a hypersurface metric. In order to keep the following argument transparent, we here discuss only the master equations that do not contain the potential C. One can then solve the invariance equations by choosing the appropriate coordinates, namely by substituting the first and second partial derivatives of the hypersurface metric by the corresponding Levi-Civita connection Γ, the Riemann tensor R and the non-tensorial variables S given by (5.45). The derivatives of all remaing fields may then be replaced by symmetrized covariant derivatives with respect to the Levi-Civita connection. After rewriting the master equations in terms of these new coordinates one has to check whether the variables $\Gamma^\alpha_{\beta\gamma}$ and $S_{\alpha\beta\gamma\delta}$ still appear. Because of equations (5.53) and (5.54) one then knows that the terms proportional to $\Gamma^\alpha_{\beta\gamma}$ and $S_{\alpha\beta\gamma\delta}$ must vanish individually. This way one can split the differential equations into several new equations that might be easier to solve. We will see in section 6.4 how this method works in a concrete example, where this will lead to a simple *algebraic* equation for parts of the coefficients $C_{B_1\ldots B_N}$ similar to equation (5.28), and whose explicit form is crucially influenced by the non-local part of the superhamiltonian. Unfortunately, we cannot make this procedure more explicit in the general case, since the form of the resulting equations highly depends on the field content chosen—we therefore refer the reader to section 6.4, where the here described strategies are implemented in vivo.


Gravitational master equations
determines
chapter 3, 4 & 5
Kinematics of tensorial spacetime geometry (M, G)
solve
examples chapter 6
chapter 2
determines
Gravitational action for geometry G
bi-hyperbolic & energy-distinguishing geometry (M, G)
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Gravitational solution (M, G)
Matter field dynamics $S[G, \Phi]$
supports
observations

Chapter 6

The road from matter actions to supporting gravitational dynamics: Summary of the technique and worked examples

In this chapter, we summarize the general recipe of how to obtain the gravitational master equations for any tensorial spacetime $(M, G, S[G, \Phi])$ whose solution is the Lagrangian of the gravity theory that is compatible with the matter dynamics $S[G, \Phi]$. Then we discuss four different examples of fundamental tensorial spacetime geometries and their gravitational master equations that illustrate the broad scope of applicability of our theory. In particular, we study (i) metric geometry probed by Maxwell electrodynamics, (ii) area metric geometry probed by general linear electrodynamics, (iii) a composite geometry (g, W) consisting of a metric g and a vector field W probed by modified Dirac matter, and finally (iv) the entire class of spacetime geometries that can carry point particles. We explicitly solve the master equations in the first and third example, recovering Einstein-Hilbert dynamics as the unique gravitational dynamics supporting Maxwell theory in four dimensions, and, as a first non-trivial extension, all gravitational dynamics supporting particularly modified Dirac dynamics in four dimensions.

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6.1 General recipe

Before we discuss the gravitational master equations and their solution in concrete examples, we first summarize the general procedure. As we have seen in the previous two chapters, the *structure* of the master equations is the same for all tensorial spacetimes $(M, G, S[G, \Phi])$. The only difference between different tensorial spacetimes consists in the set of hypersurface fields $\hat{G}^A$ parametrizing the spacetime geometry G on the one hand, and the explicit form of all the coefficients appearing in the master equations on the other hand. Hence, the only work to be done in order to obtain the gravitational master equations for any given tensorial spacetime $(M, G, S[G, \Phi])$ mainly consists of the calculation of these coefficients. Finding the gravitational dynamics for any tensorial spacetime thus proceeds simply as follows:

Step 1: Kinematics. Starting from the matter dynamics $S[G, \Phi]$, one derives the principal tensor field P associated with the matter field equations, as in (2.2). This is crucial in order to determine the algebraic restrictions one must place on the geometry G such that the latter is bi-hyperbolic and energy-distinguishing (i.e., physically speaking, such that the geometry allows for the matter dynamics $S[G, \Phi]$ to be predictive and quantizable). Point particle kinematics are already completely determined at this stage: for a massless point particle, the dynamics is encoded in the action (2.13) and for massive particles in the action (2.19).

Step 2: Setting up the geometrical phase space. In this second step, one has to choose a parametrization of the spacetime geometry G in terms of fields $\hat{G}^A$ on an observer-accessible initial data hypersurface $X(\Sigma)$. To this end, the parametrisation must satisfy the normalisation conditions $P(n) = 1$ as well as $L^a(n)\epsilon^\alpha_a = 0$, given in terms of principal tensor field P, the Legendre map L and the hypersurface frame $\{n, \epsilon^\alpha\}$. One may then pair each tensor field $\hat{G}^A$ with a conjugate momentum $\hat{\pi}_A$, which completes the geometric phase space $(\hat{G}^A, \hat{\pi}_A)$ of the spacetime geometry G on the hypersurface $X(\Sigma)$. For later use, one also calculates the particular hypersurface projection $P^{\alpha\beta}$ (defined by (3.6) with $I = 2$) of the principal tensor field P in terms of the chosen parametrization. Finally, the parametrization process yields the completeness relation that allows the reconstruction of the spacetime geometry G from the fields $\hat{G}^A$.

Step 3: Coefficients from the supermomentum. The supermomentum $\hat{\mathcal{D}}(\vec{N})$ is always given by (4.16). Setting $N = \delta_x$ and $M = \delta_y$ on the right hand side of the algebra equation (4.11), which only involves the supermomentum $\hat{\mathcal{D}}$ and the hypersurface tensor field $P^{\alpha\beta}$, yields the general expression

$$(-U^{A\,\mu\nu}\,\partial_\mu\hat{\pi}_A + V^{A\,\nu}\hat{\pi}_A)(y)\,\partial_\nu\delta_x(y) - (x \leftrightarrow y)\,, \tag{6.1}$$

from which one reads off the explicit form of the coefficients $U^{A\,\mu\nu}$ and $V^{A\,\gamma}$.

Step 4: Coefficients from the non-local superhamiltonian. Now one determines the non-local part of the superhamiltonian, which always has the form (4.20), where the coefficients $M^{A\,\gamma}$ have to be read off the equation

$$\mathcal{H}(N)G^A(z) =: N(z)K^A(z) + \partial_\gamma N M^{A\,\gamma}\,, \tag{6.2}$$

where the left hand side describes the application of the normal deformation operator $\mathcal{H}(N)$ on the fields G^A that parametrize the spacetime geometry G.

Step 5: Calculation of the remaining coefficients. The remaining coefficients, which are necessary to set up the gravitational master equations, can then be calculated as follows. The coefficients ${Q_A}^{B\,\gamma}$ arise from

$${Q_A}^{B\,\gamma} := -\frac{\partial M^{B\,\gamma}}{\partial \hat{G}^A}\,, \tag{6.3}$$

the coefficients $T^{A\,\mu\nu}$ are given by

$$T^{A\,\mu\nu} := -\,{Q_B}^{A\,\mu}M^{B\,\nu} + U^{A\,\mu\nu}\,, \tag{6.4}$$

and the coefficients $S^{A\,\gamma}$ are determined by

$$S^{A\,\gamma} := -\partial_\beta({Q_B}^{A[\beta|}M^{B\,|\gamma]}) - \partial_\beta U^{A\,(\beta\gamma)} - V^{A\,\gamma}\,. \tag{6.5}$$

Step 6: The master equations. The coefficients calculated in step 3, 4 and 5 completely determine the gravitational master equations, which, once solved, yield the expansion coefficients $C_{B_1\dots B_N}$ (for $N \geq 0$) of the Lagrangian (4.22) that describes the gravitational dynamics of the hypersurface tensor fields $\hat{G}^A$. The set of master equations decompose into two parts: those equations involving the potential C are given in section 5.3.1, while those *not* containing the potential C are given in section 5.2.1. In addition to the master equations, the coefficients $C_{B_1\dots B_N}$ must satisfy a set of invariance equations, which we discussed in section 5.3.2 for the potential C, and, in section 5.2.3, for all other coefficients $C_{B_1\dots B_N}$ with $N \geq 1$.

Step 7: Solving the master equations. Finally, one has to solve the master equations including the invariance equations in order to find the gravitational dynamics for the tensorial spacetime geometry G. Once one has found a solution of the master equations, the field equations for the hypersurface geometry $\hat{G}^A$ are given by (4.26) and (4.30) and these again must be solved in order to obtain (with the help of the completeness relation obtained from the parametrization in step two) a concrete spacetime geometry G.

We will now illustrate the above recipe by applying it to four instructive examples. In the first example, we consider metric manifolds (M, g) carrying Maxwell electrodynamics. Solution of the master equations yields Einstein-Hilbert dynamics with undetermined gravitational and cosmological constant, or rather its ADM form, for a Lorentzian metric as the unique dynamics that supports predictable and quantizable Maxwell theory. This particular result was first obtained, over 40 years ago, by Hojman, Kuchař and Teitelboim in [30, 37]. The second example is area metric geometry (M, G) carrying general linear electrodynamics [29]. We will set up, but not solve, the master equations for this theory. As a third example, we then consider a composite geometry (g, W) consisting of a metric g and a vector field W carrying a modified, Lorentz-symmetry breaking Dirac equation. We will set up and solve the master equations for this theory in four dimensions and find a whole family of possible gravitational dynamics for this spacetime geometry. As a final example, we set up the master equations for an arbitrary principal tensor field P carrying massless and massive point particles. The master equations straightforwardly provide a solution for the corresponding gravitational dynamics in the case where the rank of the principal tensor field is two, which in four dimensions is again formally equivalent to Einstein-Hilbert dynamics. For higher rank principal tensor fields, an explicit calculation of solutions of the master equations, and indeed a study of their existence and uniqueness, remains an open mathematical problem due to their complexity.

6.2 Example 1: From Maxwell theory to Einstein-Hilbert gravity theory

In this example, we consider a four-dimensional metric manifold (M, g) of a priori arbitrary signature, and as matter dynamics on it, we take Maxwell theory described by the familiar action

$$S_{\text{Maxwell}}[g, A] = -\frac{1}{4}\int_M d^4x \sqrt{|g|}\, g^{ab} g^{cd} F_{ac} F_{bd} \tag{6.6}$$

of a covector gauge field A coupled to the metric g through the field strength $F = dA$. Following the philosophy of this thesis, we now execute all steps of the aforementioned program starting with the determination of the appropriate kinematics that must underlie this matter action (as if we had never heard of Einstein's 1905 paper [18]) and ending with the determination of the dynamics for the metric g (as if we had never heard of Einstein's 1915 paper [19]).

6.2.1 Kinematics of Maxwell theory on metric manifolds

In the first step of our analysis, one has to determine, which restrictions are placed on the underlying metric geometry by requiring the non-negotiable predictivity and quantizability of Maxwell theory. To this end, one first derives the principal tensor field P of the field equations

$$\frac{1}{\sqrt{|g|}}\,\partial_a(\sqrt{|g|}\,g^{ac}g^{bd}F_{ab}) = 0 \quad \text{and} \quad \partial_{[a}F_{bc]} = 0\,, \tag{6.7}$$

the first of which arises from a variation of the action (6.6), while the second simply is the Bianchi identity enforcing $F = dA$ on any contractible domain. For the purpose of determining the principal tensor field P, it is, in fact, easier to work with the gauge invariant field F_{ab} instead of the gauge field A. For from (6.7), the principal tensor field can be calculated by observing that the field equations imply the wave equation

$$g^{ab}\nabla_a\nabla_b F_{cd} = 0\,, \tag{6.8}$$

in terms of the covariant derivative ∇ of the Levi-Civita connection of the metric g, for every of the six independent components of the field strength F. The other two of the eight initial field equations (6.7) turn out to be constraint equations[1] restricting the initial values of the fields F_{ab}. Reading off the coefficient of the highest order derivative term of (6.8), $g^{ab}\partial_a\partial_b F_{cd}$, one then finds that the principal tensor field of Maxwell theory on a metric manifold is given by[2]

$$P^{ab} = g^{ab}\,. \tag{6.9}$$

The necessary condition for the predictivity of the field equations (6.8), namely that the principal polynomial $P(x,k) = g^{ab}k_ak_b$ is hyperbolic, can now be shown to be equivalent to the condition that the inverse metric g^{ab} has Lorentzian signature (see, for example, [60]), which excludes four of the five possible signatures of the metric. All covectors k satisfying the massless dispersion relation

$$g^{ab}k_ak_b = 0 \tag{6.10}$$

[1] This can be seen by introducing coordinates (t, x^α) on M such that the surface $t = 0$ denotes an initial data surface for the fields F_{ab}. Indeed, the t-components of the field equations (6.7) then do not contain derivatives with respect to the time coordinate t, and, thus, only constrain the inital data on the initial data surface $t = 0$.

[2] One can also derive this result directly from the first order system (6.7) by fixing a particular gauge. For more details on the calculation of the principal tensor field of Maxwell theory, we refer the reader to [51, 60]

then constitute the familiar Lorentzian lightcone in each cotangent space T_x^*M of the manifold M. Moreover, our general convention to choose the overall sign of the principal tensor field such that $P(x,k)$ is positive on the convex cone of positive energy massive momenta here simply amounts to the choice of the mainly-minus signature convention $(+\,-\,-\,-)$ for the inverse metric. The set of hyperbolic covectors are thus what are usually the timelike covectors, defined by $g^{ab}k_ak_b > 0$. The dual polynomial $P^{\#}(x,v)$ is given by $P^{\#}(x,v) = g_{ab}v^av^b$, as one may check directly with the help of the definition (2.8), and, hence, the massless point particle action (2.13) takes the familiar form

$$S_{\text{massless}}[x,\mu] = \int d\tau\, \mu\, g_{ab}\dot{x}^a\dot{x}^b\,. \tag{6.11}$$

Hyperbolicity of the dual polynomial $P^{\#}(x,v)$ is, in this case, already guaranteed by the hyperbolicity of the principal polynomial, simply because the metric g_{ab} is Lorentzian if and only if its inverse g^{ab} is Lorentzian. Choosing a time orientation in terms of an everywhere timelike vector field H, it can further be shown that Lorentzian geometry is also automatically energy-distinguishing, and, thus, can indeed support predictive and quantizable Maxwell matter. The massive dispersion relation $g^{ab}p_ap_b = m^2$ defines the standard quadratic mass shell on the set of future-oriented hyperbolic covectors with $p(H) > 0$. The Legendre map (2.17) becomes in the present case simply

$$L^a(p) = \frac{g^{ab}p_a}{g^{mn}p_mp_n}\,, \tag{6.12}$$

and its inverse can easily be guessed, $L_a^{-1}(v) = g_{ab}v^b/(g_{mn}v^mv^n)$, which finally turns the action (2.19), describing the worldline of massive point particles, into the standard form

$$S_{\text{massive}}[x] = m\int d\tau\,\sqrt{g_{ab}\dot{x}^a\dot{x}^b}\,. \tag{6.13}$$

According to our general insights, future-oriented observers are in the present case simply identified by their timelike wordline tangents $\dot{o}$ with $g(\dot{o},H) > 0$ and their laboratory frames are constructed in standard fashion. Consequently, observer accessible initial data hypersurfaces are all what are usually called spacelike hypersurfaces. Thus the general abstract construction of the kinematics underlying a predictive and quantizable matter action condenses to the familiar standard constructions for the tensorial spacetime $(M, g, S_{\text{Maxwell}}[g,A])$. We are thus prepared to set up the gravitational master equations.

6.2.2 Setting up the master equations

We already discussed in section 3.2.2 that the hypersurface geometry $\hat{G}^A$—parametrizing the spacetime geometry g of the tensorial spacetime $(M, g, S_{\text{Maxwell}}[g, A])$ on an observer-accessible initial data hypersurface $X(\Sigma)$—is given in terms of a negative definite inverse hypersurface metric $g^{\alpha\beta}$. The inverse spacetime metric can then be reconstructed using the completeness relation $g^{ab} = T^a T^b + g^{\alpha\beta} e^a_\alpha e^b_\beta$ with the help of the hypersurface frame $\{T, e_\alpha\}$. The phase space

$$(\hat{g}^{\alpha\beta}, \hat{\pi}_{\alpha\beta}) \tag{6.14}$$

of the gravitational dynamics for the metric g is obtained by adjoining the weight-one tensor density $\hat{\pi}_{\alpha\beta}$ to the inverse metric $\hat{g}^{\alpha\beta}$. The capital multi-index thus only contains a symmetric index pair, ${}^A = {}^{\alpha\beta}$. Since $P^{\alpha\beta} = g^{\alpha\beta}$ according to (6.9), it is also the inverse spatial metric that appears on the right hand side of the algebra equation (4.11) between two superhamiltonians.

The supermomentum (4.16) can be cast into the form

$$\hat{\mathcal{D}}(\vec{N}) = \int dy\, 2\, N^\alpha \hat{g}^{\beta\gamma} \nabla_\beta \hat{\pi}_{\alpha\gamma}\,, \tag{6.15}$$

with ∇_α denoting the covariant derivative with respect to the Levi-Civita connection Γ induced by the hypersurface metric. In most of the standard textbooks, the components of the actual spatial metric $\hat{g}_{\alpha\beta}$ are taken as the configuration variables and their conjugate momenta are thus given by the contravariant tensor densities $\hat{\pi}^{\alpha\beta}$. This choice can of course be reached from our convention by a phase space transformation in which the momenta change according to $\hat{\pi}^{\alpha\beta} = -\hat{g}^{\alpha\gamma}\hat{g}^{\beta\delta}\hat{\pi}_{\gamma\delta}$. In these variables, the supermomentum becomes $\hat{\mathcal{D}}_\alpha = -2\hat{g}_{\alpha\beta}\nabla_\gamma \hat{\pi}^{\gamma\beta}$.

The non-local part of the superhamiltonian vanishes, as we saw in section 3.3.3. This immediately implies that $M^{A\,\gamma} = 0$ as well as $Q_A{}^{B\,\beta} = 0$. From the right hand side of the algebra equation (4.11) in the form (6.1), we obtain the coefficients

$$U^{\alpha\beta\,\mu\nu} = -2\hat{g}^{\alpha(\mu}\hat{g}^{\nu)\beta}, \tag{6.16}$$

which are automatically symmetric in both index pairs $\alpha\beta$ and $\mu\nu$. This enforces $T^{A\,[\mu\nu]} = 0$ and, finally, we can read off the coefficient S from the general expression (6.5):

$$S^{\alpha\beta\,\gamma} = -\hat{g}^{\gamma\delta}\partial_\delta \hat{g}^{\alpha\beta} + 2\hat{g}^{\delta(\alpha}\partial_\delta \hat{g}^{\beta)\gamma}\,. \tag{6.17}$$

This completes the calculation of the coefficients for the master equations for the tensorial spacetime $(M, g, S_{\text{Maxwell}}[g, A])$.

6.2.3 Solution of the master equations

We now solve the master equations for the tensorial spacetime $(M, g, S_{\text{Maxwell}}[g, A])$. In four spacetime dimensions, this directly yields general relativity as the unique solution with Newton's constant and the cosmological constant arising as undetermined integration constants. This result was first obtained by Hojman, Kuchař and Teitelboim in [30, 37], who, however, did not systematically derive the corresponding master equations. We will further improve their argument and completely solve the master equations also explicitly for the linear coefficient C_A.

First of all, we observe that, since the coefficient $M^{A\,\gamma}$ vanishes, equation (5.55) implies that also the potential C can only depend on at most the second partial derivatives of the fields $\hat{g}^{\alpha\beta}$. For the reader's convenience, we display all remaining reduced master equations here. For all coefficients $C_{\alpha_1\beta_1\ldots\alpha_N\beta_N}$ with $N \geq 1$ we have that

$$\begin{aligned} 0 &= (N+1)!\, C_{\rho\sigma\alpha_1\beta_1\ldots\alpha_N\beta_N} U^{\rho\sigma\,\mu\nu} - (N-2)(N-1)! \frac{\partial C_{\alpha_1\beta_1\ldots\alpha_{N-1}\beta_{N-1}}}{\partial\, \partial^2_{\mu\nu} \hat{g}^{\alpha_N\beta_N}}\,, && (6.18) \\ 0 &= (N+1)!\, C_{\rho\sigma\alpha_1\beta_1\ldots\alpha_N\beta_N} S^{\rho\sigma\,\mu} + (N-1)! \sum_{a=1}^{N} \frac{\partial C_{\alpha_1\beta_1\ldots\widetilde{\alpha_a\beta_a}\ldots\alpha_N\beta_N}}{\partial\, \partial_\mu \hat{g}^{\alpha_a\beta_a}} \\ &\quad + (N-1)! \partial_\gamma \frac{\partial C_{\alpha_1\beta_1\ldots\alpha_{N-1}\beta_{N-1}}}{\partial\, \partial^2_{\gamma\mu} \hat{g}^{\alpha_N\beta_N}}\,. && (6.19) \end{aligned}$$

The first order coefficient $C_{\rho\sigma}$ has to satisfy the equation

$$0 = \partial_\nu (C_{\rho\sigma} U^{\rho\sigma\,\mu\nu}) + C_{\rho\sigma} S^{\rho\sigma\,\mu} \tag{6.20}$$

and we have the symmetry condition

$$\frac{\partial C_{\alpha_1\beta_1\ldots\widetilde{\alpha_a\beta_a}\ldots\alpha_N\beta_N}}{\partial \partial^2_{\mu\nu} \hat{g}^{\alpha_a\beta_a}} = \frac{\partial C_{\alpha_1\beta_1\ldots\alpha_{N-1}\beta_{N-1}}}{\partial \partial^2_{\mu\nu} \hat{g}^{\alpha_N\beta_N}} \qquad \text{for all } N \geq 2\,. \tag{6.21}$$

The invariance equations are now the same for all coefficients $C_{B_1\ldots B_N}$ with $N \geq 0$; Explicitly these are

$$0 = \hat{P}^{\alpha(\sigma} \frac{\partial C_{B_1\ldots B_N}}{\partial \partial^2_{\mu\nu)} \hat{g}^{\alpha\rho}} \tag{6.22}$$

and

$$0 = 2\, \hat{g}^{\alpha(\mu} \frac{\partial C_{B_1\ldots B_N}}{\partial \partial_{\nu)} \hat{g}^{\alpha\rho}} - \partial_\rho \hat{g}^{\alpha\beta} \frac{\partial C_{B_1\ldots B_N}}{\partial \partial^2_{\mu\nu} \hat{g}^{\alpha\beta}} + 4\, \partial_\sigma \hat{g}^{\alpha(\mu} \frac{\partial C_{B_1\ldots B_N}}{\partial \partial^2_{\nu)\sigma} \hat{g}^{\alpha\rho}}\,. \tag{6.23}$$

Now we can solve the master equations step by step. First, we observe that equation (6.18) for $N = 2$ simply reads

$$0 = C_{\rho\sigma\alpha_1\beta_1\alpha_2\beta_2} U^{\rho\sigma\,\mu\nu}\,, \tag{6.24}$$

which may be solved to yield $C_{\rho\sigma\alpha_1\beta_1\alpha_2\beta_2} = 0$. Inserting this result back into equation (6.18), first for $N = 4$ and then repeating the procedure for all even N, we see that all coefficients with an odd number of index pairs already vanish, except for the first one, $C_{\alpha\beta}$. For our next conclusion, we temporarily change variables in favour of the metric $\hat{g}_{\alpha\beta}$. Changing the partial deriviatives of $\hat{g}^{\alpha\beta}$ accordingly, the first invariance equation (6.22) becomes

$$0 = \frac{\partial C_{\alpha_1\beta_1\ldots\alpha_N\beta_N}}{\partial \hat{g}_{\alpha(\beta,\gamma\delta)}}\,, \tag{6.25}$$

where we denote partial derivatives by a comma. Moreover, the divergence term in equation (6.19) implies

$$0 = \frac{\partial^2 C_{\alpha_1\beta_1\ldots\alpha_N\beta_N}}{\partial \hat{g}_{\alpha\beta,(\mu\nu|}\,\partial \hat{g}_{\rho\sigma,|\gamma)\delta}}\,. \tag{6.26}$$

Repeating the argument from section 5.2.2, we may conclude from the last two symmetry conditions that the second equation already holds without any symmetrizations if $\dim\Sigma = 3$. In other words, all remaining coefficients $C_{\alpha_1\beta_1\ldots\alpha_N\beta_N}$ can only depend linearly on the second derivatives of the field $\hat{g}_{\alpha\beta}$ and similarly of $\hat{g}^{\alpha\beta}$. Since, in particular, the zeroth order coefficient C depends only linearly on the second derivatives of $\hat{g}^{\alpha\beta}$, we conclude from equation (6.18) for $N = 1$ that the coefficient $C_{\alpha_1\beta_1\alpha_2\beta_2}$ must in fact be independent of the second derivatives of $\hat{g}^{\alpha\beta}$. Using this result in equation (6.18) for $N = 3$, and iterating on all odd N, we find that also all even coefficients $C_{\alpha_1\beta_1\ldots\alpha_N\beta_N}$ for $N \geq 4$ vanish. Hence, it remains to determine the coefficients C, $C_{\alpha\beta}$ and $C_{\alpha\beta\gamma\delta}$.

We may now perform a change of coordinates as described in section 5.2.3 in order to solve the invariance equations. The invariance equations (5.53) and (5.54) then imply in particular that $C = C(\hat{g}^{\alpha\beta}, R_{\alpha\beta\gamma\delta})$, where $R_{\alpha\beta\gamma\delta}$ is the Riemann-Christoffel tensor of $\hat{g}^{\alpha\beta}$. In three dimensions, we know that the Riemann tensor can be expressed in terms of the Ricci tensor $R_{\alpha\beta}$ and the metric $\hat{g}_{\alpha\beta}$ so that, actually, $C = C(\hat{g}^{\alpha\beta}, R_{\alpha\beta})$. The only such weight-one tensor density linear in the Ricci tensor is $(-\det\hat{g})^{-1/2}R$, with the Ricci scalar $R = R_{\alpha\beta}\hat{g}^{\alpha\beta}$, and the minus sign under the square root accounts for the fact that $\hat{g}^{\alpha\beta}$ must be negative definite. Thus we arrive at

$$C = -(2\kappa)^{-1}(-\det\hat{g})^{-1/2}\,(R - 2\lambda), \tag{6.27}$$

with constants κ and λ, as the only coefficient that meets all the requirements. Then we can immediately calculate, from equation (6.18) for $N = 1$, that

$$C_{\alpha\beta\mu\nu} = (16\kappa)^{-1}(-\det\hat{g})^{-1/2}\left[\hat{g}_{\alpha\mu}\hat{g}_{\beta\nu} + \hat{g}_{\beta\mu}\hat{g}_{\alpha\nu} - 2\hat{g}_{\alpha\beta}\hat{g}_{\mu\nu}\right]. \tag{6.28}$$

In the coordinates $(\hat{g}^{\alpha\beta}, \Gamma^{\alpha}_{\beta\gamma}, R_{\alpha\beta\gamma\delta})$, the coefficient $S^{\alpha\beta\,\gamma}$ can be rewritten as

$$S^{\alpha\beta\,\gamma} = U^{\alpha\beta\,\mu\nu}\Gamma^{\gamma}_{\mu\nu}\,, \tag{6.29}$$

which makes it easy to see that equation (6.20) takes the form

$$0 = \hat{g}^{\mu\rho}\hat{g}^{\sigma\nu}\nabla_{\nu}C_{\rho\sigma}\,. \tag{6.30}$$

where ∇_{γ} denotes the covariant derivative with respect to the Levi-Civita connection. Using the well-known theorem due to Lovelock [44], which also for the case of three dimensions asserts that the only divergence free second rank tensor depending only on the metric and its first and second derivatives is the Einstein tensor, and the fact that again $C_{\rho\sigma}$ can only depend linearly on the Ricci tensor, we immediately conclude that

$$C_{\alpha\beta} = \beta_1(-\det\hat{g})^{-1/2}\,(R_{\alpha\beta} - \frac{1}{2}\,\hat{g}_{\alpha\beta}\,R) + \beta_2(-\det\hat{g})^{-1/2}\,\hat{g}_{\alpha\beta}\,. \tag{6.31}$$

This is our slight improvement of the argument given by Kuchař [37], who did not determine this coefficient explicitly. The remaining master equations (6.19) and (6.21) are identically satisfied. The coefficients C, $C_{\alpha\beta}$ and $C_{\alpha\beta\gamma\delta}$ now completely determine the Lagrangian (4.22) by virtue of

$$L = C_{\alpha\beta\gamma\delta}\hat{K}^{\alpha\beta}\hat{K}^{\gamma\delta} + C_{\alpha\beta}\hat{K}^{\alpha\beta} + C \tag{6.32}$$

and thus we have found the gravitational dynamics of the geometry $\hat{g}^{\alpha\beta}$. We would now like to analyse this solution a little further.

We immediately realize that the coefficient $C_{\rho\sigma}$ can be written as the functional derivative of the scalar density

$$\Lambda = \beta_1(-\det\hat{g})^{-1/2}\,R - 2\,\beta_2\,(-\det\hat{g})^{-1/2} \tag{6.33}$$

with respect to $\hat{g}^{\alpha\beta}$. This has severe consequences for the relevance of this coefficient in the equations of motion: this part of the Lagrangian satisfies the equations of motion identically and is thus dynamically irrelevant [37]. This can be seen as follows. The first of the two Lagrangian equations of motion (4.26) of course remains untouched because it is purely kinematical and independent of the Lagrangian, so we have that

$$\dot{\hat{g}}^{\alpha\beta}(z) = N(z)\hat{K}^{\alpha\beta}(z) + (\mathcal{L}_{\vec{N}}\hat{g})^{\alpha\beta}(z)\,. \tag{6.34}$$

The second Lagrangian equation of motion (4.30) in this case reads

$$\frac{\partial}{\partial t}\left(\frac{\partial L(z)}{\partial \hat{K}^{\alpha\beta}(z)}\right) = \int_\Sigma dx\, \left[N(x)\frac{\delta L(x)}{\delta \hat{g}^{\alpha\beta}(z)}\right] + \mathcal{L}_{\vec{N}}\left(\frac{\partial L(z)}{\partial \hat{K}^{\alpha\beta}(z)}\right)\,, \tag{6.35}$$

because there is no contribution from the coefficients ${Q_A}^{B\gamma}$. We may now insert the part of the Lagrangian that is linear in the velocities $\hat{K}^{\alpha\beta}$, $L_{\text{lin}}(z) = \delta\Lambda(z)/\delta\hat{P}^{\alpha\beta}(z)\hat{K}^{\alpha\beta}(z)$, into the left hand side of this equation, and, taking into account the first equation of motion, we find

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial L_{\text{lin}}(z)}{\partial \hat{K}^{\alpha\beta}(z)}\right) &= \int_\Sigma dx\, \frac{\delta^2\Lambda(z)}{\delta\hat{g}^{\rho\sigma}(x)\delta\hat{g}^{\alpha\beta}(z)}\,\dot{\hat{g}}^{\rho\sigma}(x)\\ &= \int_\Sigma dx\, \frac{\delta^2\Lambda(z)}{\delta\hat{g}^{\rho\sigma}(x)\delta\hat{g}^{\alpha\beta}(z)}\,\Big(N(x)\hat{K}^{\rho\sigma}(x) + (\mathcal{L}_{\vec{N}}\hat{g})^{\rho\sigma}(x)\Big)\end{aligned} \tag{6.36}$$

It is then straightforward to see that these are precisely the terms that also appear on the right hand side of equation (6.35). The respective first terms cancel because the functional derivatives commute. That also the second terms cancel, one can see by writing out the Lie derivative on both sides and using the chain rule and an integration by parts on the left hand side of the equation. Since the linear term in the Lagrangian satisfies the equations of motion identically, one might be tempted to already neglect it altogether, but we still have to check whether it has an influence on the constraints (4.5). In other words, we have to transform the Lagrangian back to the local superhamiltonian by means of a Legendre transform with respect to the velocities $\hat{K}^{\alpha\beta}$.

First of all, we calculate the canonical momenta as the Legendre dual variables of the velocities,

$$\hat{\pi}_{\alpha\beta} = \frac{\partial L}{\partial \hat{K}^{\alpha\beta}} = 2C_{\alpha\beta\gamma\delta}\hat{K}^{\gamma\delta} + \frac{\delta\Lambda}{\delta\hat{g}^{\alpha\beta}}\,. \tag{6.37}$$

We now remember that we have a certain freedom to define the canonical momenta $\hat{\pi}_{\alpha\beta}$. The Poisson bracket (4.2) does not change if we add to the canonical momenta the functional derivative of a weight-one scalar density with respect to the configuration variables $\hat{G}^A$. Thus, we can redefine the canonical momenta,

$$\hat{\pi}_{\alpha\beta} \to \tilde{\pi}_{\alpha\beta} = \hat{\pi}_{\alpha\beta} - \frac{\delta\Lambda}{\delta\hat{g}^{\alpha\beta}} \tag{6.38}$$

and invert equation (6.37) to get the velocities

$$\hat{K}^{\alpha\beta} = \frac{1}{2}C^{\alpha\beta\gamma\delta}\tilde{\pi}_{\alpha\beta}\,, \tag{6.39}$$

where $C^{\alpha\beta\gamma\delta}$ is the inverse of the coefficient $C_{\alpha\beta\gamma\delta}$ and explicitly reads

$$C^{\alpha\beta\gamma\delta} = 4\kappa\,(-\det\hat{g})^{1/2}\,(\hat{g}^{\alpha\gamma}\hat{g}^{\beta\delta} + \hat{g}^{\beta\gamma}\hat{g}^{\alpha\delta} - \hat{g}^{\alpha\beta}\hat{g}^{\gamma\delta})\,, \tag{6.40}$$

which is known as the DeWitt tensor density. The local superhamiltonian then automatically becomes

$$\begin{aligned} \hat{\mathcal{H}}_{\text{local}} &= \hat{K}^{\alpha\beta}\tilde{\pi}_{\alpha\beta} - C_{\alpha\beta\gamma\delta}\hat{K}^{\alpha\beta}\hat{K}^{\gamma\delta} - C \\ &= \frac{1}{4}C^{\alpha\beta\gamma\delta}\tilde{\pi}_{\alpha\beta}\tilde{\pi}_{\gamma\delta} + (2\kappa)^{-1}(-\det\hat{g})^{-1/2}\,(R - 2\lambda)\,, \end{aligned} \tag{6.41}$$

which is the famously known Arnowitt-Deser-Misner Hamiltonian of Einstein-Hilbert dynamics [2] whose reformulation in terms of Ashtekar variables is also the starting point for a canonical quantisation of gravity [68]. Hence, that part in the Lagrangian, which is linear in the velocities, becomes completely obsolete, and we can set $\beta_1 = \beta_2 = 0$.

In summary, we arrived at the remarkable conclusion that the unique gravitational dynamics for a four-dimensional tensorial spacetime $(M, g, S_{\text{Maxwell}}[g, A])$ is given by the Einstein-Hilbert dynamics of general relativity[3]. The same is of course true if we consider the tensorial spacetime $(M, g, S_{\text{SM}}[g, \Phi])$, whose matter dynamics $S_{\text{SM}}[g, \Phi]$ include all fields of the standard model of particle physics, because their equations of motion all share the same principal tensor fields, which by deliberate construction of the standard model (taking particles to be the irreducible representations of the Poincaré group) is the principal tensor field of Maxwell electrodynamics.

From the way we have obtained the above result, we learn that the three ways to generate dynamics different from the standard ADM formulation is by (i) prescribing matter dynamics whose principal tensor field is of higher even rank or (ii) by changing the dimension of the hypersurface Σ, or (iii) by employing gravitational degrees of freedom to which matter does not couple. Following the second (less interesting) path would generate higher order terms in the Riemann tensor in the potential C and correspondingly higher powers of the velocities $\hat{K}^{\alpha\beta}$ than the second in the Lagrangian[4]. Of course, this is only true if we insist on the fact that the degrees of freedom of the theory being encoded by a metric. The third path above was taken in [15], where it was shown that it is possible to cast any gravitational theory of a metric g in D spacetime dimensions, which follows from a Lagrangian of the form $L = \sqrt{-\det g}\, f(\text{Riem})$, with f being some function of the Riemann tensor of

[3]For more details on the structure of the constraints and the Poisson algebra in the context of general relativity see, e.g., [45, 33]. The strength of finding dynamics for a hypersurface metric by finding representations to the hypersurface deformation algebra has also been studied in symmetry reduced cases taking into account quantum corrections implied by loop quantum gravity in [10, 9].

[4]The resulting Lagrangians should then be compared to those found by Lovelock in a different context [43].

g, into a Hamiltonian form that also satisfies the Poisson algebra (4.11)-(4.13). But this is only possible at the expense of introducing additional degrees of freedom and thus by enlarging the phase space. There are two reasons why this third path is not possible from the point of view we take in this thesis. First, the degrees of freedom of the theories we consider are not chosen arbitrarily, but are selected by the fact that matter couples to them. Secondly, the additional variables needed to write down the canonical description of f(Riem) theories are directly connected to some components of the Riemann tensor of the spacetime metric. In order to establish this link, one needs additional constraints in the theory [15], which are not a consequence of the correspondence of the hypersurface deformation point of view and the canonical one. One automatically obtains these constraints if one starts with the spacetime formulation of the theory, but, starting from the canonical view point, none of the canonical variables can be viewed as being part of spacetime quantities (such as the spacetime Riemann tensor) an initial value formulation of the dynamics first has to determine. Thus, the only way to generate f(Riem) theories, along the lines laid out in this thesis, is to start with an enlarged phase space in order to obtain a more general Lagrangian, solve the master equations, and only at the very end, relate part of the phase space variables through additional constraints. It seems, however, to be very difficult to obtain a particular f(Riem) theory this way.

Unfortunately, solving the master equations is not as easy for an arbitrary tensorial spacetime $(M, G, S[\Phi, G])$ as it is for $(S, g, S_{\mathrm{Maxwell}}[g, A])$. One of the complicating factors in more general cases is the non-local part of the superhamiltonian, which leads to a disadvantageous coupling of three adjacent coefficients $C_{B_1 \dots B_N}$ in the master equations. This makes it difficult to arrive at conclusions as early as in the case of general relativity, where it was simple to show that the series expansion of the Lagrangian stops after the quadratic power of the velocities. How difficult to solve the master equations can become will be illustrated in the next example.

6.3 Example 2: From general linear electrodynamics to the master equations for area metric spacetimes

In this example we consider area metric manifolds (M, G) in four dimensions [53, 54, 65, 66], whose geometry is given in terms of a fourth rank contravariant tensor field G^{abcd} featuring the symmetries $G^{abcd} = G^{cdab}$ and $G^{abcd} = -G^{bacd}$ and satisfying a non-degeneracy condition in the sense that there exists a fourth rank covariant tensor field G_{abcd} such that $G_{abmn}G^{mncd} = 4\delta^{[c}_{[a}\delta^{d]}_{b]}$. In four dimensions, such an area metric tensor field can always be decomposed into a sum $G^{abcd} = G^{abcd}_{C} + \Psi\epsilon^{abcd}$ of a cyclic part G_C with $G_C^{a[bcd]} = 0$ and a totally antisymmetric part given in terms of a scalar density $\Psi = (1/24)\, G^{abcd}\epsilon_{abcd}$ of weight -1, where ϵ^{abcd} denotes the contravariant totally antisymmetric Levi-Civita tensor density. Thus, one may define an area metric volume form ω_G by $\omega_{G\,abcd} = (1/\Psi)\, \epsilon_{abcd}$ on every four-dimensional area metric manifold (M, G) if G is non-cyclic everywhere, which we will assume in this section.

As the matter theory, coupling to area metric geometry, we choose general linear electrodynamics, which is described by the action

$$S_{\mathrm{GLED}}[G, A] = -\frac{1}{4}\int_M \omega_G\, G^{abcd}\, F_{ab}F_{cd} \tag{6.42}$$

for a covector gauge field A with field strength $F = dA$. General linear electrodynamics is a straightforward generalization of Maxwell theory on metric manifolds (M, g). Indeed, considering the special area metric tensor field $G^{abcd} = g^{a[c}g^{d]b} + (\det g)^{-1/2}\,\epsilon^{abcd}$, induced by a metric g, the resulting action (6.42) is classically equivalent to Maxwell theory on a metric manifold (M, g).

Far from being an exotic refinement of Maxwell electrodynamics, general linear electrodynamics naturally emerges as the effective background seen by gravitationally interacting photons in vacuo, see the well-known paper by Drummond and Hathrell [17]. Also the geometry of birefringent non-dissipative linear optical media is given by an area metric [64]. Pre-metric electrodynamics [29, 61] was studied before area metric electrodynamics and is virtually (but not entirely) identical to it. General linear electrodynamics has also been studied in other contexts [57, 56].

Again, we start our procedure to set up the gravitational master equations by determining the kinematics of the matter theory (6.42).

6.3.1 Kinematics of general linear electrodynamics

In order to discuss the kinematics of general linear electrodynamics, we first need the principal tensor field P of the tensorial matter field equations

$$\Psi\,\partial_a\left(\frac{1}{\Psi}\,G^{abcd}F_{cd}\right) = 0 \quad \text{and} \quad \partial_{[a}F_{bc]} = 0\,, \tag{6.43}$$

which follow from variation of the action (6.42) and use of the Bianchi identity. The calculation of the principal tensor field for these matter field equations is more involved than in the case of Maxwell theory on a metric manifold, but was found by Rubilar [61] in the context of pre-metric electrodynamics, to be given as

$$P_G^{abcd} = -\frac{1}{24\Psi^2}\,\epsilon_{mnpq}\,\epsilon_{rstu}\,G^{mnr(a}\,G^{b|ps|c}\,G^{d)qtu}\,. \tag{6.44}$$

In this case, hyperbolicity of the principal tensor field P_G alone does not automatically imply hyperbolicity of the dual polynomial. But requiring, as always, bi-hyperbolicity and energy-distinguishability directly excludes 16 of 23 possible algebraic classes [13, 64] of area metric tensors, since they do not provide a geometry on which general linear electrodynamics is predictive and quantizable. This is because bi-hyperbolicity and energy-distinguishability exclude the existence of null planes in the vanishing set of the principal polynomial $P(x,k)$ [58] and one can show that these null planes always exist for each of the 16 excluded classes of area metric tensor fields.

With the help of the algebraic classification of area metric tensors—which was achieved in the present author's bachelor thesis [64]—it is possible to directly check that the dual $P^{\#}(x,v)$ of the principal polynomial $P(x,k)$ is given by

$$P^{\#}(x,v) = -\frac{\Psi^2}{24}\epsilon^{mnpq}\epsilon^{restu}\, G_{mnra}\, G_{bpsc}\, G_{dqtu}\, v^a v^b v^c v^d\,, \tag{6.45}$$

so that the propagation of light rays can be studied using the action (2.13), see also [29, 55, 57, 61]. The corresponding Legendre map can be directly calculated from (2.17). While there is no closed expression known for the inverse L^{-1} of the Legendre map in this case, fortunately no such is necessary for setting up the corresponding master equations for the tensorial spacetime $(M, G, S_{\mathrm{GLED}}[G,A])$. Precisely this is the topic of the next section.

6.3.2 Setting up the master equations

We already discussed in section 3.2.2 that the configuration variables $\hat{G}^A$, which constitute the hypersurface geometry of the tensorial spacetime $(M, G, S_{\mathrm{GLED}}[G,A])$, are given by the tensor fields

$$\hat{G}^{\alpha\beta}, \quad \hat{G}^{\alpha}{}_{\beta} \quad \text{and} \quad \hat{G}_{\alpha\beta}, \tag{6.46}$$

where $\hat{G}^{\alpha\beta}$ and $\hat{G}_{\alpha\beta}$ are symmetric, $\hat{G}^{\alpha\beta}$ is non-degenerate and $\hat{G}^{\alpha}{}_{\beta}$ is tracefree and symmetric with respect to $\hat{G}^{\alpha\beta}$. The corresponding conjugate momenta are the weight-one tensor densities

$$\hat{\pi}_{\alpha\beta}, \quad \hat{\pi}_{\alpha}{}^{\beta} \quad \text{and} \quad \hat{\pi}^{\alpha\beta}, \tag{6.47}$$

which complete the phase space of the theory. Thus in this case the capital multi-index A has to be read as ${}^A = ({}^{\alpha\beta}, {}^{\alpha}{}_{\beta}, {}_{\alpha\beta})$ if it is a superscript and ${}_A = ({}_{\alpha\beta}, {}_{\alpha}{}^{\beta}, {}^{\alpha\beta})$ if it is a subscript index.

The supermomentum is given by (4.16), and after an integration by parts it takes the form

$$\hat{\mathcal{D}}(\vec{N}) \;=\; \int dy\, N^\gamma(y)\Big[(\partial_\gamma \hat{G}^{\alpha\beta})\hat{\pi}_{\alpha\beta} + 2\partial_\alpha(\hat{G}^{\alpha\beta}\hat{\pi}_{\beta\gamma}) + (\partial_\gamma \hat{G}_{\alpha\beta})\hat{\pi}^{\alpha\beta} - 2\partial_\alpha(\hat{\pi}^{\alpha\beta}\hat{G}_{\beta\gamma}) + (\partial_\gamma \hat{G}^\alpha{}_\beta)\hat{\pi}_\alpha{}^\beta + \partial_\alpha(\hat{G}^\alpha{}_\beta\hat{\pi}_\gamma{}^\beta) - \partial_\alpha(\hat{G}^\beta{}_\gamma\hat{\pi}_\beta{}^\alpha)\Big]\,, \tag{6.48}$$

which is suited for the localized version of the right hand side of the algebra equation (4.11), from whose abstract form (6.1) we read off the coefficients

$$U^{A\,\mu\nu} = \begin{cases} -6\hat{P}_{\hat{G}}^{(\alpha|\nu}\,\hat{G}^{\mu|\beta)}\,, & \text{for} \quad {}^A = {}^{\alpha\beta} \\ -3\hat{P}_{\hat{G}}^{\alpha\nu}\,\hat{G}^\mu{}_\beta + 3\hat{P}_{\hat{G}}^{\sigma\nu}\,\delta^\mu_\beta\,\hat{G}^\alpha{}_\sigma\,, & \text{for} \quad {}^A = {}^\alpha{}_\beta \\ +6\hat{P}_{\hat{G}}^{\sigma\nu}\,\delta^\mu_{(\alpha}\,\hat{G}_{\beta)\sigma}\,, & \text{for} \quad {}^A = {}_{\alpha\beta} \end{cases} \tag{6.49}$$

as well as the coefficients

$$V^{A\,\gamma} = \begin{cases} 6\hat{P}_{\hat{G}}^{\gamma(\alpha}\partial_\delta\hat{G}^{\beta)\delta}\,, & \text{for} \quad {}^A = {}^{\alpha\beta} \\ 3\hat{P}_{\hat{G}}^{\gamma\alpha}\partial_\delta\hat{G}^\delta{}_\beta - 3\hat{P}_{\hat{G}}^{\gamma\delta}\partial_\beta\hat{G}^\alpha{}_\delta\,, & \text{for} \quad {}^A = {}^\alpha{}_\beta \\ -6\hat{P}_{\hat{G}}^{\gamma\delta}\partial_{(\alpha}\hat{G}_{\beta)\delta}\,, & \text{for} \quad {}^A = {}_{\alpha\beta}\,, \end{cases} \tag{6.50}$$

where the function $\hat{P}_{\hat{G}}^{\alpha\beta}$ is given by

$$\hat{P}_{\hat{G}}^{\alpha\beta} = \frac{1}{6}\left(\hat{G}^{\alpha\gamma}\hat{G}^{\delta\beta}\hat{G}_{\gamma\delta} - \hat{G}^{\alpha\beta}\hat{G}^{\gamma\delta}\hat{G}_{\gamma\delta} + 2\hat{G}^{\alpha\beta}\hat{G}^\gamma{}_\delta\hat{G}^\delta{}_\gamma - 3\hat{G}^{\gamma\delta}\hat{G}^\alpha{}_\gamma\hat{G}^\beta{}_\delta\right)\,. \tag{6.51}$$

The non-local part of the superhamiltonian is given by equation (4.20) and the coefficients $M^{A\,\gamma}$ can be read off the expressions (3.35)-(3.37). Explicitly, we have

$$M^{A\,\gamma} = \begin{cases} -2\,(\omega_{\hat{G}}^{-1})^{\delta\gamma(\alpha}\,\hat{G}^{\beta)}{}_\delta\,, & \text{for} \quad {}^A = {}^{\alpha\beta} \\ -3\,\omega_{\hat{G}\,\beta\sigma\tau}\,\hat{G}^{\alpha\sigma}\,\hat{P}_{\hat{G}}^{\tau\gamma} - (\omega_{\hat{G}}^{-1})^{\gamma\alpha\sigma}\,\hat{G}_{\sigma\beta}\,, & \text{for} \quad {}^A = {}^\alpha{}_\beta \\ -6\,\omega_{\hat{G}\,\sigma\tau(\alpha}\,\hat{G}^\sigma{}_{\beta)}\,\hat{P}_{\hat{G}}^{\tau\gamma}\,, & \text{for} \quad {}^A = {}_{\alpha\beta}\,. \end{cases} \tag{6.52}$$

The coefficients $Q_B{}^{A\,\beta}$ are calculated according to $Q_B{}^{A\,\beta} = -\partial M^{A\,\beta}/\partial\hat{G}^B$. One has to be careful, though, because the function $\hat{P}_{\hat{G}}^{\alpha\beta}$ appears explicitly in the coefficients $M^{A\,\gamma}$, and has to be derived as well when calculating the coefficient Q. The coefficients $T^{A\,\mu\nu}$ are given by

$$T^{A\,\mu\nu} = -\,Q_B{}^{A\,\mu}M^{B\,\nu} + U^{A\,\mu\nu}\,. \tag{6.53}$$

This time, and in contrast to the metric case (and indeed in contrast to the other two examples we shall discuss in sections 6.4 and 6.5), one finds that these coefficients are not symmetric in $\mu\nu$. Only the coefficients $T^{\alpha\beta\,[\mu\nu]}$ vanish, but the coefficients $T_{\alpha\beta}{}^{[\mu\nu]}$ generically do not and are explicitly given by[5]

$$
\begin{aligned}
T_{\alpha\beta}{}^{[\mu\nu]} \;=\;& \operatorname{sgn}(\det \hat{G}^{\alpha\beta})\Big\{2\,\hat{G}^{\rho}{}_{\tau}\hat{G}^{\tau}{}_{(\alpha}\,\delta^{[\nu}_{\beta)}\,\hat{G}^{\mu]\sigma}\,\hat{G}_{\rho\sigma} + 6\,\hat{G}^{[\nu|}{}_{\tau}\hat{G}^{\tau}{}_{(\alpha}\,\delta^{|\mu]}_{\beta)}\,\hat{G}^{\rho\sigma}\,\hat{G}_{\rho\sigma} \\
& + 6\,\hat{G}^{\rho}{}_{\tau}\,\hat{G}^{\tau}{}_{\sigma}\,\hat{G}^{\sigma}{}_{(\alpha|}\,\hat{G}^{[\nu}{}_{\rho}\,\delta^{\mu]}_{|\beta)} + 12\,\hat{G}^{\rho}{}_{(\alpha|}\,\hat{G}^{\tau}{}_{\sigma}\,\hat{G}^{\sigma}{}_{\tau}\,\hat{G}^{[\mu}{}_{\rho}\,\delta^{\nu]}_{|\beta)} \\
& + 6\,\hat{G}^{\rho}{}_{\sigma}\,\hat{G}^{\sigma}{}_{(\alpha|}\,\hat{G}^{[\mu}{}_{\rho}\,\hat{G}^{\nu]}{}_{|\beta)} + 2\,\hat{G}^{[\nu|}{}_{(\alpha}\hat{G}^{\rho}{}_{\beta)}\hat{G}^{|\mu]\sigma}\hat{G}_{\rho\sigma} \\
& - 4\,\hat{G}^{[\nu|}{}_{\rho}\hat{G}^{\rho}{}_{(\alpha|}\hat{G}^{|\mu]\sigma}\hat{G}_{|\beta)\sigma}\Big\} \\
& - 2\,\hat{G}^{\rho}{}_{\tau}\hat{G}^{\tau}{}_{(\alpha}\,\delta^{[\nu}_{\beta)}\,\hat{G}^{\mu]\sigma}\,\hat{G}_{\rho\sigma} + 2\,\hat{G}^{[\nu|}{}_{\tau}\hat{G}^{\tau}{}_{(\alpha}\,\delta^{|\mu]}_{\beta)}\,\hat{G}^{\rho\sigma}\,\hat{G}_{\rho\sigma} \\
& + 9\,\hat{G}^{\rho}{}_{(\alpha|}\,\hat{G}^{\tau}{}_{\sigma}\,\hat{G}^{\sigma}{}_{\tau}\,\hat{G}^{[\mu}{}_{\rho}\,\delta^{\nu]}_{|\beta)} + 2\,\hat{G}^{[\nu|}{}_{(\alpha}\,\hat{G}^{\rho}{}_{\beta)}\,\hat{G}^{|\mu]\sigma}\,\hat{G}_{\rho\sigma} \\
& + 4\,\hat{G}^{[\mu}{}_{(\alpha|}\,\hat{G}^{\nu]}{}_{\rho}\,\hat{G}^{\rho\sigma}\hat{G}_{|\beta)\sigma} - 4\,\hat{G}^{[\mu|}{}_{\rho}\,\hat{G}^{\sigma}{}_{(\alpha}\,\delta^{|\nu]}_{\beta)}\,\hat{G}^{\rho\tau}\,\hat{G}_{\sigma\tau} \\
& + 4\,\hat{G}^{[\mu}{}_{(\alpha}\,\delta^{\nu]}_{\beta)}\,\hat{G}^{\rho}{}_{\sigma}\,\hat{G}^{\sigma\tau}\,\hat{G}_{\rho\tau}\,,
\end{aligned}
\tag{6.54}
$$

where $\operatorname{sgn}(\det \hat{G}^{\alpha\beta})$ denotes the overall sign of the determinant of the tensor field $\hat{G}^{\alpha\beta}$ for which we have not made any signature assumption. The sign function sgn emerges from the simplification of several products of the volume form $\omega_{\hat{G}}$ with itself. Finally, the coefficients $T^{\alpha}{}_{\beta}{}^{[\mu\nu]}$ read

$$
\begin{aligned}
T^{\alpha}{}_{\beta}{}^{[\mu\nu]} \;=\;& \operatorname{sgn}(\det \hat{G}^{\alpha\beta})\Big\{12\,\hat{G}^{\alpha}{}_{\rho}\,\hat{G}^{[\mu}{}_{(\beta}\,\hat{G}^{\nu]}{}_{\sigma)}\hat{G}^{\rho\sigma} + 3\,\hat{G}^{\sigma}{}_{\rho}\,\hat{G}^{\rho}{}_{\beta}\,\hat{G}^{[\mu}{}_{\sigma}G^{\nu]\alpha} \\
& - 6\,\hat{G}^{\sigma}{}_{\rho}\,\hat{G}^{\rho}{}_{\sigma}\,\hat{G}^{[\mu}{}_{\beta}G^{\nu]\alpha} + 2\,\hat{G}^{\alpha}{}_{\rho}\,\hat{G}^{\mu[\rho}\,\hat{G}^{\sigma]\nu}\,\hat{G}_{\sigma\beta} \\
& + 3\,\hat{G}^{[\mu}{}_{\beta}\,\hat{G}^{\nu]\alpha}\,\hat{G}^{\rho\sigma}\,\hat{G}_{\rho\sigma} - 2\hat{G}^{[\mu}{}_{\beta}\,\hat{G}^{\nu]\rho}\,\hat{G}^{\sigma\alpha}\hat{G}_{\rho\sigma} \\
& + \hat{G}^{\rho}{}_{\beta}\,\hat{G}^{\alpha[\mu}\,\hat{G}^{\nu]\sigma}\,\hat{G}_{\rho\sigma} - 3\,\hat{G}^{\alpha}{}_{\rho}\,\hat{G}^{[\mu}{}_{\sigma}\,\delta^{\nu]}_{\beta}\,\hat{G}^{\sigma}{}_{\tau}\,\hat{G}^{\rho\tau} \\
& - 6\,\hat{G}^{\alpha}{}_{\rho}\,\hat{G}^{\sigma}{}_{\tau}\,\hat{G}^{\tau}{}_{\sigma}\,\delta^{[\mu}_{\beta}\hat{G}^{\nu]\rho} + 3\,\hat{G}^{\alpha}{}_{\rho}\,\delta^{[\mu}_{\beta}\hat{G}^{\nu]\rho}\,\hat{G}^{\sigma\tau}\,\hat{G}_{\sigma\tau} \\
& - \hat{G}^{\alpha}{}_{\sigma}\,\delta^{[\mu}_{\beta}\hat{G}^{\nu]\rho}\,\hat{G}^{\sigma\tau}\,\hat{G}_{\rho\tau}\Big\} \\
& - \frac{9}{2}\,\hat{G}^{\rho}{}_{\sigma}\,\hat{G}^{\sigma}{}_{\rho}\,\hat{G}^{[\mu}{}_{\beta}\,\hat{G}^{\nu]\alpha} + \hat{G}^{[\mu}{}_{\beta}\,\hat{G}^{\nu]\alpha}\,\hat{G}^{\rho\sigma}\hat{G}_{\rho\sigma} + 2\,\hat{G}^{[\mu}{}_{\rho}\,\hat{G}^{\nu]\alpha}\,\hat{G}^{\rho\sigma}\,\hat{G}_{\beta\sigma} \\
& - \hat{G}^{\rho}{}_{\beta}\,\hat{G}^{\alpha[\mu}\,\hat{G}^{\nu]\sigma}\,\hat{G}_{\rho\sigma} - \frac{9}{2}\,\hat{G}^{\rho}{}_{\sigma}\,\hat{G}^{\sigma}{}_{\rho}\,\hat{G}^{\alpha}{}_{\tau}\delta^{[\mu}_{\beta}\hat{G}^{\nu]\tau} + \hat{G}^{\alpha}{}_{\rho}\,\delta^{[\mu}_{\beta}\hat{G}^{\nu]\rho}\,\hat{G}^{\sigma\tau}\,\hat{G}_{\sigma\tau} \\
& + \hat{G}^{\alpha}{}_{\sigma}\,\delta^{[\mu}_{\beta}\hat{G}^{\nu]\rho}\,\hat{G}^{\sigma\tau}\,\hat{G}_{\rho\tau} - 2\,\hat{G}^{[\mu}{}_{\beta}\,\delta^{\nu]}_{\beta}\,\hat{G}^{\alpha\sigma}\hat{G}^{\rho\tau}\hat{G}_{\sigma\tau} - 2\,\hat{G}^{\sigma}{}_{\beta}\,\delta^{[\mu}_{\beta}\,\hat{G}^{\nu]\alpha}\hat{G}^{\rho\tau}\hat{G}_{\sigma\tau}\,.
\end{aligned}
\tag{6.55}
$$

This complicated structure already hints at how difficult it may become to actually explicitly solve the master equations. Nevertheless, we can make an interesting observation here: although every term in the coefficients $T^{A\,[\mu\nu]}$ contains the tensor field $\hat{G}^{\alpha}{}_{\beta}$ at least once, there is no chance to restrict our attention to the constraint surface where $\hat{G}^{\alpha}{}_{\beta} = 0$ to simplify the calculation; for even if we initially set $\hat{G}^{\alpha}{}_{\beta} = 0$

[5]Since the expressions quickly become lengthy, one best uses an appropriate computer algebra system that can handle tensor manipulations in order to calculate the relevant coefficients. For the derivation of (6.54) and (6.55) the author used Cadabra for Linux systems by C. Peeters [50].

and $\hat{K}^{\alpha}{}_{\beta} = 0$, the non-local contribution in the Lagrange equation (4.26) would produce a non-vanishing change in the variable $\hat{G}^{\alpha}{}_{\beta}$, which drives the dynamics away from our would-be constraint surface. At least this is the case unless $\hat{G}_{\alpha\beta}$ is equal to the inverse of the variable $\hat{G}^{\alpha\beta}$; as one readily checks, this would bring us back to the case of a spacetime area metric G induced by a metric g where $\hat{G}^{\alpha\beta} = \hat{g}^{\alpha\beta}$. In other words, already a small deviation from the metric geometry underlying Maxwell theory highly complicates the calculation of the gravitational dynamics of the modified spacetime geometry.

Finally, the coefficient S can be calculated according to equation (6.5). We will, however, not display the result here since this example only serves illustrational purposes. In addition, the invariance equations for the coefficients $C_{B_1\dots B_N}$ are of some of interest for this particular example. We here display the invariance equations only for the coefficients $C_{B_1\dots B_N}$ with $N \geq 1$. The first invariance equation then reads

$$0 = 2\,\hat{G}^{\mu(\alpha}\frac{\partial C_{B_1\dots B_N}}{\partial\partial^2_{\beta\gamma)}\hat{G}^{\mu\rho}} + \hat{G}^{(\alpha}{}_{\mu}\frac{\partial C_{B_1\dots B_N}}{\partial\partial^2_{\beta\gamma)}\hat{G}^{\rho}{}_{\mu}} - \hat{G}^{\mu}{}_{\rho}\frac{\partial C_{B_1\dots B_N}}{\partial\partial^2_{(\alpha\beta}\hat{G}^{\mu}{}_{\gamma)}} - 2\,\hat{G}_{\rho\mu}\frac{\partial C_{B_1\dots B_N}}{\partial\partial^2_{(\alpha\beta}\hat{G}_{\gamma)\mu}}. \quad (6.56)$$

The second invariance identity can be written as

$$\begin{aligned} 0 \;=\; & 2\,\hat{G}^{\mu(\alpha}\frac{\partial C_{B_1\dots B_N}}{\partial\partial_{\beta)}\hat{G}^{\mu\rho}} + 4\,\partial_\nu\hat{G}^{\mu(\alpha}\frac{\partial C_{B_1\dots B_N}}{\partial\partial^2_{\beta)\nu}\hat{G}^{\mu\rho}} - \partial_\rho\hat{G}^{\mu\nu}\frac{\partial C_{B_1\dots B_N}}{\partial\partial^2_{\alpha\beta}\hat{G}^{\mu\nu}} \\ & + \hat{G}^{(\alpha}{}_{\mu}\frac{\partial C_{B_1\dots B_N}}{\partial\partial_{\beta)}\hat{G}^{\rho}{}_{\mu}} - \hat{G}^{\mu}{}_{\rho}\frac{\partial C_{B_1\dots B_N}}{\partial\partial_{(\alpha}\hat{G}^{\mu}{}_{\beta)}} + 2\,\partial_\nu\hat{G}^{(\alpha}{}_{\mu}\frac{\partial C_{B_1\dots B_N}}{\partial\partial^2_{\beta)\nu}\hat{G}^{\rho}{}_{\mu}} - 2\,\partial_\nu\hat{G}^{\mu}{}_{\rho}\frac{\partial C_{B_1\dots B_N}}{\partial\partial^2_{\nu(\alpha}\hat{G}^{\mu}{}_{\beta)}} \\ & - \partial_\nu\hat{G}^{\mu}{}_{\rho}\frac{\partial C_{B_1\dots B_N}}{\partial\partial^2_{\alpha\beta}\hat{G}^{\mu}{}_{\nu}} - 2\,\hat{G}_{\rho\mu}\frac{\partial C_{B_1\dots B_N}}{\partial\partial_{(\alpha}\hat{G}_{\beta)\mu}} - 4\,\partial_\nu\hat{G}_{\mu\rho}\frac{\partial C_{B_1\dots B_N}}{\partial\partial^2_{\nu(\alpha}\hat{G}_{\beta)\mu}} - \partial_\rho\hat{G}_{\mu\nu}\frac{\partial C_{B_1\dots B_N}}{\partial\partial^2_{\alpha\beta}\hat{G}_{\mu\nu}}\,, \end{aligned} \quad (6.57)$$

and we only display the contracted form of the last invariance identity:

$$\begin{aligned} -(3+n)\,C_{B_1\dots B_N} \;=\; & 2\,\hat{G}^{\mu\nu}\frac{\partial C_{B_1\dots B_N}}{\partial\hat{G}^{\mu\nu}} - 2\,\hat{G}_{\mu\nu}\frac{\partial C_{B_1\dots B_N}}{\partial\hat{G}_{\mu\nu}} \\ & + \partial_\rho\hat{G}^{\mu\nu}\frac{\partial C_{B_1\dots B_N}}{\partial\partial_\rho\hat{G}^{\mu\nu}} - \partial_\rho\hat{G}^{\mu}{}_{\nu}\frac{\partial C_{B_1\dots B_N}}{\partial\partial_\rho\hat{G}^{\mu}{}_{\nu}} - 3\,\partial_\rho\hat{G}_{\mu\nu}\frac{\partial C_{B_1\dots B_N}}{\partial\partial_\rho\hat{G}_{\mu\nu}} \\ & - 2\,\partial^2_{\rho\sigma}\hat{G}^{\mu}{}_{\nu}\frac{\partial C_{B_1\dots B_N}}{\partial\partial^2_{\rho\sigma}\hat{G}^{\mu}{}_{\nu}} - 4\,\partial^2_{\rho\sigma}\hat{G}_{\mu\nu}\frac{\partial C_{B_1\dots B_N}}{\partial\partial^2_{\rho\sigma}\hat{G}_{\mu\nu}}\,, \end{aligned} \quad (6.58)$$

where n is the difference of the total number of subscript indices and the total number of superscript indices in the coefficients $C_{B_1\dots B_N}$. The invariance equations for the potential C are, of course, even more complicated, because the latter can also depend on the third partial derivatives of the fields $\hat{G}^A$. Fortunately, the hypersurface field $\hat{G}^{\alpha\beta}$ can indeed be formally employed as a hypersurface metric, and, thus, one can immediately solve the invariance equations for all coefficients $C_{B_1\dots B_N}$ along the lines laid out in sections 5.2.3 and 5.3.2. With this observation, and the explicitly calculated coefficients above, one now has to solve the master equations in order to obtain all gravitational dynamics for the tensorial spacetime $(M, G, S_{\text{GLED}}[G, A])$. The explicit execution of this calculation appears to be a formidable—but given the physical relevance of general linear electrodynamics: a most worthwhile—problem.

But that the master equations indeed can be solved in a non-trivial example (different from general relativity) is demonstrated in the following section.

6.4 Example 3: From modified Dirac matter to its supporting gravity theories

In order to see the machinery we developed over the last chapters working at full capacity, we will now go through a non-trivial extension of metric geometry and find gravitational dynamics for it such that a modification of ordinary Dirac fermions can propagate on it. We will, in fact, see that the corresponding gravitational dynamics will have some interesting properties. It turns out that some of the degrees of freedom of the geometry do not propagate but are still fully determined by the genuine dynamics of the remaining degrees of freedom. We also further analyse the dynamics of the geometry in a particular example of our general solution. It should be kept in mind that we are *not* proposing either this particular geometry nor this particular type of matter equations as a model for any observable physics. This example simply wishes to illustrate the full derivation of a gravity action from a given matter action that is *not* of standard model type.

6.4.1 Kinematics of modified Dirac matter

In this example, we consider a modification of Dirac fermions Ψ with field equations[6]

$$(i\gamma^a + W^a)D_a\Psi = 0 \tag{6.59}$$

on a four-dimensional smooth manifold equipped with a geometry (g, W) consisting of a spacetme metric g (of a so far arbitrary but fixed signature) together with a spacetime vector field W. The spacetime γ-matrices $\gamma^a = \gamma^I e_I^a$ are constructed with the help of globally smooth frame fields e_I, which satisfy $g^{ab} = \eta^{IJ} e_I^a e_J^b$ and the flat spacetime γ-matrices γ^I, which satisfy the Clifford algebra $\{\gamma^I, \gamma^J\} = 2\eta^{IJ}$. Here,

[6]This time, we start directly from field equations rather than a matter action. In the general theory, we chose to start from an action for pure convenience in order to be sure to generate tensorial field equations for the matter fields, by variation. If the field equations are constructed in a way such they are guaranteed to transform as tensors, as is the case here, one can of course also directly start from the field equations instead of from an action functional.

η^{IJ} is the constant metric on $\mathbb{R}^4$ with the same signature as g. The spin covariant derivative D_a is induced from the torsion-free spin connection

$$^{S}\Gamma^{I}_{aJ} = -e^{b}_{J}(\partial_a \theta^{I}_{b} - \Gamma^{c}_{ab}\,, \theta^{I}_{c}) \qquad (6.60)$$

where Γ^c_{ab} are the Christoffel symbols of the metric g^{ab}, and θ^I_b denote the coframe fields dual to the frame fields e^a_I. The spin connection is antisymmetric with respect to η^{IJ}, and

$$D_a = \partial_a - \frac{i}{4}{}^{S}\Gamma^{I}_{aJ}\,\eta_{IK}\,[\gamma^K, \gamma^J] \qquad (6.61)$$

if the covariant derivative acts on spinors Ψ. Here an in the following, we will suppress all spinor indices.

First, we analyse the kinematics of the field equations (6.59) in order to determine all algebraic constraints imposed on the geometric fields g and W by requiring that the field equations (6.59) be predictive and quantizable. The principal polynomial of the matter equations (6.59) can either be calculated according to definition (2.2), or, in simpler fashion, by acting on the equations of motion with the differential operator $(i\gamma^J e^b_J - W^b)D_b$ from the left. We follow the second approach and get

$$-(\gamma^J\gamma^I e^b_J e^a_I + W^aW^b - i\gamma^J e^b_J W^a + i\gamma^I e^a_I W^b)D_bD_a\Psi + i\gamma^J e^b_J D_b W^a D_a\Psi = 0\,. \quad (6.62)$$

From the highest order derivative terms of this equation, one simply reads off the principal tensor field

$$P^{ab} = (g^{ab} + W^aW^b)\,, \qquad (6.63)$$

using the Clifford algebra relation $\{\gamma^I, \gamma^J\} = 2\eta^{IJ}$ and the fact that partial derivatives commute. This particular principal tensor field is also frequently used as a starting point in so-called tensor-vector theories of gravity [6]. In most of these theories, however, the principal polynomial is generated by coupling some matter field directly to the combination $g^{ab} + W^aW^b$. The Dirac field we consider, on the other hand truly feels both constituents of the geometry, the metric g and the vector field W.

In order for the matter field equations to admit an initial value formulation, we need the prinicipal tensor field to be hyperbolic. For the second rank principal tensor field (6.63), hyperbolicity simply amounts to the algebraic requirement that the matrix $g^{ab} + W^aW^b$ has mainly minus Lorentzian signature at every point of the manifold. However, this does by no means imply that the metric g itself has to be of Lorentzian signature. In fact, we can distinguish the following two cases: either the metric g is Lorentzian and the vector field W is timelike, null or of spacelike length $-g(W, W) < 1$ with respect to g, or the metric has minus Riemannian signature $(- - - -)$ and the vector field has length $-g(W, W) > 1$. Interestingly, the two cases differ in the way hyperbolicity is encoded in the geometry. In the first case, hyperbolicity is ensured by the metric, whereas in the second case, it is the vector field which renders the combination $g^{ab} + W^a W^b$ hyperbolic.

Since, in this case, the principal polynomial is of second degree, it is simple to calculate its dual polynomial $P^{\#}(x, v)$. It is given by the inverse of the matrix $g^{ab} + W^a W^b$, and, thus, we have that

$$P^{\#}(x,v) = \left(g_{ab} - \frac{1}{1 + W^r W^s g_{rs}} W^m W^n g_{ma} g_{nb} \right) v^a v^b \,, \tag{6.64}$$

The hyperbolicity of the dual polynomial is guaranteed by the hyperbolicity of the principal polynomial (6.63) so that, here, bi-hyperbolicity does not enforce further algebraic constraints on the values of g and W beyond what is already enforced by hyperbolicity. Now all constructions, from introducing a time orientation on the manifold M to the definition of observer frames, go through exactly as for metric geometry, as discussed in section 6.2, but with the inverse metric g^{ab} being replaced there by the tensor field (6.63). Thus, we do not need to repeat the constructions here, but continue *mutatis mutandis* with the derivation of the master equations for the tensorial spacetime $(M, g, W,$ Dirac$)$.

6.4.2 Setting up the geometric phase space

The next step in the construction of the master equations is to decide on a parametrization of the inital data for the hypersurface geometry representing the spacetime geometry (g, W) on a suitable initial data hypersurface $X(\Sigma)$. Suitable initial data hypersurfaces have conormals laying in the hyperbolicity cone of the principal polynomial $P(x, k)$ constructed from the principal tensor field (6.63). The construction of the corresponding hypersurface cotangent space frames $\{n, \epsilon^\alpha\}$ and tangent space frames $\{T, e_\alpha\}$ at every point of $X(\Sigma)$ reduce here to the standard procedure. We use these frames in order to project the tensor field g and W on the hypersurface $X(\Sigma)$ and define the quantities

$$g^{\alpha\beta} := g^{ab} \epsilon^\alpha_a \epsilon^\beta_b \,, \quad g^\alpha := g^{ab} \epsilon^\alpha_a n_b \,, \quad g := g^{ab} n_a n_b \,, \tag{6.65}$$

$$W^\alpha := W^a \epsilon^\alpha_a \quad \text{and} \quad W := W^a n_a \,. \tag{6.66}$$

However, not all of these hypersurface tensors can be independent since the frame conditions $P(n) = 1$ and $T^a \epsilon^\alpha_a = 0$ can be used to express W and W^α in terms of the projections g and g^α. Thus, the hypersurface tensor fields g, g^α and $g^{\alpha\beta}$ already constitute a possible parametrization of the spacetime geometry (g, W). One can check that the completeness relations

$$g^{ab} = g\, T^a T^b + 2g^\alpha\, T^{(a}\, e^{b)}_\alpha + g^{\alpha\beta}\, e^a_\alpha e^b_\beta \quad \text{and} \tag{6.67}$$

$$W^a = \pm(1-g)^{1/2}\, T^a \mp \frac{1}{(1-g)^{1/2}}\, g^\alpha\, e^a_\alpha \tag{6.68}$$

allow for a reconstruction of the spacetime geometry on the hypersurface $X(\Sigma)$. In principle, one could now press on and try to determine the master equations for the fields $(g, g^\alpha, g^{\alpha\beta})$ by calculating all the necessary coefficients. It is, however, not difficult to foresee that one runs into unnecessary technical difficulties when trying to solve the master equations in this form; for a straightforward calculation shows that acting with the normal deformation operator $\mathcal{H}(N)$ on the geometric variables $(g, g^\alpha, g^{\alpha\beta})$ yields

$$\mathcal{H}(N)\, g^{\alpha\beta}(z) = N(z)(\mathcal{L}_T g)^{ab}\epsilon^\alpha_a \epsilon^\beta_b + \partial_\gamma N(z)\left[2g^{(\alpha}g^{\beta)\gamma} + \frac{2}{1-g} g^\alpha g^\beta g^\gamma\right], \tag{6.69}$$

$$\mathcal{H}(N)\, g^\alpha(z) = N(z)(\mathcal{L}_T g)^{ab}\epsilon^\alpha_a n_b + \partial_\gamma N(z)\left[\frac{g}{1-g}\, g^\alpha g^\gamma - (1-g) g^{\alpha\gamma}\right], \tag{6.70}$$

$$\mathcal{H}(N)\, g(z) = N(z)(\mathcal{L}_T g)^{ab} n_a n_b - 2g^\gamma \partial_\gamma N(z)\,. \tag{6.71}$$

Thus, all these variables each contribute a term to the non-local part of the superhamiltonian in the canonical formulation of the dynamics, and this leads to complicated couplings in the master equations. A more advantageous choice of configuration variables helps to attenuate these difficulties. This is an example of what we discussed at the end of the previous chapter: we know that, for degree two principal polynomials, the projections $P^{\alpha\beta}$ of the principal tensor field P^{ab} do not aquire a term that is non-local in the lapse function when acting on them with the normal deformation operator. Moreover, we can build, from the spacetime geometry (g, W), the spacetime scalar field $\phi = g(W, W)$, whose projection on the hypersurface $X(\Sigma)$ also does not aquire a non-local contribution in the lapse function under normal deformations. Thus we can try to perform a change of the configuration variables from the set $(g, g^\alpha, g^{\alpha\beta})$ to new configuration variables $(P^{\alpha\beta}, \phi, g_\alpha)$. Indeed, this can be done by virtue of the transformations

$$P^{\alpha\beta} = g^{\alpha\beta} + \frac{1}{1-g} g^\alpha g^\beta\,, \tag{6.72}$$

$$\phi = 1 - g + \frac{g^\alpha g^\beta g_{\alpha\beta}}{1 - g - g^\alpha g^\beta g_{\alpha\beta}} \quad \text{and} \tag{6.73}$$

$$g_\alpha = -\frac{1}{1-g} P_{\alpha\gamma}\, g^\gamma\,, \tag{6.74}$$

where the form of the variable g_α, which replaces the old variable g^α, has been chosen purely for convenience. This transformation is invertible, and we can recover the old configuration variables $(g, g^\alpha, g^{\alpha\beta})$ with the help of the inverse transoformations

$$g^{\alpha\beta} = P^{\alpha\beta} - \frac{\phi}{1 + P^{\rho\sigma} g_\rho g_\sigma} P^{\alpha\gamma} g_\gamma P^{\beta\delta} g_\delta \,, \tag{6.75}$$

$$g^\alpha = -\frac{\phi}{1 + P^{\gamma\delta} g_\gamma g_\delta} P^{\alpha\rho} g_\rho \,, \tag{6.76}$$

$$g = 1 - \frac{1}{2}\phi - \sqrt{\frac{\phi^2}{4} - \phi^2 \frac{P^{\alpha\beta} g_\alpha g_\beta}{1 + P^{\gamma\delta} g_\gamma g_\delta}} \,. \tag{6.77}$$

The geometric phase space that represents the spacetime geometry (g, W) may, thus, alternatively be spanned by the tensor fields

$$\hat{P}^{\alpha\beta}, \quad \hat{\phi}, \quad \hat{g}_\alpha \tag{6.78}$$

and a set of canonically conjugate momenta

$$\hat{\pi}_{\alpha\beta}, \quad \hat{\pi}, \quad \hat{\pi}^\alpha \,, \tag{6.79}$$

which are all tensor densities of weight one. Hence, the capital multi-index A in the master equations ranges over ${}^A = ({}^{\alpha\beta}\,,\, {}^0\,,\, {}_\alpha)$ if it is a superscript and ${}_A = ({}_{\alpha\beta}\,,\, {}_0\,,\, {}^\alpha)$ if it is a subscript index. In order to avoid confusion, we use the '0' in the multi-index for quantities related to the scalar field $\hat{\phi}$. We can now determine the master equations by calculating all relevant coefficients.

6.4.3 Setting up the master equations

In order to calculate the coefficients for the master equations in the variables $\hat{G}^A = (\hat{P}^{\alpha\beta}, \hat{\phi}, \hat{g}_\alpha)$, one starts with the supermomentum $\hat{\mathcal{D}}(\vec{N})$ and the non-local part of the superhamiltonian $\hat{\mathcal{H}}_{\text{non-local}}(N)$. After an integration by parts, the supermomentum (4.16) takes the form

$$\hat{\mathcal{D}}(\vec{N}) = \int_\Sigma dy\, N^\gamma \left[\hat{\pi} \partial_\gamma \hat{\phi} + \hat{\pi}^\alpha \partial_\gamma \hat{g}_\alpha - \partial_\alpha (\hat{g}_\gamma \hat{\pi}^\alpha) + \hat{\pi}_{\alpha\beta} \partial_\gamma \hat{P}^{\alpha\beta} + 2\partial_\alpha (\hat{\pi}_{\beta\gamma} \hat{P}^{\alpha\beta}) \right] . \tag{6.80}$$

Bringing the right hand side of the algebra equation (4.11) into the general form (6.1), we read off the coefficients

$$U^{A\,\mu\nu} = \begin{cases} -2\hat{P}^{\mu(\alpha} \hat{P}^{\beta)\nu} \,, & \text{for} \quad {}^A = {}^{\alpha\beta} \\ 0 \,, & \text{for} \quad {}^A = {}^0 \\ \delta^\mu_\alpha \hat{g}^\nu \,, & \text{for} \quad {}^A = {}_\alpha \,, \end{cases} \tag{6.81}$$

where we introduced the function $\hat{g}^{\alpha} = \hat{P}^{\alpha\beta}\hat{g}_{\beta}$, which is not to be confused with the original projection of the spacetime metric in normal and tangential direction. Here, and in the following, we will use the hypersurface metric $\hat{P}^{\alpha\beta}$ to raise and lower indices. The supermomentum also determines the coefficients $V^{A\,\nu}$, which read

$$V^{A\,\nu} = \begin{cases} \hat{P}^{\nu\gamma}\hat{P}^{\alpha\beta}{}_{,\gamma} + 2\hat{P}^{\nu(\alpha}\hat{P}^{\beta)\gamma}{}_{,\gamma}\,, & \text{for} \quad {}^{A} = {}^{\alpha\beta} \\ \hat{P}^{\nu\gamma}\hat{\phi}_{,\gamma}\,, & \text{for} \quad {}^{A} = {}^{0} \\ \hat{P}^{\gamma\nu}\hat{g}_{\alpha,\gamma} - \hat{P}^{\nu\gamma}\hat{g}_{\gamma,\alpha}\,, & \text{for} \quad {}^{A} = {}_{\alpha}\,, \end{cases} \tag{6.82}$$

and where, for brevity, we denote partial derivatives by a comma.

Because of our now more advantageous choice for the configuration variables, the non-local part of the superhamiltonian assumes a particularly simple form; we already mentioned that the variables $\hat{P}^{\alpha\beta}$ and $\hat{\phi}$ do not contribute to the non-local part of the superhamiltonian. The contribution of the variable $\hat{g}_{\alpha}$ to the non-local part of the superhamiltonian can be obtained by combining the coordinate transformation (6.74) with the expressions (6.70) and (6.71) in order to calculate the action of the normal deformation operator on g_{α},

$$\mathcal{H}(N)\, g_{\alpha}(z) = N(z)K_{\alpha}(z) + \partial_{\gamma}N(z)\left[\delta^{\gamma}_{\alpha} + g_{\alpha}g_{\beta}P^{\beta\gamma}\right](z)\,. \tag{6.83}$$

Hence, the non-local part of the superhamiltonian takes the form

$$\hat{\mathcal{H}}_{\text{non-local}} = -\partial_{\gamma}(M_{\alpha}{}^{\gamma}\hat{\pi}^{\alpha}) \quad \text{with} \quad M_{\alpha}{}^{\gamma} = \delta^{\gamma}_{\alpha} + \hat{g}_{\alpha}\hat{g}^{\gamma}\,, \tag{6.84}$$

from which we immediately get the coefficients

$$Q_{B\,\alpha}{}^{\mu} = \begin{cases} -\,\hat{g}_{\alpha}\hat{g}_{(\beta}\delta^{\mu}_{\gamma)}\,, & \text{for} \quad {}_{B} = {}_{\beta\gamma} \\ 0\,, & \text{for} \quad {}_{B} = {}_{0} \\ -\delta^{\beta}_{\alpha}\hat{g}^{\mu} - \hat{g}_{\alpha}\hat{P}^{\mu\beta}\,, & \text{for} \quad {}_{B} = {}^{\beta}\,. \end{cases} \tag{6.85}$$

A direct calculation now reveals that $T^{A\,[\mu\nu]} = 0$. Finally, we can calculate the coefficients $S^{A\,\gamma}$ from the general expression (6.5):

$$S^{\alpha\beta\,\mu} = -\hat{P}^{\mu\gamma}\hat{P}^{\alpha\beta}{}_{,\gamma} + 2\hat{P}^{\gamma(\alpha}\hat{P}^{\beta)\mu}{}_{,\gamma} \tag{6.86}$$

$$S^{0\,\mu} = -\hat{P}^{\mu\gamma}\hat{\phi}_{,\gamma} \tag{6.87}$$

$$S_{\alpha}{}^{\mu} = 2P^{\mu\nu}g_{[\nu,\alpha]} - g^{\mu}{}_{,\alpha}. \tag{6.88}$$

We can now display the invariance equations for the coefficients $C_{B_1\ldots B_N}$ for

$N \geq 1$ here. The first invariance equation reads

$$0 = 2\hat{P}^{\mu(\alpha|} \frac{\partial C_{B_1 \dots B_N}}{\partial \hat{P}^{\mu\rho}{}_{,|\beta\gamma)}} - \hat{g}_\rho \frac{\partial C_{B_1 \dots B_N}}{\partial \hat{g}_{(\alpha,\beta\gamma)}} \,, \tag{6.89}$$

while the second one takes the form

$$\begin{aligned} 0 = \quad & 2\,\hat{P}^{\mu(\alpha|} \frac{\partial C_{B_1 \dots B_N}}{\partial \hat{P}^{\mu\rho}{}_{,|\beta)}} + 4\hat{P}^{\mu(\alpha|}{}_{,\nu} \frac{\partial C_{B_1 \dots B_N}}{\partial \hat{P}^{\mu\rho}{}_{,|\beta)\nu}} - \hat{P}^{\mu\nu}{}_{,\rho} \frac{\partial C_{B_1 \dots B_N}}{\partial \hat{P}^{\mu\nu}{}_{,\alpha\beta}} \\ & -\hat{g}_\rho \frac{\partial C_{B_1 \dots B_N}}{\partial \hat{g}_{(\alpha,\beta)}} - \hat{g}_{\mu,\rho} \frac{\partial C_{B_1 \dots B_N}}{\partial \hat{g}_{\mu,\alpha\beta}} - 2\,\hat{g}_{\rho,\mu} \frac{\partial C_{B_1 \dots B_N}}{\partial \hat{g}_{(\alpha,\beta)\mu}} - \hat{\phi}_{,\rho} \frac{\partial C_{B_1 \dots B_N}}{\partial \hat{\phi}_{,\alpha\beta}} \,. \end{aligned} \tag{6.90}$$

It will turn out that the invariance equations of the potential C take exactly the same form in this case. This is because one of the first results we shall obtain when solving the master equations is that the potential C can depend at most on the second partial derivatives of the fields.

With the explicit form of all the above coefficients, we can now solve the master equations to find the most general gravitational dynamics for the tensorial spacetime $(M, g, W,$ Dirac) in four dimensions.

6.4.4 Solution of the master equations

In this section, we will derive a particular solution to the master equations with the help of the general methods we discussed in the previous chapter. More precisely, the most general solution to the master equation under the assumption that the Lagrangian L does not contain terms that involve the velocities K_α is given by

$$\begin{aligned} L \;\; = \;\; & \sqrt{-\det \hat{P}_{\alpha\beta}} \Big[2 \frac{d^2 a_1(\hat{\phi})}{d\hat{\phi}^2} \hat{K}^2 - \frac{1}{2} \frac{d a_1(\hat{\phi})}{d\hat{\phi}}\, \hat{K}\, \hat{P}_{\alpha\beta} \hat{K}^{\alpha\beta} \\ & -a_1(\hat{\phi})\, C_{\alpha\beta\gamma\delta} \hat{K}^{\alpha\beta} \hat{K}^{\gamma\delta} + a_1(\hat{\phi}) R - 2 \frac{d a_1(\hat{\phi})}{d\hat{\phi}} \hat{P}^{\alpha\beta} \nabla_\alpha \nabla_\beta \hat{\phi} \\ & +\lambda (R_{\rho\sigma} - \frac{1}{2} \hat{P}_{\rho\sigma}\, R) \hat{K}^{\rho\sigma} + a_2(\hat{\phi}) \hat{P}_{\rho\sigma} \hat{K}^{\rho\sigma} - 2 \frac{d a_2(\hat{\phi})}{d\hat{\phi}}\, \hat{K} \\ & + \sum_{N=1}^{\infty} C_N \hat{K}^N + C_0(\hat{\phi}, \nabla^\alpha \hat{\phi} \nabla_\alpha \hat{\phi}, \hat{g}^\alpha \nabla_\alpha \hat{\phi}, \hat{g}_\alpha \hat{g}^\alpha) \Big] \,, \end{aligned} \tag{6.91}$$

with free functions $a_1(\hat{\phi})$ and $a_2(\hat{\phi})$ and an undetermined constant λ. The tensors R and $R_{\alpha\beta}$ denote the Ricci scalar and Ricci tensor of $\hat{P}^{\alpha\beta}$ and ∇ indicates the covariant derivative with respect to the Levi-Civita connection of $\hat{P}^{\alpha\beta}$. The coefficient $C_{\alpha\beta\gamma\delta}$ is given by

$$C_{\alpha\beta\gamma\delta} = \frac{1}{8} \left[\hat{P}_{\alpha\gamma} \hat{P}_{\beta\delta} + \hat{P}_{\beta\gamma} \hat{P}_{\alpha\delta} - 2 \hat{P}_{\alpha\beta} \hat{P}_{\gamma\delta} \right] \,. \tag{6.92}$$

Without further assumptions, also the function C_0 is freely specifiable and depends on the scalars $\hat{\phi}$, $\nabla^\alpha\hat{\phi}\nabla_\alpha\hat{\phi}$, $\hat{g}^\alpha\nabla_\alpha\hat{\phi}$ and $\hat{g}_\alpha\hat{g}^\alpha$. The scalar coefficients C_N are determined from C_0

$$C_1 = \frac{1}{\nabla^\alpha\hat{\phi}\nabla_\alpha\hat{\phi}}\frac{\partial C_0}{\partial \hat{g}_\rho}M_\rho{}^\beta\nabla_\beta\hat{\phi} \tag{6.93}$$

and the recursion formula

$$C_{N+1} = \frac{1}{\nabla_\rho\hat{\phi}\nabla^\rho\hat{\phi}}\frac{N!}{(N+1)!}\left[\frac{\partial C_N}{\partial \hat{g}_\gamma}M_\gamma{}^\beta\nabla_\beta\hat{\phi} + \frac{\partial C_{N-1}}{\partial \nabla_\beta\hat{\phi}}\nabla_\beta\hat{\phi}\right]. \tag{6.94}$$

for all coefficients C_{N+1} with $N \geq 1$.

The proof that the master equations yield the particular gravitational Lagrangian (6.91) proceeds in three steps. The first step involves the solution of the invariance equations by trading the partial derivatives of the fields for covariant quantities as demonstrated in the previous chapter. In a second step, one rewrites the remaining equations in terms of the covariant variables, which will almost instantly lead to the remarkable results that (i) the potential C can contain at most first derivatives of the variable $\hat{g}_\alpha$; (ii) that the potential C can only depend on at most second derivatives of the fields $\hat{P}^{\alpha\beta}$ and $\hat{\phi}$ and (iii) that the variable $\hat{g}_\alpha$ can, in fact, appear in the coefficients only in its undifferentiated form. In a third and last step, one then solves a reduced form of the master equations containing only the remaining coefficients.

We now execute these three steps and start by performing a change of coordinates by trading in the partial derivatives of the fields $(\hat{\phi}, \hat{g}_\alpha)$ for covariant derivatives with respect to the Levi-Civita connection $\Gamma^\alpha_{\beta\gamma}$ of the metric $\hat{P}^{\alpha\beta}$. At the same time, we substitute the variables $\hat{P}^{\alpha\beta}{}_{,\gamma\delta}$ by the Riemann tensor $R_{\mu\nu\rho\sigma}$ and the non-tensorial quantity $S_{\mu\nu\rho\sigma}$ introduced in section 5.2.3. The partial derivatives $P^{\alpha\beta}{}_{,\gamma}$ will be rewritten in terms of the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$. We will denote covariant derivatives either by a semicolon or by the symbol ∇_α. The transformation formulae into the new coordinates $\Gamma^\alpha_{\beta\gamma}$, $R_{\alpha\beta\gamma\delta}$ and $S_{\alpha\beta\gamma\delta}$ are given by equations (5.43)-(5.45) and we adopt the notation of the previous chapter. The transformation formulae for the remaining partial derivatives of the fields $\hat{\phi}$ and $\hat{g}_\alpha$ are given by

$$\begin{aligned}
\hat{\phi}_{;\rho} &= \hat{\phi}_{,\rho}\,, && (6.95)\\
\hat{\phi}_{;\rho\sigma} &= \hat{\phi}_{,\rho\sigma} - \Gamma^\mu_{\rho\sigma}\hat{\phi}_{,\mu}\,, && (6.96)\\
\hat{g}_{\alpha;\beta} &= \hat{g}_{\alpha,\beta} - \hat{g}_\mu\Gamma^\mu_{\alpha\beta}\,, && (6.97)\\
\hat{g}_{\alpha;(\beta\gamma)} &= \hat{g}_{\alpha,\beta\gamma} - 2\,\hat{g}_{\mu,(\gamma}\Gamma^\mu_{\beta)\alpha} - \hat{g}_{\alpha,\mu}\Gamma^\mu_{\beta\gamma} && \\
&\quad -\hat{g}_\mu\left(\Gamma^\mu_{\alpha(\beta,\gamma)} - \Gamma^\mu_{\alpha\nu}\Gamma^\nu_{\beta\gamma} - \Gamma^\mu_{\nu(\beta}\Gamma^\nu_{\gamma)\alpha}\right). && (6.98)
\end{aligned}$$

In order to deal with the potential C, which could in principle also depend on the third partial derivatives of the fields $\hat{G}^A$, we would also need to rewrite the latter into covariant form as discussed in section 5.3.2, if it did not turn out that it is sufficient to only rewrite the third partial derivatives of the field $\hat{g}_\alpha$ in covariant form. The corresponding transformation formula is given by

$$\hat{g}_{\alpha;(\beta\gamma\delta)} \;=\; \hat{g}_{\alpha,\beta\gamma\delta} + \hat{g}_{\mu,\nu\lambda}(-3\delta^\lambda_{(\delta}\delta^\nu_\gamma\Gamma^\mu_{\beta)\alpha} - 3\delta^\mu_\alpha\delta^\nu_{(\beta}\Gamma^\lambda_{\gamma\delta)}) + \text{lower order terms}\,, \tag{6.99}$$

where, as we will see, it will not be necessary to write out all terms of lower derivative order in $\hat{g}_\alpha$. In the following, we will not always write the symmetrization brackets, but it is understood that we always mean the symmetrized covariant derivatives of the field $\hat{g}_\alpha$. The inverse transformations for the partial derivatives of $\hat{P}^{\alpha\beta}$ are given by equations (5.51) and (5.52), and the partial derivatives of the variables $\hat{\phi}$ and $\hat{g}_\alpha$ are recovered from

$$\hat{\phi}_{,\rho\sigma} \;=\; \hat{\phi}_{;\rho\sigma} + \Gamma^\mu_{\rho\sigma}\hat{\phi}_{;\mu}\,, \tag{6.100}$$

$$\hat{g}_{\alpha,\beta} \;=\; \hat{g}_{\alpha;\beta} + \Gamma^\mu_{\alpha\beta}\hat{g}_\mu \quad \text{and} \tag{6.101}$$

$$\begin{aligned}\hat{g}_{\alpha,\beta\gamma} \;=\; & \hat{g}_{\alpha;(\beta\gamma)} + \hat{g}_{\mu;\nu}\left[2\Gamma^\nu_{\alpha(\gamma}\delta^\mu_{\beta)} + \Gamma^\mu_{\gamma\beta}\delta^\nu_\alpha\right] \\ & + \frac{1}{6}\hat{g}^\mu\Big[S_{\mu\alpha\beta\gamma} - R_{\alpha\beta\mu\gamma} - R_{\alpha\gamma\mu\beta} \\ & \qquad -2\hat{P}_{\rho\sigma}(\Gamma^\rho_{\beta\gamma}\Gamma^\sigma_{\alpha\mu} + \Gamma^\rho_{\mu\beta}\Gamma^\sigma_{\gamma\alpha} + \Gamma^\rho_{\mu\gamma}\Gamma^\sigma_{\beta\gamma})\Big]\,.\end{aligned} \tag{6.102}$$

In order to obtain the third partial derivatives of the fields $\hat{g}_\alpha$, we only need to solve equation (6.99) for them. The various derivatives of the connection coefficients Γ do not need to be rewritten, but the identity

$$\Gamma^\alpha_{\beta\mu,\nu} = \Gamma^\alpha_{(\beta\mu,\nu)} - \frac{2}{3}R^\alpha{}_{(\beta\mu)\nu} + \frac{2}{3}\Gamma^\alpha_{\rho(\beta}\Gamma^\rho_{\mu)\nu} - \frac{2}{3}\Gamma^\alpha_{\nu\rho}\Gamma^\rho_{\beta\mu} \tag{6.103}$$

involving the Riemann tensor $R^\alpha{}_{\beta\gamma\delta}$ of $\hat{P}^{\alpha\beta}$ and the non-tensorial variables $\Gamma^\alpha_{(\beta\mu,\nu)}$ and $\Gamma^\alpha_{\beta\gamma}$ will be useful.

We can now start the second step and rewrite the master equations in covariant form. We begin with the master equations (5.58)-(5.60) containing the potential C and the first order coefficient C_A. Equation (5.60) can be straightforwardly rewritten covariantly, but the chain rule in the first term in conjunction with the transformation formula (6.99), the derivative of the coefficient $M_\alpha{}^\beta$ in the second term and the divergence[7] in the last term all produce terms that are proportional to the variable

[7]One has to be careful when rewriting the divergence of terms that contain the coefficients $C_{B_1\ldots B_N}$ (for $N \geq 0$). Because the latter are tensor densities of weight one, so are their derivatives with respect to the variables $\partial^2\hat{G}^A$, for $N \geq 1$, and $\partial^3\hat{G}^A$ for the potential C. This requires an additional term proportional to $\Gamma^\gamma_{\gamma\mu}$ when taking the covariant derivative of such terms.

$\Gamma^{\epsilon}_{\kappa\lambda}$. Since none of the rewritten terms can depend explicitly on this variable in the new covariant coordinates, we must conclude that

$$0 = 2\frac{\partial C}{\partial \hat{g}_{\rho;\gamma\mu(\beta_1|}} \, M_{\rho}{}^{|\beta_2}\delta^{\beta_3)}_{\epsilon}\delta^{\kappa}_{\gamma}\delta^{\lambda}_{\mu} - 2\frac{\partial C}{\partial \hat{g}_{\rho;\gamma(\beta_1\beta_2|}} \, \delta^{|\beta_3)}_{\epsilon} \, M_{\rho}{}^{\nu}\delta^{(\kappa}_{\nu}\delta^{\lambda)}_{\gamma} \, . \tag{6.104}$$

Contracting the indices ϵ and κ then leads to the equation

$$0 = \frac{\partial C}{\partial \hat{g}_{\rho;\lambda(\beta_1\beta_2|}} \, M_{\rho}{}^{|\beta_3)} - \frac{\partial C}{\partial \hat{g}_{\rho;\beta_1\beta_2\beta_3}} \, M_{\rho}{}^{\lambda} \, . \tag{6.105}$$

The same logic can now be applied to equation (5.59). This time, however, rewriting this equation using the chain rule and (6.103) produces terms which are purely covariant and terms that are proportional to the non-covariant variables $\Gamma^{\alpha}_{(\beta\mu,\nu)}$ as well as terms that are quadratic in $\Gamma^{\alpha}_{\beta\gamma}$. Again the latter must vanish individually as we have described in section 5.3.2. Carefully extracting all information that can be deduced from the vanishing of these terms one finds that

$$0 = \frac{\partial C}{\partial \hat{g}_{\rho;\lambda\kappa(\beta_1|}} \, M_{\rho}{}^{|\beta_2)} \, . \tag{6.106}$$

Together with equation (6.105) and the fact that the coefficient $M_{\rho}{}^{\alpha}$ is invertible, this immediately implies that the potential cannot depend on $\hat{g}_{\rho;\alpha\beta\gamma}$. Reducing the master equations (5.59) and (5.60) accordingly, one may repeat these steps to conclude that the potential C also must not depend on $\hat{g}_{\rho;\alpha\beta}$.

We may now use our assumption that the Lagrangian we look for does not contain any of the velocities $\hat{K}_{\alpha}$ belonging to the variable $\hat{g}_{\alpha}$. In particular we then have that

$$C^{\mu} = 0 \tag{6.107}$$

This immediately implies that, because of equation (5.29) and (5.55), the potential C cannot depend on any of the third derivatives of the fields $\hat{G}^A$. Hence, from here on, we can treat the potential C on the same footing as all other coefficients. We anticipated this result when discussing the invariance equations for the geometry at hand. Moreover, our assumption also requires

$$C^{\mu}{}_{B_1 \dots B_N} = 0 \quad \text{for} \quad N \geq 1. \tag{6.108}$$

In other words, all coefficients $C_{B_1 \dots B_N}$ for which at least one of the capital indices takes the value ‘$^{\alpha}$’ vanish.

Keeping this in mind, we can switch back to the master equations as expressed in the old coordinates for a moment, and show that the remaining coefficients $C_{B_1 \dots B_N}$ with ${}_{B_i} = ({}_{\alpha\beta}, {}_{0})$ and the potential C cannot depend on the first and second partial derivatives of the variable $\hat{g}_\alpha$ at all. Setting ${}_{B_N} = {}^{\rho}$ in the symmetry condition (5.29), we learn that none of the coefficients $C_{B_1 \dots B_N}$ (for $N \geq 1$) can depend on $\hat{g}_{\alpha,\beta\gamma}$. For the potential C, we already concluded this from the master equations (5.59) and (5.60). Thus, the second partial derivatives of $\hat{g}_\alpha$ cannot appear in any of the coefficients. The same holds true for the first partial derivatives $\hat{g}_{\alpha,\beta}$. This can be seen from equation (5.26) and (5.57) setting ${}_{B_1} = {}^{\rho}$, which yields

$$\frac{\partial C_{B_2 \dots B_N}}{\partial \hat{g}_{\rho,\alpha}} = 0 \quad \text{for} \quad N \geq 1\,. \tag{6.109}$$

Finally, we can even show that coefficients $C_{B_1 \dots B_N}$ for which at least one of the capital indices is the symmetric pair $\alpha\beta$, cannot depend on the variable $\hat{g}_\alpha$ at all. Writing out the divergence in equations (5.26) and (5.57), and using the fact that now nothing in both equations depends on $\hat{g}_{\alpha,\gamma}$, we obtain

$$\frac{\partial^2 C_{B_1 \dots B_{N-1}}}{\partial \hat{G}^{B_N}{}_{,\alpha\gamma} \partial \hat{g}_\mu} = 0 \quad \text{for} \quad N \geq 1. \tag{6.110}$$

This result can be used right away when taking the derivative of equation (5.25) and (5.56) with respect to $\hat{g}_\sigma$ noticing that we can invert the coefficient $U^{\alpha\beta\,\mu\nu}$. This yields

$$\frac{\partial C_{B_1 \dots B_N}}{\partial \hat{g}_\sigma} = 0 \quad \text{for} \quad N \geq 2, \tag{6.111}$$

whenever at least one ${}_{B_i} = {}_{\alpha\beta}$. Actually, evaluating the divergence in the first term in equation (5.58), one can extend this result to the case $N = 1$, and, thus, none of the coefficients $C_{B_1 \dots B_N}$ (for $N \geq 1$ and some ${}_{B_i} = {}_{\alpha\beta}$) depends on $\hat{g}_\alpha$. It is, however, not possible to extend this result to all coefficients. The coefficients $C_{0\dots 0}$ (where all capital indices take the value '0') and the potential C can still depend on $\hat{g}_\alpha$.

The last step of the proof now lies in solving the reduced form of the master equations for the remaining non-vanishing coefficients. Hence, from now on, the capital indices will only take the values

$$B_i = (\alpha_i \beta_i,\, 0).$$

In order to aid the reader, we display the reduced set of the master equations expressed in covariant coordinates. For all $N \geq 1$, we have that

$$
\begin{aligned}
0 &= (N+1)! C_{\mu\nu B_1 \ldots B_N} U^{\mu\nu\,\alpha\beta} - (N-2)(N-1)! \frac{\partial C_{B_1 \ldots B_{N-1}}}{\partial \hat{G}^{B_N}{}_{,\alpha\beta}}\,, \qquad (6.112)\\
0 &= -(N+1)!\, C_{0 B_1 \ldots B_N} \nabla^\beta \hat{\phi} + q(N-1)! \frac{\partial C_{B_1 \ldots B_{q-1} B_{q+1} \ldots B_N}}{\partial \hat{\phi}_{;\beta}} \\
&\quad -(N-q)(N-1)! \frac{\partial C_{B_1 \ldots B_q (B_{q+1} \ldots B_{N-1}|}}{\partial \hat{\phi}_{;\rho\sigma}}\, {}^{\hat{P}}\Gamma^{\tau\beta}{}_{\rho\sigma|B_N)} \hat{\phi}_{;\tau} \\
&\quad -2(N-1)!\, \nabla_\gamma \frac{\partial C_{B_1 \ldots B_{N-1}}}{\partial \hat{G}^{B_N}{}_{,\gamma\beta}} + N! \frac{\partial C_{B_1 \ldots B_N}}{\partial \hat{g}_\rho} M_\rho{}^\beta\,, \qquad (6.113)
\end{aligned}
$$

where the indicator q denotes the number of capital indices taking the value '0', whereas $N - q$ is the number of capital indices B_i being symmetric pairs '$\alpha_i \beta_i$', and the coefficients ${}^{\hat{P}}\Gamma$ are defined in (5.46). In order to not make the equations appear too complicated, we have not written out the chain rule for derivatives with respect to the second partial derivatives of the fields $\hat{G}^A$. Moreover, equation (5.58), coupling the first order coefficients C_A with the zeroth order coefficient C, becomes

$$
0 = \nabla_\mu (C_{\rho\sigma} U^{\rho\sigma\,\beta\mu}) - C_0 \nabla^\beta \hat{\phi} + \frac{\partial C}{\partial \hat{g}_\rho} M_\rho{}^\beta\,, \qquad (6.114)
$$

and we also do not rewrite the symmetry condition (5.29) in the new coordinates, but keep it in its general form

$$
\frac{\partial C_{B_1 \ldots B_{N-1}}}{\partial \hat{G}^{B_N}{}_{,\gamma\delta}} = \frac{\partial C_{(B_1 \ldots B_{N-1}|}}{\partial \hat{G}^{|B_N)}{}_{,\gamma\delta}} \quad \text{for} \quad N \geq 2\,. \qquad (6.115)
$$

All other master equations are identically satisfied.

When solving these reduced master equations (6.112)-(6.115), we have to keep in mind that only the coefficients $C_{0\ldots0}$ and the potential C may depend on the variable $\hat{g}_\alpha$. In general, all unknowns $C_{B_1 \ldots B_N}$ can, in addition, only depend on the variables $(\hat{P}^{\alpha\beta}, R_{\alpha\beta}, \hat{\phi}, \hat{\phi}_{;\alpha}, \hat{\phi}_{;\alpha\beta})$ because of the invariance equations (5.53) and (5.54), and we already used the fact that the Riemann tensor in three dimensions can be expressed by the Ricci tensor $R_{\alpha\beta}$. From section 5.2.2, we also know that all coefficients $C_{B_1 \ldots B_N}$ can depend on the second derivatives of the fields $\hat{G}^A$ only up to cubic order. Actually, since the second derivative of the scalar field $\hat{\phi}$ does not appear in the invariance equation (6.89), we can conclude, just as in the metric case in section 6.2, that the Ricci tensor $R_{\alpha\beta}$ can only appear linearly. Moreover, mixed terms, which contain the second derivatives of $\hat{\phi}$ and the Ricci tensor, can only be linear in both. As a reminder, this can be deduced by combining the invariance equation (6.89) (which is now also valid for the potential C), in the non-covariant coordinates, with the symmetry condition one obtains from writing out the divergence term in equation (6.113):

$$\frac{\partial^2 C_{B_1\ldots B_N}}{\partial \hat{G}^M{}_{,\alpha(\beta}\partial \hat{G}^N{}_{,\gamma\delta)}} \quad \text{for} \quad N \geq 0\,. \tag{6.116}$$

Next, we derive an equation that only involves the potential C. To this end, we consider equation (6.112) for $N = 1$ and $q = 1$, and solve it for the coefficient $C_{\alpha\beta\,0}$, which yields

$$C_{\alpha\beta\,0} = \frac{1}{4}\hat{P}_{\gamma(\alpha}\hat{P}_{\beta)\delta}\frac{\partial C}{\partial \hat{\phi}_{;\gamma\delta}}\,. \tag{6.117}$$

On the other hand, considering equation (6.113) for $N = 1$ and $q = 0$, we have that

$$0 = 2C_{\alpha\beta\,0}\nabla^\beta\hat{\phi} - \frac{\partial C}{\partial \hat{\phi}_{;\rho\sigma}}{}^{\hat{P}}\Gamma^{\tau\gamma}{}_{\rho\sigma\alpha\beta}\hat{\phi}_{;\tau} - 2\nabla_\gamma\frac{\partial C}{\partial \hat{P}^{\alpha\beta}{}_{,\mu\beta}}\,, \tag{6.118}$$

because $C_{\alpha\beta}$ does not depend on $\hat{g}_\rho$. Combining both equations, using the explicit form (5.46) of ${}^{\hat{P}}\Gamma$, we obtain

$$0 = \frac{\partial C}{\partial \hat{\phi}_{;\rho\sigma}}\left(\delta^\tau_{(\rho}\hat{P}_{\sigma)(\alpha}\delta^\gamma_{\beta)} - \hat{P}_{\rho(\alpha}\hat{P}_{\beta)\sigma}\hat{P}^{\tau\gamma}\right)\nabla_\tau\hat{\phi} - 2\nabla_\gamma\frac{\partial C}{\partial \hat{P}^{\alpha\beta}{}_{,\mu\beta}}\,, \tag{6.119}$$

which constrains the dependence of the potential C on the second derivatives of the fields $\hat{P}$ and $\hat{\phi}$. Knowing the polynomial dependencies of the potential C on the second derivatives of $\hat{\phi}$ and the Ricci tensor $R_{\alpha\beta}$, we may now derive the form of the terms that contain the latter. First, we expand the divergence in (6.119) as

$$\nabla_\mu\frac{\partial C}{\partial \hat{P}^{\alpha\beta}{}_{,\mu\gamma}} = \frac{\partial^2 C}{\partial\hat{\phi}\,\partial \hat{P}^{\alpha\beta}{}_{,\mu\gamma}}\nabla_\mu\phi + \frac{\partial^2 C}{\partial\hat{\phi}_{;\rho}\,\partial \hat{P}^{\alpha\beta}{}_{,\mu\gamma}}\nabla_\mu\nabla_\rho\hat{\phi} + \frac{\partial^2 C}{\partial\hat{\phi}_{;\rho\sigma}\,\partial \hat{P}^{\alpha\beta}{}_{,\mu\gamma}}\nabla_\mu\nabla_\rho\nabla_\sigma\hat{\phi}\,, \tag{6.120}$$

and rewrite

$$\begin{aligned}\nabla_\mu\nabla_\rho\nabla_\sigma\hat{\phi} \;\;=\;\; & -\frac{2}{3}\left(\hat{P}_{\nu[\mu}\delta^{(\kappa}_{\rho]}\delta^{\tau)}_\sigma - \hat{P}_{\sigma[\mu}\delta^{(\kappa}_{\rho]}\delta^{\tau)}_\nu + \hat{P}_{\nu[\mu}\delta^{(\kappa}_{\sigma]}\delta^{\tau)}_\rho - \hat{P}_{\rho[\mu}\delta^{(\kappa}_{\sigma]}\delta^{\tau)}_\nu\right)R_{\kappa\tau}\nabla^\nu\hat{\phi} \\ & +\frac{1}{3}(\hat{P}_{\nu[\mu}\hat{P}_{\rho]\sigma} + \hat{P}_{\nu[\mu}\hat{P}_{\sigma]\rho})\hat{P}^{\kappa\tau}R_{\kappa\tau}\nabla^\nu\hat{\phi} + \nabla_{(\mu}\nabla_\rho\nabla_{\sigma)}\hat{\phi}\,,\end{aligned} \tag{6.121}$$

where the last term will drop out of equation (6.119), because of the symmetry condition (6.116). We can then use equation (6.119) to compare the different powers of the second derivatives of $\hat{\phi}$ and the Ricci tensor $R_{\alpha\beta}$ appearing in the potential C. Note that none of these terms can depend explicitly on $\hat{g}_\alpha$, because of equation (6.110), which simplifies matters significantly. It follows, for example, that the coefficient in the cubic part $C^{\rho\sigma\mu\nu\kappa\epsilon}_{\text{cubic}}\,\hat{\phi}_{;\rho\sigma}\,\hat{\phi}_{;\mu\nu}\,\hat{\phi}_{;\kappa\epsilon}$ of C has to satisfy

$$0 = C^{\rho\sigma\mu\nu\kappa\epsilon}_{\text{cubic}}\left(\delta^\tau_{(\rho}\hat{P}_{\sigma)(\alpha}\delta^\gamma_{\beta)} - \hat{P}_{\rho(\alpha}\hat{P}_{\beta)\sigma}\hat{P}^{\tau\gamma}\right)\nabla_\tau\hat{\phi}\,. \tag{6.122}$$

However, it is easy to see that the term in brackets can be inverted, which implies that there cannot be such a cubic term in C. For the mixed term $C^{\alpha\beta\gamma\delta}_{\text{mixed}}\, R_{\alpha\beta}\, \hat{\phi}_{;\gamma\delta}$, only the last term in (6.119) is relevant. A brute-force calculation then shows that also this term has to vanish. The remaining terms can then be investigated by making the exhaustive ansatz

$$\begin{aligned} C \;=\; \sqrt{-\det \hat{P}_{\alpha\beta}}\Big[& C_f(\hat{\phi}, \nabla_\alpha\hat{\phi}\nabla^\alpha\hat{\phi}, \hat{g}^\alpha\nabla_\alpha\hat{\phi}, \hat{g}^\alpha\hat{g}_\alpha) + R_{\alpha\beta}(a_1\, \hat{P}^{\alpha\beta} + a_2\, \nabla^\alpha\hat{\phi}\nabla^\beta\hat{\phi}) \\ & +\nabla_\alpha\nabla_\beta\hat{\phi}(a_3\, \hat{P}^{\alpha\beta} + a_4\, \nabla^\alpha\hat{\phi}\nabla^\beta\hat{\phi}) + \nabla_\alpha\nabla_\beta\hat{\phi}\nabla_\gamma\nabla_\delta\hat{\phi}\,(a_5\, \hat{P}^{\alpha\beta}\hat{P}^{\gamma\delta} \\ & +a_6\, \hat{P}^{\alpha\gamma}\hat{P}^{\beta\delta} + a_7\, \hat{P}^{\alpha\beta}\nabla^\gamma\hat{\phi}\nabla^\delta\hat{\phi} + a_8\, \hat{P}^{\alpha\gamma}\nabla^\beta\hat{\phi}\nabla^\delta\hat{\phi} \\ & +a_9\, \nabla^\alpha\hat{\phi}\nabla^\beta\hat{\phi}\nabla^\gamma\hat{\phi}\nabla^\delta\hat{\phi})\Big] \end{aligned} \tag{6.123}$$

where the scalar functions a_i may depend on $\hat{\phi}$ and $\nabla_\alpha\hat{\phi}\nabla^\alpha\hat{\phi}$ and the free function C_f depends on all scalars indicated in brackets. Extracting all information in equation (6.119), then leads to a system of linear differential equations for the functions a_i, which can be solved uniquely, leading to the most general form of the potential C:

$$C = \sqrt{-\det \hat{P}_{\alpha\beta}}\left[a_1(\hat{\phi})R - 2\frac{da_1(\hat{\phi})}{d\hat{\phi}}\hat{P}^{\alpha\beta}\hat{\phi}_{;\alpha\beta} + C_f(\hat{\phi}, \nabla_\alpha\hat{\phi}\nabla^\alpha\hat{\phi}, \hat{g}^\alpha\nabla_\alpha\hat{\phi}, \hat{g}^\alpha\hat{g}_\alpha)\right]. \tag{6.124}$$

A similar procedure can be applied to determine the coefficient $C_{\rho\sigma}$, which, as we know, cannot depend on $\hat{g}_\alpha$. We can even derive two independent equations for $C_{\rho\sigma}$. The first of these is given by equation (6.113) for $N = 2$ and $q = 0$:

$$0 = -\frac{\partial C_{\{\rho\sigma|}}{\partial\hat{\phi}_{;\mu\nu}}\, {}^{\hat{P}}\Gamma^{\tau\beta}{}_{\mu\nu|\epsilon\kappa\}}\nabla_\tau\hat{\phi} - \nabla_\gamma\frac{\partial C_{\rho\sigma}}{\partial \hat{P}^{\epsilon\kappa}{}_{,\gamma\beta}}\,, \tag{6.125}$$

where the symmetrization brackets $\{\dots\}$ are to be understood as symmetrizing the pairs $\rho\sigma$ and $\epsilon\kappa$, but not the individual indices. Here, we made use of the facts that $C_{\alpha\beta\, 0}$ does not depend on $\hat{g}_\alpha$ either, and that, from equation (6.112) with $N = 2$, we may conclude that $C_{\alpha\beta\, B_1 B_2} = 0$. The second equation can be derived from equation (6.113) with $N = 2$ and $q = 1$ using the same reasoning, which leads to

$$0 = \frac{C_{\rho\sigma}}{\partial\hat{\phi}_{;\beta}} - \frac{\partial C_0}{\partial\hat{\phi}_{;\mu\nu}}\, {}^{\hat{P}}\Gamma^{\tau\beta}{}_{\mu\nu\rho\sigma}\nabla_\tau\hat{\phi} - \nabla_\gamma\frac{\partial C_{\rho\sigma}}{\partial\hat{\phi}_{;\gamma\beta}}\,, \tag{6.126}$$

where we have already used the symmetry condition (6.115) in the last term. The coefficient C_0, which still appears in this equation, can be eliminated by solving equation (6.114):

$$C_0 = \frac{1}{\nabla_\rho\hat{\phi}\nabla^\rho\hat{\phi}}\left[\nabla_\beta\hat{\phi}\nabla_\mu(C_{\kappa\tau}U^{\kappa\tau\,\beta\mu}) + \frac{\partial\tilde{C}_f}{\partial\hat{g}_\rho}M_\rho{}^\beta\nabla_\beta\hat{\phi}\right], \tag{6.127}$$

with $\tilde{C}_f = \sqrt{-\det \hat{P}_{\alpha\beta}}\, C_f$. Inserting this back into equation (6.126), the second term in brackets vanishes, because of (6.124), and hence, we obtain

$$0 = \frac{C_{\rho\sigma}}{\partial\hat{\phi}_{;\beta}} - \frac{1}{\nabla_\rho\hat{\phi}\nabla^\rho\hat{\phi}}\; {}^{\hat{P}}\Gamma^{\tau\beta}{}_{\mu\nu\rho\sigma}\, U^{\xi\delta\,\psi\zeta}\, \nabla_\tau\hat{\phi}\nabla_\psi\hat{\phi}\nabla_\zeta C_{\xi\delta} - \nabla_\gamma\frac{\partial C_{\rho\sigma}}{\partial\hat{\phi}_{;\gamma\beta}}\,. \tag{6.128}$$

Using equation (6.125) and (6.128), we can now constrain the form of $C_{\rho\sigma}$ the same way we did for the potential C. First of all, writing out the divergence in equation (6.114), one can conclude that $C_{\rho\sigma}$ can be at most linear in $R_{\alpha\beta}$ and at most quadratic in $\hat{\phi}_{;\alpha\beta}$. This is the case because the resulting symmetry condition also involves the symmetric pair of indices of $C_{\rho\sigma}$, and, thus, strengthens the two symmetry conditions we already used for the potential C. There cannot be any terms mixing $R_{\alpha\beta}$ and $\hat{\phi}_{;\alpha\beta}$ for the same reason. Evaluating all information contained in the two equations (6.125) and (6.128), one obtains, as a preliminary result, that

$$\begin{aligned} C_{\rho\sigma} \;&=\; \sqrt{-\det \hat{P}_{\alpha\beta}}\Big[(b_1\hat{\phi} + b_2)(R_{\rho\sigma} - \frac{1}{2}\hat{P}_{\rho\sigma}R) + b_3\, R\, \hat{P}_{\rho\sigma} \\ &\qquad + \frac{1}{2}b_1(\hat{P}^{\alpha\beta}\hat{\phi}_{;\alpha\beta}\,\hat{P}_{\rho\sigma} - \hat{\phi}_{;\rho\sigma}) + a_2(\hat{\phi})\hat{P}_{\rho\sigma}\Big]\,, \end{aligned} \tag{6.129}$$

with constants b_1, b_2, b_3 and a new unknown function $a_2(\hat{\phi})$. From equation (6.127), we conclude that $b_3 = 0$, since this equation cannot contain third partial derivatives of $\hat{P}^{\alpha\beta}$. A straightforward calculation now yields

$$\begin{aligned} C_0 \;&=\; \sqrt{-\det \hat{P}_{\alpha\beta}}\Big[-\frac{b_1}{\nabla_\rho\hat{\phi}\nabla^\rho\hat{\phi}}\nabla^\alpha\hat{\phi}\nabla^\beta\hat{\phi}(R_{\alpha\beta} - \hat{P}_{\alpha\beta}R) \\ &\qquad -2\frac{da_2(\hat{\phi})}{d\hat{\phi}} + \frac{1}{\nabla^\alpha\hat{\phi}\nabla_\alpha\hat{\phi}}\frac{\partial C_f}{\partial\hat{g}_\rho}M_\rho{}^\beta\nabla_\beta\hat{\phi}\Big]\,. \end{aligned} \tag{6.130}$$

Now, we have a look at equation (6.113) for $N = 2$ and $q = 2$, which reads

$$0 = -3!\, C_{000}\nabla^\beta\hat{\phi} + 2!\frac{\partial C_{00}}{\partial\hat{g}_\rho}\, M_\rho{}^\beta + 2\frac{\partial C_0}{\partial\hat{\phi}_{;\beta}}\,. \tag{6.131}$$

Since we know that $C_{\alpha\beta\,00} = 0$, the symmetry condition (6.115) implies that the coefficient C_{000} cannot depend on $R_{\rho\sigma}$. Moreover, since $\partial C_{00}/\partial\hat{g}_\rho$ cannot contain $R_{\rho\sigma}$ either, equation (6.131) implies $b_1 = 0$. Thus, we arrive at

$$C_{\rho\sigma} \;=\; \sqrt{-\det \hat{P}_{\alpha\beta}}\left[b_2(R_{\rho\sigma} - \frac{1}{2}\hat{P}_{\rho\sigma}R) + a_2(\hat{\phi})P_{\rho\sigma}\right] \quad \text{and} \tag{6.132}$$

$$C_0 \;=\; \sqrt{-\det \hat{P}_{\alpha\beta}}\left[-2\frac{da_2(\hat{\phi})}{d\hat{\phi}} + \frac{1}{\nabla_\alpha\hat{\phi}\nabla^\alpha\hat{\phi}}\frac{\partial C_f}{\partial\hat{g}_\rho}M_\rho{}^\beta\nabla_\beta\hat{\phi}\right]\,. \tag{6.133}$$

We can now determine the remaining coefficients recursively. Using equation (6.117), we get

$$C_{\alpha\beta\,0} = -\sqrt{-\det \hat{P}_{\alpha\beta}}\,\frac{1}{2}\frac{da_1(\hat{\phi})}{d\hat{\phi}}\hat{P}_{\alpha\beta}\,. \tag{6.134}$$

From equation (6.112) with $N = 1$ and $q = 0$, we find the coefficient $C_{\alpha\beta\gamma\delta}$ given by equation (6.92). It is clear now that all other coefficients containing at least one index pair $\alpha\beta$ vanish. This can be seen recursively from equation (6.112) and the fact that all coefficients with more than two capital indices do not depend on second derivatives of the fields. Thus, only the coefficients with '0' indices remain to be determined. Using equation (6.113), we get the recursion relation (6.94), and the first term in the Lagrangian in equation (6.91) is due to the second term in brackets in equation (6.124). Renaming $C_f \to C_0$ and $b_1 \to \lambda$, finally, completes the proof that started on page 103. □

We would now like to add some comments on this result. The function $a_1(\hat{\phi})$ mediates the derivative coupling between the scalar field $\hat{\phi}$ and the metric $\hat{P}^{\alpha\beta}$ in the Lagrangian (6.91). A non-derivative coupling obviously requires $a_1(\hat{\phi}) = \text{const}$[8]. The terms involving the function $a_2(\hat{\phi})$ and the constant λ are linear in the velocities $\hat{K}^{\alpha\beta}$ and $\hat{K}$. Actually, both terms can be generated from the functional

$$\Lambda = \sqrt{-\det \hat{P}_{\alpha\beta}}(\lambda R - 2a_2(\hat{\phi})) \tag{6.135}$$

by variation with respect to $\hat{P}^{\alpha\beta}$ and $\hat{\phi}$. Just as in the metric case, we can conclude that these terms do neither have an impact on the equations of motion for the fields $\hat{\phi}$ and $\hat{P}^{\alpha\beta}$, nor will they appear in the Hamiltonian in the canonical formulation of the theory, because one may simply redefine the canonical momenta. Thus, we may set $\lambda = 0 = a_2(\hat{\phi})$. One may think that requiring the Lagrangian not to contain the velocities $\hat{K}^{\alpha}$ would leave the dynamics of the variable $\hat{g}_\alpha$ undetermined and indeed we will have to further investigate the consequences of this assumption in the next section. The only dependence on the variable $\hat{g}_\alpha$ is encoded in the coefficients C_N, for $N \geq 0$. Except for the recursion relations (6.93) and (6.94), the master equations do not restrict the form of these last terms in the Lagrangian. In principle, there are two possible ways to generate a more explicit result here. Either, one can start with an arbitrary function C_0 and calculate all other, possibly infinitely many, coefficients from there, or one requires special properties of the Lagrangian, such as that it depends only quadratically on the scalar velocity $\hat{K}$, and then determines

[8] Such a solution was, actually, found by Kuchař in [37], but in a completely different context. We have generalized his findings almost accidentally, by finding the most general solution to the master equations, where the resulting Lagrangian contains derivative couplings between a scalar field and a metric. Assuming that the function C_0 does not depend on the variable $\hat{g}_\alpha$, the recursion (6.94) can be solved to yield $C_0(\phi, \hat{K}^2 + (\nabla\hat{\phi})^2)$, which recovers the result for this part found by Kuchař.

the unique solution of the recursion under this assumption. For definitness, we will work with the second method and proceed with our analysis in a special case. It turns out that it is not possible to get rid of the variable $\hat{g}_\alpha$ altogether for dynamical reasons. This might already be obvious since, in this case, one could specify a part of the geometric information completely arbitrarily and, ultimately, the matter field equations (6.59), which were the starting point of our investigations, would lose their predictive power. A partly non-dynamical background geometry would, indeed, be highly unsatisfactory from the point of view of gravitational dynamics. Indeed, we will show that although the variable $\hat{g}_\alpha$ does not have its own 'dynamical' equations of motion, it is nevertheless fully determined by the dynamics of the other variables $\hat{P}^{\alpha\beta}$ and $\hat{\phi}$.

6.4.5 Investigating a concrete model

We would now like to analyse the dynamics for the variables $(\hat{P}^{\alpha\beta}, \hat{\phi}, \hat{g}_\alpha)$, which we derived in the previous section, further. Especially the role of the variable $\hat{g}_\alpha$ deserves a more detailed investigation. As we have already mentioned, the solutions of the master equations do not fix the dynamical content of the theory completely, but leave two functions, $a_1(\hat{\phi})$ and $C_0(\hat{\phi}, \nabla^\alpha\hat{\phi}\nabla_\alpha\hat{\phi}, \hat{g}^\alpha\nabla_\alpha\hat{\phi}, \hat{g}_\alpha\hat{g}^\alpha)$, completely undetermined. While the function $a_1(\phi)$ merely mediates the derivative coupling between the metric $\hat{P}^{\alpha\beta}$ and the scalar field $\hat{\phi}$, the role of the function C_0 can only be revealed in a concrete situation. Since, apart from the specific set of variables it depends on, the coefficient function C_0 is completely undetermined by the master equations, we can freely prescribe any additional condition that is compatible with the master equations and at the same time allows to determine C_0. The additional assumption we would like to introduce here, is that the Lagrangian depends at most quadratically on the velocities $\hat{K}$. We may then find the most general solution of the recursion (6.93) and (6.94) under this assumption. Since this calculation can be done rather straightforwardly, we only describe the principle here. First of all, we can ignore the dependence of the functions C_N on the scalar field $\hat{\phi}$ itself. There is no way to constrain this dependence in any way. We simply need to keep in mind that any integration constants, which arise when solving the recursion relations, must be turned into arbitrary functions of $\hat{\phi}$ at the end. Introducing the shorthand notations $\Omega = \nabla^\alpha\hat{\phi}\nabla_\alpha\hat{\phi}$, $\Psi = \hat{g}_\alpha\nabla^\alpha\hat{\phi}$ and $\xi = \hat{g}_\alpha\hat{g}^\alpha$ for the arguments of the functions C_N, the recursion relation (6.94) takes the form

$$C_{N+1} = \frac{1}{\Omega}\frac{N!}{(N+1)!}\left[\frac{\partial C_N}{\partial\Psi}(\Omega+\Psi^2) + 2\,\frac{\partial C_N}{\partial\xi}(\Psi+\xi\Psi) + 2\Omega\,\frac{\partial C_{N-1}}{\partial\Omega} + \Psi\,\frac{\partial C_{N-1}}{\partial\Psi}\right] \tag{6.136}$$

Now, assuming that $C_N = 0$ for all $N \geq 3$, we can immediately integrate this equation for $N = 3$, which yields

$$C_2 = A(\xi)\Omega^{2n}\Psi^{-n} + B(\xi), \tag{6.137}$$

for some constant n and so far free functions $A(\xi)$ and $B(\xi)$. Reinserting this result into the same equation for $N = 2$ determines C_1, and reinserting both into the equation for $N = 1$, yields C_0. All additional unknown functions, which arise in this process, can then be determined by inserting C_1 and C_0 into the consistency condition (6.93). This leads to the condition $n(n+1)(n+2)A(\xi) = 0$ and we may then determine all possible solutions, for which any of these factors vanish. After a fair amount of algebra, one observes that the cases $A(\xi) = 0$, $n = -1$ and $n = -2$ are actually equivalent. Finally, the most general solution for the second part L_2 (determined by the equations (6.93) and (6.94)) of the full Lagrangian (6.91), under the condition that it is at most quadratic in the velocities $\hat{K}$, is:

$$\begin{aligned} L_2 \;=\; & \sqrt{-\det \hat{P}_{\alpha\beta}}\,\Bigg[\frac{a_3(\hat{\phi})}{1+\hat{g}_\alpha\hat{g}^\alpha}\,\hat{K}^2 + \frac{a_4(\hat{\phi})}{(1+\hat{g}_\alpha\hat{g}^\alpha)^{1/2}}\,\hat{K}^2 + a_5(\hat{\phi})\,\hat{K}^2 \\ & + \frac{2a_3(\hat{\phi})}{1+\hat{g}_\alpha\hat{g}^\alpha}\,\hat{g}^\beta\nabla_\beta\hat{\phi}\,\hat{K} + \frac{a_4(\hat{\phi})}{(1+\hat{g}_\alpha\hat{g}^\alpha)^{1/2}}\,\hat{g}^\beta\nabla_\beta\hat{\phi}\,\hat{K} + \frac{a_6(\hat{\phi})}{(1+\hat{g}_\alpha\hat{g}^\alpha)^{1/2}}\hat{g}^\beta\nabla_\beta\hat{\phi}\,\hat{K} \\ & + \frac{a_3(\hat{\phi})}{1+\hat{g}_\alpha\hat{g}^\alpha}(\hat{g}^\beta\nabla_\beta\hat{\phi})^2 - \frac{a_4(\hat{\phi})}{(1+\hat{g}_\alpha\hat{g}^\alpha)^{1/2}}\nabla^\beta\hat{\phi}\nabla_\beta\hat{\phi} + \frac{a_6(\hat{\phi})}{(1+\hat{g}_\alpha\hat{g}^\alpha)^{1/2}}\hat{g}^\beta\nabla_\beta\hat{\phi} \\ & + a_5(\hat{\phi})\,\nabla^\beta\hat{\phi}\nabla_\beta\hat{\phi} + \, a_7(\hat{\phi})\Bigg]\,. \end{aligned} \tag{6.138}$$

The last two lines denote the most general form for the coefficient C_0 that leads to a Lagrangian that is at most quadratic in the scalar velocities $\hat{K}$. As we have mentioned already, the free functions $a_3(\hat{\phi}), \dots, a_7(\hat{\phi})$ cannot be further constrained, so that there is still a huge class of possible gravitational theories that can underlie the matter field equations (6.59).

In order to understand the fate of the geometric variable $\hat{g}_\alpha$, we need to investigate a special case of such a theory. For definiteness, we will specialise to a particularly simple solution for the Lagrangian in order to study the dynamical properties of the derived gravitational theory. We set $a_1(\hat{\phi}) \equiv -\kappa = \text{const}$, $a_3(\hat{\phi}) \equiv \mu = \text{const}$, and all other $a_4, \dots, a_7 \equiv 0$. Then the Lagrangian (4.22) reads

$$L = \sqrt{-\det \hat{P}_{\alpha\beta}}\left[\kappa C_{\alpha\beta\gamma\delta}\hat{K}^{\alpha\beta}\hat{K}^{\gamma\delta} - \kappa R + \mu\frac{\hat{K}^2}{1+\hat{g}_\alpha\hat{g}^\alpha} + 2\mu\frac{\hat{K}}{1+\hat{g}_\alpha\hat{g}^\alpha} + \mu\frac{(\hat{g}^\beta\nabla_\beta\hat{\phi})^2}{1+\hat{g}_\alpha\hat{g}^\alpha}\right]. \tag{6.139}$$

It is, in fact, easier to analyse the dynamics of this theory by going back to the canonical spacetime picture. In order to reconstruct the local part $\hat{\mathcal{H}}_{\text{local}}$ of the superhamiltonian, we simply have to perform the inverse Legendre transformation of the above Lagrangian with respect to the velocities $\hat{K}^A$. This poses no problem although the Lagrangian is singular in the velocity $\hat{K}_\alpha$. We only have to introduce additional Lagrange multipliers Λ_α that account for this. After performing the Legendre transformation, the complete Hamiltonian (4.4) for our particular gravity theory becomes

$$\begin{aligned} H &= \int_\Sigma dy \Big[N(y) \Big\{ \frac{1}{4\kappa\sqrt{-\hat{P}}} C^{\alpha\beta\gamma\delta} \hat{\pi}_{\alpha\beta} \hat{\pi}_{\gamma\delta} + \kappa \sqrt{-\hat{P}} R + \frac{1}{4\mu\sqrt{-\hat{P}}} \hat{\pi}^2 (1 + \hat{g}_\alpha \hat{g}^\alpha) \\ &\quad - \hat{\pi} \hat{g}^\alpha \nabla_\alpha \hat{\phi} - \sqrt{-\hat{P}} \frac{(\mu-1)^2}{\mu} \frac{(\hat{g}^\beta \nabla_\beta \hat{\phi})^2}{(1 + \hat{g}_\alpha \hat{g}^\alpha)} + \Lambda_\alpha \hat{\pi}^\alpha - \partial_\gamma (\hat{\pi}^\gamma + \hat{g}^\gamma \hat{g}_\alpha \hat{\pi}^\alpha) \Big\} (y) \\ &\quad + \Big\{ \hat{\pi}_{\alpha\beta} \, \mathcal{L}_{\vec{N}} \hat{P}^{\alpha\beta} + \hat{\pi} \, \mathcal{L}_{\vec{N}} \hat{\phi} + \hat{\pi}^\alpha \, \mathcal{L}_{\vec{N}} \hat{g}_\alpha \Big\} (y) \Big] , \end{aligned} \tag{6.140}$$

with the coefficient $C^{\alpha\beta\gamma\delta} = 4\hat{P}^{\alpha(\gamma}\hat{P}^{\delta)\beta} - 2\hat{P}^{\alpha\beta}\hat{P}^{\gamma\delta}$, and we used the shorthand $\sqrt{-\hat{P}} := \sqrt{-\det \hat{P}_{\alpha\beta}}$. For further analysis, we simplify matters by setting $\mu = 1$. The Lagrange multiplier Λ_α enforces $\pi^\alpha(y) \equiv 0$ as an additional constraint apart from (4.5). Since $\pi^\alpha(y) \equiv 0$ has to hold for all values of the evolution parameter t, this also implies that $\dot{\pi}^\alpha(y) = 0$. However, Hamilton's equations for the variable $\hat{g}_\alpha$ (using the Hamiltonian (6.140)) yield

$$\dot{\hat{\pi}}^\alpha(y) \approx -N(y) \left[\frac{1}{2\sqrt{-\hat{P}}} \hat{\pi}^2 \hat{g}^\alpha - \hat{\pi} \nabla^\alpha \hat{\phi} \right] (y) , \tag{6.141}$$

where the weak equality '$\approx$' means that we already made use of the constraint $\hat{\pi}^\alpha = 0$. Hence, the variable $\hat{g}_\alpha$ is completely determined by the solutions of the equations of motion for the scalar field $\hat{\phi}$ and the metric $\hat{P}^{\alpha\beta}$ by

$$\hat{g}_\alpha(y) = 2 \left[\sqrt{-\hat{P}} \, \frac{\nabla_\alpha \hat{\phi}}{\hat{\pi}} \right] (y). \tag{6.142}$$

Hamilton's equations for the variable $\hat{\pi}^\alpha$ can be used to determine the Lagrange multiplier Λ_α, and to eliminate the variable $\hat{g}_\alpha$ and the momentum $\hat{\pi}^\alpha$ from the action (4.6) altogether. From the remaining equations of motion, it can then be checked that the effective Hamiltonian for the dynamics of the scalar field $\hat{\phi}$ and the metric $\hat{P}^{\alpha\beta}$ is given by

$$\begin{aligned} H &= \int_\Sigma dy \Big[N \Big\{ \frac{1}{4\kappa\sqrt{-\hat{P}}} C^{\alpha\beta\gamma\delta} \hat{\pi}_{\alpha\beta} \hat{\pi}_{\gamma\delta} + \kappa \sqrt{-\hat{P}} R + \frac{1}{4\sqrt{-\hat{P}}} \hat{\pi}^2 - \sqrt{-\hat{P}} \nabla^\alpha \hat{\phi} \nabla_\alpha \hat{\phi} \Big\} \\ &\quad + \Big\{ \hat{\pi}_{\alpha\beta} \, \mathcal{L}_{\vec{N}} \hat{P}^{\alpha\beta} + \hat{\pi} \, \mathcal{L}_{\vec{N}} \hat{\phi} \Big\} \Big] (y) , \end{aligned} \tag{6.143}$$

which is mathematically equivalent to a massless scalar field non-derivatively coupled to Einstein gravity.

Our considerations show that, although the variable $\hat{g}_\alpha$ is not dynamical in the sense that it satisfies its own dynamical equations of motion, it is nevertheless completely determined by the dynamics of the other degrees of freedom of the theory. Thus, we have indeed found a possible model for the gravitational dynamics of the tensorial spacetime $(M, g, W, \mathrm{Dirac})$. Of course, we have only discussed a special case of those Lagrangians that are at most quadratic in the velocities $\hat{K}^A$, but only little changes if one considers a more general example. The main difference will be the actual form of the relation between the variable $\hat{g}_\alpha$ and the other two fields $\hat{P}^{\alpha\beta}$ and $\hat{\phi}$. In the case presented above, it was possible to obtain the explicit expression (6.142), but, in more general situations, one might only end up with an implicit algebraic equation that involves all of the field variables. However, this will merely make it more difficult to solve the equations of motion, but it does not pose a problem of principle.

In order to conclude the discussion of this example, we sketch how an explicit solution of the above dynamics for the fields $\hat{P}^{\alpha\beta}$, $\hat{\phi}$ and $\hat{g}_\alpha$ can be found.

6.4.6 Spherically symmetric solutions

In this section, we would like to discuss how one would, in principle, proceed in order to find a solution to the gravitational dynamics we have obtained in the previous section. Although it seems that the variable $\hat{g}_\alpha$ does not play an active role anymore, we will see that, in fact, it constrains the gravitational solutions one may obtain for the variables $\hat{P}^{\alpha\beta}$ and $\hat{\phi}$. In order to demonstrate this, we consider the spherically symmetric case of the dynamics given by the Hamiltonian (6.143) and equation (6.142). To this end, we choose standard spherical coordinates (r, θ, φ) on the hypersurface $X(\Sigma)$ and assume that it admits three Killing vectors ξ_i, whose commutators satisfy the algebra of the rotation group $SO(3)$:

$$\begin{aligned} \xi_1 &= \sin\varphi\,\partial_\theta + \cot\theta\cos\varphi\,\partial_\varphi\,, \\ \xi_2 &= -\cos\varphi\,\partial_\theta + \cot\theta\sin\varphi\,\partial_\varphi\,, \\ \xi_3 &= \partial_\varphi\,. \end{aligned}$$

Evaluating the corresponding Killing conditions,

$$\mathcal{L}_{\xi_i}\hat{G}^A = 0 = \mathcal{L}_{\xi_i}\hat{\pi}_A \quad \text{for} \quad i = 1, 2, 3 \tag{6.144}$$

on the geometric variables $\hat{G}^A = (\hat{P}^{\alpha\beta}, \hat{\phi})$, the conjugate momenta $\hat{\pi}_A = (\hat{\pi}_{\alpha\beta}, \hat{\pi})$, as well as on the lapse function N and the shift vector field $\vec{N}$, it is straightforward to show that the fields can be parametrized as [8]

$$
\begin{aligned}
\hat{\phi} &= \phi(r,t), \quad \hat{\pi} = \sin\theta\, \pi_\phi(r,t), \quad N = N(r,t), \quad \vec{N} = N^r(r,t)\,\partial_r, \\
\hat{P}^{\alpha\beta} &= \mathrm{diag}\left(-e^{2\mu(r,t)}, -e^{2\lambda(r,t)}, -\frac{e^{2\lambda(r,t)}}{\sin^2\theta}\right) \quad \text{and} \\
\hat{\pi}_{\alpha\beta} &= \sin\theta\,\mathrm{diag}\left(-\frac{1}{2}\pi_\mu(r,t)\,e^{-2\mu(r,t)}, -\frac{1}{4}\pi_\lambda e^{-2\lambda(r,t)}, -\frac{1}{4}\pi_\lambda e^{-2\lambda(r,t)}\sin^2\theta\right)
\end{aligned}
$$

in terms of the field variables μ, λ and ϕ and their conjugate momenta π_μ, π_λ and π_ϕ. The additional $\sin\theta$ factor in the momenta appears because the latter are tensor densities of weight one. The variable $\hat{g}_\alpha$ inherits its form from the scalar field via equation (6.142), so that only its r-component is non-vanishing. We then insert this ansatz into the action (4.6). The integration over the angles (θ,φ) can be carried out explicitly leading to a factor of 4π, which we drop in the following. Hence, the symmetry reduced action becomes

$$
\begin{aligned}
S[\mu,\lambda,\phi,\pi_\mu,\pi_\lambda,\pi_\phi,N,N^r] &= \int dt \int dr \Big\{ \dot{\mu}\pi_\mu + \dot{\lambda}\pi_\lambda + \dot{\phi}\pi_\phi \\
&\quad -N\Big[\frac{1}{8\kappa} e^{\mu+2\lambda}\left(\pi_\mu^2 - 2\pi_\mu\pi_\lambda\right) - 2\kappa\, e^{-\mu} - 2\kappa\, e^{\mu-2\lambda}(2\mu'\lambda' - 3\lambda'^2 + 2\lambda'') \\
&\qquad + \frac{1}{4}\, e^{\mu+2\lambda}\pi_\phi^2 + e^{-\mu-2\lambda}\phi'^2\Big] - \mu'\pi_\mu N^r + \pi_\mu N^{r\prime} - \lambda'\pi_\lambda N^r - \phi'\pi_\phi N^r \Big\},
\end{aligned}
\tag{6.145}
$$

where the dot and prime denote differentiation with respect to t and r, respectively. Varying the action with respect to all the variables, one obtains Hamilton's equations for the fields μ, λ and ϕ and their conjugate momenta, as well as two constraint equations. For the static case, which is characterized by $\pi_\mu = \pi_\lambda = \pi_\phi \equiv 0$, an explicit solution to the above dynamical problem has been found in [8]. However, such a solution, although mathematically viable for the isolated dynamical problem encoded by the above action, does not exist in our case since the defining equation (6.142) for the variable $\hat{g}_\alpha$ would then lead to non-predictive results.

In contrast, there is no problem if we consider cosmological solutions to the gravitational dynamics, by additionally requiring homogeneity of the hypersurface $X(\Sigma)$. Then the variable $\hat{g}_\alpha$ must vanish identically and the cosmological solutions to our gravitational dynamics will be identical to the cosmological solutions of models where a metric and a non-derivatively coupled massless scalar field present the spacetime geometry. Returning to the case of mere spherical symmetry only, we have to, however, find a solution different from the static ones.

Now, the strategy to solve the spherically symmetric problem lies in imposing appropriate coordinate conditions, which simplify the equations of motion. A possible way to do so was described in [8, 70], leading to an effectively two dimensional problem with a non-local Hamiltonian. Only recently, different gauge fixings have been found [21, 23], which lead to local effective Hamiltonians. Usually, the first step in finding a solution involves a radial gauge choice, setting

$$\lambda = -\ln r\,, \tag{6.146}$$

so that the angular part of the metric $\hat{P}^{\alpha\beta}$ is the round metric on a sphere with curvature radius r. The shift vector N^r, necessary to achieve that λ takes this form, can be determined by setting $\dot{\lambda} = 0$ in the equation of motion for λ, which results from a variation of the action (6.145) with respect to π_λ. We obtain

$$N^r = -\frac{Ne^\mu}{4\kappa r}\pi_\mu\,. \tag{6.147}$$

The momentum π_λ conjugate to λ may then be determined from the constraint, which follows from the variation with respect to N^r:

$$\pi_\lambda = r\,(\mu'\pi_\mu + \pi'_\mu + \phi'\pi_\phi). \tag{6.148}$$

Eliminating the variables λ, π_λ and N^r from the action, we obtain

$$\begin{aligned} S[\mu,\phi,\pi_\mu,\pi_\phi,N] \;=\; & \int dt \int dr \Big\{ \dot{\mu}\pi_\mu + \dot{\phi}\pi_\phi - N\Big[\frac{e^\mu}{8\kappa r^2}\,(\pi_\mu^2 - 2r\pi_\mu(\mu'\pi_\mu + \pi'_\mu \\ & +\phi'\pi_\phi)) - 2\kappa e^{-\mu} + 2\kappa e^\mu(2\mu' r + 1) + \frac{e^\mu}{4r^2}\pi_\phi + r^2 e^{-\mu}\phi'^2\Big]\Big\}. \end{aligned} \tag{6.149}$$

In order to completly fix the gauge, we have to impose yet another coordinate condition using the remaining variables μ, ϕ, π_μ and π_ϕ. The constraint that follows from the variation of the action with respect to the lapse function N may then be used to eliminate yet another variable. That we can indeed use the scalar field to impose a gauge fixing, is due to the scalar field being part of the geometric fields, instead of aquiring the role of a matter source whose behaviour one might want to study, as in [21, 23]. The particular advantage of fixing the gauge through the field ϕ lies in the fact that its conjugate momentum appears undifferentiated in the constraint, i.e., it can be determined through an algebraic equation allowing for the canonical pair μ and π_μ to satisfy a set of non-linear, but local, differential equations.

The choice of a particularly promising gauge fixing is an art in its own right, and it is probably not worth going into that for our toy theory here, once we have arrived at this point. The gauge choice not only limits the initial data one may specify for the remaining variables, before integrating the equations of motion, but it may also turn the latter more or less hyperbolic.

Having found a solution to the equations of motion for the geometry (i.e. one with $\pi_\phi \neq 0$), one can reconstruct the entire spacetime geometry (g, W) along the lines described in section 6.4.2. Finally we need to stress, that we only dicussed vacuum solutions to the gravitational dynamics here, but we have not dealt with the combined problem of the Dirac matter field (6.59) sourcing the gravitational field, which is an even more difficult problem in practice, although now entirely clear and attackable in principle. Indeed, we discuss the inclusion of matter sources in chapter 7.

6.5 Example 4: From modified dispersion relations to master equations for the underlying point particle geometry

In this final example, we derive the master equations for all tensorial spacetimes

$$(M, P, S_{\text{massless}}[P, x, \mu], S_{\text{massive}}[P, x])\,, \tag{6.150}$$

whose geometry $G = P$ is directly given by a totally symmetric, bi-hyperbolic and energy-distinguishing tensor field P of rank $\deg P$, and whose matter content solely consists of massless point particles, described by the action (2.13), and massive point particles, described by the action (2.19). Such geometries have become interesting recently, especially because they present the covariant formulation of modified dispersion relations, which are observed in many phenomenological quantum gravity scenarios [32]. In this particular case, bi-hyperbolicity and energy-distinguishability of the tensor field P are required for the massless point particle action S_{massless} to be determined by the massless dispersion relation (2.10) and the massive point particle action S_{massive} to be determined by the massive dispersion relation (2.14), as discussed in section 2.3.

We already discussed, in section 3.6, that a parametrization of the spacetime geometry P on an observer-accessible initial data hypersurface $X(\Sigma)$ can be given in terms of the tensor fields (3.6), i.e., the configuration space in this case is spanned by the fields

$$\hat{P}^{\alpha_1\dots\alpha_I} \quad \text{for } I = 2, \dots, \deg P\,, \tag{6.151}$$

to which we adjoin the canonical momenta

$$\hat{\pi}_{\alpha_1\dots\alpha_I} \quad \text{for } I = 2, \dots, \deg P\,, \tag{6.152}$$

which are tensor densities of weight one. Hence, the capital multi-indices in the master equations take the values

$$A = ((\alpha_1\alpha_2); \dots ; (\alpha_1 \dots \alpha_{\deg P})). \tag{6.153}$$

In the next section, we derive the master equations for this hypersurface geometry by calculating all the relevant coefficients.

6.5.1 Setting up the master equations

Again, in order to derive the coefficients for the master equations, we have to start with the supermomentum, which is given by (4.16), and, after an integration by parts, takes the form

$$\hat{\mathcal{D}}(\vec{N}) = \sum_{I=2}^{\deg P} \int_\Sigma dy\, N^\beta(y) \left[\partial_\beta \hat{P}^{\alpha_1\dots\alpha_I} \hat{\pi}_{\alpha_1\dots\alpha_I} + I\, \partial_{\alpha_1}(\hat{P}^{\alpha_1\dots\alpha_I} \hat{\pi}_{\alpha_2\dots\alpha_I\beta})\right] . \tag{6.154}$$

In this example, the variable $\hat{P}^{\alpha\beta}$ explicitly appears on the right hand side of the algebra equation (4.11), and from the abstract form (6.1) of the latter, we obtain the coefficients

$$U^{\alpha_1\dots\alpha_I\,\mu\nu} = -I(\deg P - 1)\hat{P}^{\nu(\alpha_1}\hat{P}^{\alpha_2\dots\alpha_I)\mu} \quad \text{for } I = 2, \dots, \deg P\,, \tag{6.155}$$

and

$$V^{\alpha_1\dots\alpha_I\,\nu} = (\deg P - 1)\,\hat{P}^{\nu\gamma}\partial_\gamma \hat{P}^{\alpha_1\dots\alpha_I} + I(\deg P - 1)\,\hat{P}^{\nu(\alpha_1}\partial_\gamma \hat{P}^{\alpha_2\dots\alpha_I)\gamma}\,, \tag{6.156}$$

for $I = 2, \dots, \deg P$. The non-local part of the superhamiltonian is given by (4.20) and the coefficients $M^{A\,\gamma}$ are given by equation (3.34):

$$M^{\alpha_1\dots\alpha_I\,\gamma} = I(\deg P - 1)\hat{P}^{(\alpha_1\dots\alpha_{I-1}}\hat{P}^{\alpha_I)\gamma} - (\deg P - I)\hat{P}^{\alpha_1\dots\alpha_I\gamma}\,, \tag{6.157}$$

for $I = 2, \dots, \deg P$, and from these it is straightforward to calculate

$$\begin{aligned} Q_{\alpha_1\dots\alpha_K}{}^{\beta_1\dots\beta_I\,\mu} \;=\; & \delta^K_{I+1}(\deg P - I)\delta^{\mu\beta_1\dots\beta_I}_{(\alpha_1\dots\alpha_{I+1})} - \delta^K_2 I(\deg P - 1)\hat{P}^{(\beta_2\dots\beta_I}\delta^{\beta_1)\mu}_{(\alpha_1\alpha_2)} \\ & -\delta^K_{I-1} I(\deg P - 1)\hat{P}^{\mu(\beta_1}\delta^{\beta_2\dots\beta_I)}_{\alpha_1\dots\alpha_{I-1}} \end{aligned} \tag{6.158}$$

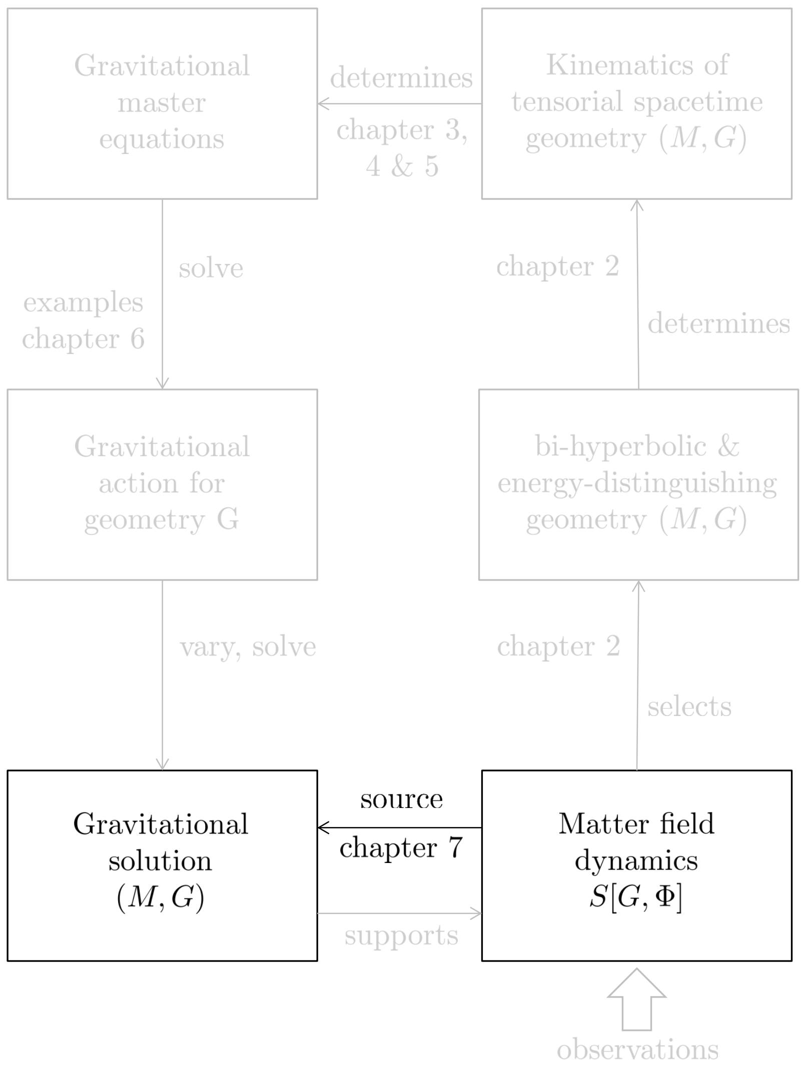
for $I, K = 2, \dots, \deg P$. Next, one checks, by direct calculation, that the coefficients $T^{A\,\mu\nu}$, abstractly given by (6.4), are symmetric in the indices $\mu\nu$. Thus, we only need to calculate the remaining coefficients $S^{A\,\gamma}$ with the help of equation (6.5).

This yields

$$\begin{aligned} S^{\alpha_1\dots\alpha_I\mu} &= I(\deg P-1)\partial_\nu \hat{P}^{\nu(\alpha_1}\hat{P}^{\alpha_2\dots\alpha_I)\mu} \\ &\quad + I(\deg P-1)\hat{P}^{\nu(\alpha_1}\partial_\nu \hat{P}^{\alpha_2\dots\alpha_I)\mu} \\ &\quad -(\deg P-1)\,\hat{P}^{\mu\nu}\partial_\nu \hat{P}^{\alpha_1\dots\alpha_I} \\ &\quad - I(\deg P-1)\,\hat{P}^{\mu(\alpha_1}\partial_\nu \hat{P}^{\alpha_2\dots\alpha_I)\nu}\,, \end{aligned} \tag{6.159}$$

for all $I=2,\dots,\deg P$. Since the hypersurface tensor field $\hat{P}^{\alpha\beta}$ can formally be employed as a hypersurface metric, which must be negative definite due to bi-hyperbolicity and energy-distinguishability, we do not need to display the invariance equations for the coefficients $C_{B_1\dots B_N}$ (for $N\geq 0$) again; they are solved along the lines laid out in section 5.2.3 for the coefficients with $N\geq 1$, and in section 5.3.2 for the potential C.

It is not difficult to see that for the special case $\deg P=2$, the above coefficients precisely reduce to the coefficients we already obtained in section 6.2. Hence, the solution of the master equations in the case of a $\deg P=2$ point particle spacetime yields the well-known ADM dynamics for the field $\hat{P}^{\alpha\beta}$ in four dimensions. The solution of the master equations in the case of a higher rank of the spacetime tensor field P so far remains an open problem. With this final example, we finish our discussion on the master equations of various tensorial spacetimes.


Gravitational master equations
determines
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Kinematics of tensorial spacetime geometry (M, G)
solve
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determines
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bi-hyperbolic & energy-distinguishing geometry (M, G)
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Chapter 7

Gravitational Sources

So far, we considered the gravitational dynamics of a certain tensorial spacetime geometry *in vacuo*. The matter field theories, which stood at the very beginning of our investigations, were only used to probe the geometry in order to reveal the relevant kinematical structure needed to construct appropriate gravitational dynamics. In this chapter, we now discuss the role that matter fields play as sources to the gravitational field. This requires, in particular, a careful construction of the the energy-momentum tensor of matter. We will exemplify our discussion by various examples.

7.1 Gotay-Marsden energy-momentum tensors

In the presence of matter, standard general relativity postulates the field equations for a Lorentzian metric g to be given by Einstein's equations[1]

$$R_{ab} - \frac{1}{2}Rg_{ab} = \frac{8\pi\kappa}{\sqrt{|g|}}\mathcal{T}_{ab} \tag{7.1}$$

where the source on the right hand side of the equations is known as the Hilbert energy-momentum tensor density

$$\mathcal{T}_{ab} := -2\,\frac{\delta\mathcal{L}_{\text{matter}}}{\delta g^{ab}}\,, \tag{7.2}$$

calculated from the matter Lagrangian density $\mathcal{L}_{\text{matter}}[\Phi, g]$ of a collection of matter fields Φ by means of the variational derivative $\delta\mathcal{L}/\delta g^{ab}$. In flat space, there is a conceptually—and depending on the matter dynamics also quantitatively—different notion of an energy-momentum tensor $\mathfrak{t}$ that arises as a collection of Noether cur-

[1] We have chosen units in which $c = 1$.

rents, one for each independent translation generator, leading to

$$\mathfrak{t}^a{}_b := \delta^a_b \mathcal{L}_{\text{matter}} - \frac{\partial \mathcal{L}_{\text{matter}}}{\partial \partial_a \phi^A} \, \partial_b \Phi^A \,, \tag{7.3}$$

which is often refered to as the canonical energy-momentum tensor. The conceptual relation between these two notions of energy-momentum tensors, the Hilbert and the canonical one, has been unclear, and indeed the matter of much debate (see e.g. [7]), for a long time. Particularly disturbing is the fact that the Hilbert tensor density is a $(0,2)$-symmetric tensor density, while the canonical energy momentum tensor must be a $(1,1)$-tensor which generically (e.g. for Maxwell theory on a metric manifold) is not even symmetric with respect to the underlying Lorentzian metric. A serious shortcoming of the definition of the Hilbert energy-momentum tensor density (7.2) is that it does not generalize to tensorial spacetime geometries other than metric manifolds. Fortunately, Gotay and Marsden derived, in their seminal paper [26], a general framework to obtain the energy-momentum tensor for almost any type of matter field on largely arbitrary spacetime geometries[2]. They show that the energy-momentum tensor of any matter field theory can be regarded as the Noether current of the matter field corresponding to complactly supported diffeomorphisms. We will not repeat their general statement here, but only use their results, which directly apply to our situation.

Let us assume that we have matter fields Φ propagating on a tensorial spacetime (M, G) with a geometry given by some tensor fields G^A. Furthermore, let the equations of motion of the matter fields be encoded by the weight-one Lagrangian density $\mathcal{L}_{\text{matter}}[\Phi, G]$. In case the geometry G is non-derivatively coupled to the matter fields Φ, Gotay and Marsden showed that the energy-momentum tensor density $\mathcal{T}^a{}_b$ of the latter can be calculated according to the generalized Hilbert formula

$$\mathcal{T}^a{}_b = C^{A\,a}{}_b \frac{\delta \mathcal{L}_{\text{matter}}}{\delta G^A} \,, \tag{7.4}$$

where the coefficients $C^{A\,a}{}_b$ can be read off the change $L_\xi G$ of the geometry G^A under infinitesimal diffeomorphisms generated by some vector field ξ, which takes the form

$$(L_\xi G)^A = C^A{}_a \xi^a + C^{A\,a}{}_b \xi^b{}_{,a} \,, \tag{7.5}$$

where L_ξ denotes the Lie-derivative along the vector field ξ. We will refer to $\mathcal{T}$ as the Gotay-Marsden energy-momentum tensor density from now on. Moreover,

[2] Their approach is based on the detailed analysis of momentum maps. Some useful background reading to better understand their constructions and notations is given in [27, 28]

Noether's theorem in this case leads to the conservation equation

$$\partial_m \mathcal{T}^m{}_b - \frac{\delta \mathcal{L}_{\text{matter}}}{\delta G^A} G^A{}_{,b} = 0\,, \tag{7.6}$$

which holds on the solutions of the matter field equations $\delta\mathcal{L}_{\text{matter}}/\delta\Phi = 0$ and follows directly from the behaviour of the Lagrangian density $\mathcal{L}_{\text{matter}}$ under infinitesimal diffeomorphisms. As one result of their study, Gotay and Marsden also reveal the connection between the Hilbert energy-momentum tensor density (7.2) and the canonical energy-momentum tensor (7.3) density. They obtain quite generally that

$$\mathcal{T}^a{}_b = \mathfrak{t}^a{}_b + \text{correction terms} \tag{7.7}$$

and, above all, provide explicit expressions for the correction terms.

The definition of the generalized Hilbert energy-momentum tensor density (7.4) reveals a salient point in generalized gravitational dynamics. The energy-momentum tensor density is always a $(1,1)$-tensor density. Moreover, it only sources parts of the gravitational field equations. For suppose we add to a matter Lagrangian density $\mathcal{L}_{\text{matter}}$ the gravitational Lagrangian density $\mathcal{L}_G$ describing the dynamics of the geometry G. Then the gravitational field equations are given by

$$\frac{\delta \mathcal{L}_G}{\delta G^A} + \frac{\delta \mathcal{L}_{\text{matter}}}{\delta G^A} = 0\,, \tag{7.8}$$

but only when contracting both sides of this equation by the coefficient $C^{A\,a}{}_b$, does the energy momentum tensor appear on the right hand side. Of course, this does not pose a problem of principle, because this only says that the gravitational field equations are not only sourced by what we interpret as energy, stress and momentum of the matter field, but by more information about the matter field, namely that encoded in the tensor $\delta\mathcal{L}_{\text{matter}}/\delta G^A$.

In the following sections, we discuss for several examples the generalized Hilbert formula (7.4) and the conservation law (7.6).

7.2 Matter sources for metric gravity

Reassuringly, the Gotay-Marsden energy-momentum tensor density (7.4) and the conservation law (7.6) reduce to the standard expressions in the case of matter theories on Lorentzian manifolds (M, g) where we take $G^A = g^{ab}$. From the Lie-derivative of the inverse metric, $(L_\xi g)^{ab} = \xi^m g^{ab}{}_{,m} - 2g^{m(a}\xi^{b)}{}_{,m}$, one obtains that $C^{mn\,a}{}_b = -2g^{a(m}\delta^{n)}_b$. Inserting this coefficient into the generalized Hilbert formula (7.4) for any matter Lagrangian $\mathcal{L}_{\text{matter}}[\Phi, g]$, which non-derivatively couples some matter fields Φ to the metric g, and lowering the index on the tensor density $\mathcal{T}^a{}_b$

with the help of the metric, one recovers the standard Hilbert tensor density (7.2). Furthermore, it is easy to show that the conservation law (7.6) in this case is equivalent to

$$\nabla_m \mathcal{T}^m{}_b = 0\,, \tag{7.9}$$

since $g^{mn}{}_{,b} = -2g^{p(m}\Gamma^{n)}_{pb}$ in terms of the Levi-Civita connection Γ of the metric g. For Maxwell electrodynamics on a Lorentzian manifold, given in terms of the Lagrangian density

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4}\sqrt{|g|}g^{ac}g^{bd}F_{ab}F_{cd}\,, \tag{7.10}$$

one obtains

$$\mathcal{T}^a{}_b = -\sqrt{|g|}\left(\frac{1}{4}\delta^a_b F_{mn}F^{mn} + F^{an}F_{nb}\right)\,. \tag{7.11}$$

Apart from this standard example, both (7.4) and (7.6) hold in much more general situations. We will now apply these findings in the context of some other tensorial geometries we considered throughout this work. We start with the energy momentum tensor of general linear electrodynamics as the straightforward generalization of Maxwell electrodynamics.

7.3 Energy momentum tensor of general linear electrodynamics

We now calculate the energy momentum tensor density of general linear electrodynamics in four dimensions. We use the Lagrangian density

$$\mathcal{L} = -\frac{1}{4\Psi}G^{abcd}F_{ab}F_{cd}\,, \tag{7.12}$$

with $\Psi = 1/24\,\epsilon_{abcd}G^{abcd}$. This time, the coefficients needed for the energy momentum tensor are given by

$$C^{mnpq\,a}{}_b = 2\,\delta_b^{[m}\,G^{n]apq} + 2\,\delta_b^{[p|}\,G^{mn|q]a}, \tag{7.13}$$

which can be read off the explicit form of $(L_\xi G)^{mnpq}$. A straightforward calculation of the Gotay-Marsden energy-momentum tensor density then shows that

$$\mathcal{T}^a{}_b = -\frac{1}{\Psi}\left[G^{anpq}F_{nb}F_{pq} + \frac{1}{24\Psi}\epsilon_{bnpq}G^{anpq}G^{rstu}F_{rs}F_{tu}\right]\,. \tag{7.14}$$

The second part can be further simplified, since we assumed the area metric tensor to be non-cyclic, $G^{abcd} = G_C^{abcd} + \Psi\epsilon^{abcd}$, which implies that $\epsilon_{bnpq}G^{anpq} = 6\Psi\,\delta^a_b$. This yields the final result for the Gotay-Marsden energy-momentum tensor density

$$\mathcal{T}^a{}_b = -\frac{1}{\Psi}\left[\frac{1}{4}\,\delta^a_b\,G^{mnpq}F_{mn}F_{pq} + G^{anpq}F_{nb}F_{pq}\right] \tag{7.15}$$

of general linear electrodynamics. Moreover, another straightforward calculation shows that the conservation equation (7.6) is indeed satisfied on-shell, i.e., on all field configurations that satisfy both field equations

$$\partial_a\left[\frac{1}{\Psi}G^{ampq}F_{pq}\right] = 0 \quad \text{and} \quad \partial_{[a}F_{bc]} = 0\,. \tag{7.16}$$

Interestingly, the energy momentum tensor of general linear electrodynamics has the property of being tracefree $\mathcal{T}^a{}_a = 0$ in four dimensions. In standard Maxwell theory, this can be traced back to the invariance of the field theory under conformal transformations. In our setting the notion of conformal transformations, however, must also be generalized. A conformal transformation of an area metric tensor must then be given by $\tilde{G}^{abcd}(x) = \Omega^2(x)G^{abcd}(x)$ with some scalar function $\Omega(x)$, so it preserves the angles between oriented areas instead of vectors., but also yields the same principal tensor field P, see [57].

Since the Lagrangian (7.12) is only valid in four dimensions, we cannot check whether the tracelessness is not also a feature of the theory in higher dimensions. Nevertheless, there exists a different Lagrangian for general linear electrodynamics that can be used in arbitrary dimension d using the determinant of the area metric (considering the latter as a bilinear form on the space of two forms) as a scalar density instead of Ψ [64]:

$$L = -\frac{1}{4}\,|\det G_{abcd}|^{1/(2d-2)}\,G^{mnpq}F_{mn}F_{pq}\,. \tag{7.17}$$

Following the same steps as above, the trace of the Gotay-Marsden energy-momentum tensor density can then be shown to take the form

$$\mathcal{T}^a{}_a = -|\det G_{abcd}|^{1/(2d-2)}\,G^{mnpq}F_{mn}F_{pq}\left(1 - \frac{d}{4}\right)\,, \tag{7.18}$$

so that, also in this case, the trace only vanishes for $d = 4$ as expected.

In the next section, we calculate the Gotay-Marsden energy-momentum tensor of point particles on generic bi-hyperbolic and energy-distinguishing point particle geometries.

7.4 Point particle sources

We would now like to consider a massive point particle propagating on a spacetime M of arbitrary dimension, whose geometry is described by a bi-hyperbolic and energy-distinguishing tensor field $P^{a_1 \dots a_{\deg P}}$. The action of such a massive point particle with worldline $x(\tau)$ and mass m was given in (2.19). However, this action is not in the form needed for a calculation of the Gotay-Marsden energy momentum tensor density (7.4) because it is not the integral over a $(\dim M)$-dimensional scalar density. Nevertheless, we can trivially rewrite the point particle Lagrangian in such a form by introducing a delta function integral. Hence, the Lagrangian density $\mathcal{L}$ of a massive point particle simply becomes

$$\mathcal{L}(y)[x,P] = \int_{\mathbb{R}} d\tau \, m \, \Big[P(L^{-1}(\dot{x}(\tau)))\Big]^{-1/\deg P} \delta_{x(\tau)}(y) \, , \tag{7.19}$$

where L denotes the Legendre map (2.17) and L^{-1} its inverse. The coefficients C in the definition (7.4) of the energy momentum tensor density are readily obtained as

$$C^{a_1 \dots a_{\deg P}\, c}{}_d = -(\deg P) \, P^{c(a_1 \dots a_{\deg P - 1}} \, \delta_d^{a_{\deg P})} , \tag{7.20}$$

so that it only remains to calculate $\delta\mathcal{L}/\delta P^{a_1 \dots a_{\deg P}}$ in order to find $\mathcal{T}^a{}_b$. First of all, we obtain from the homogeneity of the principal polynomial that

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta P^{a_1 \dots a_{\deg P}}} &= -\frac{m}{\deg P} \int_{\mathbb{R}} d\tau \, \delta_{x(\tau)}(y) \, \Big[P(L^{-1}(\dot{x}(\tau)))\Big]^{-(1+\deg P)/\deg P} \times \\ \times \Bigg[L^{-1}_{a_1}(\dot{x}) \dots L^{-1}_{a_{\deg P}}(\dot{x}) &+ (\deg P) \, P^{m_1 \dots m_{\deg P}} \, L^{-1}_{m_2}(\dot{x}) \dots L^{-1}_{m_{\deg P}}(\dot{x}) \, \frac{\partial L^{-1}_{m_1}(\dot{x})}{\partial P^{a_1 \dots a_{\deg P}}} \Bigg] . \end{aligned} \tag{7.21}$$

We now take care of the second term in the second line. We obviously need the derivative of the inverse Legendre map with respect to the tensor P, which follows from the identity $L^m(L^{-1}(\dot{x})) = \dot{x}^m$ by taking the derivative with respect to P on both sides, which leads to the equation

$$\begin{aligned} & (DDf)^{mn}(L^{-1}(\dot{x})) \, \frac{\partial L^{-1}_n(\dot{x})}{\partial P^{a_1 \dots a_{\deg P}}} \\ = \; & \frac{\delta^m_{(a_1} L^{-1}_{a_2}(\dot{x}) \dots L^{-1}_{a_{\deg P})}(\dot{x})}{P(L^{-1}(\dot{x}))} - \dot{x}^m \, \frac{L^{-1}_{a_1}(\dot{x}) \dots L^{-1}_{a_{\deg P}}(\dot{x})}{P(L^{-1}(\dot{x}))} \, , \end{aligned} \tag{7.22}$$

where the function $f(q) = (-1/\deg P) \ln P(q)$ is only defined on the positive energy hyperbolicity cones C_x of the polynomial defined by the tensor field P, and DDf denotes its second fibre derivative. It was shown in [58] that DDf is invertible for bi-hyperbolic and energy-distinguishing geometries P. The inverse $(DDf^L)_{ab}$ uses the Legendre dual function f^L of f, for which we have that $Df^L(x)(v) = -L^{-1}(v)$ for all vectors v in the image of the Legendre map L. Solving equation (7.22) for $\partial L^{-1}/\partial P^{a_1 \dots a_{\deg P}}$ and inserting the result back into equation (7.21), one can then show that the second term in the second line of the latter actually vanishes, because the inverse Legendre map L^{-1} is a homogeneous function of degree -1.

Thus, using the definition of the Legendre map L, the Gotay-Marsden energy-momentum tensor density of a massive point particle takes the simple form

$$\mathcal{T}^a{}_b(y) = m \int_{\mathbb{R}} d\tau P(L^{-1}(\dot{x}))^{-1/\deg P} \dot{x}^a L^{-1}_b(\dot{x})\, \delta_{x(\tau)}(y)\,. \tag{7.23}$$

Since the energy-momentum tensor density is a distribution this time, one evaluates the conservation law (7.6) by integrating the entire equation over a test function, which is supported on a tube-shaped region around the worldline $x(\tau)$ of the point particle. Using for τ the proper time parameter along the curve, the conservation law (7.6) is then equivalent to

$$\dot{x}^a \partial_a L^{-1}_b(\dot{x}) - \dot{x}^a \partial_b L^{-1}_a(\dot{x}) = 0\,, \tag{7.24}$$

which can be rewritten to yield the Finslerian geodesic equation

$$\ddot{x}^b g_{ab}(x,\dot{x}) + \dot{x}^m \dot{x}^n \partial_n g_{am}(x,\dot{x}) - \frac{1}{2}\dot{x}^m \dot{x}^n \partial_a g_{mn}(x,\dot{x}) = 0 \tag{7.25}$$

by using the Finsler metric $g_{am}(x,\dot{x})$ from (2.22) and the identiy $L^{-1}_a(x,\dot{x}) = g_{am}(x,\dot{x})\dot{x}^m$. This reproduces the equations of motion one obtains from variation of the massive point particle action (2.19) with respect to the worldline $x(\tau)$. Hence, the energy-momentum tensor of the massive point particle is conserved on the solutions to the equations of motion for any point particle geometry P.

Finally, since the energy-momentum tensor is evaluated on solutions $x(\tau)$, we may use coordinates $(t, \vec{x})$, with dt being an element of the hyperbolicity cone C to which also $L^{-1}(\dot{x}(\tau))$ belongs, and invert $t(\tau)$ to perform the integral over τ with the help of the delta function. Choosing τ to be the proper time parameter, we can then bring the energy momentum tensor density into the form

$$\mathcal{T}^a{}_b(t,\vec{y}) = mL^{-1}_b(\dot{x})\,\dot{x}^a \frac{d\tau}{dt}\,\delta_{\vec{x}(\tau(t))}(\vec{y})\,. \tag{7.26}$$

We note that $mL^{-1}_b(\dot{x})$ are the components of the four momentum of the point particle. This is the generalization of the well known result from general relativity where $L^{-1}_b(\dot{x}) = g_{ab}\dot{x}^a$ in proper time parametrization.

7.5 Pressureless dust

In this last example for matter sources, we generalize Brown and Kuchař's description of an incoherent dust as a source of the gravitational field [12] to bi-hyperbolic and energy-distinguishing spacetimes (M, P) where the point particle geometry $P^{a_1 \dots a_{\deg P}}$ has any arbitrary even degree. Incoherent dust has attracted particular attention in the context of quantum gravity, where it can be used as a reference system that helps to explicitly solve the constraints of the theory before quantization. This may circumvent the problem of finding implementations of the constraint equations as operator equations in the quantum theory in order to identify the physical Hilbert space of the geometric degrees of freedom [25, 41]. Moreover, pressureless dust is of course also one of the relevant matter sources in cosmological applications.

We will see in the following that pressureless dust on a spacetime (M, P) can be described by the Lagrangian density

$$\mathcal{L}_{\text{dust}}[T, W_k, Z^k, M, P] = -\frac{\mu}{\deg P} \ln P(U)\,, \tag{7.27}$$

where the covector field $U_a = T_{,a} + W_k Z^k{}_{,a}$ denotes the momentum field of the dust particles which are described by the scalar functions T, W_k and Z^k (with the label $k = 1, \dots, \dim M - 1$, and we must have that $\det Z^K{}_{,a} \neq 0$ for $Z^K = (T, Z^k)$) and the scalar weight-one density μ. The interpretation of these variables can only be revealed by the field equations that follow from this Lagrangian, which we will do in the following. Variation with respect to these variables yields the field equations

$$0 = \frac{\delta \mathcal{L}_{\text{dust}}}{\delta \mu} = -\frac{1}{\deg P} \ln P(U)\,, \tag{7.28}$$

$$0 = \frac{\delta \mathcal{L}_{\text{dust}}}{\delta W_k} = -\mu\, U^a Z^k{}_{,a}\,, \tag{7.29}$$

$$0 = \frac{\delta \mathcal{L}_{\text{dust}}}{\delta T} = (\mu\, U^a)_{,a}\,, \tag{7.30}$$

$$0 = \frac{\delta \mathcal{L}_{\text{dust}}}{\delta Z^k} = (\mu\, W_k U^a)_{,a}\,, \tag{7.31}$$

where $U^a = L^a(U)$ is the image of the dust momenta U_a under the Legendre map (2.17) induced by the geometry P. We chose the particular form of the Lagrangian (7.27) because we needed to ensure the existence of a Legendre map. Now, the first equation of motion implies that the U_a have unit norm with respect to the tensor field P and the second equation of motion implies that the scalar fields Z^k are conserved along the flow of the vector field U^a. Combining the first two equations we learn that $U^a T_{,a} = 1$, so that the scalar field T changes at a constant rate along the flow of U^a, which justifies to call T the proper time function of the dust particles.

Combining the last two equations shows that also the scalar fields W_k are conserved along the flow of U^a. With the help of the field equations, it is then easy to show that the covector $T_{,a}$ and the $(\dim M - 1)$ covectors $Z^k{}_{,a}$ form a basis of the cotangent space T_x^*M at every point x of the manifold M, which is also constant under the flow of the vector field U. This can be seen by calculating the Lie derivative of those covector fields with respect to the vector field U which vanishes in both cases. Hence, the dust variables define a reference frame at every point of the manifold. The last two field equations also imply that the currents

$$J^a = \mu\, U^a \quad \text{and} \quad J^a_k = \mu\, W_k U^a \tag{7.32}$$

are conserved. Thus, the scalar density μ can be interpreted as the local mass density of the dust particles. For more details on the interpretation of this theory in the metric context, we refer the reader to [12], and now proceed with the calculation of the Gotay-Marsden energy-momentum tensor density of the theory. The coefficients $C^{A\,a}{}_b$ required for its calculation are adopted from (7.20) and thus we find the Gotay-Marsden energy-momentum tensor density of incoherent dust,

$$\mathcal{T}^a{}_b = \mu\, U^a U_b\,. \tag{7.33}$$

Again, it can be checked with the help of the field equations that the conservation law (7.6) is satisfied. Moreover, in analogy to the case of a single point particle from the previous section, this conservation law implies, together with the conservation of the mass current $J^a = \mu\, U^a$, that

$$U^a\, U_{[b,a]} = 0\,, \tag{7.34}$$

which is equivalent to the autoparallel equation $(\nabla_U U)^a = 0$ for the velocity vectors U^a using the covariant derivative ∇ of the Chern-Rund connection $\Gamma(U)$ for the cases $\deg P \geq 4$ (see, again, section IX. in [58], which summarizes how to cast the geodesic equation into an autoparallel equation), which for $\deg P = 2$ reduces to the standard Levi-Civita connection of the metric P^{ab}. This finally justifies the term pressureless dust for this kind of matter theory since the dust particles move on autoparallels and, thus, do not interact via internal forces.

Using the Lagrangian (7.27) one may thus extend the notion of pressureless dust to arbitrary point particle geometries P that are bi-hyperbolic and energy-distinguishing.

Chapter 8

Conclusions

In this thesis, we derived a set of gravitational master equations, which, once solved, determine the gravitational dynamics of any tensorial spacetime, i.e., any pair (M, G), where M is a finite-dimensional smooth manifold and G a smooth tensor field, which supports the predictable and quantizable dynamics of matter fields one wishes to populate the spacetime with. With this result, all modifications of general relativity, whose geometry can still be described by tensorial data on smooth manifolds, have been brought under good mathematical control. Thus now—using the results of this thesis—one can *calculate* instead of having to *postulate* the gravitational dynamics of any tensorial spacetime. The only, but physically essential ingredient required to do so is the specification of the dynamics of any matter for which the to-be-calculated gravitational dynamics is supposed to provide the background geometry. These results were achieved in three steps.

First, we assumed that the spacetime geometry G is already known everywhere on the manifold M and analysed the kinematics of observer-accessible initial data hypersurfaces. The deciding element for this analysis was the Legendre map, which is constructed from the principal tensor field $P(G)$ of the matter field equations one has to specify. The Legendre map allowed us to map the naturally given normal co-directions of initial data hypersurfaces into the corresponding normal directions, which in turn enabled us to study the change of the hypersurface geometry induced by the spacetime geometry G on some initial data hypersurface if this hypersurface was deformed along its normal directions. This change of the hypersurface geometry was described in terms of linear differential operators, and we calculated their commutation relations, which led to the hypersurface deformation algebra for any tensorial spacetime.

Secondly, we required that the spacetime geometry be generated from initial data on a single initial data hypersurface by equations of motion, such that, stacking the evolved data together, one recovers a tensorial spacetime geometry G that respects the kinematics of hypersurface deformations. We showed that this compatibility requirement implies that the Hamiltonian, which describes the dynamics of the hypersurface geometry on a geometric phase space, is composed of two parts, the superhamiltonian and supermomentum, which satisfy a Poisson algebra of the same form as the hypersurface deformation algebra. Loosely speaking, the dynamics for the spacetime geometry G must be given by a 'representation' of the hypersurface deformation algebra on a geometric phase space.

Thirdly, we reformulated the Poisson algebra into a linear functional differential equation for a Lagrangian describing the gravitational dynamics of the hypersurface data. We then turned the linear functional differential equation for the Lagrangian into a countable set of linear partial differential equations, the gravitational master equations, for the coefficients of a series expansion of the Lagrangian in the velocities of the hypersurface geometry.

As a first consequence of the master equations, we showed that the zeroth order coefficient in the Lagrangian can depend on at most the third partial derivatives of the tensorial hypersurface geometry, while all higher expansion coefficients can depend on at most the second partial derivatives of the hypersurface geometry. In addition to the master equations, the expansion coefficients must satisfy another set of linear differential equations, the invariance equations, which express the behaviour of the coefficients under hypersurface diffeomorphisms. We argued that the invariance equations can be solved if (but by no means only if) one can formally employ one of tensor fields of the hypersurface geometry as a hypersurface metric, and we were able to provide their explicit solution in this case.

We discussed four examples of tensorial spacetimes, each interesting for another reason. As a litmus test of our entire construction, we rederived general relativity from the master equations for a four-dimensional metric manifold (M, g) with Maxwell elecrodynamics as its matter content. We showed that, indeed, one recovers, as the unique solution solution to the master equations, the ADM form of the dynamics of general relativity, with the cosmological constant and Newton's constant arising as integration constants. As a second, now entirely new, example, we discussed area metric manifolds (M, G) equipped with general linear electrodynamics as matter fields. We derived the corresponding master equations, whose solution however was not attempted. That solutions to the master equations for a non-trivial extension to general relativity however can be determined was then shown for the toy model of a spacetime (M, g, W), whose geometry constists of a metric g and a vector field W, and whose matter content is given by modified Dirac dynamics. We completely solved the master equations for this case in four dimensions, which led to a whole family of possible gravity theories. This is the first non-metric example of tensorial spacetime dynamics. Finally, we derived the master equations for all tensorial spacetimes that describe the geometry of modified dispersion relations for massless and massive point particles.

In the last chapter, we considered matter sources for tensorial spacetimes. We investigated these matter sources in terms of the Gotay-Marsden energy-momentum tensor densities they give rise to in terms of several examples, ranging from electrodynamics on metric and area metric spacetimes to incoherent dust particles with modified dispersion relations.

The results presented in this thesis now provide excellent control over modified matter theories and their unavoidable effect of leading to modified gravity theories. In particular, as we examplified for the dynamics of the background geometry supporting modified Dirac matter in section 6.4, our approach to modified spacetime geometry dynamics is remarkably restrictive despite the very weak and conservative assumptions we made. This shows how important it is to take into account the very intimate relationship of matter dynamics on the one hand and its supporting gravitational dynamics on the other hand. Even very innocent looking deviations in the matter sector can have (and most probably: will have) severe consequences for the dynamics of the geometric sector of the combined theory. This tightly constrains (and, in fact, forbids) the possibility of postulating both aspects, the matter dynamics and the dynamics of the underlying geometry, of a physically viable spacetime theory completely independent of each other.

Several problems can now be tackled as they now take the form of well-defined mathematical questions. For example, the apparent and truly long-standing physical question of the existence and uniqueness of a dynamical theory for the geometry underlying given matter dynamics must be answered by proving existence and uniqueness of solutions of the corresponding gravitational master equations.

But even if one is not able to find a solution to the full gravitational master equations for some candidate geometry, it poses no problem to modify our techniques to obtain a set of approximate master equations in symmetry reduced cases, which will then be easier to solve. After calculating the supermomentum and the non-local part of the superhamiltonian for the full theory, one must simply perform a symmetry reduction of the geometric degrees of freedom and then derive, in the same fashion as demonstrated in this thesis, the gravitational master equations for the yet-to-be-determined symmetry reduced local part of the superhamiltonian. Attempts in this direction have recently led to interesting results in the context of modifications of the Poisson constraint algebra from loop quantum gravity corrections [9]. Of course, performing such a symmetry reduction most likely comes at the cost of enlarging the space of solutions of the symmetry reduced master equations in comparison to the space of solutions to the full master equations. Taking the results of Palais [49] into account (see also [69]), which teach us that the symmetry reduction of an action functional and the variation of the latter do not necessarily commute, one has to, however, treat the resulting solutions of such approximate gravitational master equations, and the dynamics they describe, with particular care.

On the kinematical level, the analysis of the energy-momentum tensors has the potential to further constrain admissible tensorial spacetime geometries by imposing, in addition to predictivity and quantizability, appropriate physical energy conditions on the considered matter field dynamics (in analogy to the weak, strong and dominant energy conditions known from general relativity).

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Selbständigkeitserklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet habe.

Berlin, den

Christof Witte